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LINEAR INSTABILITY FOR INCOMPRESSIBLE INVISCID FLUID FLOWS: TWO CLASSES OF PERTURBATIONS.

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LINEAR INSTABILITY FOR INCOMPRESSIBLE INVISCID FLUID FLOWS: TWO CLASSES OF PERTURBATIONS.

by

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One approach to examining the stability of a fluid flow is to linearize the evolution equation at an equilibrium and determine (if possible) the stability of the resulting linear evolution equation. In this dissertation, the space of perturbations of the equilibrium flow is split into two classes and growth of the linear evolution operator on each class is analyzed. Our classification of perturbations is most naturally described in V.I. Arnold's geometric view of fluid dynamics. The first class of perturbations we examine are those that preserve the topology of vortex lines and the second class is the factor space corresponding to the first class. In this dissertation we establish lower bounds for the essential spectral radius of the linear evolution operator restricted to each class of perturbations.

Table of Contents

Chapter 1

Introduction

To examine the stability of a fluid flow, one can examine the evolution of linear perturbations of the flow. The spectral radius of the evolution operator associated with the linearized flow indicates how much these perturbations are stretching. The criteria for linear instability can be reduced to conditions on the spectral radius of the linear evolution operator. More specifically, we can demonstrate the instability of some flow if we find that the spectral radius of the associated linear evolution operator to be greater than 1 for some positive time, t. The approach here involves computing the radius of a subset of the spectrum known as the essential spectrum. This quantity is equal to a Lyapunov-type exponent associated with the equilibrium flow, see [16, 5, 6, 7, 13] for example.

The results here establish criteria for the instability of an equilibrium incompressible, inviscid fluid flow subject to a restricted class of perturbations. The first class we will examine are those perturbations that preserve the topology of vortex lines. This class is the closure of the image of an operator B defined in Section 2.3, so we will refer to these perturbations as belonging to $\overline{\mathrm{Im}B}$. We will also consider the growth of perturbations in the canonical factor space, $F := L^2_{sol}/\overline{\mathrm{Im}B}$.

For a given steady fluid flow $u \in C^\infty(\mathbb{T}^n)$, we establish lower bounds for the radius of the essential spectrum of the linear evolution operator, $G(t)$, on each class of perturbations for 2- and 3-dimensional flows in terms of a series of Lyapunov-type exponents based on the following bicharacteristic amplitude system:

$$\text{(BAS)} \begin{cases} \dot{x} = u(x), \\ \dot{\xi} = -\left(\frac{\partial u}{\partial x}\right)^T \xi, \\ \dot{b} = -\left(\frac{\partial u}{\partial x}\right) b + 2\left(\frac{\partial u}{\partial x} b, \xi\right)\frac{\xi}{|\xi|^2}, \\ \big(x(0), \xi(0), b(0)\big) = (x_0, \xi_0, b_0) \in \mathcal{A}, \end{cases} \tag{1.1}$$

where the set of admissible initial conditions $\mathcal{A}$ is defined by

$$\mathcal{A} := \{(x_0, \xi_0, b_0) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^n |\ \xi_0 \perp b_0,\ |\xi_0| = |b_0| = 1\}.$$

For a 3-dimensional fluid flow u, let $\omega := \mathrm{curl} u$ be the vorticity of our steady flow and define the following Lyapunov-type exponents:

$$\mu_{3I} = \lim_{t\to\infty} \frac{1}{t} \log \sup_{\substack{(x_0,\xi_0,b_0)\in\mathcal{A} \\ x_0 \in \mathrm{supp}(\omega)}} |b(x_0, \xi_0, b_0; t)|$$

$$\mu_{3F} = \lim_{t\to\infty} \frac{1}{t} \log \sup_{\substack{(x_0,\xi_0,b_0)\in\mathcal{A} \\ x_0 \notin \mathrm{supp}(\omega)}} |b(x_0, \xi_0, b_0; t)|,$$

where $b(x_0, \xi_0, b_0; t)$ denotes a solution to (BAS) at time $t > 0$ with initial conditions (x_0, ξ_0, b_0). Then we have the following lower bound for the essential spectral radius of the linear evolution operator restricted to $\overline{\mathrm{Im}B}$: $r_{ess}(G(t)|_{\overline{\mathrm{Im}B}}) \geq e^{\mu_{3I} t}$. And we have another lower bound for the essential spectral radius of the linear evolution acting on the factor space: $r_{ess}(G_F(t)) \geq e^{\mu_{3F} t}$.

For a two dimensional incompressible, inviscid fluid flow, vorticity is represented as a scalar function (ω is the third and only nonzero component of the three dimensional curl of u). It is well known that the scalar vorticity, ω, is constant along flow lines. As a result, our classes of perturbations are described differently in 2-dimensions and the resulting exponents depend on the gradient of vorticity, $\nabla\omega$, instead of the vorticity. Define the Lyapunov-type exponent μ_{2I} by

$$\mu_{2I} = \lim_{t\to\infty} \frac{1}{t} \log \sup_{\substack{(x_0,\xi_0,b_0)\in\mathcal{A} \\ x_0\in\operatorname{supp}(\nabla\omega)}} |b(x_0, \xi_0, b_0; t)|.$$

And define μ_{2F} by

$$\mu_{2F} = \lim_{t\to\infty} \frac{1}{t} \log \sup_{(x_0,\xi_0,b_0)\in\mathcal{A}_1\cup\mathcal{A}_2} |b(x_0, \xi_0, b_0; t)|,$$

where

$$\begin{aligned} \mathcal{A}_1 :=& \{(x_0, \xi_0, b_0) \in \mathcal{A} : x_0 \notin \operatorname{supp}(\nabla\omega)\}, \\ \mathcal{A}_2 :=& \{(x_0, \xi_0, b_0) \in \mathcal{A} : \nabla\omega(x_0) \neq 0,\ b_0 \perp \nabla\omega(x_0)\}. \end{aligned}$$

Then we have similar lower bounds for the essential spectral radius of the linear evolution on each class of perturbations: $r_{ess}(G(t)|_{\overline{\operatorname{Im} B}}) \geq e^{\mu_{2I}t}$ and $r_{ess}(G_F(t)) \geq e^{\mu_{2F}t}$.

1.1 Chapter summaries

Chapter 2 This chapter covers preliminary information necessary for the proofs of the main theorems in Chapters 3 and 4. In Section 2.1 we define our notion of linear instability and connect it to criteria on the spectral radius of the linear evolution operator associated with the flow. In Section 2.2 we discuss the essential spectrum and the main result from [16] which gives a method for computing the essential spectral radius of the linear evolution operator . Then we define our two classes of perturbations in Section 2.3. The last section of this chapter introduces concepts related to psuedodifferential operators that we will need.

Chapter 3 This chapter contains the main results concerning 3-dimensional flows. In Section 3.1 we introduce our high frequency vector fields and then estimate their linear evolution in Lemma 3.1.2. We also establish criteria for these vector fields to approximate our first class of perturbations. In Section 3.2 we prove the main theorem concerning 3-dimensional flows, Theorem 3.2.1. Through Corollaries 3.2.3 and 3.2.4 we relate the essential spectral radius of the linear evolution to the essential spectral radius of the linear evolution restricted to each class of perturbations. Section 3.3 gives a specific example of a flow with instability from the first class of perturbations.

Chapter 4 This chapter parallels Chapter 3, only here we deal with 2-dimensional flows. In Section 4.1 we introduce our 2-dimensional high frequency perturbations and, in Lemma 4.1.3, estimate their linear evolution. We also establish when such a perturbations is approximately in the factor space and when one is approximately in $\overline{\mathrm{Im} B}$, our first class of perturbations. In Section 4.2.3 we state and prove the main theorem for 2-dimensional flows. In Section 4.3 we demonstrate that any 2-dimensional flow with a hyperbolic stagnation point has instability from both classes of perturbations. We also give an example of a flow that indicates that our 3-dimensional lower bound for growth in the factor space may not be sharp.

1.2 Notation conventions

- Our results are for flows on $\mathbb{T}^n := \mathbb{R}^n/2\pi\mathbb{Z}^n$ for $n = 2, 3$. If the domain of a space of functions or vector fields is not specified, it is $\mathbb{T}^n$.

- Throughout this paper u will denote a C^∞ vector field solution to steady-state Euler's equation, (2.1) from Section 2.1, on the 2- or 3-dimensional torus. We let ω denote the vorticity, $\omega := \mathrm{curl} u$. For 3-dimensional flows, ω is a vector field on $\mathbb{T}^3$. For 2-dimensional flows, ω is treated as a scalar function on $\mathbb{T}^2$ defined to be the 3rd and only non-zero component of $\mathrm{curl} u$.

- We will denote a space of vector fields by the space that contains its components. For example, C^∞ is used as shorthand for $(C^\infty)^2$ or $(C^\infty)^3$ whenever the dimension of the vector field is clear from context.

- Whenever v and w are vector fields on $\mathbb{T}^n$, $v \cdot \nabla w$ is a vector field defined by $(v \cdot \nabla w)^j = \sum_{i=1}^n v^i \partial_i w^j$ in components.

- We denote the subspace of divergence free, or solenoidal, vector fields in a vector space by adding the subscript *sol*. For example, $L^2_{sol}(\mathbb{T}^n) := \{v \in L^2(\mathbb{T}^n) | \mathrm{div} v = 0\}$.

- Throughout this paper projections on L^2 will be labeled π with some appropriate subscript. For example π_{sol} is the orthogonal projection onto solenoidal vector fields, L^2_{sol} and $\pi_{\xi_0^\perp}$ is the orthogonal projection onto the space of vector fields perpendicular to a fixed vector ξ_0.

- We use the standard "big-O" and "little-o" notation for infinitesimal asymptotics. If $g(\delta) = O(\delta)$ we mean $|g(\delta)| < C\delta$ for some constant C. If $g(\delta) = o(1)$ we mean $g(\delta) \to 0$ as $\delta \to 0$.

Chapter 2

Preliminaries

This chapter covers some preliminary information necessary for the proofs of the main theorems in Chapters 3 and 4. First we introduce linear instability for steady incompressible inviscid fluid flow and develop linear instability criteria based on the spectral raduis of the linear evolution of perturbations. In Section 2.2 we give an overview of Vishik's result concerning the connection between the essential spectral radius and a Lyapunov-exponent related to (BAS). Some of the constructions and proofs from that result are required for the main result of this paper and these objects and theorems are as such. In Section 2.3 we define our two classes of perturbations and prove that the first class is invariant under the linearized flow. In Section 2.4 we introduce our definitions relating to ε-pseudodifferential operators and prove some important lemmas used to prove the main theorems.

2.1 Linear instability for incompressible inviscid fluid flow

First we define linear instability for equilibrium solutions to Euler's equation for incompressible fluid motion on the 2- or 3-dimensional torus, $\mathbb{T}^n :=$

$\mathbb{R}^n/2\pi\mathbb{Z}^n$. Our equilibrium solutions are vector field solutions to time-independent Euler's equation in $C^\infty(\mathbb{T}^n)$:

$$\text{(SE)}\begin{cases} u \cdot \nabla u = -\nabla p, \\ \text{div} u = 0, \end{cases} \tag{2.1}$$

where the pressure p is a scalar function in $C^\infty(\mathbb{T}^n)$ and is determined up to a constant.

If we linearize Euler's equation about the equilibrium solution u we get the following equation for the linear evolution of a perturbation $w_0(x)$:

$$\text{(LE)}\begin{cases} \partial_t w = -u \cdot \nabla w - w \cdot \nabla u - \nabla q, \\ w(x,0) = w_0(x), \end{cases} \tag{2.2}$$

where $\nabla q \in L^2(\mathbb{T}^n)$ is the gradient of a scalar pressure determined by the requirement that the solution $w(x,t)$ remain divergence free for all time. For our purposes, the initial perturbation w_0 is a divergence free vector field in $L^2(\mathbb{T}^n)$. W will need the following fact:

Remark 1. *The Hodge Decomposition Theorem gives us that the space of square integrable vector fields on $\mathbb{T}^n$ is the orthogonal sum of divergence free vector fields, denoted $L^2_{sol} := \{v \in L^2(\mathbb{T}^n)|\ divv = 0\}$, and the subspace of gradient vector fields, denoted $L^2_{grad} := \{v \in L^2(\mathbb{T}^n)|\ v = \nabla\alpha$ for some $\alpha \in H^1(\mathbb{T}^n)\}$. Thus we may write $L^2(\mathbb{T}^n) = L^2_{sol}(\mathbb{T}^n) \oplus L^2_{grad}(\mathbb{T}^n)$. This fact guarantees the uniqueness of the pressure gradient in (SE) and (LE) above.*

Define $G(t) : L^2_{sol}(\mathbb{T}^n) \to L^2_{sol}(\mathbb{T}^n)$ to be the solution operator to the linearized system; that is, $w(x,t) := G(t)w_0(x)$ is the unique solution to (LE) above. It is well known that the operators $G(t)$ form a strongly continuous group of bounded linear operators on $H^k_{sol}(\mathbb{T}^n)$ for each $k \in \mathbb{Z}_+$.

