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Brief paper

Model reduction with pole-zero placement and high order moment matching☆

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ABSTRACT

In this paper, we calculate a low order model of a linear system of large dimension, that matches a set of high order moments of the transfer function and achieves pole-zero placement constraints. The model satisfying all the constraints simultaneously is selected from a family of parametrized reduced order models. The parameters are computed solving an explicit linear algebraic system. Furthermore, we construct the Loewner matrices from the given data and the imposed pole-zero and first order moment constraints. The resulting approximations achieve a trade-off between good norm approximation and the preservation of the dynamics of the given system in a region of interest. The theory is illustrated on the academic example of the cart controlled by a double pendulum and the practical example of the CD player.

1. Introduction

Mathematical modelling of physical and industrial plants yields high-dimensional linear, time-invariant (LTI) systems. Model reduction, to find low-order approximations meeting desired constraints, is called for. Moment matching-based approximation techniques stand out as computationally efficient and easy to implement (Antoulas, 2005). The notion of moment is related to the unique solution of a Sylvester equation, see Gallivan, Vandendorpe, and Van Dooren (2004, 2006). For a given high-dimensional system, families of parametrized low order models are computed, based on the time-domain approach to moment matching in Astolfi (2010) and Ionescu, Astolfi, and Colaneri (2014).

Motivation and contributions. Fixing *all* the parameters in the family, provides the unique low order model that meets a *single* required constraint. For instance, in Astolfi (2010), the free parameters are selected such that the stability or the relative degree are preserved. In Ionescu et al. (2014), the free parameters are used to compute the *unique* reduced model of *minimal order* that matches all the moments of the given system. Furthermore, in Ionescu and Iftime (2012) the computation of families of stable LTI low order models for infinite-dimensional systems is addressed, using state-feedback stabilization arguments. In Anic, Beattie, Gugercin, and Antoulas (2013), Gugercin, Antoulas, and Beattie (2008), matching zero and first order moments of the system at the mirror images of the poles of the approximant yields the model with the lowest H_2-norm of the approximation error. In Ionescu (2016), the model that matches a double number of moments as well as the model that matches the moments of the given system and its first order derivative are computed. Recently, in Necoara and Ionescu (2018, 2020), using optimization algorithms, the model achieving the minimal H_2-norm of the approximation error has been found. Furthermore, in Ibrir (2017), optimization methods are used for minimizing a mixed H_2/H_∞ small-gain criterion yielding a local minimizer. However, all the aforementioned techniques inherently place the poles and/or zeros of the reduced order models at arbitrary locations in the complex plane, e.g., close to the imaginary axis, losing practical desired behaviours. The methods either focus on the placement of poles such that constraints on stability are met or such that the approximation error is minimized. To the best of our knowledge, in model reduction, the simultaneous preservation of *multiple* properties such as fixing stable poles and zeros and matching high order moments is not solved. The work in this paper was inspired

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by Datta, Chakraborty, and Chaudhuri (2012), where a state-feedback controller is designed to place some poles of the plant while the rest of the closed-loop poles are constrained at given locations. We focus on the trade-off between the preservation of desired properties and the approximation accuracy, i.e., we place some poles and zeros at prescribed locations and match a number of high(er) order moments to decrease the approximation error. The parameters of the model are computed solving an *explicit* linear algebraic system. In this paper, we seek a ν order approximation that simultaneously satisfies *multiple* properties, i.e., matches ν moments of orders $0:j_i$ at ν interpolation points of multiplicity j_i+1, has ℓ poles, k zeros and matches $\nu-(\ell+k)$ moments of orders $1:j_i+1, i=1:\varsigma$, such that $\sum_i (j_i+1)=\nu$. By $a:b$ we mean all the integers between the integers a and b. We provide a linear system that yields the sufficient condition on the free parameters to place $\ell \le \nu$ poles. For a particular canonical form of the interpolation points, we write the necessary and sufficient condition on the free parameters for the pole placement. We also derive the linear system yielding the sufficient condition to place $k<\nu$ zeros. For a particular canonical form of the interpolation points, we write the necessary and sufficient condition on the free parameters for the zero placement. Moreover, we write the linear system such that $\nu-(\ell+k)$ moments of orders $1:j_i+1$ are matched. Then, we construct the Loewner matrices (see, e.g., Beattie and Gugercin (2012), Gosea, Zhang, and Antoulas (2020) and Mayo and Antoulas (2007) for model reduction and, e.g., Kergus, Formentin, Poussot-Vassal1, and Demourant (2018) for control) that include the available data and imposed pole, zero and first order moment constraints. We compute (for a particular case) the *equivalent* Loewner-based reduced order model of order ν that matches ν zero order moments, places ℓ poles and k zeros at imposed locations and, furthermore, matches $\nu-(\ell+k)$ first order moments, achieving (partial) Hermite interpolation. The resulting reduced order models achieve a trade-off between good error norm approximation and the preservation of the dynamics in a desired region of interest.

Content. In Section 2, we recall the time-domain moment matching for linear systems. In Section 3, we solve the sets of linear constraints to place poles, zeros and match further moments, respectively. In Section 4, we provide a relation between the main results of the manuscript (presented in Section 3) and the Loewner matrices framework ($j_i=0$). In Section 5, we illustrate the theory on the academic example of the cart controlled by a double pendulum and the practical example of the CD player.

Notation. $\mathbb{R}$ is the set of real numbers and $\mathbb{C}$ is the set of complex numbers. $\mathbb{C}^-$ is the complex open left half plane. If A is a matrix, then A^T is the transpose. $\sigma(A)$ is the spectrum of A. Let $K:\mathbb{C}\to\mathbb{C}$, then $K'(s)=\mathrm{d}K(s)/\mathrm{d}s$.

2. Preliminaries

Consider a single input-single output (SISO) linear time-invariant (LTI) minimal system

$$\Sigma:\quad \dot{x}(t)=Ax(t)+Bu(t),\quad y(t)=Cx(t),\quad t\ge 0, \tag{1}$$

with the state $x(t)\in\mathbb{R}^n$, the input $u(t)\in\mathbb{R}$, the output $y(t)\in\mathbb{R}$, $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^n$ and $C\in\mathbb{R}^{1\times n}$. The transfer function of (1) is

$$K:\mathbb{C}\to\mathbb{C},\quad K(s)=C(sI-A)^{-1}B. \tag{2}$$

Throughout the rest of the paper, we assume that the system (1) is stable, i.e., $\sigma(A)\subset\mathbb{C}^-$. Note that $\sigma(A)$ is a symmetric set of complex numbers, i.e., if $\lambda\in\sigma(A)$ then $\bar{\lambda}\in\sigma(A)$, including multiplicities. We now present the notion of moments of K, given by (2), as in Astolfi (2010).

Definition 1. The 0-moment of K, as in (2), at $s^*\in\mathbb{C}\setminus\sigma(A)$ is $\eta_0(s^*)=K(s^*)\in\mathbb{C}$. The k-moment of K at $s^*\in\mathbb{C}\setminus\sigma(A)$ is $\eta_k(s^*)=(-1)^k/(k!)\left[\mathrm{d}^k K(s)/\mathrm{d}s^k\right]_{s=s^*}\in\mathbb{C}$.

Let $\{s_i\in\mathbb{C}\setminus\sigma(A)\mid i=1:\varsigma\}$, be a symmetric set of complex numbers (including multiplicities). Take $j_i\ge 0$ such that $\sum_{i=1}^{\varsigma}(j_i+1)=\nu$. For each i, let $\eta_0(s_i),\ldots,\eta_{j_i}(s_i)$ denote the ν moments of orders $0:j_i$ of K at the given points s_i. Let $S\in\mathbb{R}^{\nu\times\nu}$, with the symmetric spectrum $\sigma(S)=\{s_i\mid i=1:\varsigma\}$, be such that $\sigma(S)\cap\sigma(A)=\emptyset$. Let $L\in\mathbb{R}^{1\times\nu}$ be such that the pair (L,S) is observable.[1] Let $\Pi\in\mathbb{R}^{n\times\nu}$ be the solution of the Sylvester equation

$$A\Pi+BL=\Pi S. \tag{3}$$

Since the system is minimal and $\sigma(A)\cap\sigma(S)=\emptyset$, then Π is the unique solution of Eq. (3), with rank $\Pi=\nu$, see e.g. de Souza and Bhattacharyya (1981). Then, the moments of K are uniquely determined by the elements of the vector $C\Pi\in\mathbb{R}^{1\times\nu}$.

Proposition 1 (*Astolfi, 2010*)**.** *The ν moments $\eta_0(s_i),\ldots,\eta_{j_i}(s_i)$, $i=1:\varsigma$, of K at $\sigma(S)$ are in one-to-one relation with the elements of the vector $C\Pi$.*

Consider the LTI system $\dot{\xi}=F\xi+Gu$, $\psi=H\xi$, with $F\in\mathbb{R}^{\nu\times\nu}$, $G\in\mathbb{R}^{\nu}$ and $H\in\mathbb{R}^{1\times\nu}$, and the corresponding transfer function $K_G(s)=H(sI-F)^{-1}G$. Let $\hat{\eta}_0(s_i),\ldots,\hat{\eta}_{j_i}(s_i)$ denote the moments of orders $0:j_i$ of K_G at s_i. Then, moment matching is defined as follows.

Definition 2. K_G matches ν moments of K at $\{s_1,\ldots,s_\varsigma\}$, if $\eta_\kappa(s_i)=\hat{\eta}_\kappa(s_i)$, for all $\kappa=0:j_i$, $i=1:\varsigma$ and $\sum_i^{\varsigma}(j_i+1)=\nu$.

The next result gives the necessary and sufficient conditions for a low-order system to achieve moment matching.

