The Dissertation Committee for Sreelata Jonnalagedda
certifies that this is the approved version of the following dissertation:

Product Strategies under Durability, Lock-in and Assortment Considerations

Committee:

Stephen Gilbert, Supervisor

Dorothee Honhon

Vijay Mahajan

Anant Balakrishnan

Genaro Gutierrez

Product Strategies under Durability, Lock-in and Assortment Considerations

by

Sreelata Jonnalagedda, B.Tech.; M.S

DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN

December 2009

Chaitra , *Advait*: the most beautiful *Propositions* of my life

Acknowledgments

My mother inspired me to always push myself and reach out for more than my imagination could conceive. She always let my choices take precedence over hers, supported me through every endeavor; she will always be my source of strength. My father, with an open mind and an eagerness to learn, showed me what influential teachers are made of. I hope that I would be an eternally young student and an inspiring teacher, just like him. Professor Steve Gilbert, my advisor, is an exemplary researcher, a nurturing mentor, and a steadfast supporter of his students. His influence on me as a scholar will go well beyond this dissertation, I benefited and matured immensely as a researcher under his mentorship.

Thanks are due to Professor Dorothee Honhon for leading me through our research project together, her exuberance and cheer has been highly infectious. My deepest gratitude to Professor Vijay Mahajan for his encouragement in marketing myself in the academic world. I would also like to thank Professor Anant Balakrishnan and Professor Genaro Gutierrez for being great sources of support during my PhD program.

I am fortunate to have the support of my mother and father in-law, Bharathi and Sadanand; their readiness to help when I needed it most allowed me to focus on my research. I would like to thank my brother Raghu for cheering me along in this marathon of an effort, and all the people who made my life as a PhD student and as an Austinite very enjoyable. In particular, Sreekumar and Deepa, the cool Longhorns who lured me to Austin; Harpal Singh for warning me of the perils of going to graduate school; Amy for sharing the pleasures and pains of being a PhD mom; Saurabh Bansal for being a great classmate, neighbor, and the best baby sitter in Austin, and my neighbors, Sarah and Simran, for sharing their experiences with the kids and in the kitchen.

Chaitra and Advait, the little imps, ensured that I had an exciting life outside of school; they played a crucial part in my well being as a doctoral student. The last but never the least: Shilpi, for pretending to be terribly bored during my numerous lectures on time inconsistency; for a person who always encouraged me to look *within* for inspiration and strength, he is the one person I could not have done *without.*

Product Strategies under Durability, Lock-in and Assortment Considerations

Publication No. ______________

Sreelata Jonnalagedda, Ph.D.
The University of Texas at Austin, 2009

Supervisor: Stephen Gilbert

In this dissertation I focus on two considerations that influence the product strategy of a firm. The first is consumers' choice and its influence on a firm's product offering, and the second is the interaction between durable products and their contingent consumables. First, I study the assortment planning problem for a firm; I illustrate the complexity of solving this product selection problem, present simple solutions for some commonly used choice models, and develop heuristics for other practically motivated models. Second, I study the incentives of a durable goods monopolist when she can lock-in consumers through a contingent consumable. Adopting a lock-in strategy has two interesting effects on the incentives of a durable goods manufacturer. On one hand, by locking-in consumers to its consumable, a durable goods monopolist can curb its temptation to reduce durable prices over time, thereby mitigating the classic time inconsistency problem. On the other hand, lock-in will create a hold-up issue and adversely affect

consumers' expectations of future prices for the consumable. My research demonstrates the trade-off between time inconsistency and hold-up, and derives insights about the conditions under which a lock-in strategy can be effective. I further analyze the trade-off between time inconsistency and hold-up associated with lock-in in the presence of consumable stock-piling. My findings indicate in the presence of consumer stock-piling, lock-in has an effect similar to that of competition in the consumables market: they help to dampen the hold-up problem that arises from lock-in and at the same time increase the manufacturer's incentive to reduce durable prices over time.

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Chapter 1

Executive Summary

The value realized by a firm from selling its products is determined to a large extent by the strategy that is used to bring the products to market (Chesbrough (2003)). Firms that launched successful products such as Apple (iPod) and Microsoft (Xbox) have been credited with adopting the right product strategy for their products (Wall Street Journal (2006)) . Therefore it is important for firms to consider implications of the product strategies they adopt. In my research, I focus on two interactions that have economic consequences for a firm; The first is how consumers make choices and its influence on the optimal product assortments, and the second is the interaction between durable products and their contingent consumables.

Choosing the right assortment is an important consideration for a firm as this determines to a large extent a firm's patronage and therefore her profits. Indeed, it is also a very difficult decision for firms because of the complexity associated with consumer's choice. To understand the interaction between consumer choice and product selection issues for a firm, we model customer preferences through the definition of consumer types, where a type is a ranking of the potential products by order of preference.

A customer purchases the highest ranked product (if any) offered in the assortment. Products also differ in their cost and price parameters so they have different levels of profitability for the firm. First, we consider a general consumer choice model and show that there is very little structure to the optimal assortment(s). We find that one should always offer the most profitable product, as measured by the profit margin. Second, by placing practically motivated restrictions on the general choice model (such as the Multinomial Logit model, the locational choice model or the Markovian second choice model), we obtain an efficient method to determine the optimal assortment for several of them. In some cases the problem of finding an optimal assortment can be likened to solving a shortest path problem or a dynamic program. In others cases, we show that the optimal assortment(s) contain(s) a certain number of the most profitable products. Finally, we suggest a number of heuristics and test their performance numerically in the two models for which we do not have an efficient method for obtaining an optimal solution. We find that the best two are greedy-type heuristics, in which products are added to (or removed from) the assortment in decreasing order of their impact on expected profit.

When consumers are forward looking or strategic, they trade-off the value of purchasing the good now to the expected future value. For example, when consumers make purchase decisions on durables such as an automobile, they are more sensitive to the anticipated future price of the good than for products they purchase for instantaneous consumption. This trade-off faced by strategic consumers, has practical implications for the durable good manufacturers in choosing a suitable business model to adopt. In this dissertation, we specifically look at the interaction between the effect of durability and the firms ability to lock-in through a contingent consumable, and the implications of lock-in for durable good manufacturers. Many durable products cannot be used without a contingent consumable product, e.g. printers require ink, iPods require songs, razors require blades, etc. For such products, manufacturers may be able to lock-in consumers by making their product incompatible with consumables that are produced by other firms. We examine the effectiveness of such a strategy in the presence of strategic consumers who anticipate the future prices of both the durable product and the contingent consumable. On the one hand, by locking-in consumers to its own contingent consumable, a durable goods manufacturer can dampen its own incentive to reduce durables prices over time, thereby mitigating the classic time inconsistency problem. On the other hand, lock-in will also create a hold-up issue and will adversely affect consumers' expectations of future prices for the contingent consumable. We demonstrate the trade-off between these two issues, time inconsistency and hold-up, and derive analytical results that provide insights about the conditions under which a lock-in strategy can be effective. We also analyze the effect of competition in the contingent product market. We find that, at the same time that competition reduces the hold-up problem, presence of a large number of them erodes the firm's revenue from the contingent consumables, increasing her incentive to reduce prices on the durable. Thus, high levels of competition in the consumable market can worsen the time inconsistency issue.

When the use that a consumer can obtain from a durable is linked to the consumption of a contingent consumable, consumers are clearly concerned about the durable product manufacturer's incentive to *i)* reduce the price of durables over time (time inconsistency), and *ii)* restrict the availability of consumables in the future (lock-in). Both these concerns affect consumers decisions to purchase the durable; in particular, for consumers, lock-in potentially creates a *hold-up* problem with respect to the consumables. In anticipation of this *hold-up* problem, consumers will have an incentive to stock-pile consumables for future use. In this dissertation, we also explore the effect of stock-pile of consumables on the interaction between time inconsistency and hold-up. We find that in the presence of stock-piling, lock-in has an effect that can be likened to the presence of competition in the consumables market.

The rest of the document is organized as follows. In Chapter 2, I review the literature in assortment planning and durable products that is relevant to my work. In Chapters 3, I develop and solve a firm's assortment planning problem under a general consumer choice model. In Chapter 4, I develop a model to capture the interaction between time inconsistency and lock-in for a durable product firm that adopts a lock-in policy. In Chapter 5, I analyze the effect of stock-piling on the interaction between time inconsistency and hold-up associated with lock-in. In Chapter 6, I discuss directions for future research.

Chapter 2

Literature Review

My dissertation examines two important factors that influence the product strategy of a firm; Chapter 3 focuses on the first consideration: ′consumers′ choice and its influence on the optimal product assortments′. Chapters 4 and 5 focus on the second consideration: ′the interaction between durable products and their contingent consumables′. Below, I review the literature that is relevant to my work in *i)* Chapter 3 on assortment planning, and *ii)* in Chapters 4 and 5 on strategic considerations for a durable product firm.

2.1. Assortment Planning

Assortment planning problems generally look at a retailer who takes the position and price of the products as given and has to decide on which products to include in her assortment from a finite list of products offered by different manufacturers. In solving for the optimal assortment, researchers have adopted one of the following two assumptions as a basis for how consumers substitute from among offered products , *i)* assortment based substitution also called static substitution and *ii)* stock-out based substitution or dynamic substitution. As with our research, much of the literature on assortment planning has focused on static substitution models and I will review the relevant papers below. Identifying the structure of the optimal assortment under dynamic substitution is considerably harder ; however there a few papers such as Smith and Agrawal (2000), Mahajan and van Ryzin (2001) , Hopp and Xu (2007) and Honhon et al. (2009) , that have made strides in this area.

As our study considers assortment-based or static substitution, we review the related literature in detail. In the realm of static substitution, researchers have mainly focused on stylized models that allow for understanding the structural properties of the optimal assortments. For example, van Ryzin and Mahajan (1999) and Li (2007), adopt a Multinomial Logit model and show that the optimal assortment contains most popular products and most profitable products respectively; Gaur and Honhon (2006) show that under a locational choice model it is not necessary for the firm to offer the most popular products.

The above work identifies how the optimal assortment relates to the assumptions on how consumers make choices; in doing so, a majority of them ignore inventory considerations, and except for Li (2007), all of the papers discussed above assume that products have identical price and cost structure. However, there are a few studies that take into account inventory considerations; for example, Netessine and Rudi (2003) and Bassok et al. (1999) study the assortment planning problem under downward substitution and inventory considerations; and a few studies for example, Hopp and Xu. (2005) that allow for endogeneous pricing. Hopp and Xu. (2005) study the joint assortment planning and pricing decisions under the Multinomial Logit model and show that optimal assortment is composed of variants with equal mark-ups. Kok et al. (2008) give a comprehensive literature review on assortment planning under considerations such as consumer search costs, competition, basket shoppers, dynamic assortment rotation etc.

In our paper we consider a make-to-order setting and look at a price-taking firm who has to choose the best assortment to offer to consumers, that is, the menu of products they can choose from. We use a ranking based consumer choice model where each consumer belongs to one of the many types, where a type is represented by a ranking of all or some of the products that could potentially be offered. Mahajan and van Ryzin (2001) were the first ones to use this model to solve for the optimal assortment under dynamic consumer substitution. In addition to the ranking based preference, we allow for products to have different cost and price characteristics.

Our consumer choice model encompasses a lot of other stylized models used in literature. As we will see, by placing simple restrictions, our model can be reduced to most of models presented in the literature such as Multinomial Logit, Hotelling, Markovian second choice etc. We lose some tractability owing to the structure of our consumer model, our attempt however, is to stretch the boundaries of our knowledge under more general consumer choice models and this is infact one of our primary contributions.

We also develop sub-classes of consumer models, which can be obtained by restrictions on the order in which consumers substitute. Based on the severity of the restrictions, we can obtain methods that are of $O(n)$, $O(n^2)$ and $O(n^4)$. Some of the restricted models can be likened to solving the shortest path of a network. A shortest path based result was also obtained by Alptekinoglu (2004) under a setting in which prices of the products are endogeneous. However, we are among the first ones to obtain the shortest path result is for the situation in which the firm is a price-taker. As a special case of our general model, we solve the assortment planning problem under Markovian second choice as in Smith and Agrawal (2000). While they resort to an integer programming based approach, we provide an alternative algorithm called the In-Out algorithm that works well for Markovian second choice. Although our method is not necessarily computationally superior than the integer programming approach, it is novel and also throws light on the process in which products can be eliminated from consideration. A distinguishing feature of our work is that it is among the first studies that attempts to understand the properties of optimal assortment under a general consumer choice model.

2.2. Durable Products and Strategic Considerations

It is well known that a durable goods monopolist(DGM) loses market power because of her incentive to reduce prices of durables over time in order to increase market share. This is often referred to as time-inconsistency and was first articulated by Coase (1972). Bulow (1982) showed that, by leasing her products, the DGM can eliminate time-inconsistency and make profits comparable to that of non-durable monopolists. The question of leasing vs selling has received a lot of attention in the research on durable products. Bucovetsky and Chilton (1986) show that, although a DGM prefers to rent his durables to maximize profit, under the threat of entry, they employ a mixture of leasing and selling. Desai and Purohit (1998) find that if the units sold by the DGM depreciate faster than leased units, then the DGM is better off under selling. In a competitive setting, Desai and Purohit (1999) find that the proportion of leases is lower for less reliable products. Bhaskaran and Gilbert (2005), show that, in the presence of complementary products, selling may be more favorable for the manufacturer as it helps stimulate demand for complementary products. Bhaskaran and Gilbert (2008) also find that the intensity of competition among intermediaries that sell durables, affects the extent of leasing or selling employed by the DGM.

When leasing is not possible, as is the case for certain durables, extant research looks at credible mechanisms that implicitly or explicitly allow the DGM to make commitments about future production and ameliorate time-inconsistency. DGMs optimally choose to under invest in durability and/or employ an inefficient production technology (Bulow (1986)). Desai et al. (2004) show that, through the introduction of an intermediary it is possible to reduce time inconsistency. Arya and Mittendorf (2006) argue that DGM may benefit from double marginalization problem in the channel under certain contracts, as it naturally restricts output of the durable, thus mitigating time inconsistency.

There are many mechanisms through which vendors can lock-in consumers, we review some of the existing research in this area. A common form of lock-in is seen in markets in which there are consumption externalities. Katz and Shapiro (1985) analyze an oligopoly model in the presence of such externalities and explore the incentives for firms to produce compatible goods. They find that firms with weak networks favor compatibility and firms with large networks or good reputation are against compatibility. Farrell and Saloner (1985) look at the problem of standardization and innovation in the presence of network externalities. They find that when firms have asymmetric preferences for new technology, then a lack of coordination results in excess inertia; where firms are slow to move to new standards even if they agree on the benefits of the change. Farrell and Klemperer (2004), observe that presence of either of the two, switching costs or network effects can lock consumers in and give sellers ex-post market power. They also find that incompatible competition favors incumbent firms and results in a higher efficiency loss than that of compatible competition. It is possible for manufacturers to lock consumers into their product by tying their monopolized good to an unrelated good. When the tied good faces competition, then Whinston (1990) shows that this form of lock-in can make it unprofitable for her rival to operate. However, when the tied good and the tying good are complements, Whinston (1990) recognizes that reducing competition by tying, may not be beneficial to the monopolist, especially if the degree of complementarity between the tied goods is high. Carlton and Waldman (2002) and Choi and Stefanadis (2001) investigate how tying in the early stages of a product's life cycle can influence or deter competition from entering the market.

On the implications for product strategies for firms in the presence of network externalities, Conner (1995) finds that adopting a proprietary product strategy may not always be the best thing for a firm. Sun et al. (2004) also show that the strength of the network effects plays an important role in determining whether firms should adopt product line extensions, lump sum fee, royalty fee or free licensing strategy.

By modeling the interaction between durable products and their contingent consumables, our research captures the trade-off between the adverse affects of time-inconsistency and lock-in. As a significant number of durables are used with consumables, our work has practical implications for durable good manufacturers in deciding the right business model for their product. In Chapters 4 and 5, we focus on the durables that are used with a contingent consumable, primarily focusing on the interaction between time inconsistency which arises from durability of the product, and hold-up which arises from the durable good manufacturer's lock-in strategy. To our knowledge, this research is among the first studies to examine the interaction between time inconsistency and hold-up, and the implications of this interaction for a monopolist manufacturer of durable goods.

In Chapter 5, we extend our examination of the interaction between time inconsistency and hold-up in the presence of consumer stock-piling. In traditional models of operations management, inventory is modeled as a hedge against demand or supply uncertainty. In Chapter 5, we recognize a reason why inventory may be held in the absence of either of these forms of uncertainty. The idea of holding inventory for strategic reasons is an interesting one and has been explored only to a limited extent in the operations management literature, for example by Anand et al. (2008) and Ferguson and Koenigsberg (2007). To my knowledge this will be the first analysis to consider strategic inventories and its effect on time-inconsistency and lock-in, and will be in my view, an important contribution to the literature on strategic inventories as well as durable products.

Chapter 3

Making the most of choice: Product selection under heterogeneous consumer preferences

3.1. Introduction

Online retailers and search firms face a routine and complex task of choosing the right products to display from the multitude they offer, in a way that addresses consumers' needs and also maximizes the profits they can make from their offering. There a number of firms that dedicate their resources to improving the online display of the products in response to a search. For example, product search firms such as Nextag and travel search engines such as Kayak, have to choose the best set of products to display in response to consumers' search requests.

That this product selection process is routine is easy to understand; However, to see that it is also a task that is tremendously complex, let us consider the following example. Consider a group of consumers looking to buy a wok; a customer A, may prefer a product with a wooden handle over the one with a plastic handle over the one with a stainless steel handle. Another customer B, may prefer an anodized wok to a non-stick one to a stainless steel one, there may be a consumer whose preference order is based on price. Of course, it is possible, unlike above examples, that a consumer may have a preference ordering that is not based on any single attribute. As such, the consumer population for a product category may consist of types of consumers who rank the same set of products differently. How consumers choose from a set of products is an important consideration for firms in planning their assortments. It is possible to capture a greater segment of consumer population by making every consumer's first choice available in the product offering; but this prevents consumers from substituting to possibly more profitable items. Another extreme is to offer only the most profitable items but this may lead to reduced patronage. Firms can leverage their knowledge of consumer substitution patterns to strategically choose their assortments in order to divert the right amount of demand to more profitable products.

In this research, our objective is to understand the key influences in a firm's product selection decisions in the presence of consumers who have a heterogeneous preference ordering for different products. In particular, our interest is to investigate the existence of a simple structure for the optimal assortment and devise efficient methods to improve the product selection process for a firm. In order to achieve our objective, we use a consumer choice model based on rank ordering of products that may not be unique to the entire consumer population, and start our investigation by assessing the impact of product profitability and popularity in the process of selecting the optimal assortment.

It is known that the extent of the influence of an individual product's profitability and popularity on the optimal product offering depends on the product characteristics and certainly on the way consumers make choice. For example, a firm is better off by offering its most popular products, when all the products are equally priced and consumer preferences are represented by the Multinomial Logit model (van Ryzin and Mahajan, 1999). However, choosing products based on popularity is not necessarily the best strategy for a firm when either the consumer preferences are as in the Lancaster or Hotelling model, or when all the products have different levels of profitability under a MNL model of consumer choice(Li, 2007); In fact, Li finds that the profitability measure is suitable for choosing the optimal assortment. Given this contrast, it is of interest to further examine how popularity and profitability together determine the optimal product offering.

Our research first sheds light on the influence of profitability and popularity, we will see that when a firm caters to a heteregenous consumer population, most common measures of popularity and the profitability of a product fail to explain the characteristics of the optimal assortment, indicating that their interaction could be quite complex. Deciphering this complexity in product interaction poses the biggest challenge in arriving at an optimal assortment. We will see that when consumers preferences can be characterized by a substitution pattern that is ordered, then we find efficient methods to obtain optimal solution. In the absence of such an ordered structure some of the heuristics we develop improve the efficiency of search for the optimal solution.We find that although, under the general choice model, greedy-type methods fail to find the optimal solution but as heuristics, they perform very well.

3.1.1 Model

We use bold characters to represent vectors and subscript to denote their components, e.g., q_j is the j-th component of vector $\boldsymbol{q}$. Sets and matrices are denoted by capital letters. $|S|$ denotes the cardinality of set S.

Consumer Choice model

We consider a product category consisting of n potential products, indexed 1 to n. Let $= \{1, \ldots, n\}$. Let 0 denote the no-purchase option. Customers are heterogeneous in their choice behavior: each customer belongs to a *consumer type*. A consumer type is a vector of products that every customer of that type is willing to purchase, arranged in decreasing order of preference. For example, a customer of type $(1, 2, 4)$ has product 1 as his first choice, product 2 as his second choice, product 4 as his third choice, and he never buys products 3 and 5 to n. In general, a type $\boldsymbol{\tau}$ is a vector $(\tau_1, ..., \tau_m)$ of product indices such that $\{\tau_1, ..., \tau_m\} \subseteq \mathcal{N}$. Let $\mathcal{T}$ be the set of all possible types. Also, let $\alpha_{\boldsymbol{\tau}}$ be the proportion of customers of type $\boldsymbol{\tau} \in \mathcal{T}$ in the customer population, such that $\sum_{\boldsymbol{\tau} \in \mathcal{T}} \alpha_{\boldsymbol{\tau}} = 1$. Let $\mathcal{T}^+ = \{\tau \in \mathcal{T} : \alpha_{\boldsymbol{\tau}} > 0\}$ be the set of consumer types that exist in the population. We have $|\mathcal{T}^+| \leq |\mathcal{T}| = \sum_{j=0}^{n} C_n^j j! = \sum_{j=0}^{n} \frac{n!}{(n-j)!}$.

This consumer choice model was used previously, in a different context, by Smith and Agrawal (2000). Note that the type of a customer can result from a utility maximization procedure as in Mahajan and van Ryzin (2001). Let $U(x,j)$ be utility assigned by customer x to product j for $j = 1, ..., n$. Without loss of generality we assume that the utility from not purchasing anything is zero, i.e., $U(x,0) = 0$. Let $U(x,[k])$ be the k-th greatest value in $\{U(x,0), U(x,1), ..., U(x,n)\}$, the type of customer x is $(\tau_1, ..., \tau_m)$ if $U(x,\tau_k) = U(x,[k])$ for $k = 1, ..., m$ and $U(x,0) = U(x,[m+1])$.

Most of the consumer choice models used in the operations management literature until now, such as the Multinomial Logit model and the locational choice model, are special cases of this model which are obtained by adding constraints on $(\alpha_{\boldsymbol{\tau}}, \boldsymbol{\tau} \in \mathcal{T})$.

Let $T \subseteq \mathcal{N}$. We define $Z(T)$ to be the $|\mathcal{T}| \times n$ matrix which shows the product that customers of each type buy when faced with consideration set T, i.e., $Z_{\boldsymbol{\tau},j}(T)$ is equal to 1 if customers of type $\boldsymbol{\tau}$ choose product j from set T and zero otherwise. We have:

$$Z_{\boldsymbol{\tau},j}(T) = \begin{cases} 1 & \text{if } j \in T, \exists i : \boldsymbol{\tau_i} = j \text{ and } \tau_1, ..., \tau_{i-1} \notin T, \\ 0 & \text{otw.} \end{cases}$$

Let $R_j(T)$ denote the proportion of customers who would pick j out of consideration set T, i.e.,

$$R_j(T) = \sum_{\boldsymbol{\tau} \in \mathcal{T}^+} Z_{\boldsymbol{\tau},j}(T) \alpha_{\boldsymbol{\tau}}.$$

It follows from the definition of $Z_{\tau,j}(T)$ that $R_j(T) = 0$ if $j \notin T$. However, note that it is possible that $j \in T$ and $R_j(T) = 0$. Let $R_0(T) = 1 - \sum_{j \in T} R_j(T)$ denote the proportion of customers who do not pick anything from consideration set T.

The following properties of the $R_j(\cdot)$ function are useful:

- [(P1)] Adding a product to the consideration set does not increase the proportion of customers who pick an existing product and it does not decrease it by more than the proportion of customers picking the new product: $\forall j \notin T$, $R_i(T) \geq R_i(T \cup \{j\}) \geq R_i(T) - R_j(T \cup \{j\})$ for all $i \in T$,
- [(P2)] Adding a product to the consideration set does not increase the proportion of customers who do not pick anything $\forall j \notin T$, $R_0(T) \geq R_0(T \cup \{j\})$ or equivalently, $\sum_{i \in T \cup \{j\}} R_i(T \cup \{j\}) \geq \sum_{i \in T} R_i(T)$.

Expected profit function

Let r_j be the selling price of product j and c_j be its purchasing cost. We assume that the price and cost parameters are fixed. Let $\pi_j = (r_j - c_j)$ denote the profit margin on product j. Without loss of generality we normalize the expected number of customers who come to the store to one.

Let $S \subseteq \mathcal{N}$ be the assortment chosen by the firm, that is the menu of products customers can choose from. We assume that there is no penalty cost when customers substitute to a product which is not their first choice.

Given an assortment S, the expected demand for product j is equal to $R_j(S)$. Hence, expected profit as a function of S is given by:

$$\mathbb{E}\Pi(S) \quad = \quad \sum_{j \in S} R_j(S)\pi_j. \tag{3.1}$$

The firm's objective is to find an assortment that solves the following problem:

$$\max_{S \subseteq \mathcal{N}} \mathbb{E}\Pi(S).$$

Before we consider the optimization problem, it is useful to analyze the impact of modifying an existing assortment on expected profit. Let $C_j(S)$ be the change in expected profit obtained when adding (removing) product j to (from) set S for $j \notin S$ ($j \in S$). We have $C_j(S) = \mathbb{E}\Pi(S \cup \{j\}) - \mathbb{E}\Pi(S \backslash \{j\})$.

For $j \notin S$ it reduces to:

$$\begin{aligned}
C_j(S) &= \mathbb{E}\Pi(S \cup \{j\}) - \mathbb{E}\Pi(S), \\
&= \sum_{i \in S \cup \{j\}} R_i(S \cup \{j\})\pi_i - \sum_{i \in S} R_i(S)\pi_i, \\
&= \left(R_j(S \cup \{j\}) - \sum_{i \in S} (R_i(S) - R_i(S \cup \{j\})) \right) \pi_j + \\
&\quad \sum_{i \in S} (R_i(S) - R_i(S \cup \{j\})(\pi_j - \pi_i).
\end{aligned} \tag{3.2}$$

In the last expression, the first term is the change in expected profit due to new customers picking j (the *new demand* effect). By (P2), $\big(R_j(S \cup \{j\})\big)$

$-\sum_{i\in S}(R_i(S) - R_i(S\cup\{j\})) \geq 0$. The second term corresponds to the impact on expected profit due to customers switching from product $i \in S$ to j (the *cannibalization* effect). By (P1), $(R_i(S) - R_i(S\cup\{j\})) \geq 0$, but $(\pi_j - \pi_i)$ can be either positive or negative. Hence, while the new demand effect is always positive, the cannibalization effect can be either positive or negative. It follows that adding product j to set S increases expected profit either if (1) the cannibalization effect is positive or if (2) the new demand effect is greater than the cannibalization effect in absolute value.

3.1.2 Results for the General Model

In theory, it is always possible to find an optimal assortment by enumerating the expected profit for all the possible assortments, however, this 'brute force' method is very impractical for large values of n. Hence, we look for an efficient method to find an optimal solution, by efficient we mean a method whose complexity is less than $O(2^n)$. In identifying the optimal assortment, the firm is concerned about two main characteristics of products in the assortment: profitability and popularity; the profitability of product j is measured by its profit margin π_j. Measuring the popularity of product j, independently of the assortment offered, is more difficult. We propose two measures: $R_j(\{j\})$ and $R_j(\mathcal{N})$. The first one is the percentage of customers who choose product j if it is the only option while the second one is the percentage of customer who pick product j from a full assortment $\mathcal{N}$. Note that the two measures of popularity do not usually give the same ranking of the n products. Also, in practice, the products that score high on the profitability measure may not necessarily score high on the popularity measures and vice versa. The following example illustrates this.

Example 1: Let $n = 4$. Suppose $\pi_1 = 10$ $\pi_2 = 8$, $\pi_3 = 4$, $\pi_4 = 2$, $\alpha_{\boldsymbol{\tau}} = \frac{1}{4}$ if $\boldsymbol{\tau} \in \{(4, 1, 2), (4), (2, 4, 1), (4, 3)\}$. The following table shows the measures of popularity for the four products.

	1	2	*3*	*4*
$R_j(\{j\})$	1/2	1/2	1/4	1
$R_j(\mathcal{N})$	0	1/4	0	3/4

We see that the two popularity measures do not give the same rankings and these two rankings are different from the profitability ranking. The following table shows the expected profit associated with each possible assortment.

S	$\mathbb{E}\Pi(S)$	S	$\mathbb{E}\Pi(S)$	S	$\mathbb{E}\Pi(S)$	S	$\mathbb{E}\Pi(S)$
{1}	5	{1,2}	4.5	{2,4}	5	{1,3,4}	4
{2}	4	{1,3}	6.5	{3,4}	4	{2,3,4}	5
{3}	1.5	{1,4}	4	{1,2,3}	5.5	{1,2,3,4}	5
{4}	4	{2,3}	5.5	{1,2,4}	5		

The optimal assortment is $S^* = \{1, 3\}$ *and the optimal expected profit is* 6.5.

Example 1 illustrates that an optimal assortment need not contain the most popular product (product 4 is not in S^*). Notice that product 2 is both more profitable and more popular than product 3, since $\pi_2 > \pi_3$, $R_2(\mathcal{N}) > R_3(\mathcal{N})$ and $R_2(\{2\}) > R_3(\{3\})$ and yet the optimal solution contains product 3. This example also shows that it is not possible to eliminate some products from consideration based on some efficient frontier-type reasoning using the popularity and profitability dimensions.

Proposition 3.1.1. *Let $M = \{j \in \mathcal{N} : \pi_j = \max_{i \in \mathcal{N}} \pi_i\}$. There exists an optimal initial assortment S^* such that $M \subseteq S^*$.*

Proposition 3.1.1 shows that it is always optimal to include the most profitable products in the assortment. In Example 2, we consider a greedy algorithm on $C_j(S)$ where product j with the highest $C_j(S) > 0$ is added to an assortment iteratively and show that it does not necessarily result in an optimal solution. Example 1 and Example 2 illustrate that products interact in a complex manner in our general consumer choice model, so that there is no simple structure to the optimal assortment(s).

Example 2: Let $n = 4$. Suppose $\pi_1 = 4$, $\pi_2 = \pi_3 = 2$ and $\pi_4 = 1$, $\alpha_{\boldsymbol{\tau}} = \frac{1}{4}$ if $\boldsymbol{\tau} \in \{(1), (4,3), (3,2,1), (2)\}$. A greedy algorithm would give $S = \{1,4\}$ as $C_1(\emptyset) = 2 > C_2(\emptyset) = C_3(\emptyset) = 1 > C_4(\emptyset) = 1/4$ and $C_4(\{1\}) = 0.25 > C_2(\{1\}) = C_3(\{1\}) = 0$. We have $\mathbb{E}\Pi(\{1,4\}) = 2.25$. However stocking $\{1,2,3\}$ gives a higher expected profit, i.e. $\mathbb{E}\Pi(\{1,2,3\}) = 2.5$.

Lemma 3.1.2. *Let $T = \mathcal{N} \backslash M$. Let I and O be such that*

$$
\begin{aligned}
I &= M \cup \left\{ j \in T : \min_{\substack{S: j \notin S \\ M \subseteq S \subseteq \mathcal{N}}} C_j(S) \geq 0 \right\} \\
O &= \left\{ j \in T : \max_{\substack{S: j \notin S \\ M \subseteq S \subseteq \mathcal{N}}} C_j(S) \leq 0 \right\}
\end{aligned}
$$

There exists an optimal solution S^* such that $I \subseteq S^* \subseteq (\mathcal{N} \backslash O)$.

In other words, products in I should be included in the solution because these are products that increase expected profit when added to every assortment that contains M and products in O can be removed from consideration because these are products that always decrease expected profit when added to an existing assortment that contains M. Now, we can focus on searching for an optimal assortment amongst all sets S such that $I \subseteq S \subseteq (\mathcal{N} \backslash O)$, that is,

$$\max_{M \subseteq S \subseteq \mathcal{N}} \mathbb{E}\Pi(S) = \max_{I \subseteq S \subseteq (\mathcal{N} \backslash O)} \mathbb{E}\Pi(S). \tag{3.3}$$

But then, it is possible to update the definition of sets I and O by using this tighter condition on the set S, as shown in the following Lemma. By definition $I(O)$ is the set of those products which when added to any assortment increases(decreases) the expected profit of the assortment. This justifies the inclusion of I and exclusion of O from the optimal assortment. Let $f_j^{min}(I, O, T)$ ($f_j^{max}(I, O, T)$) be a lower (upper) bound on the minimum (maximum) change in expected profit obtained from adding product j to a set S that contains I and does not include any product in O. We provide an algorithm here to iteratively update the sets $I/O/T$ using $f_j^{min}(I, O, T)$ and $f_j^{max}(I, O, T)$.

Lemma 3.1.3. *Given sets I and O, let $T = \mathcal{N} \backslash (I \cup O)$. We update sets I and O*

using

$$I \;:=\; I \cup \left\{ j \in T : \min_{\substack{S: j \notin S \\ I \subseteq S \subseteq (\mathcal{N} \backslash O)}} C_j(S) \geq 0 \right\},$$
$$O \;:=\; O \cup \left\{ j \in T : \max_{\substack{S: j \notin S \\ I \subseteq S \subseteq (\mathcal{N} \backslash O)}} C_j(S) \leq 0 \right\}.$$

There exists an optimal solution S^* such that $I \subseteq S^* \subseteq (\mathcal{N} \backslash O)$.

It follows that the sets I and O can be updated until no product in $T = \mathcal{N} \backslash (I \cup O)$ can be added to either of the two sets. We formalize the process with the following algorithm.

In-Out Algorithm:

- Step 0: $I := M, O := \emptyset, T := \mathcal{N} \backslash M$, Change:=1.
- Step 1: While Change=1,
  - Change:=0,
  - for all $j \in T$,
    - Compute $f_j^{min}(I, O, T)$.
    - If $f_j^{min}(I, O, T) \geq 0$
      then { $I := I \cup \{j\}$, $T := T \backslash \{j\}$, Change:=1. }
      Else
        { Compute $f_j^{max}(I, O, T)$.
        If $f_j^{max}(I, O, T) \leq 0$
        then { $O := O \cup \{j\}$, $T := T \backslash \{j\}$, Change:=1. } }

where

$$f_j^{min}(I,0,T) = \sum_{\tau\in\mathfrak{T}} \alpha_{\tau} \left[\min_{\substack{I\subseteq S^{\tau}\subseteq(\mathcal{N}\setminus O)\\ j\notin S^{\tau}}} \left(\sum_{i\in S^{\tau}\cup\{j\}} Z_{\tau,i}(S^{\tau}\cup\{j\})\pi_i - \sum_{i\in S^{\tau}} Z_{\tau,i}(S^{\tau})\pi_i \right) \right] \tag{3.4}$$

$$f_j^{max}(I,O,T) = \sum_{\tau\in\mathfrak{T}} \alpha_{\tau} \left[\max_{\substack{I\subseteq S^{\tau}\subseteq(\mathcal{N}\setminus O)\\ j\notin S^{\tau}}} \left(\sum_{i\in S^{\tau}\cup\{j\}} Z_{\tau,i}(S^{\tau}\cup\{j\})\pi_i - \sum_{i\in S^{\tau}} Z_{\tau,i}(S^{\tau})\pi_i \right) \right] \tag{3.5}$$

Proposition 3.1.4. *Let I, O and T be the sets given by the* ***In-Out Algorithm****. There exists an optimal assortment S^*, such that $I \subseteq S^* \subseteq (\mathcal{N}\setminus O)$. Moreover, if $T = \emptyset$, then I is optimal.*

The following example illustrates the *In-Out Algorithm*.

