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Published in:
IEEE Control Systems Letters

On Stochastic ISS of Time-Varying Switched Systems with Generic Lévy Switching Signals

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Abstract— **Switched systems in which switching among subsystems occurs at random time instants find numerous applications in engineering. Stability analysis of such systems, however, is quite challenging. This paper investigates the stochastic input-to-state stability of this class of switched systems. The random switching instants are modeled by a non-decreasing, positive, and real-valued Lévy process, which, at every time instant, selects the active subsystem from a family of deterministic systems. No assumption on the stability of subsystems is presumed; they can be stable or unstable. Stochastic properties of the switching signal are coupled with a family of Lyapunov-like functions to obtain a sufficient condition for stochastic input-to-state stability.**

Index Terms— **Stochastic input-to-state stability, randomly switched systems, Lévy switching signal**

I. Introduction

Randomly switched systems comprise a family of deterministic subsystems and a switching signal modeled as a stochastic process. These systems can be considered *piecewise deterministic stochastic systems* because deterministic differential equations govern the dynamics between two consecutive switching instants. These systems emerge in many domains such as economic systems, communication and biological systems affected by random delays, modeling of randomly varying structures, and component failures [2]. The general framework and application of randomly switched systems are provided in [2] and [20].

Stability analysis of randomly switched systems is challenging due to their stochastic and hybrid nature. Nevertheless, a few notions of stability for these systems have been investigated in the literature. Almost sure global asymptotic stability (GAS) for randomly switched systems with stable subsystems is discussed in [2]. Motivated by the fact that component failures or abrupt disturbance may destabilize some subsystems, GAS for randomly switched systems comprising both stable and unstable subsystems is studied in [20].

In practice, dynamical systems should be minimally sensitive to external perturbations. The notion of input-to-state stability (ISS), introduced in [15], is a useful tool to characterize a system's tolerance to such perturbations. Initially developed for the analysis of continuous-time systems, ISS was later adapted to switched systems in [19]. However, the ISS framework developed therein remains inconclusive for randomly switched systems because it fails to consider information related to individual trajectories. Thus, it cannot support the ISS of every sample path. Motivated by this, it seems inevitable to introduce a stochastic framework of ISS and use tools from stochastic theory to establish corresponding stability properties for randomly switched systems.

The notion of a stochastic version of ISS traces back to the introduction of γ-ISS in [17], where ISS with respect to a deterministic perturbation is investigated, under the effect of a secondary stochastic perturbation of Wiener type. Another useful stochastic extension of ISS is the noise-to-state stability (NSS) provided in [5], where the incremental covariance of a stochastic noise is considered a perturbation. The most natural adaptation of an ISS estimate to the stochastic case referred to as *stochastic input-to-state stability (SISS)* is proposed in [16]. This notion is consistent with γ-ISS and generalizes NSS. SISS has not been investigated for randomly switched systems so far. Our paper attempts to fill this gap in the spirit of the work [16]. A preliminary step in this direction is taken in [3] by introducing *ISS in* $\boldsymbol{L_1}$ *estimate at switching instants* for randomly switched systems. However, this notion is quite restrictive and fails to conclude SISS; see [3, Section C].

In this paper, we provide a sufficient SISS condition for switched systems in which the switching among the subsystems occurs at random time instants. We model the random switching instants via a non-decreasing, positive, and real-valued Lévy process which, at each time instant, selects the active subsystem from a family of deterministic systems. We allow the subsystems to be stable or unstable. To establish a sufficient SISS condition, we capture, via a rate function, the influence of the Lévy process on the dynamics of the deterministic subsystems given by a family of Lyapunov-like functions and employ a pathwise analysis, eventually leading to a condition on the semigroup operator via Lévy symbol.

Generally, a Markov chain is employed to model the switching signal in piecewise deterministic stochastic systems [7], [9], [12], under the assumption that the parameters of the Markov chain are completely known. The stability analysis of such systems employs a martingale that relies on an infinitesimal (or extended) generator. However, this method is not easy to apply when there is little information about transition probabilities [2]. Moreover, it is not easy to get exact information about transition probabilities because of measurement inaccuracies and intrinsic random

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The work of S. Ahmed was supported by the Dutch Research Council (NWO) under Grant 201225.

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property [10]. Therefore, to relax this assumption, a Poisson process is introduced in [10], which only requires the information of the dwell time of every subsystem. Compared to Markovian switching systems, there are relatively few results on the stability of Poissonian switching systems [4], [6], [8], let alone Lévy switching systems. Motivated by this, we introduce Lévy switching systems, which are generic than the Poissonian switching systems and are not restrictive as Markovian switching systems. To the best of our knowledge, Lévy switching systems have not been studied before.

The motivation for studying switched systems with Lévy switching signals also comes from their flexibility in modeling various biological systems such as complex chemical reaction networks of a cell. In such networks, Lévy process-induced genetic toggle and genetic switches happen when a certain gene is expressed. This gene expression then results in the transcription of a relevant mRNA that serves as a template for producing certain proteins. The latter being the functional units of a cell, consequently, triggers different chemical pathways to become active. The switching dynamics of such complex reaction networks of a cell can be modeled using a Lev́y process. On the other hand, since a Poisson process is a particular case of the Lévy process, our result covers switched systems with Poisson switching that serve as suitable models for dynamic clinical trials [6] and multi-mode multi-dimensional systems with Poissonian sequencing [18].