Proposition 2 (*Ionescu, 2016*)**.** *Fix $S\in\mathbb{R}^{\nu\times\nu}$ and $L\in\mathbb{R}^{1\times\nu}$, such that the pair (L,S) is observable and $\sigma(S)\cap\sigma(A)=\emptyset$. Furthermore, assume that $\sigma(F)\cap\sigma(S)=\emptyset$. Then, the transfer function K_G matches the moments of K, at $\sigma(S)$, if and only if $HP=C\Pi$, where $P\in\mathbb{R}^{\nu\times\nu}$ is the unique solution of the Sylvester equation $FP+GL=PS$.*

The system

$$\Sigma_G:\ \dot{\xi}=(S-GL)\xi+Gu,\quad \psi=C\Pi\xi, \tag{4}$$

with the transfer function

$$K_G(s)=C\Pi(sI-S+GL)^{-1}G, \tag{5}$$

describes the family of ν order models that match ν moments of K, at $\sigma(S)$, in the sense of Definition 2, for all $G\in\mathbb{R}^{\nu}$ such that $\sigma(S-GL)\cap\sigma(S)=\emptyset$, see, e.g. Astolfi (2010).

We now formulate the moment matching-based model reduction problem to be solved.

Problem 1. Consider the system Σ as in (1) and the family of ν order models Σ_G, as in (4), matching ν moments of orders $0:j_i$ of K at s_i, $i=1:\varsigma$, with multiplicity j_i, such that $\sum_i^{\varsigma}(j_i+1)=\nu$. Find the parameter matrix $G\in\mathbb{R}^{\nu}$ such that

(i) Σ_G has ℓ poles at $\lambda_i\in\mathbb{C}\setminus\sigma(S)$, $i=1:\ell$,
(ii) Σ_G has k zeros at $z_j\in\mathbb{C}\setminus\sigma(S)$, $j=1:k$,
(iii) $\nu-(k+\ell)$ moments of orders $1:j_i+1$ of K_G and K at s_i, $i=1:\varsigma$ match.

[1] Since the controllability/observability is a generic property, in the sense that the set of controllable/observable pairs is an open and dense subset of the set of all pairs of a given size, to any matrix S there correspond (an infinity of) matrices L, such that the pair (L,S) is observable, see, e.g., Sontag (1998, p. 96) or (Murray Wonham, 1985, e.g., p. 43, Section 1.4, Theorem 1.2, Lemma 1.1, Corollary 1.1) for more detailed arguments.

3. Model reduction with pole-zero placement and matching of high order moments

In this section, we derive the linear constraints (7) or (8), (11) or (12) as well as (16), parametrized in $G \in \mathbb{R}^{\nu}$, resulting in the linear systems (17) or (18) yielding the solution to Problem 1.

3.1. Pole placement as linear constraints

In this section, we place ℓ poles of the reduced order at desired locations, by properly selecting G. Consider the system (1) and the family of ν order models Σ_G in (4) that match ν moments of K at $\sigma(S)$, for all $G \in \mathbb{R}^{\nu}$. Let $\lambda_i \in \mathbb{C}$, $i = 1:\ell$, $\ell \le \nu$ be such that $\lambda_i \notin \sigma(S)$. Then λ_i are poles of Σ_G if $\det(\lambda_i I - S + GL) = 0$, $i = 1:\ell$ and such that $\{\lambda_1, \ldots, \lambda_\ell\}$ is a symmetric set. Let $Q_{\mathbf{P}} \in \mathbb{R}^{\ell \times \ell}$ be a matrix such that $\sigma(Q_{\mathbf{P}}) = \{\lambda_1, \ldots, \lambda_\ell\}$. Furthermore, consider $C_{\mathbf{P}} \in \mathbb{R}^{1\times n}$ such that $C_{\mathbf{P}}\Pi = 0$, where $\Pi \in \mathbb{R}^{n\times\nu}$ solves (3), and let $\Upsilon_{\mathbf{P}} \in \mathbb{R}^{\ell\times n}$ be the unique solution of the Sylvester equation

$$Q_{\mathbf{P}}\Upsilon_{\mathbf{P}} = \Upsilon_{\mathbf{P}}A + R_{\mathbf{P}}C_{\mathbf{P}}, \tag{6}$$

with $R_{\mathbf{P}} \in \mathbb{R}^{\ell}$ any matrix such that the pair $(Q_{\mathbf{P}}, R_{\mathbf{P}})$ is controllable. Hence rank $\Upsilon_{\mathbf{P}} = \ell$, see. e.g., de Souza and Bhattacharyya (1981). The next result imposes linear constraints on G such that the reduced model Σ_G has ℓ poles at $\{\lambda_1, \ldots, \lambda_\ell\}$.

Theorem 1. *Let Σ_G, as in (4), be a ν order model matching the moments of K at $\sigma(S)$. Let $\Upsilon_{\mathbf{P}} \in \mathbb{R}^{\ell\times n}$ be the unique solution of (6) and assume that $\operatorname{rank}(\Upsilon_{\mathbf{P}}\Pi) = \ell$. Consider $C_{\mathbf{P}} \in \mathbb{R}^{1\times n}$ such that $C_{\mathbf{P}}\Pi = 0$ (i.e., $C_{\mathbf{P}}^T \in \ker \Pi^T$). If G is a solution of the equation*

$$\Upsilon_{\mathbf{P}}\Pi G = \Upsilon_{\mathbf{P}}B, \tag{7}$$

then $\sigma(Q_{\mathbf{P}}) = \{\lambda_1, \ldots, \lambda_\ell\} \subseteq \sigma(S - GL)$.

Proof. Let $\lambda \in \sigma(Q_{\mathbf{P}})$. Then, there exists the (left) eigenvector $v \in \mathbb{C}^{\ell}$, $v \ne 0$, such that $v^T(\lambda I - Q_{\mathbf{P}}) = 0$. Post multiplying with $\Upsilon_{\mathbf{P}}\Pi$ yields $v^T(\lambda\Upsilon_{\mathbf{P}}\Pi - Q_{\mathbf{P}}\Upsilon_{\mathbf{P}}\Pi) = 0$. Hence, by (6), we write $v^T(\lambda\Upsilon_{\mathbf{P}}\Pi - \Upsilon_{\mathbf{P}}A\Pi - R_{\mathbf{P}}C_{\mathbf{P}}\Pi) = 0$. Since assuming $C_{\mathbf{P}}\Pi = 0$ leads to $v^T(\lambda\Upsilon_{\mathbf{P}}\Pi - \Upsilon_{\mathbf{P}}A\Pi) = 0$, using (3) further yields $v^T(\lambda\Upsilon_{\mathbf{P}}\Pi - \Upsilon_{\mathbf{P}}\Pi S + \Upsilon_{\mathbf{P}}BL) = 0$. Assuming (7) holds, we get $v^T\Upsilon_{\mathbf{P}}\Pi(\lambda I - S + GL) = 0$. Since we assume that $\operatorname{rank}(\Upsilon_{\mathbf{P}}\Pi) = \ell$, then $(\Upsilon_{\mathbf{P}}\Pi)^T v = 0$ if and only if $v = 0$. Hence, $\lambda \in \sigma(S - GL)$ with the (left) eigenvector $(\Upsilon_{\mathbf{P}}\Pi)^T v$ and the claim follows. □

Remark 1. Theorem 1 yields the sufficient condition (7) on G such that $\ell \le \nu$ of the poles of K_G are fixed, when S, L and $Q_{\mathbf{P}}$ are arbitrary matrices such that the pair (L, S) is observable and the pair $(Q_{\mathbf{P}}, R_{\mathbf{P}})$ is controllable. Furthermore, if $\ell = \nu$ and $\Upsilon_{\mathbf{P}}\Pi$ is assumed invertible, then $\sigma(S - GL) = \sigma(Q_{\mathbf{P}})$, if and only if $G = (\Upsilon_{\mathbf{P}}\Pi)^{-1}\Upsilon_{\mathbf{P}}B$. Moreover, a sufficient condition to satisfy (7) is to select G as a solution of the matrix equation $\Pi G = B$. Hence, post-multiplying Eq. (6) with Π yields $Q_{\mathbf{P}}\Upsilon_{\mathbf{P}}\Pi = \Upsilon_{\mathbf{P}}A\Pi$. Using Eq. (3), one immediately gets $\Upsilon_{\mathbf{P}}A\Pi = \Upsilon_{\mathbf{P}}\Pi(S - GL)$. Furthermore, if $\Upsilon_{\mathbf{P}}\Pi$ is assumed invertible, then the ν order model Σ_G with G such that $\Pi G = B$ is written equivalently as $(\Upsilon_{\mathbf{P}}\Pi)^{-1}\Upsilon_{\mathbf{P}}A\Pi = S - GL$, $\quad G = (\Upsilon_{\mathbf{P}}\Pi)^{-1}\Upsilon_{\mathbf{P}}B$.

When $S = \operatorname{diag}(s_1, \ldots, s_\nu) \in \mathbb{R}^{\nu\times\nu}$ and the zero-order moments are considered ($j_i = 0, i = 1:\varsigma$) then (7) can be replaced by an equivalent linear system in the unknown $G \in \mathbb{R}^{\nu}$.

Proposition 3. *Let $S = \operatorname{diag}(s_1, \ldots, s_\nu) \in \mathbb{R}^{\nu\times\nu}$, $s_i \ne s_j, i \ne j$ and $L = [1 \ \ldots \ 1]$. Then $\{\lambda_1, \ldots, \lambda_\ell\} \subset \mathbb{R}$ are a set of poles of $K_G(s)$ as in (5) if and only if $G \in \mathbb{R}^{\nu}$ is the solution of the linear system*

$$1 + LD_\kappa^{-1}G = 0, \quad \forall \kappa = 1:\ell, \tag{8}$$

with $D_\kappa = \operatorname{diag}(\theta_{\kappa 1}, \ldots, \theta_{\kappa\nu})$, where $\theta_{\kappa i} = \lambda_\kappa - s_i$, $i = 1:\nu$ and $\kappa = 1:\ell$.