Example 3: Let $n = 4$. Let $\pi_1 = 7, \pi_2 = 3$ and $\pi_3 = \pi_4 = 2$.
Let $\alpha_{\tau} = \frac{1}{7}$ for $\tau \in \{(2),(3),(4),(2,1),(3,1),(3,2),(4,3)\}$ and 0 otherwise. The *In-Out Algorithm* finds an optimal assortment after 3 iterations. Table 3.1 shows the value of the I, O and T sets as well as the f_j^{min} and f_j^{max} functions in each iteration.

Table 3.1: Iterations of the *In-Out Algorithm*

Iteration	I	O	T		1	2	3	4
1	$\{1\}$	$\emptyset$	$\{2,3,4\}$	$f_j^{min}(I,O,T)$	/	-0.143	-0.571	**0.286**
				$f_j^{max}(I,O,T)$	/	0.286	0.143	0.571
2	$\{1,4\}$	$\emptyset$	$\{2,3\}$	$f_j^{min}(I,O,T)$	/	-0.143	-0.571	/
				$f_j^{max}(I,O,T)$	/	0.286	**-0.143**	/
3	$\{1,4\}$	$\{3\}$	$\{2\}$	$f_j^{min}(I,O,T)$	/	**0.286**	/	/
				$f_j^{max}(I,O,T)$	/	0.286	/	/
end	$\{1,2,4\}$	$\{3\}$	$\emptyset$					

After the third iteration, $T = \emptyset$ so the optimal solution is $I = \{1,2,4\}$.

Upper bound

We obtain upper bound on the optimal expected profit, denoted UB_1, by assuming that the firm has perfect information about each customers' types and allocates to them the product with the highest value of π_j of all products in their type.

Lemma 3.1.5. *$\mathbb{E}\Pi(S) \leq UB_1$ for all $S \subseteq \mathcal{N}$, where*

$$UB_1 = \sum_{\tau=(\tau_1,\ldots,\tau_m)\in\mathcal{T}^+} \alpha_{\boldsymbol{\tau}} \max_{i=1,\ldots,m} \pi_{\tau_i}.$$

UB_1 can be improved using the sets I and O obtained with the *In-Out Algorithm*.

Lemma 3.1.6. *$\mathbb{E}\Pi(S) \leq UB_2(I,O) \leq UB_1$ for all $S \subseteq \mathcal{N}$, where*

$$UB_2(I,O) = \sum_{\tau=(\tau_1,\ldots,\tau_m)\in\mathcal{T}^+} \alpha_{\boldsymbol{\tau}} \max_{\substack{i=1,\ldots,m \\ \tau_i\notin O \\ \nexists k<i,\tau_k\in I}} \pi_{\tau_i}.$$

These upper bounds are useful in proving the optimality of some of the methods proposed for a few special cases and also help simplify the heuristics.

3.1.3 Results for special cases

In this section we consider special cases of the consumer choice model defined in Section3.1.1.

Multinomial Logit model

In the Multinomial Logit (MNL) model (see van Ryzin and Mahajan (1999)for a complete description), each product is characterized by a preference level or *popularity index*. Let v_j be the popularity index of product j and v_0 be that of the non-purchase option. Then for $j \in S$, $R_j(S) = \frac{v_j}{\sum_{i \in S} v_i + v_0}$. Note that the ranking of products based on the popularity indices v_j matches that based on our popularity measures $R_j(\{j\})$ and $R_j(\mathcal{N})$. This model is a special case of the model described in section 3.1.1 as it is equivalent to setting

$$\alpha_{\boldsymbol{\tau}} = \frac{v_{\tau_1}}{\sum_{i=0}^{n} v_i} \frac{v_{\tau_2}}{\sum_{i=0}^{n} v_i - v_{\tau_1}} \cdots \frac{v_{\tau_m}}{\sum_{i=0}^{n} v_i - \sum_{i=1}^{m-1} v_{\tau_i}} \frac{v_0}{\sum_{i=0}^{n} v_i - \sum_{i=1}^{m} v_{\tau_i}}$$
$$\text{for all } \boldsymbol{\tau} = (\tau_1, ..., \tau_m) \in \mathcal{T}$$

The MNL model is widely used in the literature but, it is usually restricted to choice sets containing alternatives that are equally dissimilar (e.g., different colors or different sizes, but not different color-size combinations)". Li, 2007 shows that for $j \notin S$, (3.2) simplifies to

$$C_j(S) \quad = \quad \frac{v_j}{\sum_{i \in S} v_i + v_j + v_0} \left[\pi_j - \mathbb{E}\Pi(S) \right]. \tag{3.6}$$

It follows that for a given assortment S, adding product j increases expected profit if and only if $\pi_j \geq \mathbb{E}\Pi(S)$. In particular, it is better to add a very unpopular product j with $\pi_j \geq \mathbb{E}\Pi(S)$ than a very popular product k with $\pi_k < \mathbb{E}\Pi(S)$. Without loss of generality we assume that products are numbered such that $\pi_1 \geq \cdots \geq \pi_n$ with ties broken arbitrarily and let $S_{(j)} = \{1, ..., j\}$. Li, 2007 shows that there exists j^* such that $S_{(j^*)}$ is optimal and j^* satisfies $\pi_{j^*} \geq \mathbb{E}\Pi(S_{(j^*-1)})$. We strengthen this result with the following proposition.

Proposition 3.1.7. *Let j^* be the largest integer such that $\pi_{j^*} > \mathbb{E}\Pi(S_{(j^*-1)})$. $S_{(j^*)}$ is an optimal assortment. Also j^* is such that $\pi_{j^*+1} > \pi_{j^*}$.*

It follows that when looking for an optimal assortment, it is enough to look at the sets $S_{(j)}$ with j such that $\pi_j > \pi_{j-1}$, that is, we can ignore sets $S_{(j)}$ with j such that $\pi_j = \pi_{j-1}$. In other words, the products with the same value of π can be sorted arbitrarily when constructing the sets $S_{(j)}$, despite the fact that they may have different popularity indices. The complexity of the method to find an optimal assortment is $O(n \log n)$. Proposition 3.1.7 suggests that the most important dimension is the profitability of the products. The value of the popularity index of some of the products (v_j) may not even be considered in the algorithm. The following example illustrates this point.

Example 4: Let $n = 3$. Let $\pi_1 = 15, \pi_2 = 5$ and $\pi_3 = 4$. Let $v_0 = 1$, $v_1 = 1, v_2 = 2$ and $v_3 = x$. The optimal solution is $S = \{1\}$ since $C_2(\{1\}) = -1.25 < 0$ irrespective of the value of x. In fact $C_3(\{1,2\})$ and $\mathbb{E}\Pi(\{1,2,3\})$ are decreasing in x. The reason is that under the MNL model, a new product "steals" demand from the existing products proportionally to their existing demand. Specifically, for $j \notin S, i \in S$,

$$R_i(S \cup \{j\}) \;=\; R_i(S)(1 - R_j(S \cup \{j\}) \text{ and } R_0(S \cup \{j\}) = R_0(S)(1 - R_j(S \cup \{j\}).$$

This is a well-known limitation of the MNL model called the *independence of irrelevant alternatives*. The more popular a new product is, the more new demand it brings but also, the more it cannibalizes the existing products in the assortment, which are more profitable.

One-dimensional locational choice model

Let $\mathcal{L}$ be a one-dimensional attribute space. Customers are characterized by the location of their most preferred product on this attribute space. Let G be the distribution of customer locations on the attribute space. Let $b_j \in \mathcal{L}$ be the location of product j on the attribute space, $j = 1, ..., n$. Let $U(x, j)$ be the utility that a customer located at $x \in \mathcal{L}$ gets from buying product j,

$$U(x, j) \;=\; Z_j - r_j - d|x - b_j|,$$

where Z_j is the reservation price for product j and $d > 0$ is the dis-utility associated with a distance of 1 between the customer and product locations. We assume that there are no two products with identical values of b_j, Z_j and

r_j. Let $L_j = \frac{Z_j - r_j}{d}$ be the maximum distance between b_j and a customer who gets a non negative utility from product j. Without loss of generality, let the non-purchase utility be equal to zero. For each type $\boldsymbol{\tau}$, let $\mathcal{L}_{\boldsymbol{\tau}}$ be the part of the attribute space where customers which are of type $\boldsymbol{\tau} = (\tau_1, ..., \tau_m)$:

$$\mathcal{L}_{\boldsymbol{\tau}} = \{x \in \mathcal{L} : U(x, \tau_1) \geq U(x, \tau_2) \geq ... \geq U(x, \tau_m) > 0\}.$$

It can be shown that $\mathcal{L}_{\boldsymbol{\tau}}$ is a set of disjoint segments $\mathcal{L}_{\boldsymbol{\tau}} = \bigcup_{k=1}^{K_\tau} [\underline{l}_\tau^k, \bar{l}_\tau^k]$. Then we have

$$\alpha_\tau = \sum_{k=1}^{K_\tau} G(\bar{l}_\tau^k) - G(\underline{l}_\tau^k). \tag{3.7}$$

For example, in Example 5 below, $\alpha_{(1)} = G(\bar{l}_{(1)}^1) - G(\underline{l}_{(1)}^1) + G(\bar{l}_{(1)}^2) - G(\underline{l}_{(1)}^2)$ where $\underline{l}_{(1)}^1 = 0, \bar{l}_{(1)}^1 = 0.1, \underline{l}_{(1)}^2 = 0.3$ and $\bar{l}_{(1)}^2 = 0.5$.

Hence, this model constitutes a special case of our general choice model. It can be shown that if $\mathcal{T} \subseteq \{\boldsymbol{\tau} = (\tau_1, ..., \tau_m) \in \mathcal{T} : \max_{k=1,...,m} \tau_k - \min_{k=1,...,m} \tau_k + 1 = m\}$ then there exists an attribute space $\mathcal{L}$ and distribution G such that $\alpha_{\boldsymbol{\tau}}$ is obtained as in (3.7). Also, $|\mathcal{T}^+| \leq \sum_{j=1}^{n} (n - j + 1) j!$.

This model is also referred to as Lancaster demand model or Hotelling-type demand model. It applies to product categories in which product are horizontally differentiated on one attribute, e.g., shirts of different colors, yogurt with different fat content, etc. Suppose, without loss of generality, that the products are numbered such that $b_1 - L_1 \leq b_2 - L_2 \leq ... \leq b_n - L_n$ with ties broken arbitrarily. Consider $i, j \in \mathcal{N}$, such that $i < j$. If

$b_i + L_i \geq b_j + L_j$, then every customer (weakly) prefers product i to product j. If $b_i + L_i < b_j + L_j$, some customers prefer product i to j and some customers prefer product j to i. In this case, let $a_{(i,j)}$ be the location of the customer who is indifferent between buying product i and product j,

$$a_{(i,j)} = \frac{(L_i - L_j) + (b_i + b_j)}{2}.$$

Lemma 3.1.8. *There exists an optimal assortment S such that for all $i, j \in S$ with $i < j$, $b_i + L_i < b_j + L_j$.*

It follows that we can consider only assortments $S = \{s_1, ..., s_k\}$ with $s_1 < ... < s_k$, such that $b_{s_j} + L_{s_j} < b_{s_{j+1}} + L_{s_{j+1}}$ for $j = 1, ..., k-1$. In this case, we have:

$$R_{s_j}(S) = \begin{cases} G\left(\min\{a_{(s_j,s_{j+1})}, b_{s_j} + L_{s_j}\}\right) - G\left(b_{s_j} - L_{s_j}\right) \\ \text{for } j = 1, \\ G\left(\min\{a_{(s_j,s_{j+1})}, b_{s_j} + L_{s_j}\}\right) - G\left(\max\{a_{(s_{j-1},s_j)}, b_{s_j} - L_{s_j}\}\right) \\ \text{for } j = 2, ...k-1, \\ G\left(b_{s_j} + L_{s_j}\right) - G\left(\max\{a_{(s_{j-1},s_j)}, b_{s_j} - L_{s_j}\}\right) \\ \text{for } j = k. \end{cases} \tag{3.8}$$

Note that:

$$R_{s_j}(S) = R_{s_j}\left(\{s_{j-1}, s_j, s_{j+1}\}\right). \tag{3.9}$$

We model the problem of finding the best assortment as a shortest path problem. Define $\mathcal{A}$ to be the set of nodes where $\mathcal{A} = \{(i,j) : 1 \leq i < j \leq n \text{ and } b_i + L_i < b_j + L_j\} \cup \{(j, n+1); j = 1, ..., n\} \cup \{(0,0), (n+1, n+1)\}$.

The cost of arcs are defined as follows for $(i,j),(k,l) \in \mathcal{A}$:

$$c_{(i,j),(k,l)} = \begin{cases} -\left[G\left(\min\{a_{(k,l)}, b_k + L_k\}\right) - G\left(b_k - L_k\right)\right]\pi_k \\ \text{if } 0 = i = j < k < l \leq n+1, \\ -\left[G\left(\min\{a_{(k,l)}, b_k + L_k\}\right) - G\left(\max\{a_{(i,k)}, b_k - L_k\}\right)\right]\pi_k \\ \text{if } 0 < i < j = k < l \leq n+1, \\ 0 \\ \text{if } 0 < i < j = k = l = n+1, \\ +\infty \\ \text{otherwise} \end{cases}$$

where we set $a_{(j,n+1)} = +\infty$ for $j = 1, ..., n$.

Proposition 3.1.9. *a) The problem of finding an optimal assortment reduces to a shortest path problem between* $(0,0)$ *and* $(n+1, n+1)$.

b) The complexity of the method to find an optimal assortment is $O(n^3)$

Note that, in a different context, Alptekinoglu, 2004 find a similar structure for their problem of finding the optimal trade off between variety and lead time.

Example 5: Let $n = 3$, $\mathcal{L} = [0,1], d = 10$ and G be uniform on $[0,1]$. The following table shows the parameters for the three products.

j	1	2	3
b_j	0.3	0.2	0.8
Z_j	5	2	4
r_j	2	1	2
L_j	0.3	0.1	0.2
π_j	20	10	100

The attribute space is such that each product is associated with a triangle. The height of the triangle for product j is $Z_j - r_j$, which is the utility a customer located at b_j gets from product j. The intersection of the triangle with the attribute space corresponds to locations of customers who get a positive utility from it. The length of this intersection is $2L_j$.

First note that every customer prefers 1 to 2 since $b_1 + L_1 = 0.6 > b_2 + L_2 = 0.3$ (graphically, the triangle corresponding to product 2 is inside the one that corresponds to product 1) and that product 1 is more profitable than product 2 since $pi_1 > \pi_2$. Yet the optimal assortment is $\{2,3\}$ so it is not possible to eliminate products that are less preferred than a more profitable product. The optimal expected profit is equal to 60. The path that corresponds to this assortment is $(0,0) \to (2,3) \to (3,4) \to (4,4)$.

Increasing Preferences

Under the increasing preferences model, it is possible to renumber the products such that $\mathcal{T}^+ \subseteq \{\boldsymbol{\tau} = (\tau_1, ..., \tau_m) \in \mathcal{T} : \tau_1 < \tau_2 < ... < \tau_m\}$ so that $|\mathcal{T}^+| \leq 2^n$. For example if $n = 3$, customers can only be of the following types: $\{(1),(2),(3),(1,2),(1,3),(2,3),(1,2,3)\}$. This consumer choice model applies to product that differ with respect to multiple attributes, when customers are, to a certain extent, willing to accept products with a larger (but not smaller) value of each attribute e.g., wood panels with different widths and lengths. An important property is that for $j \notin S, i \in S$,

$$R_i(S \cup \{j\}) \quad = \quad R_i(S) \text{ if } j > i. \tag{3.10}$$

In other words, a new product only “steals" demand from products with a higher index. For this model we get the following result which often helps speed the search for the optimal solution.

Lemma 3.1.10. *There exists an optimal assortment that contains the set* $G = \{j : \pi_j \geq \pi_i \text{ for all } i > j\}$.

In other words, products that can only cannibalize less profitable products should be included in the assortment. The optimal solution can be found by comparing the $2^{n-|G|} - 1$ assortments in which G is included. Note that $n \in G$ and $M \subseteq G$ so $|G| \geq 1$.

Partial one-way substitution

Under partial one-way substitution, it is possible to renumber the products such that $\mathcal{T}^+ \subseteq \{\boldsymbol{\tau} = (\tau_1, ..., \tau_m) \in \mathcal{T} : \tau_k = \tau_{k-1} + 1 \text{ for } k = 2, ..., m\}$. Hence, $|\mathcal{T}+| \leq \frac{n(n+1)}{2}$. For example if $n = 3$, customers can only be of the following types:

$$\{(1), (2), (3), (1,2), (2,3), (1,2,3)\}$$

This consumer choice model applies to products that differ with respect to one attribute, when customers are, to a certain extent, willing to accept products with a larger (but not smaller) value of the attribute e.g., table-cloths of different lengths or adjustable shower rods with different maximum length. Let $S = \{s_1, ..., s_k\}$ with $s_1 < ... < s_k$, we have

$$R_{s_j}(S) \;=\; R_{s_j}\left(\{s_{j-1}, s_j\}\right) \tag{3.11}$$

We model the problem of finding the best assortment as a shortest path problem. Let $\mathcal{A}$ be the set of nodes, where $\mathcal{A} = \{0, 1, ..., n\}$. The cost of arcs are defined as follows, for $i, j \in \mathcal{A}$:

$$c_{i,j} = \begin{cases} -R_j(\{i,j\})\pi_j & \text{if } 0 < i < j \leq n, \\ -R_j(\{j\})\pi_j & \text{if } i = 0, 1 \leq j \leq n, \\ +\infty & \text{otw.} \end{cases}$$

Proposition 3.1.11. *a) The problem of finding an optimal assortment reduces to a shortest path problem between nodes* 0 *and* n*.*

b) The complexity of the method to find an optimal assortment is $O(n^2)$*.*

Full one-way substitution

Under full one-way substitution, it is possible to renumber the products such that $\mathcal{T}^+ \subseteq \{\boldsymbol{\tau} = (\tau_1, ..., \tau_m) \in \mathcal{T} : \tau_k = n - (m-k) \text{ for } k = 1, ..., m\}$. Hence, $|\mathcal{T}^+| \leq n$. For example if $n = 3$, customers can only be of the following types: $\{(1,2,3), (2,3), (3)\}$.

This consumer choice model is similar in its application to the partial one-way substitution model except that customers are always willing to accept a substitute product with a larger (but not smaller) value of the attribute.

Proposition 3.1.12. *The set* $S^* = \{j : \pi_j \geq \pi_i, \text{ for all } i > j\}$ *is an optimal assortment.*

Note that the optimal assortment(s) does (do) not depend on the actual proportion of customers that are of each type $\alpha_{\boldsymbol{\tau}}$ for $\boldsymbol{\tau} \in \mathcal{T}^+$ and is (are) such that every customer is satisfied. The complexity of the method to find an optimal assortment is $O(n)$.

Homogeneous population

Under homogeneous population model, it is possible to renumber the products such that $\mathcal{T}^+ \subseteq \{\boldsymbol{\tau} = (\tau_1, ..., \tau_m) \in \mathcal{T} : \tau_k = k \text{ for } k = 1, ..., m\}$. Hence, $|\mathcal{T}^+| \leq n$. For example if $n = 3$, customers can only be of the following types: $\{(1), (1, 2), (1, 2, 3)\}$. This consumer choice model applies to product categories for which all customers agree on the ranking of products but they differ in their willingness to substitute, *e.g.* DVDs with more or less special features and/or special packaging offered at the same price.

Lemma 3.1.13. $S^* = \{k\}$ *such that* $R_k(\{k\})\pi_k = \max_j R_j(\{j\})\pi_j$ *is an optimal assortment.*

Universal backup model

Under the universal backup model, it is possible to renumber the products such that $\mathcal{T}^+ \subseteq \{\boldsymbol{\tau} = (\tau_1, ..., \tau_m) \in MT : m \leq 2 \text{ and } \tau_m = n\}$. Hence, $|\mathcal{T}^+| \leq n$. Without loss of generality n is the backup product. For example if $n = 3$, customers can only be of the following types: $\{(1,3),(2,3),(3)\}$.

This consumer choice model applies to categories for which every customers differ in their first choice but everyone has the same second choice: such as the example of ice cream (Mahajan and van Ryzin, 2001) where "every customer may be willing to settle for vanilla ice cream if their favorite flavor is out of stock".

Proposition 3.1.14. *$S^* = \left\{j \in \mathcal{T} : \pi_j \geq \pi_n\right\}$ is an optimal assortment.*

Like in the full one-way substitution model, the optimal assortment(s) does (do) not depend on the actual proportion of customers that are of each type $\alpha_{\boldsymbol{\tau}}$ for $\boldsymbol{\tau} \in \mathcal{T}^+$ and that is (are) such that every customer is satisfied.

Markovian second choice

Under Markovian second choice model, it is possible to renumber the products such that $\mathcal{T}+ \subseteq \{\boldsymbol{\tau} = (\tau_1, ..., \tau_m) \in \mathcal{T} : m \leq 2$. For example if $n = 3$, customers can only be of the following types: $\{(1),(2),(3),(1,2),(1,3),$ $(2,1),(2,3),(3,1),(3,2)\}$. This consumer choice model applies to product categories with high brand loyalty, where every customer is willing to buy at most two products.

We are not able to find an efficient method to get this optimal solution in this model. However we are able to get good results using a simplified version of the *In-Out Algorithm*. See 3.2 for some numerical results.

Lemma 3.1.15. *Consider the* ***In-Out Algorithm*** *from 3.3 with* $f_j^{min}(I, O, U) = C_j(\mathcal{N} \backslash (O \cup \{j\}))$ *and* $f_j^{max}(I, O, U) = C_j(I)$*. There exists an optimal assortment* S^* *such that* $I \subseteq S^*$*. Moreover if* $T = \emptyset$*, then* I *is optimal.*

The complexity of the *In-Out Algorithm* with these values is $O(n^2)$. See Example 3 for an application of the *In-Out Algorithm* with the Markovian second choice model.

Summary and insights

Figure 3.1 provides a summary of our results. An arrow from one model to another indicates that the bottom one is a special case of the top one, *e.g.*, the homogeneous population model is a special case of the one-way substitution model. Boxes in gray indicate that we have an efficient method (other than enumeration) to obtain an optimal solution.

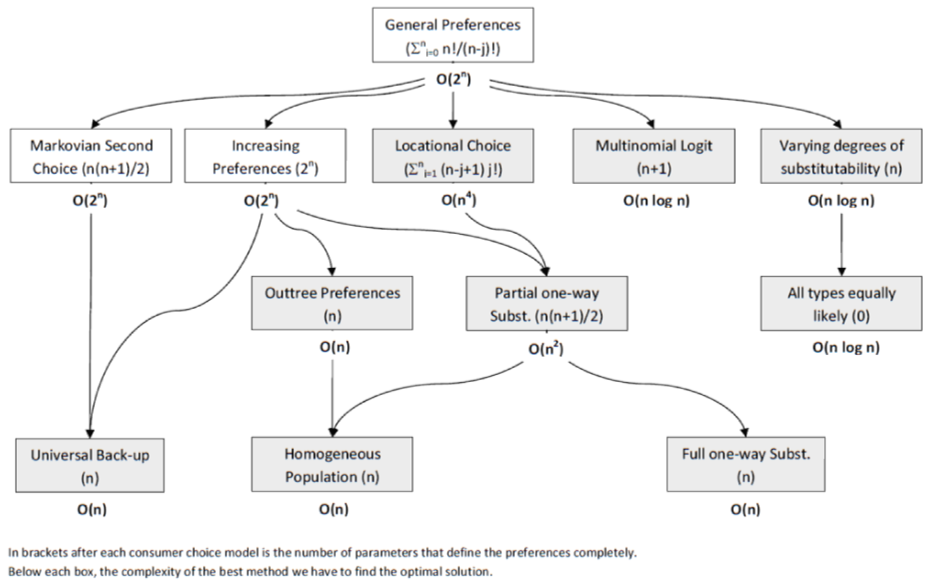

Figure 3.1: Summary of Results for Special Cases

Analyzing the special cases also teaches us some valuable insights about the assortment decision. In models where substitution occurs between every pair of products, such as the MNL model and the varying degrees of substitution model, the guiding principle for constructing the assortment is the *absolute* profitability of the products, as measured by their profit margin. If, in contrast, substitution is only localized or one-directional, as in the locational choice, out tree, partial and full one-way substitution and the homogeneous population models, then it is important to understand which products are cannibalized by the inclusion of a new product to the assortment and compare their *relative* profitability.

Also we learned that if the proportion of customers that want to buy product j from any assortment S only depends on a subset of S, as in the one-dimensional locational, out tree and partial one-way substitution models then the problem of finding an optimal assortment can be found using dynamic programming.

3.2. Heuristics and Numerical Results

3.2.1 Heuristics

For the general consumer choice model as well as the special cases for which we do not have an efficient method of finding an optimal solution (increasing preferences and Markovian second choice), we resort to heuristics. We propose the following 6 heuristics. Each heuristic requires the specification of sets I and O, which may be obtained from the *In-Out Algorithm* if it is run beforehand. If not, I and O are set equal to M and $\emptyset$ respectively, except for the increasing preference model where I is set equal to G by Lemma 3.1.10.

Greedy-add (GA) heuristic:

- Step 0: $S_1 = I$, $T = \mathcal{N} \backslash (I \cup O)$, $k = 1$.
- Step 1: While $T \neq \emptyset$,

- Find $j = \max\{i \in T : C_i(S_k) = \max_{l \in T} C_l(S_k)\}$.
- $T := T \backslash \{j\}$, $S_k = S_{k-1} \cup \{j\}$, $k = k + 1$.

- Step 2: Find S is such that $\mathbb{E}\Pi(S) = \max_k \mathbb{E}\Pi(S_k)$.

Greedy-remove (GR) heuristic:

- Step 0: $S_1 = (\mathcal{N} \backslash O)$, $T = \emptyset$, $k = 1$.

- Step 1: While $T \neq \mathcal{N} \backslash (I \cup O)$,
  - Find $j = \max\{i \in (S_k \backslash I) : C_i(S_k) = \max_{l \in (S_k \backslash I)} C_l(S_k)\}$.
  - $T := T \cup \{j\}$, $S_k = S_{k-1} \backslash \{j\}$, $k = k + 1$.

- Step 2: Find S is such that $\mathbb{E}\Pi(S) = \max_k \mathbb{E}\Pi(S_k)$.

Largest marginal profit (LMP) heuristic: This algorithm is identical to the Greedy-add Algorithm except that $C_j(S_k)$ is replaced by $B_j(S_k)$ which, for $j \notin S_k$, is the marginal benefit of product j when added to set S_k, i.e., the change in expected profit per extra unit of demand,

$$B_j(S_k) = \frac{\mathbb{E}\Pi(S_k \cup \{j\}) - \mathbb{E}\Pi(S_k)}{R_0(S_k) - R_0(S_k \cup \{j\})} = \frac{C_j(S_k)}{R_0(S_k) - R_0(S_k \cup \{j\})}.$$

This algorithm is inspired by the "largest marginal benefit" Algorithm of Talluri and van Ryzin (2004).

The most profitable ones (MP) heuristic: First renumber the products in $\mathcal{N} \backslash (I \cup O)$ such that $\pi_1 \geq \pi_2 \geq ... \geq \pi_{\tilde{n}}$ where $\tilde{n} = |\mathcal{N} \backslash (I \cup O)|$. Let $S_k = \{1, ..., k\}$. Find S is such that $\mathbb{E}\Pi(I \cup S) = \max_k \mathbb{E}\Pi(I \cup S_k)$.

The most popular ones using $R_j(\mathcal{N})$ (MPa) heuristic: This heuristic is similar to the previous one except the products in $\mathcal{N} \backslash (I \cup O)$ are renumbered with respect to $R_j(\mathcal{N})$.

The most popular ones using $R_j(\{j\})$ (MPf) heuristic: This heuristic is similar to the previous one except the products in $\mathcal{N} \backslash (I \cup O)$ are renumbered with respect to $R_j(\{j\})$.

The complexity of the GA, GR and MB heuristics is $O(n^2)$ while that of the MP, MPa and MPf heuristics is $O(n \log n)$. The following example shows that none of these heuristic always gives the optimal solution for the increasing preferences model of 3.1.3, for the Markovian second choice model of 3.1.3 and hence also for the general consumer choice model .

Example: Let $n = 6$. Let $\pi_1 = 9, \pi_2 = 8, \pi_3 = 16, \pi_4 = 14, \pi_5 = 19$ and $\pi_6 = 3$. Let $\mathcal{T}^+ =$
$\{(2), (5), (1,2), (1,4), (1,6), (2,3), (2,5), (2,6), (3,4), (3,6), (4,5), (5,6)\}$,
which satisfies the conditions of the increasing preferences and Markovian second choice models. Let α_τ is given by the following table:

τ	$\alpha_\tau = (\cdot/20)$
(2)	1
(5)	1
(1,2)	1
(1,4)	5
(1,6)	1
(2,3)	1
(2,5)	1
(2,6)	2
(3, 4)	1
(3,6)	1
(4,5)	3
(5,6)	1

The following table shows the solutions given by the 6 heuristics with $I = G = \{5, 6\}$ and $O = \emptyset$.

	S	$\mathbb{E}\Pi(S)$
GA	$\{3, 4, 5, 6\}$	12.10
GR	$\{3, 4, 5, 6\}$	12.10
LMP	$\{3, 4, 5, 6\}$	12.10
MP	$\{3, 4, 5, 6\}$	12.10
MPa	$\{1, 2, 3, 4, 5, 6\}$	11.15
MPf	$\{1, 2, 3, 4, 5, 6\}$	11.15

The optimal assortment is $S^* = \{1, 3, 5, 6\}$ and $\mathbb{E}\Pi(S) = 12.35$.

Note that if we run the **In-Out Algorithm** prior to running the 6 heuristics then we would obtain $I = \{3, 5, 6\}$ and $O = \emptyset$. Using these values in the 6 heuristics, GA, GR and MPf find the optimal solution, the other three still do not.

3.2.2 Numerical results

We test the performance of the *In-Out Algorithm* and of our six heuristics using 2 scenarios. Each scenario has 6 simulation rounds which differ in the number of potential products n. In each simulation round, we use 10,000 problem instances where the parameters are randomly generated. The values of π_j are generated using independent discrete uniform distributions between 0 and 20, for $j = 1, ...n$.

In Scenario 1, we use the Markovian second choice model 3.1.3 to represent customer preferences. In this case, the values of $\alpha_{\boldsymbol{\tau}}$ are generated as follows. First we generate a uniform random integer number x between n and n^2, because $|\mathcal{T}^+| \leq n^2$ in this model. Then we generate $\alpha_{\boldsymbol{\tau}}$ for all $\boldsymbol{\tau} = (\tau_1, ..., \tau_m)$ such that $m \leq 2$, using a multinomial distribution with x trials and a probability of success for each type equal to $1/x$. The results for Scenario 1 are presented in Table 3.2

In Scenario 2, we use the increasing preferences model. In this case, the values of $\alpha_{\boldsymbol{\tau}}$ are generated in a similar fashion. First we generate a uniform random integer number x between n and $2^n - 1$, because $|\mathcal{T}^+| \leq 2^n - 1$ in this model. Then we generate $\alpha_{\boldsymbol{\tau}}$ for all $\boldsymbol{\tau} = (\tau_1, ..., \tau_m)$ such that $\tau_1 < \tau_2 < ... < \tau_m$, using multinomial distribution with x trials and a probability of success for each type equal to $1/x$. The results for Scenario 2 are presented in Table 3.3.

In theory, it is always possible to obtain an optimal assortment for a given problem instance by enumerating all $2^{n-|M|} - 1$ possible assortments and looking for the best one. However, this method is too computationally intensive for large value of n (for $n = 20$ it takes more than 8 hours per problem instance in scenario 2). By Proposition 3.1.4, the *In-Out Algorithm* provides an optimal solution if it terminates with $T = \emptyset$. If it does not, and the number of products left in the T set is small enough, an optimal assortment can be found by enumerating all $2^{|T|}$ possible assortments which include I and finding the best one. Failing that, we know the solution of one of the 6 heuristics is optimal if its expected profit is equal to one of the upper bounds. If none of these conditions is satisfied then we do not know what the optimal solution(s) is (are). In Tables 3.2 and 3.3 we report, for each round, the percentage of instances in which we know that an optimal assortment was found (*OPT % known*).

For the *In-Out Algorithm* (IO), we report the percentage of instances in which the algorithm found an optimal solution (% *opt*) as well as average number of products that were left in the T set after the algorithm was run (*Avg left*).

For the 6 heuristics, we report the percentage of instances in which the solution obtained was optimal (% *opt*) as well as the average and maximum optimality gaps as measured by (Optimal Expected Profit - Expected Profit of heuristic)/Optimal Expected Profit (*Avg % OG* and *Max % OG*).

To better understand the benefits of the *In-Out Algorithm* we report the performance of the 6 heuristics before and after its use. In other words, we first run the 6 heuristics with $I = M$ in Scenario 1 or $I = G$ in Scenario 2 and $O = \emptyset$ (*pre*). Then we run them again using the sets I and O obtained by the *In-Out Algorithm* (*post*).

In Scenario 1, (see Table 3.2) we obtained an optimal assortment in Scenario 1 for all but 23 problem instances. The *In-Out Algorithm* reached the optimal solution in 78.63% of the problem instances but its performance worsens with the number of products. It is generally fast (thanks to Proposition 3.1.15) and it significantly improves the performance of all 6 heuristics. The best heuristic for this model is GR, it reaches an optimal solution for about 98.60% of problem instances before the *In-Out Algorithm* and 99.15% of them after. The performance of all 6 heuristics gets worse as n increases.

In Scenario 2, (see Table 3.3) it was more difficult to obtain an optimal assortment (only 84.62% of the problem instances) as the set $\mathcal{T}^+$ is generally larger in the increasing preferences model compared to the Markovian second choice model. The *In-Out Algorithm* does not perform as well as in Scenario 1 and its average CPT is higher. However it continues to improve the performance of all 6 heuristics, though not as much as in Scenario 1. The best heuristic is GA, it reaches an optimal solution for 99.69% of the problem instances before the *In-Out Algorithm* and 99.71% of them after. Surprisingly, the performance of the best three heuristics (GA, GR and LMP) improves with n.

