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The Arithmetic and Geometry of Two-Generator Kleinian Groups

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The Arithmetic and Geometry of Two-Generator Kleinian Groups

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The Arithmetic and Geometry of Two-Generator Kleinian Groups

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This thesis investigates the structure and properties of hyperbolic 3-manifold groups (particularly knot and link groups) and arithmetic Kleinian groups. In Chapter 2, we establish a stronger version of a conjecture of A. Reid and others in the arithmetic case: if two elements of equal trace (e.g., conjugate elements) generate an arithmetic two-bridge knot or link group, then the elements are parabolic (and hence peripheral). In Chapter 3, we identify all Kleinian groups that can be generated by two elements for which equality holds in Jørgensen's Inequality in two cases: torsion-free Kleinian groups and non-cocompact arithmetic Kleinian groups.

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Chapter 1

Introduction and Preliminaries

This thesis investigates the structure and properties of hyperbolic 3-manifold groups (particularly knot and link groups) and arithmetic Kleinian groups with the broader goal of better understanding the geometry and topology of 3-manifolds. The Geometrization Conjecture of W. Thurston has guided most progress in this field for nearly thirty years, and the 2003 announcement of its proof by G. Perelman reaffirmed hyperbolic 3-manifolds as the most prevalent, yet least understood, class of 3-manifolds, leaving many important directions and problems for continued research. Accordingly, a variety of techniques from diverse areas of mathematics (including topology, geometry, analysis, group theory, algebra, and number theory) have been developed to study hyperbolic 3-manifolds and Kleinian groups.

A *Kleinian group* is a discrete subgroup of $\mathrm{PSL}_2\mathbb{C}$, which is the quotient of the special linear group $\mathrm{SL}_2\mathbb{C}$ by its center $\{\pm I\}$. Identifying $\mathrm{PSL}_2\mathbb{C}$ with the orientation-preserving isometries of hyperbolic 3-space $\mathbb{H}^3$, every orientable hyperbolic 3-manifold is isometric to the quotient $\mathbb{H}^3/\Gamma$ for some torsion-free Kleinian group Γ, and, conversely, every torsion-free Kleinian group Γ yields an orientable hyperbolic 3-manifold $\mathbb{H}^3/\Gamma$. More generally, a Kleinian group Γ (with or without torsion) corresponds to a hyperbolic 3-orbifold $\mathbb{H}^3/\Gamma$. A Kleinian group Γ has *finite covolume* if $\mathbb{H}^3/\Gamma$ has finite volume with respect to the hyperbolic volume element induced from the hyperbolic metric. A finite-covolume Kleinian group is *non-cocompact* (i.e., $\mathbb{H}^3/\Gamma$ is not compact) if and only if Γ contains parabolic elements (i.e., non-trivial elements whose trace is ± 2).

Blurring the distinction between elements of $\mathrm{PSL}_2\mathbb{C}$ and their lifts to $\mathrm{SL}_2\mathbb{C}$ (a slight but common abuse of notation that will be continued tacitly henceforth), the *trace field* of a Kleinian group Γ is $\mathbb{Q}(\mathrm{tr}\Gamma) = \mathbb{Q}(\pm\mathrm{tr}\gamma : \gamma \in \Gamma)$. The following important consequence of Mostow Rigidity underlies the connection to algebraic number theory: if Γ is a finite-covolume Kleinian group, then $\mathbb{Q}(\mathrm{tr}\Gamma)$ is a finite extension of $\mathbb{Q}$ and a topological invariant of $\mathbb{H}^3/\Gamma$. This number-theoretic connection is strongest for arithmetic Kleinian groups, which we now describe.

While the trace field is a conjugacy invariant of a Kleinian group Γ, it is not, in general, an invariant of the commensurability class of Γ (two Kleinian groups Γ_1 and Γ_2 are *commensurable* if there exists $g \in \mathrm{PSL}_2\mathbb{C}$ such that $\Gamma_1 \cap g\Gamma_2 g^{-1}$ has finite index in both Γ_1 and $g\Gamma_2 g^{-1}$). The *invariant trace field* of a Kleinian group Γ is $k\Gamma = \mathbb{Q}(\mathrm{tr}\gamma^2 : \gamma \in \Gamma)$; this is an invariant of the commensurability class of Γ. For now, consider only the following characterization of non-cocompact arithmetic Kleinian groups: a finite-covolume non-cocompact Kleinian group Γ is *arithmetic* if and only if Γ is commensurable with a Bianchi group $\mathrm{PSL}_2(O_d)$, where O_d is henceforth the ring of integers in $\mathbb{Q}(\sqrt{-d})$ and $d \in \mathbb{N}$ is square-free. An invaluable reference for all of the above is [31].

The general approach of this thesis is to exploit the interplay with algebra and number theory to address conjectures and problems related to the structure and properties of hyperbolic 3-manifold groups (particularly knot and link groups) and arithmetic Kleinian groups. For instance, in Chapter 2, we establish the following stronger version of a natural conjecture posed by A. Reid and others in the arithmetic case (which includes the figure-eight knot and Whitehead link groups).

Theorem. *If two elements of equal trace (e.g., conjugate elements) generate an arithmetic two-bridge knot or link group, then the elements are parabolic (and hence peripheral).*

Maintaining this theme, in Chapter 3, we consider *Jørgensen groups*, i.e., Kleinian groups that can be generated by two elements for which equality holds in Jørgensen's Inequality and identify all such groups in the following two cases.

Theorem. *The only torsion-free Jørgensen group is the figure-eight knot group.*

Theorem. *There are exactly fourteen non-cocompact arithmetic Jørgensen groups:* $\mathrm{PGL}_2(O_1)$, $\mathrm{PGL}_2(O_3)$, $\mathrm{PSL}_2(O_1)$, $\mathrm{PSL}_2(O_2)$, $\mathrm{PSL}_2(O_3)$, $\mathrm{PSL}_2(O_7)$, $\mathrm{PSL}_2(O_{11})$, *two subgroups of index 6 and 8 respectively in* $\mathrm{PGL}_2(O_1)$*, the unique subgroup of index 10 in* $\mathrm{PSL}_2(O_3)$*, the figure-eight knot group, and three* $\mathbb{Z}_2$*-extensions of the figure-eight knot group.*

The remainder of this chapter establishes notation and recalls preliminary results for subsequent use. Following [31], [32], and [4], Section 1.1 addresses the geometry of Kleinian groups. Of utmost importance is Jørgensen's Inequality, a well known necessary condition for two elements in $\mathrm{PSL}_2\mathbb{C}$ to generate a non-elementary discrete group, and Poincaré's Polyhedron Theorem, which posits conditions sufficient for a set of elements in $\mathrm{PSL}_2\mathbb{C}$ to generate a discrete group and determines when the group has finite covolume by constructing its fundamental polyhedron. Section 1.2 treats the arithmetic of Kleinian groups, with much of the material specialized to two-generator Kleinian groups. While [31] is again the primary reference, the program to identify all two-generator arithmetic Kleinian groups initiated in [14], [29], [13], and [10] is fundamental to and provides the motivation for this thesis.

1.1 The Geometry of Kleinian Groups

An element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2\mathbb{C}$ acts on the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ via the linear fractional transformation $\gamma(z) = \dfrac{az+b}{cz+d}$. This extends to an action on the upper half 3-space $\mathbb{H}^3 = \{z + tj \mid z \in \mathbb{C},\ t > 0\}$ regarded in terms of quaternions via the Poincaré extension:

$$\gamma(z+tj) = \frac{(az+b)\overline{(cz+d)} + a\bar{c}t^2 + tj}{|cz+d|^2 + |c|^2 t^2}.$$

When equipped with the hyperbolic metric

$$ds^2 = \frac{dx^2 + dy^2 + dt^2}{t^2}$$

induced from the line element ds, $\mathbb{H}^3 = \{(x, y, t) \in \mathbb{R}^3 \mid t > 0\}$ becomes a model of hyperbolic 3-space (i.e., the unique 3-dimensional connected, simply connected Riemannian manifold with constant sectional curvature -1) and will be regarded as such henceforth. Under this metric, the geodesics in $\mathbb{H}^3$ are circles and straight lines orthogonal to $\partial\mathbb{H}^3$. The metric also induces the hyperbolic volume element

$$dV = \frac{dx\ dy\ dt}{t^3}$$

and enables identification of $\mathrm{PSL}_2\mathbb{C}$ with the subgroup of orientation-preserving isometries of $\mathbb{H}^3$. Non-identity elements $\gamma \in \mathrm{PSL}_2\mathbb{C}$ are classified as

- *elliptic* if $\mathrm{tr}\gamma \in \mathbb{R}$ and $|\mathrm{tr}\gamma| < 2$.
- *parabolic* if $\mathrm{tr}\gamma = \pm 2$.
- *loxodromic* otherwise.

If γ is loxodromic and $\mathrm{tr}\gamma \in \mathbb{R}$, then γ is frequently termed *hyperbolic*. An element $\gamma \in \mathrm{PSL}_2\mathbb{C}$ is parabolic if and only if it has exactly one fixed point in its action on $\hat{\mathbb{C}}$; otherwise, it has exactly two fixed points joined by a unique geodesic in $\mathbb{H}^3$ called the *axis* of γ.

A subgroup Γ of $\mathrm{PSL}_2\mathbb{C}$ is *reducible* if all its elements share a common fixed point in their action on $\hat{\mathbb{C}}$ and *irreducible* otherwise. A subgroup Γ of $\mathrm{PSL}_2\mathbb{C}$ is *elementary* if it has a finite orbit in its action on $\mathbb{H}^3 \cup \hat{\mathbb{C}}$ and *non-elementary* otherwise. Non-elementary groups are irreducible, but the converse is not true (e.g., non-cyclic finite groups are irreducible but elementary).

We conclude this section by recalling two fundamental results that address necessary and sufficient conditions for a subgroup of $\mathrm{PSL}_2\mathbb{C}$ to be discrete. The first, established by T. Jørgensen in [23], is the following well known necessary condition for two elements of $\mathrm{PSL}_2\mathbb{C}$ to generate a non-elementary discrete group.

Jørgensen's Inequality. *If $\langle X, Y \rangle$ is a non-elementary Kleinian group, then*

$$|\mathrm{tr}^2 X - 4| + |\mathrm{tr}[X, Y] - 2| \geq 1.$$

In Chapter 3, we investigate Kleinian groups that can be generated by two elements for which equality holds in Jørgensen's Inequality.

The second is Poincaré's Polyhedron Theorem, which posits conditions sufficient for a set of elements in $\mathrm{PSL}_2\mathbb{C}$ to generate a discrete group and which we will use in Section 3.3 to construct fundamental polyhedra for certain Kleinian groups to determine whether they have finite covolume (a Kleinian group has finite covolume if and only if its fundamental polyhedron has finite hyperbolic volume). Following Chapter IV of [32], we begin by recalling some terminology.

A (convex) polyhedron P in $\mathbb{H}^3$ is a *fundamental polyhedron* for a discrete subgroup Γ of $\mathrm{PSL}_2\mathbb{C}$ if the following four conditions hold.

(i) For every non-trivial $\gamma \in \Gamma$, $\gamma(P) \cap P = \emptyset$.

(ii) For every $z \in \mathbb{H}^3$, there is a $\gamma \in \Gamma$ with $\gamma(z) \in \overline{P}$.

(iii) For every side s of P, there is a side s' and an element $\gamma_s \in \Gamma$ such that $\gamma_s(s) = s'$, $\gamma_{s'} = \gamma_s^{-1}$, and $(s')' = s$. The element γ_s is called a *side pairing transformation.*

(iv) Any compact set meets only finitely many Γ-translates of P.

Then Γ is generated by the side pairing transformations of P. If there is a side s with $s' = s$, then the condition that $\gamma_{s'} = \gamma_s^{-1}$ implies the relation $\gamma_s^2 = 1$, which is called a *reflection relation.*

Let e_1 be an edge of P and s_1 one of the two sides of P having e_1 as a boundary. Then there there is a side s_1' and a side pairing transformation γ_1 with $\gamma_1(s_1) = s_1'$. Let $e_2 = \gamma_1(e_1)$. Then the edge e_2 lies on the boundary of s_1' and another side s_2. Again, there is a side s_2' and a side pairing transformation γ_2 with $\gamma_2(s_2) = s_2'$. Continuing thusly, we generate sequences $\{e_m\}$ of edges and $\{\gamma_m\}$ of side pairing transformations. By condition (iv) above, the sequence of edges is periodic; let k be the least period. The cyclically ordered sequence of edges $\{e_1, \ldots, e_k\}$ is a *cycle* of edges, and k is its *period.* Let $\gamma_1, \ldots, \gamma_k$ be the corresponding side pairing transformations. Then $\gamma_k \cdots \gamma_1(e_1) = e_1$, and $h = \gamma_k \cdots \gamma_1$ is the *cycle transformation* at e_1. There is a positive integer t so that $h^t = 1$, which is called a *cycle relation.*

Poincaré's Polyhedron Theorem. *Let P be a polyhedron with side pairing transformations satisfying the following five conditions.*

(i) *For each side s of P, there is a side s' and an element $\gamma_s \in \Gamma$ such that $\gamma_s(s) = s'$, $\gamma_{s'} = \gamma_s^{-1}$, and $\gamma_s(P) \cap P = \emptyset$.*

(ii) *For every point $z \in P^*$, $p^{-1}(z)$ is a finite set, where P^* is the space of equivalence classes induced by the side pairing transformations so that the projection $p : \overline{P} \to P^*$ is continuous and open.*

(iii) *For each edge e, there is a positive integer t so that $h^t = 1$, where h is the cycle transformation at e.*

(iv) *If $\{e_1, \ldots, e_k\}$ is any cycle of edges of P, $\alpha(e_m)$ the angle measured from inside P at the edge e_m, and q the smallest positive integer such that $h^q = 1$, where h is the cycle transformation at e_m, then*

$$\sum_{m=1}^{k} \alpha(e_m) = 2\pi/q.$$

(v) *P^* is complete.*

Then the group Γ generated by the side pairing transformations is discrete, P is a fundamental polyhedron for Γ, and the reflection relations and cycle relations form a complete set of relations for Γ.

We also recall the notation of [26], [27], and [28], which we borrow when using Poincaré's Polyhedron Theorem in Section 3.3. Let F_X and F_X^{-1} be two sides of a polyhedron P such that F_X is mapped onto F_X^{-1} by the side pairing transformation X. Denote by $e_{(m,n),\theta}$ the nth edge of the mth cycle transformation such that the angle measured from the polyhedron P at the edge is θ. The following diagram represents the mth cycle transformation

$$e_{(m,1),\theta_1} \xrightarrow{X_1} e_{(m,2),\theta_2} \xrightarrow{X_2} \cdots \xrightarrow{X_{m-1}} e_{(m,n),\theta_n} \xrightarrow{X_n} \circlearrowleft_{\theta}^{p}$$

where the initial edge $e_{(m,1),\theta_1}$ is mapped to the second edge $e_{(m,2),\theta_2}$ by the side pairing transformation X_1, then the edge $e_{(m,2),\theta_2}$ is mapped to the edge $e_{(m,3),\theta_3}$ by the side pairing transformation X_2, and so on. The final edge $e_{(m,n),\theta_n}$ is mapped to the initial edge $e_{(m,1),\theta_1}$ by the side pairing transformation X_n and the sum of all angles at the edges in this sequence is equal to θ. The cycle transformation $X_n X_{n-1} \cdots X_1$ is either the identity transformation, in which case $p = 1$, or an elliptic transformation of order p.

1.2 The Arithmetic of Kleinian Groups

In this section, we define several algebraic invariants associated to Kleinian groups, leading to the definition of an arithmetic Kleinian group. We also recall or establish several results for use in Chapters 2 and 3. While [31] remains the primary reference, much of the material is specialized to two-generator Kleinian groups in the spirit of [14], [29], [13], and [10], such as the following useful combination of (3.25) and Lemmas 3.5.7, 3.5.8, and 8.5.2 in [31].

Lemma 1.1. *Let $\Gamma = \langle A, B\rangle$ be a non-elementary Kleinian group with $\mathrm{tr}A \neq 0$.*

- *The trace field is*
$$\mathbb{Q}(\mathrm{tr}\Gamma) = \mathbb{Q}(\mathrm{tr}A, \mathrm{tr}B, \mathrm{tr}AB).$$

- *If $\mathrm{tr}B = 0$, then the invariant trace field is*
$$k\Gamma = \mathbb{Q}(\mathrm{tr}^2 A, \mathrm{tr}[A, B]).$$

- *If $\mathrm{tr}B \neq 0$, then the invariant trace field is*
$$k\Gamma = \mathbb{Q}(\mathrm{tr}^2 A, \mathrm{tr}^2 B, \mathrm{tr}A\mathrm{tr}B\mathrm{tr}AB).$$

- *If $\mathrm{tr}A$, $\mathrm{tr}B$, and $\mathrm{tr}AB$ are algebraic integers, then $\mathrm{tr}\Gamma$ consists of algebraic integers.*

One more ingredient is required to define what it means for a Kleinian group to be arithmetic. If Γ is a finitely generated non-elementary Kleinian group, then

$$A\Gamma = \left\{ \sum a_i \gamma_i : a_i \in k\Gamma, \gamma_i \in \Gamma^{(2)} \right\}$$

is a quaternion algebra over $k\Gamma$ called the *invariant quaternion algebra* of Γ. Below is Theorem 3.6.2 of [31], which determines the invariant quaternion algebra for most Kleinian groups in terms of the Hilbert symbol.

Theorem 1.2. *If $\langle X, Y \rangle$ is an irreducible subgroup of a non-elementary group Γ in $\mathrm{PSL}_2\mathbb{C}$ such that X is not parabolic and neither X nor Y is elliptic of order two, then the invariant quaternion algebra is*

$$A\Gamma = \left(\frac{\mathrm{tr}^2 X(\mathrm{tr}^2 X - 4),\ \mathrm{tr}^2 X \mathrm{tr}^2 Y(\mathrm{tr}[X, Y] - 2)}{k\Gamma} \right).$$

We now define an *arithmetic Kleinian group* by stating Theorem 8.3.2 of [31], together with Lemma 4.1 in [14].

Theorem 1.3. *A finite-covolume Kleinian group Γ is arithmetic if and only if the following three conditions hold.*

1. *$k\Gamma$ is a number field with exactly one complex place.*

2. *$\mathrm{tr}\Gamma$ consists of algebraic integers.*

3. *$A\Gamma = \left(\frac{a,\ b}{k\Gamma} \right)$ is ramified at all real places of $k\Gamma$, i.e., $\tau(a)$ and $\tau(b)$ are negative for all real embeddings $\tau : k\Gamma \to \mathbb{R}$.*

Theorem 1.3 provides the following convenient description of non-cocompact arithmetic Kleinian groups (see Theorem 8.2.3 of [31]).

Theorem 1.4. *A finite-covolume non-cocompact Kleinian group Γ is arithmetic if and only if Γ is commensurable with a Bianchi group $\mathrm{PSL}_2(O_d)$.*

When Γ is a finite-covolume non-cocompact Kleinian group, the conditions of Theorem 1.4 are equivalent to $\mathrm{tr}\Gamma$ consisting of algebraic integers and $k\Gamma = \mathbb{Q}(\sqrt{-d})$ for some $d \in \mathbb{N}$.

We conclude this section by collecting several results that will be useful in Chapters 2 and 3. The first is Lemma 7.1, together with the comments and definitions that precede it, in [14].

Lemma 1.5. *Let Γ be a finite-covolume Kleinian group whose traces lie in R, the ring of integers in $\mathbb{Q}(\mathrm{tr}\Gamma)$. If $\langle X, Y\rangle$ is a non-elementary subgroup of Γ, then $\mathcal{O} = R[1, X, Y, XY]$ is an order in the quaternion algebra*

$$A\Gamma = \left\{\sum a_i\gamma_i : a_i \in \mathbb{Q}(\mathrm{tr}\Gamma), \gamma_i \in \Gamma\right\}$$

over $\mathbb{Q}(\mathrm{tr}\Gamma)$. Its discriminant $d(\mathcal{O})$ is the ideal $\langle 2 - \mathrm{tr}[X, Y]\rangle$ in R.

The second is Theorem 6.3.4 in [31].

Theorem 1.6. *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be orders in a quaternion algebra over a number field. If $\mathcal{O}_1 \subset \mathcal{O}_2$, then $d(\mathcal{O}_2)\,|\,d(\mathcal{O}_1)$, and $\mathcal{O}_1 = \mathcal{O}_2$ if and only if $d(\mathcal{O}_1) = d(\mathcal{O}_2)$.*

Our application is the following.

Corollary 1.7. *Let Γ be a finite-covolume Kleinian group whose traces lie in R, the ring of integers in $\mathbb{Q}(\mathrm{tr}\Gamma)$. If $\langle A, B\rangle = \Gamma = \langle X, Y\rangle$, then $2-\mathrm{tr}[X, Y]$ is a unit multiple of $2 - \mathrm{tr}[A, B]$ in R.*

Proof. By Lemma 1.5, $\mathcal{O}_1 = R[1, A, B, AB]$ and $\mathcal{O}_2 = R[1, X, Y, XY]$ are orders in $A\Gamma$. Furthermore, $d(\mathcal{O}_1) = \langle 2 - \mathrm{tr}[A, B]\rangle$ and $d(\mathcal{O}_2) = \langle 2 - \mathrm{tr}[X, Y]\rangle$ are ideals in R. The Cayley-Hamilton Theorem yields the identity

$$X + X^{-1} = \mathrm{tr}X \cdot 1$$

which implies $A^{-1}, B^{-1} \in \mathcal{O}_1$ and $X^{-1}, Y^{-1} \in \mathcal{O}_2$. Since $\mathcal{O}_1$ and $\mathcal{O}_2$ are ideals that are also rings with 1, we have

$$\Gamma = \langle A, B \rangle \subset \mathcal{O}_1 \text{ and } \Gamma = \langle X, Y \rangle \subset \mathcal{O}_2.$$

Clearly, $R \subset \mathcal{O}_1, \mathcal{O}_2$, so $R\Gamma \subseteq \mathcal{O}_1, \mathcal{O}_2$, and $\mathcal{O}_1, \mathcal{O}_2 \subseteq R\Gamma$ by definition. Therefore,

$$\mathcal{O}_1 = R\Gamma = \mathcal{O}_2.$$

Hence, $d(\mathcal{O}_1) = d(\mathcal{O}_2)$ by Theorem 1.6, and the result follows. ■

Chapter 2

Conjugate Generators of Knot and Link Groups

By a knot or link group, we will mean the fundamental group of the knot or link complement in S^3. It is well known that a two-bridge knot or link group is generated by two meridians of the knot or link ([9]). The converse is also known; it is proved for link complements in S^3 in [7] (Corollary 3.3), for hyperbolic 3-manifolds of finite volume in [1] (Theorem 4.3), and for the most general class of 3-manifolds in [6] (Corollary 5):

Theorem 2.1. *If M is a compact, orientable, irreducible 3-manifold with incompressible boundary and $\pi_1 M$ is generated by two peripheral elements, then M is homeomorphic to the exterior of a two-bridge knot or link in S^3.*

Arising from work on Simon's Conjecture (see Section 2.4 for statement and Problem 1.12 of [25]), A. Reid and others proposed the following conjecture, which for convenience we will call:

Reid's Conjecture. *Let K be a knot for which $\pi_1(S^3 \smallsetminus K)$ is generated by two conjugate elements. Then the elements are peripheral (and hence the knot is two-bridge by above).*

A knot in S^3 is a hyperbolic, satellite, or torus knot (Corollary 2.5 of [43]). By Proposition 17 of [16], the (p, q)-torus knot group can be generated by two conjugate elements only when $p = 2$, i.e., when the torus knot is two-bridge. In Section 2.3, we establish Reid's Conjecture for the $(2, 3)$-torus knot group (i.e., the trefoil knot group), and we prove in Section 2.4 that Reid's Conjecture implies Simon's Conjecture for two-bridge knots. We also note that Simon's Conjecture for two-bridge knots has recently been proved in [5].