Proof. Note that λ is a pole of $K_G(s)$ if $\det(\lambda I - S + GL) = 0$. Then, explicitly writing the determinant yields:

$$\begin{vmatrix} \theta_{\kappa 1} + g_1 & g_1 & \ldots & g_1 \\ g_2 & \theta_{\kappa 2} + g_2 & \ldots & g_2 \\ \vdots & \vdots & \ddots & \vdots \\ g_\nu & g_\nu & \ldots & \theta_{\kappa\nu} + g_\nu \end{vmatrix} = 0,$$

$\theta_{\kappa i} = \lambda_\kappa - s_i$, $i = 1:\nu$, $\kappa = 1:\ell$.

Equivalently, in matrix form $\det(D_\kappa + GL) = 0$, where $D_\kappa = \operatorname{diag}(\theta_{\kappa 1}, \ldots, \theta_{\kappa\nu})$, for each $\kappa = 1:\ell$. Using the well-known Sherman–Morrison–Woodbury formula (Horn & Johnson, 1985), the claim follows immediately. □

3.2. Zero placement as linear constraints

Consider the system (1) and the family of ν order models Σ_G in (4) that match ν moments of K at $\sigma(S)$, for all $G \in \mathbb{R}^{\nu}$. Let $\{z_1, \ldots, z_k\} \subset \mathbb{C}$ be a symmetric set, with $k < \nu$ and $z_i \ne s_j$, $i = 1:k, j = 1:\nu$. By, e.g., Astolfi (2010), Iftime and Ionescu (2013), Ionescu and Iftime (2012), there exists a subfamily of models Σ_G, such that the set of zeros of each model contains $z_1, \ldots, z_k$. Equivalently, there exists G such that

$$\det\begin{bmatrix} z_i I - S & G \\ C\Pi & 0 \end{bmatrix} = 0, \quad i = 1:k. \tag{9}$$

Now let $Q_{\mathbf{Z}} \in \mathbb{R}^{k\times k}$ be such that $\sigma(Q_{\mathbf{Z}}) = \{z_1, \ldots, z_k\}$ and $R_{\mathbf{Z}} \in \mathbb{R}^{k}$ be any matrix such that the pair $(Q_{\mathbf{Z}}, R_{\mathbf{Z}})$ is controllable. Let $\Upsilon_{\mathbf{Z}} \in \mathbb{R}^{k\times n}$ be the unique solution of the Sylvester equation

$$Q_{\mathbf{Z}}\Upsilon_{\mathbf{Z}} = \Upsilon_{\mathbf{Z}}A + R_{\mathbf{Z}}C. \tag{10}$$

Note that rank $\Upsilon_{\mathbf{Z}} = k$, see, e.g., de Souza and Bhattacharyya (1981). The moments of K at z_i are given by $\Upsilon_{\mathbf{Z}}B = 0$. The next result imposes linear constraints on G such that the reduced model Σ_G has k zeros at $\{z_1, \ldots, z_k\}$.

Proposition 4. *Let Σ_G, as in (4), be a ν order model that matches the moments of K at $\sigma(S)$. Furthermore, let $\Upsilon_{\mathbf{Z}} \in \mathbb{R}^{k\times n}$ be the unique solution of (6), such that $\Upsilon_{\mathbf{Z}}B = 0$ and assume that $\operatorname{rank}(\Upsilon_{\mathbf{Z}}\Pi) = k$. If G is a solution of the equation*

$$\Upsilon_{\mathbf{Z}}\Pi G = 0, \tag{11}$$

then $z_i \in \sigma(Q_{\mathbf{Z}})$, $i = 1:k$, are zeros of the system Σ_G.

Proof. Let $z \in \sigma(Q_{\mathbf{Z}})$. Then, there exists $w \in \mathbb{C}^k, w \ne 0$ such that $w^T(zI - Q_{\mathbf{Z}}) = 0$. Postmultiplying with $\Upsilon_{\mathbf{Z}}\Pi$ yields $w^T(z\Upsilon_{\mathbf{Z}}\Pi - Q_{\mathbf{Z}}\Upsilon_{\mathbf{Z}}\Pi) = 0$. Hence, by (10), we get $w^T(z\Upsilon_{\mathbf{Z}}\Pi - \Upsilon_{\mathbf{Z}}A\Pi - R_{\mathbf{Z}}C\Pi) = 0$. Assuming $\Upsilon_{\mathbf{Z}}B = 0$ and using (6) yield $w^T[\Upsilon_{\mathbf{Z}}\Pi(zI - S) - R_{\mathbf{Z}}C\Pi] = 0$, equivalent to

$$\begin{bmatrix} w^T\Upsilon_{\mathbf{Z}}\Pi & w^T R_{\mathbf{Z}} \end{bmatrix}\begin{bmatrix} zI - S \\ C\Pi \end{bmatrix} = 0.$$

Now, note that

$$\begin{bmatrix} w^T\Upsilon_{\mathbf{Z}}\Pi & w^T R_{\mathbf{Z}} \end{bmatrix}\begin{bmatrix} zI - S & G \\ C\Pi & 0 \end{bmatrix} = \begin{bmatrix} 0 & \Upsilon_{\mathbf{Z}}\Pi G \end{bmatrix}.$$

Hence, if $\Upsilon_{\mathbf{Z}}\Pi G = 0$, then

$$u^T = \begin{bmatrix} w^T\Upsilon_{\mathbf{Z}}\Pi & w^T R_{\mathbf{Z}} \end{bmatrix} \ne 0,$$

satisfies

$$u^T\begin{bmatrix} zI - S & G \\ C\Pi & 0 \end{bmatrix} = 0,$$

yielding (9) and the claim follows. □

Now, let $G = [g_1\, g_2\, \dots\, g_\nu]^T \in \mathbb{R}^\nu$. Then, condition (9) is equivalent to a system of k equations with ν unknowns $g_1, \dots, g_\nu$, given by $(-1)^\nu [-g_1\zeta_1(z_1) + g_2\zeta_2(z_1) + \dots + (-1)^\nu g_\nu \zeta_\nu(z_1)] = 0, \dots, (-1)^\nu [-g_1\zeta_1(z_k) + g_2\zeta_2(z_k) + \dots + (-1)^\nu g_\nu\zeta_\nu(z_k)] = 0$, with $\zeta_j(s)$ polynomials of degree $\nu - 1$, $j = 1 : \nu$. When $S = \mathrm{diag}(s_1, \dots, s_\nu) \in \mathbb{R}^{\nu\times\nu}$, $s_i \neq s_j$, $i \neq j$, $L = [1\ 1\ \dots\ 1]$ and the zero-order moments are considered, the polynomial equations can be replaced by a linear system in the unknown $G \in \mathbb{R}^\nu$, equivalent to (11).

Proposition 5. *Let $S = \mathrm{diag}(s_1, \dots, s_\nu) \in \mathbb{R}^{\nu\times\nu}$, $s_i \neq s_j$, $i \neq j$, $L = [1\ \dots\ 1]$ and let $C\Pi = [\eta_1\ \dots\ \eta_\nu]$. Then Σ_G, as in (4), is a model with the set $\{z_j, j = 1 : k\} \subset \mathbb{R}$, $z_j \neq s_i$, $i, j = 1 : k$, among the zeros of the transfer function $K_G(s)$, if and only if the elements of $G = [g_1\ \dots\ g_\nu]^T$ satisfy*

$$\sum_{i=1}^{\nu} \frac{\eta_i}{\gamma_{ji}} g_i = 0, \quad j = 1 : k, \tag{12}$$

where $\gamma_{ji} = z_j - s_i,\ i = 1 : \nu,\ j = 1 : k$.

Proof. The numbers $z_1, \dots, z_k$ are zeros of $K_G(s)$ if and only if (9) is satisfied, i.e.,

$$\begin{vmatrix} \gamma_{j1} & 0 & 0 & \dots & 0 & g_1 \\ 0 & \gamma_{j2} & 0 & \dots & 0 & g_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \gamma_{j\nu} & g_\nu \\ \eta_1 & \eta_2 & \eta_3 & \dots & \eta_\nu & 0 \end{vmatrix} = 0,$$

$\gamma_{ji} = z_j - s_i,\ i = 1 : \nu,\ j = 1 : k$.

Note that $\gamma_{ji} \neq 0$ for all i, j. Then, successively decomposing the determinant by the last column and computing the resulting minors through row decomposition yield $\sum_{i=1}^{\nu} \eta_i g_i \prod_{l=1:\nu, l\neq i} \gamma_{jl} = 0$, $j = 1 : k$. Dividing by $\prod_{l=1:\nu} \gamma_{jl} \neq 0, j = 1 : k$, leads to the claim. □

3.3. Matching high order moments as linear constraints

In this section, we explicitly determine the matrix $G \in \mathbb{R}^\nu$ yielding the subfamily of models that match ν moments of orders $0 : j_i$ and $\mu \leq \nu$ moments of orders $1 : j_i + 1$ of K at $\sigma(S)$. Without loss of generality, let $S = \mathrm{diag}(S_p, S_{\mathbf{D}}) \in \mathbb{R}^{\nu\times\nu}$, with $S_{\mathbf{D}} \in \mathbb{R}^{\mu\times\mu}$ and S_p any matrix such that $\sigma(S) = \sigma(S_p) \cup \sigma(S_{\mathbf{D}}) = \{s_i \mid i = 1 : \varsigma\}$, $\sum_i (j_i + 1) = \nu$, with $\sigma(S)$ and $\sigma(S_{\mathbf{D}})$ symmetric (including multiplicities). Let $L = [L_p\ L_{\mathbf{D}}] \in \mathbb{R}^{1\times\nu}$, with $L_{\mathbf{D}} \in \mathbb{R}^{1\times\mu}$ be such that (L, S) is observable. Let $\Pi = [\Pi_1, \dots, \Pi_\nu]$ be the unique solution of the Sylvester equation (3) and $\Upsilon_{\mathbf{D}}$ be the unique solution of the Sylvester equation

$$S_{\mathbf{D}}\Upsilon_{\mathbf{D}} = \Upsilon_{\mathbf{D}} A + R_{\mathbf{D}} C, \tag{13}$$