It is also worth emphasizing that some notable results for SISS of stochastic switched systems have been proposed in [11], [21], and [22]. However, two substantial differences exist between these results and the present paper. The first one is the *difference in the model*. The stochastic switched systems in [11], [21], and [22] are driven by Brownian motion and do not consider the random nature of switching instants. In the present paper, the subsystems are deterministic, and the stochastic effects arise from the randomness of the switching instants. The second one is the *difference in the method*. To conclude SISS, stochastic analysis methods based on Itô formula have been proposed in [11], [21], and [22]. In the present paper, we first identify the stability characteristics of the deterministic subsystems via multiple Lyapunov-like functions and then couple them with the properties of the Lévy switching signal to conclude SISS.

Contributions of this study: We provide SISS of randomly switched systems. To the best of our knowledge, SISS for this class of switched systems has not been studied before. We model the switching signals as a generic Lévy process. Therefore, our result does not require exact transition probability information between subsystems compared to the Markovian switching systems and it is generic than the Poissonian switching systems. Our analysis holds for randomly switched systems that may contain some unstable subsystems. This is motivated by the fact that component failures or abrupt disturbances may destabilize some subsystems of the switched system. Moreover, we allow the parameters of the subsystems to be time-varying and uncertain.

The paper is organized as follows. Section II presents some preliminaries, including necessary notation and the notion of SISS for randomly impulsive systems. The main result appears in Section III. A numerical example is provided in Section IV to illustrate the result. Section V concludes the paper and provides some future research directions.

II. Preliminaries and SISS

The sets of natural numbers, integers, and nonnegative integers are denoted by $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, respectively. The sets of real numbers, non-negative real numbers, and positive real numbers are denoted by $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, and $\mathbb{R}_{>0}$, respectively. The set of complex numbers is denoted by $\mathbb{C}$. Let $\mathcal{C}(D)$ denote the set of all continuous functions with domain D. The cardinality of a set A is denoted by $\#A$. The identity matrix of an appropriate dimension is denoted by I. The usual Euclidean norm of vectors, and the induced norm of matrices, are denoted by $|\cdot|$. Let $|f|_{\mathcal{I}}$ denote the supremum of any real-valued piecewise continuous function f on any interval $\mathcal{I}$ in its domain. The inner product of two vectors x and y is denoted by $\langle x, y \rangle$. The indicator of an event E is denoted by $\mathbf{1}_E$ which is equal to 1 if the event E holds. The indicator function of the event E is denoted by $\mathbf{1}_E(\omega)$ which is equal to 1 if $\omega \in E$ and 0 otherwise. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$; it belongs to class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and $\alpha(t) \to \infty$ as $t \to \infty$; it belongs to class $\mathcal{L}$ if it is continuous, strictly decreasing, and $\alpha(t) \to 0$ as $t \to \infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{KL}$ if the function $\beta(\cdot, t) \in \mathcal{K}$ for each fixed $t \geq 0$, the function $\beta(s, \cdot)$ is non-increasing for each fixed $s \geq 0$, and $\beta(s, t) \to 0$ as $t \to \infty$.

Consider a time-varying switched nonlinear system

$$\dot{x}(t) \quad = \quad f_{\sigma(t)}(t, x(t), u(t)), \ t \geq 0, \tag{1}$$

where $x : [0, \infty) \to \mathbb{R}^n$ is the state, $x(0) = x_0 \in \mathbb{R}^n$ is the initial condition, and the piecewise continuous bounded function $u : [0, \infty) \to \mathbb{R}^m$ is the input. Let

$$\pi = (\pi_k)_{k \in \mathbb{N}_0} = \Big\{ (i_k, t_k) \Big\}_{k \in \mathbb{N}_0, i_k \in \{1, 2, \ldots, q\}}$$

be a switching sequence, where $(t_k)_{k \in \mathbb{N}_0}$ is a sequence of random variables given by the realization of a non-decreasing, positive, and real-valued Lévy process $S := (T_t)_{t \geq 0}$, i.e., S is a subordinator process that models the random switching instants of the system (1). The subordinator S is defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with Ω being the event space, $\mathcal{F}$ being its sigma algebra, and $\mathbb{P}$ being a probability measure. Moreover, let $(\mathcal{F}_t)_{t \geq 0}$ denote the filtration generated by S and $\mathbb{E}[\cdot]$ denote the expectation functional with respect to the probability measure $\mathbb{P}$. The piecewise continuous function $\sigma : [0, \infty) \to \{1, 2, \ldots, q\}$ such that $\sigma(t) = i_k$ when $t \in [t_k, t_{k+1})$ is called a *switching signal*. The sequence π specifies the order in which the j-th subsystem, $j \in \{1, 2, \ldots, q\}$, is active between the switching instants $t_k = T_k$ and $t_{k+1} = T_{k+1}$. We assume that the function f_j, $j \in \{1, 2, \ldots, q\}$, is locally Lipschitz with respect to its second argument and piecewise continuous with respect to its other arguments.