Table 3.2: Performance Analysis for Markovian Second Choice

		n=5		n=10		n=15		n=20	
OPT	% known	100		100		100		99.77	
	% opt	87.14		78.08		74.43		72.61	
IO	Avg left	0.30		0.68		0.91		1.06	
	Avg CPT	0.00		0.05		0.18		0.44	
		pre	post	pre	post	pre	post	pre	post
	% opt	97.83	99.67	96.22	99.02	94.68	98.41	94.24	98.18
GA	Avg % OG	0.07	0.01	0.05	0.01	0.05	0.01	0.04	0.01
	Max % OG	23.38	8.28	10.59	3.24	11.66	2.31	8.28	1.92
	Avg CPT	0.00	0.00	0.04	0.00	0.24	0.00	0.73	0.01
	% opt	99.69	99.76	98.68	99.16	97.96	98.77	97.65	98.62
GR	Avg % OG	0.01	0.00	0.01	0.01	0.01	0.00	0.01	0.00
	Max % OG	5.34	5.34	3.56	3.53	4.37	2.62	1.92	1.61
	Avg CPT	0.00	0.00	0.05	0.00	0.27	0.00	0.83	0.01
	% opt	96.56	98.62	94.11	96.96	92.81	96.29	92.21	95.89
LMP	Avg % OG	0.06	0.02	0.04	0.02	0.03	0.01	0.03	0.01
	Max % OG	10.00	6.67	3.30	3.30	3.28	3.28	2.38	1.88
	Avg CPT	0.00	0.00	0.05	0.00	0.24	0.00	0.73	0.02
	% opt	75.04	95.22	52.53	89.84	40.62	86.19	33.67	83.89
MP	Avg % OG	1.00	0.14	1.01	0.14	0.93	0.11	0.86	0.09
	Max % OG	25.00	16.18	18.37	8.89	14.14	6.91	15.81	4.20
	Avg CPT	0.00	0.00	0.01	0.00	0.04	0.00	0.08	0.00
	% opt	26.25	93.27	1.53	86.29	0.00	82.85	0	80.53
MPa	Avg % OG	8.44	0.28	11.57	0.28	12.38	0.24	13.03	0.09
	Max % OG	63.83	51.04	52.76	20.38	43.59	11.66	37.84	7.30
	Avg CPT	0.00	0.00	0.02	0.00	0.06	0.00	0.14	0.01
	% opt	24.15	92.19	1.39	85.12	0.00	81.97	0	80.09
MPf	Avg % OG	9.65	0.40	12.62	0.35	13.20	0.28	13.69	0.21
	Max % OG	73.33	51.04	58.05	28.03	46.15	14.93	48.50	13.30
	Avg CPT	0.00	0.00	0.02	0.00	0.08	0.00	0.18	0.01

Table 3.3: Performance Analysis for Increasing Preferences

		n=5		n=7		n=9		n=11	
OPT	% known	100		100		100		71.27	
IO	% opt	43.81		10.26		1.87		0.35	
	Avg left	1.61		3.81		5.86		7.65	
	Avg CPT	0.00		0.01		0.06		0.31	
		pre	post	pre	post	pre	post	pre	post
GA	% opt	99.73	99.78	99.37	99.40	99.57	99.57	99.79	99.80
	Avg % OG	0.01	0.01	0.01	0.01	0.00	0.00	0.00	0.00
	Max % OG	16.67	16.67	14.29	6.90	6.35	6.35	1.75	0.67
	Avg CPT	0.00	0.00	0.01	0.01	0.05	0.05	0.31	0.31
GR	% opt	99.57	99.66	99.30	99.34	98.56	99.59	99.65	99.68
	Avg % OG	0.02	0.02	0.02	0.02	0.01	0.01	0.00	0.00
	Max % OG	50.33	42.48	16.67	16.67	12.77	12.77	1.97	1.97
	Avg CPT	0.00	0.00	0.01	0.01	0.05	0.05	0.31	0.31
LMP	% opt	99.14	99.38	98.81	98.96	99.01	99.03	99.40	99.41
	Avg % OG	0.02	0.01	0.01	0.01	0.01	0.01	0.00	0.00
	Max % OG	12.20	12.20	20.72	8.98	4.95	4.95	6.64	6.64
	Avg CPT	0.00	0.00	0.01	0.01	0.05	0.05	0.31	0.31
MP	% opt	88.94	93.23	79.92	82.46	70.30	71.19	60.46	60.80
	Avg % OG	0.39	0.22	0.43	0.37	0.49	0.47	0.48	0.48
	Max % OG	25.00	18.29	16.98	12.05	12.95	12.95	11.06	11.06
	Avg CPT	0.00	0.00	0.00	0.00	0.02	0.02	0.08	0.08
MPa	% opt	62.71	75.09	36.62	42.76	23.02	24.75	20.56	21.17
	Avg % OG	3.59	2.36	5.27	4.74	6.02	5.90	3.93	3.87
	Max % OG	68.33	68.33	53.74	53.74	47.36	47.36	41.18	29.33
	Avg CPT	0.00	0.00	0.00	0.00	0.03	0.03	0.14	0.14
MPf	% opt	60.19	72.82	34.17	40.28	21.74	23.41	19.91	20.54
	Avg % OG	3.59	2.99	6.51	5.94	7.14	7.00	4.17	4.12
	Max % OG	68.33	68.33	57.39	53.74	52.13	52.13	46.15	38.35
	Avg CPT	0.00	0.00	0.00	0.00	0.03	0.03	0.14	0.14

In general, we recommend using the *In-Out Algorithm* first to obtain the sets I, O and T. Then, if set T is still large, we recommend using the GA and GR heuristics, otherwise we suggest using the brute force approach. We never recommend using the ranking-based heuristics (MP, MPa and MPf) as they generally perform badly, especially when n is large.

3.3. Managerial Implications and Concluding Remarks

We have shown that the problem of finding an optimal assortment in an efficient manner for a make-to-order firm in the face of a heterogeneous customer population is not an easy task. With a general consumer choice model, *i.e.*, without any constraint on the number of customers of each type, we can only show that the optimal assortment(s) include(s) the most profitable product(s). However, for a number of commonly used special cases, we are able to find an optimal assortment with an efficient method. Our results suggest that for models where substitution occurs between every pair of products the guiding principle for constructing the assortment is the *absolute* profitability of the products, as measured by their profit margin. If, in contrast, substitution is only localized or one-directional, then the guiding principle is that of *relative* profitability. For the increasing preferences and Markovian second choice models, we did not find an efficient method to get an optimal solution but we recommend using the *In-Out Algorithm* first, then the GA or GR heuristics as they proved to perform well numerically.

Our consumer choice model captures a wide range of substitution patterns amongst products. However one of our limiting assumptions is that we assume that customer preferences can be fully described independently of the assortment offered by the firm. As a result, we are not able to capture situations in which customers are more likely to buy a given product if it is offered in a large assortment than in a small assortment because the large assortment provides them with the reassurance that their choice is the right one. Also, our model does allow a new product to increase the purchase probability of the existing products, for example, because the existing products compare favorably to the new one. Finally we do not capture situations in which adding new products beyond a certain point creates confusion in the mind of customers who become less likely to make a purchase. These more complex product interactions will be the subject of our future research.

Another limiting assumption is the make-to-order setting. However, finding an optimal assortment in a make-to-stock setting is even more difficult for two reasons. First, the inventory of some products may run out during the selling season and this affects the choice behavior of subsequent customers who have to choose from a subset of the initial assortment. This phenomenon is called *stock-out based substitution*. In this case, the demand for a product not only depends on the initial assortment, but also on the inventory level of the other products. Also, the profit value depends on the sequence in which customers come to the store, and this greatly complicates the model (see Mahajan and van Ryzin, 2001 for more details). Second, even if we ignore stock-out based substitution by forcing customers to choose from the initial assortment (a.k.a. assuming *assortment-based substitution*), the problem is more complex because inventory costs depend on the variability of the demand. To illustrate this, let D be the random variable of the number of customers coming to the store and assume that it is Poisson with mean λ. For a given assortment S, each customer has a probability $R_j(S)$ of picking product j, therefore the demand for product j is Poisson with mean $\lambda R_j(S)$. In a News vendor setting, the inventory costs increase proportionately to the standard deviation of demand, that is $\sqrt{\lambda R_j(S)}$. Hence the profit contribution of a product is an increasing convex function of its demand. Unfortunately all of our (positive) results no longer hold in this case because there is an incentive to stock fewer products and increase their demand in order to save on inventory costs. However, if we assume that demand for product j is equal to $R_j(S)D$ (which is equivalent to assuming that the coefficient of variation of the demand is 1), Li, 2007 shows that expected profit function becomes:

$$\mathbb{E}\Pi(S) \quad = \quad \sum_{j \in S} R_j(S) \left[r_j \mathbb{E}[\min(q_j^*, D)] - c_j q_j^* \right], \qquad (3.12)$$

where q_j^* is in the optimal inventory for product j. Note that (3.12) is equivalent to (3.1) with $\pi_j = \left[r_j \mathbb{E}[\min(q_j^*, D)] - c_j q_j^* \right]$. Hence, it follows that all of our results apply to a make-to-stock setting if (1) there is no stock-out based substitution and (2) demand has a coefficient of variation equal to 1. To the best of our knowledge, finding an optimal assortment in a make-to-stock setting with stock-out based substitution and a coefficient of variation less than one is still an open problem.

Another possible extension of our model is to include a penalty cost for customers who do not get their first choice product. Let $r_{\boldsymbol{\tau},j}$ be the price paid by a customer of type $\boldsymbol{\tau}$ for product j. For example, one can set:

$$r_{\boldsymbol{\tau},j} = \begin{cases} r_j & \text{if } \tau_1 = j, \\ r_j - b & \text{otw.} \end{cases}$$

where b is some fixed discount given to customers who have to substitute. In this case we have:

$$\mathbb{E}\Pi(S) = \sum_{j \in S} \sum_{\tau \in \mathcal{T}} (r_{\boldsymbol{\tau},j} - c_j) \alpha_{\boldsymbol{\tau}} Z_{\boldsymbol{\tau},j}(S).$$

All of our (positive) results no longer hold in this case as well because the profit margin on a product now depends on the assortment offered. Hence, this problem is considerably harder to solve and we leave it for future research.

Chapter 4

Durable Products, Time inconsistency, and Lock-in

4.1. Introduction

There are many durable products for which manufacturers have devised clever ways of charging consumers based on the amount of use that they derive from the product. Typically, this is done by selling contingent products or services. For example, printers do not print without ink cartridges, commercial aircraft do not fly without replacement parts, etc. Even some sophisticated business application software is nearly impossible to use without expensive consulting and maintenance services. Of course, this strategy which has often been colorfully referred to as, *giving away the razor to make money from selling blades,* has been widely recognized as a viable business strategy.

There are many ways in which the strategy of locking consumers into contingent products and services can be implemented in practice. When Iomega revolutionized the information storage industry by introducing its Zip drive in 1994, it initially monopolized the market for Zip disks, which were the contingent storage medium that was required to use the Zip drive.

But later, these disks were also sold by Fuji, Verbatim, Toshiba, and Maxell. This strategy of initially monopolizing the market for contingent consumables and later allowing competition is not uncommon. In fact, while Gillette typically monopolizes the sales of blades for its most recently introduced razor, it is not uncommon to find generic blade suppliers for older razors. Yet many other firms maintain monopoly control over contingent consumables for extended periods. For example, Abbott Labs remains the sole source of test strips for its FreeStyle glucose monitors; Apple makes it difficult for consumers to down-load music from web-sites other than iTunes; and many consumer electronics products are compatible with only proprietary peripheral add-ons and accessories.

Our interest is in understanding when a firm should employ a lock-in strategy and, if so, how it can be tailored to particular circumstances. We focus on how a strategy of locking-in consumers to a contingent consumable product / service affects the interactions between a durable goods manufacturer and strategic consumers. In particular, we consider the following two related concerns that affect the purchasing decisions of strategic consumers: First, as is the case for many durable products, consumers worry about the potential for the price of the durable to decline after they purchase. This concern is related to the manufacturer's incentive to reduce prices after selling to the consumers with the highest valuations. In the extreme, it implies that a monopolist's ability to set a price above marginal cost is inconsistent with her own incentives and the passage of time. Consequently, it is often referred to as *time inconsistency*. Second, consumers are concerned about the extent to which they will be able to derive utility from the durable product after they purchase; for example, consumers of Amazon's reading device Kindle are concerned about the availability of e-books at reasonable prices (WIRED (2009)). When consumers are locked-in to purchasing a contingent consumable each time they use the durable, they will be concerned about their exposure to being *held-up* with respect to the price at which consumables are sold.

It is well known that a durable good manufacturer can mitigate time inconsistency by leasing her product to consumers. Under a lease, consumers pay a lease fee for the right to use the product for a given period of time but the manufacturer is the residual claimant who owns the product at the end of the lease. Because this eliminates the durable good manufacturer's incentive to reduce price over time it mitigates time inconsistency and allows the manufacturer to earn rents comparable to those in a non-durable good monopoly.

The strategy of locking consumers into a contingent consumable product bears some superficial similarity to a lease since both approaches endow the manufacturer with the ability to charge consumers based on their use of a durable. However, there is a significant distinction: With a lease, consumers pay according to the amount of *time* for which they have access to the durable, whereas with lock-in, they pay according to their *utilization* of the durable. Thus, a lock-in policy allows a manufacturer to charge consumers according to their frequency of use. Because of this distinction, and because of the fact that lock-in policies are observed frequently in practice, it is of interest to better understand the role of these policies in the presence of strategic consumers.

As we will see, a policy of locking consumers into a contingent consumable creates an interesting balance between the two main concerns of strategic consumers, i.e. time inconsistency and hold-up. On the one hand, the opportunity to sell consumables to the highest valuation consumers can reduce the manufacturer's incentive to reduce the price of its durable over time. Thus, lock-in can help to mitigate the time inconsistency effect. On the other hand, in order for the manufacturer to make its durable attractive to consumers with lower valuations, she will have an incentive to continue to offer contingent consumables at reasonable prices. Thus, the presence of differentiated consumers, which gives rise to time inconsistency, may also help to mitigate hold-up issues with respect to the contingent consumable product.

4.2. Model Description

In order to focus on products whose physical durability outlasts their technical relevance, we adopt a variation of Bulow's two period durable goods model in which the product does not depreciate between periods 1 and 2. As in the original Bulow (1982) model, we require that either the same set of consumers are present in both periods or there is a perfect second-hand market. Either of these assumptions is sufficient to ensure that all the units available in the market in period 2 are allocated to the consumers with highest valuations.

Where our model differs from that of Bulow (1982) is that we assume that consumers derive utility from the durable only by using it in conjunction with a contingent consumable. Specifically, we assume that one unit of a contingent consumable is required for each instance of use of the durable and that in each period a consumer's marginal utility is decreasing in the amount that he uses the durable. Let z be the amount of use that a consumer derives from the durable product in a given period. We allow for two types of consumers (High (h) and Low (l)) such that their marginal utilities are $U_h'(z) = 1 - z$ and $U_l'(z) = \alpha - z$ respectively, where $\alpha < 1$. Note that this assumption of linearly decreasing marginal utility is similar to the one made by Bhaskaran and Gilbert (2005) in their micro model of the utility that consumers derive from multiple units of a product that is complementary to a durable. By construction, we can see that the utilities for both type of consumers are concave and are given as follows.

$$U_i(z) = \begin{cases} \int_0^z \quad (\alpha - z)dz = \alpha z - \dfrac{z^2}{2} \text{ if } \quad i = l \\ \int_0^z \quad (1 - z)dz = z - \dfrac{z^2}{2} \text{ if } \quad i = h \end{cases}$$

We normalize the market size to 1 and denote by θ the fraction of consumers who are type h consumers. In any period in which a consumer has access to the durable, he maximizes his utility by consuming it until the point that his marginal utility is equal to the price at which the contingent consumable can be obtained. Thus, if the consumable is available at price p_c, the amount of use that a consumer derives from the durable is as follows:

$$z_i(p_c) = \begin{cases} [1 - p_c]^+ & \text{if } i = h \\ [\alpha - p_c]^+ & \text{if } i = l \end{cases} \tag{4.1}$$

and the net utility associated with his use can be represented as follows

$$V_i(p_c) = U_i(z_i(p_c)) - z_i(p_c)p_c \tag{4.2}$$

In order to focus on the interaction between time inconsistency and hold-up, we assume that the manufacturer sells, rather than leases, the durable product. There are many reasons why manufacturers cannot lease durable products, including the moral hazard issues that arise when the actions taken by the user of a product are not observable. This may help to explain why leasing is not commonly observed in consumer electronics. We recognize that if the manufacturer were to lease her durable product, then time inconsistency could be eliminated, and the trade-off that we are to examine would not exist. However, because we often do not observe leasing of durables that require contingent consumables, e.g. the Kindle, iPhone, iPod, etc., our assumption that durables are sold is justified. In addition to assuming that the durables are sold, rather than leased, we assume that consumables are sold according to a simple linear pricing mechanism. In practice, there are many obstacles to the use of more sophisticated forms of pricing (e.g. quantity discounts / bundling), including the difficulty of preventing the resale of "broken" bundles, and consumers' preference for purchasing consumables as they need them instead of making a single, one-time purchase. Finally, we assume that the contingent consumables must be consumed in the period in which they are sold. That is, we do not allow for the possibility that consumers will stockpile consumables that they can either consume or sell in the future. This is most easily justified for situations in which the contingent consumables are intangible, e.g. the songs or e-books that a consumer may want to down-load in the future are yet to be created.

Throughout our analysis we assume marginal costs are zero. We do so only for ease of exposition, relaxing this to include positive marginal costs for consumables is straight forward and does not change the fundamental nature of the insights we obtain in the model. However, with positive marginal cost for durables, the monopolist's incentive to produce additional durables is decreasing in the cost of production, eventually eliminating such incentive altogether. For our results to be applicable, it is sufficient to assume that the marginal cost of production of durables is low enough that the low-type consumers obtain some net utility from consumption. As is common in the durable goods literature, we assume that the performance of the durable does not deteriorate, but we allow for second period profits/utilities to be discounted at $\rho \leq 1$.

In order to characterize the relationship among the prices and demands, we begin by deriving the inverse demand for consumables as a function of the quantity of durables (Q) in use. Consumers have demand for consumables only if they have access to a durable, and the price is below their maximum marginal utility, i.e. α for type l and 1 for type h . The cumulative quantity of durables $Q \in [0,1]$, but it is sufficient to restrict our attention to quantities $Q \in \{0, \theta, 1\}$. The demand for consumables $y\left(Q,\, p_c\right)$ at a given price p_c given that there are Q units of the durable in the market is as follows:

$$y\left(Q,\, p_c\right) = \begin{cases} 0 & \text{if } Q = 0 \\ \theta z_h\left(p_c\right) & \text{if } Q = \theta \\ \theta z_h\left(p_c\right) + (1-\theta) z_l\left(p_c\right) & \text{if } Q = 1 \end{cases} \tag{4.3}$$

It follows that the market clearing price for consumables is the following function of the quantity, Q, of durables in use and the quantity of consumables, y:

$$p_c\left(Q,\, y\right) = \begin{cases} 0 & \text{if } Q = 0 \\ 1 - \frac{y}{\theta} & \text{if } Q = \theta \text{ or } y \leq (1-\alpha)\theta \\ \theta + \alpha(1-\theta) - y & \text{otherwise} \end{cases} \tag{4.4}$$

In order to determine the price at which durables can be sold, we define the *implicit rental price* as the market clearing price at which a given quantity of the durable could be rented in a given period. It represents the maximum amount that the marginal consumer(s) would pay for access to the durable for a single period of use. In each period, the implicit rental price, r, is a function of the quantity (Q) of durables available and the consumables price (p_c) and is expressed as follows:

$$r(Q,\ p_c) = \begin{cases} V_h(p_c) = \frac{(1-p_c)^2}{2} & \text{if } Q = \theta \\ V_l(p_c) = \frac{(\alpha-p_c)^2}{2} & \text{if } Q = 1 \end{cases} \tag{4.5}$$

In period 2, the market clearing price at which the durable can be sold is simply the implicit rental price, since a consumer who buys it in period 2 obtains exactly one period of service from it. In period 1, the market clearing price at which the durable can be sold is equal to its implicit rental price in period 1 as given by (4.5), plus the discounted anticipated second period price.

4.2.1 Unrestricted Access to Consumables

We begin our analysis by considering durables for which consumers who have access to a unit of the durable have unrestricted access to consumption. A consumer has unrestricted access to consumption after the purchase of the durable if either no consumable is tied to consumption of the durable or if a competitive market supplies the consumable. Durables such as televisions and automobiles are examples in which consumables are freely available. Note that in most of the literature on durable products, it is typically assumed that access to the durable allows unrestricted access to its use. By unrestricted access we mean that the consumables are obtainable at marginal costs, which we have normalized to zero. By setting $p_c = 0$ in equation (5.2), the net utility for consumers under unrestricted access can be simplified as follows:

$$V_i(0) = \begin{cases} \frac{\alpha^2}{2} & \text{if } i = l \\ \frac{1}{2} & \text{if } i = h \end{cases}$$

When consumers have unrestricted access to consumables, the problem reduces to the classic problem of a durable goods monopolist, first recognized by Coase (1972) and further analyzed by Bulow (1982). The only decision that a monopolist manufacturer has to make is the quantity of output of the durable in each period. Bulow (1982) shows that the manufacturer has an incentive to continue to produce over time, driving down the market price of the durable. Because strategic consumers anticipate this, their willingness to pay decreases. This effect has been referred to as *time inconsistency*, in reference to the fact that the manufacturer's ability to extract monopolist rents is inconsistent with her own incentives over time. To provide a basis of comparison, we will now demonstrate the effect of time inconsistency in our framework.

Let Q_t be the quantity of durables in use in period t so that $Q_1 = \theta$ or 1. In period 2, the quantity Q_2 includes the Q_1 units that were in use in period 1 plus any additional sales made by the manufacturer in the second period. The market clearing price of the durable in period t is the price at which the lowest valuation consumer to purchase is indifferent between buying and not buying. Thus the market clearing price in period 2 is the implicit rental price $r(Q_2, 0)$. The manufacturer's profit in period 2 under unrestricted access can be expressed as:

$$\pi_2^U (Q_2, Q_1) = (Q_2 - Q_1)\, r\, (Q_2, 0)$$

Regardless of whether $Q_1 = 1$ or $Q_1 = \theta$, we will have that $\pi_2^U (Q_2, Q_1)$ is maximized when $Q_2 (Q_1) = 1$. That is in period 2 the manufacturer always sells to the low-type consumers if she has not done so earlier, and her conditionally optimal second period profit is:

$$\pi_2^U(Q_1) = \begin{cases} 0 & \text{if } Q_1 = 1 \\ (1-\theta)\frac{\alpha^2}{2} & \text{if } Q_1 = \theta \end{cases}$$

In period 1, consumers anticipate reduced durable prices in period 2. The market clearing price for the durable in period 1 is the implicit rental price of the durable in period 1 plus the discounted second period rental price $p_{d1}^U (Q_1) = r (Q_1, 0) + \rho r (Q_2 (Q_1), 0)$, which can be expressed as:

$$p_{d1}^U (Q_1) = \begin{cases} \frac{1}{2}\left(1+\rho\alpha^2\right) & \text{if } Q_1 = \theta \\ (1+\rho)\frac{\alpha^2}{2} & \text{if } Q_1 = 1 \end{cases}$$

First period profit of the manufacturer is $\pi_1^U (Q_1) = Q_1 p_{d1}^U (Q_1) + \pi_2^U (Q_1)$, which can be expressed as follows:

$$\pi_1^U (Q_1) = \begin{cases} (1+\rho)\frac{\alpha^2}{2} & \text{if } Q_1 = 1 \\ \frac{\theta}{2} + \rho\frac{\alpha^2}{2} & \text{if } Q_1 = \theta \end{cases} \tag{4.6}$$

Lemma 4.2.1. *There exists a threshold, $\alpha^U = \sqrt{\theta}$, such that:*

a) If the manufacturer could pre-commit to her second period output, she would sell to both groups if and only if $\alpha \geq \alpha^U$, and sell only to the high-type otherwise. All sales will be made in period 1, and no sales would be made in period 2.

b) If the manufacturer cannot pre-commit to her second period output, then she will sell to both groups in period 1 if and only if $\alpha \geq \alpha^U$. Otherwise, she will sell only to the high-type consumers in period 1, but will have an unavoidable incentive to sell to the low-type consumers in period 2.

The first part of the Lemma is obtained by using (4.5) to compare the per-period rental income, i.e. $r(Q, 0)$, for $Q = 1$ versus $Q = \theta$. Similarly, the second part can be shown from comparing the upper and lower branches of (4.6) to see that $\pi_1^U(1) \geq \pi_1^U(\theta)$ only if $(1+\rho)\frac{\alpha^2}{2} \geq \frac{\theta}{2} + \rho\frac{\alpha^2}{2}$, which is equivalent to $\alpha \geq \sqrt{\theta}$.

Thus, Lemma 4.2.1 demonstrates how time inconsistency plays out in our specific framework: When the manufacturer sells her product in period 1, her inability to pre-commit to future output levels causes her to sell to consumers with lower valuations than she would if she could pre-commit. Because this results in consumers anticipating a decline in price over time, it erodes the manufacturer's ability to extract rents from consumers who purchase the product in the first period. Stokey (1981) and Bulow (1982) recognize that this issue can be eliminated if the manufacturer leases her product, and Desai and Purohit (1998) point out that any credible commitment to avoid future output can serve as a means of mitigating the time inconsistency issue. In practice however, it is not always possible for the manufacturer to lease her product or to find a credible way to commit to future output levels.

It is of interest to compare the results of our model under unrestricted access to the results of Mussa and Rosen (1978) regarding the optimal design of a product line. Recall that they obtain a classical mechanism design result in the context of designing a product line for a set of consumers who are differentiated according to their valuation for quality. Specifically, they find that the optimal product line will be characterized by *efficiency at the top*, and *downward distortion of quality* at the bottom. In our context, since unrestricted access implies that the consumable can be obtained at its marginal cost of production, all consumers derive an efficient level of use from the durable product in the periods for which they have access to it. However, the manufacturer may postpone the sale of her durable to low-type consumers to prevent cannibalization of demand from the high-type consumers, and this is a form of downward distortion of quality. Note also that this is similar to the results of Moorthy and Png (1992) who analyzed the question of whether vertically differentiated products should be introduced simultaneously or sequentially. They find that, once the manufacturer sells to the high valuation consumers, she will have no incentive to distort the physical quality that she offers to consumers with lower valuations. However, as in our model, simply postponing the availability of the product for these low valuation consumers serves as a conceptual form of downward distortion.

So far, when we have considered only the situation in which consumers have free access to the consumable, our model demonstrates very close parallels to these classic product-line design results. However, as we will see below, under a lock-in policy in which consumers can obtain the consumables exclusively from the durable goods manufacturer, the optimal product line may not retain the *efficiency at the top* characteristic.

4.2.2 Controlled Consumption Through Lock-in

At the opposite end of the spectrum from providing unrestricted access to consumables is a monopoly over the consumables market. Here, we analyze the problem faced by a monopolist manufacturer of durables who can lock-in(LI) consumers to her product through a contingent consumable. In doing so we assume that the manufacturer cannot pre-commit to its output (or the price) of either the durable or the contingent consumable. In practice, there are examples of manufacturers who make explicit guarantees about the future availability of contingent consumables, e.g. replacement parts for aircraft, but there are also many examples in which it would be highly impractical for a manufacturer to do so, particularly in situations where the content of consumables that will be demanded by consumers in the future has yet to be determined, e.g. e-books, video-games, music downloads, etc.

As before, Q_t denotes the quantity of consumables in use in period t

. Let y_t denote the quantity of consumables available in period t . In period 2, the market clearing price for the consumables depends upon Q_2 and y_2 according to the function $p_c(Q_2, y_2)$ as defined in (4.3), and the price of the durable in period 2, p_{d2}, is given by the implicit rental price $r(Q_2, p_c(Q_2, y_2))$ as defined in (4.5). In the second period, the manufacturer solves the following problem:

$$\max_{Q_2, y_2} \pi_2^{LI}(Q_1, Q_2, y_2) = (Q_2 - Q_1)\, r(Q_2, p_c(Q_2, y_2)) + y_2\, p_c(Q_2, y_2) \quad (4.7)$$

The solution to this problem is characterized in the following Lemma:

Lemma 4.2.2. *a) If $Q_1 = 1$, then $Q_2^*(Q_1) = 1$, and the quantity (y_2^*) of consumables that is produced is adequate to induce both types of consumers to purchase it if and only if $\alpha \geq \frac{\sqrt{\theta}}{1+\sqrt{\theta}}$. Otherwise, the market clearing price for consumables exceeds the low-type consumers' maximum marginal utility, i.e. $p_c(Q_2^*, y_2^*) \geq \alpha$, and low-type consumers do not consume any units of the consumable.*

b) If $Q_1 = \theta$ then $Q_2^*(Q_1) = 1$ if and only if $\alpha \geq \sqrt{\frac{\theta}{2(1+\theta)}}$, while $Q_2^*(Q_1) = \theta$ otherwise. In either case, the manufacturer produces enough units of the consumable so that each consumer who owns a durable consumes a positive quantity of the consumable.

c) The second period consumables price at the conditionally optimal solution, $p_c(Q_2^*, y_2^*)$, is non-decreasing in Q_1.

d) For any Q_1, the second period consumables price at the conditionally optimal solution, $p_c(Q_2^*, y_2^*)$, is non-decreasing in θ.

The equilibrium quantities, prices of consumable and durables, and the profits of the manufacturer are summarized in Tables 4.1 and 4.2.

Table 4.1: Second period quantities $Q_1 = 1$

$Q_1 = 1$	$\alpha \leq \frac{\sqrt{\theta}}{1+\sqrt{\theta}}$	$\alpha \geq \frac{\sqrt{\theta}}{1+\sqrt{\theta}}$
$y_2^*(Q_1)$	$\frac{\theta}{2}$	$\frac{\alpha+\theta-\alpha\theta}{2}$
$p_c(Q_2^*, y_2^*)$	$\frac{1}{2}$	$\frac{\alpha+\theta-\alpha\theta}{2}$
$Q_2^*(Q_1)$	1	1
$r(Q_2^*, p_c(Q_2^*, y_2^*))$	0	$\frac{(\alpha-\theta+\alpha\theta)^2}{8}$
$\pi_2^L(Q_2^*, y_2^*)$	$\frac{\theta}{4}$	$\frac{(\alpha+\theta-\alpha\theta)^2}{4}$

Based on Lemma 4.2.2 we can see that, when the low-type consumers have a high maximum marginal utility for the consumable, i.e. $\alpha > \frac{\sqrt{\theta}}{1+\sqrt{\theta}}$, then in the second period the manufacturer sells durables to all consumers

Table 4.2: Second period quantities $Q_1 = \theta$

$Q_1 = \theta$	$\alpha \leq \sqrt{\frac{\theta}{2(1+\theta)}}$	$\alpha \geq \sqrt{\frac{\theta}{2(1+\theta)}}$
$y_2^*(Q_1)$	$\frac{\theta}{2}$	$\alpha(1-\theta) + \frac{\theta^2}{(1+\theta)}$
$p_c(Q_2^*, y_2^*)$	$\frac{1}{2}$	$\frac{\theta}{1+\theta}$
$Q_2^*(Q_1)$	θ	1
$r(Q_2^*, p_c(Q_2^*, y_2^*))$	$\frac{1}{8}$	$\frac{(\alpha-\theta+\alpha\theta)^2}{2(1+\theta)^2}$
$\pi_2^L(Q_2^*, y_2^*)$	$\frac{\theta}{4}$	$\frac{\alpha^2+\theta^2-\alpha^2\theta^2}{2(1+\theta)}$

who do not own it already and produces enough consumables that the market clearing price falls low enough to attract purchases from the low-type consumer, i.e. $p_c(Q_2^*, y_2^*) < \alpha$. However, the implications of the Lemma are more intriguing when the maximum marginal utility (α) of low-type consumers falls into the intermediate or low range. When maximum marginal utility is intermediate, i.e. $\sqrt{\frac{\theta}{2(1+\theta)}} \leq \alpha \leq \frac{\sqrt{\theta}}{1+\sqrt{\theta}}$, the manufacturer sells durables to the low-type consumers if they have not purchased the durable in period 1. However, the output of consumables is large enough to attract purchases from low-type consumers only when they have to wait until period 2 to acquire the durable. In this intermediate range of α, the manufacturer maximizes his income from consumables by restricting output so that $p_c(Q_2^*, y_2^*) \geq \alpha$. If both groups of consumers already hold the durable, this is precisely what she does. However, if the manufacturer postpones selling durables to low-type consumers until period 2, then she is concerned about her income from these additional durable sales in addition to her income from consumables. Because she can extract the net utility from the low-type consumers through her durable price, she has greater incentive to produce consumables when she waits until the second period to sell durables to low-type consumers. This is why, as shown in Figure 4.1, for $\alpha \geq \sqrt{\frac{\theta}{2(1+\theta)}}$ the second period consumables price is lower when $Q_1 = \theta$ than when $Q_1 = 1$. A similar explanation can be provided for part d) of the Lemma, which recognizes that the market clearing consumables price is non-decreasing in the portion of high-type consumers: When there are more high-type consumers, i.e. larger θ, a larger portion of her potential income in period 2 is driven by consumables sales to this group. This weakens the manufacturer's incentive to increase output of consumables beyond the quantity that maximizes the revenues from consumables sales to these high-type consumers in order to attract additional durables (and consumables) sales to low-type consumers.

Finally, when the maximum marginal utility for low-type consumers is very low, i.e. $\alpha \leq \sqrt{\frac{\theta}{2(1+\theta)}}$, the manufacturer has no reason to sell durables to low-type consumers. As was the case, for intermediate α, if low-type consumers already hold the durable, the manufacturer restricts

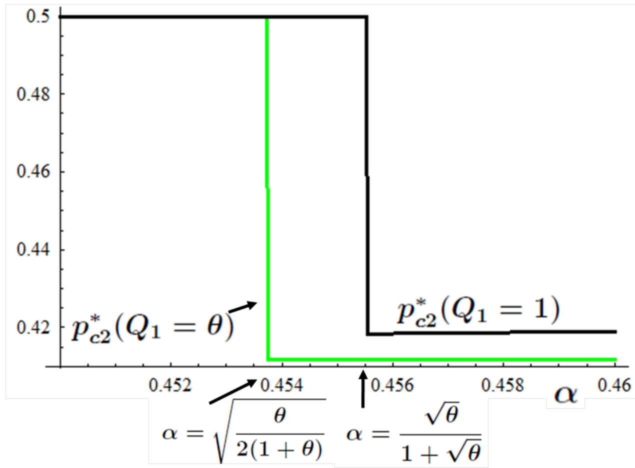

Figure 4.1: Conditionally optimal price of consumables in Period 2

output of consumables so that these low-type consumers do not purchase any units. The important thing to notice here is that, in contrast to the manufacturer's optimal strategy under unrestricted access, the manufacturer may not sell durables to the low-type consumers in the second period. In order to protect the revenues generated by consumable sales to the high-type consumers, the manufacturer avoids selling durables to the low-type consumers. Thus, the manufacturer's lock-in strategy not only generates revenues from consumables sales, it also helps to mitigate time-inconsistency. On the other hand, because the high-type consumers can anticipate this rationing of consumables, this creates a hold-up problem that, on its own, should tend to decrease their willingness to pay for the durable in period 1.