These next two definitions give criteria for a solution to (SE), $u \in C^\infty(\mathbb{T}^n)$ with associated linear evolution operator $G(t)$, to be linearly unstable.

Definition 1. *The growth bound ω_0 for an evolution equation with solution operator $G(t)$ is defined by*

$$\omega_0 = \lim_{t\to\infty} \frac{1}{t} \log \|G(t)\|_{L^2}.$$

Definition 2. *We say that a steady state solution to Euler's equation (SE) is hydrodynamically unstable if the growth bound associated with the linearization about u is positive.*

We may connect the growth bound to the spectral radius of the linear evolution operator, $r(G(t))$, as follows: Fix $t_0 > 0$ and compute

$$\begin{aligned}
\omega_0 &= \lim_{N\to\infty} \frac{1}{Nt_0} \log \|G(Nt_0)\|_{L^2} \\
&= \lim_{N\to\infty} \frac{1}{t_0} \log \|G(Nt_0)\|_{L^2}^{1/N} \\
&= \frac{1}{t_0} \log r(G(t_0)).
\end{aligned}$$

Functionally, this is the criteria we will use to determine linear instability:

Remark 2. *If $r(G(t_0)) > 1$ for any $t_0 > 0$, then we have linear instability.*

2.2 Determining the essential spectral radius

In this section we present a result of Vishik [16] which provides a method of determining the radius of the essential spectrum of $G(t)$. From Remark 2 above, we see that if the essential spectral radius of $G(t_0)$ is greater than one at any time $t_0 > 0$, then we have linear instability. Thus, Vishik's result gives us criteria for linear instability.

To begin, we define the essential spectrum for a bounded linear operator in an indirect way. We may introduce following classification of points in the spectrum of a bounded linear operator T:

$$\sigma(T) = \sigma_{disc}(T) \cup \sigma_{ess}(T),$$

where we define σ_{disc} and σ_{ess} below.

Definition 3. *For any bounded linear operator T on a separable Hilbert space $\mathcal{H}$ we define the discrete spectrum of T, $\sigma_{disc}(T)$, to be the set of $\lambda \in \sigma(T)$ such that following conditions holds:*

- *λ is isolated in $\sigma(T)$,*
- *The Riesz projector $P = \frac{1}{2\pi i}\oint_\gamma \frac{dz}{z-T}$, where γ is a small circle around λ, has finite rank,*
- *$\lambda - T$ is invertible on the invariant subspace $KerP$ = Im(I-P),*

The essential spectrum of T is defined by $\sigma_{ess}(T) := \sigma(T) \setminus \sigma_{disc}(T)$.

We denote the essential spectral radius of an bounded linear operator T by $r_{ess}(T) := \sup\{|\lambda| : \lambda \in \sigma_{ess}(T)\}$.

In [16], Vishik developes a method for computing the essential spectral radius of the evolution operator $G(t)$ in terms of a Lyapunov-type exponent based on the bicharacteristic amplitude system, (1.1), which we repeat here:

$$\text{(BAS)}\begin{cases} \dot{x} = u(x), \qquad x(0) = x_0; \\ \dot{\xi} = -\left(\frac{\partial u}{\partial x}\right)^T \xi, \qquad \xi(0) = \xi_0; \\ \dot{b} = -\left(\frac{\partial u}{\partial x}\right) b + 2\left(\frac{\partial u}{\partial x} b, \xi\right)\frac{\xi}{|\xi|^2}, \qquad b(0) = b_0. \end{cases}$$

Theorem 2.2.1 (Vishik)**.** *Let* $\mathcal{A} := \{(x_0, \xi_0, b_0) : |\xi_0| = |b_0| = 1,\ b_0 \perp \xi_0\}$ *and define the following Lyapunov-type exponent:*

$$\mu := \lim_{t\to\infty} \frac{1}{t} \log \sup_{(x_0,\xi_0,b_0)\in\mathcal{A}} |b(x_0, \xi_0, b_0; t)|,$$

where $b(x_0, \xi_0, b_0; t)$ *is a solution to (BAS) above with intial conditions* (x_0, ξ_0, b_0)*. Then* $r_{ess}(G(t)) = e^{\mu t}$*.*

The approach to proving the main theorems of this dissertation is very much in the spirit of Vishik's work in [16]. As a result, we use much of the same machinery. Just as in Vishik's paper, we make a high-frequency ansatz on our perturbations, which leads to an approximation of $G(t)$ on high-frequencies by a pseudodifferential operator composed with parallel transport along the flow. We then estimate lower bounds for the norm of $G(t)$ by looking at the size of the symbol of our pseudodifferential operator (computed from a_0, a solution to (2.4) below).

We introduce an ε-psuedodifferential operator to separate vector fields into their high- and low-frequency parts. Let $\varepsilon > 0$, for any amplitude $\sigma \in C^\infty(\mathbb{T}^n \times \mathbb{R}^n)$ (satisfying appropriate conditions to be specified later) define

$$(\mathrm{op}_\varepsilon[\sigma]w)(x) := \frac{1}{(2\pi\varepsilon)^3} \int \sigma(x, \xi) e^{i\xi\cdot(x-y)/\varepsilon} w(y)\, dy d\xi.$$

Let $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ be a function of $|\xi|$ only, with $0 \le \chi(\xi) \le 1$, and

$$\chi(\xi) = \begin{cases} 1 & \text{if } |\xi| \le \frac{1}{2}, \\ 0 & \text{if } |\xi| \ge \frac{2}{3}. \end{cases}$$

Then

$$G(t) = G(t) \circ \mathrm{op}_\varepsilon\Big[1 - \chi\big(\frac{\xi}{\sqrt{\varepsilon}}\big)\Big] + G(t) \circ \mathrm{op}_\varepsilon\Big[\chi\big(\frac{\xi}{\sqrt{\varepsilon}}\big)\Big].$$

To approximate the linear evolution operator acting on high frequency vector fields, $G(t) \circ \mathrm{op}_\varepsilon\Big[1 - \chi\big(\frac{\xi}{\sqrt{\varepsilon}}\big)\Big]$, we introduce the parallel transport operator. Let $g^t : \mathbb{T}^n \to \mathbb{T}^n$ denote the flow map defined by trajectories of the following ODE:

$$\frac{d}{dt} g^t x = u(g^t x), \qquad g^0 = \mathrm{Id}.$$

Define $\mathfrak{g}_u(t)$ to be the evolution operator for the equation

$$\begin{cases} \dot{Y} = -u \cdot \nabla Y, \\ Y(x, 0) = Y_0(x) \in L^2(\mathbb{T}^n). \end{cases} \tag{2.3}$$

Solutions to (2.3) are parallel transport of the initial data Y_0 along the flow trajectories:

$$\mathfrak{g}_u(t) Y_0(x) = Y_0(g^{-t} x).$$

Let $a_0(x, \xi, t) \in M_{n\times n}$, for $(x, \xi, t) \in \mathbb{T}^n \times \mathbb{R}^n \backslash \{0\} \times \mathbb{R}$ where $n = 2, 3$, be a solution to

$$\begin{cases} \dot{a_0} = -\nabla_u a_0 - \frac{\partial u}{\partial x} a_0 + 2 \frac{\xi \otimes \xi}{|\xi|^2} \big(\frac{\partial u}{\partial x} a_0\big), \\ a_0(x, \xi, 0) = \big(1 - \frac{\xi \otimes \xi}{|\xi|^2}\big) \cdot \big(1 - \chi\big(\frac{\xi}{\sqrt{\varepsilon}}\big)\big), \end{cases} \tag{2.4}$$

where ∇_u is the Lie derivative computed in the cotangent bundle $T^*(\mathbb{T}^n)$ along flow trajectories:

$$\nabla_u := \frac{d}{dt}\Big|_{t=0} (g^t, (g_*^{-t})^*).$$

In coordinates $\nabla_u = (u, -\frac{\partial u}{\partial x}^T \xi)$. The matrix-valued function a_0 is used to compute the symbol of a pseudodifferential operator that, when composed with parallel transport along the flow, approximates $G(t)$ on high frequencies:

Definition 4. *Let $G^s_\varepsilon(t) : L^2_{sol} \to L^2_{sol}$ be defined by*

$$G^s_\varepsilon(t)w_0 = op^s_\varepsilon[a_0] \circ \mathfrak{g}_u(t)w_0,$$

where in $\mathbb{R}^3$,

$$(op^s_\varepsilon[a_0]w)(x) = \nabla \times \frac{\varepsilon}{(2\pi\varepsilon)^3} \int \frac{i\xi}{|\xi|^2} \times a_0(x, \xi, t) e^{i\xi \frac{x-y}{\varepsilon}} w(y) d^3y d^3\xi.$$

In [16], Vishik proves that $G^s_\varepsilon(t)$ approximates $G(t)$ on high frequencies in the following sense:

Theorem 2.2.2. *Let $G(t)$ be the evolution operator associated with Euler's equation linearized at u. Then for all $t \geq 0, G^s_\varepsilon(t)$ is a bounded operator in L^2_{sol} and for any fixed $T > 0$*

$$\|G(t) \circ op_\varepsilon\Big[1 - \chi\big(\frac{\xi}{\sqrt{\varepsilon}}\big)\Big] - G^s_\varepsilon(t)\|_{\mathcal{L}(L^2_{sol}, L^2)} = O(\sqrt{\varepsilon}), \quad 0 \leq t \leq T, \tag{2.5}$$

with the constant in O uniform over the interval $[0, T]$.

In order to explain the connection between solutions a_0 to (2.4) and solutions to (BAS), we must introduce a decomposition of our symbol a_0:

$$a_0(x, \xi, t) = A_0(x, \xi, t)\big(1 - X\big(x, \frac{\xi}{\sqrt{\varepsilon}}, t\big)\big), \tag{2.6}$$

where A_0 is a solution to the following system:

$$\begin{cases} \partial_t A_0 = -\nabla_u A_0 - \frac{\partial u}{\partial x} A_0 + 2\frac{\xi \otimes \xi}{|\xi|^2} \frac{\partial u}{\partial x} A_0, \\ A_0(x, \xi, 0) = 1 - \frac{\xi \otimes \xi}{|\xi|^2}. \end{cases} \tag{2.7}$$

And X satisfies

$$\begin{cases} \partial_t X = -\nabla_u X, \\ X(x,\xi,0) = \chi(\xi). \end{cases} \tag{2.8}$$

The matrix symbol $a_0(x,\xi,t)$ forms a strongly continuous cocycle over the flow $(g^t\cdot, (g_*^{-t}(x))^*\cdot)$ on the cotangent bundle $T^*(\mathbb{T}^n)$. Similarly, $A_0(x,\xi,t)$ forms a strongly continuous cocyle on $\mathbb{T}^n \times \mathbb{R}P^{n-1}$. An important consequence of this fact is that the Lyapunove-type exponent in Theorem 2.2.1 is well defined. See [3] for a detailed discussion of cocycles and their properties. Solutions to (BAS) are solutions to (2.7) for $A_0(\cdot,\cdot,t)$ along characteristics which are the flow lines $(g^t\cdot, (g_*^{-t}(x))^*\cdot)$ in $\mathbb{T}^n \times \mathbb{R}P^{n-1}$. Thus it follows that for any $(x_0,\xi_0,b_0) \in T^*(\mathbb{T}^n) \times \mathbb{R}^n$ we have

$$b(x_0,\xi_0,b_0;t) = A_0(g^t x_0, (g_*^{-t}(x))^*\xi_0, t)b_0. \tag{2.9}$$

There is one more fact about solutions to (BAS) that we will need. If we let $b(t) := b(x_0,\xi_0,\xi_0^\perp;t)$ and $\xi(t) := (g_*^{-t}(x_0))^*\xi_0$ then we have

$$\dot{b} = -\Big(\frac{\partial u}{\partial x}\Big)b + 2\Big(\frac{\partial u}{\partial x}b,\xi\Big)\frac{\xi}{|\xi|^2} = -\pi_{\xi^\perp}\Big(\frac{\partial u}{\partial x}b\Big) + \pi_\xi\Big(\frac{\partial u}{\partial x}b\Big).$$

Hence we may compute

$$\begin{aligned} \frac{d}{dt}\big(b(t),\xi(t)\big) &= \big(\dot{b}(t),\xi(t)\big) + \big(b(t),\dot{\xi}(t)\big) \\ &= \big(\frac{\partial u}{\partial x}b(t),\xi(t)\big) + \big(b(t), -\big(\frac{\partial u}{\partial x}\big)^T\xi(t)\big) = 0. \end{aligned}$$

Thus, whenever $b_0 \perp \xi_0$ we have

$$(b(x_0,\xi_0,b_0;t), (g_*^{-t}(x_0))^*\xi_0) = 0. \tag{2.10}$$

Nussbaum's formula for computing the essential spectral radius of a bounded linear operator is last key piece of machinery we carry over from [16]. A proof can be found in Nussbaum's original paper, [11]. Let X be a separable Hilbert space. We define an appropriate norm on the quotient space $\mathcal{L}(X)/\mathfrak{S}_\infty$ where $\mathfrak{S}_\infty$ is the ideal of compact operators.