Definition 1: Given a subordinator $S = (T_t)_{t\geq 0}$, system (1) is SISS if, for all $\epsilon \in (0,1)$, there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for every $x_0 \in \mathbb{R}^n$ and every piecewise continuous bounded input function $u : [0,\infty) \to \mathbb{R}^m$, the estimate

$$\mathbb{P}\{|x(t)| \leq \beta(|x_0|, t) + \gamma(|u|_{[0,t]})\} \geq 1 - \epsilon$$

holds for all $t \geq 0$ along the solutions of (1).

Definition 2 ([1]): Let $L = (L_t)_{t\geq 0}$ be a Lévy process then the characteristic exponent η (also referred to as *Lévy symbol*) of the process L is a function $\eta : \mathbb{R}^n \to \mathbb{C}$ given by the mapping $z \mapsto e^{t\eta(z)}$. It can be obtained from the characteristic function of L, i.e., $\mathbb{E}\Big[e^{i\langle z, L_t\rangle}\Big] = e^{t\eta(z)}$ for all $t \geq 0$ and $z \in \mathbb{R}^n$.

The following lemmas are requisite to prove our main result in the next section.

Lemma 1 ([13]): Let $(L_t)_{t\geq 0}$ be an $\mathbb{R}^n$-valued pure-jump Lévy process and let $f : [0,\infty) \to \mathbb{R}^n$ be a continuous function. Then the integral process $I_{s,t} = \int_s^t f(r) dL_r$ can be written as $I_{s,t} = \sum_{k=1}^{\infty} \mathbf{1}_{s\leq r_k \leq t} f(r_k)\Delta L_{r_k}$, where $\Delta L_{r_k} = L_{r_k} - L_{r_k^-}$.

Lemma 2: Let $(L_t)_{t\geq 0}$ be a Lévy process and let $\xi_{s,t} = h(t)\int_s^t f(r)dL_r$ be an integral process. Then the (double) integral process $I_{s,t} = \int_s^t \xi_{s,r} dr$ can be written as $I_{s,t} = \int_s^t g_{r,t} dL_r$ with

$$g_{r,t} := f(r)\int_r^t h(p)dp. \tag{2}$$

Proof: Observe that

$$\begin{aligned} I_{s,t} &= \int_s^t h(p)\int_s^p f(r)\, dL_r\, dp \\ &= \int_s^t \int_s^t f(r)h(p)\mathbf{1}_{s\leq r\leq p}(r)\, dL_r\, dp \\ &\overset{\text{Fubini}}{=} \int_s^t \int_s^t f(r)h(p)\mathbf{1}_{s\leq r\leq p}(p)\, dp\, dL_r \\ &= \int_s^t g_{r,t}\, dL_r \end{aligned}$$

with $g_{r,t}$ as defined in (2). This concludes the proof. ■

Lemma 3: Let $(L_t)_{t\geq 0}$ be a Lévy process with symbol η such that the mapping $z \mapsto \eta(z)$ is analytically extendible to the imaginary axis $\{-iz \,|z \in \mathbb{R}^n\}$. In particular, let η be such that

$$\mathbb{E}(e^{\langle z, L_t\rangle}) = e^{t\eta(-iz)}, \quad z \in \mathbb{R}^n. \tag{3}$$

Let $I_{s,t} = \int_s^t h(p)\int_s^p f(r)\, dL_r\, dp$ be the (double) integral process. Then

$$\mathbb{E}(e^{I_{s,t}}) = e^{\int_s^t \eta(-ig)dr}. \tag{4}$$

Proof: Using first Lemma 2 and then Lemma 1, it follows that

$$\begin{aligned} \mathbb{E}\big(e^{I_{s,t}}\big) &= \mathbb{E}\big(e^{\int_s^t g_{r,t} dL_r}\big) \\ &= \mathbb{E}\big(\lim_{n\to\infty} e^{\sum_{k=1}^n \mathbf{1}_{s\leq r_k\leq t} g_{r_k,t}\Delta L_{r_k}}\big) \\ &= \mathbb{E}\big(\lim_{n\to\infty}\prod_{k=1}^n e^{\mathbf{1}_{s\leq r_k\leq t} g_{r_k,t}\Delta L_{r_k}}\big) \\ &= \lim_{n\to\infty}\prod_{k=1}^n \mathbb{E}(e^{\mathbf{1}_{s\leq r_k\leq t} g_{r_k,t}\Delta L_{r_k}}). \end{aligned}$$

Using (3) and defining $\Delta r_k := r_k - r_k^-$, it follows that

$$\begin{aligned} \mathbb{E}(e^{I_{s,t}}) &= \lim_{n\to\infty}\prod_{k=1}^n \mathbf{1}_{s\leq r_k\leq t} e^{\Delta r_k \eta(-ig_{r_k,t})} \\ &= \lim_{n\to\infty} e^{\sum_{k=1}^n \mathbf{1}_{s\leq r_k\leq t}\eta(-ig_{r_k,t})\Delta r_k} \\ &= e^{\int_s^t \eta(-ig_{r,t})dr}. \end{aligned}$$