To explore this issue further, let us turn our attention to the problem that is faced by the manufacturer in the first period:

$$\max_{Q_1, y_1} \pi_1^{LI}(Q_1, y_1) = Q_1 \, p_{d1}^{LI}(Q_1, y_1) + y_1 \, p_c(Q_1, y_1) + \rho \pi_2^{LI}(Q_2^*(Q_1), y_2^*(Q_1)) \tag{4.8}$$

where the price of durable in period 1 is:

$$p_{d1}^{LI}(Q_1, y_1) = r(Q_1, y_1) + \rho r(Q_2^*(Q_1), y_2^*(Q_1)) \tag{4.9}$$

Note that this price depends not only on the quantities of durables and consumables that are made available in period 1, but also upon the consumers' anticipation of the manufacturer's optimal decisions in period 2. By substituting (4.4), (4.5), and the results from Table 4.2 into (4.9), we have that, when $Q_1 = \theta$, the first period durable price can be expressed as:

$$p_{d1}^{LI}(\theta, y_1) = \begin{cases} V_h(p_c(\theta, y_1)) + \frac{\rho}{8} & \text{for } \alpha \leq \sqrt{\frac{\theta}{2(1+\theta)}} \\ V_h(p_c(\theta, y_1)) + \rho \frac{(\alpha - \theta + \alpha\theta)^2}{2(1+\theta)^2} & \text{otherwise} \end{cases} \tag{4.10}$$

where, regardless of the value of α, the implicit rental price in period 1 is equal to the net utility, $V_h(p_c(\theta, y_1))$, that high-type consumer obtains when y_1 consumables are made available exclusively for high-type consumers. However, the implicit rental price for period 2 does depend upon the value of α. For $\alpha \leq \sqrt{\frac{\theta}{2(1+\theta)}}$, no additional durables are sold in the second period, and the implicit rental price reflects the net utility, $V_h(p_c)$, that a high-type consumer will receive from owning the product in the second period. As shown in Lemma 4.2.2, the manufacturer will restrict her second period output of consumables to maximize her revenue from selling to the locked-in high-type consumers, which will result in a market price of $p_c = \frac{1}{2}$. On the other hand, for $\alpha \geq \sqrt{\frac{\theta}{2(1+\theta)}}$, consumers anticipate that durables will be sold to low-type consumers in the second period, and the implicit rental price in period 2 reflects the net utility, $V_l\left(p_c\right)$, that a low-type consumer will receive from owning the durable. Recall that, when the manufacturer sells to the low-type consumers in period 2, she increases the availability of the consumable and extracts the low-type's net utility through the price of the durable, i.e. $r\left(Q_2^*\left(\theta\right), y_2^*\left(\theta\right)\right)$.

If $Q_1 = 1$, the logic is similar, but since all consumers will hold the durable in period 2, the implicit rental price for the second period always reflects the net utility that will be received by a low-type consumer. Substituting (4.4), (4.5), and the results from Table 4.1 into (4.9), we have that, when $Q_1 = 1$, the first period durables price can be expressed as:

$$p_{d1}^{LI}\left(1, y_1\right) = \begin{cases} V_l\left(p_c(1, y_1)\right) & \text{for } \alpha \leq \frac{\sqrt{\theta}}{1+\sqrt{\theta}} \\ V_l\left(p_c(1, y_1)\right) + \rho \frac{(\alpha-\theta+\alpha\theta)^2}{8} & \text{otherwise} \end{cases} \tag{4.11}$$

where, regardless of the value of α, the implicit rental price in period 1 is now equal to the net utility, $V_l\left(p_c(1, y_1)\right)$, that low-type consumer obtains when y_1 consumables are made available and both types of consumers have access to durables. Note that, for this case of $Q_1 = 1$, we will have $p_c\left(y_1\right) < \alpha$ if and only if $y_1 > (1-\alpha)\theta$. Thus, for any $y_1 \leq (1-\alpha)\theta$, we will have $V_l\left(p_c\left(y_1\right)\right) = 0$. The implicit rental price for the second period can be explained as follows: Recall from Lemma 4.2.2, that when $\alpha \leq \frac{\sqrt{\theta}}{1+\sqrt{\theta}}$ the manufacturer restricts her second period output of consumables to such an extent that the market clearing price exceeds the maximum marginal utility of the low-type consumers. This discourages them from consuming it and gives them a net utility of $V_l\left(p_c\left(Q_2^*, y_2^*\right)\right) = 0$. Otherwise, when $\alpha > \frac{\sqrt{\theta}}{1+\sqrt{\theta}}$, the manufacturer's maximization of second period consumables revenue results in a sufficient quantity of consumables that low-type consumers obtain a net utility of $V_l\left(p_c\left(Q_2^*, y_2^*\right)\right) = \frac{(\alpha-\theta+\alpha\theta)^2}{8}$.

We can now characterize the optimal solution to the manufacturer's problem when she *locks-in* consumers of her durable to purchasing the consumable exclusively from her.

Proposition 4.2.3. *(a) For every $\theta \in (0,1)$, there exists $\underline{\alpha}^{LI}(\theta)$ and $\overline{\alpha}^{LI}(\theta)$, where $\underline{\alpha}^{LI}(\theta) = \sqrt{\frac{\theta}{2(1+\theta)}} < \frac{\sqrt{\theta}}{1+\sqrt{\theta}} < \overline{\alpha}^{LI}(\theta) < 1$, such that:*

- *If $\alpha \leq \underline{\alpha}^{LI}(\theta)$, the manufacturer sells the durable to high-type consumers in period 1 and does not sell to low-type consumers at all , i.e. $Q_1^*(\alpha,\theta) = Q_2^*(\alpha,\theta) = \theta$.*

- *If $\underline{\alpha}^{LI}(\theta) \leq \alpha \leq \overline{\alpha}^{LI}(\theta)$ the manufacturer sells to high-type consumers in period 1 and low-type in period 2, i.e. $Q_1^*(\alpha,\theta) = \theta < Q_2^*(\alpha,\theta) = 1$.*

- *If $\alpha \geq \overline{\alpha}^{LI}(\theta)$, the monopolist sells to all consumers in period 1, i.e. $Q_1^*(\alpha,\theta) = Q_2^*(\alpha,\theta)) = 1$.*

b) In every period, the consumable is available in sufficient quantity that the market *clearing price falls below the maximum marginal utility of all consumers who hold the durable.*

The equilibrium prices of durable and consumables are given in Table 3. While the manufacturer always sells durables to high-type consumers in period 1, she may sell them to low-type consumers either in period 1, in period 2, or not at all. When $\alpha \leq \underline{\alpha}^{LI}(\theta)$, the manufacturer does not sell to the low-type consumers in either period, which contrasts sharply with the result that we obtained for the case of unrestricted access to the consumable. Recall that under unrestricted access, the manufacturer always sells to low-type consumers in period 2 if she has not already done so in period 1 because otherwise she would not obtain any income in period 2. However, under lock-in, if she does not sell durables to the low-type consumers in period 2, the manufacturer can focus exclusively upon consumables sales to high-type consumers. In fact, if she does sell additional durables in the second period, she faces a trade-off between the consumables revenue from high-type consumers versus the extraction of net utility from low-type consumers. Recall from Figure 4.1 that, when α increases to $\underline{\alpha}^{LI} = \sqrt{\frac{\theta}{2(1+\theta)}}$, the equilibrium price of consumables, $p^*_{c2}\left(Q_1 = \theta\right)$ falls dramatically as a consequence of the manufacturer's sudden willingness to sell durables to low-type consumers in period 2 in order to extract their net utility. Only when $\alpha \geq \underline{\alpha}^{LI}(\theta)$ is the manufacturer willing to compromise her revenue stream from selling consumables alone to just the high-type consumers in order to extract the net utility of low-type consumers by selling them durables in the second period.

Table 4.3: Equilibrium Prices under Lock-in

	Low $\alpha \leq \underline{\alpha}^{LI}(\theta)$	Medium $\underline{\alpha}^{LI}(\theta) \leq \alpha \leq \overline{\alpha}^{LI}(\theta)$	High $\alpha \geq \overline{\alpha}^{LI}(\theta)$
Period 1			
p_{d1}	$\frac{1}{8}(4+\rho)$	$\frac{1}{2}(1+\rho\frac{(\alpha+(1-\alpha)\theta)^2}{2(1+\theta)^2})$	$\frac{(\alpha+(1-\alpha)\theta)^2}{8}(4+\rho)$
p_{c1}	0	0	$\theta(1-\alpha)$
Period 2			
p_{d1}	$\frac{1}{8}$	$\frac{(\alpha+(1-\alpha)\theta)^2}{2(1+\theta)^2}$	$\frac{(\alpha+(1-\alpha)\theta)^2}{8}$
p_{c1}	$\frac{1}{2}$	$\frac{\theta}{1+\theta}$	$\frac{\alpha+\theta-\alpha\theta}{2}$

When the marginal utility of low-type consumers is not sufficiently high, i.e. $\alpha < \overline{\alpha}^{LI}$, then the manufacturer sells durables to only high-type consumers in period 1. Otherwise, for $\alpha \geq \overline{\alpha}^{LI}$, she sells to both groups in period 1 because the cannibalization effect is sufficiently weak that the manufacturer prefers not to postpone sales to the low-type consumers until the second period.

In Figure 4.2, we can see the relationship between the equilibrium first period consumables price, $p^*_{c1}(Q^*_1, y^*_1)$, and the maximum marginal utility (α) of low-type consumers. When $\alpha \leq \overline{\alpha}^{LI}$ so that $Q^*_1 = \theta$, the manufacturer can use the price of the durable product to extract the full net utility of the high-type consumers for their first period use. In order to maximize this net utility, she produces enough consumables to drive the market clearing price to zero, i.e. $p^*_{c1}(Q^*_1, y^*_1) = 0$ for $\alpha \leq \overline{\alpha}^{LI}$. However, when $\alpha \geq \overline{\alpha}^{LI}$

so that $Q_1^* = 1$, the durable price can extract only the net utility of the low-type consumers first period use. If she restricts output of consumables, she lowers the size of this extractable net utility, but she more than makes up for it in her revenues from consumables sales to both groups. As a result, we have $p_{c1}^*(Q_1^*, y_1^*) > 0$ when $\alpha \in [\overline{\alpha}^{LI}, 1)$. Observe that, in this range where $p_{c1}^*(Q_1^*, y_1^*) > 0$, the price of consumables is linearly decreasing in α.

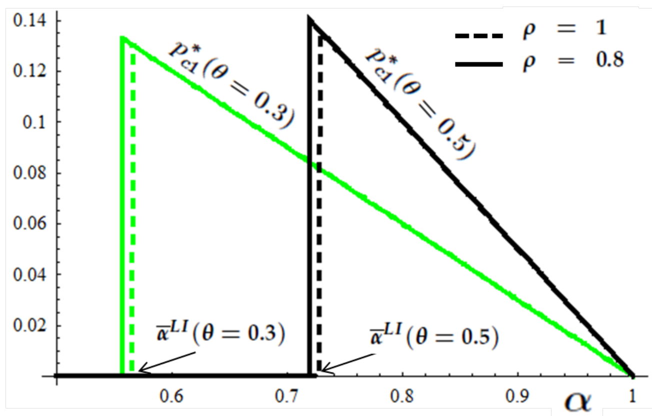

Figure 4.2: Equilibrium price of consumables in period 1

The reason for this is that, as α increases, the low-type consumers become more like the high-types and there is more potential net utility that can be extracted from them. Indeed as $\alpha \to 1$, the manufacturer can extract the full net utility from both high-and low-types, and maximizes these net utilities by driving the consumables price to zero. A second, closely related observation is that, within the range of α for which $Q_1^* = 1$ for both $\theta = 0.3$

and $\theta = 0.5$, the first period price of consumables is larger for $\theta = 0.5$ than for $\theta = 0.3$. As the portion of high-type consumers increases, the manufacturer shifts her emphasis away from the net utility that can be extracted from low-type consumers toward the revenues that she earns from consumables sales to high-type consumers. A third observation is that an increase in the proportion (θ) of high-type consumers from 0.3 to 0.5 increases the threshold, $\overline{\alpha}^{LI}$, above which the manufacturer sells durables to both groups in period 1. This reflects the fact that a larger portion of high-type consumers makes the manufacturer less willing to sacrifice any of the net utility that can be extracted from these consumers. Finally, we can see that increasing the discount factor from $\rho = 0.8$ (the solid lines) to $\rho = 1$ (the dashed lines), shifts $\bar{\alpha}^{LI}$ slightly higher. This is a consequence of the fact that a larger value of ρ implies less discounting of future cash flows and this increases the manufacturer's willingness to postpone the sales revenue that she might earn from low-type consumers until period 2.

It is of interest to compare the structure of our results for lock-in to our earlier results for the unrestricted access. Recall from our discussion at the end of Section 4.2.1 that, under unrestricted access, we retain the main features of the optimal mechanism design problem: efficiency at the top and downward distortion of quality. However, under lock-in this is no longer the case. Under lock-in, the price of consumables is positive in period 2, and may also be positive in period 1 (for $\alpha > \overline{\alpha}^{LI}$). This creates downward distortion for the low-type consumers, but it also implies that the high-type consumers obtain less than their efficient level of use from the durable in at least one period. Thus, while a lock-in strategy preserves the downward distortion property, it does not preserve efficiency at the top. Let us now evaluate the conditions under which the strategy of lock-in dominates that of providing unrestricted access.

Proposition 4.2.4. *For a given value of θ, there exists thresholds $\underline{\alpha} \leq \overline{\alpha}$ such that for $\alpha \geq \overline{\alpha}$ unrestricted access dominates lock-in and for $\alpha \leq \underline{\alpha}$ lock-in dominates unrestricted access.*

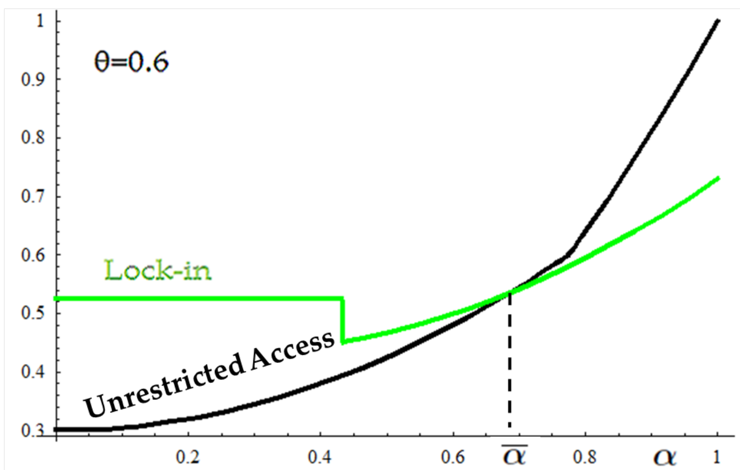

Figure 4.3: Profits under Unrestricted Access vs Lock-in

Although we have been unable to prove that the two thresholds in Proposition 4.2.4 are identical, i.e. $\underline{\alpha} = \bar{\alpha}$, we have been unable to identify a counter-example in an extensive set of numerical experiments. A representative comparison of the profits for the two strategies is depicted in Figure 4.3. Recall that the primary advantage of the lock-in strategy is the role that it plays in mitigating time inconsistency by reducing the manufacturer's incentive to sell durables to low-type consumers. On the other hand, a lock-in strategy also creates a hold-up issue with respect to consumers. When α is large, the two types of consumers are relatively homogeneous so that time inconsistency, which is a form of demand cannibalization, is not a big concern. Thus, the manufacturer is better off if she can allow consumers to have unrestricted access to consumables to eliminate the hold-up issue. On the other hand, as α decreases, the time inconsistency issue becomes increasingly important. At sufficiently low values of α, the time inconsistency issue dominates hold-up and the manufacturer is better off using a lock-in strategy.

4.2.3 Imperfect Competition in Consumables

Thus far, we have considered two opposing ends of a spectrum of strategies: unrestricted access and lock-in. Yet it is not uncommon to find situations in which manufacturers adopt strategies that involve limited competition in the market for consumables. For example, as mentioned earlier, Iomega eventually allowed Fuji, Verbatim, Toshiba, and Maxell to sell Zip disks, the contingent storage medium for its Zip drive. More recently, Amazon has been leaving strong hints that it will permit non-proprietary formats for digital books to be read on its Kindle (WIRED (2009)). To assess the question of whether some intermediate level of competition in the consumables market can be beneficial to a durable goods manufacturer, we now extend our model to allow for imperfect competition in the consumables market.

In our extended model, we allow for competition in the consumables market in only the second period. Note that, based on our comparison between *lock-in* and *unrestricted access* in Section 4.2.1, the main role that lock-in plays in influencing strategic consumers is in how it affects their anticipation of the output quantities in the second period. As we have defined *unrestricted access*, the manufacturer either designs her product so that it does not require a contingent consumable or she designs it to be compatible with an existing consumable that is provided by a competitive market. However, all of the potential benefits from doing this arise as a result of the way it eliminates the hold-up issue in the second period. At best, any competition in the first period does no harm to the manufacturer. Thus, by allowing for imperfect competition in period 2 only, we look at competition in the most favorable light. However, as observed at Iomega and at Amazon, it is not uncommon for a durable goods manufacturer to use a lock-in strategy early in a product's life-cycle while sending signals that it will permit competition in the consumables market later on. We also note that this assumption that competition occurs only in the second period is similar to the assumptions of Choi and Stefanadis (2001) and Carlton and Waldman

(2002).

The sequence of events in our extended model is as follows: In period 1, the manufacturer chooses Q_1 and y_1, the quantities of durables and consumables to produce without competition. The market clearing price for the durable depends upon consumers' anticipation of future durable sales and availability of complements. In period 2, the manufacturer faces competition in the consumables market from n rivals, whose products are perfect substitutes for the manufacturer's consumable. The manufacturer acts as a Stackelberg leader by choosing the cumulative quantity of the durable Q_2 and a quantity of consumables y_{20} to be made available in the market. In response to the manufacturer's actions each of the n rival suppliers of consumables simultaneously choose the output quantities $y_{2i}(y_{20})$ for $i = 1, 2, .., n$. As in our original model, Q_2 represents the total quantity of durables that are available in period 2, including those units sold in period 1. Hence, by definition, $Q_2 \geq Q_1$.

It is worth noting that, because our model of imperfect competition involves competition in only period 2, the limiting case ($n \rightarrow \infty$) will not converge to our model of *unrestricted access* that was presented in Section 4.2.1 where consumers have access to the complement at marginal cost in both periods. This distinction is intentional because our objective in this section is to identify the most advantageous form of competition from the perspective of the manufacturer. As discussed above, it is only the (delayed) competition in the second period that has the potential to benefit the manufacturer.

The prices of the consumables and durables are determined by their inverse demand functions given in (4.4), where the total consumables output is $y_2 = y_{20} + y_{21} + .. + y_{2n}$. In period 2, each of the n rivals choose y_{2i} in response to the manufacturers actions to maximize:

$$\Pi_i \left(Q_2 \left(Q_1\right), Y, y_{20}\right) = y_{2i} p_c \left(Q_2, y_{20} + y_N\right) \tag{4.12}$$

where $y_N = y_{21} + .. + y_{2n}$ and Y is the n-dimensional vector of y_{2i}, where $i = 1, 2, .., n$. The manufacturer chooses Q_2 and y_{20} to maximize:

$$\begin{aligned} \pi_2^n \left(Q_2 \left(Q_1\right), y_{20}, Y^* \left(Q_2, y_{20}\right)\right) &= \left(Q_2 - Q_1\right) r \left(Q_2, p_c \left(Q_2, y_{20} + \sum_{i=1}^{n} y_{2i}^* \left(Q_2, y_{20}\right)\right)\right) \\ &\quad + y_{20}\, p_c \left(Q_2, y_{20} + \sum_{i=1}^{n} y_{2i}^* \left(Q_2, y_{20}\right)\right) \end{aligned} \tag{4.13}$$

where $Y^* \left(Q_2, y_{20}\right)$ is the n-dimensional vector of the optimal response of the rivals $y_{2i}^* \left(Q_2, y_{20}\right)$, resulting from the simultaneous maximization of (4.12) for $i = 1, 2, .., n$ rivals. The equilibrium to this second period game is conditional upon Q_1 and can be characterized as follows:

Lemma 4.2.5. *a) If $Q_1 = 1$, then $Q_2^* \left(Q_1\right) = 1$, and there exists two thresholds, $\alpha_1(\theta, n) \geq \alpha_r(\theta, n) = \frac{\sqrt{\theta}}{(n+1)\left(1+\sqrt{\theta}\right)}$, such that the quantity (y_2^*) of consumables that is produced is adequate to induce both types of consumers to purchase it if and only if $\alpha \geq \alpha_1(\theta, n)$. Otherwise, the market clearing price for consumables exceeds the low-type consumers' maximum marginal utility, i.e. $p_c \left(Q_2^*, y_2^*\right) \geq \alpha$, and low-type consumers do not consume any units of the consumable.*

b) If $Q_1 = \theta$ *then there exists a threshold,* $\alpha_\theta(\theta, n) = \sqrt{\frac{\theta(2n+1)}{2(n+1)(1+2n+\theta)}}$*, such that* $Q_2^*(Q_1) = 1$ *if* $\alpha \geq \alpha_\theta(\theta, n)$*, while* $Q_2^*(Q_1) = \theta$ *otherwise. In either case, the quantity* (y_2^*) *of consumables that is produced is adequate to ensure that each consumer who owns a durable consumes a positive quantity of the consumable.*

c) For any Q_1*, the second period consumables price,* $p_c(Q_2^*, y_2^*)$*, is non-decreasing in* θ*.*

In Tables 4.4 and 4.5, we summarize the second period equilibrium that is conditional upon Q_1. Note that all of the values in these tables converge to the corresponding values in Tables 4.1 and 4.2 when $n = 0$, reflecting the fact that lock-in is a special case of imperfect competition.

For high values of α, the structure of this second period sub-game is quite similar to the corresponding sub-game for lock-in. For both settings, when α is sufficiently large, in this case $\alpha \geq \alpha_\theta(\theta, n)$, the manufacturer sells durables to all consumers who do not own it already and enough consumables are produced so that $p_c(Q_2^*, y_2^*) < \alpha$ and low-type consumers purchase positive amounts of consumable. The threshold $\alpha_\theta(\theta, n)$ is decreasing in n, which implies that, as the number of competitors increases, the manufacturer becomes increasingly willing to sell durables to consumers with low marginal valuations if she has not done so already. Because it is this increased willingness to sell to consumers with low valuations that gives rise to time inconsistency, we can see how increased amounts of competition in the consumables market can aggravate consumers' concerns about the price of the durable decreasing over time.

Similarly, for low values of α the structure of this second period sub-game is quite similar to the corresponding sub-game for lock-in. When α is very low, the manufacturer does not sell durables to low-type consumers who do not already own them, and the equilibrium price of consumables would preclude their consumption by a low-type consumer even if he or she had access to a durable.

Table 4.4: Second period quantities $Q_1 = \theta$

$Q_1 = \theta$	$\alpha \leq \alpha_\theta(\theta, n)$	$\alpha \geq \alpha_\theta(\theta, n)$
$y^*_{20}(Q_1)$	$\frac{\theta}{2}$	$\alpha(1-\theta) + \frac{\theta^2}{(1+\theta)}$
$y^*_{2i}\left(Q_1\right)$	$\frac{\theta}{2(n+1)}$	$\frac{\theta}{1+2n+\theta}$
$p_c\left(Q^*_2, y^*_{20}, Y^*\right)$	$\frac{1}{2(n+1)}$	$\frac{\theta}{1+2n+\theta}$
$Q^*_2\left(Q_1\right)$	θ	1
$r\left(Q^*_2 p_c\left(Q^*_2, y^*_{20}, Y^*\right)\right)$	$\frac{(2n+1)^2}{8(n+1)^2}$	$\frac{(\alpha(1+2n+\theta)-\theta)^2}{2(1+2n+\theta)^2}$
$\pi^n_2\left(Q^*_2, y^*_{20}, Y^*\right)$	$\frac{\theta}{4(n+1)}$	$\frac{\alpha^2(1-\theta)(1+2n+\theta)+\theta^2}{2(1+2n+\theta)}$
$\Pi_i\left(Q^*_2, y^*_{20}, Y^*\right)$	$\frac{\theta}{4(n+1)^2}$	$\frac{\theta^2}{(1+2n+\theta)^2}$

It is also of interest to observe from Tables 4.4 and 4.5 how the second period consumables price is affected by the number, n, of rivals in the consumables market. First, we can see that, for sufficiently large values of α, the second period consumables price is lower when $Q_1 = \theta$ than when $Q_1 = 1$. This can be confirmed by observing that $\frac{\theta}{1+2n+\theta} < \frac{\alpha+\theta-\alpha\theta}{2(n+1)}$ so long as $\alpha > \frac{\theta}{1+2n+\theta}$. The reason for this difference in consumables price is the same as it was under lock-in; when the manufacturer sells durables to low valuation consumers in the second period, she can extract their surplus through the price of the durable and she can increase the size of this extractable surplus by increasing output (driving down prices) of consumables. We can also see from the tables, that the second period consumables price is decreasing in n with the following exception: As shown in Table 4.5, when $Q_1 = 1$ and $\alpha \in (\alpha_r(\theta, n), \alpha_1(\theta, n))$, the price of consumables is independent of n. The reason for this is that, in this interval, the manufacturer reduces her own output to the maximum amount that eliminates the incentive for any one of the rivals to unilaterally increase its own output by enough to drive the consumables price below α. Although the amount by which the manufacturer must reduce her own output certainly does depend upon the number of rivals, the aggregate quantity, and consequently the price, of consumables is (interestingly) independent of n. However, as we will soon see, this situation does not arise in the equilibrium.

Table 4.5: Second period quantities $Q_1 = 1$

$Q_1 = 1$	$\alpha \leq \alpha_r(\theta, n)$	$\alpha \in (\alpha_r(\theta, n), \alpha_1(\theta, n))$	$\alpha \geq \alpha_1(\theta, n)$
$y^*_{20}(Q_1)$	$\frac{\theta}{2}$	$\theta - \frac{(n+1)\sqrt{\theta}\alpha\left(1+\sqrt{\theta}\right)}{2}$	$\frac{\alpha+\theta-\alpha\theta}{2}$
$y^*_{2i}\left(Q_1\right)$	$\frac{\theta}{2(n+1)}$	$\frac{\alpha\sqrt{\theta}\left(1+\sqrt{\theta}\right)}{2}$	$\frac{\alpha+\theta-\alpha\theta}{2(n+1)}$
$p_c\left(Q^*_2, y^*_{20}, Y^*\right)$	$\frac{1}{2(n+1)}$	$\frac{\alpha\left(1+\sqrt{\theta}\right)}{2\sqrt{\theta}}$	$\frac{\alpha+\theta-\alpha\theta}{2(n+1)}$
$Q^*_2\left(Q_1\right)$	1	1	1
$r\left(Q^*_2 p_c\left(Q^*_2, y^*_{20}, Y^*\right)\right)$	0	0	$\frac{(\alpha(1+2n+\theta)-\theta)^2}{8(1+n)^2}$
$\pi^n_2\left(Q^*_2, y^*_{20}, Y^*\right)$	$\frac{\theta}{4(n+1)}$	$\frac{\alpha\left(1+\sqrt{\theta}\right)^2}{4}$	$\frac{(\alpha+\theta-\alpha\theta)^2}{4(n+1)}$
$\Pi_i\left(Q^*_2, y^*_{20}, Y^*\right)$	$\frac{\theta}{4(n+1)^2}$	$\frac{\alpha\left(1+\sqrt{\theta}\right)\left(2\sqrt{\theta}-(n+1)\left(1+\sqrt{\theta}\right)\right)}{4}$	$\frac{(\alpha+\theta-\alpha\theta)^2}{4(n+1)^2}$

Let us now turn our attention to the manufacturer's first period problem under imperfect competition. In the first period, the manufacturer seeks to maximize her profit which can be expressed as:

$$\begin{aligned} \pi^n_1\left(Q_1, y_1\right) &= Q_1\, p^C_{d1}\left(Q_1, y_1\right) + y_1\, p_c\left(Q_1, y_1\right) \\ &\quad + \rho\pi^n_2\left(Q^*_2\left(Q_1\right), y^*_{20}\left(Q_1\right), Y^*\left(Q^*_2\left(Q_1\right), y^*_{20}\left(Q_1\right)\right)\right) \end{aligned} \tag{4.14}$$

where the price of durable in period 1 is:

$$p_{d1}^{C}\left(Q_1,y_1\right)=r\left(Q_1,y_1\right)+\rho r\left(Q_2^*\left(Q_1\right),y_{20}^*\left(Q_1\right),Y^*\left(Q_2^*\left(Q_1\right),y_{20}^*\left(Q_1\right)\right)\right) \tag{4.15}$$

As under lock-in, the price of the durable depends on the quantities of durables and consumables that are made available in period 1, and the consumers anticipation of the manufacturer's and the rivals' optimal decisions in period 2. By substituting (4.4), (4.5), and the results from Table 4.4 into (4.15), we have that, when $Q_1 = \theta$, the first period durable price can be expressed as:

$$p_{d1}^{C}\left(\theta,y_1\right)=\begin{cases} V_h\left(p_c\left(\theta,y_1\right)\right)+\rho\frac{(2n+1)^2}{8(n+1)^2} & \text{for } \alpha\leq\alpha_\theta(\theta,n) \\ V_h\left(p_c\left(\theta,y_1\right)\right)+\rho\frac{(\alpha(1+2n+\theta)-\theta)^2}{2(1+2n+\theta)^2} & \text{otherwise} \end{cases} \tag{4.16}$$

where the implicit rental price in period 1 is equal to $V_h\left(p_c\left(\theta,y_1\right)\right)$, the net utility that a high-type consumer obtains when y_1 consumables are made available in period 1 for high-type consumers only. The implicit rental price for period 2 depends upon the value of α; for $\alpha\leq\alpha_\theta(\theta,n)$, no additional durables are sold in the second period and therefore $V_h(p_c)$ represents the implicit rental price. As shown in Table 4.4, when the manufacturer does not sell additional durables, the total output of consumables is such that the market price of consumables is $p_c=\frac{1}{2(n+1)}$. For $\alpha\geq\alpha_\theta(\theta,n)$, consumers anticipate that the durables will be sold to low-type consumers in the second period, and therefore $V_l(p_c)$ represents the implicit rental price in period 2.

If $Q_1 = 1$, the implicit rental price for the second period always reflects the net utility that will be received by a low-type consumer. Substituting (4.4), (4.5), and the results from Table 4.5 into (4.15), we have that, when $Q_1 = 1$, the first period durables price can be expressed as:

$$p_{d1}^{C}(1, y_1) = \begin{cases} V_l\left(p_c\left(1, y_1\right)\right) & \text{for } \alpha \leq \alpha_1(\theta, n) \\ V_l\left(p_c\left(1, y_1\right)\right) + \rho \frac{(\alpha(1+2n+\theta)-\theta)^2}{8(1+n)^2} & \text{otherwise} \end{cases} \tag{4.17}$$

where the implicit rental price in period 1 is equal to $V_l\left(p_c\left(1, y_1\right)\right)$, the net utility that a low-type consumer obtains when y_1 consumables are made available in period 1 and both types have access to durables. Recall that for the case of $Q_1 = 1$, we will have $p_c\left(1, y_1\right) < \alpha$ if and only if $y_1 > (1-\alpha)\theta$ and for any $y_1 \leq (1-\alpha)\theta$, we will have $V_l\left(p_c\left(1, y_1\right)\right) = 0$. From Lemma 4.2.5, that when $\alpha \leq \alpha_1(\theta, n)$ the second period output of consumables is sufficiently low so that the market clearing price exceeds the maximum marginal utility of the low-type consumers, resulting in a net utility of $V_l\left(p_c\left(Q_2^*, y_2^*\right)\right) = 0$. Otherwise, when $\alpha > \alpha_1(\theta, n)$, the second period consumables output is sufficiently high so that low-type consumers obtain a net utility of $V_l\left(p_c\left(Q_2^*, y_2^*\right)\right) = \frac{(\alpha(1+2n+\theta)-\theta)^2}{8(1+n)^2}$.

We can now characterize the optimal solution to the game in which the manufacturer is by herself in period 1 but is a Stackelberg leader in period 2.

Proposition 4.2.6. *(a) There exists two thresholds, $\underline{\alpha}^C(n, \theta)$ and $\overline{\alpha}^C(n, \theta)$, such that $\overline{\alpha}^C(n, \theta) \geq \underline{\alpha}^C(n, \theta) = \alpha_\theta(\theta, n)$ where:*

- *If* $\alpha \leq \underline{\alpha}^C(n,\theta)$*, the manufacturer sells the durable to high-type consumers in period 1 and does not sell to low-type consumers at all , i.e.* $Q_1^*(\alpha,\theta) = Q_2^*(\alpha,\theta) = \theta$*.*

- *If* $\alpha \in \left(\underline{\alpha}^C(n,\theta), \overline{\alpha}^C(n,\theta)\right)$ *, the manufacturer sells to high-type consumers in period 1 and low-type in period 2, i.e.* $Q_1^*(\alpha,\theta) = \theta < Q_2^*(\alpha,\theta) = 1$*.*

- *If* $\alpha \geq \overline{\alpha}^C(n,\theta)$*, the monopolist sells to all consumers in period 1, i.e.* $Q_1^*(\alpha,\theta) = Q_2^*(\alpha,\theta) = 1$*.*

b) In both periods, the consumable is available in sufficient quantity that the market clearing price falls below the maximum marginal utility of all consumers who hold the durable.

Table 4.6: Equilibrium Prices under Stackelberg Competition

	Low α	Medium α	High α
	$\alpha \leq \underline{\alpha}^C(n,\theta)$	$\underline{\alpha}^C(n,\theta) \leq \alpha \leq \overline{\alpha}^C(n,\theta)$	$\alpha \geq \overline{\alpha}^C(n,\theta)$
Period 1			
p_d	$\frac{1}{2}\left(1+\rho\left(\frac{2n+1}{2n+2}\right)^2\right)$	$\frac{1}{2}\left(1+\rho\left(\alpha-\frac{\theta}{1+2n+\theta}\right)^2\right)$	$\frac{1}{2}\left((\alpha-(1-\alpha)\theta)^2+\rho\left(\alpha-\frac{\alpha+\theta-\alpha\theta}{2(n+1)}\right)^2\right)$
p_c	0	0	$\theta(1-\alpha)$
Period 2			
p_d	$\frac{(2n+1)^2}{8(n+1)^2}$	$\frac{(\alpha(1+2n+\theta)-\theta)^2}{2(1+2n+\theta)^2}$	$\frac{(\alpha(1+2n+\theta)-\theta)^2}{8(1+n)^2}$
p_c	$\frac{1}{2(n+1)}$	$\frac{\theta}{1+2n+\theta}$	$\frac{\alpha+\theta-\alpha\theta}{2(n+1)}$

The equilibrium prices of the durable and the consumable under imperfect competition are given in Table 4.6. The basic structure is similar to that of lock-in. Let us now consider the conditions under which imperfect competition dominates a lock-in strategy.

Proposition 4.2.7. *For every $\theta \in (0,1)$ and $n \geq 1$, there exists thresholds, $0 < \underline{\alpha}(n,\theta) < \overline{\alpha}(n,\theta) < 1$, such that the manufacturer benefits from competition in the consumables market for all $\alpha < \underline{\alpha}(n,\theta)$ and $\alpha > \overline{\alpha}(n,\theta)$.*

The above result is illustrated in Figures 4.4 and 4.5. Recall from Section 4.2.2 that unrestricted access dominates lock-in only when α sufficiently high. This is because, while unrestricted access completely mitigates the hold-up problem, it exacerbates time inconsistency to the extent that the manufacturer always sells to low-type consumers in period 2, regardless of how low their marginal utility (α) is. However, as we can see in the figures, imperfect competition can dominate lock-in either when α is very high or when it is very low. While imperfect competition does not completely eliminate the hold-up issue, neither does it necessarily induce the manufacturer to sell to low-type consumers when their marginal utility is very low. Consequently, when α is sufficiently low that the imperfect competition would not induce durable sales to low-type consumers, the manufacturer can still benefit from the role that is played by the competition in mitigating hold-up.