Definition 5. *For any* $T \in \mathcal{L}(X)$

$$\|T\|_{\mathcal{K}} = \inf_{K \in \mathfrak{S}_\infty} \|T + K\|_{\mathcal{L}(X)}. \tag{2.11}$$

The seminorm $\|\cdot\|_{\mathcal{K}}$ on $\mathcal{L}(X)$ is the canonic norm on the quotient space $\mathcal{L}(X)/\mathfrak{S}_\infty$. We can compute the essential spectral radius of a bounded operator with this norm:

Theorem 2.2.3 (Nussbaum)**.** *For any* $T \in \mathcal{L}(X)$

$$r_{ess}(T) = \lim_{n \to \infty} (\|T^n\|_{\mathcal{K}})^{\frac{1}{n}}. \tag{2.12}$$

Since $G(t) \circ \mathrm{op}_\varepsilon\Big[\chi\big(\frac{\xi}{\sqrt{\varepsilon}}\big)\Big]$ is a compact operator,

$$r_{ess}(G(t)) = r_{ess}\Big(G(t) \circ \mathrm{op}_\varepsilon\Big[1 - \chi\big(\frac{\xi}{\sqrt{\varepsilon}}\big)\Big]\Big).$$

Thus, it suffices to consider linear evolution on high frequencies to determine the essential spectral radius.

2.3 Two classes of perturbations

In his book, *Mathematical Methods of Classical Mechanics*, V.I. Arnold characterizes the dynamics of an incompressible, inviscid fluid in a geometric way.

We may think of the motion of a fluid as a family of volume preserving diffeomorphisms of the fluid domain indexed by time. These diffeomorphisms form an infinite-dimensional Lie group. If we let kinetic energy be the right-invariant metric for our Lie group, then geodesics will correspond to flows that minimize kinetic energy. Thus we may view fluid dynamics as motion along geodesics in our group of volume preserving diffeomorphisms with the kinetic energy metric. The two classes of perturbations that are studied in this dissertation are most naturally described in this geometric view of fluid dynamics.

Let SDiff(Ω) denote the space of C^∞ volume preserving diffeomorphisms of a compact fluid domain, Ω. The corresponding Lie algebra, $\mathfrak{g}$, is the collection of C^∞, divergence free vector fields on Ω. The Lie commutator is defined by the bracket:

$$[v_1, v_2] := (v_2 \cdot \nabla)v_1 - (v_1 \cdot \nabla)v_2.$$

The metric here is the L^2-inner product of vector fields. Through this metric, we can identify $\mathfrak{g}$ and its dual, denoted $\mathfrak{g}^*$. One of the essential differences between 2-dimesional and 3-dimensional hydrodynamics is the difference in the geometry of orbits of the co-adjoint representation in the two cases, see [1]. Formally, we may compute the co-adjoint representation of SDiff(Ω), denoted $Ad^* : \text{SDiff}(\Omega) \to \text{End}(\mathfrak{g}^*)$ to be defined by

$$Ad^*_g v = \text{curl}^{-1} g_* \text{curl} v \qquad g \in \text{SDiff}(\Omega), v \in \mathfrak{g}^*.$$

Two vector fields v_1 and v_2 are isovorticial if there exists some $g \in \text{SDiff}(\Omega)$

such that

$$g_*\text{curl}v_1 = \text{curl}v_2,$$

where g_* denotes the pushforward by the diffeomorphism g. It follows that the orbit of our steady solution u under the co-adjoint action is the collection of vector fields isovorticial to u.

The first space of perturbations we study is the tangent space to the co-adjoint orbit of u. The derivative of Ad^*_g with respect to g evaluated at the identity gives an operator $ad^* : \mathfrak{g} \to \text{End}(\mathfrak{g}^*)$. Elements in the tangent space to the co-adjoint orbit of u at u are of the form $ad^*_w u$, for some $w \in \mathfrak{g}$. This leads to the mapping $B : L^2_{sol} \to L^2_{sol}$ defined by

$$Bw := \text{curl}u \times w - \nabla\alpha,$$

where the pressure term $\nabla\alpha$ is uniquely defined by the requirement that B map into divergence free vector fields. Notice that whenever $w \in C^\infty_{sol}$, we have $Bw = ad^*_w u$. Thus $\overline{\text{Im}B}$ forms our first class of perturbations.

Since B is skew adjoint, we have

$$L^2_{sol} = \overline{\text{Im}B} \oplus \text{Ker}B.$$

We also observe that $\overline{\text{Im}B}$ is invariant under the linearized flow $G(t)$:

Proposition 2.3.1. *Let $u \in C^\infty_{sol}(\mathbb{T}^3)$ be a steady solution to Euler's equation and $G(t)$ the evolution operator for linear perturbations of u. Then for any $w \in L^2_{sol}(\mathbb{T}^3)$,*

$$G(t)Bw = BG(-t)^*w.$$

Proof. First we endow the space $C^\infty_{sol}(\mathbb{T}^3)$ with a Lie algebra structure via the Poisson bracket on vector fields, define $[\cdot,\cdot] : C^\infty_{sol}(\mathbb{T}^3) \times C^\infty_{sol}(\mathbb{T}^3) \to C^\infty_{sol}(\mathbb{T}^3)$ by

$$[v_1, v_2] := (v_2 \cdot \nabla)v_1 - (v_1 \cdot \nabla)v_2.$$

Since v_1 and v_2 are divergence free,

$$[v_1, v_2] = \text{curl}(v_1 \times v_2).$$

Hence $[v_1, v_2]$ is always divergence free. Now define the bilinear form B : $C^\infty_{sol}(\mathbb{T}^3) \times L^2_{sol}(\mathbb{T}^3) \to L^2_{sol}(\mathbb{T}^3)$ by

$$B(a, b) := \text{curl}a \times b - \nabla\alpha,$$

where $\nabla\alpha$ is uniquely defined by the requirement that $B(a, b)$ be divergence free. If we denote by $\langle , \rangle$ the L^2-inner product on $\mathbb{T}^3$, then for any $a, b, c \in C^\infty_{sol}(\mathbb{T}^3)$,

$$\langle [a, b], c\rangle = \langle B(c, a), b\rangle.$$

To see this, compute

$$\langle [a, b], c\rangle - \langle B(c, a), b\rangle = \int_{\mathbb{T}^3} \big(\text{curl}(a \times b)\big) \cdot c - (\text{curl}c \times a) \cdot b + b \cdot \nabla\alpha \; dV.$$

Since b is divergence free, $\int_{T^3} b \cdot \nabla\alpha \; dV = 0$, so we have

$$\begin{aligned}\langle [a, b], c\rangle - \langle B(c, a), b\rangle &= \int_{\mathbb{T}^3} \big(\text{curl}(a \times b)\big) \cdot c - (a \times b) \cdot \text{curl}c \; dV \\ &= \int_{\mathbb{T}^3} \nabla \cdot \big((a \times b) \times c\big) \; dV.\end{aligned}$$

Then if we apply Stokes' Theorem to the RHS of our equation, we have

$$\langle [a, b], c\rangle - \langle B(c, a), b\rangle = \int_{\partial\mathbb{T}^3} \big((a \times b) \times c\big) \; dS = 0,$$

where the last equality follows from the assumption that a, b and c are periodic. Notice that for any $v \in C^\infty_{sol}(\mathbb{T}^3)$,

$$\operatorname{curl}\big((-v\cdot\nabla)v + \nabla p\big) = \operatorname{curl}\big(v \times (\operatorname{curl} v)\big) = \operatorname{curl}\big(-B(v,v)\big).$$

Since both $(-v \cdot \nabla)v + \nabla p$ and $-B(v,v)$ are divergence free, it follows that our solution to steady Euler's equation, u, satisfies

$$-B(u,u) = 0.$$

It follows that we can also write linearized Euler's equation (LE) as

$$\begin{cases} \partial_t w = -B(u,w) - B(w,u), \\ w(x,0) = w_0(x), \end{cases} \tag{2.13}$$

where u is our solution to steady Euler's and w_0, $w \in L^2_{sol}(\mathbb{T}^3)$. We define the operator L by

$$L = -(u\cdot\nabla)w - (w\cdot\nabla)u - \nabla q = -B(u,w) - B(w,u).$$

The unbounded operator L is the generator for the strongly continuous group, or C_0 group of bounded operators that define the evolution of linear perturbations: G(t). It is straightforward to compute the adjoint operator L^*:

$$\begin{aligned} \langle Lw, v\rangle &= \langle -B(u,w), v\rangle + \langle -B(w,u), v\rangle \\ &= \langle B(u,v), w\rangle - \langle [u,v], w\rangle \qquad\qquad v, w \in C^\infty_{sol}(\mathbb{T}^3). \end{aligned}$$

Which implies that for any vector field $v \in C^\infty(\mathbb{T}^3)$,

$$L^* v := B(u,v) - [u,v].$$

It follows that L^* generates a C_0 group, $G(t)^*$ on $L^2_{sol}(\mathbb{T}^3)$. In fact, for all $t \in \mathbb{R}$, $G(t)^*$ is the adjoint operator of $G(t)$. We also have that $G(t)^*$ is a C_0 group on $H^k_{sol}(\mathbb{T}^3)$ for any $k \in \mathbb{Z}_+$. This implies $G(t)^* : C^\infty_{sol}(\mathbb{T}^3) \to C^\infty_{sol}(\mathbb{T}^3)$. Let $v_0 \in C^\infty_{sol}(\mathbb{T}^3)$. We must show that $w(t) = B(u, G(-t)^* w_0) \in C^\infty_{sol}(\mathbb{T}^3)$ satisfies (LE). Since images of L are divergence free vector fields, it is equivalent for $w(t)$ to satisfy the following system of equations:

$$\nabla \times (\text{LE}) \begin{cases} \partial_t \text{curl} w = -[\text{curl} u, w] - [\text{curl} w, u], \\ \text{curl} w(x, 0) = \text{curl} B(u, v_0). \end{cases}$$

Compute

$$\partial_t \text{curl} w = \frac{d}{dt} \text{curl} B(u, G(-t)^* v_0).$$

Let $v(t) := G(-t)^* v_0$. Since L^* is the generator of the C_0 group $G(t)^*$, taking the derivative gives

$$\begin{aligned} \partial_t \text{curl} w &= [\text{curl} u, -L^* v(t)] \\ &= \big[\text{curl} u, [u, v(t)] - B(u, v(t))\big]. \end{aligned}$$

If we apply the Jacobi identity, noting that $[\text{curl} u, u] = 0$, we have

$$\partial_t \text{curl} w = -\big[[\text{curl} u, v(t)], u\big] - \big[\text{curl} u, B(u, v(t))\big].$$

Since $\text{curl} B(u, v(t)) = [\text{curl} u, v(t)]$, we have

$$\frac{d}{dt} \text{curl} B(u, G(-t)^* v_0) = -\big[\text{curl} B(u, v(t)), u\big] - \big[\text{curl} u, B(u, v(t))\big].$$

Hence $B(u, G(-t)^* v_0)$ satisfies (LE) with initial condition $w_0 = B(u, v_0)$. Therefore, we have for any $v \in C^\infty_{sol}(\mathbb{T}^3)$,

$$G(t) B v = B G(-t)^* v.$$

For any $t \in \mathbb{R}$ both $G(t)B$ and $BG(-t)^*$ are bounded operators on $L^2_{sol}(\mathbb{T}^3)$, so we may extend the result to any $v \in L^2_{sol}(\mathbb{T}^3)$. This completes the proof. $\square$

Remark 3. *This proposition holds for 2-dimensional vector fields as well. That is, if the vector field $u \in C^\infty(\mathbb{T}^2)$ is a solution to steady Euler's equation, then for any $w \in L^2_{sol}(\mathbb{T}^3)$, we have $G(t)Bw = BG(-t)^*w$. To see this, just consider u and w to be 3-dimensional planar vector fields and the proof of Proposition 2.3.1 holds.*

It follows from Proposition 2.3.1 that $\overline{\mathrm{Im} B}$ is an invariant subspace under the linearized flow. Now it makes sense to consider the essential spectral radius of the evolution of perturbations in $\overline{\mathrm{Im} B}$ under the linear flow about our steady equilibrium u.

We also consider the linearized flow on the factor space $F := L^2_{sol}/\overline{\mathrm{Im} B}$ with the canonical factor space norm:

$$\|v\|_F := \inf_{w \in \overline{\mathrm{Im} B}} \|v + w\|_{L^2_{sol}}.$$

This factor space forms our second class of perturbations.