with $R_{\mathbf{D}} = L_{\mathbf{D}}^T \in \mathbb{R}^\mu$ such that the pair $(S_{\mathbf{D}}, R_{\mathbf{D}})$ is controllable. We assume that the pair $(S_{\mathbf{D}}, R_{\mathbf{D}})$ is controllable such that rank $\Upsilon_{\mathbf{D}} = \mu$. The moments of orders $1 : j_i + 1$ of K at $\sigma(S_{\mathbf{D}})$, are the moments of orders $0 : j_i$ of K' at $\sigma(S_{\mathbf{D}})$. Define $\widetilde{\Sigma} : \dot{x} = Ax + Bu$, $\dot{z} = Az + x$, $y = -Cz$, where $z \in \mathbb{R}^n$ and $y \in \mathbb{R}$, with the transfer function K'. Interconnecting $\widetilde{\Sigma}$ to the signal generator

$$\dot{\omega} = S\omega, \quad \theta = L\omega, \ \omega(0) \neq 0, \ \omega(t) \in \mathbb{R}^\nu, \tag{14}$$

by $u = \theta$ and to the generalized signal generator

$$\dot{\pi} = S_{\mathbf{D}}\pi + R_{\mathbf{D}} w, \ d = \pi + \Upsilon_{\mathbf{D}} z, \ \pi(0) = 0, \ \pi(t) \in \mathbb{R}^\mu, \tag{15}$$

by $w = y$, where $\Upsilon_{\mathbf{D}}$ is the unique solution of (13) and $R_{\mathbf{D}} = L_{\mathbf{D}}^T$, yields the output signal d. Then, by Ionescu (2016, Theorem 2), the $0 : j_i$ order moments of K' at $\sigma(S_{\mathbf{D}})$ are given by the steady-state behaviour of the signal d.

We now impose matching properties at the first order derivative of $K(s)$ in the sense of matching the relation defining signal d. Let Σ_G be as in (4), with the transfer function $K_G(s)$ as in (5) and the state–space representation of $K_G'(s)$, Ionescu (2016), $\widetilde{\Sigma}_G : \dot{\xi} = (S - GL)\xi + Gu$, $\dot{\chi} = (S - GL)\chi + \xi$, $\eta = -C\Pi\chi$, with $\chi(t) \in \mathbb{R}^\nu$. Considering the interconnection of $\widetilde{\Sigma}_G$ with the signal generators (14) by $u = \theta$ and (15) by $v = \tilde{\eta}$, respectively, yields an output $\zeta(t)$, parametrized as $\zeta(t) = \pi(t) + P\chi(t)$, with $P \in \mathbb{R}^{\nu\times\nu}$ any (invertible) matrix. We say that the moments of orders $1 : j_i + 1$ of K and K_G at $\sigma(S_{\mathbf{D}})$ match if the dynamics of $\zeta(t)$ are similar to the dynamics of $d(t)$ in (15), i.e., $\dot{\zeta} = S_{\mathbf{D}}\zeta + \Upsilon_{\mathbf{D}}\Pi\xi$, with $\Upsilon_{\mathbf{D}}$ the solution of (13) and Π the solution of (3). The next result presents the closed form of $G \in \mathbb{R}^\nu$ such that K_G matches ν moments of orders $0 : j_i$ of K at $\sigma(S)$ and K_G matches $\mu \leq \nu$ moments of orders $1 : j_i + 1$ of K at $\sigma(S_{\mathbf{D}})$.

Theorem 2. *Consider the system (1). Let Π be the unique solution of (3) and $\Upsilon_{\mathbf{D}}$ be the unique solution of (13). Let Σ_G be as in (4). Then the μ moments of orders $1 : j_i + 1$ of K_G at $\sigma(S_{\mathbf{D}})$ match the μ moments of orders $1 : j_i + 1$ of K at $\sigma(S_{\mathbf{D}}) \subset \sigma(S)$, if and only if*

$$\Upsilon_{\mathbf{D}}\Pi G = \Upsilon_{\mathbf{D}} B. \tag{16}$$

Proof. We prove the necessity. Since $\zeta = \pi + P\chi$, then $\dot{\zeta} = \dot{\pi} + P\dot{\chi}$. The moments of the transfer function of $\widetilde{\Sigma}_G$ match the moments of the transfer function of $\widetilde{\Sigma}$ at $\sigma(S_{\mathbf{D}})$ if $P \in \mathbb{R}^{\nu\times\nu}$ is such that $\dot{\zeta} = S_{\mathbf{D}}\zeta + \Upsilon_{\mathbf{D}}\Pi\xi$. Since $\dot{\pi} = S_{\mathbf{D}}\pi + R_{\mathbf{D}} w$ and $w = \eta$, where η is the output of $\widetilde{\Sigma}_G$, we write $S_{\mathbf{D}}\pi - R_{\mathbf{D}} C\Pi\chi + P(S - GL)\chi + P\xi = S_{\mathbf{D}}\pi + S_{\mathbf{D}} P\chi + \Upsilon_{\mathbf{D}}\Pi\xi$, for all ξ, χ. Then, $P = \Upsilon_{\mathbf{D}}\Pi$ and $PS - S_{\mathbf{D}} P = R_{\mathbf{D}} C\Pi + PGL$. Equivalently, $S_{\mathbf{D}}\Upsilon_{\mathbf{D}}\Pi - \Upsilon_{\mathbf{D}}\Pi S = \Upsilon_{\mathbf{D}}\Pi GL + R_{\mathbf{D}} C\Pi$. Hence, $\Upsilon_{\mathbf{D}}\Pi GL = S_{\mathbf{D}}\Upsilon_{\mathbf{D}}\Pi - \Upsilon_{\mathbf{D}}\Pi S - R_{\mathbf{D}} C\Pi$. By (13), $Q\Upsilon_{\mathbf{D}}\Pi = (\Upsilon_{\mathbf{D}} A + R_{\mathbf{D}} C)\Pi$. Then, $\Upsilon_{\mathbf{D}}\Pi GL = \Upsilon_{\mathbf{D}}\Pi S - \Upsilon_{\mathbf{D}} A\Pi$. By (3), $\Upsilon_{\mathbf{D}} A\Pi = \Upsilon_{\mathbf{D}}(\Pi S - BL)$ yielding the claim. The proof of the sufficiency uses similar arguments. □

Remark 2. If $\mu = \nu$, the result in Ionescu (2016) is a particular case of (16). Hence, selecting $G = (\Upsilon_{\mathbf{D}}\Pi)^{-1}\Upsilon_{\mathbf{D}} B$, all the ν moments of orders $1 : j_i + 1$ of K are matched at $\sigma(S_{\mathbf{D}}) = \sigma(S)$, where $\Upsilon_{\mathbf{D}}\Pi \in \mathbb{R}^{\nu\times\nu}$ is assumed invertible.

3.4. Problem 1 As a linear system

Let Σ_G, as in (4), define a family of ν order models that match ν moments of (1) at $\{s_i \in \mathbb{C} \setminus \sigma(A) \mid i = 1 : \varsigma\}$, s_i of multiplicities $j_i \geq 0$ such that $\sum_{i=1}^{\varsigma} (j_i + 1) = \nu$, parametrized in $G \in \mathbb{R}^\nu$. Let $\{\lambda_1, \dots, \lambda_\ell\} \subset \mathbb{C} \setminus \{s_1, \dots, s_\nu\}$ and $\{z_1, \dots, z_k\} \subset \mathbb{C}$, $\ell + k \leq \nu$, symmetric sets (including multiplicities). To write Problem 1 as a linear system, we collect the constraints (7), (11) and (16) yielding the system of three matrix equations in $G \in \mathbb{R}^\nu$,

$$\begin{cases} \Upsilon_{\mathbf{P}}\Pi G = \Upsilon_{\mathbf{P}} B, \\ \Upsilon_{\mathbf{Z}}\Pi G = 0, \\ \Upsilon_{\mathbf{D}}\Pi G = \Upsilon_{\mathbf{D}} B, \end{cases} \Leftrightarrow \Upsilon\Pi G = \Upsilon B, \tag{17}$$

with $\Upsilon = \begin{bmatrix} \Upsilon_{\mathbf{P}}^T & \Upsilon_{\mathbf{Z}}^T & \Upsilon_{\mathbf{D}}^T \end{bmatrix}^T \in \mathbb{R}^{\nu\times n}$. Furthermore, assuming $\Upsilon\Pi$ is invertible, then $G = (\Upsilon\Pi)^{-1}\Upsilon B$. For $j_i = 0, i = 1 : \varsigma$, tf $S = \mathrm{diag}(s_1, \dots, s_\nu) \in \mathbb{R}^{\nu\times\nu}$ and $\lambda_i, z_j \in \mathbb{R}, i = 1 : \ell, j = 1 : k$, distinct, collecting the constraints (8), (12) and (16) yields a linear system, equivalent to (17), parametrized in $G = [g_1\ \dots\ g_\nu]^T \in \mathbb{R}^\nu$,

$$\begin{cases} 1 + LD_\kappa^{-1} G = 0, & \kappa = 1 : \ell, \\ \sum_{i=1}^{\nu} \frac{\eta_i}{\gamma_{ji}} g_i = 0, & j = 1 : k, \\ \Upsilon_{\mathbf{D}}\Pi G = \Upsilon_{\mathbf{D}} B, \end{cases} \tag{18}$$

with $D_\kappa = \mathrm{diag}(\theta_{\kappa 1}, \dots, \theta_{\kappa\nu})$, $\theta_{\kappa i} = \lambda_\kappa - s_i$, $i = 1 : \nu$, $\kappa = 1 : \ell$, $\gamma_{ji} = z_j - s_i$, $i = 1 : \nu$, $j = 1 : k$, $\Upsilon_{\mathbf{D}}$ is the solution of (13) and Π is the solution of (3). The solution of Problem 1 can be computed solving the linear system (18).

3.5. Algorithm to solve Problem 1

We now summarize the results in an algorithm to solve Problem 1, i.e., calculate the reduced order model Σ_G of (1), with G the solution of the linear system (17) (or (18) in particular instances).

Algorithm 1 (*Solution Σ_G of Problem 1*).Consider the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$ and $C \in \mathbb{R}^{1\times n}$ of (1).