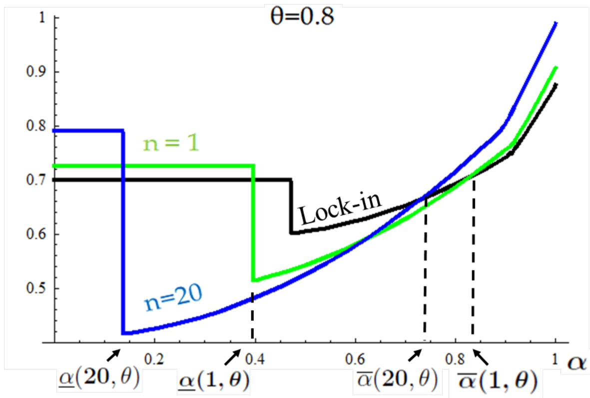

Figure 4.4: Profits of Manufacturer under Competition

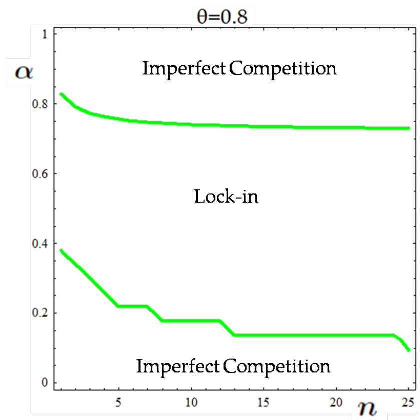

Figure 4.5: Benefit for the Manufacturer as a Function of Competition (n) and Differentiation (α)

In Figure 4.4, we can also see that, the benefits of competition are not necessarily monotone in the intensity of competition as measured by the number, n, of rivals. For example, for α in the interval $(\underline{\alpha}(20,\theta), \underline{\alpha}(1,\theta))$, we can see that although a single rival $(n = 1)$ would be beneficial, many rivals $(n = 20)$ would be disastrous. The reason for this is that a single rival would help to mitigate the hold-up problem without eroding the manufacturer's consumables revenue by so much that she will sell to low-type consumers in period 2. However, for $n = 20$, the cannibalization of consumables revenue would induce sales of durables to low-type consumers in period 2, and the combined loss of consumables revenue as well as the implications for time inconsistency would dominate any benefits from mitigating hold-up.

It is of some interest to compare how the three different strategic approaches that we have considered, i.e. unrestricted access, lock-in, and imperfect competition, compare in terms of the total quantity of durable products that are produced by the manufacturer. Because the issue of time inconsistency is based on the idea that a durable goods manufacturer's own incentives over time cause her to produce more durables than she would if she could pre-commit to her output quantity, the total quantity produced is a reasonable proxy for the strength of the time inconsistency effects. The larger the quantity, the stronger the time inconsistency effects.

Proposition 4.2.8. *The total quantity of durables sold is lowest under lock-in, next lowest under imperfect second period competition, and highest under unrestricted access: i.e.* $Q_2^{LI} \leq Q_2^C \leq Q_2^U$.

This result confirms that, as the manufacturer faces more competition in the consumables market, she has stronger incentive to sell to low-type consumers. When the manufacturer controls consumption through lock-in, it is beneficial to sell durables to low-type consumers only if their net utility is sufficiently high. However, the manufacturer can increase the net utility of low-type consumers only by driving down the price of consumables through increased output, and this interferes with the maximization of the revenue that she can earn from selling consumables to high-type consumers who are already locked-in. Consequently, a lock-in strategy decreases the manufacturer's temptation to reduce the price of her durable over time. Because competition in the consumables market erodes the potential profit that the manufacturer can make from selling consumables to locked-in consumers, it increases her temptation to sell additional durables. However, even with imperfect competition in the consumables market, the manufacturer has less incentive to sell durables to consumers with lower marginal utilities than under unrestricted use. Recall that, as we have modeled imperfect competition, not only does it involve a lower intensity of competition, it also protects the manufacturer from competition in period 1, and both of these differences will tend to reduce the manufacturer's incentive to sell durables to consumers with low marginal utilities.

4.3. Managerial Implications and Concluding Remarks

We demonstrate that a manufacturer's lock-in strategy has two opposing effects upon strategic consumers. The first effect is the hold-up issue that results from consumers' anticipation that, once they have purchased a durable, the manufacturer's incentive to maximize profits from selling consumables will interfere with their efficient utilization of the durable. This effect tends to reduce consumers' willingness to pay for the durable. However, the second effect is that the manufacturer's revenue stream from consumable sales to existing consumers may dampen her incentive to reduce durables prices over time. Thus, while a lock-in strategy creates a hold-up problem with respect to consumers, it can also partially mitigate time inconsistency, and these two effects impose opposing forces upon consumers' willingness to pay for the durable.

In the absence of lock-in, where either no consumable is required or where consumables are supplied by a competitive industry, consumers anticipate unrestricted use of the durable, but they worry about time inconsistency, i.e. the manufacturer's incentive to reduce the price of its durable over time. With lock-in, once the manufacturer has sold durables to consumers with high marginal utilities, she has two potential sources of revenue in the second period: sales of consumables to the locked-in consumers, and sales of both consumables and durables to the consumers with lower marginal utilities, and this creates a trade-off. If the manufacturer were concerned only with the revenue generated from selling consumables to high marginal utility consumers, she would restrict availability to keep the market clearing price high. Alternatively, if she sells durables to consumers with lower marginal utilities, she will have two reasons to reduce the price of consumables: First, the greater price sensitivity of the new consumers will alone provide an incentive for her to reduce prices. Second, because the manufacturer can extract the net utility of these new consumers through the price of her durable, lower consumables prices increase the size of the net utility that can be extracted. Of course, lower consumables prices also reduce the revenue that she earns from selling consumables to the consumers who are already locked-in, and this serves as a deterrent to selling durables to consumers whose marginal utilities are too low.

Our results indicate that, in order for a manufacturer to benefit from a lock-in strategy, consumers need to be sufficiently differentiated in terms of their marginal utilities. When consumers are relatively homogeneous, then time inconsistency is only a minor concern. Thus, a lock-in policy would introduce the pernicious effects of hold-up without significant offsetting benefits. Indeed, in the extreme where every consumer has the same marginal utility for the use of the durable, the manufacturer should make every effort to provide them with unrestricted access, either by designing her product to not require a consumable or by designing it to be compatible with a standardized consumable that may already exist in the market.

On the other hand, in situations where consumers are more differentiated in terms of their marginal utilities, the manufacturer has more to gain from having some way to restrict access to consumables. Specifically, by using either a pure lock-in strategy or by allowing limited competition, the manufacturer can use the complement to mitigate time inconsistency. Our results also provide some guidelines for how a manufacturer should interact with potential competitors in the market for consumables. First, competition is beneficial only because of the role that it can play in providing assurance to consumers about *future* availability of consumables. Therefore, it is vital that consumers *believe* that competition will exist in the future. Of course, one way for a manufacturer to provide credible assurance of such competition is to make her product compatible with consumables that are already an industry standard. In our analysis, we referred to this as the *unrestricted access* approach. While this approach eliminates the hold-up problem, it leaves the manufacturer completely exposed to time inconsistency and vulnerable to the adverse effects of competition at the beginning of the product life cycle. Nevertheless, this can be an appropriate strategy when consumers are very homogeneous. Second, the effect of competition upon the manufacturer's profit is not necessarily monotone in the intensity of competition. While increased competition reduces hold-up, it also cannibalizes the manufacturer's sales of consumables and increases the effects of time inconsistency by increasing the manufacturer's incentive to sell durables to consumers with lower marginal utilities. Thus, when consumers are moderately or highly differentiated, it may be most advantageous to encourage a few selected rivals to supply consumables to the market after the initial stage of the product life cycle. For example, Amazon's recent announcements about the possibility of providing alternative sources of e-books for its Kindle reading device seem to be aimed at reassuring consumers of the future availability of consumables from alternative suppliers. Similarly, although the manufacturers of smart-phones, such as Apple's iPhone, do not produce the consumables (minutes on telecommunications networks) for their products themselves, their strategies of requiring consumers to use one or two selected telecommunications carriers may help them to balance the trade-off between hold-up and time inconsistency.

Chapter 5

Durable Products and Contingent Consumables: The effect of Consumable Stock-piling on the Interaction between Time Inconsistency and Lock-in.

5.1. Introduction

Most durable equipment need contingent consumables in order for consumers to obtain continual use of the product, and it is fairly common for durable product manufacturers to lock-in consumers by controlling the supply of these consumables. For example, manufacturers of expensive machines such as aircrafts, construction equipment, etc., often are the sole suppliers of product-specific replacement parts; Abbot Labs, which manufactures FreeStyle glucose monitors, is an exclusive supplier of the disposable strips needed for the glucose tests. Inkjet manufacturers such as HP recommend the use of proprietary cartridges for their inkjets and it is widely believed that the revenues from toner cartridges form a significant portion of the profits for an inkjet manufacturer.

The strategy of locking into its consumable empowers the durable goods manufacturer in the consumables market; one way that a durable product manufacturer can profit from locked-in consumers is by raising the prices of consumables. AT&T's refusal to supply replacement switches to one of its customers (Bell Atlantic) at reasonable prices is often cited as an example of an abuse of power resulting from lock-in. When consumers are strategic, the difference in prices of consumables over time that arises from the manufacturer's propensity to exploit locked-in consumers creates an incentive for consumers to stock-pile consumables for future use.

In this chapter we investigate how consumers' incentive to counter lock-in by stockpiling consumables, interacts with the manufacturer's policy for the sale of durables and consumables. The durable product manufacturer, as such faces, the notorious problem of *time inconsistency*, which is her disincentive to commit to future prices that will allow her to preserve the market value of the durable. In addition, the stockpile of consumables interferes with the manufacturer's ability to profit from locked-in consumers, which in turn, influences the manufacturer's strategy for selling the durables.

Thus, a durable product manufacturer's lock-in policy will induce consumers to stock-pile consumables for strategic reasons thereby affecting the manufacturer's incentives for future production of the durables. Our analysis shows that the effect of consumable stock-piling upon the incentives of the durable good manufacturer depend on the extent of differentiation among consumers when consumer inventory is transferable. In addition, we find that the effect of consumer stockpiling can be likened to the presence of competition in the consumables market.

5.2. Model

We adopt and enhance the consumer model developed in Chapter 4 to account for consumer stock-piling. Specifically, as in Chapter 4 we assume that consumers derive utility from the durable only by using it in conjunction with a contingent consumable. As in Chapter 4, let z be the amount of use that a consumer derives from the durable in a given period. Recall that consumers are heterogeneous in their valuation for the use of the durable; the high (h) and the low (l) types, whose marginal utilities from consuming z units of the consumable are $U_h^{'}(z) = 1 - z$ and $U_l^{'}(z) = \alpha - z$ respectively, where $\alpha < 1$. By construction, we can see that the utilities for both type of consumers are concave and are given as follows.

$$U_i(z) = \begin{cases} \int_0^z \quad (\alpha - z)dz = \alpha z - \dfrac{z^2}{2} \text{ if } \quad i = l \\ \int_0^z \quad (1 - z)dz = z - \dfrac{z^2}{2} \text{ if } \quad i = h \end{cases}$$

We normalize the market size to 1 and denote by θ the fraction of consumers who are type h consumers. In any period in which a consumer has access to the durable, he maximizes his utility by consuming it until the point that his marginal utility is equal to the price at which the contingent consumable can be obtained. Recall that if the consumable is available at price p_c, the amount of use that a consumer derives from the durable is as follows:

$$z_i(p_c) = \begin{cases} [1 - p_c]^+ & \text{if } i = h \\ [\alpha - p_c]^+ & \text{if } i = l \end{cases} \tag{5.1}$$

and the net utility associated with his use can be represented as follows

$$V_i(p_c) = U_i(z_i(p_c)) - z_i(p_c)p_c \tag{5.2}$$

Consistent with the assumptions in Chapter 4, we assume that the manufacturer sells, rather than leases, the durable product. Further, we allow for the possibility that consumers will purchase consumables that they can either consume now or stock-pile for future use. The sequence of events is as follows: In period 1, the manufacturer chooses Q_1 and y_1 , the quantities of durables and consumables respectively. The prices at which the market clears depend on the response of rational consumers, who purchase consumables for consumption in the current period, and stock-pile consumables for future use in anticipation of the second period equilibrium. In period 2, the manufacturer responds to consumers' stock-pile of consumables by choosing a cumulative quantity of the durables, Q_2 and the quantity of consumables y_2 . To keep the exposition simple, we assume that marginal costs of production of the consumable and the durable are zero. Allowing for positive marginal costs of the consumable, does not alter the insights obtained from our model. However, sufficiently high marginal costs of production for the durable eliminates time inconsistency, as the manufacturer's incentive to produce additional durables decreases. For our results to be applicable, it is sufficient to assume that the marginal cost of production of durable is low enough so that the low type consumers obtain some surplus from consumption.

In order to avoid notational density, we also make the following assumptions ; $i)$ the manufacturer and consumers do not discount future utilities and profits and $ii)$ the holding cost incurred by consumers for the stock-piled consumables is zero. Although, it is desirable to relax the model to allow for discounting and non-zero costs of holding inventory, this relaxation clutters the model, hindering the interpretation of the results. For the holding costs, the consumers' incentive to stock-pile decreases as the cost increases so that in the extreme case of very high holding costs, consumers do not stock-pile consumables and our model reduces to that of lock-in analyzed in Chapter 4; we will use this extreme case as a benchmark to compare our results.

In order to characterize the relationship among the prices and demands in a given period t, we begin by deriving the inverse demand for consumables as a function of the quantity of durables (Q_t) in use. Let p_{ct} represent the price of consumables in period t . The supply of consumables in period t is $y_t + x_{t-1}$, where y_t is quantity of consumables supplied by the manufacturer in period t and x_{t-1} is the stock-pile of consumables from period $t - 1$. Consumers will purchase consumables in a given period only if they have access to a durable, and the price is below their maximum marginal utility, i.e. α for type l and 1 for type h . In addition, the cumulative quantity of durables $Q_t \in [0, 1]$, but it is sufficient to restrict our attention to quantities 0, θ or 1. The demand for consumables in a given period has two components, *i*) the consumption in the current period $D_t(Q_t, p_{ct})$, where D_t is given below and *ii*) the quantity stockpiled by consumers x_t , in anticipation of the manufacturer's response in the subsequent period $t + 1$.

$$D_t(Q_t,\ p_{ct}) = \begin{cases} 0 & \text{if } Q_t = 0 \\ z_h(p_{ct}) & \text{if } Q_t = \theta \text{ or } p_{ct} \leq \alpha \\ \theta z_h(p_{ct}) + (1-\theta) z_l(p_{ct}) & \text{otherwise} \end{cases} \tag{5.3}$$

Therefore, in a given period t, the market clearing price for consumables is the solution of the following equation:

$$y_t(Q_t,\ p_{ct}) + x_{t-1} = D_t(Q_t,\ p_{ct}) + x_t(Q_{t+1}(Q_t,\ p_{ct}),\ p_{ct+1}(Q_t,\ p_{ct})) \tag{5.4}$$

Recall that the *implicit rental price* is the price at which the durable could be rented in a given period. In each period, the implicit rental price, r is a function of the quantity (Q) of durables available and the consumables price (p_c) and is expressed as follows:

$$r(Q_t,\ p_{ct}) = \begin{cases} V_h(p_{ct}) = \frac{(1-p_{ct})^2}{2} & \text{if } Q_t = \theta \\ V_l(p_{ct}) = \frac{(\alpha-p_{ct})^2}{2} & \text{if } Q_t = 1 \end{cases} \tag{5.5}$$

In period 2, the price of the durable is simply the implicit rental price. The most that the manufacturer can charge for the durable in period 1 is the implicit rental price in period 1 as given by (5.5), plus the discounted anticipated second period durable price. We established the manufacturer's strategy under the benchmark case of lock-in in Chapter 4, where consumers do not hold stock-pile consumables; we now analyze our model when consumers strategically stock-pile consumables for future use. In analyzing this model we assume that inventories carried over are perfect substitutes for current production. In addition, we assume that consumer inventories are perfectly transferable; transfer is possible, for example, when the market for consumables is highly efficient.

An equivalent way to model transferable consumer inventory is to assume that there are sufficiently large number of competitive firms (hoarders) that hold inventory of consumables. When the manufacturer makes consumables available in period 1, the hoarders buy up consumables and hold them in inventory for sale in period 2. Consistent with the assumptions stated earlier, all the hoarders have zero holding cost and have no ability to produce additional consumables. We solve the game between the manufacturer, consumers and the hoarders using standard backward induction.

The market clearing price for the consumables in period 2 depends upon Q_2, y_2 and x_1, according to the equation 5.4. Solving 5.4 for $x_2 = 0$, we can obtain $p_{c2}(Q_2, y_2, x_1)$ as defined follows:

$$p_{c2}(Q_2,\ y_2,\ x_1) = \begin{cases} 0 & \text{if } Q_2 = 0 \\ 1 - \frac{y_2 + x_1}{\theta} & \text{if } Q_2 = \theta \text{ or } y_2 + x_1 \leq (1-\alpha)\theta \\ \theta + \alpha(1-\theta) - (y_2 + x_1) & \text{otherwise} \end{cases} \tag{5.6}$$

Then, the price of the durable in period 2, p_{d2}, is given by the implicit rental price $r(Q_2, p_c(Q_2, y_2))$ as defined in equation 5.5. In the presence of stockpiling, the manufacturer solves the following problem in period 2:

$$\max_{y_2, Q_2} \pi_2^S(Q_1,\ Q_2,\ y_2,\ x_1) = (Q_2 - Q_1)\, r(Q_2, p_c(Q_2,\ y_2,\ x_1)) + y_2\, p_c(Q_2,\ y_2,\ x_1) \tag{5.7}$$

Lemma 5.2.1. *a) If $Q_1 = 1$, then $Q_2^*(Q_1) = 1$, and the quantity $(y_2^* + x_1)$ of consumables available in period 2 is adequate to induce both types of consumers to purchase if and only if if $x_1 \geq \overline{x}(Q_1 = 1) = \theta(1-\alpha) - \alpha\sqrt{\theta}$. Otherwise, the market clearing price for consumables exceeds the low-type consumers' maximum marginal utility, i.e. $p_c(Q_2^*, y_2^*) \geq \alpha$, and low-type consumers do not consume any units of the consumable.*

b) If $Q_1 = \theta$ then $Q_2^*(Q_1) = 1$ if and only if
$x_1 \geq \overline{x}(Q_1 = \theta) = \theta - \alpha\sqrt{2\theta(1+\theta)}$, while $Q_2^*(Q_1) = \theta$ otherwise. In either case, the availability of consumables is sufficiently high so that each consumer who owns a durable consumes a positive quantity of the consumable.

c) The threshold inventory $\overline{x}(Q_1)$, above which the manufacturer serves both consumers in period 2 is non-decreasing in Q_1

Table 5.1: Optimal second period quantities $Q_1 = 1$

$Q_1 = 1$	$x_1 \leq -\alpha\sqrt{\theta} + \theta(1-\alpha)$	$x_1 \geq -\alpha\sqrt{\theta} + \theta(1-\alpha)$
$y_2^*(Q_1^*, x_1)$	$\frac{\theta - x_1}{2}$	$\frac{1}{2}(\alpha + \theta - \alpha\theta - x_1)$
$p_c(Q_2^*, y_2^*, x_1)$	$\frac{\theta - x_1}{2\theta}$	$\frac{1}{2}(\alpha + \theta - \alpha\theta - x_1)$
$Q_2^*(Q_1, x_1)$	1	1
$r(Q_2^* p_c(Q_2^*, y_2^*, x_1))$	0	$\frac{1}{8}(x_1 + \alpha - \theta + \alpha\theta)^2$
$\pi_2^I(Q_2^*, Q_1, y_2^*, x_1)$	$\frac{(\theta - x_1)^2}{4\theta}$	$\frac{1}{4}(\alpha + \theta - \alpha\theta - x_1)^2$

The optimal quantities, prices of consumable and durables, and the optimal profits of the manufacturer as a function of x are summarized in Tables 5.1 and 5.2.

Based on Lemma 5.2.1 it is easy to see that, when the quantity of stock-piled inventory is very high maximum, i.e. $x_1 \geq \theta(1-\alpha) - \alpha\sqrt{\theta}$, then in the second period the manufacturer sells durables to all consumers who do not own it already and the market clearing price of consumables falls low enough to attract purchases from the low-type consumer, i.e. $p_c\left(Q_2^*, y_2^*, x_1\right) < \alpha$. When the stock-pile of consumables is in the intermediate range, that

Table 5.2: Optimal second period quantities $Q_1 = \theta$

$Q_1 = \theta$	$x_1 \leq \theta - \alpha\sqrt{2\theta(1+\theta)}$	$x_1 \geq \theta - \alpha\sqrt{2\theta(1+\theta)}$
$y_2^*(Q_1, x_1)$	$\frac{\theta - x_1}{2}$	$\alpha(1-\theta) + \frac{\theta^2}{(1+\theta)} - \frac{x_1\theta}{1+\theta}$
$p_c(Q_2^*, y_2^*, x_1)$	$\frac{\theta - x_1}{2\theta}$	$\frac{\theta - x_1}{1+\theta}$
$Q_2^*(Q_1, x_1)$	θ	1
$r(Q_2^* p_c(Q_2^*, y_2^*, x_1))$	$\frac{(\theta + x_1)^2}{8\theta}$	$\frac{(\alpha - \theta + \alpha\theta + x_1)^2}{2(1+\theta)^2}$
$\pi_2^I(Q_2^*, Q_1, y_2^*, x_1)$	$\frac{(\theta - x_1)^2}{4\theta}$	$\frac{\alpha^2 + (x_1 - (1-\alpha)\theta)(x_1 - (1+\alpha)\theta)}{2(1+\theta)}$

is $\theta - \alpha\sqrt{2\theta(1+\theta)} \leq x_1$ and $x_1 \leq \theta(1-\alpha) - \alpha\sqrt{\theta}$, the quantity of consumables in inventory is high enough to induce the manufacturer to sell durables to the low-type consumers if they have not purchased the durable in period 1. However, the inventory of consumables is not large enough to induce consumable sales to low-type consumers if they already own the durable. In this intermediate range of x_1, in order to sell to low end consumers who already own the durable, the firm will have to produce large enough quantity of consumables so as to induce a low enough price on consumables for the low type consumers to purchase. When the stock-pile of consumables is very low, i.e. $x_1 \leq \theta - \alpha\sqrt{2\theta(1+\theta)}$, the manufacturer does not sell durables to the low end consumers if they do not own them already. From 5.2.1 we can infer that the inventory held by hoarders should be sufficiently high to induce the manufacturer to serve the low type consumers in period 2.

In the presence of stock-piled consumables in period 2, the additional output produced by the manufacturer's is driven by whether or not the manufacturer can sell additional durables. When $Q_1 = 1$, the optimal decision of the manufacturer in period 2 is driven by maximization of consumable profits alone. However, when $Q_1 = \theta$, the manufacturer maximizes joint profits from the sale of consumables as well as durables; the additional durables sold to low type consumers in period 2 will allow the manufacturer to extract all the surplus that is created for the low type consumers through rents on the durable. The opportunity to sell additional durables in period 2 increases the manufacturer's incentive to increase availability of consumables thereby lowering the price of the consumables. Therefore, as part c) of Lemma 5.2.1 indicates, for a lower Q_1, lower levels of inventory is sufficient to trigger sales to the low type consumers in period 2.

So far, we derived the optimal response of the manufacturer in period 2, given the quantity of consumables held by the hoarders, x_1 . We now derive the equilibrium stock-piling behavior, as a function of the quantity of durables sold in period 1, Q_1 and the quantity of consumables, y_1 made available by the manufacturer in period 1. In deriving the demand function $x_1(Q_1, y_1)$, we take into account the fact that consumers fully anticipate the response of the manufacturer in period 2 to the inventory they stock-pile in period 1. At a given quantity of durables, Q_1 and consumables y_1, made available by the manufacturer in period 1, the discount factor ρ and the per unit holding cost h , the equilibrium demand for stock-piling, x_1 , to be used in period 2 can be determined from the following equation:

$$p_{c1}(Q_1, y_1, x_1) + h = \rho p_{c2}(Q_2^*(Q_1, y_1, x_1), y_2^*(Q_1, x_1), x_1) \qquad (5.8)$$

The equation 5.8 states that hoarders hold a quantity x_1 in inventory such that the price they pay in the current period to purchase them plus the holding cost(h) is equal to the discounted price in period 2, where the discount factor is ρ. To see why the equation 5.8 holds, suppose $p_{c1}(Q_1, y_1, x_1) + h < \rho p_{c2}(Q_2^*(Q_1, y_1, x_1), y_2^*(Q_1, x_1), x_1)$, that is price of consumables in period 1 is less than in period 2. Then the hoarders can make a positive margin from holding an additional unit, therefore the hoarders have an incentive to increase their stock-piled quantity x_1. Now suppose $p_{c1}(Q_1, y_1, x_1) + h \geq \rho p_{c2}(Q_2^*(Q_1, y_1, x_1), y_2^*(Q_1, y_1, x_1), x_1)$, *i.e.* the price of consumables in period 1 plus the holding cost is greater than the anticipated price of consumables in period 2, then the hoarders do not have an incentive to stock-pile consumables. Therefore, we can see that in equilibrium, the equation 5.8 holds. Although we use a general holding cost h , and discount factor ρ to illustrate how they influence the stock-piling behavior above, for tractability we will set $h = 0$ and $\rho = 1$ for the rest of the analysis.

Solving the (5.8) we obtain the price of consumables in period 1 as follows:

$$p_{c1}(Q_1,\ y_1,\ x_1) = \begin{cases} 0 & \text{if } Q_1 = 0 \\ 1 - \frac{y_1 - x_1}{\theta} & \text{if } Q_1 = \theta \text{ or } y_1 - x_1 \leq (1-\alpha)\theta \\ \theta + \alpha(1-\theta) - (y_1 - x_1) & \text{otherwise} \end{cases} \tag{5.9}$$

The equilibrium stock-pile of consumables conditional on the manufacturer's actions in period 1 is characterized in the following Lemma:

Lemma 5.2.2. *The equilibrium quantity that is stock-piled,* $x_1^*(Q_2^*(Q_1, y_1), y_2^*(Q_1, y_1), \alpha, \theta)$, *(we use* $x_1^*(Q_1, y_1, \alpha, \theta)$ *for brevity) can be characterized as follows:*

$$x_1^*(Q_1, y_1, \alpha, \theta) = \begin{cases} max\{0,\ \frac{1}{3}(2y_1 - (\alpha(1-\theta) + \theta)\} \\ \qquad \text{if } y_1 \geq \frac{1}{2}\left(\alpha\left(1 - 3\sqrt{\theta}\right) + 4\theta(1-\alpha)\right) \text{ and } Q_1 = 1 \\ max\{0,\ \frac{1}{3}(2y_1 - \theta)\} \\ \qquad \text{if } y_1 \leq \frac{\alpha\sqrt{2\theta}(1+2\theta)}{\sqrt{1+\theta}} - 2\theta \text{ and } Q_1 = \theta \\ max\{0,\ \frac{1}{1+2\theta}(y_1(1+\theta) - \theta)\} \\ \qquad \text{otherwise} \end{cases} \tag{5.10}$$

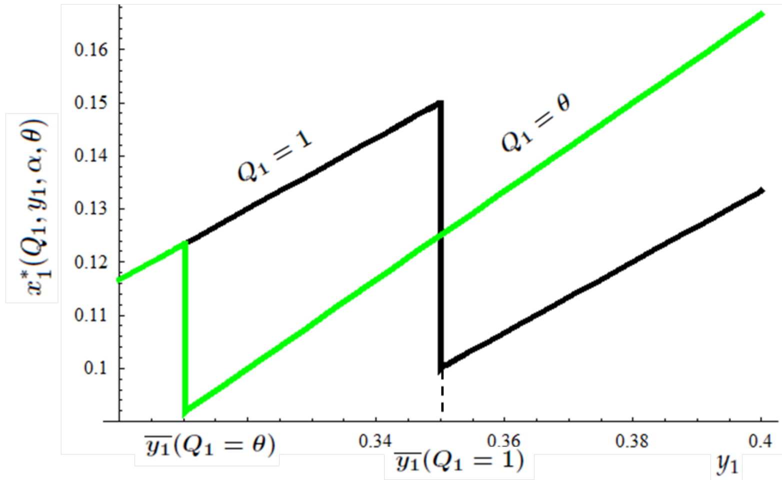

Figure 5.1: An illustration of equilibrium stock-piling

The equilibrium quantity of consumables held in inventory depends on consumers expectation of future durable sales; Specifically, the quantity of consumables stock-piled is inversely related to the expected quantity of durables that will be sold in period 2. For a given quantity of consumables, y_1 made available in period 1, the consumers ration a lower proportion of consumables for future consumption when they expect that a higher quantity of durables will be sold in period 2; this explains the drop in the equilibrium holding quantity x_1^*, at $y_1 = \overline{y_1}$, in Figure 5.1. Further, higher the Q_1, higher is the threshold quantity of consumables $\overline{y_1}$, that induces the manufacturer to serve to both consumer groups in period 2.

Having established the second period equilibrium we solve the first period problem in which the manufacturer maximizes the following profits:

$$\pi_1^S(Q_1, y_1) = Q_1\, p_{d1}^S(Q_1, y_1,\ x_1^*(Q_1, y_1)) + y_1\, p_c(Q_1, y_1,\ x_1^*(Q_1, y_1)) \tag{5.11}$$
$$+\rho\pi_2^S(\ Q_2^*(Q_1, y_1),\ y_2^*(Q_1, y_1),\ x_1^*(Q_1, y_1))$$

where the price of durable (p_{d1}^S) in period 1 is given by

$$p_{d1}^S(Q_1, y_1) = r(Q_1, y_1,\ x_1^*(Q_1, y_1)) + \tag{5.12}$$
$$\rho r(Q_2^*(Q_1,\ x_1^*(Q_1, y_1)), y_2^*(Q_1,\ x_1^*(Q_1, y_1)))$$

Clearly the price of the durable depends not only on the quantities of durables and consumables produced in period 1, but also upon the consumers' anticipation of the manufacturer's optimal decisions in period 2. Further, note that as a result of stock-piling and our assumption that consumers incur no holding costs and the discount factor $\rho = 1$, from 5.8, we have that the price of consumables in period 1 equals the price in period 2. From this logic, (5.10), (5.5), and the results from Table 5.2 we can derive the price of the durable in period 1. When $Q_1 = \theta$, the first period durable price can be expressed as:

$$p_{d1}^S\,(\theta, y_1) = \begin{cases} \frac{(y_1+\theta)^2}{9\theta^2} & \text{for } y_1 \leq \frac{\alpha\sqrt{2\theta}(1+2\theta)}{\sqrt{1+\theta}} - 2\theta \\ \frac{1+2y_1^2+(\alpha-2(1-\alpha)\theta)^2+2y_1(1+\alpha-2(1-\alpha)\theta)}{2(1+\theta)^2} & \text{otherwise} \end{cases} \tag{5.13}$$

If $Q_1 = 1$, and the manufacturer makes very few consumables available ($y_1 < \frac{1}{2}\left(\alpha\left(1-3\sqrt{\theta}\right)+4\theta(1-\alpha)\right)$), then she induces a price of consumables greater than α. In such a situation the low type consumers would choose not to purchase any consumables and the only way that the firm can sell durables to all the consumers is by giving the durable away. However, such a choice would not be optimal for the manufacturer as she will clearly be better off if she sold the durable to high type consumers alone. Therefore we can conclude that if the manufacturer chooses to sell the durable to both types of consumers then she would simultaneously choose a quantity of consumables at least as big as $\frac{1}{2}\left(\alpha\left(1-3\sqrt{\theta}\right)+4\theta(1-\alpha)\right)$. From this logic, (5.10), (5.5), and the results from Table 5.2 we can derive the price of the durable in period 1 when $Q_1 = 1$.

$$p_{d1}^{S}(1, y_1) = \frac{(y_1 + \alpha - 2(1-\alpha)\theta)^2}{9} \tag{5.14}$$

We can now characterize the optimal solution to the manufacturer's problem when she *locks-in* consumers and consumers strategically stockpile consumables.

Proposition 5.2.3. *(a) For every $\theta \in (0, 1)$, there exists $\underline{\alpha}^S(\theta)$ and $\overline{\alpha}^S(\theta)$, where $\underline{\alpha}^S(\theta) < \overline{\alpha}^s(\theta) < 1$, such that:*

- *If $\alpha \leq \underline{\alpha}^S(\theta)$, the manufacturer sells the durable to high-type consumers in period 1 and does not sell to low-type consumers at all , i.e. $Q_1^*(\alpha, \theta) = Q_2^*(\alpha, \theta) = \theta$.*

- *If* $\underline{\alpha}^S(\theta) \leq \alpha \leq \overline{\alpha}^S(\theta)$ *the manufacturer sells to high-type consumers in period 1 and low-type in period 2, i.e.* $Q_1^*(\alpha,\theta) = \theta < Q_2^*(\alpha,\theta) = 1$.

- *If* $\alpha \geq \overline{\alpha}^S(\theta)$, *the monopolist sells to all consumers in period 1, i.e.* $Q_1^*(\alpha,\theta) = Q_2^*(\alpha,\theta)) = 1$.

Table 5.3: Equilibrium Prices under Lock-in with Stock-Piling

	Low α	Medium α	High α
	$\alpha \leq \underline{\alpha}^S(\theta)$	$\underline{\alpha}^S(\theta) \leq \alpha \leq \overline{\alpha}^S(\theta)$	$\alpha \geq \overline{\alpha}^S(\theta)$
Period 1			
p_d	$\frac{(1+2\theta)}{9\theta(1+\theta)}\left(2\alpha^2(1+2\theta)+9\theta-6\alpha\sqrt{2\theta(1+\theta)}\right)$	$\frac{1+\theta^2(1-4\alpha)+\alpha^2(1+4\theta+5\theta^2)}{2(1+\theta)^2}$	$(\alpha+(1-\alpha)\theta)^2$
p_c	$\frac{\sqrt{2}\alpha(1+2\theta)}{3\sqrt{\theta(1+\theta)}}$	$\frac{\theta(1-\alpha)}{1+\theta}$	$\theta(1-\alpha)$

The optimal policy for selling durables and consumables under lock-in with stock-piling mirrors the structure of lock-in without stock-piling. The equilibrium prices of the durable and consumable in period 1 are given in Table 5.3. We numerically compare below the effect of consumer stock-piling on the total quantity of durables sold and the profits of the manufacturer. Although, I have been unable to prove analytically, numerical results indicate that 'lock-in with stock piling' yields higher profits than 'lock-in without stock-piling' when consumers are either homogeneous or highly heterogeneous. As can be seen in Figure 5.2 we can see that the manufacturer benefits from stock-piling for all $\alpha \geq \overleftrightarrow{\alpha}(\theta)$ and $\alpha \leq \underleftrightarrow{\alpha}(\theta)$.