2.4 ε-pseudodifferential operators

In this section we define our ε-pseudodifferential operators and prove several technical lemmas that will be necessary for the main results of this dissertation.

Definition 6. *For $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ the class of symbols $S^m_{\rho,\delta}(\mathbb{T}^n)$ denotes the space of functions $\sigma \in C^\infty(\mathbb{T}^n \times \mathbb{R}^n)$ such that for all α, $\beta \in \mathbb{Z}^n$ there is a* *constant $C_{\alpha,\beta}$ such that for any $(x, \xi) \in \mathbb{T}^n \times \mathbb{R}^n$*

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-\rho|\beta|+\delta|\alpha|}.$$

We often write $S^m_{\rho,\delta}(\mathbb{T}^n)$ or $S^m_{\rho,\delta}$ to denote the same class of symbols.

Remark 4. *It follows directly from the definition above that if $\sigma \in C^\infty(\mathbb{T}^n \times \mathbb{R}^n)$ is positively homogeneous of degree m in the region $|\xi| \geq R$ for some $R > 0$ (that is, $\sigma(x, \lambda\xi) = \lambda^m \sigma(x, \xi), \lambda \geq 1, |\xi| \geq R$), then $\sigma \in S^m_{1,0}(\mathbb{T}^n)$.*

Definition 7. *For any $\varepsilon > 0$ and $\sigma \in S^0_{\rho,\delta}(\mathbb{T}^n)$ where $0 \leq \delta < \rho \leq 1$ define $op_\varepsilon[\sigma(x, \xi)] : \mathcal{D}(\mathbb{T}^n) \to \mathcal{D}(\mathbb{T}^n)$ by*

$$(op_\varepsilon[\sigma(x, \xi)]w)(x) := \frac{1}{(2\pi\varepsilon)^n} \int \sigma(x, \xi) e^{i\xi\cdot(x-y)/\varepsilon} w(y) d^n y d^n \xi.$$

Remark 5. *If $\sigma \in S^m_{1,0}(\mathbb{T}^n)$ for $m \leq 0$, the psuedodifferential operator $op_\varepsilon[\sigma(x, \xi)]$ is a bounded linear operator on $L^2_{sol}(\mathbb{T}^n)$. A proof for periodic operators is given in [14] for example.*

We will need the following variant of the Calderon and Vaillancourt theorem [2] for x-periodic amplitudes to estimate the norms of some ε-pseudodifferential operators (see also [4]).

Theorem 2.4.1 (Calderon-Vaillancourt)**.** *Let $\sigma(x, \xi) \in C^\infty(\mathbb{T}^n \times \mathbb{R}^n)$ for $0 \leq \rho < 1$, satisfy the following inequalities.*

$$\left|\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi)\right| \leq C_{\alpha\beta}(1 + |\xi|)^{\rho(|\alpha|-|\beta|)},$$

for all $(x, \xi) \in \mathbb{T}^n \times \mathbb{R}^n$, and $(\alpha, \beta) \in \mathbb{Z}^n$. Then the pseudodifferential operator $op_1[\sigma(x, \xi)]$ extends from Schwartz space $\mathcal{D}(\mathbb{T}^n) = C^\infty(\mathbb{T}^n)$ to $L^2(\mathbb{T}^n)$ and *defines a bounded operator there, moreover:*

$$\| op_1[\sigma]\|_{\mathcal{L}(L^2)} \leq C(n) \sum_{\substack{|\alpha|\leq 2((n/2)+1) \\ |\beta|\leq 2((n/(1-\rho))+1)}} C_{\alpha\beta}.$$

Lemma 2.4.2. *Let* $\sigma_\varepsilon(x,\xi) \in S^{-m}_{1,0}(\mathbb{T}^n)$ *for* $m > 0$*. Suppose* $\sigma_\varepsilon(x,\xi) = 0$ *whenever* $|\xi| < \frac{c_0}{\sqrt{\varepsilon}}$*. Then* $\| op_1[\sigma_\varepsilon]\|_{\mathcal{L}(L^2)} = O(\sqrt{\varepsilon}^m)$*.*

Proof. We will use the Calderon-Vaillancourt inequality to estimate the L^2-operator norm of $\mathrm{op}_1[\sigma_\varepsilon(x,\xi)]$. Let β, $\gamma \in \mathbb{Z}^n$. Since $\sigma_\varepsilon(x,\xi) \in S^{-m}_{1,0}$, there is some constant $C_{\beta,\gamma}$ such that for any $x \in \mathbb{T}^n$

$$|\partial_x^\beta \partial_\xi^\gamma \sigma_\varepsilon(x,\xi)| \leq C_{\beta,\gamma}(1+|\xi|)^{-m-|\gamma|}.$$

Multiply this inequality by $(1+|\xi|)^{1/2(|\gamma|-|\beta|)}$ to get

$$\begin{aligned} |\partial_x^\beta \partial_\xi^\gamma \sigma_\varepsilon(x,\xi)|(1+|\xi|)^{1/2(|\gamma|-|\beta|)} \leq & C_{\beta,\gamma}(1+|\xi|)^{-(m+1/2(|\beta|+|\gamma|))} \\ \leq & C_{\beta,\gamma}(c_0\sqrt{\varepsilon})^{m+1/2(|\beta|+|\gamma|)}. \end{aligned}$$

This last inequality follows from the fact that the symbol $\sigma_\varepsilon(x,\xi) = 0$ for $|\xi| < \frac{c_0}{\sqrt{\varepsilon^m}}$. So for any $(x,\xi) \in \mathbb{T}^n \times \mathbb{R}^n$ we have

$$|\partial_x^\beta \partial_\xi^\gamma \sigma_\varepsilon(x,\xi)| \leq C_{\beta,\gamma}(c_0\sqrt{\varepsilon})^{m+1/2(|\beta|+|\gamma|)}(1+|\xi|)^{1/2(|\beta|-|\gamma|)}.$$

Thus, we may use the Calderon-Vaillancourt inequality for $\rho = 1/2$ to estimate the norm of our operator. The most substantial contribution to the norm is the $\beta = \gamma = 0$ summand. Therefore we have

$$\|\mathrm{op}_1[\sigma_\varepsilon(x,\xi)]\|_{\mathcal{L}(L^2_{sol})} = O(\sqrt{\varepsilon}^m).$$

□

Next we give the proofs for two lemmas originally proved in [16]. For the first lemma we will need the following formula, proved in Grigis and Sjostrand, [8].

Lemma 2.4.3 (Stationary Phase Formula)**.** *Let Q be a symmetric nondegenerate matrix, then for $g \in C_0^\infty(\mathbb{R}^N)$*

$$\int e^{ix\cdot\ Qx/\delta} g(x)\ dx = g(0) + R_1(g, \delta), \tag{2.14}$$

where for $D_x^\alpha := (-i\partial_{x_1})^{\alpha_1} ... (-i\partial_{x_N})^{\alpha_N}$,

$$|R_1(g,\delta)| \le C_Q \delta^{n/2} \sum_{|\alpha| \le 2[n/2]+2} \|D_x^\alpha \big(\frac{1}{4}(D_x, Q^{-1}D_x)\big) g\|_{L^1(\mathbb{R}^N)}.$$

Lemma 2.4.4 (Vishik)**.** *Suppose $S \in C^\infty(\mathbb{R}^n)$ and for any $m \in \mathbb{Z}^n$, there exists some $\xi_0 \in \mathbb{Z}^n$ such that*

$$S(x + 2\pi m) = S(x) + 2\pi m \xi_0. \tag{2.15}$$

Let $\sigma \in S^0_{0,0}(\mathbb{T}^n)$ and $\delta^{-1} \in \mathbb{Z}_+$. Then there exists a constant $C(n, \sigma, S)$ depending only on n, σ and S and an index $k \in \mathbb{Z}_+$ depending only on the dimension n such that for any $f_0 \in C^\infty(\mathbb{T}^n)$ and any $x \in \mathbb{T}^n$,

$$\big|\big(op_\delta[\sigma] f_0 e^{iS/\delta}\big)(x) - \sigma(x, \nabla S) f_0(x) e^{iS(x)/\delta}\big| \le C\delta \sum_{|\alpha|\le k} \sup_{x\in\mathbb{T}^n} |\partial_x^\alpha f_0(x)|.$$

Remark 6. *Notice that S and ∇S are well defined and smooth on $\mathbb{T}^n$, so $e^{iS(x)/\delta}$ and $\sigma(x, \nabla S(x))$ are both functions in $C^\infty(\mathbb{T}^n)$.*

Proof. Define

$$I(x) := \big(\mathrm{op}_\delta[\sigma] f_0 e^{iS/\delta}\big)(x) = \frac{1}{(2\pi\delta)^n} \int \sigma(x,\xi) e^{i[\xi\cdot(x-y)+S(y)]/\delta} f_0(y)\ dy d\xi.$$

Let $\Delta_\xi := \sum_{i\leq n} \partial^2_{\xi_i}$. We will use the fact that for any $M \in \mathbb{Z}_+$

$$(1 + \delta^{2M}(-\Delta_\xi)^M)e^{i\xi(x-y)/\delta} = (1 + |x-y|^{2M})e^{i\xi\cdot(x-y)/\delta}, \tag{2.16}$$

and integrate by parts to get

$$I(x) = \frac{1}{(2\pi\delta)^n}\int \frac{(1+\delta^{2M}(-\Delta_\xi)^M)\sigma(x,\xi)}{1+|x-y|^{2M}} e^{i[\xi\cdot(x-y)+S(y)]/\delta} f_0(y)\ dyd\xi.$$

Let $\tilde{\sigma}(x,\xi) := (1+\delta^{2M}(-\Delta_\xi)^M)\sigma(x,\xi)$. Then we have

$$I(x) = \frac{1}{(2\pi\delta)^n}\int \frac{\tilde{\sigma}(x,\xi)}{1+|x-y|^{2M}} e^{i\Psi_x(y,\xi)/\delta} f_0(y)\ dyd\xi, \tag{2.17}$$

where the phase Ψ_x depends on x as a parameter:

$$\Psi_x(y,\xi) := \xi(x-y) + S(y).$$

The goal here is to use the Stationary Phase Formula (2.14), so we first remark that the only critical point of Ψ_x with respect to (y,ξ) is $(y,\xi) = (x, \nabla S(x))$. We make a change of variables in the integral (2.17) to move the critical point to the origin in $\mathbb{T}^n \times \mathbb{R}^n$:

$$z = y - x, \qquad \theta = \xi - \nabla S(x).$$

This gives us

$$I(x) = \frac{1}{(2\pi\delta)^n}\int \frac{\tilde{\sigma}(x,\theta+\nabla S(x))}{1+|z|^{2M}} e^{i[S(x+z)-(\theta+\nabla S(x))\cdot z]/\delta} f_0(x+z)\ dzd\theta. \tag{2.18}$$

To use the Stationary Phase Formula, we must have a quadratic phase function, so we now transform the phase in (2.18). We begin using the Taylor expansion about x:

$$\begin{aligned} S(x+z) - (\nabla S(x),\ z) - (\theta,\ z) \\ &= S(x) + \int_0^1 (1-\tau)\frac{d^2}{d\tau^2} S(x+\tau z)\ d\tau - (\theta,\ z) \\ &= S(x) + \Big(\int_0^1 (1-\tau)\frac{d}{d\tau}\nabla S(x+\tau z)\ d\tau,\ z\Big) - (\theta,\ z) \end{aligned}$$

Let

$$\rho(z) := \int_0^1 (1-\tau)\frac{d}{d\tau}\nabla S(x+\tau z)\ d\tau, \tag{2.19}$$

and we make another change of variables:

$$\eta := \theta - \rho(z).$$

Then we have

$$I(x) = \frac{1}{(2\pi\delta)^n}\int \frac{\tilde{\sigma}(x, \eta + \rho(z) + \nabla S(x))}{1+|z|^{2M}} e^{i[S(x)-\eta]\cdot z/\delta} f_0(x+z)\ d\eta dz.$$

Now we integrate by parts again using the same type of identity as in (2.16), only here $J \in \mathbb{Z}_+$ is our index. We have

$$I(x) = \frac{e^{iS(x)/\delta}}{(2\pi\delta)^n}\int \frac{(1+\delta^{2J}(-\Delta_\eta)^J)\tilde{\sigma}(x, \eta + \rho(z) + \nabla S(x))}{(1+|\eta|^{2J})(1+|z|^{2M})} e^{-i\eta\cdot z/\delta} f_0(x+z)\ d\eta dz.$$

Let

$$\overline{\sigma}(z, \eta) := \frac{(1+\delta^{2J}(-\Delta_\eta)^J)\tilde{\sigma}(x, \eta + \rho(z) + \nabla S(x))}{(1+|\eta|^{2J})(1+|z|^{2M})} f_0(x+z),$$

and we get

$$I(x) = \frac{e^{iS(x)/\delta}}{(2\pi\delta)^n}\int \overline{\sigma}(z, \eta) e^{-i\eta\cdot z/\delta}\ d\eta dz.$$