This concludes the proof. ■

Example 1: Let L_t be a Poisson process with rate λ, its symbol $\eta(\cdot)$ is given by $\eta(z) = \lambda(e^{iz} - 1)$. Then we have

$$e^{\int_0^t \eta(-ig)\, ds} = e^{-\lambda t} e^{\lambda \int_0^t e^g ds}. \tag{5}$$

Example 2: When L_t is a compound Poisson process with rate λ and associated Lévy measure μ, its symbol is given by $\eta(u) = \int_{\mathbb{R}} \lambda(e^{iuz} - 1)\mu(dz)$. Then we have

$$\eta(-ig) = \lambda\int_{\mathbb{R}} (e^{gz} - 1)\mu(dz) = \lambda\Big(\int_{\mathbb{R}} e^{gz}\mu(dz) - 1\Big).$$

(a) For $\mu \sim \mathcal{U}(\Theta)$, where $\Theta \subset \mathbb{R}\backslash\{0\}$ is a discrete countable and finite set, $\eta(-ig) = \lambda\Big(\frac{1}{\#\Theta}\sum_{k=1}^{\#\Theta} e^{gz_k} - 1\Big)$. We have

$$e^{\int_0^t \eta(-ig)\, ds} = e^{-\lambda t}\prod_{k=1}^{\#\Theta} e^{\frac{\lambda}{\#\Theta}\int_0^t e^{gz_k} ds}. \tag{6}$$

(b) For $\mu \sim \mathcal{N}(\vartheta, \varsigma^2)$, $\eta(-ig) = \lambda\Big(e^{\frac{\varsigma^2 g^2}{2} + \vartheta\varsigma g} - 1\Big)$,

$$e^{\int_0^t \eta(-ig)\, ds} = e^{-\lambda t} e^{\lambda\int_0^t \exp\left(\frac{\varsigma^2 g^2}{2} + \vartheta\varsigma g\right) ds}. \tag{7}$$

III. Main Result

This section provides the main result, i.e., a sufficient SISS condition for the randomly switched system (1), but before stating and proving it, we provide some assumptions.

Assumption 1: There exist q locally absolutely continuous functions $V_j : [0,\infty) \times \mathbb{R}^n \to [0,\infty)$ for $j \in \{1, 2, \ldots, q\}$, class $\mathcal{K}_\infty$ functions α_1 and α_2, and class $\mathcal{K}_\infty$ functions $\delta_1, \delta_2, \ldots, \delta_q$ such that

$$\alpha_1(|x|) \leq V_j(t, x) \leq \alpha_2(|x|) \tag{8}$$

hold almost surely for $t \in [0,\infty)$, $j \in \{1, 2, \ldots, q\}$, and given a Lévy process $(L_t)_{t\geq 0}$ with symbol η and $h, f \in \mathcal{C}(\mathbb{R})$, there exists an $\mathcal{F}_{t-s}$ adapted process $\xi_{s,t} = h(t)\int_s^t f(r)dL_r$, $t > s \geq 0$ such that

$$\dot{V}_{\sigma(t_k)}(t, x) \leq \kappa_{\sigma(t_k)} V_{\sigma(t_k)}(t, x) + \delta_{\sigma(t_k)}(|u(t)|) \tag{9}$$

holds almost surely along all trajectories of the system (1) for almost all $t \in [t_k, t_{k+1})$, all $k \in \mathbb{N}_0$, and all choices of the piecewise continuous bounded function $u : [0,\infty) \to \mathbb{R}^m$, where $\kappa_{\sigma(t)}$ is a realization of the process $\xi_t := \xi_{0,t}$, $t \geq 0$.

Remark 1: We model the jump distribution of $\kappa_{\sigma(t)}$ by the background Lévy process $(L_t)_{t\geq 0}$ in Assumption 1. The switching instants $(t_k)_{k\in\mathbb{N}_0} = (T_k)_{k\in\mathbb{N}_0}$ of the Lévy process $(L_t)_{t\geq 0}$, driving the integral process $k_{\sigma(t)} = \xi_{0,t}$ in Assumption 1, are precisely determined by the subordinator process $S = (T_t)_{t\geq 0}$. Moreover, the value $\kappa_{\sigma(t_k)} = \kappa_j$, $j \in \{1, 2, \ldots, q\}$, is random during $t \in [T_k, T_{k+1})$, which allows us to study systems with time-varying parameters, as illustrated in Section IV.

Assumption 2: The functions V_j from Assumption 1 admit a constant $\mu > 0$ such that

$$V_i(t,x) \leq e^{\mu} V_j(t,x) \tag{10}$$

holds almost surely for all $i, j \in \{1, 2, \ldots, q\}$ and all $t \in [0, \infty)$.

Assumption 3: For $[s,t] \subset [0,\infty)$ with $t - s < \infty$ and $f, h \in \mathcal{C}(\mathbb{R})$ with $|f|, |h| < \tilde{\varphi}$, let $\xi_{s,t} := h(t)\int_s^t f(r) dL_r$ and $I_{s,t} = \int_s^t \xi_{s,r} dr = \int_s^t g_{r,t} dL_r$ be such that

$$\mathbb{E}[\max(e^{\varphi_2^2 (t-s)|L_{t-s}|}, 1)] < c < \infty,$$

where $\varphi_2 \geq c_1(1+\tilde{\varphi})$ for $c_1 > 0$. Moreover, for $\mu > 0$ as in Assumption 2, let $\xi^{\mu}_{s,t} := h(t)\int_s^t [f(r) + \mu]\, dL_r$ and $I^{\mu}_{s,t} = \int_s^t \xi^{\mu}_{s,r} dr = \int_s^t g^{\mu}_{r,t} dL_r$ be such that

$$\mathbb{E}(e^{I^{\mu}_{s,t}}) \leq \varphi_1 e^{-\zeta\varphi_2(t-s)}$$

for some fixed $\varphi_1, \zeta > 0$.