Step 1: Choose $\nu < n \in \mathbb{N}$, the reduced order of the models approximating (1).

Step 2: Consider the symmetric sets $\{s_i \in \mathbb{C} \setminus \sigma(A) \mid i = 1 : \varsigma\}$, s_i of multiplicities $j_i \geq 0$ such that $\sum_{i=1}^{\varsigma}(j_i + 1) = \nu$, $\{\lambda_1, \ldots, \lambda_\ell\} \subset \mathbb{C}$, $s_j \neq \lambda_j, j = 1 : \ell$ and $\{z_1, \ldots, z_k\} \subset \mathbb{C}$, $\ell + k \leq \nu$.

Step 3: Construct the matrix $S = \mathrm{diag}(S_p, S_z, S_{\mathbf{D}}) \in \mathbb{R}^{\nu\times\nu}$, with $S_p \in \mathbb{R}^{\ell\times\ell}$, $S_z \in \mathbb{R}^{k\times k}$, $S_{\mathbf{D}} \in \mathbb{R}^{\nu-(\ell+k)\times\nu-(\ell+k)}$, such that $\sigma(S) = \sigma(S_p)\cup\sigma(S_z)\cup\sigma(S_{\mathbf{D}}) = \{s_i \mid i = 1 : \varsigma\}$ and $\sum_i(j_i + 1) = \nu$. Pick any matrix $L = [L_p \ L_z \ L_{\mathbf{D}}] \in \mathbb{R}^{1\times\nu}$, $L_{\mathbf{D}} \in \mathbb{R}^{1\times\nu-(\ell+k)}$ such that (L, S) is observable.

Step 4: Compute $\Pi \in \mathbb{R}^{n\times\nu}$, the solution of (3).

Step 5: Compute the family of ν order models Σ_G as in (4), matching the ν moments of orders $0 : j_i$ at $\sigma(S)$, parametrized in $G = [g_1 \ \ldots \ g_\nu]^T \in \mathbb{R}^\nu$.

Step 6: Compute $\Upsilon_{\mathbf{D}}$, the solution of (13), for $R_{\mathbf{D}} = L_{\mathbf{D}}^T$.

Step 7: Compute G, the solution of (17). In particular, for $j_i = 0, i = 1 : \varsigma$, if $S = \mathrm{diag}(s_1, \ldots, s_\nu) \in \mathbb{R}$ and $\lambda_i, z_j \in \mathbb{R}, i = 1 : \ell, j = 1 : k$, compute G, the solution of (18).

Step 8: Substituting G in Σ_G from **Step 5** yields the solution of Problem 1.

4. Loewner matrices model reduction with pole-zero placement and Hermite interpolation

In this section, we provide the solution of Problem 1, with $j_i = 0$, using Loewner matrices. We first construct the Loewner matrices to match ν *zero* order moments of K at $\{s_1, s_2, \ldots, s_\ell, \ldots, s_{\ell+k}, \ldots, s_\nu\} \subset \mathbb{R}$, $s_i \neq s_j$, place ℓ distinct poles at $\{\lambda_1, \ldots, \lambda_\ell\} \subset \mathbb{R}$, with $\ell \leq \nu$, $\lambda_i \neq s_j, i = 1 : \ell, j = 1 : \nu$, place k distinct zeros at $\{z_1, \ldots, z_k\} \subset \mathbb{R}$, with $k < \nu$, $z_i \neq s_j, i = 1 : k, j = 1 : \nu$ and match $\nu - (\ell + k)$ *first* order moments of K at $\{s_{\ell+k+1}, \ldots, s_\nu\}$,

$$\mathbb{L}_{ij} = \begin{cases} \dfrac{K_{\mathbf{P}}(\lambda_i)}{\lambda_i - s_j}, & i = 1 : \ell, j = 1 : \nu, \\ \dfrac{-K(s_j)}{z_i - s_j}, & i = 1 : k, j = 1 : \nu, \\ \dfrac{K(s_i) - K(s_j)}{s_i - s_j}, & i \neq j = \ell + k + 1 : \nu, \\ -K'(s_i), & i = j = \ell + k + 1 : \nu, \end{cases} \tag{19a}$$

$$\sigma\mathbb{L}_{ij} = \begin{cases} \dfrac{\lambda_i K_{\mathbf{P}}(\lambda_i)}{\lambda_i - s_j}, & i=1 : \ell, j=1 : \nu, \\ \dfrac{-s_j K(s_j)}{z_i - s_j}, & i=1 : k, j=1 : \nu, \\ \dfrac{s_i K(s_i) - s_j K(s_j)}{s_i - s_j}, & i \neq j=\ell+k+1:\nu \\ -s_i K'(s_i), & i=j=\ell+k+1:\nu, \end{cases} \tag{19b}$$

where $K_{\mathbf{P}}(\lambda) = C_{\mathbf{P}}(\lambda_i I - A)^{-1}B$, with $C_{\mathbf{P}} \in \mathbb{R}^{1\times n}$ any matrix such that $C_{\mathbf{P}}(s_j I - A)^{-1}B = 0$, $\forall s_j \neq \lambda_i$, $i = 1 : \ell, j = 1 : \nu$. Let

$$\begin{aligned} S &= \mathrm{diag}(s_1, s_2, \ldots, s_\ell, s_{\ell+1}, \ldots, s_{\ell+k}, s_{\ell+k+1}, \ldots, s_\nu) \\ &= \mathrm{diag}(S_p, S_z, S_{\mathbf{D}}), \end{aligned} \tag{20}$$

with $S_p = \mathrm{diag}(s_1, \ldots, s_\ell)$, $S_z = \mathrm{diag}(s_{\ell+1}, \ldots, s_{\ell+k})$, $S_{\mathbf{D}} = \mathrm{diag}(s_{\ell+k+1}, \ldots, s_\nu)$ and let $L = [1 \ 1 \ \ldots \ 1] = [L_p \ L_z \ L_{\mathbf{D}}] \in \mathbb{R}^{1\times\nu}$, $L_{\mathbf{D}} \in \mathbb{R}^{1\times(\nu-\ell-k)}$. Furthermore, let

$$\begin{aligned} Q &= \mathrm{diag}(\lambda_1, \ldots, \lambda_\ell, z_1, \ldots, z_k, s_{\ell+1}, \ldots, s_\nu) \\ &= \mathrm{diag}(Q_{\mathbf{P}}, Q_{\mathbf{Z}}, S_{\mathbf{D}}). \end{aligned} \tag{21}$$

with $Q_{\mathbf{P}} = \mathrm{diag}(\lambda_1, \ldots, \lambda_\ell)$ and $Q_{\mathbf{Z}} = \mathrm{diag}(z_1, \ldots, z_k)$. Let Π be the solution of (3). Furthermore, construct $\Upsilon = [\Upsilon_{\mathbf{P}}^T \ \Upsilon_{\mathbf{Z}}^T \ \Upsilon_{\mathbf{D}}^T]^T \in \mathbb{R}^{\nu\times n}$, where $\Upsilon_{\mathbf{P}}$ is the unique solution of (6), $Q_{\mathbf{P}}\Upsilon_{\mathbf{P}} = \Upsilon_{\mathbf{P}}A + R_{\mathbf{P}}C_{\mathbf{P}}$, where $C_{\mathbf{P}} \in \mathbb{R}^{1\times n}$ such that $C_{\mathbf{P}}\Pi = 0$, $\Upsilon_{\mathbf{Z}}$ is the unique solution of (10), $Q_{\mathbf{Z}}\Upsilon_{\mathbf{Z}} = \Upsilon_{\mathbf{Z}}A + R_{\mathbf{Z}}C$ and $\Upsilon_{\mathbf{D}}$ is the unique solution of (13), $S_{\mathbf{D}}\Upsilon_{\mathbf{D}} = \Upsilon_{\mathbf{D}}A + R_{\mathbf{D}}C$, where $R_{\mathbf{D}} = L_{\mathbf{D}}^T$. Note that, in matrix form, Υ is the unique solution of the Sylvester equation

$$Q\Upsilon = \Upsilon A + \mathbf{R}(C_{\mathbf{P}}, C), \tag{22}$$

where $\mathbf{R}(C_{\mathbf{P}}, C) = [(R_{\mathbf{P}}C_{\mathbf{P}})^T \ (R_{\mathbf{Z}}C)^T \ (R_{\mathbf{D}}C)^T]^T$. We present the main result stating that the Loewner matrices given by (19) can be written directly in terms of Υ and Π and that they are the solutions of two Sylvester equations.

One can use the real Jordan forms for real matrices to get similar results for complex conjugate points.

Theorem 3. *Let the Loewner matrices be as in (19) and S and Q as in (20) and (21). Let Π be the unique solution of (3) and Υ be the unique solution of (22) and assume that the matrix $\Upsilon\Pi$ is invertible. Consider the statements*

(1) $\mathbb{L}$ is defined by Eq. (19a),
(2) $\mathbb{L} = -\Upsilon\Pi$ and satisfies the Sylvester equation $\mathbb{L}S - Q\mathbb{L} = \mathbf{R}(C_{\mathbf{P}}, C)\Pi - \Upsilon BL$,
(3) $\sigma\mathbb{L}$ is defined by Eq. (19b),
(4) $\sigma\mathbb{L} = \mathbb{L}[S-(\Upsilon\Pi)^{-1}\Upsilon BL]$ and satisfies the Sylvester equation $\sigma\mathbb{L}S - Q\sigma\mathbb{L} = \mathbf{R}(C_{\mathbf{P}}, C)\Pi S - Q\Upsilon BL$.

Then (1) ⇔ (2) and (3) ⇔ (4).