We need a compactly supported integrand to use the Stationary Phase Formula, so we introduce the cutoff function $\kappa \in C_0^\infty(R^{2n})$ such that

$$\kappa(z, \eta) := \begin{cases} 1, & \text{if } \sqrt{|z|^2 + |\eta|^2} \leq 1, \\ 0, & \text{if } \sqrt{|z|^2 + |\eta|^2} \geq 2. \end{cases}$$

Then we have

$$I(x) = \frac{e^{iS(x)/\delta}}{(2\pi\delta)^n}\Big[\int \kappa(z,\eta)\overline{\sigma}(z,\eta)e^{-i\eta\cdot z/\delta}\ d\eta dz \qquad (2.20)$$
$$+ \int (1-\kappa(z,\eta))\overline{\sigma}(z,\eta)e^{-i\eta\cdot z/\delta}\ d\eta dz\Big].$$

First we will deal with the non-compact integrand. Using the following identity:

$$\big(z\partial_\eta + \eta\partial_z\big)e^{-i\eta\cdot z/\delta} = \frac{-i}{\delta}(|z|^2+|\eta|^2)e^{-i\eta\cdot z/\delta},$$

we integrate by parts

$$\begin{aligned}&\frac{1}{(2\pi\delta)^n}\int (1-\kappa(z,\eta))\overline{\sigma}(z,\eta)e^{-i\eta\cdot z/\delta}\ d\eta dz\\ &=\frac{1}{(2\pi\delta)^n}\int (1-\kappa(z,\eta))\overline{\sigma}(z,\eta)\Big[\frac{i\delta}{|z|^2+|\eta|^2}\big(z\partial_\eta+\eta\partial_z\big)\Big]^{n+1}e^{-i\eta\cdot z/\delta}\ d\eta dz\\ &=\frac{-(i)^{n+1}\delta}{(2\pi)^n}\int e^{-i\eta\cdot z/\delta}\big(z\partial_\eta+\eta\partial_z\big)^{n+1}\frac{(1-\kappa)\overline{\sigma}(z,\eta)}{(|z|^2+|\eta|^2)^{n+1}}\ d\eta dz\end{aligned}$$

Thus we have a bound on the second term of (2.20)

$$\Big|\frac{1}{(2\pi\delta)^n}\int (1-\kappa)\overline{\sigma}e^{-i\eta\cdot z/\delta}\ d\eta dz\Big| \le \delta\frac{C}{(2\pi)^n}\|\overline{\sigma}\|_{W^{n+1,1}}, \qquad (2.21)$$

where $\|\cdot\|_{W^{n+1,1}}$ denotes the norm in the Sobolev space $W^{n+1,1}(\mathbb{R}^{2n})$. Now we use the Stationary Phase Formula (2.14) on the first term of (2.20). We integrate in coordinates $(\eta_1,..\eta_n,z_1,..,z_n)$ and the matrix $Q\in M_{2n\times 2n}$ is defined by

$$Q := \frac{1}{2}\begin{pmatrix} 0 & \mathbb{I}\\ -\mathbb{I} & 0\end{pmatrix},$$

where $\mathbb{I}$ denotes the $n\times n$ identity matrix. Thus (2.14) and the estimate (2.21) implies

$$I(x) = e^{iS(x)/\delta}\Big[\overline{\sigma}(0,0) + R_1(\kappa\overline{\sigma},\delta) + O(\delta\|\overline{\sigma}\|_{W^{n+1,1}})\Big]. \qquad (2.22)$$

It remains to bound $|R_1(\kappa\overline{\sigma},\delta)|$ and $\|\overline{\sigma}\|_{W^{n+1,1}}$. From (2.14)

$$\begin{aligned}|R_1(\kappa\overline{\sigma},\delta)| &\le C_Q\delta^n \sum_{|\alpha+\beta|\le 2n+2} \|\partial_z^\alpha\partial_\eta^\beta\big(\sum_{j=1}^n \partial_{z_j}\partial_{\eta_j}\kappa\overline{\sigma}\big)\|_{L^1(\mathbb{R}^{2n})}\\ &\le \sum_{|\alpha+\beta|\le 2n+4} \|\partial_z^\alpha\partial_\eta^\beta\kappa\overline{\sigma}\|_{L^1(\mathbb{R}^{2n})}.\end{aligned}$$

Then because we may choose the M and J from our definition of $\overline{\sigma}$ we have

$$\begin{aligned}|R_1(\kappa\overline{\sigma},\delta)| \le C &\sup_{\substack{|\alpha+\beta|\le 2n+4+2J\\ x\in\mathbb{T}^n,(z,\eta)\in\mathbb{R}^{2n}}} \big|\partial_z^\alpha\partial_\eta^\beta\tilde{\sigma}(x,\eta+\rho(z)+\nabla S(x))\big| \\ &\cdot \sum_{|\alpha|\le 2n+4+2J} \sup_{x\in\mathbb{T}^n}\big|\partial_z^\alpha f_0(x+z)\big|,\end{aligned} \tag{2.23}$$

and

$$\begin{aligned}\|\overline{\sigma}\|_{W^{n+1,1}} \le C &\sup_{\substack{|\alpha+\beta|\le n+1+2J\\ x\in\mathbb{T}^n,(z,\eta)\in\mathbb{R}^{2n}}} \big|\partial_z^\alpha\partial_\eta^\beta\tilde{\sigma}(x,\eta+\rho(z)+\nabla S(x))\big| \\ &\cdot \sum_{|\alpha|\le n+1+2J} \sup_{x\in\mathbb{T}^n}\big|\partial_z^\alpha f_0(x+z)\big|\end{aligned} \tag{2.24}$$

Here we use that since the original symbol $\sigma \in S^0_{0,0}(\mathbb{T}^n)$, so is $\tilde{\sigma} \in S^0_{0,0}(\mathbb{T}^n)$. Also, $\nabla S(x) \in C^\infty(\mathbb{T}^n)$. In order to estimate the constant in (2.23) we use that all the derivatives $\partial_z^\alpha\rho(z)$ (defined in (2.19))are bounded:

$$\sup_{z\in\mathbb{R}^n}\big|\partial_z^\alpha\rho(z)\big| \le C_\alpha,$$

where again we use $\nabla S(x) \in C^\infty(\mathbb{T}^n)$. From (2.22), (2.23) and (2.24) we have

$$|I(x)-\overline{\sigma}(0,0)| \le \delta C(n,\sigma,S) \sum_{|\alpha|\le 2n+2[n/2]+6} \sup_{z\in\mathbb{T}^n}\big|\partial_z^\alpha f_0(z)\big|, \tag{2.25}$$

and

$$\overline{\sigma}(0,0) = \sigma(x,\nabla S(x))f_0(x) + O\Big(\delta^{2J}\sum_{|\alpha|\le 2J}\sup_{z\in\mathbb{T}^n}\big|\partial_z^\alpha f_0(z)\big|\Big). \tag{2.26}$$

Thus (2.25) and (2.26) prove the lemma. □

Lemma 2.4.5 (Vishik)**.** *Let* $f_0 \in C^\infty(\mathbb{T}^n)$ *and let* $S \in C^\infty(\mathbb{R}^n)$ *satisfy (2.15) such that for some* $c_S > 0$,

$$|\nabla S(x)| \geq c_S \qquad \textit{for all } x \in supp(f_0).$$

Suppose $\delta^{-1} \in \mathbb{Z}_+$ *and let* $\sigma \in S^0_{0,0}(\mathbb{T}^n)$ *and* $\kappa \in C^\infty(\mathbb{T}^n \times \mathbb{R}^n)$ *such that* $\kappa = 0$ *for* $|\xi| > c_\kappa$*. Then for any* $0 < c_0 < \frac{c_S}{c_\kappa}$*, there exists a constant* $C = C(n, \sigma, S) > 0$ *such that*

$$\left|\left(op\left[\kappa(x, \frac{\delta}{c_0}\xi)\sigma(x,\xi)\right]f_0 e^{iS/\delta}\right)(x)\right| \leq C\delta \sum_{|\alpha|\leq 1} \sup_{x\in\mathbb{T}^n} \left|\partial_x^\alpha f_0(x)\right|.$$

Proof. Let

$$\begin{aligned} I(x) :=& \left(\mathrm{op}[\kappa(x, \frac{\delta}{c_0}\xi)\sigma(x,\xi)]f_0 e^{iS/\delta}\right)(x) \\ =& \frac{1}{(2\pi)^n}\int \kappa(x, \frac{\delta}{c_0}\xi)\sigma(x,\xi)f_0(y)e^{i\xi(x-y)+iS(y)/\delta}dyd\xi \qquad (2.27) \\ =& \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathrm{supp}(f_0)} \kappa(x, \frac{\delta}{c_0}\xi)\sigma(x,\xi)f_0(y)e^{i\xi(x-y)+iS(y)/\delta}dyd\xi \qquad (2.28) \end{aligned}$$

Notice that

$$\partial_{y_j} e^{i\xi(x-y)+iS(y)/\delta} = -\left(i\xi_j - \frac{i}{\delta}\partial_{y_j}S(y)\right)e^{i\xi(x-y)+iS(y)/\delta},$$

and

$$\sum_{j=1}^n \left(i\xi_j + \frac{i}{\delta}\partial_{y_j}S(y)\right)\left(i\xi_j - \frac{i}{\delta}\partial_{y_j}S(y)\right) = -\left|\xi - \frac{1}{\delta}\nabla S(y)\right|^2.$$

Then it follows that for the operator L_y defined by

$$L_y := \sum_{j=1}^n \frac{(i\xi_j - \frac{i}{\delta}\partial_{y_j}S(y))}{|\xi - 1/\delta\nabla S(y)|^2},$$

we have

$$e^{i\xi(x-y)+iS(y)/\delta} = L_y e^{i\xi(x-y)+iS(y)/\delta}.$$

Observe that the coefficients of L_y are small where $\kappa(x, \frac{\delta}{c_0}\xi) \neq 0$. For any $1 < j < n$, $y \in \text{supp}(f_0)$ and ξ such that $|\xi| \leq \frac{c_0 c_\kappa}{\delta}$, we have

$$\frac{i\xi_j - \frac{i}{\delta}\partial_{y_j} S(y)}{|\xi - 1/\delta \nabla S(y)|^2} \leq \frac{1}{|\xi - 1/\delta \nabla S(y)|} \leq \frac{1}{c_S/\delta - c_\kappa c_0/\delta} = \frac{\delta}{c_S - c_\kappa c_0}.$$

Now from (2.27) we have for any $M \in \mathbb{Z}_+$,

$$I(x) := \frac{1}{(2\pi)^n} \int \frac{1 + (-\Delta_\xi)^M}{1 + |x - y|^{2M}} \kappa(x, \frac{\delta}{c_0}\xi) \sigma(x, \xi) f_0(y) L_y \{e^{i\xi(x-y) + iS(y)/\delta}\} dy d\xi.$$

Integrate by parts using L_y and take $M = [n/2] + 1$ to get

$$|I(x)| \leq \frac{\delta C}{c_S - c_\kappa c_0} \Big(\sum_{|\beta| \leq 2M} \sup_{\substack{x \in \mathbb{T}^n \\ \xi \in \mathbb{R}^n}} \partial_\xi^\beta \big(\kappa(x, \frac{\delta}{c_0}\xi) \sigma(x, \xi)\big)\Big) \Big(\sum_{|\alpha| \leq 1} \sup_{y \in \mathbb{T}^n} |\partial_y^\alpha f_0(y)| \Big).$$

$\square$

Chapter 3

Lower bounds for growth of perturbations in 3-Dimensions

This chapter contains the main result for 3-dimensional fluid flows. The proof requires that we approximate our evolution operator on high frequencies by a psuedodifferential operator composed with parallel transport: $\mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}_u(t)$. We then examine the evolution of carefully constructed high frequency perturbations under the action of $\mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}_u(t)$ to compute a lower bound for the growth of perturbations in each class.

3.1 Classifying 3-dimensional fast oscillating vector fields

This section contains several lemmas regarding fast oscillating vector fields to be used in computing the lower bound. The goal is to establish criteria for these perturbations so they approximate perturbations in $\overline{\mathrm{Im}B}$ or that we may estimate their growth in the factor space $F := L^2_{sol}/\overline{\mathrm{Im}B}$.

Lemma 3.1.1. *Let v be a vector field in $H^1(\mathbb{T}^n)$ for $n = 2, 3$. Then*

$$\|\pi_{sol}(v(x)e^{ix\cdot\xi_0/\delta}) - \pi_{\xi_0^\perp}(v(x))e^{ix\cdot\xi_0/\delta}\|_{L^2} \leq \delta\frac{C}{|\xi_0|}\|v\|_{H^1},$$

where π_{sol} denotes the orthogonal projection of L^2 onto L^2_{sol}.