When a knot or link complement in S^3 is hyperbolic, peripheral elements such as meridians map to parabolic elements under the discrete faithful representation of the knot or link group into $\mathrm{PSL}_2\mathbb{C}$. Conjugate elements have equal trace, and we prove in Section 2.1 a stronger version of Reid's Conjecture for the figure-eight knot (whose complement in S^3 is well known to be hyperbolic; see Theorem 2.4 below):

Theorem 2.2. *If two elements of equal trace generate the figure-eight knot group, then the elements are parabolic.*

As we will see in Section 2.1, however, the figure-eight knot group can be generated by three conjugate loxodromic elements, so this result is, in some sense, sharp.

The proof of Theorem 2.2 relies heavily on the arithmeticity of the figure-eight knot complement in S^3. Since the figure-eight knot is the only knot with arithmetic hyperbolic complement in S^3 ([33]), extending our result to all hyperbolic knot groups would require new techniques. By Section 5 of [13], however, there are exactly four arithmetic Kleinian groups generated by two parabolic elements; each is the fundamental group of a hyperbolic two-bridge knot or link complement in S^3: the figure-eight knot, the Whitehead link, and the links 6^2_2 and 6^2_3. In Section 2.2, we again exploit arithmeticity to extend our result to these:

Theorem 2.3. *If two elements of equal trace generate an arithmetic two-bridge knot or link group, then the elements are parabolic.*

2.1 The Figure-Eight Knot

The universal cover of an orientable hyperbolic 3-manifold M is isometric to $\mathbb{H}^3$, and $\pi_1 M$ can be identified with a torsion-free Kleinian group Γ such that M is isometric to $\mathbb{H}^3/\Gamma$. We will regard this identification as the *discrete faithful representation* of $\pi_1 M$ into $\mathrm{PSL}_2\mathbb{C}$. The following is due to R. Riley ([36]).

Theorem 2.4. *The discrete faithful representation of the figure-eight knot group*

$$\pi_1(S^3 \smallsetminus K) = \langle a, b \mid a^{-1}bab^{-1}a = ba^{-1}bab^{-1} \rangle$$

into $\mathrm{PSL}_2\mathbb{C}$ *is generated by*

$$a \mapsto A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } b \mapsto B = \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix}$$

where $\omega = e^{\pi i/3}$. *Thus,* $S^3 \smallsetminus K$ *is isometric to* $\mathbb{H}^3/\Gamma$ *where* $\Gamma = \langle A, B \rangle$.

Then $\Gamma = \langle A, B \rangle$ is a finite-covolume Kleinian subgroup of $\mathrm{PSL}_2(\mathbb{Z}[\omega])$, so $\mathrm{tr}\Gamma \subset \mathbb{Z}[\omega]$, the ring of integers in the trace field $\mathbb{Q}(\omega)$. The invariant quaternion algebra is $M_2(\mathbb{Q}(\omega))$.

To prove Theorem 2.2, we first establish two technical lemmas which will also be used in Section 2.2. Also, we let $\omega = e^{\pi i/3}$ henceforth.

Lemma 2.5. *Let* Γ *be a Kleinian group whose traces lie in* R*, the ring of integers in* $\mathbb{Q}(\mathrm{tr}\Gamma)$*, and* $X, Y \in \Gamma$ *with* $\mathrm{tr}X = \mathrm{tr}Y$*. If*

$$x = \mathrm{tr}X = \mathrm{tr}Y,\ y = \mathrm{tr}XY - 2,\ \text{and } z = 2 - \mathrm{tr}[X, Y],$$

then $y \,|\, z$ *in* R*, and*

$$x^2 = \frac{z}{y} + y + 4.$$

Proof. Standard trace relations (e.g., relation 3.15 in Section 3.4 of [31]) yield

$$\begin{aligned} z &= 4 + x^2 \mathrm{tr} XY - \mathrm{tr}^2 XY - 2x^2 \\ &= (\mathrm{tr} XY - 2)x^2 - (\mathrm{tr} XY - 2)(\mathrm{tr} XY + 2) \\ &= y(x^2 - (y+4)) \end{aligned}$$

The result now follows. ■

Lemma 2.6. *Let $x = a + b\omega \in \mathbb{Z}[\omega]$, and*

$$x^2 = a^2 - b^2 + (2ab + b^2)\omega = n + m\omega \in \mathbb{Z}[\omega].$$

If $-4 \leq m \leq 4$, then the following are the only possibilities for x^2.

- *If $m = 0$, then $x^2 = a^2 \in \mathbb{Z}^2$ or $x^2 \leq 0$.*
- *If $m = \pm 1$, then $x^2 = -1 + \omega$ or $x^2 = -\omega$.*
- *If $m = \pm 2$, then $x^2 \notin \mathbb{Z}[\omega]$, i.e., $m = \pm 2$ is not possible.*
- *If $m = \pm 3$, then $x^2 \in \{3\omega, -8 + 3\omega, 3 - 3\omega, -5 - 3\omega\}$.*
- *If $m = \pm 4$, then $x^2 = -4 + 4\omega$ or $x^2 = -4\omega$.*

Proof. For each case, we have the following.

- If $m = 2ab + b^2 = 0$, then $b = 0$ or $a = -\dfrac{b}{2}$, so $x^2 = a^2 \in \mathbb{Z}^2$ or

$$x^2 = -\frac{3b^2}{4} \leq 0.$$

- If $m = 2ab + b^2 = \pm 1$, then $a = \dfrac{\pm 1 - b^2}{2b}$, so $b \mid 1$. Thus, $b = \pm 1$, and

$$(a, b) \in \{(0, \pm 1), \pm(1, -1)\}.$$

Therefore, $x^2 = a^2 - b^2 + (2ab + b^2)\omega = -1 + \omega$ or $x^2 = -\omega$.

- If $m = 2ab + b^2 = \pm 2$, then $a = \dfrac{\pm 2 - b^2}{2b}$, so $b \,|\, 2$ and $2 \,|\, b$. Thus, $b = \pm 2$, and

$$(a, b) \in \left\{ \pm \left(\frac{1}{2}, -2 \right), \pm \left(\frac{3}{2}, -2 \right) \right\}.$$

Therefore, $x^2 = a^2 - b^2 + (2ab + b^2)\omega \notin \mathbb{Z}[\omega]$.

- If $m = 2ab + b^2 = \pm 3$, then $a = \dfrac{\pm 3 - b^2}{2b}$, so $b \,|\, 3$. Thus, $b \in \{\pm 1, \pm 3\}$, and

$$(a, b) \in \{\pm(1, 1), \pm(1, -3), \pm(2, -1), \pm(2, -3)\}.$$

Therefore, $x^2 = a^2 - b^2 + (2ab + b^2)\omega \in \{3\omega, -8 + 3\omega, 3 - 3\omega, -5 - 3\omega\}$.

- If $m = 2ab + b^2 = \pm 4$, then $a = \dfrac{\pm 4 - b^2}{2b}$, so $b \,|\, 4$ and $2 \,|\, b$. Thus, $b \in \{\pm 2, \pm 4\}$, and

$$(a, b) \in \left\{ (0, \pm 2), \pm (2, -2), \pm \left(\frac{3}{2}, -4 \right), \pm \left(\frac{5}{2}, -4 \right) \right\}.$$

Of these, the only values of $x^2 = a^2 - b^2 + (2ab + b^2)\omega \in \mathbb{Z}[\omega]$ are $x^2 = -4 + 4\omega$ and $x^2 = -4\omega$.

■

Theorem 2.2 now follows immediately from:

Theorem 2.7. *Let Γ denote the image of the discrete faithful representation of the figure-eight knot group into $\mathrm{PSL}_2\mathbb{C}$ as above. If $\Gamma = \langle X, Y \rangle$ with $\mathrm{tr}X = \mathrm{tr}Y$, then X and Y are parabolic.*

Proof. By Corollary 1.7, $2 - \mathrm{tr}[X, Y]$ is a unit multiple of $2 - \mathrm{tr}[A, B] = -\omega^2$ in $\mathbb{Z}[\omega]$. The complete group of units in $\mathbb{Z}[\omega]$ is given by

$$(\mathbb{Z}[\omega])^* = \{1,\ \omega,\ \omega^2 = \omega - 1,\ \omega^3 = -1,\ \omega^4 = -\omega,\ \omega^5 = 1 - \omega\}.$$

Since Γ is torsion-free, $\mathrm{tr}[X,Y] \notin (-2,2)$, so

$$2 - \mathrm{tr}[X,Y] = \omega^n \text{ for some } n \in \{1,2,3,4,5\}.$$

Let

$$x = \mathrm{tr}X = \mathrm{tr}Y = a + b\omega \in \mathbb{Z}[\omega],\ y = \mathrm{tr}XY - 2,\ \text{and } z = 2 - \mathrm{tr}[X,Y].$$

Lemma 2.5 implies $y \mid z$ in $\mathbb{Z}[\omega]$, so y is also a unit in $\mathbb{Z}[\omega]$. Since $\mathrm{tr}XY \notin (-2,2)$,

$$y = \omega^m \text{ for some } m \in \{0,1,2,4,5\}.$$

Varying m and n as above generates the following table of values for

$$x^2 = \frac{z}{y} + y + 4 = \omega^{n-m} + \omega^m + 4.$$

	$m=0$	$m=1$	$m=2$	$m=4$	$m=5$
$n=1$	$5+\omega$	$5+\omega$	4	$3-\omega$	4
$n=2$	$4+\omega$	$4+2\omega$	$4+\omega$	$4-2\omega$	$4-\omega$
$n=3$	4	$3+2\omega$	$3+2\omega$	$5-2\omega$	$5-2\omega$
$n=4$	$5-\omega$	$3+\omega$	$2+2\omega$	$5-\omega$	$6-2\omega$
$n=5$	$6-\omega$	4	$2+\omega$	4	$6-\omega$

Of these, by Lemma 2.6, the only possible value for x^2 is 4, i.e., X and Y must be parabolic if they are to generate Γ and have equal trace. ■

Remark 2.8. *The figure-eight knot group can be generated by three conjugate loxodromic elements.*

Proof. Let $\alpha = a^{-1}b^2$, $\beta = b\alpha b^{-1} = ba^{-1}b$, and $\gamma = b^{-1}\alpha b = b^{-1}a^{-1}b^3$. Then

$$\begin{aligned}
\beta^{-1}\alpha\gamma^{-1}\alpha\beta^{-1}\alpha^2 &= b^{-1}ab^{-1}a^{-1}b^2b^{-3}aba^{-1}b^2b^{-1}ab^{-1}a^{-1}b^2a^{-1}b^2 \\
&= b^{-1}ab^{-1}a^{-1}b^{-1}a(ba^{-1}bab^{-1})a^{-1}b^2a^{-1}b^2 \\
&= b^{-1}ab^{-1}a^{-1}b^{-1}a(a^{-1}bab^{-1}a)a^{-1}b^2a^{-1}b^2 \\
&= b
\end{aligned}$$

which implies $b \in \langle \alpha, \beta, \gamma \rangle$, so $b^2\alpha^{-1} = a \in \langle \alpha, \beta, \gamma \rangle$. Hence, $\langle \alpha, \beta, \gamma \rangle = \pi_1(S^3 \smallsetminus K)$, and

$$\alpha \mapsto \begin{pmatrix} 1-2\omega & -1 \\ 2\omega & 1 \end{pmatrix}, \beta \mapsto \begin{pmatrix} 1-\omega & -1 \\ 1+\omega & 1-\omega \end{pmatrix}, \text{ and } \gamma \mapsto \begin{pmatrix} 1-3\omega & -1 \\ -3+5\omega & 1+\omega \end{pmatrix}.$$

Thus, the figure-eight knot group is generated three conjugate loxodromic elements.

■

2.2 The Arithmetic Two-Bridge Links

For the remainder of this chapter, let $\theta = (1 + i\sqrt{7})/2$. The discrete faithful representation of an arithmetic two-bridge link group into $\mathrm{PSL}_2\mathbb{C}$ is generated by the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix}$$

where $\xi = 1 + i$, $1 + \omega$, and θ for the Whitehead link and the links 6_2^2 and 6_3^2 respectively (Section 5 of [13]). Then $\Gamma = \langle A, B \rangle$ is a finite-covolume Kleinian group whose traces lie in $\mathbb{Z}[i]$, $\mathbb{Z}[\omega]$, and $\mathbb{Z}[\theta]$ respectively. These are the rings of integers in the respective trace fields $\mathbb{Q}(i)$, $\mathbb{Q}(\omega)$, and $\mathbb{Q}(\theta)$. The respective invariant quaternion algebras are $M_2(\mathbb{Q}(i))$, $M_2(\mathbb{Q}(\omega))$, and $M_2(\mathbb{Q}(\theta))$. We now establish Theorem 2.3 for each arithmetic two-bridge link group.

Theorem 2.9. *Let* Γ *denote the image of the discrete faithful representation of the Whitehead link group into* $\mathrm{PSL}_2\mathbb{C}$ *as above. If* $\Gamma = \langle X, Y\rangle$ *with* $\mathrm{tr}X = \mathrm{tr}Y$*, then* X *and* Y *are parabolic.*

Proof. By Corollary 1.7, $2 - \mathrm{tr}[X, Y]$ is a unit multiple of $2 - \mathrm{tr}[A, B] = -2i$ in $\mathbb{Z}[i]$. Since $(\mathbb{Z}[i])^* = \{\pm 1, \pm i\}$ and $\mathrm{tr}[X, Y] \notin (-2, 2)$,

$$2 - \mathrm{tr}[X, Y] \in \{-2, \pm 2i\}.$$

Let

$$x = \mathrm{tr}X = \mathrm{tr}Y = a + bi \in \mathbb{Z}[i],\ y = \mathrm{tr}XY - 2,\ \text{and } z = 2 - \mathrm{tr}[X, Y].$$

By Lemma 2.5, $y \mid z$ in $\mathbb{Z}[i]$, so, since $\mathrm{tr}XY \notin (-2, 2)$, we have

$$y \in \{1, \pm i, \pm(1 + i), \pm(1 - i), 2, \pm 2i\}.$$

Varying y and z as above generates the following table of possible values for

$$x^2 = \frac{z}{y} + y + 4.$$

	$z=-2$	$z=2i$	$z=-2i$
$y=1$	3	$5+2i$	$5-2i$
$y=i$	$4+3i$	$6+i$	$2+i$
$y=-i$	$4-3i$	$2-i$	$6-i$
$y=1+i$	$4+2i$	$6+2i$	4
$y=1-i$	$4-2i$	4	$6-2i$
$y=-1+i$	$4+2i$	4	$2+2i$
$y=-1-i$	$4-2i$	$2-2i$	4
$y=2$	5	$6+i$	$6-i$
$y=2i$	$4+3i$	$5+2i$	$3+2i$
$y=-2i$	$4-3i$	$3-2i$	$5-2i$

We now check which of these values have the form $x^2 = a^2 - b^2 + 2abi \in \mathbb{Z}[i]$ based on 0, ± 1, ± 2, and ± 3 being the only imaginary parts that arise in the table.

Case 1. The imaginary part of x^2 is 0; that is,

$$\begin{aligned} 2ab = 0 &\implies a = 0 \text{ or } b = 0 \\ &\implies x^2 = -b^2 \text{ or } x^2 = a^2 \in \mathbb{Z}^2 \end{aligned}$$

The only value in the table with imaginary part 0 that can be expressed in either of these forms is $x^2 = 4$, i.e., X and Y are parabolic.

Case 2. The imaginary part of x^2 is ± 1; that is, $2ab = \pm 1$, which is impossible for $a, b \in \mathbb{Z}$. Hence, x^2 cannot have imaginary part ± 1.

Case 3. The imaginary part of x^2 is ± 2; that is,

$$\begin{aligned} 2ab = \pm 2 &\implies a^2 = b^2 = 1 \\ &\implies x^2 = \pm 2i \end{aligned}$$

But $\pm 2i$ does not appear in the table, so x^2 cannot have imaginary part ± 2.

Case 4. The imaginary part of x^2 is ± 3; that is, $2ab = \pm 3$, which is impossible for $a, b \in \mathbb{Z}$. Hence, x^2 cannot have imaginary part ± 3.

This exhausts the table of possible values for x^2; therefore, X and Y must be parabolic if they are to generate Γ and have equal trace. ■

Theorem 2.10. *Let Γ denote the image of the discrete faithful representation of the 6_2^2 link group into* $\mathrm{PSL}_2\mathbb{C}$ *as above. If $\Gamma = \langle X, Y \rangle$ with $\mathrm{tr}X = \mathrm{tr}Y$, then X and Y are parabolic.*

Proof. By Corollary 1.7, $2 - \mathrm{tr}[X,Y]$ is a unit multiple of $2 - \mathrm{tr}[A,B] = -3\omega$ in $\mathbb{Z}[\omega]$. Since $\mathrm{tr}[X,Y] \notin (-2,2)$, we have

$$2 - \mathrm{tr}[X,Y] \in \{\pm 3\omega, \pm 3(1-\omega), -3\} \subset 3(\mathbb{Z}[\omega])^*.$$

Let

$$x = \mathrm{tr}X = \mathrm{tr}Y = a + b\omega \in \mathbb{Z}[\omega],\ y = \mathrm{tr}XY - 2,\ \text{and } z = 2 - \mathrm{tr}[X,Y].$$

Lemma 2.5 implies $y \,|\, z$, and hence $y \,|\, 3$, in $\mathbb{Z}[\omega]$. Therefore, since $\mathrm{tr}XY \notin (-2,2)$, we have

$$y \in \{\pm(3-3\omega), \pm(2-\omega), \pm(1+\omega), \pm(1-\omega), \pm(1-2\omega), \pm\omega, \pm 3\omega, 1, 3\}.$$

Varying y and z as above generates the following table of values for

$$x^2 = \frac{z}{y} + y + 4.$$

	$z = 3\omega$	$z = -3\omega$	$z = 3 - 3\omega$	$z = -3 + 3\omega$	$z = -3$
$y = 3 - 3\omega$	$6 - 2\omega$	$8 - 4\omega$	$8 - 3\omega$	$6 - 3\omega$	$7 - 4\omega$
$y = -3 + 3\omega$	$2 + 2\omega$	4ω	3ω	$2 + 3\omega$	$1 + 4\omega$
$y = 2 - \omega$	$5 + \omega$	$7 - 3\omega$	$8 - 2\omega$	4	$5 - 2\omega$
$y = -2 + \omega$	$3 - \omega$	$1 + 3\omega$	2ω	4	$3 + 2\omega$
$y = 1 + \omega$	$6 + 2\omega$	4	$6 - \omega$	$4 + 3\omega$	$3 + 2\omega$
$y = -1 - \omega$	$2 - 2\omega$	4	$2 + \omega$	$4 - 3\omega$	$5 - 2\omega$
$y = 1 - \omega$	$2 + 2\omega$	$8 - 4\omega$	$8 - \omega$	$2 - \omega$	$5 - 4\omega$
$y = -1 + \omega$	$6 - 2\omega$	4ω	ω	$6 + \omega$	$3 + 4\omega$
$y = 1 - 2\omega$	$3 - \omega$	$7 - 3\omega$	$6 - \omega$	$4 - 3\omega$	$6 - 4\omega$
$y = -1 + 2\omega$	$5 + \omega$	$1 + 3\omega$	$2 + \omega$	$4 + 3\omega$	$2 + 4\omega$
$y = \omega$	$7 + \omega$	$1 + \omega$	$4 - 2\omega$	$4 + 4\omega$	$1 + 4\omega$
$y = -\omega$	$1 - \omega$	$7 - \omega$	$4 + 2\omega$	$4 - 4\omega$	$7 - 4\omega$
$y = 3\omega$	$5 + 3\omega$	$3 + 3\omega$	$4 + 2\omega$	$4 + 4\omega$	$3 + 4\omega$
$y = -3\omega$	$3 - 3\omega$	$5 - 3\omega$	$4 - 2\omega$	$4 - 4\omega$	$5 - 4\omega$
$y = 1$	$5 + 3\omega$	$5 - 3\omega$	$8 - 3\omega$	$2 + 3\omega$	2
$y = 3$	$7 + \omega$	$7 - \omega$	$8 - \omega$	$6 + \omega$	6

Of these, by Lemma 2.6, the only possible values for x^2 are 4, 3ω, and $3 - 3\omega$. If

$$x^2 = \mathrm{tr}^2 X = \mathrm{tr}^2 Y = 3\omega,$$

then $\mathrm{tr}X = \mathrm{tr}Y = \pm(1+\omega)$, so the axes of X and Y project closed geodesics in $\mathbb{H}^3/\Gamma = S^3 \setminus 6_2^2$ of length

$$\mathrm{Re}\left(2\cosh^{-1}\left(\pm\frac{1+\omega}{2}\right)\right) \approx 1.087070145.$$

Similarly, if

$$x^2 = \mathrm{tr}^2 X = \mathrm{tr}^2 Y = 3 - 3\omega,$$

then $\mathrm{tr}X = \mathrm{tr}Y = \pm(2-\omega)$, so the axes of X and Y project closed geodesics in $\mathbb{H}^3/\Gamma = S^3 \setminus 6_2^2$ of length

$$\mathrm{Re}\left(2\cosh^{-1}\left(\pm\frac{2-\omega}{2}\right)\right) \approx 1.087070145.$$

But rigorous computation of the length spectrum in SnapPea ([44], [22]) shows that the shortest two closed geodesics in $S^3 \setminus 6_2^2$ have length

$$0.86255462766206 \text{ and } 1.66288589105862.$$

Thus, the only possible value for x^2 is 4, i.e., X and Y must be parabolic if they are to generate Γ and have equal trace. ■

Theorem 2.11. *Let Γ denote the image of the discrete faithful representation of the 6_3^2 link group into $\mathrm{PSL}_2\mathbb{C}$ as above. If $\Gamma = \langle X, Y\rangle$ with $\mathrm{tr}X = \mathrm{tr}Y$, then X and Y are parabolic.*

Proof. By Corollary 1.7, $2 - \mathrm{tr}[X,Y]$ is a unit multiple of $2 - \mathrm{tr}[A,B] = -\theta^2 = 2-\theta$ in $\mathbb{Z}[\theta]$, i.e., $2 - \mathrm{tr}[X,Y] = \pm(2-\theta)$. Let

$$x = \mathrm{tr}X = \mathrm{tr}Y = a + b\theta \in \mathbb{Z}[\theta],\ y = \mathrm{tr}XY - 2, \text{ and } z = 2 - \mathrm{tr}[X,Y].$$

Lemma 2.5 implies $y \mid z$ in $\mathbb{Z}[\theta]$. Since $\mathrm{tr}XY \notin (-2,2)$, we have

$$y \in \{1, \pm(2-\theta), \pm\theta\}.$$

Therefore,

$$x^2 = \frac{z}{y} + y + 4 \in \{4, 1+\theta, 3+\theta, 4\pm 2\theta, 5-\theta, 7-\theta\}.$$

Of these, arguing as before, the only possible value for

$$x^2 = a^2 - 2b^2 + (2ab + b^2)\theta \in \mathbb{Z}[\theta]$$

is 4, i.e., X and Y must be parabolic if they are to generate Γ and have equal trace. ■

2.3 Torus Knots

Throughout this section, we assume $\gcd(p, q) = 1$ and $2 \leq p < q$. As is well known, the (p, q)-torus knot group admits the presentation

$$\pi_1(S^3 \setminus K_{p,q}) = \langle c, d \mid c^p = d^q \rangle,$$

which clearly surjects $\mathbb{Z}_p * \mathbb{Z}_q = \langle s, t \mid s^p = t^q = 1 \rangle$ via $c \mapsto s$, $d \mapsto t$ (see [9]).