Proof. We first prove statement (1) ⇔ (2). Note that (19a) can be equivalently written as

$$\begin{aligned} \lambda_i\mathbb{L} - \mathbb{L}s_j &= \mathbf{C}(\lambda_i I - A)^{-1}B - \mathbf{C}(s_j I - A)^{-1}B, \\ s_i\mathbb{L} - \mathbb{L}s_j &= \mathbf{C}(s_i I - A)^{-1}B - \mathbf{C}(s_j I - A)^{-1}B, \end{aligned} \tag{23}$$

where $\mathbf{C} = C_{\mathbf{P}}$ for all $i, j = 1 : \ell$ and $\mathbf{C} = C$ for all $i \neq j = \ell+1 : \nu$, respectively. Note that, for any $\alpha \neq \beta \in \mathbb{R}$,

$$\begin{aligned} &\frac{\mathbf{C}(\alpha I - A)^{-1}B - \mathbf{C}(\beta I - A)^{-1}B}{\alpha - \beta} \\ &= \frac{\mathbf{C}[(\alpha I - A)^{-1} - (\beta I - A)^{-1}]B}{\alpha - \beta} \\ &= \frac{\mathbf{C}(\alpha I - A)^{-1}[\beta I - A - \alpha I + A](\beta I - A)^{-1}B}{\alpha - \beta} \\ &= -\mathbf{C}(\alpha I - A)^{-1}(\beta I - A)^{-1}B, \end{aligned}$$

with $\mathbf{C} \in \{C, C_{\mathbf{P}}\}$. Hence, substituting $\alpha = \lambda_i$, $\beta = s_j$, $\mathbf{C} = C_{\mathbf{P}}$, for all $i, j = 1 : \ell$, substituting $\alpha = z_i$, $\beta = s_j$, $\mathbf{C} = C$, $i, j = \ell + 1 : \ell + k$ and substituting $\alpha = s_i$, $\beta = s_j$, $\mathbf{C} = C$, for all $i \neq j = \ell+k+1 : \nu$, yields $\mathbb{L}_{ij} = -\mathbf{C}(\lambda_i I - A)^{-1}(s_j I - A)^{-1}B$, $\forall i, j = 1 : \ell$, $\mathbb{L}_{ij} = -\mathbf{C}(z_i I - A)^{-1}(s_j I - A)^{-1}B$, $\forall i, j = \ell + 1 : \ell + k$, and $\mathbb{L}_{ij} = -\mathbf{C}(s_i I - A)^{-1}(s_j I - A)^{-1}B$, $\forall i \neq j = \ell + k + 1 : \nu$. Moreover, by (19a), $\mathbb{L}_{ii} = -K'(s_i) = C(s_i I - A)^{-2}B = C(s_i I - A)^{-1}(s_i I - A)^{-1}B, i = \ell + 1 : \nu$. Let $\Upsilon_i = C(s_i I - A)^{-1}$ and $\Pi_j = (\lambda_j I - A)^{-1}B, i, j = 1 : \nu$. Then $\Upsilon = [\Upsilon_1^T \ \Upsilon_2^T \ \ldots \ \Upsilon_\nu^T]^T \in \mathbb{R}^{\nu\times n}$ and $\Pi = [\Pi_1 \ \Pi_2 \ \ldots \ \Pi_\nu] \in \mathbb{R}^{n\times\nu}$ are the (unique) solutions of the Sylvester equations (3) and (22), respectively. Hence, $\mathbb{L}_{ij}$ as in (19a) can be written equivalently as $\mathbb{L}_{ij} = -\Upsilon_i\Pi_j$, $\forall i, j = 1 : \ell$ and $\forall i, j = \ell + 1 : \nu, i \neq j$, and $\mathbb{L}_{ii} = -\Upsilon_i\Pi_i$, $\forall i = \ell + 1 : \nu$.

Table 1
Input–output data sets for the data-driven Loewner matrices.

Input data	Output data
$\{s_1, \dots, s_\ell, \dots, s_{\ell+k}, s_{\ell+k+1}, \dots, s_\nu\}$	$\{\eta_1, \dots, \eta_\nu\}$
$\{\lambda_1, \dots, \lambda_\ell\}$, $\lambda_i \neq s_j$, $i = 1:\ell, j = 1:\nu$	$\{\eta_{\lambda_1}, \dots, \eta_{\lambda_\ell}\}$
$\{z_1, \dots, z_k\}$, $z_i \neq s_j$, $i = 1:k, j = 1:\nu$	$\underbrace{\{0, \dots, 0\}}_{k \text{ times}}$
$\{s_{\ell+k+1}, \dots, s_\nu\}$	$\{\eta'_{\ell+k+1}, \dots, \eta'_\nu\}$

Furthermore, writing (23) for each i, j yields $\mathbb{L} = -\Upsilon\Pi$. Hence $\mathbb{L}S - Q\mathbb{L} = Q\Upsilon\Pi - \Upsilon\Pi S$. Then, employing (3) and (22) yields $\mathbb{L}S - Q\mathbb{L} = -\Upsilon(A\Pi + BL) + (\Upsilon A + \mathbf{R}(C_\mathbf{P}, C)\Pi)$ and the claim follows.
Moreover, note that for any $\alpha \neq \beta \in \mathbb{R}$,

$$\frac{\alpha\mathbf{C}(\alpha I - A)^{-1}B - \beta\mathbf{C}(\beta I - A)^{-1}B}{\alpha - \beta} = -\mathbf{C}(\alpha I - A)^{-1}A(\beta I - A)^{-1}B,$$

with $\mathbf{C} \in \{C, C_\mathbf{P}\}$. Hence, substituting $\alpha = \lambda_i, \beta = s_j$ and $\mathbf{C} = C_\mathbf{P}$, for all $i, j = 1 : \ell$, substituting $\alpha = z_i, \beta = s_j, \mathbf{C} = C$, $i, j = \ell + 1 : \ell + k$ and substituting $\alpha = s_i, \beta = s_j$ and $\mathbf{C} = C$, for all $i \neq j = \ell + 1 : \nu$, eventually yields $\sigma\mathbb{L} = -\Upsilon A\Pi$. By (3), $\sigma\mathbb{L} = \Upsilon BL - \Upsilon\Pi S$. Finally, note that $\sigma\mathbb{L}S - Q\sigma\mathbb{L} = Q\Upsilon A\Pi - \Upsilon A\Pi S$ and using (3) and (13) yields the claim. □

We now write the approximation Σ_G matching ν zero order moments of K, satisfying ℓ pole constraints, k zero constraints and matching $\nu - (\ell + k)$ first order moments of K, simultaneously.

Theorem 4. *Let Σ_G be a model described by the Eqs. (4) with the transfer function (5). Then, for*

$$G = -\mathbb{L}^{-1}\Upsilon B, \tag{24}$$

with $\mathbb{L}$ given by (19a) assumed invertible and Υ the solution of (22), the model $\Sigma_{-\mathbb{L}^{-1}\Upsilon B}$ matches ν zero order moments of K at $\sigma(S) = \{s_1, \dots, s_\nu\} \subset \mathbb{R}$, has ℓ poles at $\{\lambda_1, \dots, \lambda_\ell\} \subset \sigma(Q) \subset \mathbb{R}$, $\lambda_i \neq s_j, i = 1 : \ell, j = 1 : \nu$, has k zeros at $\{z_1, \dots, z_k\} \subset \mathbb{R}, z_i \neq s_j, i = 1 : k, j = 1 : \nu$ and matches $\nu - (\ell + k)$ first order moments of K at $\{s_{\ell+k+1}, \dots, s_\nu\} \subset \sigma(S)$. Furthermore,

$$K_{-\mathbb{L}^{-1}\Upsilon B}(s) = C\Pi(\sigma\mathbb{L} - s\mathbb{L})^{-1}\Upsilon B. \tag{25}$$

Proof. Consider $K_G(s)$ as in (4). If $G = -\mathbb{L}^{-1}\Upsilon B$, then, by Theorem 3, $K_{-\mathbb{L}^{-1}\Upsilon B}(s) = C\Pi(sI - \mathbb{L}^{-1}\sigma\mathbb{L})^{-1} \cdot (-\mathbb{L}^{-1}) \cdot \Upsilon B$. Inserting $\mathbb{L}^{-1}$ back in the term $(sI - \mathbb{L}^{-1}\sigma\mathbb{L})$ yields the claim. □

The next result is a direct consequence of Theorem 4.

Corollary 1. *For a model Σ_G, as in (4), G in (24) is the unique solution of the system (17), or, equivalently, (18). The same reduced order model is also yielded by (25).*

Proof. By Theorem 4, since $\mathbb{L} = -\Upsilon\Pi$, then G in (24) is identical to the G in Section 3.4, solution of (17) and the claim follows. □

The following statement generalizes the result of Theorem 3 showing that, for any non-derogatory matrices Q and S, with R and L such that the pair (L, S) is observable and the pair (Q, R) is controllable, the matrix $\widehat{\mathbb{L}} = -\Upsilon\Pi$, with Υ and Π the unique solutions of (3) and (22), respectively, is equivalent to the Loewner matrix in (19).

Theorem 5. *Consider the system (1). Let $S \in \mathbb{R}^{\nu\times\nu}$ be any matrix with $\sigma(S) = \{s_1, s_2, \dots, s_\ell, \dots, s_{\ell+k}, \dots, s_\nu\} \subset \mathbb{C}$, a symmetric set of distinct points, not poles of (2) and $L \in \mathbb{R}^{1\times\nu}$ such that the pair (L, S) is observable. Let $Q \in \mathbb{R}^{\nu\times\nu}$ be any matrix with $\sigma(Q) = \{\lambda_1, \dots, \lambda_\ell, z_1, \dots z_k, s_{\ell+k+1}, \dots, s_\nu\} \subset \mathbb{C}$, a symmetric set of distinct points, with $\lambda_i \neq s_j, i = 1 : \ell, j = 1 : \nu$, $z_i \neq s_j, i = 1 : k, j = 1 : \nu$. Furthermore, let $\widehat{\Pi}$ be the unique solution of the Sylvester equation (3), and $\widehat{\Upsilon}$ be the unique solution of (22). Then, the matrices*

$$\widehat{\mathbb{L}} = -\widehat{\Upsilon}\widehat{\Pi}, \tag{26a}$$

$$\widehat{\sigma\mathbb{L}} = \widehat{\mathbb{L}}[S - (\widehat{\Upsilon}\widehat{\Pi})^{-1}\widehat{\Upsilon}BL] \tag{26b}$$

satisfy the equations

$$\widehat{\mathbb{L}}S - Q\widehat{\mathbb{L}} = \mathbf{R}(C_\mathbf{P}, C)\widehat{\Pi} - \widehat{\Upsilon}BL, \tag{27a}$$

$$\widehat{\sigma\mathbb{L}}S - Q\widehat{\sigma\mathbb{L}} = \mathbf{R}(C_\mathbf{P}, C)\Pi S - Q\widehat{\Upsilon}BL. \tag{27b}$$

Moreover, $\widehat{\mathbb{L}} = T_Q^{-1}\mathbb{L}T_S$, where $T_Q \in \mathbb{C}^{\nu\times\nu}$ is such that $T_Q Q T_Q^{-1} = \Lambda_Q = \mathrm{diag}(\lambda_1, \dots, \lambda_\ell, z_1, \dots z_k, s_{\ell+k+1}, \dots, s_\nu)$ and $T_S \in \mathbb{C}^{\nu\times\nu}$ is such that $T_S S T_S^{-1} = \Lambda_S = \mathrm{diag}(s_1, \dots, s_\nu)$.