Proof. First we will prove the lemma for a 3-dimensional vector field, then consider the special case of planar vector fields to take care of the 2-dimensional case.

Assume $v \in \big(H^1(\mathbb{T}^3)\big)^3$. Define a vector field, $\alpha \in L^2_{sol}(\mathbb{T}^3)$, that approximates the projection of $v(x)e^{ix\cdot\xi_0/\delta}$ onto $\xi_0^\perp$:

$$\begin{aligned}\alpha(x) :=& \delta\nabla\times\Big(\frac{i\xi_0\times v(x)}{|\xi_0|^2}e^{ix\cdot\xi_0/\delta}\Big)\\ =& \delta\nabla\times\Big(\frac{i\xi_0\times\pi_{\xi_o^\perp}(v(x))}{|\xi_0|^2}e^{ix\cdot\xi_0/\delta}\Big)\\ =& \delta\Big[-\frac{i\xi_0\times\pi_{\xi_o^\perp}(v(x))}{|\xi_0|^2}\times\nabla(e^{ix\cdot\xi_0/\delta})+\Big(\nabla\times\frac{i\xi_0\times\pi_{\xi_o^\perp}(v(x))}{|\xi_0|^2}\Big)e^{ix\cdot\xi_0/\delta}\Big].\end{aligned}$$

Since $(\xi_0\times\pi_{\xi_o^\perp}(v))\times\xi_0 = \pi_{\xi_o^\perp}(v)$ we have

$$\alpha(x) = \pi_{\xi_o^\perp}(v(x))e^{ix\cdot\xi_0/\delta} + \delta\Big[\Big(\nabla\times\frac{i\xi_0\times\pi_{\xi_o^\perp}(v(x))}{|\xi_0|^2}\Big)e^{ix\cdot\xi_0/\delta}\Big].$$

It follows that $\|\alpha - \pi_{\xi_o^\perp}(v)e^{i(\cdot)\cdot\xi_0/\delta}\|_{L^2} \leq \delta\frac{1}{|\xi_0|}\|v\|_{H^1}$.

Now we define an gradient vector field, $\beta \in L^2_{grad}(\mathbb{T}^3)$, that approximates the projection of $v(x)e^{ix\cdot\xi_0/\delta}$ in the direction of ξ_0:

$$\begin{aligned}\beta(x) :=& -\frac{i\delta}{|\xi_0|^2}\nabla((\xi_0,v(x))e^{ix\cdot\xi_0/\delta})\\ =& -\frac{i\delta}{|\xi_0|^2}\big[(\xi_0,v(x))\frac{i\xi_0}{\delta}e^{ix\cdot\xi_0/\delta}+\nabla(\xi_0,v(x))e^{ix\cdot\xi_0/\delta}\big]\\ =& \pi_{\xi_0}(v(x))e^{ix\cdot\xi_0/\delta}-\frac{i\delta}{|\xi_0|^2}\nabla(\xi_0,v(x))e^{ix\cdot\xi_0/\delta}.\end{aligned}$$

Thus, $\|\beta - \pi_{\xi_0}(v)e^{i(\cdot)\cdot\xi_0/\delta}\|_{L^2} \leq \delta\frac{1}{|\xi_0|}\|v\|_{H^1}$. From the Hodge decomposition (see Remark 1) we know $L^2(\mathbb{T}^3) = L^2_{sol}(\mathbb{T}^3)\oplus L^2_{grad}(\mathbb{T}^3)$ and, from the computations above, $\pi_{\xi_o^\perp}(v)e^{i(\cdot)\cdot\xi_0/\delta}$ is approximately soleniodal and $\pi_{\xi_0}(v)e^{i(\cdot)\cdot\xi_0/\delta}$ is approximately a gradient, it follows that

$$\|\pi_{sol}(v(x)e^{ix\cdot\xi_0/\delta}) - \pi_{\xi_0^\perp}(v(x))e^{ix\cdot\xi_0/\delta}\|_{L^2(\mathbb{T}^3)} \leq \delta\frac{C}{|\xi_0|}\|v\|_{H^1(\mathbb{T}^3)}. \tag{3.1}$$

Next we consider the 2-dimensional case. Let $v \in \left(H^1(\mathbb{T}^2)\right)^2$ and assume $\xi_0 \in \mathbb{Z}^2$. Then $v(x)e^{ix\cdot\xi_0/\delta}$ can be viewed as a planar vector field on $\mathbb{T}^3$ and the estimate (3.1) above applies. Note that $\pi_{sol}(v(x)e^{ix\cdot\xi_0/\delta})$ and $\pi_{\xi_0^\perp}(v(x))e^{ix\cdot\xi_0/\delta}$ in 3-dimensions have 0 third component. Hence they are still planar in the same sense that $v(x)e^{ix\cdot\xi_0/\delta}$ is planar. It follows that as 2-dimensional vector fields on $\mathbb{T}^2$,

$$\|\pi_{sol}(v(x)e^{ix\cdot\xi_0/\delta}) - \pi_{\xi_0^\perp}(v(x))e^{ix\cdot\xi_0/\delta}\|_{L^2(\mathbb{T}^2)} \leq \delta\frac{C}{|\xi_0|}\|v\|_{H^1(\mathbb{T}^2)}.$$

□

Recall from Section 2.3, $B : L^2_{sol} \to L^2_{sol}$ is defined by

$$Bv := \omega \times v - \nabla\alpha,$$

where $u \in C^\infty_{sol}(\mathbb{T}^n)$ is a stationary solution to Euler's equation, $\omega := \mathrm{curl} u$ is the vorticity and the pressure gradient $\nabla\alpha$ is uniquely determined by the requirement that Bv is divergence free. Hence, from Remark 1 in Section 2.1, we have an equivalent formulation for B:

$$Bv = \pi_{sol}(\omega \times v),$$

where π_{sol} is the projection onto L^2_{sol}.

Here we define the basic structure of our fast oscillating vector fields. In Chapter 4 we will discuss the special case of 2-dimensional fast oscillating vector fields, but for now we are working in 3-dimensions. Define $\psi_\delta \in \left(L^2_{sol}(\mathbb{T}^3)\right)^3$ by

$$\psi_\delta(x) = \delta\nabla \times \left(\frac{i\xi_0 \times P}{|\xi_0|^2} h_0(x) e^{ix\cdot\xi_0/\delta}\right), \tag{3.2}$$

where $\xi_0 \in \mathbb{Z}^3, \delta^{-1} \in \mathbb{Z}_+, P \perp \xi_0$ is a constant vector and $h_0 \in C^\infty(\mathbb{T}^3)$ is an arbitrary smooth scalar function. Notice that we can expand the expression for ψ_δ to get

$$\psi_\delta(x) = h_0(x)Pe^{ix\cdot\xi_0/\delta} + \delta\big[\nabla h_0(x)\times\Big(\frac{i\xi_0 \times P}{|\xi_0|^2}\Big)e^{ix\cdot\xi_0/\delta}\big]. \tag{3.3}$$

The advantage of looking at vector fields such as ψ_δ is that we can estimate $G_\varepsilon(t)\psi_\delta$ explicitly, which we will see in this next lemma. We omit the proof, which can be found in Vishik's paper, [16]. A similar statement is made for slightly less general fast oscillating vector fields in Section 4.1 and the proof given in Section 4.1 uses the same techniques as Vishik's original proof.

Lemma 3.1.2. *Let ψ_δ be defined as in line (3.2) above. Then for any fixed $t > 0$, we have the following approximation for $G_\varepsilon(t)\psi_\delta(x) := (op_\varepsilon[a_0]\circ\mathfrak{g}_u(t)\psi_\delta)(x)$:*

$$(op_\varepsilon[a_0] \circ \mathfrak{g}_u(t)\psi_\delta)(x) = h_0(g^{-t}x)A_0(x, (g_*^{-t}(x))^*\xi_0, t)Pe^{ig^{-t}x\cdot\xi_0/\delta} + r_\delta(x),$$

where A_0 is the homogeneous part of a_0 defined by (2.7) and $\|r_\delta\|_{L^2} = O(\delta)$.

Remark 7. *From equation (2.9) we have*

$$h_0(g^{-t}x)A_0(x, (g_*^{-t}(x))^*\xi_0, t)Pe^{ig^{-t}x\cdot\xi_0/\delta} = h_0(g^{-t}x)b(g^{-t}x, \xi_0, P; t)e^{ig^{-t}x\cdot\xi_0/\delta},$$

where $b(g^{-t}x, \xi_0, P; t)$ *is the solution to our (BAS) with initial conditions* $(g^{-t}x, \xi_0, P)$.

This next lemma gives criteria for these fast oscillating vector fields to be close to $\overline{\mathrm{Im} B}$ in 3-dimensions. The criteria requires that we introduce a parameter ζ that localizes the support of the fast oscillating vector field.

Lemma 3.1.3. *Let* $x_0 \in \mathbb{T}^3$, $\xi_0 \in \mathbb{Z}^3$ *such that* $(\omega(x_0), \xi_0) \neq 0$. *Let* $h_0 \in C^\infty(\mathbb{T}^3)$ *such that* $\mathrm{supp} h_0 \subset B_1(0)$, *the ball centered at* 0 *of radius* 1. *Let* $0 < \zeta < 1$ *and define* h_ζ *by*

$$h_\zeta := h_0\Big(\frac{x - x_0}{\zeta}\Big).$$

And let

$$\psi_{\zeta,\delta}(x) := \delta \nabla \times \left(\frac{i\xi_0 \times P}{|\xi_0|^2} h_\zeta(x) e^{ix\cdot\xi_0/\delta} \right),$$

where $P \perp \xi_0$ *is a constant vector and* $\delta^{-1} \in \mathbb{Z}_+$. *Then there exists* $\overline{\psi}_{\zeta,\delta} \in L^2_{sol}$ *such that*

$$\psi_{\zeta,\delta} - B(\overline{\psi}_{\zeta,\delta}) = r_\zeta + r_\delta,$$

where $\|r_\zeta\|_{L^2} \leq c_0 \zeta^{5/2}$ *for some constant* $c_0 > 0$ *that does not depend on* δ *and* $\|r_\delta\|_{L^2} = O(\delta)$.

Proof. First we find an appropriate constant vector $Q \perp \xi_0$ to play the role of P in our preimage $\overline{\psi}_{\zeta,\delta}$. Let $T : \pi_{\xi_o^\perp}(\mathbb{R}^3) \to \pi_{\xi_o^\perp}(\mathbb{R}^3)$ be defined by

$$Tv := \pi_{\xi_o^\perp}(\omega(x_0) \times v).$$

Our assumption that $(\omega(x_0), \xi_0) \neq 0$ implies that T is a bijection on $\pi_{\xi_o^\perp}(\mathbb{R}^3)$. To see this suppose $v \in \pi_{\xi_o^\perp}(\mathbb{R}^3)$ such that $v \neq 0$ and $Tv = 0$. Then, since

$\omega(x_0) \not\perp \xi_0$, it is impossible for $\omega(x_0)$ and v to be parallel. Hence, the nonzero vector $(\omega(x_0) \times v)$ must be parallel to ξ_0. This implies that $\omega(x_0) \perp \xi_0$, which contradicts our assuption. Thus, T is a bijection and there is a constant vector $Q \perp \xi_0$ such that $P = \pi_{\xi_o^\perp}(\omega(x_0) \times Q)$.