In the presence of lock-in, the manufacturer's incentive to restrict future availability of consumables creates a hold-up problem with respect to the price of the consumables. In order to counter this hold-up, consumers have an incentive to stock-pile consumables for future use. To the extent that this strategic stock-piling alleviates the hold-up problem for consumers it also decreases the manufacturers ability to profit from selling consumables to locked-in consumers, thereby increasing her incentive to sell additional durables.

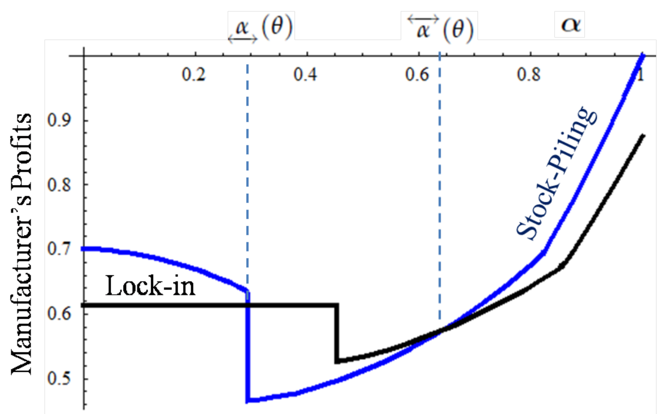

Figure 5.2: Profits under Stock-Piling vs Lock-in

To the extent that stock-piling cannibalizes manufacturer's sales of consumables, it helps to mitigate the hold-up problem with respect to consumers who purchase the durables in period 1 and, under certain conditions, increases the profits of the manufacturer. For example, in Figure 5.2 , for $\alpha \geq \overleftrightarrow{\alpha}(\theta)$ and $\alpha \leq \underset{\longleftrightarrow}{\alpha}(\theta)$, 'lock-in with stock-piling' results in profits higher than the benchmark case of 'lock-in without stock-piling'. In a relatively homogeneous market time-inconsistency is less important as the manufacturer sells durables to all her consumers in period 1; however, hold-up is a concern. Similarly, in a highly differentiated market time inconsistency is irrelevant as the manufacturer avoids selling to the low type consumers. Therefore, when α is either sufficiently high or sufficiently low, the ability to stock-pile mitigates the hold-up problem yielding higher profits than the benchmark case (also reflected in Figure 5.2).

5.3. Managerial Implications and Concluding Remarks

In this chapter, we continue our investigation of the effectiveness of a lock-in strategy for a durable product manufacturer in the presence of consumer stock-piling. It is useful to recognize that stock-piling of consumables introduces an inefficiency in the system because of the holding costs associated with stock-piling. Thus, even though stock-piling makes little sense from a consumer's perspective our findings indicate that there are strategic benefits associated with stock-piling for both the consumers as well as the manufacturer. From both a consumer's point of view as well as the manufacturer's, stock-piling is an effective strategic tool to address the potential hold-up problem that arises from the durable good manufacturer's lock-in strategy. When consumers are highly homogeneous, by softening the hold-up problem, stock-piling increases extractable rents for the manufacturer.

Ofcourse, the best strategy that a firm could adopt would be for the manufacturer to commit to future prices of the durables and consumables so as to maximize the rents that she can extract from consumers. However, the lack of a credible commitment mechanism results in the durable firm to behave in a time inconsistent manner. In Chapter 4, we recognized that by locking-in consumers to its contingent product, the durable good firm can soften the time inconsistency problem when consumers are either highly homogeneous or highly heterogeneous. Recall however, that lock-in also introduces a hold-up problem for consumers with respect to the usage of the product.

In this chapter we explored how consumers' propensity to stock-pile consumables in order to address the hold-up problem associated with lock-in, affects the strategy of the durable product manufacturer. Our results indicate that consumer stock-piling addresses the hold-up problem to a certain extent and could be beneficial when consumers are sufficiently homogeneous. However, for moderately differentiated consumer population, lock-in in the presence of consumer stock-piling may hurt the manufacturer by worsening the time inconsistency problem. The rationale is that as the stock-piled consumables erode the manufacturer's ability to profit from selling consumables to locked-in consumers, it results in an increased incentive to sell additional durables to the low type consumers. By allowing consumers to stock-pile the manufacturer creates competition for herself in the consumable market in period 2, and thus stock-piling can be likened to the presence of competition in the consumables market.

As recognized in Chapter 4, extremely high levels of competition completely takes away the firms revenues from the consumables in period 2, inducing the manufacturer to sell additional durables to continue to make profits. However, locking-in consumers and at the same time allowing consumers to stock-pile consumables may be better than allowing infinite number of competitors in the consumables market. Because, under lock-in, by definition, the manufacturer is the sole producer of consumables. Although stock-piling of consumables creates competition, she still retains market power in the consumable market because of the fact that she is the sole supplier of consumables.

In conclusion, when a durable good manufacturer adopts a lock-in strategy through a contingent consumable, allowing consumers to stock-pile the consumables will help to mitigate consumer's anxiety about being held-up with respect to the price of consumables. However, it also erodes the manufacturer's revenue stream from consumables and increasing her incentive to sell additional durables thus exacerbating time inconsistency. We show that lock-in with consumer stock-piling is most beneficial when consumers are homogeneous so that hold-up is a bigger concern than is time inconsistency or when consumers are highly heterogeneous that the lower segment is ignored. In essence, a durable good manufacturer when adopting a lock-in strategy, can mitigate her own potential for opportunistic behavior in the future by allowing consumers to stock-pile consumables.

Chapter 6

Concluding Remarks and Future Work

My dissertation focuses on two issues that affect the product strategy of a firm; the first is consumer choice and its influence on optimal product assortments and the second is the interaction between durable products and their contingent consumables.

On-line search engines such as Kayak (a travel search firm) and Nextag (a product search firm) cannot influence the pricing of their products, but they can influence their profits by choosing the right set of products to display. Motivated by the product choice of search firms, in Chapter 3 titled: *'Making the Most of Choice: Product Selection under Heterogeneous Consumer Preferences,'* I study the assortment planning problem of a firm under a general consumer choice model. I find that solving for the optimal assortment for a firm when consumer's choice has very little structure, is indeed a complex task. However, it is one that deserves attention and has wide practical implications for a number of retailers as well as e-tailers.

In Chapter 3, I illustrate the complexity of solving the product selection problem in the most general case. By placing simple restrictions I show that our model can be reduced to commonly used choice models such as Multinomial Logit, locational choice and other substitution models. I present simple solutions for some commonly used choice models such as locational choice and downward substitution models, and develops heuristics for other practically motivated models that do not render themselves to simple approaches.

This work has exciting applications for search firms to enhance their market reach and for retailers to enhance their product selection processes. My work is as also an excellent starting point for studying more complex consumer choice processes. There are several important directions that can be pursued in from here; one of them is to study the influence of fixed and substitution costs. Another, is to understand the interaction between product substitution and competition and how it influences the optimal assortments. Finally, developing some bounds on the loss that a retailer incurs by assuming simple consumer choice models may be a promising direction to pursue.

While, the primary challenge for firms that deal with a large number of products is to decipher the complexity of consumer choice, a durable goods monopolist is mainly concerned about the strategic implications of its potential for opportunistic behavior in the future. Chapter 4 and 5 specifically study the incentives of a durable goods monopolist when it can lock-in consumers through a contingent consumable.

It is possible for durable product firms to lock-in consumers by requiring them to use a consumable that is proprietary to the firm. For example, Amazon, which is the manufacturer of the reading device Kindle, locks-in consumers by supporting e-books on its machine only in a restricted number of formats (primarily in the proprietary AZW format). Such a lock-in has two interesting effects on the incentives of a durable goods manufacturer. On one hand, by locking-in consumers to its consumable, a durable goods monopolist can curb its temptation to reduce durable prices over time, thereby mitigating the classic time inconsistency problem. On the other hand, lock-in will create a hold-up issue and adversely affect consumers' expectations of future prices for the consumable. In Chapter 4 titled: *'Durable Products, Time Inconsistency and Lock-in'*, we demonstrate the trade-off between time inconsistency and hold-up, and derive insights about the conditions under which a lock-in strategy can be effective. In addition, I analyze competition in the consumables market and its effects on the manufacturers incentives for lock-in.

I find that time inconsistency and hold-up problem associated with lock-in interact in an interesting and non-evident fashion. At the same time that lock-in creates a hold-up problem with respect to consumables it also helps to address the time inconsistency problem. For a traditional durable goods monopolist, in the absence of revenues from consumable sales, the incentive to lower the price of the durable to attract more price sensitive consumers prevails, hurting the profits of the manufacturer. However, with lock-in, the revenues that the durable firm can make from selling consumables to locked-in high valuation consumers prevents the firm from dropping the price of the durable too low to sell to the low end consumers. Thus lock-in offers some protection from time inconsistency for the durable good firm. It also can be viewed as a way for firms to place their high end innovations in the hands of those consumers who value and pay for it the most, without resulting in a quick death of the market for the durable.

In the example above of Amazon's Kindle, consumers cannot hold inventory of consumables such as e-books. However, for certain products such as replacement parts, it is possible for consumers to buy-up consumables for future use. In Chapter 5, we identify the consumers' incentive to stock-pile consumables in anticipation of a hold-up problem that arises from the manufacturer's lock-in strategy. Further, we investigate the implications of these consumable inventories, held for strategic reasons, on the interaction between time-inconsistency and hold-up. Our findings indicate that strategic inventories have an effect similar to that of competition, as they help to dampen the hold-up problem that arises from lock-in. This research on durability and lock-in has implications for durable product manufacturers in choosing an appropriate business model for their products and is therefore of practical significance.

My work on durable products and contingent consumables, can be extended to analyze the market for consoles and video games. More specifically: incorporating network effects and entry control mechanisms in the video game market to understand the incentives of durable product firms and investigating strategic issues associated with assortment planning and product line design in the context of intermediaries and inventories are interesting directions to pursue. It would also be of interest to understand the effect of competition in durables as well as new product introductions on the interaction between lock-in and time inconsistency. My studies so far contribute to the rich literature on assortment planning, product line design and the marketing of durable products, and have excellent potential to be further studied to improve our understanding of how firms can enhance their product strategies.

Appendices

Appendix A

Making the most of choice: Product selection under heterogeneous consumer preferences

<u>Proof of Proposition 3.1.4:</u> Given Lemmas 3.1.2 and 3.1.2, it is sufficient to show that, for any set I, O and T,

$$f_j^{min}(I,O,T) \leq \min_{\substack{S:j\notin S \\ I\subseteq S\subseteq(\mathcal{N}\setminus O)}} C_j(S), \tag{A.1}$$

$$f_j^{max}(I,O,T) \geq \max_{\substack{S:j\notin S \\ I\subseteq S\subseteq(\mathcal{N}\setminus O)}} C_j(S). \tag{A.2}$$

We hav

$$\min_{\substack{S:j\notin S \\ I\subseteq S\subseteq(\mathcal{N}\setminus O)}} C_j(S) = \min_{\substack{S:j\notin S \\ I\subseteq S\subseteq(\mathcal{N}\setminus O)}} \left(\sum_{i\in S\cup\{j\}} R_i(S\cup\{j\})\pi_i - \sum_{i\in S} R_i(S)\pi_i \right),$$

$$= \min_{\substack{S:j\notin S \\ I\subseteq S\subseteq(\mathcal{N}\setminus O)}} \left(\sum_{i\in S\cup\{j\}} \sum_{\boldsymbol{\tau}\in\mathcal{T}} \alpha_{\boldsymbol{\tau}} Z_{\boldsymbol{\tau},i}(S\cup\{j\})\pi_i - \sum_{i\in S} \sum_{\boldsymbol{\tau}\in\mathcal{T}} \alpha_{\boldsymbol{\tau}} Z_{\boldsymbol{\tau},i}(S)\pi_i \right)$$

$$= \min_{\substack{S:j\notin S \\ I\subseteq S\subseteq(\mathcal{T}\setminus O)}} \sum_{\boldsymbol{\tau}\in\mathcal{T}} \alpha_{\boldsymbol{\tau}} \left(\sum_{i\in S\cup\{j\}} Z_{\boldsymbol{\tau},i}(S\cup\{j\})\pi_i - \sum_{i\in S} Z_{\boldsymbol{\tau},i}(S)\pi_i \right)$$

$$\geq \sum_{\boldsymbol{\tau}\in\mathcal{T}} \alpha_{\boldsymbol{\tau}} \min_{\substack{S^{\boldsymbol{\tau}}:j\notin S^{\boldsymbol{\tau}} \\ I\subseteq S^{\boldsymbol{\tau}}\subseteq(\mathcal{N}\setminus O)}} \left(\sum_{i\in S^{\boldsymbol{\tau}}\cup\{j\}} Z_{\boldsymbol{\tau},i}(S^{\boldsymbol{\tau}}\cup\{j\})\pi_i - \sum_{i\in S^{\boldsymbol{\tau}}} Z_{\boldsymbol{\tau},i}(S^{\boldsymbol{\tau}})\pi_i \right),$$

$$= f_j^{min}(I,O,T).$$

(A.2) can be shown in a similar fashion. ■

Proof of Lemma 3.1.5: We have

$$\begin{aligned}
\max_{S\subseteq\mathcal{N}} \sum_{j\in S} R_j(S)\pi_j &= \max_{S\subseteq\mathcal{N}} \sum_{j\in S}\sum_{\boldsymbol{\tau}\in\mathcal{T}^+} Z_{\boldsymbol{\tau},j}(S)\alpha_{\boldsymbol{\tau}}\pi_j, \\
&= \max_{S\subseteq\mathcal{N}} \sum_{\boldsymbol{\tau}\in\mathcal{T}^+} \alpha_{\boldsymbol{\tau}} \sum_{j\in S} Z_{\boldsymbol{\tau},j}(S)\pi_j, \\
&\leq \sum_{\boldsymbol{\tau}\in\mathcal{T}^+} \alpha_{\boldsymbol{\tau}} \max_{S^{\boldsymbol{\tau}}\subseteq\mathcal{N}} \left(\sum_{j\in S^{\boldsymbol{\tau}}} Z_{\boldsymbol{\tau},j}(S^{\boldsymbol{\tau}})\pi_j \right), \\
&= UB_1.
\end{aligned}$$

Hence proved. ■

Proof of Lemma 3.1.6: From (3.3),

$$\begin{aligned}
\max_{S\subseteq\mathcal{N}} \sum_{j\in S} R_j(S)\pi_j &= \max_{I\subseteq S\subseteq(\mathcal{N}\setminus O)} \sum_{j\in S} R_j(S)\pi_j, \\
&\leq \sum_{\boldsymbol{\tau}\in\mathcal{T}} \alpha_{\boldsymbol{\tau}} \max_{I\subseteq S^{\boldsymbol{\tau}}\subseteq(\mathcal{N}\setminus O)} \left(\sum_{j\in S^{\boldsymbol{\tau}}} Z_{\boldsymbol{\tau},j}(S^{\boldsymbol{\tau}})\pi_j \right), \\
&= UB_2(I,O), \\
&\leq \sum_{\boldsymbol{\tau}\in\mathcal{T}} \alpha_{\boldsymbol{\tau}} \max_{S^{\boldsymbol{\tau}}\subseteq\mathcal{N}} \left(\sum_{j\in S^{\boldsymbol{\tau}}} Z_{\boldsymbol{\tau},j}(S^{\boldsymbol{\tau}})\pi_j \right), \\
&= UB_1.
\end{aligned}$$

Hence proved. ■

Proof of Proposition 3.1.7: From Li (2007) we know that one of the sets $S_{(1)}, ...S_{(n)}$ is optimal. We show that $\pi_j \leq \mathbb{E}\Pi(S_{(j-1)})$ implies that $\pi_{j+1} \leq \mathbb{E}\Pi(S_{(j)})$. We do this by contradiction: assume that $\pi_j \leq \mathbb{E}\Pi(S_{(j-1)})$ and $\pi_{j+1} > \mathbb{E}\Pi(S_{(j)})$. Because $\pi_j \geq \pi_{j+1}$, we also have that $\pi_j > \mathbb{E}\Pi(S_{(j)})$, and

$$\pi_j > \mathbb{E}\Pi(S_{(j)}) \Leftrightarrow \pi_j > \sum_{i=1}^{j} \frac{v_i}{1 + \sum_{j=1}^{j-1} v_j + v_j} \pi_i \Leftrightarrow \tag{A.3}$$

$$\pi_j > \sum_{i=1}^{j-1} \frac{v_i}{1 + \sum_{j=1}^{j-1} v_j} \pi_i = \mathbb{E}\Pi(S_{(j-1)}),$$

which is a contradiction. Let j^* be the largest integer such that $\pi_{j^*} > \mathbb{E}\Pi(S_{(j^*-1)})$. It follows that $\pi_j > \mathbb{E}\Pi(S_{(j-1)})$ for $j = 1, ..., j^*$ and $\pi_j \leq \mathbb{E}\Pi(S_{(j-1)})$ for all $j = j^* + 1, ..., n$. By (3.6), this implies that $C_j(S_{(j-1)}) > 0$ for $j = 1, ..., j^*$ and $C_j(S_{(j-1)}) \leq 0$ for $j = j^* + 1, ..., n$, which in turns implies that $\mathbb{E}\Pi(S_{(1)}) < ... < \mathbb{E}\Pi(S_{(j^*-1)}) < \mathbb{E}\Pi(S_{(j^*)}) \geq \mathbb{E}\Pi(S_{(j^*+1)}) \geq ...\mathbb{E}\Pi(S_{(n)})$. Hence, $S_{(j^*)}$ is optimal.

Now we show that $\pi_{j^*+1} > \pi_{j^*}$. Suppose (contradiction) that $\pi_{j^*+1} = \pi_{j^*}$. From (A.3) we see that when $\pi_{j+1} = \pi_j$, we have $\pi_j > \mathbb{E}\Pi(S_{(j-1)})$ if and only if $\pi_{j+1} > \mathbb{E}\Pi(S_{(j)})$. Hence, we would have $\pi_{j^*+1} > \mathbb{E}\Pi(S_{(j^*)})$, which is a contradiction to the definition of j^*. ■

Proof of Lemma 3.1.8: Suppose (contradiction) that all the optimal assortments are such that there exists $i, j \in S$ with $i < j$ and $b_i + L_i \geq b_j + L_j$. Let S^* be one of them. In this case every customer prefers product i to product j so that $R_j(S^*) = 0$. By (P1), $R_i(S^* \backslash \{j\}) \leq R_i(S^*) \leq R_i(S^* \backslash \{j\}) - R_j(S^*)$

for all $i \neq j$ and $i \in S^*$. Hence, $R_i(S^*\backslash\{j\}) = R_i(S^*)$ for all $i \neq j$ and $i \in S^*$ and therefore $\mathbb{E}\Pi(S^*) = \mathbb{E}\Pi(S^*\backslash\{j\})$ so that $S^*\backslash\{j\}$ is also optimal, which is a contradiction. ■

Proof of Proposition 3.1.9: a) Let $S = \{s_1, \ldots, s_k\}$ with $s_1 < ... < s_k$, such that $b_{s_j} + L_{s_j} < b_{s_{j+1}} + L_{s_{j+1}}$ for $j = 1, ..., k-1$. Let $p(S)$ be the path that corresponds to assortment S, where $p(S) = (0,0) \rightarrow (s_1, s_2) \rightarrow (s_2, s_3) \rightarrow ... \rightarrow (s_{k-1}, s_k) \rightarrow (s_k, n+1) \rightarrow (n+1, n+1)$. Let P be the set of all admissible paths, *i.e.*, the paths with finite costs. Every assortment that satisfies the condition of Lemma 3.1.8 corresponds to a path in P and vice versa.

The cost of path $p(S)$ is equal to:

$$
\begin{aligned}
C(p(S)) &= c_{(0,0),(s_1,s_2)} + c_{(s_1,s_2),(s_2,s_3)} + \ldots + c_{(s_{k-1},s_k),(s_k,n+1)} + c_{(s_k,n+1),(n+1,n+1)}, \\
&= \left[G\left(\min\{a_{(s_1,s_2)}, b_{s_1} + L_{s_1}\}\right) - G\left(b_{s_1} - L_{s_1}\right)\right]\pi_{s_1}, \\
&\quad - \sum_{j=1}^{k-1} [G\left(\min\{a_{(s_j,s_{j+1})}, b_{s_j} + L_{s_j}\}\right) \\
&\quad - G\left(\max\{a_{(s_{j-1},s_j)}, b_{s_j} - L_{s_j}\}\right)]\pi_{s_j}, \\
&\quad - \left[G\left(b_{s_k} + L_{s_k}\right) - G\left(\max\{a_{(s_{k-1},s_k)}, b_{s_k} - L_{s_k}\}\right)\right]\pi_{s_j} + 0, \\
&= -\sum_{j=1}^{k} R_{s_j}(S)\pi_{s_j}, \\
&= -\mathbb{E}\Pi(S).
\end{aligned}
$$

where the third equality is by (3.8). Hence, $\min_{p \in P} C(p(S)) = \min_{S \in \mathcal{N}} -E\Pi(S) = \max_{S \in \mathcal{N}} E\Pi(S)$.

b) The complexity of a shortest path problem in an acyclic network is bounded by the number of arcs . The graph has a special structure, because there is possibly an arc between two nodes (i, j) to (k, l) only if $j = k$. There are at most j nodes that end with product $j \in \{1, 2, .., n\}$ and these are connected to at most $n + 1 - j$ nodes that start with product j. Therefore, the maximum number of arcs is equal to $2n + \sum_{j=1}^{n}(n + 1 - j)j$, where $2n$ is the maximum number of nodes leaving the source or ending in the destination node. Therefore the total number of arcs and hence the complexity of the algorithm is $O(n^3)$.■

Proof of Proposition 3.1.10: Suppose S^* is an optimal assortment but suppose there exists $j \in G$ but $j \notin S^*$. From (3.2),

$$\begin{aligned}
C_j(S^*) &= \sum_{i \in S^*} (R_i(S^*) - R_i(S^* \cup \{j\})(\pi_j - \pi_i) \\
&\quad + \left(R_j(S^* \cup \{j\}) - \sum_{i \in S^*} (R_i(S^*) - R_i(S^* \cup \{j\}) \right) \pi_j, \\
&= \sum_{\substack{i \in S^* \\ i > j}} (R_i(S^*) - R_i(S^* \cup \{j\})(\pi_j - \pi_i) \qquad \text{(A.4)} \\
&\quad + \left(R_j(S^* \cup \{j\}) - \sum_{\substack{i \in S^* \\ i > j}} (R_i(S^*) - R_i(S^* \cup \{j\}) \right) \pi_j
\end{aligned}$$

Using (3.10). The first term in (A.4) is non negative by (P1) and the fact that $\pi_j \geq \pi_i$ for $i > j$ and the second term is non negative because of (P2). Hence $C_j(S^*) \geq 0$. We get the result by repeating the same procedure for each k in G and not in S^*. ■

Proof of Proposition 3.1.11: a) First, note that adding product n cannot decrease expected profit: for all S such that $n \notin S$, we have $C_n(S) = R_n(S \cup \{n\})\pi_n \geq 0$ because $R_j(S) = R_j(S \cup \{n\})$ for $j < n$. Second, consider an assortment $S = \{s_1, s_2, .., s_k, n\}$, such that $s_1 < s_2 < \ldots < s_k < n$. Let $p(S)$ be the path that corresponds to set S, i.e., $p(S) = 0 \rightarrow s_1 \rightarrow s_2 \rightarrow \ldots \rightarrow s_k \rightarrow n$. It follows that every path with finite cost corresponds to an assortment that contains product n and vice versa. Let P be the set of all admissible paths.

The cost of path $p(S)$ is equal to

$$
\begin{aligned}
C(p(S)) &= c_{0,s_1} + c_{s_1,s_2} + ... + c_{s_k,n}, \\
&= -R_{s_1}(\{s_1\})\pi_{s_1} - \sum_{j=2,i=j-1}^{k} R_{s_i}(\{s_i, s_j\})\pi_{s_j} + -R_n(\{s_k, n\})\pi_n, \\
&= -\sum_{i=1}^{n} R_i(S)\pi_i, \\
&= -E\Pi(S).
\end{aligned}
$$

where the third equality is by (3.11). Hence, $\min_{p \in P} C(p) = \min_{S \in \mathcal{N}} -E\Pi(S) = \max_{S \in \mathcal{N}} E\Pi(S)$.

b) The complexity of a shortest path problem in an acyclic network is bounded by the number of arcs. There are $\frac{(n+1)(n+2)}{2}$ arcs. ■

Proof of Proposition 3.1.12: First note that $n \in S^*$. Let $S^* = \{s_1, ..., s_k, n\}$

with $s_1 < ... < s_k < n$. By definition of S^*, $\pi_{s_1} \geq \pi_{s_2} \geq ... \geq \pi_{s_k} \geq \pi_n$. Also,

$$\begin{aligned}
\mathbb{E}\Pi(S^*) &= (\alpha_{(1,...,n)} + ... + \alpha_{(s_1,...,n)})\pi_{s_1} + (\alpha_{(s_1+1,...,n)} + ... + \alpha_{(s_2,...,n)})\pi_{s_2} \\
&\quad + ... + (\alpha_{(s_k+1,...,n)} + ... + \alpha_{(n)})\pi_n, \\
&= \sum_{j=1}^{n} \alpha_{(j,...,n)} \max_{k=j,...,n} \pi_k, \\
&= \sum_{\boldsymbol{\tau} \in \mathcal{T}^+} \max_i \pi_{\tau_i} = UB_1.
\end{aligned}$$

where UB_1 is the upper bound defined in 3.1.2. Hence S^* is optimal.■

<u>Proof of Lemma 3.1.13:</u> First suppose that all the optimal assortments contain more than one product. Let S be one of them. Let $\hat{i}$ be such that $\hat{i} = \min\{j : j \in S\}$. Note that $R_j(S) = 0$ for all $j \in S$ and $j > \hat{i}$. By (P1), $R_{\hat{i}}(S) = R_{\hat{i}}(S \backslash \{j\})$ for all $j \in S, j > \hat{i}$. Hence, we can remove all such products without affecting expected profit and $\mathbb{E}\Pi(S) = R_{\hat{i}}(\{\hat{i}\})\pi_{\hat{i}}$. Hence it is optimal to offer only product k such that $R_k(\{k\})\pi_k = \max_j R_j(\{j\})\pi_j$.
■

<u>Proof of Proposition 3.1.14:</u>We show that S^* achieves the upper bound UB_1 defined in 3.1.2.

$$\begin{aligned}
\mathbb{E}\Pi(S^*) &= \sum_{j:\pi_j \geq \pi_n} \alpha_{(j,n)}\pi_j + \left(\sum_{j:\pi_j < \pi_n} \alpha_{(j,n)} + \alpha_{(n)} \right) \pi_n, \\
&= \sum_{\boldsymbol{\tau}=(\tau_1,...,\tau_m) \in \mathcal{T}^+} \alpha_{\boldsymbol{\tau}} \max_{i=1,...,m} \pi_{\tau_i} = UB_1.
\end{aligned}$$

Hence proved.■

Proof of Lemma 3.1.15:As we can see below

$$\max_{\substack{j\notin S\\ I\subseteq S\subseteq(\mathcal{N}\setminus O)}} C_j(S) = C_j(I), \tag{A.5}$$

$$\min_{\substack{j\notin S\\ I\subseteq S\subseteq(\mathcal{N}\setminus O)}} C_j(S) = C_j\left(\mathcal{N}\setminus(O\cup\{j\})\right). \tag{A.6}$$

For $j, k \notin S$,

$$\begin{aligned} C_j(S) &= \sum_{i\in S}\alpha_{(j,i)}(\pi_j-\pi_i) \\ &\quad + \sum_{\substack{i\neq j,k\\ i\notin S}}(\alpha_{(j,i)}+\alpha_{(i,j)})\pi_j + (\alpha_{(j,k)}+\alpha_{(k,j)})\pi_j + \alpha_{(j)}\pi_j, \\ C_j(S\cup\{k\}) &= \sum_{i\in S}\alpha_{(j,i)}(\pi_j-\pi_i) \\ &\quad + \alpha_{(j,k)}(\pi_j-\pi_k) + \sum_{\substack{i\neq j,k\\ i\notin S}}(\alpha_{(j,i)}+\alpha_{(i,j)})\pi_j + \alpha_{(j)}\pi_j. \end{aligned}$$

We have $C_j(S) \geq C_j(S\cup\{k\})$ because $(\alpha_{(j,k)}+\alpha_{(k,j)})\pi_j \geq \alpha_{(j,k)}(\pi_j-\pi_k)$ for all $k \neq j$ and $k \notin S$. Hence, the maximum is achieved with the smallest set S that satisfies the conditions in (A.5), *i.e.* I, and the minimum is achieved with the the largest set that satisfies the conditions in (A.6), *i.e.* $\mathcal{N}\setminus(O\cup\{j\})$.

■

Appendix B

Durable Products, Time inconsistency, and Lock-in

Proof of Lemma 4.2.2

a) When $Q_1 = 1$, we must have $Q_2 = 1$ also. Thus, (4.7) reduces to maximizing the following second period profit function with respect to the quantity of consumables:

$$\pi_2^{LI}(1,1,y_2) = \begin{cases} y_2\left(1 - \frac{y_2}{\theta}\right) & \text{if } y_2 \le (1-\alpha)\theta \\ y_2\left(\theta + \alpha(1-\theta) - y_2^c\right) & \text{if } y_2 \ge (1-\alpha)\theta \end{cases}$$

where we have simply substituted (4.4) and (4.5) into (4.7). The above expression is continuous and piece-wise concave, and the first-order conditions for the upper and lower branches are satisfied at $y_2^{c1} = \frac{\theta}{2}$ and $y_2^{c2} = \frac{\theta+\alpha(1-\theta)}{2}$ respectively. It is easy to show that $y_2^{c2} < (1-\alpha)\theta$ if and only if $\alpha < \frac{\theta}{1+\theta}$, while $y_2^{c1} > (1-\alpha)\theta$ if and only if $\alpha > \frac{1}{2}$. It follows from this and the continuity of $\pi_2^{LI}(1,1,y_2)$ that $\alpha < \frac{\theta}{1+\theta}$ implies that the optimal quantity of consumables is $y_2^* = y_2^{c1}$, while $\alpha > \frac{1}{2}$ implies that $y_2^* = y_2^{c2}$.

When $\alpha \in \left[\frac{\theta}{1+\theta}, \frac{1}{2}\right]$, then both $y_2^{c1} \le (1-\alpha)\theta$ and $y_2^{c2} \ge (1-\alpha)\theta$, so both of them are candidates to be the optimal quantity. By substituting into $\pi_2(1,1,y_2)$, it is easy to show that $\pi_2^{LI}(1,1,y_2^{c1}) = \frac{\theta}{4} \ge \pi_2^{LI}(1,1,y_2^{c2}) =$

$\left(\frac{\theta+\alpha(1-\theta)}{2}\right)^2$ if and only if $\alpha \leq \frac{\sqrt{\theta}}{1+\sqrt{\theta}} \leq \frac{1}{2}$. The result follows from the fact that, in order for the market clearing price of the consumable to fall below the low-type consumers' maximum marginal utility, we must have $y_2 \geq (1-\alpha)\theta$.

b) When $Q_1 = \theta$ then the monopolist has two options with respect to the sales of the durable in period 2: She can either set $Q_2 = 1$ and sell additional durables to the low-type consumers or she can set $Q_2 = \theta$, not selling any more durables. If she sets $Q_2 = \theta$, then she produces the quantity of consumables that maximizes her revenues from selling consumables to high-type consumers only, i.e. $\pi_2^{LI}(\theta, \theta, y_2) = y_2 p_c(\theta, y_2) = y_2\left(1 - \frac{y_2}{\theta}\right)$. Alternatively, if she sets $Q_2 = 1$ then the market clearing price from the additional durable sales will be positive if and only if $y_2 \geq (1-\alpha)\theta$. Otherwise, the price of consumables would be prohibitively high, i.e. $p_c\left(1, y_2\right) > \alpha$, and the low-type consumers would obtain no benefit from owning the durable. Given the quantity of consumables that she makes available, the market clearing price for the durable will be equal to the total utility that low-type consumers derive from ownership, i.e. $V_l\left(p_c\left(1, y_2\right)\right)$. Thus, we can write:

$$\pi_2^{LI}\left(\theta, 1, y_2\right) = y_2\left(\theta + \alpha(1-\theta) - y_2\right) + (1-\theta)V_l\left(p_c\left(1, y_2\right)\right)$$

The first order condition for $\pi_2^{LI}\left(\theta, \theta, y_2\right)$ is $y_2^{c3} = \frac{\theta}{2}$, and the first-order-condition for $\pi_2^{LI}\left(\theta, 1, y_2\right)$ is $y_2^{c4} = \frac{\alpha+(1-\alpha)\theta^2}{(1+\theta)}$. The result follows from the fact that:

$$\pi_2\left(\theta, 1, y_2^{c4}\right) = \frac{\alpha^2+\theta^2-\alpha^2\theta^2}{2(1+\theta)} \geq \pi_2\left(\theta, \theta, y_2^{c3}\right) = \frac{\theta}{4}$$

if and only if $\alpha \geq \sqrt{\frac{\theta}{2(1+\theta)}}$.

c) When $\alpha \leq \sqrt{\frac{\theta}{2(1+\theta)}}$, $p_c\left(Q_2^*, y_2^*\right) = \frac{1}{2}$ regardless of whether $Q_1 = 1$ or $Q_1 = \theta$.

If $\sqrt{\frac{\theta}{2(1+\theta)}} \leq \alpha \leq \frac{\sqrt{\theta}}{1+\sqrt{\theta}}$ then $p_c\left(Q_2^*(1), y_2^*\right) = \frac{1}{2} \geq \frac{\theta}{1+\theta} = p_c\left(Q_2^*(\theta), y_2^*(\theta)\right)$

If $\alpha \geq \frac{\sqrt{\theta}}{1+\sqrt{\theta}}$, $p_c\left(Q_2^*(1), y_2^*(1)\right) = \frac{1}{2}(\alpha+\theta-\alpha\theta) \geq \frac{\theta}{1+\theta} = p_c\left(Q_2^*(\theta), y_2^*(\theta)\right)$.

Observe that the above inequality holds so long as $\alpha \geq \frac{\theta}{1+\theta}$ which is true for $\alpha \geq \frac{\sqrt{\theta}}{1+\sqrt{\theta}}$.

d) This is easily observed in Table 4.2 and Table 4.1. ■

Proof of Proposition 4.2.3 Let us first consider the conditionally optimal solution for $Q_1 = 1$. By substituting (4.11) into (4.8) the FOC w.r.t. y is $y_1^{c*}(1) = \alpha$. Observe that, for $Q_1 = 1$, we will have $p_c\left(y_1^{c*}(1)\right) \geq \alpha$ for all $\alpha < \frac{\theta}{1+\theta}$. This implies that, if $Q_1 = 1$, the implicit rental price in period 1 would be zero since $V_l(p_c) = 0$ for $p_c \geq \alpha$. Thus, $Q_1 = \theta$ dominates $Q_1 = 1$ for any $\alpha < \frac{\theta}{1+\theta}$. For larger values of α, we can substitute $y_1 = y_1^{c*}(1) = \alpha$ and (4.11) into (4.8) and from Lemma 4.2.2 we have the following optimal profits, conditional on $Q_1 = 1$.

$$\pi_1\left(1, y_1^{c*}(1)\right) = \begin{cases} \frac{1}{2}\left(\alpha^2 + (1-\alpha)^2\theta^2\right) + \frac{\rho\theta}{4} & \text{if } \dfrac{\theta}{1+\theta} \leq \alpha \leq \dfrac{\sqrt{\theta}}{1+\sqrt{\theta}} \\ \frac{1}{2}\left(\alpha^2 + (1-\alpha)^2\theta^2\right) & \\ + \frac{\rho\left(3\alpha^2+2(1-\alpha)\alpha\theta+3(1-\alpha)^2\theta^2\right)}{8} & \text{if } \alpha \geq \frac{\sqrt{\theta}}{1+\sqrt{\theta}} \end{cases} \tag{B.1}$$

where the upper branch of the above function represents the case in which, given that $Q_1 = 1$, the manufacturer will restrict consumables output in the second period to such an extent that the market clearing price will not induce any consumption from low-type consumers, i.e. $p_c\left(Q_2^*, y_2^*\right) \geq \alpha$. The lower branch represents the case in which $p_c\left(Q_2^*, y_2^*\right) \leq \alpha$.