Remark 2: Assumption 3 provides the sufficient dwell-time condition in terms of the properties (Lévy symbol η and function g) of the Lévy-driven process $\xi_{s,t}$ that models the rate function $\kappa_{\sigma(t)}$ of the family of Lyapunov-like functions to ensure stability of the switched system. Various examples of $\xi_{s,t}$, for different cases of Lévy processes, that fulfill Assumption 3 have been provided in Section IV.

Remark 3: For some arbitrary $c > 0$, taking $\kappa_{\sigma(t)} = -c + \xi_{0,t}$, we get $e^{I^{\mu}_{0,t}} = e^{-c(t)} e^{\int_0^t g^{\mu}_{r,t} dL_r}$. We use this relation in the illustrative example Case 1 provided in Section IV. However, we ignore it here in the analysis since a negative constant is a trivial extension.

For the sake of brevity, we use the following notation:

$$v(t,x) := V_{\sigma(t)}(t,x),\ \xi_t = \kappa_{\sigma(t)},\ \bar{\delta}(r) := \max_{1\leq j\leq q} \delta_j(r),$$

$$\Lambda(\ell,t) := e^{I_{\ell,t}},\ \Gamma_k := e^{\mu}\Lambda(t_{k-1},t_k),\ \Lambda^{\mu}(\ell,t) := e^{I^{\mu}_{\ell,t}}\ .$$

Now we formally state and prove the main result.

Theorem 1: Let Assumptions 1-3 hold. Then, system (1) is SISS.

Proof: Observe from Assumption 1 that for unique choices of $k \in \mathbb{N}_0$ and $r = \{1, 2, \ldots, q\}$ such that $\sigma(t_k) = r$, the inequality

$$\dot{V}_r(p,x) \leq \kappa_r V_r(p,x) + \delta_r(|u(p)|) \tag{11}$$

is satisfied almost surely for every $p \in [t_k, t_{k+1})$. Multiplying both sides of (11) by the integrating factor $e^{-\kappa_r p}$, then moving the resulting term $e^{-\kappa_r p}\kappa_r V_r(p,x)$ from right side to left side, and then integrating both sides between two instants $t_* \in [t_k, t]$ and $t \in [t_k, t_{k+1})$, we get

$$V_r(t,x) \leq e^{\kappa_r(t-t_*)} V_r(t_*,x) + \int_{t_*}^t e^{\kappa_r(t-s)} \delta_r(|u(s)|) ds\ .$$

With the choice of $t_* = t_k$ and using simplifying notation $v(t)$ to mean $v(t,x)$, it follows that for all $k \in \mathbb{N}_0$ and all $t \in [t_k, t_{k+1})$,

$$\begin{aligned} v(t) &\leq \Lambda(t_k,t)v(t_k) + \int_{t_k}^t \Lambda(s,t)\bar{\delta}(|u(s)|)ds \\ &\leq e^{\mu}\Lambda(t_k,t)v(t_k^-) + \int_{t_k}^t \Lambda(s,t)\bar{\delta}(|u(s)|)ds\ , \end{aligned} \tag{12}$$

where the last inequality is a consequence of (10). Now, let us consider $t \geq 0$ such that $t \in [t_k, t_{k+1})$ for some $k \in \mathbb{N}_0$ and $\rho \in \mathbb{N}$ such that $[0, t_0) = [t_{k-\rho-1}, t_{k-\rho})$. We deduce that

$$\begin{aligned} v(t_k^-) &\leq \Gamma_k v(t_{k-1}^-) + \int_{t_{k-1}}^{t_k} \Lambda(s,t_k)\bar{\delta}(|u(s)|)ds \\ &\ \ \vdots \\ v(t_{k-\rho+1}^-) &\leq \Gamma_{k-\rho+1} v(t_{k-\rho}^-) \\ &\quad + \int_{t_{k-\rho}}^{t_{k-\rho+1}} \Lambda(s,t_{k-\rho+1})\bar{\delta}(|u(s)|)ds \end{aligned} \tag{13}$$

with

$$\begin{aligned} v(t_{k-\rho}^-) &\leq \Lambda(0,t_{k-\rho})v(0) \\ &\quad + \int_0^{t_{k-\rho}} \Lambda(s,t_{k-\rho})\bar{\delta}(|u(s)|)ds\ . \end{aligned} \tag{14}$$

We deduce from (13) that

$$\begin{aligned} v(t_k^-) &\leq \Lambda^{\mu}(t_{k-\rho},t_k)v(t_{k-\rho}^-) \\ &\quad + \int_{t_{k-\rho}}^{t_k} \Lambda^{\mu}(s,t_k)\bar{\delta}(|u(s)|)ds\ . \end{aligned} \tag{15}$$