Define the vector field $\overline{\psi}_{\zeta,\delta} \in C^\infty_{sol}(\mathbb{T}^3)$ by

$$\overline{\psi}_{\zeta,\delta}(x) := \delta \nabla \times \left(\frac{i\xi_0 \times Q}{|\xi_0|^2} h_\zeta(x) e^{ix\cdot\xi_0/\delta} \right).$$

Then from the expansion (3.3) and the linearity of B we have

$$B(\overline{\psi}_{\zeta,\delta}) = B(h_\zeta Q e^{i(\cdot)\cdot\xi_0}) + \delta B\big[\nabla h_\zeta \times \Big(\frac{i\xi_0 \times Q}{|\xi_0|^2}\Big) e^{ix\cdot\xi_0/\delta}\big].$$

We may also expand $\psi_{\zeta,\delta}$ as in (3.3) to get

$$\psi_{\zeta,\delta} - B(\overline{\psi}_{\zeta,\delta}) = h_\zeta P e^{i(\cdot)\cdot\xi_0/\delta} - B(h_\zeta Q e^{i(\cdot)\cdot\xi_0}) + \delta R_1, \tag{3.4}$$

where

$$R_1 = \big[\nabla h_\zeta \times \Big(\frac{i\xi_0 \times P}{|\xi_0|^2}\Big)\big] - B\big[\nabla h_\zeta \times \Big(\frac{i\xi_0 \times Q}{|\xi_0|^2}\Big) e^{ix\cdot\xi_0/\delta}\big]. \tag{3.5}$$

Hence, $\|R_1\|_{L^2} \leq \Big(\frac{|P|}{|\xi_0|}\|h_\zeta\|_{H^1} + \frac{|Q|}{|\xi_0|}\|B\|_{\mathcal{L}(L^2)}\|h_\zeta\|_{H^1}\Big)$. Notice that $\|B\|_{\mathcal{L}(L^2)} \leq \|\omega\|_{L^\infty}$, so

$$\|R_1\|_{L^2} \leq \Big(\frac{|P|}{|\xi_0|} + \frac{|Q|}{|\xi_0|}\|\omega\|_{L^\infty}\Big)\|h_\zeta\|_{H^1}. \tag{3.6}$$

To get a bound on the main term of the RHS of (3.4) we first use Lemma 3.1.1 to compute

$$\begin{aligned} B(h_\zeta Q e^{i(\cdot)\cdot\xi_0/\delta}) &:= \pi_{sol}(\omega \times h_\zeta Q e^{i(\cdot)\cdot\xi_0/\delta}) \\ &= h_\zeta \pi_{\xi_o^\perp}(\omega \times Q) e^{i(\cdot)\cdot\xi_0/\delta} + R_\delta, \end{aligned}$$

where

$$\|R_\delta\|_{L^2} \le \delta \frac{C}{|\xi_0|} \|h_\zeta \omega\|_{H^1}. \tag{3.7}$$

Define

$$r_\zeta := h_\zeta P e^{i(\cdot)\cdot\xi_0/\delta} - h_\zeta \pi_{\xi_o^\perp}(\omega \times Q) e^{i(\cdot)\cdot\xi_0/\delta}.$$

Then we may write the main term from the RHS of (3.4) as

$$h_\zeta P e^{i(\cdot)\cdot\xi_0/\delta} - B(h_\zeta Q e^{i(\cdot)\cdot\xi_0/\delta}) = r_\zeta + R_\delta. \tag{3.8}$$

We will demonstrate that $\|r_\zeta\|_{L^2} \le c_0 \zeta^{5/2}$ where the constant c_0 is positive and does not depend on δ. Since $P = \pi_{\xi_o^\perp}(\omega(x_0) \times Q)$ and supph_ζ is contained in the ball of radius ζ centered at x_0, $B_\zeta(x_0)$, we have

$$\begin{aligned} \|r_\zeta\|_{L^2} &= \|h_\zeta P e^{i(\cdot)\cdot\xi_0/\delta} - h_\zeta \pi_{\xi_o^\perp}(\omega \times Q) e^{i(\cdot)\cdot\xi_0/\delta}\|_{L^2} \\ &\le \|h_\zeta\|_{L^2} \|\pi_{\xi_o^\perp}(\omega(x_0) \times Q) - \pi_{\xi_o^\perp}(\omega \times Q)\|_{L^\infty(B_\zeta(x_0))} \\ &\le \|h_\zeta\|_{L^2} \|\pi_{\xi_o^\perp}\big(\omega(x_0) - \omega(\cdot)) \times Q)\big)\|_{L^\infty(B_\zeta(x_0))}. \end{aligned} \tag{3.9}$$

Since $\omega(x)$ is Lipschitz and for any $x \in \text{supp}h_\zeta$, $|x - x_0| \le \zeta$, it follows that

$$|\omega(x_0) - \omega(x)| \le \zeta \|\omega\|_{Lip} \text{ for any } x \in \text{supp}h_\zeta.$$

This implies

$$\|\pi_{\xi_o^\perp}\big(\omega(x_0) - \omega(\cdot)) \times Q)\big)\|_{L^\infty(B_\zeta(x_0))} \le \zeta \|\omega\|_{Lip} |Q|.$$

And since $\|h_\zeta\|_{L^2} = \zeta^{3/2} \|h_0\|_{L^2}$, we have from estimate (3.9) that

$$\|r_\zeta\|_{L^2} \le \zeta^{5/2} \|h_0\|_{L^2} \|\omega\|_{Lip} |Q|.$$

Let $c_0 = \|h_0\|_{L^2}\|\omega\|_{Lip}|Q|$, which is independent of δ. Now define $r_\delta := \delta R_1 + R_\delta$. Therefore, from (3.4) and (3.8) we have

$$\psi_{\zeta,\delta} - B(\overline{\psi}_{\zeta,\delta}) = r_\zeta + r_\delta.$$

From (3.6) and (3.7) we have $\|r_\delta\|_{L^2} = O(\delta)$. ☐

Remark 8. *For fast oscillating vector fields like ψ_δ in 2-dimensions, $(\omega, \xi_0) \equiv 0$, so this lemma does not give us any information about $\overline{ImB}$ in 2-dimensions.*

3.2 Main theorem for 3-dimensional flows

In this section we prove the main theorem concerning 3-dimensional flows, Theorem 3.2.1. Through Corollaries 3.2.3 and 3.2.4 we relate the essential spectral radius of the linear evolution to the essential spectral radius of the linear evolution restricted to each class of perturbations.

Theorem 3.2.1. *Let $u \in C^\infty_{sol}(\mathbb{T}^3)$ be a solution to steady Euler's equation (SE) in 3-dimensions with vorticity, $\omega := curl u$ and let $G(t)$ denote the solution operator to Euler's equation linearized about u. Define*

$$\mathcal{A} := \{(x_0, \xi_0, b_0) \in \mathbb{T}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 | \ \xi_0 \perp b_0, \ |\xi_0| = |b_0| = 1\}.$$

Then the following statements hold:

(i) Let $\mu_{3I} \in \mathbb{R}$ be defined by

$$\mu_{3I} = \lim_{t\to\infty} \frac{1}{t} \log \sup_{\substack{(x_0.\xi_0,b_0)\in\mathcal{A} \\ x_0 \in supp(\omega)}} |b(x_0, \xi_0, b_0; t)|,$$

where $b(x_0, \xi_0, b_0; t)$ *denotes a solution to (BAS) with initial conditions* (x_0, ξ_0, b_0). *Then we have the following lower bound for the essential spectral radius of our evolution operator restricted to* $\overline{ImB}$*:*

$$e^{\mu_{3I} t} \leq r_{ess}(G(t)|_{\overline{ImB}}).$$

(ii) If $supp(\omega)$ *is a proper subset of the fluid domain* $\mathbb{T}^3$*, let* $\mu_{3F} \in \mathbb{R}$ *be defined by*

$$\mu_{3F} = \lim_{t\to\infty} \frac{1}{t} \log \sup_{\substack{(x_0,\xi_0,b_0)\in\mathcal{A} \\ x_0 \notin supp(\omega)}} |b(x_0, \xi_0, b_0; t)|,$$

where $b(x_0, \xi_0, b_0; t)$ *denotes a solution to (BAS) with initial conditions* (x_0, ξ_0, b_0). *Then we have another lower bound for the essential spectral radius of the evolution operator acting on the factor space,* $F := L^2_{sol}/\overline{ImB}$*:*

$$e^{\mu_{3F} t} \leq r_{ess}(G_F(t))$$

where $G_F(t)$ *denotes* $G(t)$ *on the factor space and* $\| \cdot \|_F$ *denotes the canonical factor space norm.*

To prove Theorem 3.2.1 we first prove the following proposition:

Proposition 3.2.2. *Let* $u \in C^\infty(\mathbb{T}^3)$ *be a steady solution to Euler's equation with vorticity* $\omega := curlu$. *Define*

$$\mathcal{A} := \{(x_0, \xi_0, b_0) \in \mathbb{T}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 |\ \xi_0 \perp b_0,\ |\xi_0| = |b_0| = 1\}.$$

Fix $T > 0$.

(i) Let $\Theta_I(t)$ denote the following quantitity:

$$\Theta_I(t) = \sup_{\substack{(x_0,\xi_0,b_0)\in\mathcal{A} \\ x_0\in supp(\omega)}} |b(x_0, \xi_0, b_0; t)|,$$

where $b(x_0, \xi_0, b_0; t)$ is a solution to (BAS) corresponding to u. Then for any $\varepsilon > 0$ and $t \in [0, T]$

$$\|G^s_\varepsilon(t)\|_{\mathcal{L}(\overline{ImB},L^2)} + O(\sqrt{\varepsilon}) \geq \Theta_I(t),$$

where the constant in O is uniform for $t \in [0, T]$.

(ii) Whenever $supp(\omega)$ is a proper subset of the fluid domain, $\mathbb{T}^3$, define $\Theta_F(t)$ by

$$\Theta_F(t) = \sup_{\substack{(x_0,\xi_0,b_0)\in\mathcal{A} \\ x_0\notin supp(\omega)}} |b(x_0, \xi_0, b_0; t)|,$$

where $b(x_0, \xi_0, b_0; t)$ is a solution to (BAS) corresponding to u. Then for any $\varepsilon > 0$ and $t \in [0, T]$

$$\|G^s_\varepsilon(t)\|_{\mathcal{L}(F)} + O(\sqrt{\varepsilon}) \geq \Theta_F(t).$$

where the constant in O is uniform for $t \in [0, T]$.

Proof of Proposition 3.2.2. To prove this proposition we will choose appropriate sequences of fast oscillating vector fields (one that is almost in $\overline{\mathrm{Im}B}$ and one that is in Ker$B$) and show that the sizes of their images under $G^s_\varepsilon(t)$ approach $\Theta_I(t)$ and $\Theta_F(t)$, respectfully, from below.

The Image: Now we will consider a fast oscillating vector field that is almost in $\overline{\mathrm{Im}B}$: Choose $x_0 \in \mathbb{T}^3$ and $\xi_0 \in \mathbb{Z}^3$ such that $(\omega(x_0), \xi_0) \neq 0$ and

$h_0 \in C^\infty(\mathbb{T}^3)$ with supp$h_0 \subset B_0(1)$ and $h_0(0) = 1$. Let $0 < \zeta < 1$ and define $h_\zeta \in C^\infty(\mathbb{T}^3)$ by

$$h_\zeta := h_0\Big(\frac{x - x_0}{\zeta}\Big).$$

Then by Lemma 3.1.3 there exists $\overline{\psi}_{\zeta,\delta} \in L^2_{sol}$ such that

$$B(\overline{\psi}_{\zeta,\delta})(x) = \psi_{\zeta,\delta}(s) + r_\zeta + r_\delta, \tag{3.10}$$

where the $\|r_\zeta\|_{L^2} \leq c_0\zeta^{5/2}$ for c_0 independent of δ and $\|r_\delta\|_{L^2} = O(\delta)$. Then if we expand $\psi_{\zeta,\delta}$ as in line (3.3), we have

$$B(\overline{\psi}_{\zeta,\delta})(x) = h_\zeta(x)Pe^{ix\cdot\xi_0/\delta} + r_\zeta + \overline{r}_\delta, \tag{3.11}$$

where

$$\overline{r}_\delta = r_\delta + \delta\big[\nabla h_\zeta(x) \times \Big(\frac{i\xi_0 \times P}{|\xi_0|^2}\Big)e^{ix\cdot\xi_0/\delta}\big].$$

It follows that $\|\overline{r}_\delta\|_{L^2} = O(\delta)$. Apply Lemma 3.1.2 to the main order term in the expansion (3.10) for $B(\overline{\psi}_{\zeta,\delta})$ to estimate

$$\begin{aligned}(\mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}_u(t)B(\overline{\psi}_{\zeta,\delta}))(x)\\ = h_\zeta(g^{-t}x)b(g^{-t}x, \xi_0, P; t)e^{ig^{-t}x\cdot\xi_0/\delta} + \tilde{r}_\zeta(x) + \tilde{r}_\delta(x),\end{aligned}$$

where $\tilde{r}_\delta = \mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}_u(t)\overline{r}_\delta$ and $\tilde{r}_\zeta = \mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}_u(t)r_\zeta$. Hence $\|\tilde{r}_\delta\|_{L^2} = O(\delta)$ and $\|\tilde{r}_\zeta\|_{L^2} \leq \tilde{c}_0\zeta^{5/2}$ where $\tilde{c}_0 := c_0\|\mathrm{op}_\varepsilon[a_0]\|_{\mathcal{L}(L^2)}$ does not depend on δ. It follows that

$$\lim_{\delta\to 0}\|(\mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}_u(t)B(\overline{\psi}_{\zeta,\delta}))(x)\|_{L^2} = \|h_\zeta(g^{-t}x)b(g^{-t}x, \xi_0, P; t)\|_{L^2} + O(\zeta^{5/2}).$$