We begin our investigation of conjugate generators for torus knot groups by paraphrasing Proposition 17 of [16]:

Proposition 2.12. *If $\mathbb{Z}_p * \mathbb{Z}_q = \langle s, t \mid s^p = t^q = 1 \rangle$ can be generated by two conjugate elements, then $p = 2$.*

Thus, via the surjection above, the (p, q)-torus knot group can be generated by two conjugate elements only when $p = 2$, i.e., when the torus knot is two-bridge with normal form $(q/1)$. The $(2, q)$-torus knot group $\langle c, d \mid c^2 = d^q \rangle$ has a parabolic representation into $\mathrm{PSL}_2\mathbb{C}$ generated by

$$c \mapsto C = \begin{pmatrix} 0 & (2\cos(\pi/q))^{-1} \\ -2\cos(\pi/q) & 0 \end{pmatrix} \text{ and } d \mapsto D = \begin{pmatrix} 1 - 4\cos^2(\pi/q) & 1 \\ -4\cos^2(\pi/q) & 1 \end{pmatrix}$$

(Theorem 6 of [35]). Then $\Gamma_q = \langle C, D \rangle$ is a finite-coarea subgroup of $\mathrm{PSL}_2\mathbb{R}$ and has a presentation $\langle C, D \mid C^2 = D^q = 1 \rangle = \mathbb{Z}_2 * \mathbb{Z}_q$. Furthermore, the traces of Γ_q are algebraic integers in $\mathbb{Q}(\mathrm{tr}\Gamma) = \mathbb{Q}(\cos(\pi/q))$ (cf. Section 8.3 of [31]).

To analyze the case $q = 3$ (i.e., the trefoil knot), we recall the classification of generating pairs for the groups $\mathbb{Z}_p * \mathbb{Z}_q$ up to Nielsen equivalence (Corollary 4.14 of [45] and Theorem 13 of [16]):

Theorem 2.13. *Every generating pair for the group $\mathbb{Z}_p * \mathbb{Z}_q = \langle s, t \mid s^p = t^q = 1 \rangle$ is Nielsen equivalent to exactly one generating pair of the form (s^m, t^n) where*

$$\gcd(m, p) = \gcd(n, q) = 1,\ 0 < 2m \leq p,\ \textit{and}\ 0 < 2n \leq q.$$

Corollary 2.14. *If two conjugate elements generate the $(2,3)$-torus knot group (i.e., the trefoil knot group), then the elements are peripheral.*

Proof. Suppose $\langle \alpha, \beta \rangle = \pi_1(S^3 \smallsetminus K_{2,3})$ with α conjugate to β. Let $\alpha \mapsto X$ and $\beta \mapsto Y$ via the representation above. Then $\langle X, Y \rangle = \Gamma_3 = \mathbb{Z}_2 * \mathbb{Z}_3$, and X is conjugate to Y, so $\mathrm{tr}X = \mathrm{tr}Y$. Thus, by Theorem 2.13 with $p = 2$ and $q = 3$, (X, Y) is Nielsen equivalent to (C, D). Since commutators of Nielsen equivalent pairs have equal trace, we have

$$2 - \mathrm{tr}[X, Y] = 2 - \mathrm{tr}[C, D] = -1.$$

Lemma 2.5 then implies $\mathrm{tr}XY - 2 = \pm 1$, and so $\mathrm{tr}^2 X = \mathrm{tr}^2 Y = 4$. Hence, α and β are peripheral. ■

As defined in this section, Γ_q is a discrete faithful representation into $\mathrm{PSL}_2\mathbb{R}$ of the $(2, q, \infty)$-triangle group. Thus, the proof of Corollary 2.14 shows that if two elements of equal trace (e.g., conjugate elements) generate the $(2, 3, \infty)$-triangle group (i.e., the modular group), then the elements are parabolic. Following Section 13.3 of [31] (cf. [41]), the $(2, q, \infty)$-triangle group is arithmetic only when $q = 3$, 4, 6, or ∞. Similar methods can then be used to show that if two elements of equal trace generate an arithmetic $(2, q, \infty)$-triangle group, then the elements are parabolic. The $(2, q, \infty)$-triangle group can, however, be generated by two conjugate hyperbolic elements when $q > 3$ is odd: with $\Gamma_q = \langle C, D \rangle$ as above, let $X = CD$ and $Y = C^{-1}XC = DC$; then

$(YX)^{(q+1)/2} = D$, so $\langle X, Y \rangle = \Gamma_q$, and $\mathrm{tr}X = \mathrm{tr}Y = -4\cos(\pi/q) < -2$ since $q > 3$, so X and Y are hyperbolic.

2.4 Simon's Conjecture

The following is attributed to J. Simon (cf. Problem 1.12 of [25]) and has recently been proved for two-bridge knots in [5].

Simon's Conjecture. *A knot group can surject only finitely many other knot groups.*

To show Reid's Conjecture implies Simon's Conjecture for two-bridge knots, we first note Theorem 5.2 of [34] (recall that a knot complement is called *small* if it does not contain a closed embedded essential surface):

Theorem 2.15. *If M is a small hyperbolic knot complement in S^3, then there exist only finitely many hyperbolic 3-manifolds N for which there is a peripheral preserving epimorphism $\pi_1 M \twoheadrightarrow \pi_1 N$.*

Theorem 2.16. *Reid's Conjecture implies Simon's Conjecture for two-bridge knots.*

Proof. Let K be a two-bridge knot and $\varphi_i : \pi_1(S^3 \smallsetminus K) \twoheadrightarrow \pi_1(S^3 \smallsetminus K_i)$ a collection of epimorphisms φ_i and knots K_i.

If K_i is a (p, q)-torus knot, then its Alexander polynomial has degree $(p-1)(q-1)$ by Example 9.15 in [9] and divides the Alexander polynomial of K since φ_i is an epimorphism by exercise 9 of Chapter VII in [11] (cf. the proof of Proposition 17 in [17]). This can occur for only finitely many integer pairs (p, q); hence, only finitely many K_i are torus knots.

Now assume K_i is not a torus knot. Since K is two-bridge, $\pi_1(S^3 \smallsetminus K)$ is generated by two conjugate meridians, a and b. Then $\varphi_i(a)$ and $\varphi_i(b)$ are conjugate elements that generate $\pi_1(S^3 \smallsetminus K_i)$, so Reid's Conjecture implies that they are peripheral. Hence, by Theorem 2.1, K_i is two-bridge and therefore hyperbolic since it is not a torus knot.

Let $\lambda \in \pi_1(S^3 \setminus K)$ such that $\langle a, \lambda \rangle$ is a peripheral subgroup of $\pi_1(S^3 \setminus K)$. Then λ commutes with a in $\pi_1(S^3 \setminus K)$, so $\varphi_i(\lambda)$ commutes with $\varphi_i(a)$ in $\pi_1(S^3 \setminus K_i)$ and hence is a peripheral element of $\pi_1(S^3 \setminus K_i)$. Therefore, φ_i is peripheral preserving, so the result follows from Theorem 2.15 since hyperbolic two-bridge knot complements in S^3 are small ([21]). ■

Chapter 3

Jørgensen Number and Jørgensen Groups

We begin by recalling T. Jørgensen's well known necessary condition for two elements of $\mathrm{PSL}_2\mathbb{C}$ to generate a non-elementary discrete group ([23]).

Jørgensen's Inequality. *If $\langle X, Y\rangle$ is a non-elementary Kleinian group, then*

$$|\mathrm{tr}^2 X - 4| + |\mathrm{tr}[X, Y] - 2| \geq 1.$$

Accordingly, the *Jørgensen number* of an ordered pair of elements in $\mathrm{PSL}_2\mathbb{C}$ is

$$J(X, Y) = |\mathrm{tr}^2 X - 4| + |\mathrm{tr}[X, Y] - 2|,$$

and the *Jørgensen number* of a rank-two non-elementary Kleinian group Γ is

$$J(\Gamma) = \inf\{J(X, Y) : \langle X, Y\rangle = \Gamma\}.$$

Jørgensen's Inequality guarantees $J(\Gamma) \geq 1$, and if $J(\Gamma) = 1$, then Γ is a *Jørgensen group* and has been the subject of much study. Among the first such results is the following combination of Theorems 1 and 2 in [24].

Theorem 3.1. *If* $\langle X, Y\rangle$ *is a non-elementary Kleinian group with* $J(X, Y) = 1$*, then the following two statements hold.*

1. $\langle X, YXY^{-1}\rangle$ *is also non-elementary and* $J(X, YXY^{-1}) = 1$.
2. X *is parabolic, or* X *is elliptic of order at least seven and* $\mathrm{tr}XYXY^{-1} = 1$.

A Jørgensen group $\Gamma = \langle X, Y\rangle$ with $J(X, Y) = 1$ is then *of parabolic* or *elliptic type* according to whether X is parabolic or elliptic as a consequence of Theorem 3.1.

The following problem has attracted much attention (e.g., [24], [15], [39], [37], [26], [27], [28], and [18]).

Problem 3.2. *Identify all Jørgensen groups.*

It is observed in [24] that there are uncountably many non-conjugate Jørgensen groups in general, so most work on Problem 3.2 has entailed restriction to more tractable cases. For instance, all Jørgensen subgroups of the Picard group $\mathrm{PSL}_2(O_1)$ are found in [18], and Theorem 3 of [24] solves Problem 3.2 in the Fuchsian case (recall that a Kleinian group is *Fuchsian* if it is conjugate to a subgroup of $\mathrm{PSL}_2\mathbb{R}$):

Theorem 3.3. *The only Fuchsian Jørgensen groups are the* $(2, 3, q)$*-triangle groups where* $q \geq 7$ *or* $q = \infty$.

This chapter solves Problem 3.2 in two cases: torsion-free Kleinian groups and non-cocompact arithmetic Kleinian groups.

Recall that every torsion-free Kleinian group Γ yields an orientable hyperbolic 3-manifold $\mathbb{H}^3/\Gamma$, and, conversely, every orientable hyperbolic 3-manifold is isometric to $\mathbb{H}^3/\Gamma$ for some torsion-free Kleinian group Γ. If $\mathbb{H}^3/\Gamma$ is the figure-eight knot complement in S^3, then, as established in Theorem 2.4, Γ is generated by a well known pair of conjugate parabolic elements that are easily seen to have Jørgensen number one. In Section 3.1, we identify this as the only torsion-free Jørgensen group by proving the following result.

Theorem 3.4. *The figure-eight knot complement in S^3 is the only orientable hyperbolic 3-manifold $\mathbb{H}^3/\Gamma$ with $J(\Gamma) = 1$.*

We observe in Section 3.2 that the main theorems of [26], [27], and [28] combine to find all Jørgensen groups of parabolic type (θ, k), meaning that the group can be generated by the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_{\theta,k} = \begin{pmatrix} ke^{i\theta} & ik^2e^{i\theta} - ie^{-i\theta} \\ -ie^{i\theta} & ke^{i\theta} \end{pmatrix}$$

where $0 \leq \theta \leq 2\pi$ and $k \in \mathbb{R}$. It is conjectured in [26], [27], and [28] that every Jørgensen group of parabolic type is conjugate in $\mathrm{PSL}_2\mathbb{C}$ to a Jørgensen group of parabolic type (θ, k). If this conjecture were true, then Problem 3.2 would be solved for Jørgensen groups of parabolic type.

In Section 3.3, however, we disprove this conjecture by finding four Jørgensen groups of parabolic type that are not conjugate to any Jørgensen group of parabolic type (θ, k) found in [26], [27], or [28]; these counterexamples are $\mathrm{PGL}_2(O_3)$, $\mathrm{PSL}_2(O_3)$, $\mathrm{PSL}_2(O_7)$, and $\mathrm{PSL}_2(O_{11})$. This is part of our second contribution to Problem 3.2: we identify all non-cocompact arithmetic Jørgensen groups (and hence all arithmetic Jørgensen groups of parabolic type) by proving the following result.

Theorem 3.5. *There are exactly fourteen non-cocompact arithmetic Jørgensen groups:* $\mathrm{PGL}_2(O_1)$*,* $\mathrm{PGL}_2(O_3)$*,* $\mathrm{PSL}_2(O_1)$*,* $\mathrm{PSL}_2(O_2)$*,* $\mathrm{PSL}_2(O_3)$*,* $\mathrm{PSL}_2(O_7)$*,* $\mathrm{PSL}_2(O_{11})$*, two subgroups of index 6 and 8 respectively in* $\mathrm{PGL}_2(O_1)$*, the unique subgroup of index 10 in* $\mathrm{PSL}_2(O_3)$*, the figure-eight knot group, and three $\mathbb{Z}_2$-extensions of the figure-eight knot group.*

Since Jørgensen groups are two-generator by definition, Theorem 3.5 can also be regarded as a solution to the following problem in the case of Jørgensen groups of parabolic type.

Problem 3.6. *Identify all two-generator arithmetic Kleinian groups.*

Problem 3.6 has also attracted much attention, including [29], which proves that only finitely many arithmetic Kleinian groups can be generated by two elliptic elements; [13], which identifies all arithmetic Kleinian groups generated by two parabolic elements; and [10], which identifies all two-generator arithmetic Kleinian groups with one generator parabolic and the other elliptic.

In proving Theorem 3.5, we demonstrate that $\mathrm{PSL}_2(O_1)$, $\mathrm{PSL}_2(O_2)$, $\mathrm{PSL}_2(O_7)$, $\mathrm{PSL}_2(O_{11})$, the unique subgroup of index 10 in $\mathrm{PSL}_2(O_3)$, a $\mathbb{Z}_2$-extension of the figure-eight knot group by an involution that conjugates each parabolic generator to its own inverse, and the two subgroups of index 6 and 8 respectively in $\mathrm{PGL}_2(O_1)$ are two-generator arithmetic Kleinian groups with one generator parabolic and the other loxodromic. None of these were identified as two-generator arithmetic Kleinian groups in [29], [13], or [10] and can therefore be regarded as a contribution to Problem 3.6.

We now turn our attention to Jørgensen groups of elliptic type. The following useful result is observed in [15].

Theorem 3.7. *Every Jørgensen group of elliptic type contains a* $(2, 3, q)$*-triangle group where* $q \geq 7$.

Exploiting this and the enumeration of all arithmetic triangle groups in [41], we establish in Section 3.4 a characterization of arithmetic Jørgensen groups of elliptic type (and hence of cocompact arithmetic Jørgensen groups), which shows that no arithmetic Jørgensen group is of both parabolic and elliptic type.

We conclude this chapter in Section 3.5 by bounding $J(\Gamma)$ if $\mathbb{H}^3/\Gamma$ is the complement of a two-bridge knot or link in S^3 and in Section 3.6 by computing $J(\Gamma)$ for seven such groups, including that of the Whitehead link, whose Jørgensen number was shown to be two in [38]; the others have not yet appeared in publication.

3.1 Torsion-Free Jørgensen Groups

Before proving Theorem 3.4, we recall the notion of waist size for cusps in hyperbolic 3-manifolds introduced in [2]. The *waist size* $w(M, \mathcal{C})$ of a cusp $\mathcal{C}$ in an orientable hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ is the length of the shortest translation in $\mathcal{C}$ after expanding $\mathcal{C}$ until it first touches itself on its boundary, i.e., expanding $\mathcal{C}$ to a *maximal cusp.* Note that if Γ contains parabolic elements, then M is noncompact and contains one or more cusps. The following is a combination of Lemma 2.4 and Theorem 3.1, both from [2].

Theorem 3.8. *The waist size of any cusp in an orientable hyperbolic 3-manifold is at least one, and the only hyperbolic 3-manifold with a cusp of waist size one is the figure-eight knot complement in S^3.*

We begin by bounding waist size (cf. Lemma 2.5 of [3]).

Lemma 3.9. *Let $M = \mathbb{H}^3/\Gamma$ be an orientable hyperbolic 3-manifold containing a cusp $\mathcal{C}$. Conjugating Γ as necessary, assume that $\mathcal{C}$ lifts to a disjoint set of horoballs, one of which, $\mathcal{H}$, is based at ∞, and that Γ contains the parabolic element $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then*

$$1 \leq w(M, \mathcal{C}) \leq \inf \left\{ |c| : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \text{ and } c \neq 0 \right\}.$$

Proof. After expanding $\mathcal{C}$ to a maximal cusp, its height h equals that of the expanded horoball $\mathcal{H}$. Let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ with $c \neq 0$. Then $T(\mathcal{H})$ is a horoball based at $\frac{a}{c}$ of diameter $\frac{1}{h|c|^2}$ whose interior is disjoint from that of $\mathcal{H}$. Hence, $h \geq \frac{1}{h|c|^2}$, so $h \geq \frac{1}{|c|}$. The translation length of A at height h is $\frac{1}{h} \leq |c|$, so $w(M, \mathcal{C}) \leq |c|$. Since

T was arbitrary,

$$w(M,\mathcal{C}) \leq \inf\left\{ |c| : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \text{ and } c \neq 0 \right\}.$$

Theorem 3.8 establishes $1 \leq w(M,\mathcal{C})$. ■

To prove Theorem 3.4, we define the *generalized Jørgensen number* of an arbitrary subgroup Γ of $\mathrm{PSL}_2\mathbb{C}$ to be

$$\widetilde{J}(\Gamma) = \inf\{J(X,Y) : \langle X,Y\rangle \leq \Gamma \text{ is discrete and non-elementary}\}.$$

Clearly, Jørgensen's Inequality guarantees $\widetilde{J}(\Gamma) \geq 1$, and if Γ is rank-two, non-elementary, and discrete, then $J(\Gamma) \geq \widetilde{J}(\Gamma)$ by definition.

Theorem 3.10. *The only orientable hyperbolic 3-manifold $\mathbb{H}^3/\Gamma$ with $\widetilde{J}(\Gamma) = 1$ is the figure-eight knot complement in S^3.*

Proof. As established in [36], if $\mathbb{H}^3/\Gamma$ is the figure-eight knot complement in S^3, then Γ is generated by the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ e^{\pi i/3} & 1 \end{pmatrix}.$$

A simple computation shows that $J(A,B) = |e^{2\pi i/3}| = 1$, so $\widetilde{J}(\Gamma) = J(\Gamma) = 1$ by definition.

Conversely, assume $M = \mathbb{H}^3/\Gamma$ is an orientable hyperbolic 3-manifold with $\widetilde{J}(\Gamma) = 1$. By definition, there exist A, B in Γ such that $\langle A,B\rangle$ is non-elementary and $J(A,B) = 1$. Since Γ is torsion-free, Theorem 3.1 implies A is parabolic. Conjugate Γ so that

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$$

Then M contains a cusp $\mathcal{C}$ that lifts to a disjoint set of horoballs, one of which is based at ∞. By assumption,

$$J(A,B) = |\text{tr}^2 A - 4| + |\text{tr}[A,B] - 2| = |c_0|^2 = 1,$$

so $|c_0| = 1$, and $w(M, \mathcal{C}) = 1$ by Lemma 3.9. Hence, $\mathbb{H}^3/\Gamma$ is the figure-eight knot complement in S^3 by Theorem 3.8. ■

Since $J(\Gamma) \geq \widetilde{J}(\Gamma) \geq 1$, Theorem 3.4 now follows easily.