Using (14) to upper bound $v(t_k^-)$ in (15), we get

$$v(t_k^-) \leq \Lambda^{\mu}(0,t_k)v(0) + \int_0^{t_k} \Lambda^{\mu}(s,t_k)\bar{\delta}(|u(s)|)ds. \tag{16}$$

Similarly, using (16) to upper bound $v(t)$ in (12), we get

$$v(t) \leq \Lambda^{\mu}(0,t)v(0) + \int_0^t \Lambda^{\mu}(s,t)\bar{\delta}(|u(s)|)ds\ . \tag{17}$$

Since the initial condition is deterministic, taking expectation on both sides of (17) with with $v_0 := v(0)$, we get

$$\begin{aligned} \mathbb{E}[v(t)] &\leq \mathbb{E}\big[\Lambda^{\mu}(0,t)\big]v_0 + \int_0^t \mathbb{E}\big[\Lambda^{\mu}(s,t)\big]\mathbb{E}\big[\bar{\delta}(|u(s)|)\big]ds \\ &\leq \varphi_1 e^{-\zeta\varphi_2 t}v_0 + \bar{\delta}(|u|_{[0,t]})\int_0^t \varphi_1 e^{-\zeta\varphi_2(t-s)}ds \\ &\leq \varphi_1 e^{-\zeta\varphi_2 t}v_0 + \frac{\varphi_1\bar{\delta}(|u|_{[0,t]})}{\zeta\varphi_2}. \end{aligned} \tag{18}$$

Using (8) and applying Markov's inequality [14, Chapter II, 18.1] to (18), we have

$$\mathbb{P}\Big\{|x(t)| \le \alpha_1^{-1}\big(\tfrac{2\varphi_1 e^{-\zeta\varphi_2 t} v_0}{\epsilon}\big) + \alpha_1^{-1}\big(\tfrac{2\varphi_1 \bar{\delta}(|u|_{[0,t]})}{\zeta\varphi_2 \epsilon}\big)\Big\} \ge 1-\epsilon \tag{19}$$

for an arbitrary $\epsilon \in (0,1)$. To reach the inequality (19), we employ the property $\alpha_i(a+b) \le \alpha_i(2a) + \alpha_i(2b)$ of our functions $\alpha_i \in \mathcal{K}_\infty$ for all $a \ge 0$ and $b \ge 0$ (which follows by separately considering the cases $a \ge b$ and $a < b$).

Using $v_0 \le \alpha_2(|x_0|)$ from (8) in (19), we arrive at

$$\mathbb{P}\big\{|x(t)| \le \beta(|x_0|, t) + \gamma(|u|_{[0,t]})\big\} \ge 1-\epsilon,$$

where

$$\beta(r,s) = \alpha_1^{-1}\big(\tfrac{2\varphi_1 e^{-\zeta\varphi_2 s}\alpha_2(r)}{\epsilon}\big),\ \gamma(p) = 2\alpha_1^{-1}\big(\tfrac{2\varphi_1 \bar{\delta}(p)}{\zeta\varphi_2 \epsilon}\big).$$

This concludes the proof. ■

IV. Illustration

In this section, we provide a numerical example to illustrate the effectiveness of the main result. To this end, consider a switched nonlinear system

$$\dot{x}(t) = f_{\sigma(t)}(x(t), u(t)), \quad \sigma : [0,\infty) \to \{1,2,3\} \tag{20}$$

with

$$\begin{aligned} f_1(x(t),u(t)) &= \begin{bmatrix} a_1(t)x_1(t) + u(t)\sin^2 x_2(t) \\ b_1(t)x_2(t) \end{bmatrix} \\ f_2(x(t),u(t)) &= \begin{bmatrix} a_2(t)x_1(t) - x_1^3(t) + u(t) \\ b_2(t)x_2(t) \end{bmatrix} \\ f_3(x(t),u(t)) &= \begin{bmatrix} a_3(t)x_1(t) + 3u(t) \\ b_3(t)x_2(t) \end{bmatrix}, \end{aligned}$$

where $a_\ell : [0,\infty) \to \mathbb{R}$ and $b_\ell : [0,\infty) \to \mathbb{R}$ for $\ell \in \{1,2,3\}$ are bounded piecewise continuous functions.
Let $V_1(x) = \frac{1}{2}(x_1^2 + x_2^2)$, $V_2(x) = x_1^2 + x_2^2$, and $V_3(x) = \frac{1}{4}(x_1^2 + x_2^2)$ be positive definite quadratic functions. Then using Young's inequality, it can be shown for $V_\ell(t) := V_\ell(x(t))$, $\ell \in \{1,2,3\}$ that the inequality

$$\dot{V}_\ell(t) \le \kappa_\ell V_\ell(t) + \delta_\ell ||u(t)||^2, \quad t \in [t_k, t_{k+1}),\ k \in \mathbb{N}_0$$

holds almost surely, along the solutions of (20), for some real numbers $\delta_\ell > 0$ and

$$\kappa_\ell = \max_{t\in[t_k,t_{k+1}]} [1 + \max(a_\ell(t), b_\ell(t))],\ \ \ell \in \{1,2,3\}.$$