Now let us consider the conditionally optimal solution for $Q_1 = \theta$. By substituting (4.10) into (4.8), the FOC w.r.t. y is $y_1^{c*}(\theta) = \theta$. We can now we can substitute $y_1 = y_1^{c*}(\theta) = \theta$ and (4.16) into (4.8) and from Lemma 4.2.2 we have the following optimal profits, conditional on $Q_1 = \theta$:

$$\pi_1\left(\theta, y_1^{c*}(\theta)\right) = \begin{cases} \frac{\theta}{8}(4+3\rho) & \text{if } \alpha \leq \sqrt{\dfrac{\theta}{2(1+\theta)}} \\ \frac{\alpha^2\rho+\theta\left(1+2\alpha^2\rho+\theta(2+\theta+(1-\alpha)(1-\alpha+2\theta)\rho)\right)}{2(1+\theta)^2} & \text{if } \alpha \geq \sqrt{\dfrac{\theta}{2(1+\theta)}} \end{cases} \tag{B.2}$$

where the upper branch represents the case in which, given that $Q_1 = \theta$, the manufacturer's optimal second period response does not include selling durables to the low-type consumers. The lower branch represents the case in which it does.

Let us now consider $\Delta^{LI}(\alpha) = \pi_1\left(1, y_1^{c*}(1)\right) - \pi_1\left(\theta, y_1^{c*}(\theta)\right)$ for various values of α. It is easy to confirm that $\frac{\theta}{1+\theta} \leq \sqrt{\frac{\theta}{2(1+\theta)}} \leq \frac{\sqrt{\theta}}{1+\sqrt{\theta}}$. For $\alpha \in \left[\frac{\theta}{1+\theta} \leq \sqrt{\frac{\theta}{2(1+\theta)}}\right]$, the function $\Delta^{LI}(\alpha)$ is obtained by taking the difference between the upper branch of (B.1) and the upper branch of (C.1), to get:

$$\Delta^{LI}(\alpha) = \frac{\alpha^2 - \theta + (1-\alpha)^2\theta^2}{2} - \frac{\theta\rho}{8}$$

It is easy to confirm that this is increasing and convex, and that $\Delta^{LI}(\alpha) < 0$ at the point, $\alpha = \sqrt{\frac{\theta}{2(1+\theta)}}$. Thus, $Q_1^* = \theta$ for $\alpha \leq \sqrt{\frac{\theta}{2(1+\theta)}}$.

For $\alpha \in \left[\sqrt{\frac{\theta}{2(1+\theta)}}, \frac{\sqrt{\theta}}{1+\sqrt{\theta}}\right]$, the function $\Delta^{LI}(\alpha)$ is obtained by taking the difference between the upper branch of (B.1) and the lower branch of (C.1), to get:

$$\Delta^{LI}(\alpha) = \frac{2(1+\theta)^2\left(\alpha^2 - \theta + (1-\alpha)^2\theta^2\right)}{4(1+\theta)^2} \tag{B.3}$$

$$+\frac{\left(\theta - 3\theta^3 + 4\alpha\theta^2(1+\theta) - 2\alpha^2(1+\theta)^2\right)\rho}{4(1+\theta)^2} \tag{B.4}$$

It is easy to confirm that the second derivative of this function is positive, so that it is convex. The sign of the above difference is determined by its numerator. At the the upper limit of the range for which this difference is valid, i.e. $\alpha = \frac{\sqrt{\theta}}{1+\sqrt{\theta}}$, we have that the numerator of the above expression is equal to the following:

$$\frac{-4\theta^{3/2}(1+\theta)^2 - \theta\left(1-\sqrt{\theta}\right)^2(1+(2-\theta)\theta))\rho}{\left(1+\sqrt{\theta}\right)^2} < 0$$

To see that $\Delta^{LI}(\alpha)$ is increasing at $\alpha = \frac{\sqrt{\theta}}{1+\sqrt{\theta}}$, we first note that it will be increasing so long as its numerator is increasing. The first derivative of the numerator of (B.3) is equal to:

$$4(1+\theta)\left[\alpha\left(1+\theta\right)\left(1+\theta^2-\rho\right)-\theta^2(1+\theta-\rho)\right]$$

Evaluating the term in brackets at the point $\alpha = \sqrt{\frac{\theta}{2(1+\theta)}}$, we have:

$$\sqrt{\frac{\theta}{2(1+\theta)}}\left(1+\theta\right)\left(1+\theta^2-\rho\right)-\theta^2(1+\theta-\rho)$$

$$\geq \frac{\theta}{1+\theta}\left(1+\theta\right)\left(1+\theta^2-\rho\right)-\theta^2(1+\theta-\rho)$$
$$=\theta\left(1+\theta^2-\rho\right)-\theta^2(1+\theta-\rho)=\theta\left(\theta\left(1-\rho\right)\left(1-\theta\right)\right)>0$$

where we have used the fact that $\frac{\theta}{1+\theta} \leq \sqrt{\frac{\theta}{2(1+\theta)}}$. It follows that $Q_1^* = \theta$ for all $\alpha \in \left[\sqrt{\frac{\theta}{2(1+\theta)}}, \frac{\sqrt{\theta}}{1+\sqrt{\theta}}\right]$.

Finally, for $\alpha > \frac{\sqrt{\theta}}{1+\sqrt{\theta}}$, the difference, $\Delta^{LI}(\alpha)$ is obtained by taking the difference between the lower branches of (B.1) and (C.1) to obtain:

$$\Delta^{LI}\left(\alpha\right)=\frac{4\left(1+\theta\right)^2\left(\alpha-\theta+\left(1-\alpha\right)^2\theta^2\right)-\left(1-\theta\right)\rho\left(1+3\theta\right)\left(\alpha-\left(1-\alpha\right)\theta\right)^2}{8\left(1+\theta\right)^2} \tag{B.5}$$

The second derivative of (B.5) is positive, so it is convex. The sign of (B.5) is the same as the sign of its numerator. Evaluating the numerator at the lower limit of its valid range, i.e. $\alpha = \frac{\sqrt{\theta}}{1+\sqrt{\theta}}$, we have that it can be expressed as:

$$\frac{\theta}{\left(1+\sqrt{\theta}\right)^2}\left(\left(1+2\theta\right)2\sqrt{\theta}\left(-4+\rho\right)-8\theta^{5/2}-\rho\left(1+3\theta-\theta^2-3\theta^3+6\theta^{5/2}\right)\right)$$

$$< 0$$

Similarly, if we evaluate the numerator of (B.5) at the point, $\alpha = 1$, we have that it can be expressed as:

$$(-1+\theta)\left(\rho\,(1+3\theta) - 4\,(1+\theta)^2\right) > 0$$

which implies that there must be a point, $\bar{\alpha}^{LI} \in \left(\frac{\sqrt{\theta}}{1+\sqrt{\theta}}, 1\right)$,such that, within this interval, (B.5) is positive if and only if $\alpha > \bar{\alpha}^{LI}$. Thus, $Q_1^* = 1$ if and only if $\alpha > \bar{\alpha}^{LI}$. Otherwise, $Q_1^* = \theta$. From the results in Lemma 4.2.2, we now that, conditional upon $Q_1 = \theta$, we will have $Q_2^*(Q_1) = 1$ if and only if $\alpha \geq \sqrt{\frac{\theta}{2(1+\theta)}}$, so it follows that $\underline{\alpha}^{LI} = \sqrt{\frac{\theta}{2(1+\theta)}}$.

b) This follows from the fact that $Q_1^* = 1$ only if $\alpha \geq \bar{\alpha}^{LI} > \frac{\sqrt{\theta}}{1+\sqrt{\theta}}$, and from Lemma 4.2.2 that says that for $Q_1 = 1$ we will have $p_c\left(Q_2^*, y_2^*\right) < \alpha$ so long as $\alpha > \frac{\sqrt{\theta}}{1+\sqrt{\theta}}$. ■

Proof of Proposition 4.2.4 Define $\Delta_{diff}^{UA}(\alpha) = \pi_1^U - \pi_1^{LI}$, that is, $\Delta_{diff}^{UA}(\alpha)$ is the difference between the optimal profits under unrestricted access and lock-in, for a given α. Note that for $\alpha \in \left[0, \underline{\alpha}^{LI}(\theta)\right)$ profit under unrestricted access π_1^U is increasing in α and profit under lock-in π_1^{LI} is a constant. Further:

$$\Delta_{diff}^{UA}(0) = -\frac{3\theta\rho}{8} \leq 0$$

$$\lim_{\alpha \to \underline{\alpha}^{LI}(\theta)^-} \Delta_{diff}^{UA}(\alpha) = -\frac{\theta\rho(1+3\theta)}{8(1+\theta)} \leq 0$$

From above we can conclude that for $\alpha \in \left[0, \underline{\alpha}^{LI}(\theta)\right)$, the difference $\Delta^{UA}_{diff}(\alpha) < 0$. Note that $\Delta^{UA}_{diff}(\alpha)$ is continuous in α over the interval $\left(\underline{\alpha}^{LI}(\theta), 1\right]$. Further:

$$\Delta^{UA}_{diff}(1) = \frac{1}{2}(1+\rho) - \frac{1}{8}(4+3\rho) = \frac{\rho}{8} \geq 0$$

$$\lim_{\alpha \to \underline{\alpha}^{LI}(\theta)^+} \Delta^{UA}_{diff}(\alpha) = -\frac{\theta^2}{(1+\theta)^2}\left((1+2\theta) - \sqrt{2\theta(1+\theta)}\right) \leq 0$$

Then, there exists a set of points $\hat{\alpha}$, where

$$\hat{\alpha} = \left\{\alpha : \alpha \in \left(\underline{\alpha}^{LI}(\theta), 1\right] \text{ and } \Delta^{UA}_{diff}(\alpha) = 0\right\}$$

, where $\hat{\alpha}$ contains an odd number of elements. Let $\overline{\alpha} = max\, \hat{\alpha}$ and $\underline{\alpha} = min\, \hat{\alpha}$. Therefore there exists thresholds $\underline{\alpha} \leq \bar{\alpha}$ such that for $\alpha \geq \overline{\alpha}$ unrestricted access dominates lock-in and for $\alpha \leq \underline{\alpha}$ lock-in dominates unrestricted access. ■

Proof of Lemma 4.2.5 Each of the n rivals maximizes its second period profits below in response to the manufacturer's actions Q_2 and y_{20}.

$$\Pi_i\left(Q_2, y_{20}, Y_{2i}\right) = \begin{cases} y_{2i}\left(1 - \frac{y_{20} + \sum_{i=1}^{n} y_{2i}}{\theta}\right) \\ \qquad \text{if } y_{20} + \sum_{i=1}^{n} y_{2i} \leq (1-\alpha)\theta \text{ or } Q_2 = \theta \\ y_{2i}\left(\theta + \alpha(1-\theta) - \left(y_{20} + \sum_{i=1}^{n} y_{2i}\right)\right) \\ \qquad \text{if } y_{20} + \sum_{i=1}^{n} y_{2i} \geq (1-\alpha)\theta \text{ and } Q_2 = 1 \end{cases} \tag{B.6}$$

where we have simply substituted (4.4) into (4.12), where $y = y_{20} + \sum_{i=1}^{n} y_{2i}$. The above expression is continuous and piece-wise concave, and the first-order conditions for the upper and lower branches are satisfied at $y_{2i}^{r1} = \frac{\theta - y_{20}}{n+1}$ and $y_{2i}^{r2} = \frac{\theta + \alpha(1-\theta) - y_{20}}{n+1}$ respectively.

a) When $Q_1 = 1$, we must have $Q_2 = 1$ also. Thus, the second period model reduces to a Stackelberg game in consumables alone. The manufacturer can induce two types of equilibria among the rivals: one in which the total output is low enough that $p_c < \alpha$ so that both types of consumers purchase the consumable in period 2, and one in which the total output is high enough that $p_c \geq \alpha$ so that only high type consumers purchase positive quantities of the consumable. Let y_{20}^h be the manufacturer's optimal second period consumable output conditional upon her inducing a competitive response among the rivals in which $p_c \geq \alpha$. Similarly, let y_{20}^b be her optimal second period output conditional upon her inducing a response in which $p_c < \alpha$. The following two Lemmas both focus on the situation that arises when all consumers hold the durable at the beginning of period 2. Lemma B.0.1 (B.0.2) characterizes the manufacturer's conditionally optimal consumables output, given that she restricts her attention to inducing an outcome in which $p_c \geq \alpha$ ($p_c < \alpha$).

Lemma B.0.1. *If the manufacturer restricts attention to inducing an outcome in which $p_c \geq \alpha$, then her conditionally optimal policy can be characterized in terms of two thresholds: $\alpha_r(\theta, n) = \frac{\sqrt{\theta}}{(n+1)\left(1+\sqrt{\theta}\right)}$ and $\alpha_{r2}(\theta, n) = \frac{2\theta}{(n+1)\left(\theta+\sqrt{\theta}\right)}$ $\geq \alpha_r(\theta, n)$. For $\alpha \leq \alpha_r(\theta, n)$, her conditionally optimal output quantity is*

$y_{20}^{f1}= \frac{\theta}{2}$, *and her corresponding profit is* $\pi_2\left(1,1,y_{20}^{f1}\right) = \frac{\theta}{4(n+1)}$. *For* $\alpha \in [\alpha_r(\theta,n), \alpha_1(\theta,n)]$, *her conditionally optimal output*

$$y_{20}^{restr} = \frac{1}{2}\left(2\theta - (n+1)\,\alpha\left(\theta+\sqrt{\theta}\right)\right)$$

and her corresponding profit is

$$\pi_2\left(1,1,y_{20}^{restr}\right) = \frac{\alpha}{4}\left(2\left(\theta+\sqrt{\theta}\right) - (n+1)\,\alpha\left(1+\sqrt{\theta}\right)\right)$$

For $\alpha > \alpha_1(\theta,n)$ *, she cannot induce an equilibrium in which* $p_c \geq \alpha$ *even if she reduces her own output to zero.*

Proof of Lemma B.0.1 For $p_c \geq \alpha$, the profit of each rival in the consumables market is as shown in the upper branch of (B.6). Solving for the first-order conditions, we have that each rival's response to y_{20} is $y_{2i}^{r1} = \frac{\theta - y_{20}}{n+1}$. Substituting this response into (4.13) we can obtain the first-order condition for the manufacturer as $y_{20}^{f1} = \frac{\theta}{2}$. By substituting $y_{20}^{f1} = \frac{\theta}{2}$ and $y_{2i}^{r1} = \frac{\theta - y_{20}}{n+1}$ into (4.13), her corresponding profits are:

$$\pi_2\left(1, y_{20}^{f1}, Y^*\left(y_{20}^{f1}\right)\right) = \frac{\theta}{4(n+1)}$$

Similarly, by substituting into (B.6) that, at the quantities, $y_{20}^{f1} = \frac{\theta}{2}$ and $y_{2i}^{r1} = \frac{\theta - y_{20}}{n+1}$, each rival's profit is $\Pi^{r1} = \frac{\theta}{4(n+1)^2}$.The maximum output of the manufacturer that can induce $p_c\left(1, y_{20} + ny_{2i}^{r1}\right) \geq \alpha$ is $y_{max}^{f1} = (1-(n+1)\alpha)\theta$. Further, $y_{max}^{f1} = (1-(n+1)\alpha)\theta \geq \frac{\theta}{2}$ only if $\alpha \leq \frac{1}{2(n+1)}$. It remains to be shown that no rival has a unilateral incentive to deviate from the quantity

y_{2i}^{r1}. Given the output of the manufacturer, $y_2^{f1} = \frac{\theta}{2}$, and the output of each of $n-1$ rivals, $y_{2i}^{r1} = \frac{\theta}{2(n+1)}$, a given consumables supplier (j) may have an incentive to deviate by unilaterally producing enough to cause $p_c < \alpha$. Substituting $y_2^{f1} = \frac{\theta}{2}$ and $y_{2i}^{r1} = \frac{\theta}{2(n+1)}$ for $i \neq j$ into the lower branch of (B.6), we can see that if rival j were to increase her own output by enough to induce $p_c < \alpha$ her profit would be:

$$y_{2j}\left(\theta + \alpha(1-\theta) - \left(\frac{\theta}{2} + (n-1)\frac{\theta}{2(n+1)} + y_{2j}\right)\right)$$

The first-order condition for this profit is $y^{dev1} = \frac{\alpha(n+1)+\theta(1-\alpha)-n\alpha\theta}{2(n+1)}$. In order for $y^{dev1} = \frac{\alpha(n+1)+\theta(1-\alpha)-n\alpha\theta}{2(n+1)}$ to induce $p_c < \alpha$, we need to have $\theta + \alpha(1-\theta) - \left(\frac{\theta}{2} + (n-1)\frac{\theta}{2(n+1)} + y^{dev1}\right) < \alpha$, which is equivalent to $\alpha > \frac{\theta}{(n+1)(1+\theta)}$. Further, in order for the deviation to be profitable for the rival, we need:

$$y^{dev1}\left(\theta + \alpha(1-\theta) - \left(\frac{\theta}{2} + (n-1)\frac{\theta}{2(n+1)} + y^{dev1}\right)\right)$$

$$= \frac{(\alpha(1-\theta)(n+1)+\theta)^2}{4(1+n)^2} \geq \frac{\theta}{4(n+1)^2}$$

which is true only if $\alpha > \alpha_r(\theta, n) = \frac{\sqrt{\theta}}{(n+1)(1+\sqrt{\theta})} \geq \frac{\theta}{(n+1)(1+\theta)}$. That is for $\alpha \leq \alpha_r(\theta, n)$, the output quantities, $y_{20}^{f1} = \frac{\theta}{2}$ and $y_{2i}^{r1} = \frac{\theta - y_{20}}{n+1}$, define a unique equilibrium and result in $p_c \geq \alpha$. For $\alpha \in [\alpha_r(\theta, n), \alpha_1(\theta, n)]$, the most that the manufacturer can produce without creating an incentive for at least one rival to deviate from $y_{2i}^{r1} = \frac{\theta - y_{20}}{n+1}$ is $y_{20} = y_{20}^{restr} =$

$\frac{1}{2}\left(2\theta-(n+1)\alpha(\theta+\sqrt{\theta}\right) \leq \frac{\theta}{2}$. Hence, in this range, this is her conditionally optimal output quantity, and her corresponding profit can be obtained by substituting $y_{20}^{restr}(\alpha)$ and $y_{2i}^{r1}=\frac{\theta-y_{20}^{restr}}{n+1}$ into (4.13). Finally, when $\alpha>\alpha_1^h(\theta,n)$, we have that $y_{20}^{rest}<0$, which implies that the manufacturer cannot induce an equilibrium in which $p_c\geq\alpha$.▲

Lemma B.0.2. *If the manufacturer restricts attention to inducing an outcome in which $p_c<\alpha$, then for $\alpha\geq\frac{\theta}{(n+2\theta+n\theta)}$, she can induce such an outcome and her conditionally optimal output quantity is $y_{20}^{f2}=\frac{\alpha+\theta-\alpha\theta}{2}$, and her corresponding profit is $\pi_2\left(1,1,y_{20}^{f2}\right)=\frac{(\alpha+\theta-\alpha\theta)^2}{4(n+1)}$. If $\alpha\leq\frac{\theta}{(n+2\theta+n\theta)}$ then the manufacturer may be able to induce an outcome in which $p_c<\alpha$. When she can, her conditionally optimal profit is: $\pi_2\left(1,1,y_{20}^{inc}\right)\leq\frac{(\alpha+\theta-\alpha\theta)^2}{4(n+1)}$.*

Proof of Lemma B.0.2 For $p_c<\alpha$, the profit of each rival in the consumables market is as shown in the lower branch of (B.6). Solving for the first-order conditions, we have that each rival's response to y_{20} is $y_{2i}^{r2}=\frac{\alpha+\theta-\alpha\theta-y_{20}}{n+1}$. Substituting this response into (4.13), we can obtain the first-order condition for the manufacturer as $y_{20}^{f2}=\frac{\alpha+\theta-\alpha\theta}{2}$. The minimum output of the manufacturer for which $p_c\left(1,y_{20}+ny_{2i}^{r2}\right)<\alpha$ is $y_{min}^{f2}=\theta-(n+\theta)\alpha$. Further, $y_{min}^{f2}=\theta-(n+\theta)\alpha\leq\frac{\alpha+\theta-\alpha\theta}{2}$ only if $\alpha\geq\frac{1}{2n+1+\theta}$. It follows that, for $\alpha\geq\frac{1}{2n+1+\theta}$, the manufacturer's conditionally optimal output quantity (given that all rivals will respond according to $y_{2i}^{r2}=\frac{\alpha+\theta-\alpha\theta-y_{20}}{n+1}$) is $y_{20}^{f2}=\frac{\alpha+\theta-\alpha\theta}{2}$. By substituting $y_{20}^{f2}=\frac{\alpha+\theta-\alpha\theta}{2}$ and $y_{2i}^{r2}=\frac{\alpha+\theta-\alpha\theta-y_{20}}{n+1}$ into (4.13), her corresponding profits can be obtained. Similarly, by substituting into (B.6) it can be confirmed that, at the quantities,$y_{20}^{f2}=\frac{\alpha+\theta-\alpha\theta}{2}$ and

$y_{2i}^{r2} = \frac{\alpha+\theta-\alpha\theta-y_{20}}{n+1}$, each rival's profit is $\Pi_i^{r2} = \frac{(\alpha+\theta-\alpha\theta)^2}{4(n+1)^2}$. It remains to be shown that no rival has a unilateral incentive to deviate. Given the output of the manufacturer, y_{20}^{f2}, and the output of each of $n-1$ rivals, y_{2i}^{r2}, a given consumables supplier (j) may have an incentive to deviate by unilaterally reducing its own output by enough to cause $p_c \geq \alpha$. Substituting $y_{20}^{f2} = \frac{\alpha+\theta-\alpha\theta}{2}$ and $y_{2i}^{r2} = \frac{\alpha+\theta-\alpha\theta-y_{20}}{n+1}$ for $i \neq j$ into the upper branch of (B.6), we can see that the profit earned by supplier j from a unilateral deviation would be:

$$y_{2j}\left(1-\frac{1}{\theta}\left(\frac{n\left(\alpha+\theta-\alpha\theta\right)}{n+1}-y_{2j}\right)\right) \tag{B.7}$$

The first-order condition for the deviant rival is $y_{2j} = y^{dev2} = \frac{n\alpha\theta-n\alpha+\theta}{2(n+1)}$. In order for $y_{2j} = y^{dev2}$ to induce $p_c \geq \alpha$, we need to have $1-\frac{1}{\theta}\left(\frac{n(\alpha+\theta-\alpha\theta)}{n+1}-y^{dev2}\right)$ $\geq \alpha$, which is equivalent to $\alpha \geq \frac{\theta}{(n+2\theta+n\theta)}$. Further, by substituting $y_{2j} = y^{dev2}$ and $y_{2j} = y_{2i}^{r2}$ into the deviant supplier's profit function (as shown in (B.7)) , we can see that he is better off producing y^{dev2}instead of y_{2i}^{r2}only if $\alpha < \frac{\theta}{(n+1)\sqrt{\theta}+(n+\theta)} < \frac{\theta}{(n+2\theta+n\theta)}$. That is for $\alpha \geq \frac{\theta}{(n+2\theta+n\theta)}$, the output quantities, $y_{20}^{f2} = \frac{\alpha+\theta-\alpha\theta}{2}$ and $y_{2i}^{r2} = \frac{\alpha+\theta-\alpha\theta-y_{20}}{n+1}$, define a unique equilibrium and result in $p_c < \alpha$. Note that $\frac{1}{2n+1+\theta} > \frac{\theta}{(n+2\theta+n\theta)}$,which ensures that for $\alpha \geq \frac{\theta}{(n+2\theta+n\theta)}$, we will have that $p_c\left(1, y_{20}^{f2}+ny_{2i}^{r2}\right) < \alpha$. Finally, when $\alpha \leq \frac{\theta}{(n+2\theta+n\theta)}$, the manufacturer may be able to induce an equilibrium in which $p_c < \alpha$, but she will need to produce $y_{20}^{inc} > y_{20}^{f2} = \frac{\alpha+\theta-\alpha\theta}{2}$ to do so. Because $y_{20}^{f2}$is the quantity that maximizes her profits given each rival responds according to $y_{2i}^{r2} = \frac{\alpha+\theta-\alpha\theta-y_{20}}{n+1}$. It follows that $\pi_2\left(1,1,y_{20}^{inc}\right) \leq \frac{(\alpha+\theta-\alpha\theta)^2}{4(n+1)}$. ▲

We can now use the results of Lemmas B.0.1 and B.0.2 to complete the proof of part a of Lemma *4.2.5* . From Lemma B.0.1 , we know that, for $\alpha \leq \alpha_r(\theta, n) = \frac{\sqrt{\theta}}{(n+1)\left(1+\sqrt{\theta}\right)}$, the manufacturer's conditionally optimal profits are $\pi_2\left(1, y_{20}^{f1}, Y^*\left(y_{20}^{f1}\right)\right) = \frac{\theta}{4(n+1)}$, given that she induces an outcome in which $p_c \geq \alpha$. From Lemma B.0.2, we know that, for $\alpha \geq \frac{\theta}{(n+2\theta+n\theta)}$, the manufacturer's conditionally optimal profits are $\pi_2^n\left(1, y_{20}^{f2}, Y^*\left(y_{20}^{f2}\right)\right) = \frac{(\alpha+\theta-\alpha\theta)^2}{4(n+1)}$, conditional upon the restriction that she induce an equilibrium with $p_c < \alpha$. Moreover, for $\alpha < \frac{\theta}{(n+2\theta+n\theta)}$, the manufacturer's profits from inducing an equilibrium with $p_c < \alpha$ are bounded above by $\frac{(\alpha+\theta-\alpha\theta)^2}{4(n+1)}$. Comparing these two conditionally optimal profits, we can see that $\pi_2^n\left(1, y_{20}^{f2}, Y^*\left(y_{20}^{f2}\right)\right) \geq \pi_2\left(1, y_{20}^{f1}, Y^*\left(y_{20}^{f1}\right)\right)$ is equivalent to $\alpha \geq \frac{\sqrt{\theta}}{1+\sqrt{\theta}}$. It is easy to see that $\frac{\theta}{(n+2\theta+n\theta)} \leq \alpha_r(\theta, n) \leq \frac{\sqrt{\theta}}{1+\sqrt{\theta}}$, where $\alpha_r(\theta, n) = \frac{\sqrt{\theta}}{(n+1)\left(1+\sqrt{\theta}\right)}$. Thus, it is obvious that for $\alpha \leq \alpha_r(\theta, n)$ the manufacturer's optimal output quantity will be: $y_{20}^{f1} = \frac{\theta}{2}$, and her corresponding profit is $\pi_2\left(1, 1, y_{20}^{f1}\right) = \frac{\theta}{4(n+1)}$, and this will result in $p_c \geq \alpha$. It is also obvious that, for $\alpha \geq \alpha_{r2}(\theta, n)$, the manufacturer's optimal output quantity will be: $y_{20}^{f2} = = \frac{\alpha+\theta-\alpha\theta}{2}$, and her corresponding profit is $\pi_2\left(1, 1, y_{20}^{f2}\right) = \frac{(\alpha+\theta-\alpha\theta)^2}{4(n+1)}$, and this will result in $p_c < \alpha$.

For $\alpha \in [\alpha_r(\theta, n), \alpha_{r2}(\theta, n)]$, the manufacturer can induce an equilibrium in which $p_c \geq \alpha$, and her conditionally optimal profit from doing so is $\pi_2\left(1, 1, y_{20}^{restr}(\alpha)\right) = \frac{\alpha}{4}\left(2\left(\theta + \sqrt{\theta}\right) - (n+1)\,\alpha\left(1 + \sqrt{\theta}\right)\right)$. It can be shown that $\pi_2\left(1, 1, y_{20}^{f2}\right) > \pi_2\left(1, 1, y_{20}^{restr}(\alpha)\right)$ if and only if $\alpha > \alpha_{r3}(\theta, n)$

where $\alpha_{r3}(\theta, n)$ is:

$$\frac{\theta^2}{\theta^2 + n\theta + (n+1)\sqrt{\theta} - \sqrt{(n-1)\left(1-\sqrt{\theta}\right)\left(\sqrt{\theta}+\theta\right)^2\left((n-1)\sqrt{\theta}+n+1\right)}} \tag{B.8}$$

Let $\alpha_1(\theta, n) = Max\{\alpha_r(\theta, n), \alpha_{r3}(\theta, n)\}$. It follows that the manufacturer will produce $y_{20}^{restr}(\alpha)$ if and only if $\alpha_{r3}(\theta, n) > \alpha_r(\theta, n)$ and $\alpha \in (\alpha_r(\theta, n), \alpha_1(\theta, n))$.

b) When $Q_1 = \theta$ then the monopolist has two options with respect to the sales of the durable in period 2: She can either set $Q_2 = 1$ and sell additional durables to the low type consumers or she can set $Q_2 = \theta$, not selling any more durables. If she sets $Q_2 = \theta$, and produces a quantity of consumables y_{20}, then the rivals maximizes their revenues as given in the upper branch of (B.6) . Recall that the first order conditions are satisfied at $y_{2i}^{r1} = \frac{\theta - y_{20}}{n+1}$. The manufacturer then maximizes $\pi_2^n(\theta, \theta, y_{20}) = y_{20} p_c\left(\theta, y_{20} + n y_{2i}^{r1}\right) = y_{20}\left(1 - \frac{y_{20}+n\theta}{(n+1)\theta}\right)$. The first-order condition the manufacturer is $y_{20}^{f1} = \frac{\theta}{2}$ and her corresponding profit is:

$$\pi_2\left(\theta, \theta, y_{20}^{f1}\right) = \frac{\theta}{4(n+1)}$$

Alternatively, if she sets $Q_2 = 1$ then the market clearing price from the additional durable sales will be positive if and only if $y_{20} + \sum_{i=1}^{n} y_{2i}^{r2} = \frac{y_{20}+n(\theta+\alpha(1-\theta))}{n+1} \geq (1-\alpha)\theta$. Otherwise, we would have $p_c\left(1, y_{20} + \sum_{i=1}^{n} y_{2i}^{r2}\right) > \alpha$, and the low type consumers would obtain no benefit from owning the durable. Given the quantity of consumables that she makes available, the market clearing price for the durable will be equal to the net utility that low type consumers derive from ownership, i.e. $V_l\left(p_c\left(1, y_{20}+\sum_{i=1}^{n} y_{2i}^{r2}\right)\right)$. Thus, we can write:

$$\pi_2^n(\theta, 1, y_{20}) = y_{20}\left(\frac{1}{n+1}(\theta+\alpha(1-\theta)-y_{20})\right)+(1-\theta)V_l(p_c(1, y_2))$$

Given that each rival responds according to y_{2i}^{r2}, the manufacturer's first-order condition is: $y_{20}^{f3} = \frac{\alpha(1-\theta^2)+\theta^2}{(1+\theta)}$ and her profits are:

$$\pi_2\left(\theta, 1, y_{20}^{f3}\right) = \frac{\alpha^2(1-\theta)(1+2n+\theta)+\theta^2}{2(1+2n+\theta)} \geq \frac{\theta}{4(n+1)} = \pi_2\left(\theta, \theta, y_{20}^{f1}\right)$$

where the above inequality is true if and only if $\alpha \geq \alpha_\theta(\theta, n) = \sqrt{\frac{\theta(2n+1)}{2(n+1)(1+2n+\theta)}}$. It remains to be shown that none of the rivals will have an incentive to unilaterally deviate to a sufficiently low amount of output that would result in $p_c \geq \alpha$. Substituting y_{20}^{f3} and y_{2i}^{r2} for $i \neq j$ into the upper branch of (B.6) , we can see that if rival j if he were to unilaterally reduce his output by enough to cause $p_c \geq \alpha$, his profit would be:

$$y_{2j}\left(1-\frac{1}{\theta}\left(\frac{\alpha\left(1-\theta^2\right)+\theta^2}{1+\theta}+(n-1)\frac{\theta}{1+2n+\theta}+y_{2j}\right)\right) \tag{B.9}$$

The above is maximized at $y^{dev3} = \frac{-\alpha-2n\alpha+2\theta+2n\alpha\theta+\alpha\theta^2}{2(1+2n+\theta)}$. It is easy to confirm that this deviation results in a price of consumable $p_c > \alpha$ only if $\alpha < \frac{2\theta}{(1+\theta)(1+2n+\theta)}$. By substituting $y_{2j} = y^{dev3}$ and $y_{2j} = y_{2i}^{r2}$ into rival j's profit function (as shown in (B.9)) and comparing, we can see that he is better off producing y^{dev3}instead of y_{2i}^{r2}, only if: $\alpha \leq \frac{2\theta}{\left(1+\sqrt{\theta}\right)(1+2n+\theta)}$. Note that $\frac{2\theta}{\left(1+\sqrt{\theta}\right)(1+2n+\theta)} < \frac{2\theta}{(1+\theta)(1+2n+\theta)} < \sqrt{\frac{\theta(2n+1)}{2(n+1)(1+2n+\theta)}} = \alpha_\theta(\theta, n)$. Therefore, for any αin which the manufacturer sells durables in period 2, there is an equilibrium in which $p_c < \alpha$.

c) This is easily observed in Table 4.4 and Table 4.5. ■

Proof of Proposition 4.2.6

Because the manufacturer does not face competition in the first period, for a given output, Q_1, of durables, her conditionally optimal output of consumables will be the same as it was under lock-in. As shown in the proof of Proposition 4.2.3 the conditionally optimal solution for $Q_1 = 1$ is $y_1^{c*}(1) = \alpha$. Therefore, just as under lock-in, conditional upon $Q_1 = 1$, $p_c\left(y_1^{c*}(1)\right) < \alpha$ only if $\alpha \geq \frac{\theta}{1+\theta}$. Thus, for any $\alpha < \frac{\theta}{1+\theta}$, the manufacturer's conditionally optimal consumables output, for $Q_1 = 1$, would result in $p_c \geq \alpha$. Not only would this drive the implicit rental price for durables to zero (since $V_l(p_c) = 0$ for $p_c \geq \alpha$), it would also mean that the low valuation consumers would not contribute any consumables revenue to the manufacturer. It follows that $Q_1^* = \theta$ for any $\alpha < \frac{\theta}{1+\theta}$. In order to determine what happens for $\alpha \geq \frac{\theta}{1+\theta}$, it will be helpful to introduce the following Lemma:

Lemma B.0.3. *For any $\alpha \geq \frac{\theta}{1+\theta}$, we must have either that: $\alpha \leq \alpha_r(\theta, n)$ or that $\alpha_r(\theta, n) = \alpha_1(\theta, n)$. In equilibrium, we will never have $Q_1 = 1$ when $\alpha_r(\theta, n) < \alpha_1(\theta, n)$.*

Proof: Recall that, by definition, $\alpha_1(\theta, n) = Max\left\{\alpha_r(\theta, n), \alpha_{r3}(\theta, n)\right\}$ where $\alpha_r(\theta, n) = \frac{\sqrt{\theta}}{(n+1)\left(1+\sqrt{\theta}\right)}$ and $\alpha_{r3}(\theta, n)$ is as defined in (B.8). It is easy to confirm that $\alpha_{r3}(\theta, n)$ is decreasing in n and that, for $n = 1$, we have$\alpha_{r3}(\theta, 1) <$

$\frac{\theta}{1+\theta}$. Thus, for any $n \geq 1$ such that $\alpha_r(\theta, n) \geq \frac{\theta}{1+\theta}$, we must have that either: $\alpha_1(\theta, n) = \alpha_{r3}(\theta, 1) < \frac{\theta}{1+\theta}$; or that: $\alpha_1(\theta, n) = \alpha_r(\theta, n)$. ▲

The implication of this result is that, for any $\alpha \geq \frac{\theta}{1+\theta}$, we cannot have $\alpha \in \{\alpha_r(\theta, n), \alpha_1(\theta, n)\}$. Since in equilibrium, $\alpha \geq \frac{\theta}{1+\theta}$ is a necessary condition to have $Q_1^* = 1$, the manufacturer's second period consumables output must be either $y_{20}^{f1} = \frac{\theta}{2}$, or $y_{20}^{f2} = \frac{\alpha+\theta-\alpha\theta}{2}$ whenever we consider the possibility of $Q_1 = 1$. Thus, by substituting $y_1 = y_1^{c*}(1) = \alpha$ and (4.17) into (4.14) , from Lemma 4.2.5 and Lemma B.0.3 we can obtain the following expression for the manufacturer's conditionally optimal profit given that $Q_1 = 1$ for any $\alpha \geq \frac{\theta}{1+\theta}$:

$$\pi_1^n\left(1, y_1^{c*}(1)\right) = \begin{cases} \frac{1}{2}\left(\alpha^2 + (1-\alpha)^2\theta^2\right) \\ +\rho\frac{\theta}{4(n+1)} & \text{if } \alpha \leq \alpha_r(\theta, n) \\ \frac{1}{2}\left(\left(\alpha - (1-\alpha)\theta\right)^2 + \rho\left(\alpha - \frac{\alpha+\theta-\alpha\theta}{2(n+1)}\right)^2\right) + \\ \alpha\theta(1-\alpha) + \rho\frac{(\alpha+\theta-\alpha\theta)^2}{4(n+1)} & \text{if } \alpha \geq \alpha_r(\theta, n) \end{cases} \tag{B.10}$$

where the upper branch of the above function represents the case in which, given that $Q_1 = 1$, the consumables output in the second period is such that the market clearing price will not induce any consumption from low-type consumers, i.e. $p_c\left(Q_2^*, y_2^*\right) \geq \alpha$. The lower branch represents the case in which $p_c\left(Q_2^*, y_2^*\right) \leq \alpha$.