Corollary 3.11. *The only orientable hyperbolic 3-manifold* $\mathbb{H}^3/\Gamma$ *with* $J(\Gamma) = 1$ *is the figure-eight knot complement in* S^3*.*

3.2 Jørgensen Groups of Parabolic Type (θ, k)

Following [26], [27], and [28], a group of parabolic type (θ, k) is denoted $G_{\theta,k}$ and is generated by the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_{\theta,k} = \begin{pmatrix} ke^{i\theta} & ik^2e^{i\theta} - ie^{-i\theta} \\ -ie^{i\theta} & ke^{i\theta} \end{pmatrix}$$

where $0 \leq \theta \leq 2\pi$ and $k \in \mathbb{R}$. The main theorems of [26], [27], and [28] combine to find all Jørgensen groups of parabolic type (θ, k), and the following conjecture is made.

Conjecture 3.12. *Every Jørgensen group of parabolic type is conjugate in* $\text{PSL}_2\mathbb{C}$ *to a Jørgensen group of parabolic type* (θ, k)*.*

Thus, if Conjecture 3.12 were true, then Problem 3.2 is solved for Jørgensen groups of parabolic type. We will see in Section 3.3 that this is not the case by exhibiting four arithmetic Jørgensen groups of parabolic type that are not conjugate to any Jørgensen group of parabolic type (θ, k) found in [26], [27], or [28]; these counterexamples are $\mathrm{PGL}_2(O_3)$, $\mathrm{PSL}_2(O_3)$, $\mathrm{PSL}_2(O_7)$, and $\mathrm{PSL}_2(O_{11})$. Since arithmetic Kleinian groups necessarily have finite covolume, we restrict our attention to finite-covolume Jørgensen groups of parabolic type (θ, k) and state the main theorems of [26], [27], and [28], together with Corollary 3.5, Lemma 3.6, and Lemma 3.8 of [27] (cf. Corollary 6.3 and Lemma 6.4 of [37]), as follows.

Theorem 3.13. *For $0 \leq \theta \leq \pi/2$ and $k \geq 0$, $G_{\theta,k}$ is a finite-covolume Jørgensen group if and only if (θ, k) is one of the following pairs.*

• $(\frac{\pi}{6}, \frac{\sqrt{3}}{2}n)$ *for* $n \in \mathbb{Z}$	• $(\frac{\pi}{4}, \frac{1}{2})$	• $(\frac{\pi}{4}, 1)$	• $(\frac{\pi}{4}, \frac{3}{2})$
• $(\frac{\pi}{4}, 1 + \frac{\sqrt{2}}{2})$	• $(\frac{\pi}{4}, \frac{5+\sqrt{5}}{4})$	• $(\frac{\pi}{4}, 1 + \frac{\sqrt{3}}{2})$	• $(\frac{\pi}{3}, \frac{\sqrt{3}}{2}n)$ *for* $n \in \mathbb{Z}$
• $(\frac{\pi}{2}, \frac{1}{2})$	• $(\frac{\pi}{2}, \frac{\sqrt{2}}{2})$	• $(\frac{\pi}{2}, \frac{1+\sqrt{5}}{4})$	• $(\frac{\pi}{2}, \frac{\sqrt{3}}{2})$

Furthermore,

- *For $0 \leq \theta \leq \pi/2$ and $k \in \mathbb{R}$, $G_{\theta,k}$ is a Jørgensen group if and only if $G_{\pi-\theta,k}$ is a Jørgensen group.*

- *For $0 \leq \theta \leq \pi$ and $k \in \mathbb{R}$, $G_{\pi+\theta,k} = G_{\theta,k}$.*

- *For $0 \leq \theta \leq 2\pi$ and $k \in \mathbb{R}$, $G_{\theta,k}$ is a Jørgensen group if and only if $G_{\theta,-k}$ is a Jørgensen group.*

This completes the identification of all finite-covolume Jørgensen groups of parabolic type (θ, k). To determine which of these are arithmetic using Theorem 1.4, we recall from Section 5 of [10] the presentations of several Bianchi groups (cf. [20] and [40]).

Theorem 3.14. *For $d \in \{1, 2, 3, 7, 11\}$, $\mathrm{PSL}_2(O_d)$ is generated by the three matrices*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; \textit{and } T = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

where $\alpha = \sqrt{-d}$ *if* $d \not\equiv 3 \mod 4$ *and* $\alpha = (1+\sqrt{-d})/2$ *if* $d \equiv 3 \mod 4$. *The full presentations of these Bianchi groups are as follows.*

- $\mathrm{PSL}_2(O_1) = \langle A, S, T \mid S^2 = (AS)^3 = [A,T] = (T^2ST^{-1}S)^2 =$
$$(TST^{-1}STS)^2 = (ATST^{-1}STS)^2 = 1\rangle$$
- $\mathrm{PSL}_2(O_2) = \langle A, S, T \mid S^2 = (AS)^3 = [A,T] = [S,T]^2 = 1\rangle$
- $\mathrm{PSL}_2(O_3) = \langle A, S, T \mid S^2 = (AS)^3 = [A,T] = (TST^{-1}AT^{-1}AS)^2 =$
$$(TST^{-1}AS)^3 = A^{-1}T^{-1}SA^{-1}TSAT^{-1}SAT^{-1}SA^{-1}TS = 1\rangle$$
- $\mathrm{PSL}_2(O_7) = \langle A, S, T \mid S^2 = (AS)^3 = [A,T] = (SAT^{-1}ST)^2 = 1\rangle$
- $\mathrm{PSL}_2(O_{11}) = \langle A, S, T \mid S^2 = (AS)^3 = [A,T] = (SAT^{-1}ST)^3 = 1\rangle$

We also recall Theorem 1.1 from [10], which identifies all two-generator arithmetic Kleinian groups with one generator parabolic and the other elliptic.

Theorem 3.15. *Suppose* $\Gamma = \langle A, B\rangle$ *is an arithmetic Kleinian group with* A *parabolic and* B *elliptic. Then* B *has order 2, 3, 4, or 6, and there are exactly fourteen such groups:*

- *If* B *has order 2, then there are six groups:*
  1. *Two* $\mathbb{Z}_2$*-extensions of the figure eight knot group each with index 6 in* $\mathrm{PSL}_2(O_3)$.
  2. *A* $\mathbb{Z}_2$*-extension of the Whitehead link group with* $\Gamma \cap \mathrm{PSL}_2(O_1)$ *of index 2 in* Γ *and 12 in* $\mathrm{PSL}_2(O_1)$.
  3. *Two* $\mathbb{Z}_2$*-extensions of the* 6_2^2 *link group each with* $\Gamma \cap \mathrm{PSL}_2(O_3)$ *of index 2 in* Γ *and 24 in* $\mathrm{PSL}_2(O_3)$.

4. *A $\mathbb{Z}_2$-extension of the 6_3^2 link group with $\Gamma \cap \mathrm{PSL}_2(O_7)$ of index 2 in Γ and 12 in $\mathrm{PSL}_2(O_7)$.*

- *If B has order 3, then there are three groups:*

1. $[\mathrm{PSL}_2(O_1) : \Gamma] = 8$.
2. $\Gamma = \mathrm{PSL}_2(O_3)$.
3. $[\mathrm{PSL}_2(O_7) : \Gamma] = 2$.

- *If B has order 4, then there are three groups:*

1. $\Gamma = \mathrm{PGL}_2(O_1)$.
2. $\Gamma \cap \mathrm{PSL}_2(O_2)$ *has index 4 in* Γ *and 24 in* $\mathrm{PSL}_2(O_2)$.
3. $\Gamma \cap \mathrm{PSL}_2(O_3)$ *has index 2 in* Γ *and 30 in* $\mathrm{PSL}_2(O_3)$.

- *If B has order 6, then there are two groups:*

1. $\Gamma = \mathrm{PGL}_2(O_3)$.
2. $\Gamma \cap \mathrm{PSL}_2(O_{15})$ *has index 6 in* Γ *and 6 in* $\mathrm{PSL}_2(O_{15})$.

We are now prepared to identify which Jørgensen groups of parabolic type (θ, k) in Theorem 3.13 are arithmetic. Note that for our purposes we conjugate $G_{\theta,k}$ by $\begin{pmatrix} 1 & -ik \\ 0 & 1 \end{pmatrix}$, henceforth regarding $G_{\theta,k}$ as the group generated by the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_{\theta,k} = \begin{pmatrix} 0 & -ie^{-i\theta} \\ -ie^{i\theta} & 2ke^{i\theta} \end{pmatrix}$$

where $0 \leq \theta \leq 2\pi$ and $k \in \mathbb{R}$. We use Theorems 3.13, 3.14, and 3.15 to prove the following result.

Proposition 3.16. *For $0 \leq \theta \leq \pi/2$ and $k \geq 0$, $G_{\theta,k}$ is an arithmetic Jørgensen group if and only if (θ, k) is one of the following pairs.*

- $(\frac{\pi}{6}, \frac{\sqrt{3}}{2}n)$ *for $n \in \mathbb{Z}$, in which case $G_{\theta,k}$ is the figure-eight knot group if n is odd and a $\mathbb{Z}_2$-extension of the figure-eight group by an involution that conjugates one parabolic generator to the other (or its inverse) if n is even.*
- $(\frac{\pi}{4}, \frac{1}{2})$, *in which case $G_{\theta,k} = \mathrm{PGL}_2(O_1)$.*
- $(\frac{\pi}{4}, 1)$, *in which case $G_{\theta,k}$ is a subgroup of index 8 in $\mathrm{PGL}_2(O_1)$.*
- $(\frac{\pi}{4}, \frac{3}{2})$, *in which case $G_{\theta,k}$ is a subgroup of index 6 in $\mathrm{PGL}_2(O_1)$.*
- $(\frac{\pi}{3}, \frac{\sqrt{3}}{2}n)$ *for $n \in \mathbb{Z}$, in which case $G_{\theta,k}$ is a $\mathbb{Z}_2$-extension of the figure-eight group by an involution that conjugates one parabolic generator to the other (or its inverse) if n is even and by an involution that conjugates each parabolic generator to its own inverse if n is odd.*
- $(\frac{\pi}{2}, \frac{1}{2})$, *in which case $G_{\theta,k} = \mathrm{PSL}_2(O_1)$.*
- $(\frac{\pi}{2}, \frac{\sqrt{2}}{2})$, *in which case $G_{\theta,k} = \mathrm{PSL}_2(O_2)$.*
- $(\frac{\pi}{2}, \frac{\sqrt{3}}{2})$, *in which case $G_{\theta,k}$ is the unique subgroup of index 10 in $\mathrm{PSL}_2(O_3)$.*

Proof. Let $G_{\theta,k}$ be a finite-covolume Jørgensen group with $0 \leq \theta \leq \pi/2$ and $k \geq 0$. Then (θ, k) must be one of the pairs listed in Theorem 3.13. By Lemma 1.1, we see that $\mathrm{tr}G_{\theta,k}$ consists of algebraic integers for each of these pairs.

Suppose first that $k = 0$. Then $(\theta, k) = (\frac{\pi}{6}, 0)$ or $(\frac{\pi}{3}, 0)$, so

$$kG_{\theta,k} = \mathbb{Q}(e^{2i\theta}) = \mathbb{Q}(\sqrt{-3})$$

by Lemma 1.1. Thus, $G_{\theta,k}$ is arithmetic by Theorem 1.4 in either case. Furthermore, $\langle A, B_{\theta,k}AB_{\theta,k}^{-1}\rangle$ is the figure-eight knot group (cf. [36]) and has index 2 in $G_{\theta,k}$ since

$B_{\theta,k}$ is elliptic of order 2. By Lemma 1.1,

$$\mathbb{Q}(\mathrm{tr}G_{\frac{\pi}{6},0}) = \mathbb{Q}(e^{\pi i/3}) \neq \mathbb{Q}(\mathrm{tr}G_{\frac{\pi}{3},0}) = \mathbb{Q}(e^{\pi i/6}),$$

so $G_{\frac{\pi}{6},0}$ and $G_{\frac{\pi}{3},0}$ are nonconjugate $\mathbb{Z}_2$-extensions of the figure eight knot group by an involution that conjugates one parabolic generator to the other (or its inverse).

Now suppose $k \neq 0$. Then $kG_{\theta,k} = \mathbb{Q}(ki, e^{2i\theta})$ by Lemma 1.1. Hence, $G_{\theta,k}$ is not arithmetic if $(\theta, k) = (\frac{\pi}{4}, 1+\frac{\sqrt{2}}{2})$, $(\frac{\pi}{4}, \frac{5+\sqrt{5}}{4})$, $(\frac{\pi}{4}, 1+\frac{\sqrt{3}}{2})$, or $(\frac{\pi}{2}, \frac{5+\sqrt{5}}{4})$ as $\mathbb{Q}(ki, e^{2i\theta})$ is not a quadratic imaginary extension of $\mathbb{Q}$ in these cases, contradicting Theorem 1.4. If (θ, k) is one of the remaining pairs from Theorem 3.13, then $\mathbb{Q}(ki, e^{2i\theta}) = \mathbb{Q}(\sqrt{-d})$ for some $d \in \mathbb{N}$, so $G_{\theta,k}$ is arithmetic by Theorem 1.4. We now identify each of these groups.

Case 1. Suppose $(\theta, k) = (\frac{\pi}{6}, \frac{\sqrt{3}}{2}n)$ for some integer n. Let

$$C = B_{\theta,k}AB_{\theta,k}^{-1} = \begin{pmatrix} 1 & 0 \\ e^{\pi i/3} & 1 \end{pmatrix} \text{ and}$$

$$T = C^{-1}ACA^{-2}CAC^{-1} = \begin{pmatrix} 1 & 2\sqrt{3}i \\ 0 & 1 \end{pmatrix}.$$

If n is even, then let

$$D = B_{\theta,k}T^{-n/2} = \begin{pmatrix} 0 & -e^{\pi i/3} \\ e^{-\pi i/3} & 0 \end{pmatrix}.$$

Thus, $B_{\theta,k} = DT^{n/2}$ so, since $T \in \langle A, C\rangle$ and $DAD^{-1} = C$, we have $G_{\theta,k} = \langle A, D\rangle$, which is $G_{\theta,k}$ with $(\theta, k) = (\frac{\pi}{6}, 0)$ and is a $\mathbb{Z}_2$-extension of the figure eight knot group as above.

If n is odd, then observe that

$$B_{\theta,k}^{-1}A^{-1}CAC^{-1}T^{(n-1)/2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so $B_{\theta,k} \in \langle A, C\rangle$ since $T \in \langle A, C\rangle$. Thus $G_{\theta,k} = \langle A, C\rangle$, which is clearly the figure-eight knot group (cf. [36]).

Case 2. Suppose $(\theta, k) = (\frac{\pi}{4}, \frac{1}{2})$. Then $kG_{\theta,k} = \mathbb{Q}(i)$ and $G_{\theta,k} = \langle A, B_{\theta,k}A\rangle$. Since $B_{\theta,k}A$ has order 4, we conclude that $G_{\theta,k} = \mathrm{PGL}_2(O_1)$ by Theorem 3.15.

Case 3. Suppose $(\theta, k) = (\frac{\pi}{4}, 1)$. Then $kG_{\theta,k} = \mathbb{Q}(i)$. By Section 6.4.2 of [27],

$$G_{\frac{\pi}{4},\frac{1}{2}} = \langle S, T, U \mid U^2 = S^4 = T^4 = (US)^2 = (U^{-1}T)^3 = (TS)^2 = 1\rangle$$

where

$$\begin{aligned} S &= AB^2_{\frac{\pi}{4},\frac{1}{2}}AB^{-1}_{\frac{\pi}{4},\frac{1}{2}}, \\ T &= A^2B^2_{\frac{\pi}{4},\frac{1}{2}}AB^{-1}_{\frac{\pi}{4},\frac{1}{2}}, \text{ and} \\ U &= AB^2_{\frac{\pi}{4},\frac{1}{2}}A^{-1}B^{-2}_{\frac{\pi}{4},\frac{1}{2}}A^{-1}B_{\frac{\pi}{4},\frac{1}{2}}. \end{aligned}$$

Then $A = TS^{-1}$ and $B_{\frac{\pi}{4},1} = SUTS^2T^{-1}S^{-1}$, so, using Magma ([8]), we find that $G_{\frac{\pi}{4},1}$ is a subgroup of index 8 in $G_{\frac{\pi}{4},\frac{1}{2}} = \mathrm{PGL}_2(O_1)$.

Case 4. Suppose $(\theta, k) = (\frac{\pi}{4}, \frac{3}{2})$. Then $kG_{\theta,k} = \mathbb{Q}(i)$. Again using the presentation of $G_{\frac{\pi}{4},\frac{1}{2}}$ above, we see that

$$A = TS^{-1} \text{ and } B_{\frac{\pi}{4},\frac{3}{2}} = SUTS^2T^{-1}ST^{-1}S^{-1},$$

so, using Magma ([8]), we find that $G_{\frac{\pi}{4},\frac{3}{2}}$ is a subgroup of index 6 in $G_{\frac{\pi}{4},\frac{1}{2}} = \mathrm{PGL}_2(O_1)$.

Case 5. Suppose $(\theta, k) = (\frac{\pi}{3}, \frac{\sqrt{3}}{2}n)$ for some integer n. Let

$$C = B_{\theta,k}A^{-1}B_{\theta,k}^{-1} = \begin{pmatrix} 1 & 0 \\ e^{-\pi i/3} & 1 \end{pmatrix} \text{ and}$$

$$T = CA^{-1}C^{-1}A^2C^{-1}A^{-1}C = \begin{pmatrix} 1 & 2\sqrt{3}i \\ 0 & 1 \end{pmatrix}.$$

If n is even, then let

$$D = B_{\theta,k}T^{-n/2} = \begin{pmatrix} 0 & -e^{\pi i/6} \\ e^{-\pi i/6} & 0 \end{pmatrix}.$$

Thus, $B_{\theta,k} = DT^{n/2}$ so, since $T \in \langle A, C\rangle$ and $DAD^{-1} = C$, we have $G_{\theta,k} = \langle A, D\rangle$, which is $G_{\theta,k}$ with $(\theta, k) = (\frac{\pi}{3}, 0)$ and is a $\mathbb{Z}_2$-extension of the figure eight knot group as above.

If n is odd, then let

$$D = B_{\theta,k}^{-1}A^{-1}CAC^{-1}T^{-(n-1)/2} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

so $B_{\theta,k} \in \langle A, C, D\rangle$ since $T \in \langle A, C\rangle$. As D conjugates A and C to their inverses and $\langle A, C\rangle$ is the figure-eight knot group, we conclude that $G_{\theta,k} = \langle A, C, D\rangle$ is a $\mathbb{Z}_2$-extension of the figure eight knot group by an involution that conjugates each parabolic generator to its own inverse.

Case 6. Suppose $(\theta, k) = (\frac{\pi}{2}, \frac{1}{2})$. Then let

$$S = A^{-1}B_{\theta,k}A^{-1}B_{\theta,k}^{-1}A^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and}$$

$$T = AB_{\theta,k}AB_{\theta,k}^{-1}AB_{\theta,k} = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}.$$

Then $ST = B_{\theta,k}$, so $G_{\theta,k} = \langle A, S, T\rangle = \mathrm{PSL}_2(O_1)$ by Theorem 3.14.

Case 7. Suppose $(\theta, k) = (\frac{\pi}{2}, \frac{\sqrt{2}}{2})$. Then let

$$S = A^{-1}B_{\theta,k}A^{-1}B_{\theta,k}^{-1}A^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and}$$

$$T = AB_{\theta,k}AB_{\theta,k}^{-1}AB_{\theta,k} = \begin{pmatrix} 1 & \sqrt{-2} \\ 0 & 1 \end{pmatrix}.$$

Then $ST = B_{\theta,k}$, so $G_{\theta,k} = \langle A, S, T\rangle = \mathrm{PSL}_2(O_2)$ by Theorem 3.14.

Case 8. Suppose $(\theta, k) = (\frac{\pi}{2}, \frac{\sqrt{3}}{2})$. Then let

$$S = A^{-1}B_{\theta,k}A^{-1}B_{\theta,k}^{-1}A^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and}$$

$$T = AB_{\theta,k}AB_{\theta,k}^{-1}AB_{\theta,k} = \begin{pmatrix} 1 & \sqrt{-3} \\ 0 & 1 \end{pmatrix}.$$

Then $ST = B_{\theta,k}$, so $G_{\theta,k} = \langle A, S, T\rangle$. Using Theorem 3.14 and Magma ([8]), we conclude that Γ is the unique subgroup of index 10 in the Bianchi group $\mathrm{PSL}_2(O_3)$ (cf. Section 6 of [20]).

■

3.3 Arithmetic Jørgensen Groups of Parabolic Type

To prove Theorem 3.5, we first identify all arithmetic Jørgensen groups of parabolic type by proving the following theorem. Throughout the proof, we note whenever a group of parabolic type (θ, k) is encountered and refer to its identification in Proposition 3.16. We use Theorems 3.14 and 3.15 similarly. To complete the proof of Theorem 3.5, we will show in Section 3.4 that arithmetic Jørgensen groups of elliptic type are cocompact and hence the fourteen arithmetic Jørgensen groups of parabolic type listed below are precisely all non-cocompact arithmetic Jørgensen groups.

Theorem 3.17. *There are exactly fourteen arithmetic Jørgensen groups of parabolic type; they are:*
$\mathrm{PGL}_2(O_1)$, $\mathrm{PGL}_2(O_3)$, $\mathrm{PSL}_2(O_1)$, $\mathrm{PSL}_2(O_2)$, $\mathrm{PSL}_2(O_3)$, $\mathrm{PSL}_2(O_7)$, $\mathrm{PSL}_2(O_{11})$, *two subgroups of index 6 and 8 respectively in* $\mathrm{PGL}_2(O_1)$, *the unique subgroup of index 10 in* $\mathrm{PSL}_2(O_3)$, *the figure-eight knot group, and three* $\mathbb{Z}_2$*-extensions of the figure-eight knot group.*

Proof. Let Γ be an arithmetic Jørgensen group of parabolic type. By definition, there exists a pair of generators (A, B) for Γ with $J(A, B) = 1$ and A parabolic. Conjugate Γ so that

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & -1/\sigma \\ \sigma & \lambda \end{pmatrix} \text{ where } \sigma, \lambda \in \mathbb{C}, \sigma \neq 0$$

(cf. proof of Theorem 3.3 in [10]). Working in $\mathrm{PSL}_2\mathbb{C}$, we may replace B with $-B$ (and λ with $-\lambda$). Hence we assume $\mathrm{Re}\,\sigma > 0$ or $\mathrm{Re}\,\sigma = 0$ and $\mathrm{Im}\,\sigma > 0$.