Let $S = (T_t)_{t\ge 0}$ denote a Gamma subordinator, i.e. a subordinator process with $T_t \sim \Gamma(t;1,\lambda)$. Let L_t be a pure-jump Lévy process that takes a certain value, based on its jump distribution, at the jump instant given by T_t. Based on Assumption 1, $\kappa_{\sigma(t)}$ is given by $\xi_{0,t}$ that satisfies Assumption 3. For system (20), it has the following implication: let $T_t^\ell = (T_j^\ell)_{j\in\mathbb{N}_0} = (T_{k_j})_{j\in\mathbb{N}_0} \subset S$ be the sequence of random switch instants when the ℓ^{th} sub-systems become active, then $\kappa_{\sigma(t)} = \kappa_\ell = \kappa_{T_t^\ell} = \kappa_{T_{k_j}}$, where

$$\kappa_{T_{k_j}} = \max_{t\in[T_{k_j},T_{k_j+1}]} [1 + \max(a_\ell(t), b_\ell(t))].$$

Thus, due to the random switching instants and time dependence of a_ℓ, b_ℓ, the upper-bound κ_ℓ is also random. Consequently, κ_ℓ and $\kappa_{\sigma(t)}$ are piecewise constant processes.

According to Assumption 1, the jump process $\kappa_{\sigma(t)}$ is driven by a Lévy process, and it is equal to $\xi_{T_{k_j}}$ for $t \in [T_{k_j}, T_{k_j+1})$.

Now we consider various types of switching signals σ by considering different functions f, h and different jump processes (L_t) for which Assumption 3 applies and show that the above family of randomly switched systems is SISS. We choose $\lambda = 4$ and $\mu = 2$ for rest of the example.

Let the switching signal σ be defined as

$$\sigma(t) := \begin{cases} 1, & \text{if } \xi_{0,t} \in (-\infty, -3/2), \\ 2, & \text{if } \xi_{0,t} \in [-3/2, 0), \\ 3, & \text{if } \xi_{0,t} \in [0, \infty). \end{cases}$$

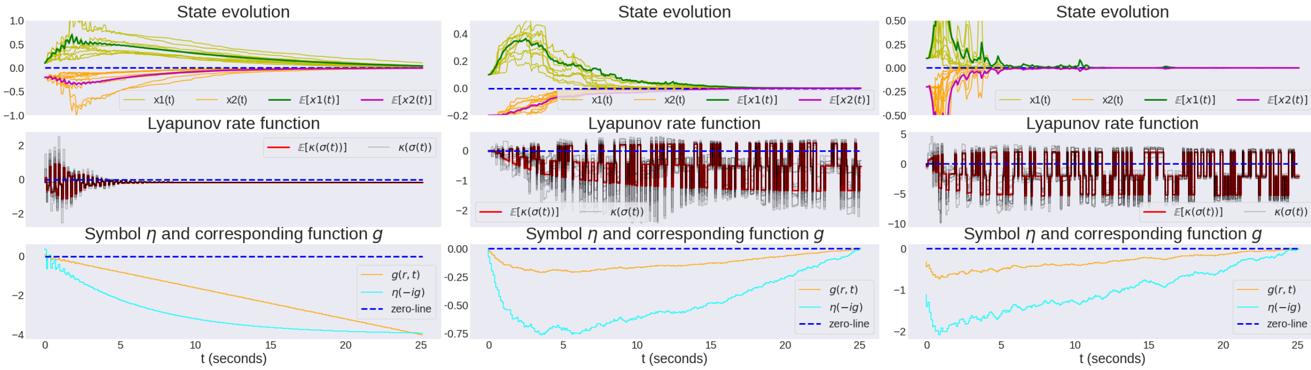
Case 1: Let $h(t) = \sin(nt)e^{-at}$, $f(t) := e^{bt}$ with $a > b > 0$, $(L_t)_{t\ge0}$ be a Poisson process $(N_t)_{t\ge0}$, and define $\kappa_{\sigma(t)} := -c + \xi_{0,t}$ with $c > 0$. Then, for any $\mu \ge 0$, we have

$$\begin{aligned} g_{r,t} &= (\mu + f(r)) \textstyle\int_r^t h(s)ds \\ &= \tfrac{(\phi(nr)e^{-(a-b)r} - \phi(nt)e^{br-at})}{(1+n^2)} + \tfrac{(\mu\phi(nr)e^{-ar} - \mu\phi(nt)e^{-at})}{(1+n^2)} \\ &\le \tfrac{1+n}{(1+n^2)}\Big(1 - e^{b(r-t)} + \mu(e^{-ar} - e^{-at})\Big) < \tfrac{(1+n)(1+\mu)}{(1+n^2)} \end{aligned}$$

for all $0 < r < t$, where $\phi(nt) := \sin(nt) + n\cos(nt)$. Now using the relation (5) and Remark 3 for $\lambda > 1$ and $c \in (0,1)$, choosing $n > \mu + 2$ such that $\frac{(1+n)(1+\mu)}{(1+n^2)} < c$ we see that Assumption 3 is fulfilled. For the simulation, we have chosen $c = 0.16, n = 20, \lambda = 4, a = 1.5, b = 1$. The simulation results are as shown in Figure 1a. From the plots, we observe that initial κ is oscillating because of which the solutions are unstable, however, as time evolves κ converges to $-c$, thus indicating a stabilization effect. Also looking at the cyan curve in the lower subplot, we see that symbol η is consistently below zero, thus indicating the fulfillment of sufficient condition for SISS.