Proof. Pre-multiplying (3) with $\widehat{\Upsilon}$ yields $\widehat{\Upsilon}A\widehat{\Pi} + \widehat{\Upsilon}BL = \widehat{\Upsilon}\widehat{\Pi}S$. By (22), $\widehat{\Upsilon}A = Q\widehat{\Upsilon} - \mathbf{R}(C_\mathbf{P}, C)$. Hence, $(Q\widehat{\Upsilon} - \mathbf{R}(C_\mathbf{P}, C))\widehat{\Pi} + \widehat{\Upsilon}BL = \widehat{\Upsilon}\widehat{\Pi}S \Leftrightarrow Q\widehat{\Upsilon}\widehat{\Pi} - \widehat{\Upsilon}\widehat{\Pi}S = \mathbf{R}(C_\mathbf{P}, C)\widehat{\Pi} - \widehat{\Upsilon}BL$, equivalent to the Sylvester equation in Mayo and Antoulas (2007, equation (12)). Then, $\widehat{\Upsilon}\widehat{\Pi}(S - (\widehat{\Upsilon}\widehat{\Pi})^{-1}\widehat{\Upsilon}BL) = (Q - \mathbf{R}(C_\mathbf{P}, C)\widehat{\Pi}(\widehat{\Upsilon}\widehat{\Pi})^{-1})(\widehat{\Upsilon}\widehat{\Pi})$. Hence, $Q\widehat{\mathbb{L}} + \mathbf{R}(C_\mathbf{P}, C)\widehat{\Pi} = -S + (\widehat{\Upsilon}\widehat{\Pi})^{-1}\widehat{\Upsilon}BL$ and then, (27) follows. Since $\Lambda_S = T_S^{-1}ST_S$, then $\Pi = \widehat{\Pi}T_S \in \mathbb{C}^{n\times\nu}$ satisfies $A\Pi + BLT_S = \Pi\Lambda_S$. Similarly, $\Upsilon = T_Q^{-1}\widehat{\Upsilon} \in \mathbb{C}^{n\times\nu}$, where Υ satisfies $Q\Upsilon = \Upsilon A + T_Q^{-1}\mathbf{R}(C_\mathbf{P}, C)$. Then, $\widehat{\mathbb{L}} = T_Q^{-1}\mathbb{L}T_S$. □

Remark 3. In practice, the models (1) are not known, motivating the extension of the results to the data-driven model order reduction using Loewner matrices, as in Mayo and Antoulas (2007). The Loewner matrices (19) can also be constructed when A, B, C or the transfer function K are not available, but data sets are available, as in Mayo and Antoulas (2007). Consider the input data sets partitioned as in Table 1, such that $\ell + k \leq \nu$.

Then the data-driven Loewner matrices are

$$\mathbb{L}_{ij} = \begin{cases} \frac{\eta_{\lambda_i}}{\lambda_i - s_j}, & i, j = 1 : \ell, \\ \frac{-\eta_j}{z_i - s_j}, & i, j = 1 : k, \\ \frac{\eta_i - \eta_j}{s_i - s_j}, & i \neq j = \ell + k + 1 : \nu, \\ -\eta'_i, & i = j = \ell + k + 1 : \nu, \end{cases} \tag{28a}$$

$$\sigma\mathbb{L}_{ij} = \begin{cases} \frac{\lambda_i\eta_{\lambda_i}}{\lambda_i - s_j}, & i, j = 1 : \ell, \\ \frac{-s_j\eta_j}{z_i - s_j}, & i, j = 1 : k, \\ \frac{s_i\eta_i - s_j\eta_j}{s_i - s_j}, & i \neq j = \ell + k + 1 : \nu \\ -s_i\eta'_i, & i = j = \ell + k + 1 : \nu. \end{cases} \tag{28b}$$

Computational complexity. The approximations that match ν zero order moments at s_i, $i = 1 : \ell$, place ℓ prescribed poles, k prescribed zeros and match $\nu - (\ell + k)$ first order moments are computed employing (4) and then solving the linear system (18) with complexity $\mathcal{O}(\nu^3)$. Using Theorem 4, the storage of the $\nu \times \nu$ Loewner matrices and the inversion of $\mathbb{L}$ with complexity $\mathcal{O}(\nu^3)$ are required. The low order Sylvester equations can be solved efficiently using Krylov techniques, see, e.g., Antoulas (2005). The simulations have been performed under Maple 2018 and Matlab R2015b, on a desktop equipped with 4 GB RAM, 2.2MHz CPU and Windows 10.

Table 2
Simulation results for Σ_G of order ν with ℓ poles and k zeros and $\nu-(\ell+k)$ derivatives matched versus the BT and the IRKA.

ν	ℓ	k	Max Re$(p(K_G))$	$\|K-K_G\|_2$	$K_G(0)$	Max Re$(p(K_{BT}))$	$\|K-K_{BT}\|_2$	$K_{BT}(0)$	Max Re$(p(K_{IRKA}))$	$\|K-K_{IRKA}\|_2$	$K_{IRKA}(0)$
	3	0	$-7.4\cdot 10^{-1}$	1.523							
3	2	1	$-2.91\cdot 10^{-4}$	1.10		$-2.26\cdot 10^{-5}$	2.08	4.7206	$-2.26\cdot 10^{-5}$	$2.09\cdot 10^{-3}$	4.6395
	0	0	$2\cdot 10^{-1}$	∞							
	6	0	$-7.05\cdot 10^{-1}$	1.22							
6	4	2	$-7.9\cdot 10^{-4}$	1.09	4.5661	$-2.26\cdot 10^{-5}$	1.86	4.6554	$-2.26\cdot 10^{-5}$	$5.28\cdot 10^{-5}$	4.657
	0	0	$2.12\cdot 10^{-2}$	∞							
	12	0	$-5.4\cdot 10^{-3}$	8.07							
12	5	3	$-1.86\cdot 10^{-6}$	$4.16\cdot 10^{-3}$		$-2.26\cdot 10^{-5}$	1.81	4.653	$-2.26\cdot 10^{-5}$	$1.37\cdot 10^{-5}$	4.655
	0	0	$8.64\cdot 10^{-1}$	∞							

5. Illustrative examples

5.1. Cart controlled by a double pendulum

Consider the system (1) of the cart controlled by a double-pendulum, with

$$A=\begin{bmatrix}0&1&0&0&0&0\\-1&-1&\frac{98}{5}&1&0&0\\0&0&0&1&0&0\\1&1&\frac{-19}{5}&-2&\frac{4}{5}&1\\0&0&0&0&0&1\\0&0&\frac{98}{5}&1&\frac{-98}{5}&-2\end{bmatrix},B=\begin{bmatrix}0\\1\\0\\-1\\0\\0\end{bmatrix},C^T=\begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix}. \tag{29}$$

The poles of the *stable* system are $\{-1.6\pm 6.63j, -0.74\pm 3.48j, -0.16\pm 0.55j\}$. We follow Algorithm 1 to compute a reduced order model of (29).

Step 1: Let $\nu=3$.
Step 2: Choose the interpolation points $\{s_1,s_2,s_3\}=\{0, 1/4, 1/2\}$ and the poles $\lambda_{1,2}=\{-0.16\pm 0.55j\}$.
Step 3: Let $S=\mathrm{diag}(0, 1/4, 1/2)$, $L=[1\ 1\ 1]$. The pair (L,S) is observable.
Step 4: We compute Π, the unique solution of (3) yielding $C\Pi=[1\ 0.69\ 0.45]$.
Step 5: The family of third order models Σ_G matching three zero order moments at $\sigma(S)=\{0, 1/4, 1/2\}$, parametrized in $G=[g_1\ g_2\ g_3]^T\in\mathbb{R}^3$ is given by (4), with

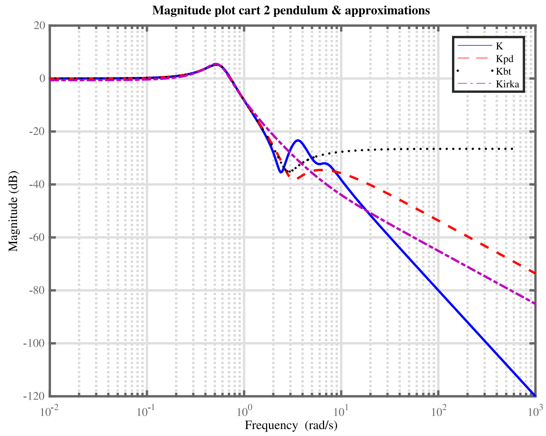
$$F=\begin{bmatrix}-g_1&-g_1&-g_1\\-g_2&0.5-g_2&-g_2\\-g_3&-g_3&0.25-g_3\end{bmatrix}, G=\begin{bmatrix}g_1\\g_2\\g_3\end{bmatrix}, H^T=\begin{bmatrix}1\\0.69\\0.45\end{bmatrix}.$$

Step 6: Solving (10) yields
$\Upsilon_{\mathbf{D}}=[1.104\ 1.312\ 0.421\ 0.864\ 0.203\ 0.427]$.
Step 7: Solving (17) yields
$G=[-12.591\ 0.992\ \ -31.316]^T$.
Step 8: $K_{pd}(s)=(0.1698s^2+0.275s+1.574)/(s^3+5.166s^2+1.849s+1.574)$.