By Theorem 1.4, $k\Gamma = \mathbb{Q}(\sqrt{-d})$ for some $d \in \mathbb{N}$, so Lemma 1.1 implies $\mathrm{tr}[A, B] - 2 = \sigma^2 \in O_d$. As $J(A, B) = |\sigma^2| = 1$, we see that σ^2 is a unit in O_d. Therefore, given our assumption on σ, we have $\sigma \in \{1, i, e^{\pm\pi i/4}\}$ if $d = 1$,

$\sigma \in \{1, i, e^{\pm \pi i/6}, e^{\pm \pi i/3}\}$ if $d = 3$, and $\sigma \in \{1, i\}$ otherwise.

Suppose $\lambda = 0$. Then $k\Gamma = \mathbb{Q}(\sigma^2)$ by Lemma 1.1, so we must have $\sigma \in \{e^{\pm \pi i/4}\}$ and $d = 1$ or $\sigma \in \{e^{\pm \pi i/6}, e^{\pm \pi i/3}\}$ and $d = 3$ since $k\Gamma = \mathbb{Q}(\sqrt{-d})$.

If $\sigma = e^{-\pi i/4}$ and $d = 1$, then $\Gamma = G_{\theta,k}$ with $(\theta, k) = (\frac{\pi}{4}, 0)$ and thus is not a Jørgensen group by Proposition 3.16.

If $\sigma = e^{\pi i/4}$ and $d = 1$, then $\Gamma = G_{\pi-\theta,k}$ with $(\theta, k) = (\frac{\pi}{4}, 0)$ and thus is not a Jørgensen group by Theorem 3.13 and Proposition 3.16.

If $\sigma = e^{-\pi i/6}$ and $d = 3$, then $\Gamma = G_{\theta,k}$ with $(\theta, k) = (\frac{\pi}{3}, 0)$ and thus is a $\mathbb{Z}_2$-extension of the figure-eight group by Proposition 3.16.

If $\sigma = e^{\pi i/6}$ and $d = 3$, then $\Gamma = G_{\pi-\theta,k}$ with $(\theta, k) = (\frac{\pi}{3}, 0)$ and is a $\mathbb{Z}_2$-extension of the figure-eight group by Theorem 3.15.

If $\sigma = e^{-\pi i/3}$ and $d = 3$, then $\Gamma = G_{\theta,k}$ with $(\theta, k) = (\frac{\pi}{6}, 0)$ and thus is a $\mathbb{Z}_2$-extension of the figure-eight group Proposition 3.16.

If $\sigma = e^{\pi i/3}$ and $d = 3$, then $\Gamma = G_{\pi-\theta,k}$ with $(\theta, k) = (\frac{\pi}{6}, 0)$ and is a $\mathbb{Z}_2$-extension of the figure-eight group by Theorem 3.15.

Now suppose $\lambda \neq 0$. Then $k\Gamma = \mathbb{Q}(\lambda^2, \lambda\sigma)$ by Lemma 1.1, and, as above, $\sigma \in \{1, i, e^{\pm \pi i/4}\}$ if $d = 1$, $\sigma \in \{1, i, e^{\pm \pi i/6}, e^{\pm \pi i/3}\}$ if $d = 3$, and $\sigma \in \{1, i\}$ otherwise. We divide the remainder of the proof into parts according to these possible values.

Part 1. Suppose $\sigma = 1$. Then $k\Gamma = \mathbb{Q}(\lambda) = \mathbb{Q}(\sqrt{-d})$ and $\mathrm{tr}B = \lambda \in O_d$ by Theorem 1.4.

Case 1.1. If $d \not\equiv 3 \mod 4$, then $\lambda = m + n\sqrt{-d}$ for some integers m and n. Replacing B with BA^{-m} (as $\langle A, BA^{-m}\rangle = \langle A, B\rangle$), we assume $\lambda = n\sqrt{-d}$. Then $\Gamma = G_{\theta,k}$ with $(\theta, k) = (\frac{\pi}{2}, \frac{\sqrt{d}}{2}n)$, so, by Proposition 3.16, $(\theta, k) = (\frac{\pi}{2}, \frac{1}{2})$ or $(\frac{\pi}{2}, \frac{\sqrt{2}}{2})$, which corresponds to $\mathrm{PSL}_2(O_1)$ or $\mathrm{PSL}_2(O_2)$ respectively.

Case 1.2. Now suppose $d \equiv 3 \mod 4$. Then $\lambda = m + n\frac{1+\sqrt{-d}}{2}$ for some integers m and n.

Subcase 1.2.1. If n is even, replace B with $BA^{-m-n/2}$ and assume $\lambda = \frac{n\sqrt{-d}}{2}$. Then $\Gamma = G_{\theta,k}$ with $(\theta, k) = (\frac{\pi}{2}, \frac{\sqrt{d}}{4}n)$, so, by Proposition 3.16, $(\theta, k) = (\frac{\pi}{2}, \frac{\sqrt{3}}{2})$, which corresponds to the unique subgroup of index 10 in $\mathrm{PSL}_2(O_3)$.

Subcase 1.2.2. If n is odd, replace B with $BA^{-m-\frac{n-1}{2}}$ and assume $\lambda = \frac{1+n\sqrt{-d}}{2}$. We now see that there exist such Γ that are arithmetic Jørgensen groups of parabolic type but are not identified in Proposition 3.16 (and hence Γ is not of the form $G_{\theta,k}$). Let

$$\begin{aligned} S &= A^{-1}BA^{-1}B^{-1}A^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ T &= S^{-1}B = \begin{pmatrix} 1 & \frac{1+n\sqrt{-d}}{2} \\ 0 & 1 \end{pmatrix}, \text{ and} \\ U &= TA^{-1} = \begin{pmatrix} 1 & \frac{-1+n\sqrt{-d}}{2} \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Then $ST = B$ and $UA = T$, so $\Gamma = \langle A, S, T\rangle = \langle A, S, U\rangle$. If $n < 0$, replace T with U^{-1}, so assume $n > 0$.

If $n = 1$ and $d = 3$, 7, or 11, then Theorem 3.14 identifies Γ as $\mathrm{PSL}_2(O_3)$, $\mathrm{PSL}_2(O_7)$, or $\mathrm{PSL}_2(O_{11})$ respectively. Thus, these Bianchi groups are arithmetic Jørgensen groups of parabolic type but are not identified as such in Proposition 3.16. Hence, $\mathrm{PSL}_2(O_3)$, $\mathrm{PSL}_2(O_7)$, and $\mathrm{PSL}_2(O_{11})$ are not of the form $G_{\theta,k}$.

Now assume $n > 1$ or $d > 11$ and consider the polygon P in Figure 3.1 with sides

$$F_A = \left\{(x,y,t) \in \mathbb{H}^3 \mid x = -\frac{1}{2},\ \frac{1-n^2d}{4n\sqrt{d}} \le y \le \frac{n^2d-1}{4n\sqrt{d}},\ x^2+y^2+t^2 \ge 1\right\}$$

$$F_{A^{-1}} = \left\{(x,y,t) \in \mathbb{H}^3 \mid x = \frac{1}{2},\ \frac{1-n^2d}{4n\sqrt{d}} \le y \le \frac{n^2d-1}{4n\sqrt{d}},\ x^2+y^2+t^2 \ge 1\right\}$$

$$F_S = \left\{(x,y,t) \in \mathbb{H}^3 \mid -\frac{1}{2} \le x \le 0,\ x^2+y^2+t^2 = 1\right\}$$

$$F_{S^{-1}} = \left\{ (x,y,t) \in \mathbb{H}^3 \mid 0 \leq x \leq \frac{1}{2},\ x^2+y^2+t^2 = 1 \right\}$$

$$F_T = \left\{ (x,y,t) \in \mathbb{H}^3 \mid -\frac{1}{2} \leq x \leq 0,\ y = \frac{1-n^2d-4x}{4n\sqrt{d}} \right\}$$

$$F_{T^{-1}} = \left\{ (x,y,t) \in \mathbb{H}^3 \mid 0 \leq x \leq \frac{1}{2},\ y = \frac{n^2d-1-4x}{4n\sqrt{d}} \right\}$$

$$F_U = \left\{ (x,y,t) \in \mathbb{H}^3 \mid 0 \leq x \leq \frac{1}{2},\ y = \frac{1+n^2d+4x}{4n\sqrt{d}} \right\}$$

$$F_{U^{-1}} = \left\{ (x,y,t) \in \mathbb{H}^3 \mid -\frac{1}{2} \leq x \leq 0,\ y = \frac{n^2d+1+4x}{4n\sqrt{d}} \right\}$$

and edges

$$e_{(1),\pi} = F_S \cap F_{S^{-1}} = \left\{ (x,y,t) \in \mathbb{H}^3 \mid x = 0,\ y^2+t^2 = 1 \right\}$$

$$e_{(2,1),\pi/3} = F_A \cap F_S = \left\{ (x,y,t) \in \mathbb{H}^3 \mid x = -\frac{1}{2},\ y^2+t^2 = \frac{3}{4} \right\}$$

$$e_{(2,2),\pi/3} = F_{A^{-1}} \cap F_{S^{-1}} = \left\{ (x,y,t) \in \mathbb{H}^3 \mid x = \frac{1}{2},\ y^2+t^2 = \frac{3}{4} \right\}$$

$$e_{(3,1),\theta} = F_A \cap F_{U^{-1}} = \left\{ (x,y,t) \in \mathbb{H}^3 \mid x = -\frac{1}{2},\ y = \frac{n^2d-1}{4n\sqrt{d}} \right\}$$

$$e_{(3,2),\theta} = F_{A^{-1}} \cap F_{T^{-1}} = \left\{ (x,y,t) \in \mathbb{H}^3 \mid x = \frac{1}{2},\ y = \frac{n^2d-1}{4n\sqrt{d}} \right\}$$

$$e_{(3,3),\phi} = F_T \cap F_U = \left\{ (x,y,t) \in \mathbb{H}^3 \mid x = 0,\ y = -\frac{n^2d+1}{4n\sqrt{d}} \right\}$$

$$e_{(4,1),\theta} = F_A \cap F_T = \left\{ (x,y,t) \in \mathbb{H}^3 \mid x = -\frac{1}{2},\ y = \frac{1-n^2d}{4n\sqrt{d}} \right\}$$

$$e_{(4,2),\theta} = F_{A^{-1}} \cap F_U = \left\{ (x,y,t) \in \mathbb{H}^3 \mid x = \frac{1}{2},\ y = \frac{1-n^2d}{4n\sqrt{d}} \right\}$$

$$e_{(4,3),\phi} = F_{T^{-1}} \cap F_{U^{-1}} = \left\{ (x,y,t) \in \mathbb{H}^3 \mid x = 0,\ y = \frac{n^2d+1}{4n\sqrt{d}} \right\}.$$

Then $X(F_X) = F_{X^{-1}}$ for $X \in \{A,S,T,U\}$ are side pairing transformations

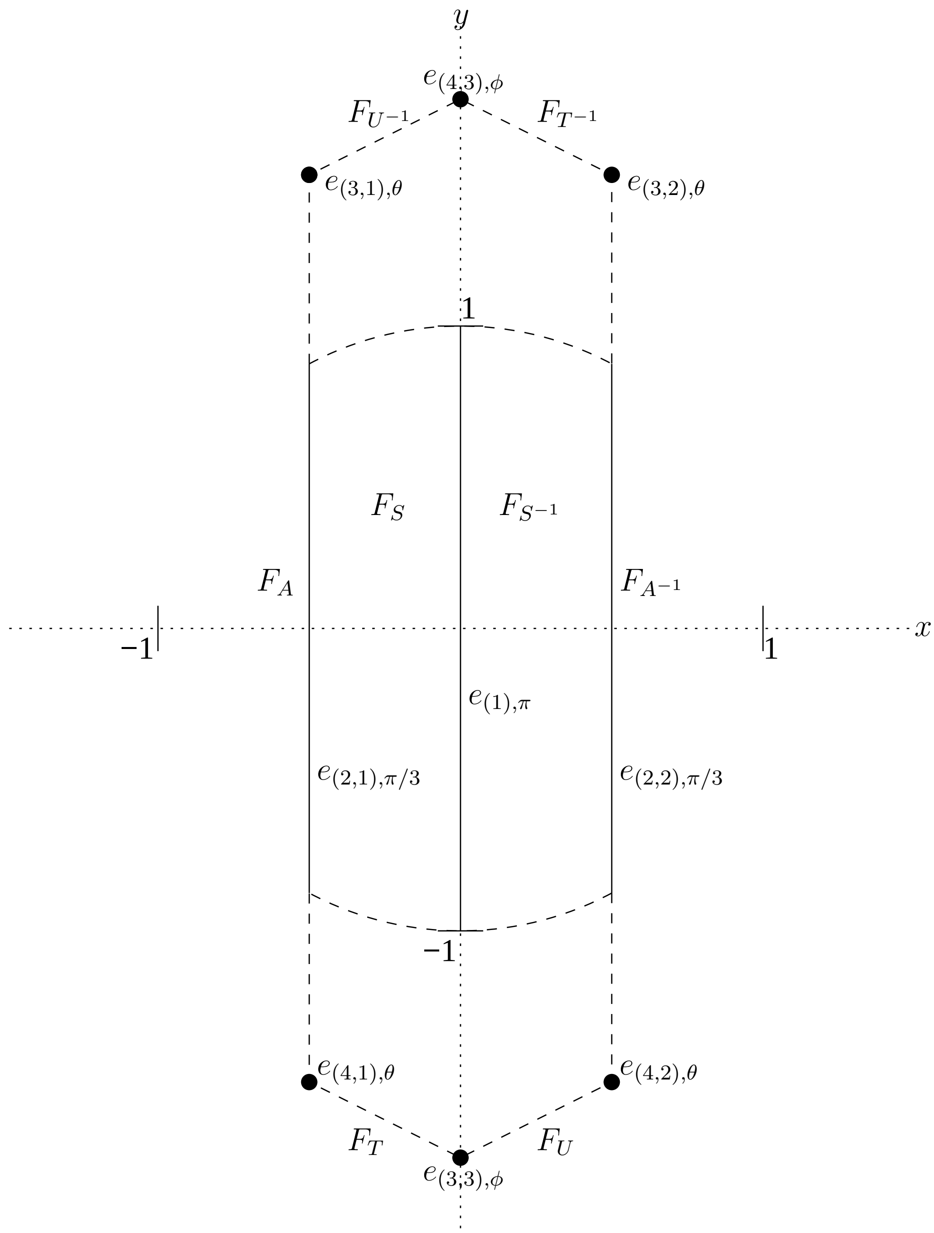

Figure 3.1: The polygon P viewed from the point at ∞ above. Solid lines denote edges, solid dots denote edges parallel to the t-axis, dashed lines are in $\partial\mathbb{H}^3$, and dotted lines denote the x- and y-axes in $\partial\mathbb{H}^3$. The sides F_S and $F_{S^{-1}}$ are on the unit sphere centered at the origin; all other sides are perpendicular to $\partial\mathbb{H}^3$.

with the following four cycle transformations:

1. $e_{(1),\pi} \xrightarrow{S} \circlearrowleft^{2}_{\pi}$

2. $e_{(2,1),\pi/3} \xrightarrow{A} e_{(2,2),\pi/3} \xrightarrow{S^{-1}} \circlearrowleft^{3}_{2\pi/3}$

3. $e_{(3,1),\theta} \xrightarrow{A} e_{(3,2),\theta} \xrightarrow{T^{-1}} e_{(3,3),\phi} \xrightarrow{U} \circlearrowleft^{1}_{2\pi}$

4. $e_{(4,1),\theta} \xrightarrow{A} e_{(4,2),\theta} \xrightarrow{U} e_{(3,3),\phi} \xrightarrow{T^{-1}} \circlearrowleft^{1}_{2\pi}$

Therefore, by Poincaré's Polyhedron Theorem, Γ has the presentation

$$\Gamma = \langle A, S, T, U \mid S^2 = (S^{-1}A)^3 = UT^{-1}A = T^{-1}UA = 1 \rangle,$$

and P is the fundamental polyhedron for Γ. Clearly P has infinite volume, so Γ is not arithmetic if $n > 1$ or $d > 11$.

Part 2. Suppose $\sigma = i$. Then $k\Gamma = \mathbb{Q}(\lambda i) = \mathbb{Q}(\sqrt{-d})$ and

$$\mathrm{tr}B\mathrm{tr}AB - \mathrm{tr}^2 B = \lambda i \in O_d$$

by Theorem 1.4.

Case 2.1. If $d \not\equiv 3 \mod 4$, then $\lambda i = m + n\sqrt{d}i$ for some integers m and n, so $\lambda = n\sqrt{d} - mi$. Replacing B with BA^m, we assume $\lambda = n\sqrt{d}$. Then $\Gamma = G_{\pi+\theta,k} = G_{\theta,k}$ with $(\theta, k) = (0, \frac{n\sqrt{d}}{2})$, so Γ is not a Jørgensen group in this case by Theorem 3.13 and Proposition 3.16.

Case 2.2. Now suppose $d \equiv 3 \mod 4$. Then $\lambda i = m + n\frac{1+\sqrt{d}i}{2}$ for some integers m and n, so $\lambda = \frac{n\sqrt{d}}{2} - (m + n/2)i$.

Subcase 2.2.1. If n is even, replace B with $BA^{m+n/2}$ and assume $\lambda = \frac{n\sqrt{d}}{2}$. Then $\Gamma = G_{\pi+\theta,k} = G_{\theta,k}$ with $(\theta, k) = (0, \frac{n\sqrt{d}}{4})$, so Γ is not a Jørgensen group in this case by Theorem 3.13 and Proposition 3.16.

Subcase 2.2.2. If n is odd, replace B with $BA^{m+\frac{n+1}{2}}$ and assume $\lambda = \frac{n\sqrt{d}+i}{2}$. We now see that no such Γ is an arithmetic Jørgensen group of parabolic type. Let

$$\begin{aligned} S &= A^{-1}BAB^{-1}A^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ T &= S^{-1}B = \begin{pmatrix} i & \frac{n\sqrt{d}+i}{2} \\ 0 & -i \end{pmatrix}, \text{ and} \\ U &= TA^{-1} = \begin{pmatrix} i & \frac{n\sqrt{d}-i}{2} \\ 0 & -i \end{pmatrix}. \end{aligned}$$

Then $ST = B$ and $UA = T$, so $\Gamma = \langle A, S, T\rangle = \langle A, S, U\rangle$. If $n < 0$, replace T with U^{-1}, so assume $n > 0$.

First assume $n > 1$ or $d > 11$ and consider the polygon P in Figure 3.2 with sides

$$F_A = \left\{(x,y,t) \in \mathbb{H}^3 \mid x = -\frac{1}{2},\ y \leq \frac{n^2d-1}{4n\sqrt{d}},\ x^2+y^2+t^2 \geq 1\right\}$$

$$F_{A^{-1}} = \left\{(x,y,t) \in \mathbb{H}^3 \mid x = \frac{1}{2},\ y \leq \frac{n^2d-1}{4n\sqrt{d}},\ x^2+y^2+t^2 \geq 1\right\}$$

$$F_S = \left\{(x,y,t) \in \mathbb{H}^3 \mid -\frac{1}{2} \leq x \leq 0,\ x^2+y^2+t^2 = 1\right\}$$

$$F_{S^{-1}} = \left\{(x,y,t) \in \mathbb{H}^3 \mid 0 \leq x \leq \frac{1}{2},\ x^2+y^2+t^2 = 1\right\}$$

$$F_T = \left\{(x,y,t) \in \mathbb{H}^3 \mid -\frac{1}{2} \leq x \leq -\frac{1}{4},\ y = \frac{1+n^2d+4x}{4n\sqrt{d}}\right\}$$

$$F_{T^{-1}} = \left\{(x,y,t) \in \mathbb{H}^3 \mid -\frac{1}{4} \leq x \leq 0,\ y = \frac{1+n^2d+4x}{4n\sqrt{d}}\right\}$$

$$F_U = \left\{(x,y,t) \in \mathbb{H}^3 \mid \frac{1}{4} \leq x \leq \frac{1}{2},\ y = \frac{n^2d+1-4x}{4n\sqrt{d}}\right\}$$

$$F_{U^{-1}} = \left\{(x,y,t) \in \mathbb{H}^3 \mid 0 \leq x \leq \frac{1}{4},\ y = \frac{n^2d+1-4x}{4n\sqrt{d}}\right\}$$

and edges

$$e_{(1),\pi} = F_S \cap F_{S^{-1}} = \left\{(x,y,t) \in \mathbb{H}^3 \mid x = 0,\ y^2 + t^2 = 1\right\}$$

$$e_{(2,1),\pi/3} = F_A \cap F_S = \left\{(x,y,t) \in \mathbb{H}^3 \mid x = -\frac{1}{2},\ y^2 + t^2 = \frac{3}{4}\right\}$$

$$e_{(2,2),\pi/3} = F_{A^{-1}} \cap F_{S^{-1}} = \left\{(x,y,t) \in \mathbb{H}^3 \mid x = \frac{1}{2},\ y^2 + t^2 = \frac{3}{4}\right\}$$

$$e_{(3,1),\theta} = F_A \cap F_T = \left\{(x,y,t) \in \mathbb{H}^3 \mid x = -\frac{1}{2},\ y = \frac{n^2 d - 1}{4n\sqrt{d}}\right\}$$

$$e_{(3,2),\theta} = F_{A^{-1}} \cap F_U = \left\{(x,y,t) \in \mathbb{H}^3 \mid x = \frac{1}{2},\ y = \frac{n^2 d - 1}{4n\sqrt{d}}\right\}$$

$$e_{(3,3),\phi} = F_{T^{-1}} \cap F_{U^{-1}} = \left\{(x,y,t) \in \mathbb{H}^3 \mid x = 0,\ y = \frac{n^2 d + 1}{4n\sqrt{d}}\right\}$$

$$e_{(4),\pi} = F_T \cap F_{T^{-1}} = \left\{(x,y,t) \in \mathbb{H}^3 \mid x = -\frac{1}{4},\ y = \frac{n\sqrt{d}}{4}\right\}$$

$$e_{(5),\pi} = F_U \cap F_{U^{-1}} = \left\{(x,y,t) \in \mathbb{H}^3 \mid x = \frac{1}{4},\ y = \frac{n\sqrt{d}}{4}\right\}.$$

Then $X(F_X) = F_{X^{-1}}$ for $X \in \{A, S, T, U\}$ are side pairing transformations with the following five cycle transformations:

1. $e_{(1),\pi} \xrightarrow{S} \circlearrowleft^2_\pi$
2. $e_{(2,1),\pi/3} \xrightarrow{A} e_{(2,2),\pi/3} \xrightarrow{S^{-1}} \circlearrowleft^3_{2\pi/3}$
3. $e_{(3,1),\theta} \xrightarrow{A} e_{(3,2),\theta} \xrightarrow{U} e_{(3,3),\phi} \xrightarrow{T^{-1}} \circlearrowleft^1_{2\pi}$
4. $e_{(4),\pi} \xrightarrow{T} \circlearrowleft^2_\pi$
5. $e_{(5),\pi} \xrightarrow{U} \circlearrowleft^2_\pi$

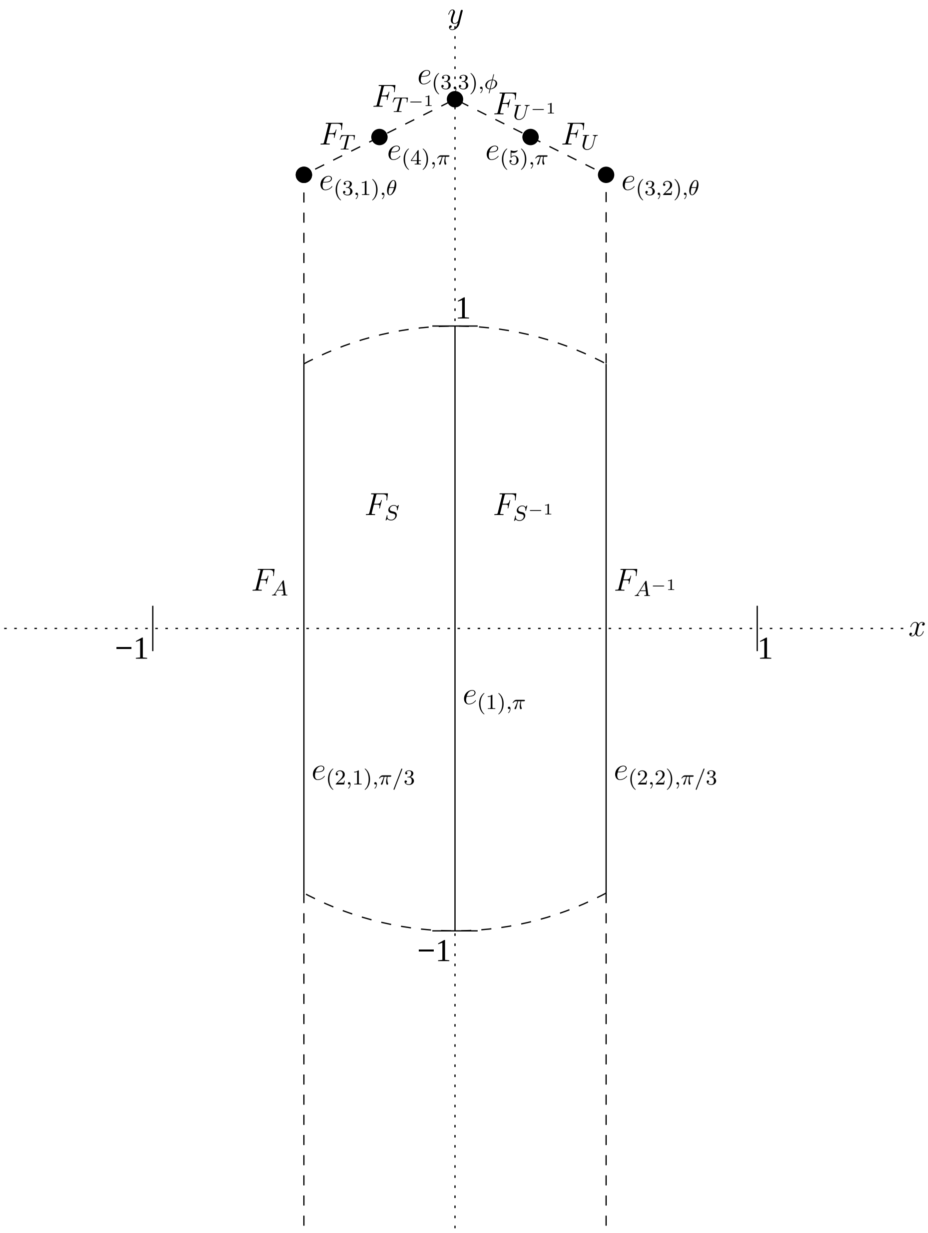

Figure 3.2: The polygon P viewed from the point at ∞ above. Solid lines denote edges, solid dots denote edges parallel to the t-axis, dashed lines are in $\partial\mathbb{H}^3$, and dotted lines denote the x- and y-axes in $\partial\mathbb{H}^3$. The sides F_S and $F_{S^{-1}}$ are on the unit sphere centered at the origin; all other sides are perpendicular to $\partial\mathbb{H}^3$.

Therefore, by Poincaré's Polyhedron Theorem, Γ has the presentation

$$\Gamma = \langle A, S, T, U \mid S^2 = (S^{-1}A)^3 = T^{-1}UA = T^2 = U^2 = 1 \rangle,$$

and P is the fundamental polyhedron for Γ. Clearly P has infinite volume, so Γ is not arithmetic if $n > 1$ or $d > 11$.

Now assume $n = 1$ and $d = 3, 7$, or 11 and consider the polygon P in Figure 3.3 with sides

$$F_A = \left\{ (x, y, t) \in \mathbb{H}^3 \mid x = -\frac{1}{2},\ y \leq \frac{n^2 d - 1}{4n\sqrt{d}},\ x^2 + y^2 + t^2 \geq 1 \right\}$$

$$F_{A^{-1}} = \left\{ (x, y, t) \in \mathbb{H}^3 \mid x = \frac{1}{2},\ y \leq \frac{n^2 d - 1}{4n\sqrt{d}},\ x^2 + y^2 + t^2 \geq 1 \right\}$$

$$F_S = \left\{ (x, y, t) \in \mathbb{H}^3 \mid -\frac{1}{2} \leq x \leq 0,\ y \leq \frac{n^2 d + 1 + 4x}{4n\sqrt{d}},\ x^2 + y^2 + t^2 = 1 \right\}$$

$$F_{S^{-1}} = \left\{ (x, y, t) \in \mathbb{H}^3 \mid 0 \leq x \leq \frac{1}{2},\ y \leq \frac{n^2 d + 1 - 4x}{4n\sqrt{d}},\ x^2 + y^2 + t^2 = 1 \right\}$$

$$F_T = \left\{ (x, y, t) \in \mathbb{H}^3 \mid -\frac{1}{2} \leq x \leq -\frac{1}{4},\ y = \frac{1 + n^2 d + 4x}{4n\sqrt{d}},\ x^2 + y^2 + t^2 \geq 1 \right\}$$

$$F_{T^{-1}} = \left\{ (x, y, t) \in \mathbb{H}^3 \mid -\frac{1}{4} \leq x \leq 0,\ y = \frac{1 + n^2 d + 4x}{4n\sqrt{d}},\ x^2 + y^2 + t^2 \geq 1 \right\}$$

$$F_U = \left\{ (x, y, t) \in \mathbb{H}^3 \mid \frac{1}{4} \leq x \leq \frac{1}{2},\ y = \frac{n^2 d + 1 - 4x}{4n\sqrt{d}},\ x^2 + y^2 + t^2 \geq 1 \right\}$$

$$F_{U^{-1}} = \left\{ (x, y, t) \in \mathbb{H}^3 \mid 0 \leq x \leq \frac{1}{4},\ y = \frac{n^2 d + 1 - 4x}{4n\sqrt{d}},\ x^2 + y^2 + t^2 \geq 1 \right\}$$

and edges

$$\begin{aligned} e_{(1),\pi} &= F_S \cap F_{S^{-1}} \\ &= \left\{ (x, y, t) \in \mathbb{H}^3 \mid x = 0,\ y \leq \frac{n^2 d + 1}{4n\sqrt{d}},\ y^2 + t^2 = 1 \right\} \end{aligned}$$

$$\begin{aligned} e_{(2,1),\pi/3} &= F_A \cap F_S \\ &= \left\{(x,y,t) \in \mathbb{H}^3 \mid x = -\frac{1}{2},\ y \leq \frac{n^2 d - 1}{4n\sqrt{d}},\ y^2 + t^2 = \frac{3}{4}\right\} \end{aligned}$$

$$\begin{aligned} e_{(2,2),\pi/3} &= F_{A^{-1}} \cap F_{S^{-1}} \\ &= \left\{(x,y,t) \in \mathbb{H}^3 \mid x = \frac{1}{2},\ y \leq \frac{n^2 d - 1}{4n\sqrt{d}},\ y^2 + t^2 = \frac{3}{4}\right\} \end{aligned}$$

$$\begin{aligned} e_{(3,1),\theta} &= F_A \cap F_T \\ &= \left\{(x,y,t) \in \mathbb{H}^3 \mid x = -\frac{1}{2},\ y = \frac{n^2 d - 1}{4n\sqrt{d}},\ x^2 + y^2 + t^2 = 1\right\} \end{aligned}$$

$$\begin{aligned} e_{(3,2),\theta} &= F_{A^{-1}} \cap F_U \\ &= \left\{(x,y,t) \in \mathbb{H}^3 \mid x = \frac{1}{2},\ y = \frac{n^2 d - 1}{4n\sqrt{d}},\ x^2 + y^2 + t^2 = 1\right\} \end{aligned}$$

$$\begin{aligned} e_{(3,3),\phi} &= F_{T^{-1}} \cap F_{U^{-1}} \\ &= \left\{(x,y,t) \in \mathbb{H}^3 \mid x = 0,\ y = \frac{n^2 d + 1}{4n\sqrt{d}},\ x^2 + y^2 + t^2 = 1\right\} \end{aligned}$$

$$\begin{aligned} e_{(4),\pi} &= F_T \cap F_{T^{-1}} \\ &= \left\{(x,y,t) \in \mathbb{H}^3 \mid x = -\frac{1}{4},\ y = \frac{n\sqrt{d}}{4},\ x^2 + y^2 + t^2 = 1\right\} \end{aligned}$$

$$\begin{aligned} e_{(5),\pi} &= F_U \cap F_{U^{-1}} \\ &= \left\{(x,y,t) \in \mathbb{H}^3 \mid x = \frac{1}{4},\ y = \frac{n\sqrt{d}}{4},\ x^2 + y^2 + t^2 = 1\right\} \end{aligned}$$

$$\begin{aligned} e_{(6,1),\psi} &= F_T \cap F_S \\ &= \left\{(x,y,t) \in \mathbb{H}^3 \mid -\frac{1}{2} \leq x \leq -\frac{1}{4},\ y = \frac{1+n^2d+4x}{4n\sqrt{d}},\ x^2+y^2+t^2=1\right\} \end{aligned}$$

$$\begin{aligned} e_{(6,2),\psi} &= F_{T^{-1}} \cap F_S \\ &= \left\{(x,y,t) \in \mathbb{H}^3 \mid -\frac{1}{4} \leq x \leq 0,\ y = \frac{1+n^2d+4x}{4n\sqrt{d}},\ x^2+y^2+t^2=1\right\} \end{aligned}$$

$$\begin{aligned} e_{(6,3),\psi} &= F_{U^{-1}} \cap F_{S^{-1}} \\ &= \left\{(x,y,t) \in \mathbb{H}^3 \mid 0 \leq x \leq \frac{1}{4},\ y = \frac{n^2d+1-4x}{4n\sqrt{d}},\ x^2+y^2+t^2=1\right\} \end{aligned}$$

$$\begin{aligned} e_{(6,4),\psi} &= F_U \cap F_{S^{-1}} \\ &= \left\{(x,y,t) \in \mathbb{H}^3 \mid \frac{1}{4} \leq x \leq \frac{1}{2},\ y = \frac{n^2d+1-4x}{4n\sqrt{d}},\ x^2+y^2+t^2=1\right\}. \end{aligned}$$

Then $X(F_X) = F_{X^{-1}}$ for $X \in \{A,S,T,U\}$ are side pairing transformations with the following six cycle transformations:

1. $e_{(1),\pi} \xrightarrow{S} \circlearrowleft_{\pi}^{2}$
2. $e_{(2,1),\pi/3} \xrightarrow{A} e_{(2,2),\pi/3} \xrightarrow{S^{-1}} \circlearrowleft_{2\pi/3}^{3}$
3. $e_{(3,1),\theta} \xrightarrow{A} e_{(3,2),\theta} \xrightarrow{U} e_{(3,3),\phi} \xrightarrow{T^{-1}} \circlearrowleft_{2\pi}^{1}$
4. $e_{(4),\pi} \xrightarrow{T} \circlearrowleft_{\pi}^{2}$
5. $e_{(5),\pi} \xrightarrow{U} \circlearrowleft_{\pi}^{2}$
6. $e_{(6,1),\psi} \xrightarrow{T} e_{(6,2),\psi} \xrightarrow{S} e_{(6,3),\psi} \xrightarrow{U} e_{(6,4),\psi} \xrightarrow{S} \circlearrowleft_{2\pi/q}^{q}$ where $q = 3$ if $d = 3$ or 11 and $q = 2$ if $d = 7$.

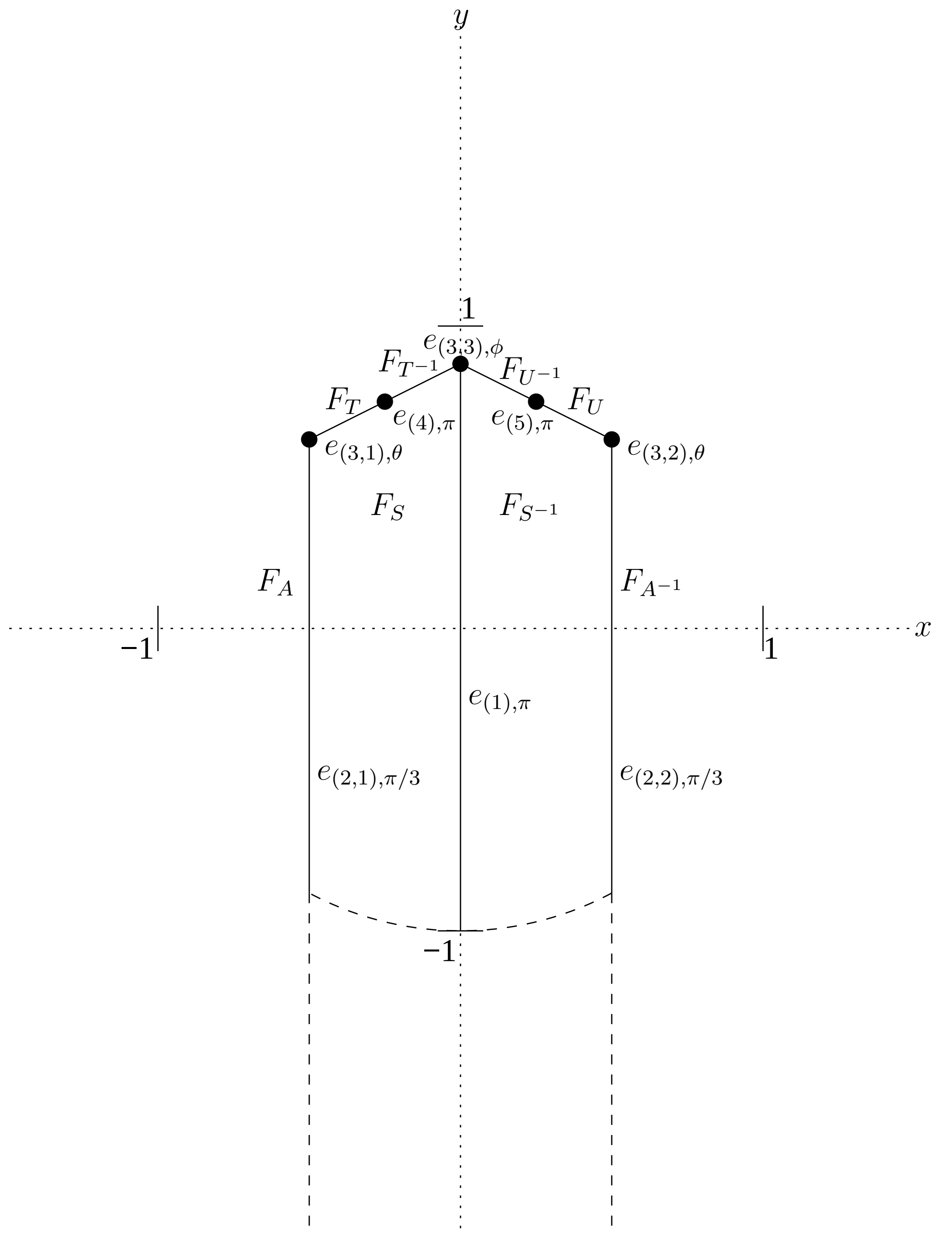

Figure 3.3: The polygon P viewed from the point at ∞ above. Solid lines denote edges, solid dots denote edges parallel to the t-axis, dashed lines are in $\partial\mathbb{H}^3$, and dotted lines denote the x- and y-axes in $\partial\mathbb{H}^3$. The sides F_S and $F_{S^{-1}}$ are on the unit sphere centered at the origin; all other sides are perpendicular to $\partial\mathbb{H}^3$.

Therefore, by Poincaré's Polyhedron Theorem, Γ has the presentation

$$\Gamma = \langle A, S, T, U \mid S^2 = (S^{-1}A)^3 = T^{-1}UA = T^2 = U^2 = (SUST)^q = 1\rangle,$$

and P is the fundamental polyhedron for Γ. Clearly P has infinite volume, so Γ is not arithmetic if $n = 1$ and $d = 3, 7$, or 11.

Part 3. If $\sigma = e^{-\pi i/4}$ and $d = 1$, then $k\Gamma = \mathbb{Q}(\lambda^2, \lambda e^{-\pi i/4}) = \mathbb{Q}(i)$ and

$$\mathrm{tr}B\mathrm{tr}AB - \mathrm{tr}^2B = \lambda e^{-\pi i/4} \in O_1$$

by Theorem 1.4. Thus, $\lambda e^{-\pi i/4} = m + ne^{-\pi i/2}$ for some integers m and n, so $\lambda = me^{\pi i/4} + ne^{-\pi i/4}$. Replacing B with BA^{-n}, we assume $\lambda = me^{\pi i/4}$. Then $\Gamma = G_{\theta,k}$ with $(\theta, k) = (\frac{\pi}{4}, \frac{m}{2})$, so, by Proposition 3.16, $m = 1$, 2, or 3, in which case Γ is $\mathrm{PGL}_2(O_1)$, a subgroup of index 8 in $\mathrm{PGL}_2(O_1)$, or a subgroup of index 6 in $\mathrm{PGL}_2(O_1)$ respectively.

Part 4. If $\sigma = e^{\pi i/4}$ and $d = 1$, then $k\Gamma = \mathbb{Q}(\lambda^2, \lambda e^{\pi i/4}) = \mathbb{Q}(i)$ and

$$\mathrm{tr}B\mathrm{tr}AB - \mathrm{tr}^2B = \lambda e^{\pi i/4} \in O_1$$

by Theorem 1.4. Thus, $\lambda e^{\pi i/4} = m + ne^{\pi i/2}$ for some integers m and n, so $\lambda = me^{-\pi i/4} + ne^{\pi i/4}$. Replacing B with BA^{-n}, we assume $\lambda = me^{-\pi i/4}$. Using Theorem 3.13, we have $\Gamma = G_{\pi+\theta,k} = G_{\theta,k}$ with $(\theta, k) = (\frac{\pi}{4}, -\frac{m}{2})$, so, by Proposition 3.16, $m = 1$, 2, or 3, in which case Γ is $\mathrm{PGL}_2(O_1)$, a subgroup of index 8 in $\mathrm{PGL}_2(O_1)$, or a subgroup of index 6 in $\mathrm{PGL}_2(O_1)$ respectively.

Part 5. Suppose $\sigma = e^{-\pi i/6}$ and $d = 3$. Then $k\Gamma = \mathbb{Q}(\lambda^2, \lambda e^{-\pi i/6}) = \mathbb{Q}(\sqrt{-3})$ and

$$(\mathrm{tr}B\mathrm{tr}AB - \mathrm{tr}^2B)(\mathrm{tr}[A, B] - 2) = \lambda e^{-\pi i/2} \in O_3$$

by Theorem 1.4. Hence, $\lambda e^{-\pi i/2} = m + ne^{-2\pi i/3}$ for some integers m and n, so $\lambda = mi + ne^{-\pi i/6}$. Replacing B with BA^{-n}, we assume $\lambda = mi$.

Case 5.1. If m is even, then replace B by $BA^{m/2}$ and assume

$$\lambda = mi + \frac{m}{2}e^{-\pi i/6} = \frac{m\sqrt{3}}{2}e^{\pi i/3}.$$

Hence, $\Gamma = G_{\theta,k}$ with $(\theta, k) = (\frac{\pi}{3}, \frac{m\sqrt{3}}{4})$, so m can be any even integer by Theorem 3.13 and Proposition 3.16, and Γ is a $\mathbb{Z}_2$-extension of the figure-eight knot group by an involution that conjugates one parabolic generator to the other (or its inverse) if $m \equiv 0 \mod 4$ and by an involution that conjugates each parabolic generator to its own inverse if $m \equiv 2 \mod 4$.

Case 5.2. Suppose m is odd. We now see that such Γ are arithmetic Jørgensen groups of parabolic type but are not identified in Proposition 3.16 (and hence Γ is not of the form $G_{\theta,k}$). Let

$$C = BA^{-1}B^{-1} = \begin{pmatrix} 1 & 0 \\ e^{-\pi i/3} & 1 \end{pmatrix} \text{ and}$$

$$T = C^{-1}ACA^{-2}CAC^{-1} = \begin{pmatrix} 1 & -2\sqrt{-3} \\ 0 & 1 \end{pmatrix}.$$

Subcase 5.2.1. If $m \equiv 1 \mod 4$, then let

$$D = BA^{\frac{m+3}{2}}T^{\frac{m-1}{4}} = \begin{pmatrix} 0 & -e^{\pi i/6} \\ e^{-\pi i/6} & \sqrt{3} \end{pmatrix},$$

so $B \in \langle A, D\rangle$ since $T \in \langle A, C\rangle$ and $C = DAD^{-1}$. Since D is elliptic of order 6, we conclude that $\Gamma = \langle A, D\rangle$ is $\mathrm{PGL}_2(O_3)$ by Theorem 3.15. Thus, $\mathrm{PGL}_2(O_3)$ is an arithmetic Jørgensen group of parabolic type but is not identified in Proposition 3.16. Hence, $\mathrm{PGL}_2(O_3)$ is not of the form $G_{\theta,k}$.