Case 2: Let $h(t) = \frac{1}{(a+t)^2}$ with $a \ge 1$, $f(t) := t$, and $(L_t)_{t\ge0}$ a compound Poisson process with a jump distribution $\mathcal{U}(\Theta)$, $\Theta := \{5/6, -3, -3/2, 1/10, -2\}$. Let us define $\kappa_{\sigma(t)} := \phi(t)\xi_{0,t}$, where ϕ being a piecewise constant function such that $\int_r^t \phi^+ > \int_r^t \phi^-$ for all $0 < r < t$ and $g_{r,t} = (f(r) + \mu)\int_r^t \phi(s)h(s)ds \le \tilde{\phi}(r,t)\Big(\frac{(r+\mu)(t-r)}{a^2 + a(t+r) + rt}\Big)$. Then using the relation (6), we get that $\mathbb{E}_{\mathcal{U}}[e^{gz}] = \frac{1}{5}\sum_{z\in\Theta} e^{gz} \Rightarrow \mathbb{E}_{\mathcal{U}}[e^{gz}] < 1$, which consequently ensures that Assumption 3 is fulfilled. The simulation results are as shown in Figure 1b. For the simulation, we set $a = 1$, and ϕ is obtained in the following way: Let $(s_k)_{k\in\mathbb{N}}$ be the uniform discretization of the simulation time interval $[0,t]$, then $\phi(s_k) \in \mathcal{U}(\{1.3, 0.5, -0.25\})$ and $\phi(s) = \phi(s_k)$ for $s \in [s_k, s_{k+1})$. The simulation results are as shown in Figure 1b. From the plots, we observe that due to the oscillatory signal ϕ, the mean κ (as well as its sample paths) also shows oscillation, with jumps going above zero. However, because the symbol η (cyan curve in lower subplot) is consistently below zero, the solutions eventually converge to equilibrium.

Case 3: Let h be as in Case 2 with $a = 2$ and $f(t) = 1 + t$. Let $(L_t)_{t\ge0}$ a compound Poisson process with a jump



(a) Simulation results for case 1.(IV).

(b) Simulation results for case 2.(IV).

(c) Simulation results for case 3.(IV).

Fig. 1: Simulation results for all three examples. The top plot shows the evolution of the state variables (x_1, x_2) as a function of time. The mean values are depicted as bold curves (green for x_1 and magenta for x_2), while sample paths are denoted in yellow (x_1) and orange (x_2) lines. The middle plot shows the rate function $\kappa_{\sigma(t)}$ as a function of time. Both mean (in bold red) and sample paths (faint black) are depicted. In the lower subplots, we see the corresponding g variable appearing in the symbol of the Lévy process. This plots clearly shows the sufficient condition (Assumption 3) for SISS.

distribution $\mathcal{N}(\vartheta, \varsigma^2)$ with $\vartheta < -\frac{\varsigma g}{2}$. Let us define $\kappa_{\sigma(t)} := \phi(t)\xi_{0,t}$ with $\phi(t)$ as in Case 2, then again we get that $\mathbb{E}_{\mathcal{N}}[e^{gz}] < 1$, which consequently ensures that Assumption 3 is fulfilled. The function ϕ is obtained analogous to Case 2 with $\phi(s_k) \in \mathcal{U}(\{1.3, .5, -.5\})$ and $\phi(s) = \phi(s_k)$ for $s \in [s_k, s_{k+1})$. The other involved constants are chosen as $\vartheta = -2$, $\varsigma = 1.5$. The simulation results are as shown in Figure 1c. From the plots, we observe that due to oscillatory signal ϕ and the Gaussian jump distribution of the driving Lévy process, there are more instabilities in the solution. The effects of the oscillatory signal ϕ can be seen as oscillations in the mean κ (and also its sample paths), where there are jumps going above zero, depicting unstable subsystems. Despite of these oscillations, because the symbol η (cyan curve in lower subplot) is consistently below zero, thus the solutions eventually converge to equilibrium.

V. Conclusions

We provided a sufficient condition for SISS of randomly switched systems for the challenging case when some of the subsystems may be unstable. We modeled the random switching instants by a non-decreasing, positive, and real-valued Lévy process. To establish SISS, we relied on the theory of the Lévy process and its characterization via the Lévy symbol η to establish an upper bound on the expected value of the semigroup operator. This approach enabled us to generalize the applicability of the result to systems with time-varying and uncertain parameters. We illustrated our result via simulation-based examples for three different types of underlying Lévy processes.

Some future extensions of this work include: (i) characterizing SISS of randomly switched systems driven by a stochastic noise, (ii) studying SISS of impulsive switched systems with Lévy switching signals, (iii) exploring SISS of infinite-dimensional randomly switched systems, (iv) investigating SISS for randomly switched systems with time-varying delays, and (v) applying the results of this work to real-world applications.

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