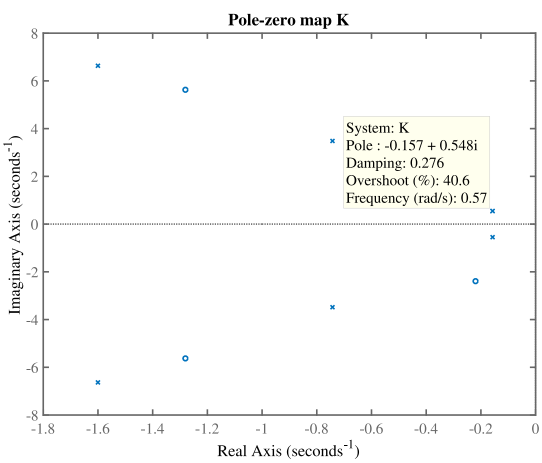
The third order approximation K_{pd} of (29) has the properties that K_{pd} matches the moments at $\sigma(S)=\{0, 1/4, 1/2\}$, $K'_{pd}(0)=K'(0)$ and K_{pd} has poles at $-0.16\pm 0.55j$, see Figs. 1(b) and 1(c). The resulting H_2-error norm of the approximation is $5.62\cdot 10^{-2}$. We compare with the third order balanced truncation-based model $K_{BT}=(-0.066s^3+0.001s^2+2.336\cdot 10^{-5}s+1.472\cdot 10^{-6})/(s^3+0.062s^2+7.895\cdot 10^{-6}s+3.161\cdot 10^{-7})$. The H_2 norm of the approximation error achieved by K_{BT} is 1.301. The poles of K_{BT} are arbitrarily placed at $\{-2.257e{-}05\pm 0.002j, -0.062\}$ and $K'_{BT}(0)\neq K'(0)$. We compare the result with the third order IRKA model, see, e.g., Gugercin et al. (2008). The IRKA is initialized in $\sigma(S)$. The resulting approximation is given by $K_{IRKA}(s)=(0.056s^2+0.246s-0.302)/(s^3-0.78s^2-0.002s-0.323)$, with poles at $1.066, -0.143\pm 0.532j$ and the H_2-error norm of order $7\cdot 10^-3$. Fig. 1(a) shows that all the approximations behave well at low frequency. Note that the model K_{pd} exhibits almost identical responses to the harmonic inputs of frequencies up to approximately 6 rad/sec., whereas the rest preserve similar behaviours on smaller frequency sets, even if they appear more accurate. Figs. 1(b) and 1(c) show that the model K_{pd} satisfies the imposed pole constraints.

5.2. CD player

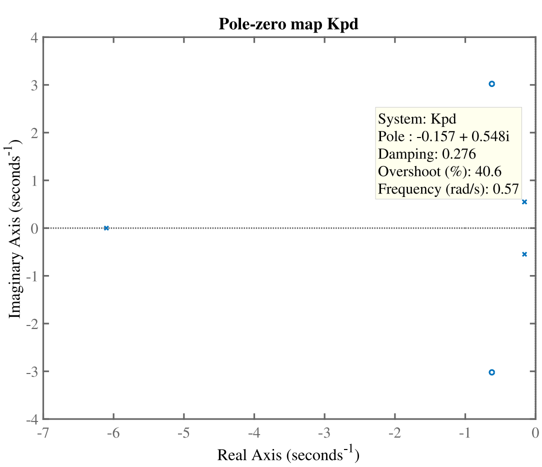
Consider a single input single output LTI system of the CD player, with $n=120$, see, e.g., Antoulas (2005), Gugercin et al. (2008). Let $S=\mathrm{diag}(s_1,\ldots,s_\nu)$ such that $s_i, i=1:\nu$ is not an eigenvalue of A and let $L=[1\ 1\ \ldots\ 1]=[L_1\ L_2\ L_3]$. Note that the matrix pair (L,S) is observable. Furthermore, arbitrarily fix the sets of numbers $\{\lambda_1,\ldots,\lambda_\ell\}$, such that $s_j\neq\lambda_j, j=1:\ell$ and $\{z_1,\ldots,z_k\}$, such that $\ell+k\le\nu$. Let Π be the solution of the Sylvester equation (3). We now write the family of ν order models Σ_G as in (4), parametrized in $G\in\mathbb{R}^\nu$, that match the moments of the transfer function of the CD player system at $\{s_1,\ldots,s_\nu\}$. Build the matrix $D_\kappa=\mathrm{diag}(\theta_{\kappa 1},\ldots,\theta_{\kappa\nu})$, $\theta_{\kappa i}=\lambda_\kappa-s_i,\ i=1:\nu,\ \kappa=1:\ell$, and the numbers $\gamma_{ji}=z_j-s_i,\ i=1:\nu,\ j=1:k$. Also consider $\Upsilon_{\mathbf{D}}$, the unique solution of the Sylvester equation $S_{\mathbf{D}}\Upsilon_{\mathbf{D}}=\Upsilon_{\mathbf{D}}A+RC$, where $R=L_3^T$ and $S_{\mathbf{D}}=\mathrm{diag}(s_1,\ldots,s_\nu)$. In the sequel, we compute the approximations Σ_G of several orders ν that have ℓ poles at $\lambda_1,\ldots,\lambda_\ell$, k zeros at $z_1,\ldots,z_k$ and satisfy the property that the first order moments of K and K_G, at $s_{(\ell+k)+1},\ldots,s_\nu$, match. We compute G for $\nu=3,6,12$, for different values of ℓ and k. Note that, based on the results of Theorem 4 and Corollary 1, instead of computing the family Σ_G, we can compute the Loewner matrices (19) and obtain identical results. We compare the results of the proposed method with the ν order balanced truncation approximation K_{BT} and the ν order Iterative Rational Krylov Algorithm (IRKA) approximation, K_{IRKA}. The results of the simulations are presented in Table 2. Note that the set of interpolation points is chosen *arbitrarily* and it contains zero for DC-gain preservation. Moreover, the selected interpolation points are used to initialize the IRKA algorithm. Due to the lack of other constraints in the choice of the interpolation points, the approximation that matches ν first order moments of the given system at these points may yield unstable approximations. Furthermore, matching a significant number of higher order moments numerically/practically ensures the decrease of the H_2/H_∞-norm of the approximation error. Fig. 2 illustrates the matching of the low frequency behaviour by the proposed models. The example illustrates that the proposed approach yields models that allow for a trade-off between the good H_2/H_∞-norm of the approximation error and the desired pole-zero placement.



(a) The models

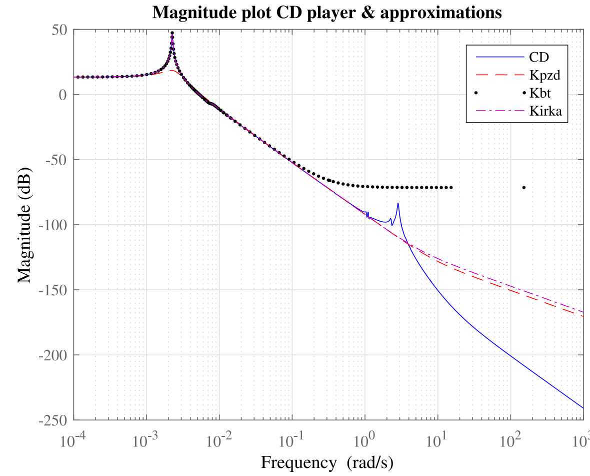


(b) Pole-zero map of K, poles at $-0.16 \pm 0.55j$

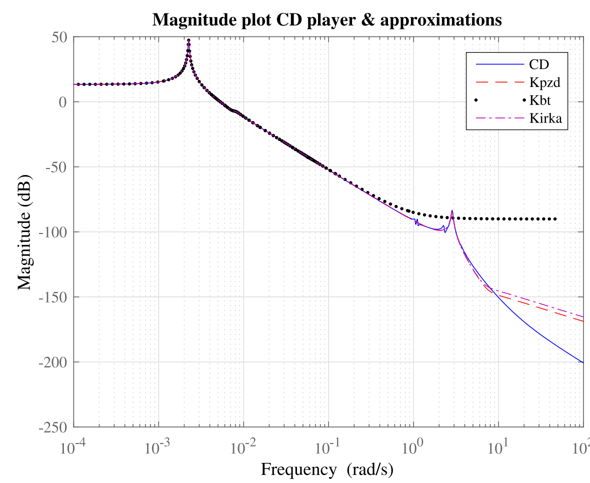


(c) Pole-zero map of $K_{\rm pd}$, poles *imposed* at $-0.16 \pm 0.55j$

Fig. 1. Magnitude plots of the models K-order 6 and $K_{\rm pd}$, $K_{\rm BT}$ and $K_{\rm IRKA}$-order 3 and the pole-zero maps of K and the third order approximation $K_{\rm pd}$.



(a) $\nu = 6, \ell = 4, k = 2$



(b) $\nu = 12, \ell = 5, k = 3$

Fig. 2. Magnitude plots of the models of the 120th order CD player model (solid blue), the proposed models (dashed red), the ν order BT model (dotted black) and the ν order IRKA model (dash-dotted magenta)

6. Conclusions

In this paper, we have computed a low order approximation that matches the moments of a given large LTI system, has certain poles and zeros fixed and matches a number of selected high order moments. The model meeting the imposed constraints is obtained solving an explicit linear system. We have also provided a relation between the main results of the manuscript (presented in Section 3) and the Loewner matrices framework, for the case when zero and first order moments are matched (Hermite interpolation). The theory has been illustrated on the academic example of a cart controlled by a double pendulum and the practical example of a CD player.

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