Subcase 5.2.2. If $m \equiv 3 \mod 4$, then let

$$D = BA^{\frac{m-3}{2}}T^{\frac{m+1}{4}} = \begin{pmatrix} 0 & -e^{\pi i/6} \\ e^{-\pi i/6} & -\sqrt{3} \end{pmatrix},$$

so $B \in \langle A, D\rangle$ since $T \in \langle A, C\rangle$ and $C = DAD^{-1}$. Since D is elliptic of order 6, we conclude that $\Gamma = \langle A, D\rangle$ is $\mathrm{PGL}_2(O_3)$ by Theorem 3.15. Thus, $\mathrm{PGL}_2(O_3)$ is an arithmetic Jørgensen group of parabolic type but is not identified in Proposition 3.16. Hence, $\mathrm{PGL}_2(O_3)$ is not of the form $G_{\theta,k}$.

Part 6. Suppose $\sigma = e^{\pi i/6}$ and $d = 3$. Then $k\Gamma = \mathbb{Q}(\lambda^2, \lambda e^{\pi i/6}) = \mathbb{Q}(\sqrt{-3})$ and

$$(\mathrm{tr}B\mathrm{tr}AB - \mathrm{tr}^2B)(\mathrm{tr}[A, B] - 2) = \lambda e^{\pi i/2} \in O_3$$

by Theorem 1.4. Hence, $\lambda e^{\pi i/2} = m + ne^{2\pi i/3}$ for some integers m and n, so $\lambda = ne^{\pi i/6} - mi$. Replacing B with BA^{-n}, we assume $\lambda = -mi$.

Case 6.1. If m is even, then replace B by $BA^{m/2}$ and assume

$$\lambda = \frac{m}{2}e^{\pi i/6} - mi = -\frac{m\sqrt{3}}{2}e^{2\pi i/3}.$$

Hence, $\Gamma = G_{\pi-\theta,-k}$ with $(\theta, k) = (\frac{\pi}{3}, \frac{m\sqrt{3}}{4})$, so m can be any even integer by Theorem 3.13 and Proposition 3.16, and Γ is again a $\mathbb{Z}_2$-extension of the figure-eight knot group in this case.

Case 6.2. Suppose m is odd. We now see that such Γ are arithmetic Jørgensen groups of parabolic type but are not identified in Proposition 3.16 (and hence Γ is not of the form $G_{\theta,k}$). Let

$$C = BA^{-1}B^{-1} = \begin{pmatrix} 1 & 0 \\ e^{\pi i/3} & 1 \end{pmatrix} \text{ and}$$

$$T = C^{-1}ACA^{-2}CAC^{-1} = \begin{pmatrix} 1 & 2\sqrt{-3} \\ 0 & 1 \end{pmatrix}.$$

Subcase 6.2.1. If $m \equiv 1 \mod 4$, then let

$$D = BA^{\frac{m+3}{2}}T^{\frac{m-1}{4}} = \begin{pmatrix} 0 & -e^{-\pi i/6} \\ e^{\pi i/6} & \sqrt{3} \end{pmatrix},$$

so $B \in \langle A, D\rangle$ since $T \in \langle A, C\rangle$ and $C = DA^{-1}D^{-1}$. Since D is elliptic of order 6, we conclude that $\Gamma = \langle A, D\rangle$ is $\mathrm{PGL}_2(O_3)$ by Theorem 3.15. Thus, $\mathrm{PGL}_2(O_3)$ is an arithmetic Jørgensen group of parabolic type but is not identified in Proposition 3.16. Hence, $\mathrm{PGL}_2(O_3)$ is not of the form $G_{\theta,k}$.

Subcase 6.2.2. If $m \equiv 3 \mod 4$, then let

$$D = BA^{\frac{m-3}{2}}T^{\frac{m+1}{4}} = \begin{pmatrix} 0 & -e^{-\pi i/6} \\ e^{\pi i/6} & -\sqrt{3} \end{pmatrix},$$

so $B \in \langle A, D\rangle$ since $T \in \langle A, C\rangle$ and $C = DA^{-1}D^{-1}$. Since D is elliptic of order 6, we conclude that $\Gamma = \langle A, D\rangle$ is $\mathrm{PGL}_2(O_3)$ by Theorem 3.15. Thus, $\mathrm{PGL}_2(O_3)$ is an arithmetic Jørgensen group of parabolic type but is not identified in Proposition 3.16. Hence, $\mathrm{PGL}_2(O_3)$ is not of the form $G_{\theta,k}$.

Part 7. Suppose $\sigma = e^{-\pi i/3}$ and $d = 3$. Then $k\Gamma = \mathbb{Q}(\lambda^2, \lambda e^{-\pi i/3}) = \mathbb{Q}(\sqrt{-3})$ and

$$(\mathrm{tr}B\mathrm{tr}AB - \mathrm{tr}^2B)(\mathrm{tr}[A,B] - 2) = -\lambda \in O_3$$

by Theorem 1.4. Hence, $\lambda = m + ne^{-\pi i/3}$ for some integers m and n. Replacing B with BA^{-n}, we assume $\lambda = m$.

Case 7.1. If m is even, then replace B by $BA^{-m/2}$ and assume

$$\lambda = m - \frac{m}{2}e^{-\pi i/3} = \frac{m\sqrt{3}}{2}e^{\pi i/6}.$$

Hence, $\Gamma = G_{\theta,k}$ with $(\theta, k) = (\frac{\pi}{6}, \frac{m\sqrt{3}}{4})$, so m can be any even integer by Proposition 3.16, and Γ is a $\mathbb{Z}_2$-extension of the figure-eight knot group by an involution that conjugates one parabolic generator to the other (or its inverse) if $m \equiv 0 \mod 4$ and is the figure-eight knot group if $m \equiv 2 \mod 4$.

Case 7.2. Suppose m is odd. We now see that such Γ are arithmetic Jørgensen groups of parabolic type but are not identified in Proposition 3.16 (and hence Γ is not of the form $G_{\theta,k}$). Let

$$C = BAB^{-1} = \begin{pmatrix} 1 & 0 \\ e^{\pi i/3} & 1 \end{pmatrix} \text{ and}$$

$$T = C^{-1}ACA^{-2}CAC^{-1} = \begin{pmatrix} 1 & 2\sqrt{-3} \\ 0 & 1 \end{pmatrix}.$$

Subcase 7.2.1. If $m \equiv 1 \mod 4$, then let

$$D = A^{-1}CAC^{-1}B^{-1}A^{-1}CAC^{-1}A^{\frac{m-1}{2}}T^{\frac{m-5}{4}} = \begin{pmatrix} 0 & -e^{\pi i/3} \\ e^{-\pi i/3} & -1 \end{pmatrix},$$

so $B \in \langle A, C, D\rangle$ since $T \in \langle A, C\rangle$, and $C \in \langle A, D\rangle$ since $C = DAD^{-1}$. Since D is elliptic of order 3, we conclude that $\Gamma = \langle A, D\rangle$ is $\mathrm{PSL}_2(O_3)$ by Theorem 3.15. Thus, $\mathrm{PSL}_2(O_3)$ is an arithmetic Jørgensen group of parabolic type but is not identified in Proposition 3.16. Hence, $\mathrm{PSL}_2(O_3)$ is not of the form $G_{\theta,k}$.

Subcase 7.2.2. If $m \equiv 3 \mod 4$, then let

$$D = A^{-1}CAC^{-1}B^{-1}A^{-1}CAC^{-1}A^{\frac{m+1}{2}}T^{\frac{m-3}{4}} = \begin{pmatrix} 0 & -e^{\pi i/3} \\ e^{-\pi i/3} & 1 \end{pmatrix},$$

so $B \in \langle A, C, D\rangle$ since $T \in \langle A, C\rangle$, and $C \in \langle A, D\rangle$ since $C = DAD^{-1}$. Since D is elliptic of order 3, we conclude that $\Gamma = \langle A, D\rangle$ is $\mathrm{PSL}_2(O_3)$ by Theorem 3.15. Thus,

$\mathrm{PSL}_2(O_3)$ is an arithmetic Jørgensen group of parabolic type but is not identified in Proposition 3.16. Hence, $\mathrm{PSL}_2(O_3)$ is not of the form $G_{\theta,k}$.

Part 8. Suppose $\sigma = e^{\pi i/3}$ and $d = 3$. Then $k\Gamma = \mathbb{Q}(\lambda^2, \lambda e^{\pi i/3}) = \mathbb{Q}(\sqrt{-3})$ and

$$(\mathrm{tr}B\mathrm{tr}AB - \mathrm{tr}^2 B)(\mathrm{tr}[A,B] - 2) = -\lambda \in O_3$$

by Theorem 1.4. Hence, $\lambda = m + ne^{\pi i/3}$ for some integers m and n. Replacing B with BA^{-n}, we assume $\lambda = m$.

Case 8.1. If m is even, then replace B by $BA^{-m/2}$ and assume

$$\lambda = m - \frac{m}{2}e^{\pi i/3} = -\frac{m\sqrt{3}}{2}e^{5\pi i/6}.$$

Hence, $\Gamma = G_{\pi-\theta,-k}$ with $(\theta, k) = (\frac{\pi}{6}, \frac{m\sqrt{3}}{4})$, so m can be any even integer by Theorem 3.13 and Proposition 3.16. Thus, Γ is again a $\mathbb{Z}_2$-extension of the figure-eight knot group if $m \equiv 0 \mod 4$ and is the figure-eight knot group if $m \equiv 2 \mod 4$.

Case 8.2. Suppose m is odd. We now see that such Γ are arithmetic Jørgensen groups of parabolic type but are not identified in Proposition 3.16 (and hence Γ is not of the form $G_{\theta,k}$). Let

$$C = BAB^{-1} = \begin{pmatrix} 1 & 0 \\ e^{-\pi i/3} & 1 \end{pmatrix} \text{ and } T = C^{-1}ACA^{-2}CAC^{-1} = \begin{pmatrix} 1 & -2\sqrt{-3} \\ 0 & 1 \end{pmatrix}.$$

Subcase 8.2.1. If $m \equiv 1 \mod 4$, then let

$$D = A^{-1}CAC^{-1}B^{-1}A^{-1}CAC^{-1}A^{\frac{m-1}{2}}T^{\frac{m-5}{4}} = \begin{pmatrix} 0 & -e^{-\pi i/3} \\ e^{\pi i/3} & -1 \end{pmatrix},$$

so $B \in \langle A, C, D\rangle$ since $T \in \langle A, C\rangle$, and $C \in \langle A, D\rangle$ since $C = DAD^{-1}$. Since D is elliptic of order 3, we conclude that $\Gamma = \langle A, D\rangle$ is $\mathrm{PSL}_2(O_3)$ by Theorem 3.15. Thus, $\mathrm{PSL}_2(O_3)$ is an arithmetic Jørgensen group of parabolic type but is not identified in

Proposition 3.16. Hence, $\mathrm{PSL}_2(O_3)$ is not of the form $G_{\theta,k}$.

Subcase 8.2.2. If $m \equiv 3 \mod 4$, then let

$$D = A^{-1}CAC^{-1}B^{-1}A^{-1}CAC^{-1}A^{\frac{m+1}{2}}T^{\frac{m-3}{4}} = \begin{pmatrix} 0 & -e^{-\pi i/3} \\ e^{\pi i/3} & 1 \end{pmatrix},$$

so $B \in \langle A, C, D\rangle$ since $T \in \langle A, C\rangle$, and $C \in \langle A, D\rangle$ since $C = DAD^{-1}$. Since D is elliptic of order 3, we conclude that $\Gamma = \langle A, D\rangle$ is $\mathrm{PSL}_2(O_3)$ by Theorem 3.15. Thus, $\mathrm{PSL}_2(O_3)$ is an arithmetic Jørgensen group of parabolic type but is not identified in Proposition 3.16. Hence, $\mathrm{PSL}_2(O_3)$ is not of the form $G_{\theta,k}$.

■

3.4 Arithmetic Jørgensen Groups of Elliptic Type

Before discussing arithmetic Jørgensen groups of elliptic type, we note the following combination of Theorem 9.5.2 and Corollary 9.5.3 in [31] (cf. Theorems 4 and 5 of [30]) regarding Fuchsian subgroups of arithmetic Kleinian groups.

Theorem 3.18. *If F is a non-elementary Fuchsian subgroup of an arithmetic Kleinian group Γ, then F is a subgroup of an arithmetic Fuchsian group G such that*

$$kG = k\Gamma \cap \mathbb{R},\ [k\Gamma : kG] = 2,\ \textit{and}\ A\Gamma \cong AG \otimes_{kG} k\Gamma.$$

We now establish the following characterization of arithmetic Jørgensen groups of elliptic type, which completes the proof of Theorem 3.5 as a corollary.

Theorem 3.19. *A finite-covolume Kleinian group Γ is an arithmetic Jørgensen group of elliptic type if and only if $\Gamma = \langle A, B\rangle$ such that the following six conditions hold.*

1. *A is elliptic of order $n \in \{7, 8, 9, 10, 11, 12, 14, 16, 18, 24, 30\}$.*

2. B *is loxodromic or hyperbolic with* $\mathrm{tr}^2 B > \dfrac{2}{1-\cos\frac{2\pi}{n}} > 4$.

3. $\mathrm{tr}[A,B] = 2\cos\frac{2\pi}{n} + 1$.

4. $k\Gamma$ *is a complex quadratic extension of* $\mathbb{Q}(\cos\frac{2\pi}{n})$ *such that*

$$-1 < \tau\left(\cos\frac{2\pi}{n}\right) < \frac{1}{2} \textit{ and } 0 < \tau(\mathrm{tr}^2 B) < \frac{2}{1-\tau(\cos\frac{2\pi}{n})} < 4$$

for each of the $2([\mathbb{Q}(\cos\frac{2\pi}{q}) : \mathbb{Q}] - 1)$ *real embeddings* τ *of* $k\Gamma$.

5. $A\Gamma \cong \left(\dfrac{-1,\ 2\cos\frac{2\pi}{n} - 1}{k\Gamma}\right) \cong \left(\dfrac{-1,\ 2(2\cos^2(\frac{2\pi}{n}) + \cos(\frac{2\pi}{n}) - 1)\mathrm{tr}^2 B}{k\Gamma}\right)$.

6. $\mathrm{tr}B$ *and* $\mathrm{tr}AB$ *are algebraic integers.*

Proof. We first prove necessity. By definition, Γ can be generated by a pair of elements (A,B) with $J(A,B) = 1$ and A elliptic. By Theorem 3.1, A has order $n \geq 7$ and $\mathrm{tr}ABAB^{-1} = 1$. Thus, $\mathrm{tr}^2 A = 2\cos\frac{2\pi}{n} + 2$, and standard trace relations (3.14 of [31], for instance) yield

$$1 = \mathrm{tr}ABAB^{-1} = (\mathrm{tr}A)(\mathrm{tr}BAB^{-1}) - \mathrm{tr}A(BAB^{-1})^{-1} = \mathrm{tr}^2 A - \mathrm{tr}[A,B],$$

so $\mathrm{tr}[A,B] = 2\cos\frac{2\pi}{n} + 1$, which establishes Condition (3) of the theorem.

Also by Theorem 3.1, $\Delta = \langle A, BAB^{-1}\rangle$ is a non-elementary subgroup of Γ with $J(A, BAB^{-1}) = 1$, so $\mathrm{tr}[A, BAB^{-1}] = 2\cos\frac{2\pi}{n} + 1$ as above. Since

$$\mathrm{tr}A = \mathrm{tr}BAB^{-1} = \pm 2\cos\frac{\pi}{n} \text{ and } \mathrm{tr}ABAB^{-1} = 1,$$

Lemma 1.1 yields $k\Delta = \mathbb{Q}(\cos\frac{2\pi}{n})$ and $\mathbb{Q}(\mathrm{tr}\Delta) = \mathbb{Q}(\cos\frac{\pi}{n}) \subset \mathbb{R}$, so Δ is a Fuchsian subgroup of Γ (cf. Corollary 3.2.5 of [31]). Therefore, by Theorem 3.3, Δ must be a $(2,3,q)$-triangle group, which has trace field $\mathbb{Q}(\cos\frac{\pi}{q})$ (see, for instance, Section 4.9 of [31]), forcing $n = q$.

Theorem 3.18 further implies that Δ is a subgroup of an arithmetic Fuchsian group G. But, by Theorem 3B of [19], $(2,3,n)$-triangle groups cannot be subgroups of strictly lager Fuchsian groups. Therefore, $\Delta = G$, and thus Δ is an arithmetic $(2,3,n)$-triangle group. Following the enumeration of all arithmetic triangle groups in Section 13.3 of [31] (cf. [41]) and noting that $n \neq \infty$ since A is elliptic of order n, the $(2,3,n)$-triangle group is arithmetic if and only if

$$n \in \{7, 8, 9, 10, 11, 12, 14, 16, 18, 24, 30\},$$

thereby establishing Condition (1).

By Theorems 1.3 and 3.18, $\mathbb{Q}(\cos\frac{2\pi}{n}) = k\Gamma \cap \mathbb{R}$, $[k\Gamma : \mathbb{Q}(\cos\frac{2\pi}{n})] = 2$, and $k\Gamma$ has exactly one complex place, so $k\Gamma$ must be a complex quadratic extension of $\mathbb{Q}(\cos\frac{2\pi}{n})$. Applying Theorem 1.2 to $\langle A, BAB^{-1}\rangle$ yields

$$\begin{aligned} A\Gamma &= \left(\frac{(2\cos\frac{2\pi}{n}+2)(2\cos\frac{2\pi}{n}-2),\ (2\cos\frac{2\pi}{n}+2)^2(2\cos\frac{2\pi}{n}-1)}{k\Gamma}\right) \\ &\cong \left(\frac{-1,\ 2\cos\frac{2\pi}{n}-1}{k\Gamma}\right) \end{aligned}$$

where we use Lemma 2.1.2 in [31] to remove squares of elements in $k\Gamma^*$. Similarly, applying Theorem 1.2 to $\langle A, B\rangle$ yields

$$\begin{aligned} A\Gamma &= \left(\frac{(2\cos\frac{2\pi}{n}+2)(2\cos\frac{2\pi}{n}-2),\ (2\cos\frac{2\pi}{n}+2)\mathrm{tr}^2B(2\cos\frac{2\pi}{n}-1)}{k\Gamma}\right) \\ &\cong \left(\frac{-1,\ 2(2\cos^2(\frac{2\pi}{n})+\cos(\frac{2\pi}{n})-1)\mathrm{tr}^2B}{k\Gamma}\right) \end{aligned}$$

again using Lemma 2.1.2 in [31] to remove squares of elements in $k\Gamma^*$. This establishes Condition (5).

Clearly $k\Gamma$ has $2([\mathbb{Q}(\cos\frac{2\pi}{n}):\mathbb{Q}]-1)$ real places, each of which corresponds to a real embedding. Let $\tau : k\Gamma \to \mathbb{R}$ be one such real embedding. Then, since

$$A\Gamma \cong \left(\frac{(2\cos\frac{2\pi}{n}+2)(2\cos\frac{2\pi}{n}-2),\ 2\cos\frac{2\pi}{n}-1}{k\Gamma}\right)$$

is ramified at all real places of $k\Gamma$ by Theorem 1.3, we have

$$\tau(\cos\frac{2\pi}{n})-1<0<\tau(\cos\frac{2\pi}{n})+1 \text{ and } 2\tau(\cos\frac{2\pi}{n})-1<0$$

which yields $-1<\tau(\cos\frac{2\pi}{n})<\frac{1}{2}$. Similarly, for

$$A\Gamma \cong \left(\frac{-1,(2\cos\frac{2\pi}{n}+2)\mathrm{tr}^2B(2\cos\frac{2\pi}{n}-1)}{k\Gamma}\right)$$

to be ramified at all real places of $k\Gamma$, we must have $\tau(\mathrm{tr}^2B)>0$ since

$$\tau(2\cos\frac{2\pi}{n}-1)<0<\tau(2\cos\frac{2\pi}{n}+2)$$

as above.

By Lemma 1.1, $k\Gamma=\mathbb{Q}(\mathrm{tr}^2A,\mathrm{tr}^2B,\mathrm{tr}A\mathrm{tr}B\mathrm{tr}AB)$. Following Section 4 of [29], the standard trace relation (3.15 in [31], for instance)

$$\mathrm{tr}[A,B]=\mathrm{tr}^2A+\mathrm{tr}^2B+\mathrm{tr}^2AB-\mathrm{tr}A\mathrm{tr}B\mathrm{tr}AB-2$$

implies that $\mathrm{tr}A\mathrm{tr}B\mathrm{tr}AB$ satisfies the quadratic equation

$$x^2-(\mathrm{tr}^2A)(\mathrm{tr}^2B)x-(\mathrm{tr}^2A)(\mathrm{tr}^2B)(\mathrm{tr}[A,B]-\mathrm{tr}^2A-\mathrm{tr}^2B+2)=0,$$

so

$$k\Gamma=\mathbb{Q}(\mathrm{tr}^2A,\mathrm{tr}^2B,\mathrm{tr}[A,B])(\sqrt{\delta})$$

where

$$\begin{aligned}\delta &= (\mathrm{tr}^2 A)(\mathrm{tr}^2 B)((\mathrm{tr}^2 A - 4)(\mathrm{tr}^2 B - 4) + 4(\mathrm{tr}[A,B] - 2)) \\ &= (2\cos\frac{2\pi}{n} + 2)(\mathrm{tr}^2 B)((2\cos\frac{2\pi}{n} - 2)(\mathrm{tr}^2 B - 4) + 4(2\cos\frac{2\pi}{n} - 1)).\end{aligned}$$

As τ is a real embedding of $k\Gamma$, we must have $\tau(\delta) > 0$. Having already shown that

$$-1 < \tau(\cos\frac{2\pi}{n}) < \frac{1}{2} \text{ and } \tau(\mathrm{tr}^2 B) > 0,$$

we conclude that

$$\tau(\mathrm{tr}^2 B) < \frac{2}{1 - \tau(\cos\frac{2\pi}{n})} < 4.$$

Condition (4) is now established.