Now let us consider the conditionally optimal solution for $Q_1 = \theta$. Recall that the FOC w.r.t. y is $y_1^{c*}(\theta) = \theta$. We can now we can substitute

$y_1 = y_1^{c*}(\theta) = \theta$ and (4.16) into (4.14) and from Lemma 4.2.5 we have the following optimal profits, conditional on $Q_1 = \theta$:

$$\pi_1^n\left(\theta, y_1^{c*}(\theta)\right) = \begin{cases} \frac{\theta}{2}\left(1+\rho\left(\frac{2n+1}{2n+2}\right)^2\right) & \\ +\rho\frac{\theta}{4(n+1)} & \text{if } \alpha \leq \alpha_\theta(\theta, n) \\ \frac{\theta}{2}\left(1+\rho\left(\alpha - \frac{\theta}{1+2n+\theta}\right)^2\right)+ & \\ \rho\frac{\alpha^2(1-\theta)(1+2n+\theta)+\theta^2}{2(1+2n+\theta)} & \text{if } \alpha \geq \alpha_\theta(\theta, n) \end{cases} \tag{B.11}$$

where the upper branch represents the case in which, given that $Q_1 = \theta$, the second period response output is such that the manufacturer does not sell additional durables to the low-type consumers. The lower branch represents the case in which it does. To determine the equilibrium values, Q_1^* and Q_2^*, we need to consider three thresholds: $\frac{\theta}{1+\theta}$, $\alpha_r(\theta, n)$, and $\alpha_\theta(\theta, n)$. To see that $\alpha_r(\theta, n) \leq \alpha_\theta(\theta, n)$, we can multiply both sides of the inequality by $\frac{n+1}{\sqrt{\theta}}$ to get:

$$\frac{1}{1+\sqrt{\theta}} \leq \sqrt{\frac{(2n+1)(n+1)}{2(1+2n+\theta)}}$$

It is easy to confirm that the right hand side of the above expression is increasing in n for $n \geq 1$, and that the inequality is valid at $n = 1$.

However, although we have just confirmed that $\alpha_r(\theta, n) \leq \alpha_\theta(\theta, n)$, there are three possibilities for the ordering of $\frac{\theta}{1+\theta}$ with respect to $\alpha_r(\theta, n)$ and $\alpha_\theta(\theta, n)$: *i)* $\alpha_r(\theta, n) \leq \alpha_\theta(\theta, n) < \frac{\theta}{1+\theta}$, *ii)* $\alpha_r(\theta, n) \leq \frac{\theta}{1+\theta} < \alpha_\theta(\theta, n)$ and *iii)* $\frac{\theta}{1+\theta} \leq \alpha_r(\theta, n) < \alpha_\theta(\theta, n)$. We consider each of these possibilities in evaluating the optimal strategies for the manufacturer in period 1.

Case i) $\alpha_r(\theta,n) \leq \alpha_\theta(\theta,n) < \frac{\theta}{1+\theta}$

Because $\alpha_r(\theta,n) < \frac{\theta}{1+\theta}$, the threshold $\alpha_r(\theta,n)$ plays no role in the comparison of profits under $Q_1 = \theta$ and $Q_1 = 1$. As discussed above, for any $\alpha \leq \frac{\theta}{1+\theta}$, the manufacturer will set $Q_1^* = \theta$. From Lemma 4.2.5, we know that for $\alpha \leq \alpha_\theta(\theta,n)$, she will also set $Q_2^* = \theta$, while for $\alpha \in \left(\alpha_\theta(\theta,n), \frac{\theta}{1+\theta}\right)$ she will set $Q_2^* = 1$.

For $\alpha \geq \frac{\theta}{1+\theta}$ we can evaluate the additional profit, denoted $\Delta_1^C(\alpha,n)$, that the manufacturer would earn from increasing her output from $Q_1 = \theta$ to $Q_1 = 1$ by subtracting the the lower branch of (B.11) from the lower branch of (B.10) . We have

$$\Delta_1^C(\alpha,n) = \frac{\left(\alpha^2 - \theta + (1-\alpha)^2\theta^2\right)}{2}$$

$$-\frac{(1-\theta)\rho\left(1+3\theta+2n(1+\theta)\right)\left(\theta-\alpha(1+2n+\theta)\right)^2}{8(n+1)^2(1+2n+\theta)^2}$$

Observe that $\Delta_1^C(\alpha,n)$ is quadratic in α. Evaluating $\Delta_1^C(\alpha,n)$ at $\alpha = 0$ and $\alpha = 1$, it is easy to confirm that $\Delta_1^C(0,n) \leq 0$ while $\Delta_1^C(1,n) \geq 0$. It follows that there exists a threshold $\overline{\alpha}^C \in [0,1]$ such that $\Delta_1^C\left(\overline{\alpha}^C\right) = 0$. Evaluating $\Delta_1^C(\alpha,n)$ at $\alpha = \frac{\theta}{1+\theta}$, we can confirm that $\Delta_1^C\left(\frac{\theta}{1+\theta}, n\right) \leq 0$, which implies that $\overline{\alpha}^C \geq \frac{\theta}{1+\theta}$ and that $Q_1^* = 1$ for $\alpha \geq \overline{\alpha}^C$.

Case ii) $\alpha_r(\theta,n) \leq \frac{\theta}{1+\theta} < \alpha_\theta(\theta,n)$

As in case i) , the threshold $\alpha_r(\theta, n)$ plays no role in the comparison of profits under $Q_1 = \theta$ and $Q_1 = 1$. As described above, for $\alpha \leq \frac{\theta}{1+\theta}$, the manufacturer sets $Q_1 = \theta$, and from Lemma 4.2.5, it follows that $Q_2 = \theta$. For $\alpha \in \left(\frac{\theta}{1+\theta}, \alpha_\theta(\theta, n)\right)$, we can evaluate the additional profit, denoted $\Delta_2^C(\alpha, n)$, that the manufacturer would earn from increasing her output from $Q_1 = \theta$ to $Q_1 = 1$ by subtracting the upper branch of (B.11) from the lower branch of (B.10) We have:

$$
\begin{aligned}
\Delta_2^C(\alpha, n) \;\; = \;\; & \frac{1}{8(n+1)^2}\left(4(n+1)^2\left(\alpha^2 - \theta + (1-\alpha)^2\theta^2\right)\right) \\
& + \frac{\rho}{8(n+1)^2}\left(\left(3 + 6n + 4n^2\right)\alpha^2\right) \\
& - \frac{\rho}{8(n+1)^2}\left(\theta\left(3 + 6n + 4n^2 - 2\alpha(1-\alpha)\right) + (3+2n)(1-\alpha)^2\theta^2\right)
\end{aligned}
$$

By differentiating twice with respect to α, It can be confirmed that $\Delta_2^C(\alpha, n)$ convex in α, and it is easy to confirm that $\Delta_2^C(0, n) \leq 0$ while $\Delta_2^C(1, n) \geq 0$. By substituting $\alpha = \alpha_\theta(\theta, 0)$ into the above, it can readily be confirmed that $\Delta_2^C(\alpha_\theta(\theta, 0), n) \leq 0$. Because $\alpha_\theta(\theta, n)$ is decreasing in n it follows that $\Delta_2^C(\alpha_\theta(\theta, n), n) \leq 0$, and that $Q_1 = \theta$ dominates $Q_1 = 1$ for $\alpha \in \left(\frac{\theta}{1+\theta}, \alpha_\theta(\theta, n)\right)$. From Lemma 4.2.5, it follows that $Q_2 = \theta$.

For $\alpha \geq \alpha_\theta(\theta, n)$, we again compare profits in the lower branch of (B.10) to the profits in the lower branch of (B.11) as we did for case i). Recall that $\Delta_1^C(0, n) \leq 0$, $\Delta_1^C(1, n) \geq 0$ and that there is a unique value of $\alpha \in (0, 1)$ for which $\Delta_1^C(\alpha, n) = 0$. Recall also that $\alpha_\theta(\theta, n)$ is decreasing in n.

Thus, if $\Delta_1^C(\alpha, n) \leq 0$ when evaluated at $\alpha = \alpha_\theta(\theta, 0)$, then this will imply that $\Delta_1^C(\alpha, n) \leq 0$ when evaluated at $\alpha = \alpha_\theta(\theta, n)$ for any $n \geq 0$. From substituting into the lower branch of (B.10) and the lower branch of (B.11) we have:

$$\begin{aligned}
\Delta_1^C\left(\alpha_\theta(\theta,0),n\right) &= \frac{\theta}{2}\left(\theta\left(1-\sqrt{\frac{\theta}{2(1+\theta)}}\right)^2 - \frac{1+2\theta}{2(1+\theta)}\right) \\
&+ \frac{(\theta-1)\left(1+3\theta+2n(1+\theta)\right)\rho\left(\theta-\sqrt{\frac{\theta}{2(1+\theta)}}(1+2n+\theta)\right)^2}{8(1+n)^2(1+2n+\theta)^2} \\
&\leq 0
\end{aligned}$$

To see that the inequality is valid, it is obvious that the final term is negative. In the first term, we can use the fact that

$$\left(1-\frac{\sqrt{\theta}}{1+\sqrt{\theta}}\right)^2 \leq \left(1-\sqrt{\frac{\theta}{2(1+\theta)}}\right)^2$$

to obtain:

$$\theta\left(1-\sqrt{\frac{\theta}{2(1+\theta)}}\right)^2-\frac{1+2\theta}{2(1+\theta)} \;\leq\; \theta\left(1-\frac{\sqrt{\theta}}{1+\sqrt{\theta}}\right)^2-\frac{1+2\theta}{2(1+\theta)}$$

$$=\theta\left(\frac{1}{1+\sqrt{\theta}}\right)^2-\frac{1+2\theta}{2(1+\theta)}$$

$$= \frac{2\theta(1+\theta)-(1+2\theta)\left(1+\theta+2\sqrt{\theta}\right)}{2(1+\theta)\left(1+\sqrt{\theta}\right)^2}$$

$$=\frac{-\left(1+\sqrt{\theta}\right)^2-4\theta\sqrt{\theta}}{2(1+\theta)\left(1+\sqrt{\theta}\right)^2}\leq 0$$

Hence $\Delta_1^C\left(\alpha_\theta(\theta,n),n\right) \leq 0$, and it follows that $\overline{\alpha}^C \geq \alpha_\theta(\theta,n)$ such that, for $\alpha \in \left[\alpha_\theta(\theta,n),\overline{\alpha}^C\right]$, we will have, $Q_1=\theta$ while for $\alpha \geq \overline{\alpha}^C$, we will have $Q_1=1$. In either case, we know from Lemma 4.2.5, that $Q_2=1$.

Case iii) $\frac{\theta}{1+\theta} \leq \alpha_r(\theta,n) < \alpha_\theta(\theta,n)$

For $\alpha \leq \frac{\theta}{1+\theta}$, we have previously determined that $Q_1=\theta$ and since $\frac{\theta}{1+\theta} \leq \alpha_\theta(\theta,n)$, Lemma 4.2.5 implies that $Q_2=\theta$. For For $\alpha \in \left[\frac{\theta}{1+\theta},\alpha_r(\theta,n)\right]$ we can evaluate the additional profit, denoted $\Delta_3^C(\alpha,n)$, that the manufacturer would earn from increasing her output from $Q_1=\theta$ to $Q_1=1$ by subtracting the upper branch of (B.11)from the upper branch of (B.10). to the upper branch of (B.11) . We have:

$$\Delta_3^C(\alpha,n) \;=\; \frac{1}{2}\left(-\theta(1-\theta)-\alpha\left((2-\alpha)\theta^2-\alpha\right)-\theta\rho\left(\frac{2n+1}{2n+2}\right)^2\right)$$

By differentiating twice with respect to α, it is easy to confirm that $\Delta_3^C(\alpha)$ is quadratic and convex in α. There are at most two roots of the equation $\Delta_3^C(\alpha) = 0$. Further, note that $\Delta_3^C(0) \leq 0$ and that $\Delta_3^C(\alpha) \leq 0$ when evaluated at $\alpha = \alpha_r(\theta, n)$. It follows that $\Delta_3^C(\alpha) \leq 0$ for all $\alpha \in \left[\frac{\theta}{1+\theta}, \alpha_r(\theta, n)\right]$. Thus, in this region, $Q_1 = \theta$ and Lemma4.2.5 implies that $Q_2 = \theta$. For $\alpha \in [\alpha_r(\theta, n), \alpha_\theta(\theta, n)]$, we evaluate $\Delta_2^C(\alpha, n)$. However, from our analysis above, we know that $\Delta_2^C(\alpha_\theta(\theta, n), n) \leq 0$, which implies that $Q_1 = \theta$ in this region. From Lemma4.2.5, we will also have $Q_2 = \theta$. For $\alpha \geq \alpha_\theta(\theta, n)$, we evaluate $\Delta_1^C(\alpha)$. However, from our analysis above, we know that $\Delta_1^C(\alpha_\theta(\theta, n) \leq 0$. Therefore there exists a threshold $\overline{\alpha}^C \in [\alpha_\theta(\theta, n), 1]$ such that for $\alpha \in \left[\alpha_\theta(\theta, n), \overline{\alpha}^C(n, \theta)\right]$ we will have $Q_1 = \theta$ while for $\alpha \geq \overline{\alpha}^C$, we will have $Q_1 = 1$. In both cases, we will have $Q_2 = 1$.

In each of the three mutually exclusive and collectively exhaustive cases above, $\alpha \leq \alpha_\theta(\theta, n)$ is both a necessary and a sufficient condition for $Q_1 = Q_2 = \theta$ in equilibrium. It follows that $\underline{\alpha}^C(n, \theta) = \alpha_\theta(\theta, n)$. Similarly, in each of the three cases, there exists a threshold $\overline{\alpha}^C(n, \theta) \geq \underline{\alpha}^C(n, \theta)$ such that for $\alpha \in \left(\underline{\alpha}^C(n, \theta), \overline{\alpha}^C(n, \theta)\right)$ we will have $Q_1 = \theta$and $Q_2 = 1$ while for $\alpha \geq \underline{\alpha}^C(n, \theta)$, we will have $Q_1 = Q_2 = 1$. ■

<u>Proof of Proposition 4.2.7</u> Let us define $\Delta_{diff}(\alpha) = \pi_1^n - \pi_1^{LI}$, that is, $\Delta_{diff}(\alpha)$ is the difference in profits of the manufacturer under competition and lock-in. It can be confirmed that $\Delta_{diff}(\alpha)$ is a constant for $\alpha \in \left[0, \underline{\alpha}^C(n, \theta)\right)$ and continuous for $\alpha \in \left(\underline{\alpha}^{LI}(\theta), 1\right]$. Evaluating $\Delta_{diff}(\alpha)$ at $\alpha = 0$ and $\alpha = 1$, it is easy to confirm that $\Delta_{diff}(\alpha) > 0$ at both of these values of α for any

$\theta \in (0,1)$ and any $n \geq 1$. The result follows from the fact that for every $\theta \in (0,1)$, we have $\underline{\alpha}^C(n,\theta) > 0$ and $\underline{\alpha}^{LI}(\theta) < 1$.■

<u>Proof of Proposition 4.2.8</u> Recall that $Q_2^U = 1$ for all α and θ . We know from Propositions 4.2.3 and 4.2.6, that the threshold α above which the manufacturer sells to both groups under lock-in as well as competition ($\underline{\alpha}$) is less than 1 . Therefore $Q_2^U \geq Q_2^{LI}$ and $Q_2^U \geq Q_2^C$. Further, as we noted earlier, $\underline{\alpha}^C(n,\theta) = \sqrt{\frac{\theta(2n+1)}{2(n+1)(1+2n+\theta)}}$ is decreasing in n. That is $\underline{\alpha}^C(n,\theta) \leq \underline{\alpha}^C(0,\theta) = \underline{\alpha}^{LI}(\theta)$. Therefore $Q_2^{LI} \leq Q_2^C$. ■

Appendix C

Durable Products and Contingent Consumables: The effect of Consumable Stock-piling on the Interaction between Time Inconsistency and Lock-in

<u>Proof of Lemma 5.2.1</u>

a)When $Q_1 = 1$, we must have $Q_2 = 1$ also. Thus, (5.7) reduces to maximizing the following second period profit function with respect to the quantity of consumables:

$$\pi_2^S(1,1,y_2,x_1) = \begin{cases} y_2\left(1 - \frac{y_2+x_1}{\theta}\right) & \text{if } y_2 + x_1 \leq (1-\alpha)\theta \\ y_2\left(\theta + \alpha(1-\theta) - y_2 - x_1\right) & \text{if } y_2 + x_1 \geq (1-\alpha)\theta \end{cases}$$

where we have simply substituted (5.9) and (5.5) into (5.7). The above expression is continuous and piece-wise concave, and the first-order conditions for the upper and lower branches are satisfied at $y_2^{s1} = \frac{\theta - x_1}{2}$ and $y_2^{s2} = \frac{\theta+\alpha(1-\theta)-x_1}{2}$ respectively. It is easy to show that $y_2^{s2} < (1-\alpha)\theta - x_1$ if and only if $\alpha < \frac{\theta - x_1}{1+\theta}$, while $y_2^{s1} > (1-\alpha)\theta$ if and only if $\alpha > \frac{1-x_1}{2}$. It follows from this and the continuity of $\pi_2^S(1,1,y_2,x_1)$ that $\alpha < \frac{\theta-x_1}{1+\theta}$ implies that the optimal quantity of consumables is $y_2^* = y_2^{s1}$, while $\alpha > \frac{\theta-x_1}{2\theta}$ implies that $y_2^* = y_2^{s2}$.

When $\alpha \in \left[\frac{\theta - x_1}{1+\theta}, \frac{\theta - x_1}{2\theta}\right]$, then both $y_2^{s1} \leq (1-\alpha)\theta - x_1$ and $y_2^{s2} \geq (1-\alpha)\theta - x_1$, so both of them are candidates to be the optimal quantity. By substituting into $\pi_2(1,1,y_2,x_1)$, it is easy to show that $\pi_2^S(1,1,y_2^{s1}) = \frac{(\theta-x)^2}{4\theta} \geq \pi_2^S(1,1,y_2^{s2}) = \left(\frac{\theta+\alpha(1-\theta)-x}{2}\right)^2$ if and only if $x \leq -\alpha\sqrt{\theta} + \theta(1-\alpha)$.

b) When $Q_1 = \theta$ then the monopolist has two options with respect to the sales of the durable in period 2: She can either set $Q_2 = 1$ and sell additional durables to the low-type consumers or she can set $Q_2 = \theta$, not selling any more durables. If she sets $Q_2 = \theta$, then she produces the quantity of consumables that maximizes her revenues from selling consumables to high-type consumers only, i.e. $\pi_2^S(\theta,\theta,y_2,x_1) = y_2 p_c(\theta,y_2,x_1) = y_2\left(1 - \frac{y_2+x_2}{\theta}\right)$. Alternatively, if she sets $Q_2 = 1$ then the market clearing price from the additional durable sales will be positive if and only if $y_2 + x_2 \geq (1-\alpha)\theta$. Otherwise, the price of consumables would be prohibitively high, i.e. $p_c(1,y_2,x_1) > \alpha$, and the low-type consumers would obtain no benefit from owning the durable. Given the quantity of consumables that she makes available, the market clearing price for the durable will be equal to the total utility that low-type consumers derive from ownership, i.e. $V_l(p_c(1,y_2+x_1))$. Thus, we can write:

$$\pi_2^S(\theta,1,y_2,x_1) = y_2(\theta + \alpha(1-\theta) - y_2 - x_1) + (1-\theta)V_l(p_c(1,y_2+x_1))$$

The first order condition for $\pi_2^S(\theta,\theta,y_2,x_1)$ is $y_2^{s3} = \frac{\theta - x_1}{2}$, and the first-order-condition for $\pi_2^S(\theta,1,y_2,x_1)$ is $y_2^{s4} = \frac{\alpha+(1-\alpha)\theta^2-\theta x}{(1+\theta)}$. The result follows from the fact that:

$$\pi_2\left(\theta, 1, y_2^{s4}, x_1\right) =\geq \pi_2\left(\theta, \theta, y_2^{s3}, x_1\right)$$

if and only if $x_1 \geq \theta - \alpha\sqrt{2\theta(1+\theta)}$.

c) $\overline{x}(1) - \overline{x}(\theta) = -\alpha\sqrt{\theta} + \theta(1-\alpha) - (\theta - \alpha\sqrt{2\theta(1+\theta)}) = \alpha\sqrt{\theta}(\sqrt{2(1+\theta} - (1+\sqrt{\theta}))$. It is easy to see that $\overline{x}(1) - \overline{x}(\theta) \geq 0$ only if $1 + \theta - 2\sqrt{\theta} \geq 0$, which is always true .■

<u>Proof of Lemma 5.2.2</u>

The quantity of consumables that are stock-piled in period 1 depends on the amount of durables sold in period 1, Q_1, the output of consumables y_1 and consumers anticipation of the manufacturer's response in period 2. The amount consumables stock-piled in period 1 is the solution of the following equation:

$$p_{c1}(Q_1, y_1, x_1) = p_{c2}(Q_2^*(Q_1, y_1, x_1), y_2^*(Q_1, x_1), x_1)$$

When $Q_1 = \theta$ and consumers anticipate that $Q_2 = 1$, substituting the results obtained in Table 5.2, 5.4 and the inverse demand function obtained from 5.3 into the above equation, we obtain the following equation:

$$1 - \frac{y_1 - x_1}{\theta} = \frac{\theta - x_1}{1+\theta}$$

Solving the above equation for x_1 gives us $x_1 = \frac{1}{1+2\theta}(y_1(1+\theta)-\theta)$. Note that from Lemma 5.2.1 $x_1 = \frac{1}{1+2\theta}(y_1(1+\theta)-\theta) \geq \overline{x}(\theta) = \theta - \alpha\sqrt{2\theta(1+\theta)}$ so that the stock-piled consumables are in a large enough quantity so as to induce sales to the low end consumers in period 2, which results in the condition $y_1 \geq \alpha(1+2\theta)\sqrt{\frac{2\theta}{1+\theta}} - 2\theta$.

When $Q_1 = \theta$ and consumers anticipate that $Q_2 = \theta$, substituting the results obtained in Table 5.2, 5.4 and the inverse demand function obtained from 5.3 into 5.8, we obtain the following equation:

$$1 - \frac{y_1 - x_1}{\theta} = \frac{\theta - x_1}{2\theta}$$

Solving the above equation for x_1 gives us $x_1 = \frac{1}{3}(2y_1 - \theta)$.

When $Q_1 = 1$, note that if the manufacturer serves both types in period 1 then by the equality of price of consumables in both periods, the manufacturer will also serve both types in period 2. Substituting the results obtained in Table 5.1, 5.4 and the inverse demand function obtained from 5.3 into 5.8, we obtain the following equation:

$$\alpha + \theta - \alpha\theta - (y_1 - x_1) = \frac{\alpha + \theta - \alpha\theta - x_1}{2}$$

Solving the above equation for x_1 gives us $x_1 = \frac{1}{3}(2y_1 - \theta - \alpha(1-\theta))$. Note that from Lemma 5.2.1 $x_1 = \frac{1}{3}(2y_1 - \theta - \alpha(1-\theta)) \geq \overline{x}(1) = -\alpha\sqrt{\theta} + \theta(1-\alpha)$ so that the stock-piled consumables are high enough to induce a price of consumables $p_c < \alpha$, which results in the condition $y_1 \geq \frac{1}{2}\left(\alpha\left(1 - 3\sqrt{\theta}\right) + 4\theta(1-\alpha)\right)$. ■

Proof of Proposition 5.2.3

Let us first consider the manufacturer's problem for $Q_1 = \theta$. The profits of the manufacturer as a function of $y_1$conditional on $Q_1 = \theta$ are given as follows:

$$\pi_1(\theta, y_1) = \begin{cases} \frac{(5\theta - y_1)(\theta + y_1)}{9\theta} \\ \quad \text{if } y_1 \leq \alpha(1+2\theta)\sqrt{\frac{2\theta}{1+\theta}} - 2\theta \\ \frac{\theta - y_1^2(1+\theta) + 2y_1\theta(1+\alpha+2\alpha\theta) + (1+2\theta)(\alpha^2(1+2\theta)+4(1-\alpha)\theta^2)}{2(1+2\theta)^2} \\ \quad \text{otherwise} \end{cases} \tag{C.1}$$

where the upper branch represents the case in which, given that $Q_1 = \theta$, the manufacturer's optimal second period response does not include selling durables to the low-type consumers. The lower branch represents the case in which it does. The slope of the profits represented by the upper branch at $y_1 = \alpha(1+2\theta)\sqrt{\frac{2\theta}{1+\theta}} - 2\theta$ is $\frac{2\sqrt{2}\alpha(1+2\theta)}{9\sqrt{\theta(1+\theta)}} > 0$. In order to sell durables to high type consumers only the manufacturer will produce no more than $y_1 = \alpha(1+2\theta)\sqrt{\frac{2\theta}{1+\theta}} - 2\theta$. The FOC w.r.t y_1 for the lower branch yields $y_1^{s1*} = \frac{\theta(1+\alpha(1+2\theta))}{1+\theta}$. $y_1^{s1*} > \alpha(1+2\theta)\sqrt{\frac{2\theta}{1+\theta}} - 2\theta$ only if $\alpha > \frac{\theta}{\theta+\sqrt{2\theta(1+\theta)}}$. We can now summarize the conditionally optimal profits of the manufacturer when $Q_1 = \theta$ is given below:

$$\pi_1^*(\theta) = \begin{cases} \theta - \frac{2\alpha^2(1+2\theta)^2}{9(1+\theta)} & \text{if } \alpha \leq \underline{\alpha}^S(\theta) = \dfrac{\theta}{\theta + \sqrt{2\theta(1+\theta)}} \\ \frac{\alpha^2(1+\theta)+\theta+(2-\alpha(2-\alpha))\theta^2}{2(1+\theta)} & \text{otherwise} \end{cases} \tag{C.2}$$

Now let us consider the manufacturer's problem for $Q_1 = 1$. The profits of the manufacturer as a function of $y_1$conditional on $Q_1 = \theta$ are given as follows:

We can now we can substitute $y_1 = y_1^{c*}(\theta) = \theta$ and (5.13) into (4.8) and from Lemma 4.2.2 we have the following optimal profits, conditional on $Q_1 = 1$:

$$\pi_1(\theta, y_1) \;=\; \frac{(-y_1^2 + 4y_1\alpha + 5\alpha^2 - 2(y_1 - 2\alpha)(1-\alpha)\theta + 8(1-\alpha)^2\theta^2)}{9}$$
$$\text{if } y_1 \geq \frac{1}{2}\left(\alpha\left(1 - 3\sqrt{\theta}\right) + 4\theta(1-\alpha)\right)$$

The FOC w.r.t. y_1 for the above function gives us the optimal quantity $y_1^{s2*} = 2\alpha - \theta + \alpha\theta$. $y_1^{s2*} > \frac{1}{2}\left(\alpha\left(1 - 3\sqrt{\theta}\right) + 4\theta(1-\alpha)\right)$ only if $\alpha > \frac{2\theta}{1+2\theta+\sqrt{\theta}}$. The conditionally optimal profits of the manufacturer when $Q_1 = 1$ is summarized below:

$$\pi_1^*(1) = \begin{cases} \frac{\alpha}{4}(\alpha(3 - 2\sqrt{\theta}) + (4 - 5\alpha)\theta - 4(1-\alpha)\theta^{\frac{3}{2}}) & \text{if } \alpha \leq \hat{\alpha_S} = \dfrac{2\theta}{1 + 2\theta + \sqrt{\theta}} \\ \alpha^2 + (1-\alpha)^2\theta^2 & \text{otherwise} \end{cases} \tag{C.3}$$

There are two possibilities for the ordering of the thresholds $\underline{\alpha}^S(\theta) = \frac{\theta}{\theta + \sqrt{2\theta(1+\theta)}}$ and $\hat{\alpha_S} = \frac{2\theta}{1+2\theta+\sqrt{\theta}}$. Consider the first possibility $\underline{\alpha}^S(\theta) < \hat{\alpha_S}$. Let us define $\Delta_1^S(\alpha)$ as the difference in profits between the upper branch of $\pi_1^*(\theta)$ and the upper branch of $\pi_1^*(1)$. It is easy to see that $\Delta_1^S(\alpha)$ is quadratic in α . Further $\Delta_1^S(\alpha = 0) > 0$ and

$$\Delta_1^S(\alpha=1)=\frac{1}{36}(-27-\frac{8}{1+\theta}+18\sqrt{\theta}+13\theta)<0$$

Therefore, there exists $\alpha \in (0,1)$ such that $\Delta_1^S(\alpha)=0$. It can be easily verified that $\Delta_1^S(\alpha=\frac{2\theta}{1+2\theta+\sqrt{\theta}})>0$ and $\Delta_1^S(\alpha=\frac{\theta}{\theta+\sqrt{2\theta(1+\theta)}})>0$.

Now let us define $\Delta_2^S(\alpha)$ as the difference in profits between the lower branch of $\pi_1^*(\theta)$ and the upper branch of $\pi_1^*(1)$. It is easy to see that $\Delta_2^S(\alpha)$ is convex and decreasing in α . Further $\Delta_2^S(\alpha=0)>0$. It can be easily verified that $\Delta_2^S(\alpha=\frac{2\theta}{1+2\theta+\sqrt{\theta}})>0$ and $\Delta_2^S(\alpha=\frac{\theta}{\theta+\sqrt{2\theta(1+\theta)}})>0$. Therefore the upper branch of $\pi_1^*(1)$ is not an equilibrium outcome. Clearly then, for $\alpha \leq \underline{\alpha}^S(\theta)$, the manufacturer sells durables to high type consumers in period 1 and does not sell additional durables in period 2.

Now let us define $\Delta_3^S(\alpha)$ as the difference in profits between the upper branch of $\pi_1^*(\theta)$ and the lower branch of $\pi_1^*(1)$. It is easy to see that $\Delta_3^S(\alpha)$ is quadratic in α . Further $\Delta_3^S(\alpha=0)>0$ and

$$\Delta_3^S(\alpha=1)=\frac{1}{9}(-9-\frac{2}{1+\theta}+\theta)<0$$

Therefore, there exists $\alpha \in (0,1)$ such that $\Delta_3^S(\alpha)=0$. It can be easily verified that $\Delta_3^S(\alpha=\frac{2\theta}{1+2\theta+\sqrt{\theta}})>0$ and $\Delta_3^S(\alpha=\frac{\theta}{\theta+\sqrt{2\theta(1+\theta)}})>0$ .Now let us define $\Delta_4^S(\alpha)$ as the difference in profits between the lower branch of $\pi_1^*(\theta)$ and the lower branch of $\pi_1^*(1)$. It is easy to see that $\Delta_4^S(\alpha)$ is quadratic in α . Further $\Delta_4^S(\alpha=1)=-\frac{1}{4}(1-\theta)<0$ and

$$\Delta_4^S(\alpha=0)=\theta(\frac{2(1-\theta^2)}{1+\theta})>0$$

Therefore, $\overline{\alpha}^s(\theta)$ solves $\Delta_4^S(\alpha) = 0$. Solving for this equation we obtain $\overline{\alpha}^s(\theta) = \frac{\theta^2+2\theta^3+\sqrt{\theta(1+\theta)(1+\theta^2(1+2\theta))}}{1+\theta+\theta^2+2\theta^3}$ ■

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Vita

Sreelata(Sree) Jonnalagedda was born in Hyderabad in 1978, the daughter of Kameswara Rao and Lakshmi Jonnalagedda. She graduated from St. Ann's High School, and St. Francis Junior College, Secunderabad in 1993 and 1995 respectively. She received a Bachelor of Technology degree in Civil engineering from the Indian Institute of Technology, Madras in July 1999, and a Master's degree in Transportation from New Jersey Institute of Technology, Newark, New Jersey in August 2000. She worked as a consultant at Supply Chain Consultants Inc. in Wilmington, Delaware and Antwerpen, Belgium from September 2000 to July 2004. Sree entered the doctoral program at the McCombs School of Business at the University of Texas, Austin in August 2004.

Permanent address: Plot No: 132, Dandamudi Enclave
Hyderabad 500065, India

This dissertation was typeset with LaTeX† by the author.

†LaTeX is a document preparation system developed by Leslie Lamport as a special version of Donald Knuth's TeX Program.