Suppose $\mathrm{tr}B$ is real. Then $\mathrm{tr}^2 B \in \mathbb{R} \cap k\Gamma = \mathbb{Q}(\cos\frac{2\pi}{n})$, so $k\Gamma = \mathbb{Q}(\cos\frac{2\pi}{n})(\sqrt{\delta})$ as above. If $\mathrm{tr}^2 B = 0$, then $\delta = 0$, which implies $k\Gamma \subset \mathbb{R}$, a contradiction, so $\mathrm{tr}^2 B > 0$. Since $k\Gamma$ has exactly one complex place, we conclude that $\delta < 0$, and so

$$(\mathrm{tr}^2 A - 4)(\mathrm{tr}^2 B - 4) + 4(\mathrm{tr}[A,B] - 2) = (2\cos\frac{2\pi}{n} - 2)(\mathrm{tr}^2 B - 4) + 4(2\cos\frac{2\pi}{n} - 1) < 0$$

since $\mathrm{tr}^2 A > 0$ as well. Thus,

$$\mathrm{tr}^2 B > \frac{2}{1 - \cos\frac{2\pi}{n}} > 4$$

since $n > 6$, so B cannot be elliptic or parabolic. Condition (2) is now established, and Condition (6) follows directly from Theorem 1.3, so necessity has been proved.

We now verify that sufficiency follows from construction. Suppose $\Gamma = \langle A, B\rangle$ is a finite-volume Kleinian group such that the six conditions of the theorem hold. Then Condition (4) ensures that $k\Gamma$ is a number field with exactly one complex place, Conditions (1) and (6) imply that $\mathrm{tr}\Gamma$ consists of algebraic integers by Lemma 1.1, and Conditions (4) and (5) guarantee that $A\Gamma$ is ramified at all real places of $k\Gamma$, so Γ is arithmetic by Theorem 1.3. Finally, by Conditions (1) and (3),

$$\begin{aligned} J(A,B) &= |\mathrm{tr}^2 A - 4| + |\mathrm{tr}[A,B] - 2| \\ &= |2\cos\frac{2\pi}{n} - 2| + |2\cos\frac{2\pi}{n} - 1| \\ &= 2 - 2\cos\frac{2\pi}{n} + 2\cos\frac{2\pi}{n} - 1 \\ &= 1 \end{aligned}$$

since $n > 6$. Thus, Γ is an arithmetic Jørgensen group of elliptic type. ■

Corollary 3.20. *No arithmetic Jørgensen group of elliptic type is commensurable with any non-compact arithmetic Kleinian group.*

Proof. By Theorem 1.4, the invariant trace field of a non-cocompact arithmetic Kleinian group has degree 2 over $\mathbb{Q}$, whereas by Condition (4) of Theorem 3.19, the invariant trace field of an arithmetic Jørgensen group of elliptic type has degree $2[\mathbb{Q}(\cos\frac{2\pi}{n}) : \mathbb{Q}]$ over $\mathbb{Q}$, where n is as in Condition (1) of the theorem. Since the invariant trace field is an invariant of the commensurability class of a finitely generated non-elementary Kleinian group, the result follows. ■

Thus, the non-cocompact arithmetic Jørgensen groups are precisely the arithmetic Jørgensen groups of parabolic type identified in Theorem 3.17, thereby completing the proof of Theorem 3.5.

3.5 Bounds on Jørgensen Number

We begin with a simple bound on Jørgensen numbers of orientable hyperbolic 3-manifolds with at least one cusp.

Proposition 3.21. *Let $\mathbb{H}^3/\Gamma$ be an orientable hyperbolic 3-manifold such that Γ contains the parabolic element $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then*

$$1 \leq \widetilde{J}(\Gamma) \leq \inf \left\{ |c|^2 : T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, c \neq 0, \text{ and } \langle A, T \rangle \text{ is non-elementary} \right\}.$$

If Γ is non-elementary and can be generated by A and another element, then

$$1 \leq J(\Gamma) \leq \inf \left\{ |c|^2 : T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, c \neq 0, \text{ and } \langle A, T \rangle = \Gamma \right\}.$$

Proof. As already noted, $1 \leq \widetilde{J}(\Gamma) \leq J(\Gamma)$. Let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ with $c \neq 0$. If $\langle A, T \rangle$ is non-elementary, then $\widetilde{J}(\Gamma) \leq J(A, T) = |c|^2$. If $\langle A, T \rangle = \Gamma$, then $J(\Gamma) \leq J(A, T) = |c|^2$. Since T was arbitrary, the bounds follow. ∎

Now suppose that $\mathbb{H}^3/\Gamma$ is a hyperbolic two-bridge knot complement $S^3 \setminus K$. Following Section 3 of [35] and Section 4.5 of [31], K is determined by a pair of relatively prime odd integers (p/q) with $0 < q < p$, and $\Gamma = \langle A, B \mid AW = WB \rangle$ where

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}, \; W = B^{e_1} A^{e_2} \cdots B^{e_{p-2}} A^{e_{p-1}},$$

and $e_i = (-1)^{\lfloor \frac{iq}{p} \rfloor}$.

The relation $AW = WB$ forces $W = \begin{pmatrix} * & 1/\sqrt{z} \\ -\sqrt{z} & 0 \end{pmatrix}$ where z satisfies the integral monic polynomial equation

$$d_n = 1 + \left(\widehat{\sum_{i_1 \text{ odd}}} e_{i_1} e_{i_2} \right) z + \cdots + (e_1 e_2 \cdots e_{2n}) z^n = 0$$

where $n = \dfrac{p-1}{2}$ and $\widehat{\sum}$ denotes summation over $i_1 < i_2 < \cdots < i_k$ with alternating parity. Then $k\Gamma = \mathbb{Q}(\mathrm{tr}\Gamma) = \mathbb{Q}(z)$, and since two-bridge knot groups are not free, a well known result establishes the bound $|z| < 4$ (see, for instance, Theorem B of [35], which is attributed to J. Brenner).

Corollary 3.22. *Suppose $M = \mathbb{H}^3/\Gamma$ is the complement in S^3 of a hyperbolic two-bridge knot K with all notation as above, and let $\mathcal{C}$ be the single cusp in M. If K is the figure-eight knot, then*

$$\widetilde{J}(\Gamma) = J(\Gamma) = 1 = w(M, \mathcal{C}).$$

Otherwise,

$$1 < \widetilde{J}(\Gamma) \leq J(\Gamma) \leq |z| < 4 \text{ and } 1 < w(M, \mathcal{C}) \leq |\sqrt{z}| < 2.$$

Proof. Results for the figure-eight knot have already been established in Section 3.1, so let K be any other hyperbolic two-bridge knot (p/q) with notation as above. The relation $AW = WB$ yields $B = W^{-1}AW$, so $\langle A, W\rangle = \Gamma$, and $J(A, W) = |z|$. Hence,

$$1 < \widetilde{J}(\Gamma) \leq J(\Gamma) \leq |z| < 4.$$

The bounds on $w(M, \mathcal{C})$ follow from Theorem 3.8 and Lemma 3.9 applied to W. ∎

Similarly, suppose that $\mathbb{H}^3/\Gamma$ is a hyperbolic two-bridge link complement $S^3 \setminus L$ where L has two components. Again following Section 4.5 of [31], L is determined by a pair of relatively prime integers (p/q) with $0 < q < p$, $p = 2n$ even, and

$\Gamma = \langle A, B \mid AW = WA \rangle$ where

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}, \ W = B^{e_1} A^{e_2} \cdots B^{e_{p-1}},$$

and $e_i = (-1)^{\lfloor \frac{iq}{p} \rfloor}$.

The relation $AW = WA$ forces z to satisfy the integral monic polynomial equation

$$c_n = \left(\sum_{i=1}^{n} e_{2i-1} \right) z + \left(\widehat{\sum_{i_1 \text{ odd}}} e_{i_1} e_{i_2} e_{i_3} \right) z^2 + \cdots + (e_1 e_2 \cdots e_{2n-1}) z^n = 0$$

where $\widehat{\sum}$ again denotes summation over $i_1 < i_2 < \cdots < i_k$ with alternating parity. As before, $k\Gamma = \mathbb{Q}(\mathrm{tr}\Gamma) = \mathbb{Q}(z)$, and $|z| < 4$.

Corollary 3.23. *Suppose $M = \mathbb{H}^3/\Gamma$ is the complement in S^3 of a hyperbolic two-bridge link L with all notation as above, and let $\mathcal{C}$ be the cusp in M whose lift contains a horoball based at ∞. Then*

$$1 < \widetilde{J}(\Gamma) \le J(\Gamma) \le |z|^2 < 16 \text{ and } 1 < w(M, \mathcal{C}) \le |z| < 4.$$

Proof. Since L has two components, it is not the figure-eight knot, so the bounds follow as before with Proposition 3.21 and Lemma 3.9 applied to B this time. ∎

3.6 Computations of Jørgensen Numbers

We first establish a technical lemma to be used in computing Jørgensen numbers for several knots and links. The axis of every loxodromic element in a Kleinian group Γ projects to a closed geodesic on $\mathbb{H}^3/\Gamma$. Conversely, the length of any closed geodesic on $\mathbb{H}^3/\Gamma$ coincides with the translation length of a corresponding loxodromic element in Γ. The following result relates the length of a closed geodesic on $\mathbb{H}^3/\Gamma$ to the trace of a corresponding loxodromic element in Γ.

Lemma 3.24. *Let $\mathbb{H}^3/\Gamma$ be a hyperbolic 3-orbifold. If γ is a loxodromic element of Γ whose axis projects to a closed geodesic of length ≥ 3, then*

$$|\mathrm{tr}^2\gamma - 4| > 18.13532399.$$

Proof. The length of the corresponding closed geodesic coincides with the translation length ℓ_γ of γ, which is related to $\mathrm{tr}\gamma$ via

$$|\mathrm{tr}\gamma| = 2\cosh\frac{\ell_\gamma}{2}.$$

Since hyperbolic cosine is an increasing function, $\ell_\gamma \geq 3$ implies

$$|\mathrm{tr}\gamma| \geq 2\cosh\frac{3}{2} = 4.704819230.$$

Thus, $|\mathrm{tr}^2\gamma| \geq 22.13532399$, and so

$$|\mathrm{tr}^2\gamma - 4| \geq ||\mathrm{tr}^2\gamma| - 4| \geq 18.13532399.$$

■

We now establish two propositions that yield computations of Jørgensen numbers for several two-bridge knots and links as corollaries. For convenience, if $\mathbb{H}^3/\Gamma$ is the complement of a knot or link K in S^3, then the generalized Jørgensen number of K is $\widetilde{J}(K) = \widetilde{J}(\Gamma)$, and the Jørgensen number of K is $J(K) = J(\Gamma)$.

Proposition 3.25. *Let $\mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold of finite volume and R^* the* *group of units in R, the ring of integers in $\mathbb{Q}(\mathrm{tr}\Gamma)$. Suppose* $\mathrm{tr}\Gamma \subset R$ *and*

$$R^* \cong W \times \langle u \rangle$$

where W is a finite cyclic group consisting of roots of unity and u is a fundamental unit with $1 < |u| < 2$. If there exists a pair of generators (A, B) for Γ with $J(A, B) = |u|^2$, and if

$$\inf\{|\mathrm{tr}^2 X - 4| + |\mathrm{tr} Y - 2| : X, Y \in \Gamma \text{ are loxodromic}\} > |u|^2,$$

then $1 < \widetilde{J}(\Gamma) \leq J(\Gamma) = |u|^2$.

Proof. Note that $\mathbb{H}^3/\Gamma$ is not the complement of the figure-eight knot in S^3 since R^* is finite in this case. Hence, $1 < \widetilde{J}(\Gamma) \leq J(\Gamma)$ by Theorem 3.10. Suppose there exists a pair of generators (X, Y) for Γ with $J(X, Y) \leq |u|^2 < 4$. By hypothesis, one of X or $[X, Y]$ is parabolic. But $\mathrm{tr}[X, Y] = 2$ implies $\Gamma = \langle X, Y \rangle$ is reducible (see, for instance, Lemma 1.2.3 in [31]), a contradiction, and $\mathrm{tr}[X, Y] = -2$ implies $J(X, Y) \geq |\mathrm{tr}[X, Y] - 2| = 4$, also a contradiction. Hence, X is parabolic. Then

$$1 < J(X, Y) = |\mathrm{tr}[X, Y] - 2| \leq |u|^2.$$

Similarly $J(A, B) = |\mathrm{tr}[A, B] - 2| = |u|^2$, so $\mathrm{tr}[A, B] - 2 = \xi_1 u^2$ for some root of unity ξ_1.

By Corollary 1.7, $\mathrm{tr}[X, Y] - 2$ is a unit multiple of $\mathrm{tr}[A, B] - 2$ in R and thus has the form $\xi_2 u^a$ for some root of unity ξ_2 and integer a. But $|u| > 1$, so $|\mathrm{tr}[X, Y] - 2| \leq |u|^2$ implies $a = 1$ or 2.

Since X is parabolic, $\mathrm{tr} X = \pm 2$. Combining this with standard trace relations (see, for instance, 3.15 in Section 3.4 of [31]) yields

$$\mathrm{tr}[X, Y] - 2 = \mathrm{tr}^2 Y \pm 2\mathrm{tr} Y \mathrm{tr} XY + \mathrm{tr}^2 XY = (\mathrm{tr} Y \pm \mathrm{tr} XY)^2.$$

Hence, $\mathrm{tr}[X,Y]-2$ is a square in R^*, so $a=2$ and $J(X,Y)=|\mathrm{tr}[X,Y]-2|=|u|^2$. The result now follows. ■

Corollary 3.26. *The Jørgensen numbers of several two-bridge knots are given in the following table.*

K	$J(K)$
5_2	1.32471796
6_1	1.55603019
7_4	2.20556943
7_7	1.55603019

Proof. The knots 5_2, 6_1, 7_4, and 7_7 are the two-bridge knots (7/3), (9/5), (15/11), and (21/13) respectively. Let K be one of these knots and Γ the Kleinian group such that $\mathbb{H}^3/\Gamma = S^3 \setminus K$. Following the procedure outlined in Section 3.5, we see that Lemma 1.1 yields $\mathrm{tr}\Gamma \subset R$, the ring of integers in $\mathbb{Q}(\mathrm{tr}\Gamma)=\mathbb{Q}(z)$, where z is as follows.

K	minimum polynomial of z	numerical value of z
5_2	$1+2z+z^2+z^3$	$-0.21507985+1.307141279i$
6_1	$1-2z+3z^2-z^3+z^4$	$0.104876618-1.552491820i$
7_4	$1+4z-4z^2+z^3$	$2.10278472+0.665456952i$
7_7	$1-z+3z^2-2z^3+z^4$	$0.95668457-1.227185638i$

As verified by PARI ([42]), $J(A,W)=|z|=|u|^2$, where A and W are as in Section 3.5 and u is the fundamental unit in $\mathbb{Q}(\mathrm{tr}\Gamma)$. Thus, we see that $S^3\setminus K$ satisfies all hypotheses of Proposition 3.25 but the last. To verify it, we enumerate squares of traces of loxodromic elements corresponding to closed geodesics of length less than three in SnapPea ([44], [22]) to find the value of

$$\alpha = \inf\{|\mathrm{tr}^2 X - 4| : X \in \Gamma \text{ is loxodromic}\}.$$

Lemma 3.24 ensures that we only need to check elements corresponding to closed geodesics of length less than three. The values of $|z| = |u|^2$ and α for the aforementioned knots are listed in the table below.

K	$\lvert z\rvert = \lvert u\rvert^2$	α
5_2	1.32471796	4.219276205
6_1	1.55603019	3.955211258
7_4	2.20556943	4.434378815
7_7	1.55603019	5.105997169

Thus, in each case we have

$$|u|^2 < \alpha \leq \inf\{|\mathrm{tr}^2 X - 4| + |\mathrm{tr} Y - 2| : X, Y \in \Gamma \text{ are loxodromic}\},$$

so the result now follows from Proposition 3.25. ■

Proposition 3.27. *Let* $\mathbb{H}^3/\Gamma$ *be a hyperbolic 3-manifold of finite volume such that* $\mathbb{Q}(\mathrm{tr}\Gamma)$ *is a quadratic imaginary field and* $\mathrm{tr}\Gamma \subset R$*, the ring of integers in* $\mathbb{Q}(\mathrm{tr}\Gamma)$*. If there exists a pair of generators* (A, B) *for* Γ *with* $J(A, B) < 4$ *and*

$$\inf\{|\mathrm{tr}^2 X - 4| + |\mathrm{tr} Y - 2| : X, Y \in \Gamma \textit{ are loxodromic}\} > J(A, B),$$

then $1 \leq \widetilde{J}(\Gamma) \leq J(\Gamma) = J(A, B)$*.*

Proof. Suppose there exists a pair of generators (X, Y) for Γ with

$$J(X, Y) \leq J(A, B) < 4.$$

By hypothesis, one of X or $[X, Y]$ is parabolic. But $\text{tr}[X, Y] = 2$ once again implies $\Gamma = \langle X, Y \rangle$ is reducible, a contradiction, and $\text{tr}[X, Y] = -2$ implies

$$J(X, Y) \geq |\text{tr}[X, Y] - 2| = 4,$$

also a contradiction. Hence, X is parabolic. Then

$$1 \leq J(X, Y) = |\text{tr}[X, Y] - 2| \leq J(A, B).$$

Similarly $J(A, B) = |\text{tr}[A, B] - 2|$.

By Corollary 1.7, $\text{tr}[X, Y] - 2$ is a unit multiple of $\text{tr}[A, B] - 2$. Since the only units in a quadratic imaginary field are roots of unity, which have norm one, we have

$$J(X, Y) = |\text{tr}[X, Y] - 2| = |\text{tr}[A, B] - 2| = J(A, B).$$

The result now follows. ∎

Corollary 3.28. *The Jørgensen numbers of several two-bridge links are given in the following table.*

L	$J(L)$
5_1^2	2
6_2^2	3
6_3^2	2

Proof. The links 5_1^2, 6_2^2, and 6_3^2 are the two-bridge links (8/3), (10/3), and (12/5) respectively. Let L be one of these links. As observed in [13], if $\mathbb{H}^3/\Gamma = S^3 \setminus L$, then $\Gamma = \langle A, B \rangle$ is an arithmetic Kleinian group, where

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$$

with z as in the following table.

L	z
5_1^2	$1+i$
6_2^2	$\dfrac{3+i\sqrt{3}}{2}$
6_3^2	$\dfrac{1+i\sqrt{7}}{2}$

As noted in Section 4.5 of [31], $\mathbb{Q}(\mathrm{tr}\Gamma) = \mathbb{Q}(z)$, a quadratic imaginary field, and, by Theorem 1.3, the arithmeticity of Γ guarantees that $\mathrm{tr}\Gamma \subset R$, the ring of integers in $\mathbb{Q}(\mathrm{tr}\Gamma)$. Thus, $\mathbb{H}^3/\Gamma$ satisfies all hypotheses of Proposition 3.27 but the last. To verify it, we again enumerate squares of traces of loxodromic elements corresponding to closed geodesics of length less than three in SnapPea ([44], [22]) to find the value of

$$\alpha = \inf\{|\mathrm{tr}^2 X - 4| : X \in \Gamma \text{ is loxodromic}\}.$$

As before, Lemma 3.24 ensures that we only need to check elements corresponding to closed geodesics of length less than three. The values of $J(A,B) = |z|^2$ and α for the aforementioned links are listed in the table below.

L	$J(A,B) = \lvert z\rvert^2$	α
5_1^2	2	4.472135955
6_2^2	3	4.582575695
6_3^2	2	5.291502622

Thus, in each case we have

$$J(A,B) < \alpha \leq \inf\{|\mathrm{tr}^2 X - 4| + |\mathrm{tr} Y - 2| : X, Y \in \Gamma \text{ are loxodromic}\},$$

so the result now follows from Proposition 3.27. ∎

3.7 Further Directions

A foremost goal is to complete the identification of all arithmetic Jørgensen groups by explicitly finding all arithmetic Jørgensen groups of elliptic type (and hence all cocompact arithmetic Jørgensen groups). The characterization of arithmetic Jørgensen groups of elliptic type established in Theorem 3.19 rests on the observation that every arithmetic Jørgensen group of elliptic type contains an arithmetic Fuchsian triangle group (this was first observed in [15], and [41] identifies all arithmetic triangle groups). This in turn determines the invariant trace field (up to an imaginary quadratic extension), the invariant quaternion algebra (up to isomorphism), and the ramification set of the invariant quaternion algebra of every arithmetic Jørgensen group of elliptic type. Since the invariant trace field and invariant quaternion algebra together completely determine the commensurability class of a Kleinian group, further consideration of the restrictions imposed by Theorem 3.19 may complete the identification of all arithmetic Jørgensen groups of elliptic type and hence solve Problem 3.2 in the case of arithmetic Kleinian groups. We also note that Conditions (1) and (3) of Theorem 3.19 show that arithmetic Jørgensen groups of elliptic type are Kleinian groups with real parameters and thus subject to constraints established in [12].

A broader aim is to identify all two-generator arithmetic Kleinian groups with one generator parabolic and the other loxodromic by generalizing the proof of Theorem 3.5. Such a result would naturally mark the fourth installment of the program undertaken in [29], [13], and [10] to solve Problem 3.6. In proving Theorem 3.5, to establish when a Jørgensen group of parabolic type has finite covolume (a necessary condition for arithmeticity), techniques from [26], [27], and [28] are modified to use Poincaré's Polyhedron Theorem to construct the fundamental polyhedra of the Jørgensen groups. That a group has Jørgensen number one imposes strong restrictions on the entries of its generating matrices and hence on the construction of its fundamental polyhedron. This approach has already produced several examples of two-generator arithmetic Kleinian groups with one generator parabolic and the other loxodromic not previously identified. Relaxing these restrictions on the entries of the generating matrices with a more careful construction of the fundamental polyhedra may yield more such examples and a method for identifying all such groups.

Also of interest is extending Theorem 2.3 to non-arithmetic two-bridge knot and link groups by modifying the methods developed in Section 3.6 to compute the Jørgensen numbers of four non-arithmetic two-bridge knots (5_2, 6_1, 7_4, and 7_7). These methods include use of Dirichlet's Unit Theorem to consider fundamental units in the invariant trace fields and use of SnapPea ([44]) to consider the length spectra of closed geodesics in the knot complements in S^3 ([22]).

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Vita

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