Then from line (3.11) we have $\|B(\overline{\psi}_{\zeta,\delta})\|_{L^2} = \|h_\zeta P\|_{L^2}$, thus we may estimate

$$\|\mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}_u(t)\|_{\mathcal{L}(\overline{\mathrm{Im}B},L^2)} + O(\zeta^{5/2})$$

$$\geq \sup_{\substack{x_0\in\mathbb{T}^3,\xi_0\in\mathbb{Z}^3 \\ (\omega(x_0),\xi_0)\neq 0 \\ P\perp\xi_0}} \frac{\|h_\zeta(g^{-t}x)b(g^{-t}x,\xi_0,P;t)\|_{L^2}}{\|h_\zeta P\|_{L^2}} \tag{3.12}$$

$$= \sup_{\substack{x_0\in\mathrm{supp}(\omega),\xi_0\in\mathbb{Z}^3 \\ P\perp\xi_0}} \frac{\|h_\zeta(g^{-t}x)b(g^{-t}x,\xi_0,P;t)\|_{L^2}}{\|h_\zeta P\|_{L^2}} \tag{3.13}$$

where the equality in the second line comes from taking the closure of the pairs $(x_0, \xi_0) \in \mathbb{T}^2 \times \mathbb{Z}^3$ such that $(\omega(x_0), \xi_0) \neq 0$. Next we take the limit as $\zeta \to 0$. The flow map g^t is measure preserving, so composition with it will not affect the norm in L^2. Also $h_\zeta(x_0) = 1$ and $b(\cdot, \cdot, P; t)$ depends linearly on P, so if we take the limit in ζ of the expression in (3.13) we have

$$\lim_{\zeta\to 0} \frac{\|h_\zeta(g^{-t}x)b(g^{-t}x,\xi_0,P;t)\|_{L^2}}{\|h_\zeta P\|_{L^2}} = |b(x_0,\xi_0,\frac{P}{|P|};t)|.$$

We can approximate any $\xi \in \mathbb{R}^3$ by $\xi_0 \in \mathbb{Z}^3$ and b is homogeneous of degree 0 in ξ_0, so it suffices to take the supremum in the RHS of (3.13) over $\xi_0 \in \mathbb{R}^3$ with $|\xi_0| = 1$. Hence

$$\|\mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}_u(t)\|_{\mathcal{L}(\overline{\mathrm{Im}B},L^2)} \geq \sup_{\substack{|\xi_0|=|b_0|=1 \\ b_0\perp\xi_0 \\ x_0\in\mathrm{supp}(\omega)}} |b(x_0,\xi_0,b_0;t)| = \Theta(t).$$

Therefore,

$$\|\mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}_u(t)\|_{\mathcal{L}(\overline{\mathrm{Im}B},L^2)} \geq \Theta_I(t).$$

The Factor Space: Recall, the factor space $F := L^2_{sol}/\overline{\mathrm{Im}B}$. Consider a vector field $\psi_\delta \in C^\infty_{sol}(\mathbb{T}^3)$, defined as in (3.2) with the extra condition that ensures it will be in Ker$B$:

$$\psi_\delta(x) = \delta\nabla \times \left(\frac{i\xi_0 \times P}{|\xi_0|^2} h_0(x) e^{ix\cdot\xi_0/\delta} \right), \tag{3.14}$$

where $\xi_0 \in \mathbb{Z}^3, \delta^{-1} \in \mathbb{Z}_+, P \perp \xi_0$ is a constant vector and $h_0 \in C^\infty(\mathbb{T}^3)$ is an arbitrary smooth scalar function with supph_0 disjoint from supp(ω). This implies that supp(ψ_δ) is disjoint from supp(ω). We may write the action of B as

$$B\psi_\delta := \pi_{sol}(\omega \times \psi_\delta).$$

It follows that $\psi_\delta \in$ Ker$B$. If we apply Lemma 3.1.2 to ψ_δ we have

$$\|\mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}_u(t)\psi_\delta - h_0(g^{-t}\cdot) b(g^{-t}\cdot, \xi_0, P; t) e^{i(\cdot)\cdot\xi_t/\delta}\|_{L^2} = O(\delta).$$

Which implies

$$\|\mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}_u(t)\psi_\delta\|_F = \|h_0(g^{-t}\cdot) b(g^{-t}\cdot, \xi_0, P; t) e^{i(\cdot)\cdot\xi_t/\delta}\|_F + O(\delta), \tag{3.15}$$

where $\|\cdot\|_F$ denotes the canonical factor space norm. The complement of supp(ω) is invariant under the flow g^t. Since supph_0 is disjoint from supp(ω), we have supp$(h \circ g^{-t})$ is also disjoint from supp(ω). Hence

$$h_0(g^{-t}x) b(g^{-t}x, \xi_0, P; t) e^{ig^{-t}x\cdot\xi_0/\delta} \in \mathrm{Ker}B.$$

It follows that

$$\frac{\|h_0(g^{-t}x) b(g^{-t}x, \xi_0, P; t) e^{ig^{-t}x\cdot\xi_0/\delta}\|_F}{\|\psi_\delta\|_F} = \frac{\|h_0(g^{-t}x) b(g^{-t}x, \xi_0, P; t) e^{ig^{-t}x\cdot\xi_0/\delta}\|_{L^2}}{\|\psi_\delta\|_{L^2}}.$$

Now consider equation (3.15) and take the limit as $\delta \to 0$ and we have

$$\|\mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}_u(t)\|_{\mathcal{L}(F)} \geq \sup_{\substack{h_0 \in C^\infty(\mathbb{T}^3), \xi_0 \in \mathbb{Z}^3\setminus\{0\} \\ \text{supp } h_0 \cap \text{supp}(\omega) = \emptyset \\ P \perp \xi_0}} \frac{\|h_0(g^{-t}x) b(g^{-t}x, \xi_0, P; t)\|_{L^2}}{\|h_0 P\|_{L^2}}. \tag{3.16}$$

We are taking a supremum over all $h_0 \in C^\infty(\mathbb{T}^3)$ with $\mathrm{supp} h_0$ disjoint from $\mathrm{supp}(\omega)$ and $\mathbb{T}^3 \setminus \mathrm{supp}(\omega)$ is invariant under the flow map, so we can restrict our consideration to $x_0 \notin \mathrm{supp}(\omega)$. The flow map g^{-t} is measure preserving, so that change of coordinates will not affect the L^2-norm. Also, since b is homogeneous of degree 0 in ξ_0 and linear in P we have

$$\|\mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}_u(t)\|_{\mathcal{L}(F)} \geq \sup_{\substack{|\xi_0|=|b_0|=1 \\ b_0 \perp \xi_0 \\ x_0 \notin \mathrm{supp}(\omega)}} |b(x_0, \xi_0, b_0; t)| = \Theta_F(t).$$

To finish the proof we must estimate the difference:

$$\|\big(G^s_\varepsilon(t) - \mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}_u(t)\big)\psi_\delta\|_{L^2}.$$

Recall that for any vector field $v \in L^2_{sol}$

$$(G^s_\varepsilon(t)v)(x) = \nabla_x \times \frac{\varepsilon}{(2\pi\varepsilon)^3} \int \frac{i\xi}{|\xi|^2} \times a_0(x, \xi, t)\mathfrak{g}_u(t)v(y)e^{i(x-y)\cdot\xi/\varepsilon} dy d\xi.$$

Notice that the matrix $a_0(x, \xi, t)$ maps into $\xi^\perp$ for all t. Since $i\xi \times (i\xi \times w) = w$ whenever $w \perp \xi$, this implies

$$\begin{aligned}
&\nabla_x(e^{ix\cdot\xi/\varepsilon}) \times \frac{\varepsilon}{(2\pi\varepsilon)^3} \int \frac{i\xi}{|\xi|^2} \times a_0(x, \xi, t)\mathfrak{g}_u(t)v(y)e^{-iy\cdot\xi/\varepsilon} dy d\xi \\
&\quad = \frac{e^{ix\cdot\xi/\varepsilon}}{(2\pi\varepsilon)^3} \int i\xi \times \Big(\frac{i\xi}{|\xi|^2} \times a_0(x, \xi, t)\mathfrak{g}_u(t)v(y)\Big)e^{-iy\cdot\xi/\varepsilon} dy d\xi \\
&\quad = \mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}_u(t)v.
\end{aligned}$$

Hence,

$$\begin{aligned}
G^s_\varepsilon(t) - \mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}_u(t) =& \varepsilon \mathrm{op}_\varepsilon\Big[\nabla_x \times \Big(\frac{i\xi}{|\xi|^2} \times a_0\Big)\Big] \circ \mathfrak{g}_u(t) \\
=& \mathrm{op}_1\Big[\nabla_x \times \Big(\frac{i\xi}{|\xi|^2} \times a_0(x, \varepsilon\xi, t)\Big)\Big] \circ \mathfrak{g}_u(t). \qquad (3.17)
\end{aligned}$$

Consider the symbol $D(x, \xi, t)$ defined by

$$D(x, \xi, t) = \nabla_x \times \Big(\frac{i\xi}{|\xi|^2} \times a_0(x, \varepsilon\xi, t) \Big).$$

For large $|\xi|$ we see that $D(x, \xi, t)$ has homogeneity of order -1. Since $a_0(x, \varepsilon\xi, t) = \big(1 - X(x, \sqrt{\varepsilon}\xi, t)\big) A_0(x, \xi, t)$, we see that there is some constant $c(T)$ that depends on T only such that for any $t \in [0, T]$, $D(x, \xi, t) = 0$ whenever $|\xi| < \frac{c(T)}{\sqrt{\varepsilon}}$. We also note that for any $\beta,\ \gamma \in \mathbb{Z}^3$, there exists a constant $C_{\beta,\gamma}(T)$ such that

$$|\partial_x^\beta \partial_\xi^\gamma D(x, \xi, t)| \leq C_{\beta,\gamma}(T)(1 + |\xi|)^{-1-|\gamma|} \quad \text{for any } t \in [0, T].$$

Now we may apply Lemma 2.4.2 to get

$$\|\mathrm{op}_1[D(x, \xi, t)]\|_{L^2} = O(\sqrt{\varepsilon}).$$

We remark that in the proof of Lemma 2.4.2 the constant in O depends only on the constants $C_{\beta,\gamma}(T)$ and $c(T)$, so $O(\sqrt{\varepsilon})$ is uniform for $t \in [0, T]$. Then from equation (3.17) we have

$$\|G_\varepsilon^s(t) - \mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}_u(t)\|_{\mathcal{L}(L^2)} = O(\sqrt{\varepsilon}), \tag{3.18}$$

and

$$\|G_\varepsilon^s(t) - \mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}_u(t)\|_{\mathcal{L}(F)} = O(\sqrt{\varepsilon}),$$

where in both estimates, the constants in O are uniform for $t \in [0, T]$. This completes the proof.

□

Now we prove the main theorem of this Chapter:

Proof of Theorem 3.2.1. Let $C \in \mathcal{L}(L^2)$ be an arbitrary operator of finite rank. Then we get the following inequality for any $\varepsilon > 0$.

$$\|G(t) + C\|_{\mathcal{L}(L^2)} \geq \|(G(t) + C) \circ \mathrm{op}_\varepsilon\big[1 - \chi\big(\frac{\xi}{\sqrt{\varepsilon}}\big)\big]\|_{\mathcal{L}(L^2)}. \tag{3.19}$$

Since C has finite rank, we may write

$$C = \sum_{j=1}^{M} g_j(f_j, \cdot),$$

for some $\{g_j\}_{j=1}^M, \{f_j\}_{j=1}^M \subset L^2$. Since $\mathrm{op}_\varepsilon\big[1 - \chi\big(\frac{\xi}{\sqrt{\varepsilon}}\big)\big]$ is self-adjoint, it follows that

$$\begin{aligned}\|C \circ \mathrm{op}_\varepsilon\big[1 - \chi\big(\frac{\xi}{\sqrt{\varepsilon}}\big)\big]\|_{\mathcal{L}(L^2)} =& \|\sum_{j=1}^{M} g_j(\mathrm{op}_\varepsilon\big[1 - \chi\big(\frac{\xi}{\sqrt{\varepsilon}}\big)\big] f_j, \cdot)\|_{\mathcal{L}(L^2)} \\ =& o(1) \text{ as } \varepsilon \to 0,\end{aligned}$$

since for each $j = 1, 2 \ldots M$,

$$\|\mathrm{op}_\varepsilon\big[1 - \chi\big(\frac{\xi}{\sqrt{\varepsilon}}\big)\big] f_j\|_{\mathcal{L}(L^2)} = o(1) \text{ as } \varepsilon \to 0.$$

This implies

$$\|C \circ \mathrm{op}_\varepsilon\big[1 - \chi\big(\frac{\xi}{\sqrt{\varepsilon}}\big)\big]\|_{\mathcal{L}(\overline{\mathrm{Im} B}, L^2)} = o(1) \text{ as } \varepsilon \to 0. \tag{3.20}$$

Let $N \in \mathbb{N}$ and Replace t with Nt in inequality (3.19). Then by equation (3.20) above

$$\|G(Nt) + C\|_{\mathcal{L}(\overline{\mathrm{Im} B}, L^2)} \geq \|G(Nt) \circ \mathrm{op}_\varepsilon\big[1 - \chi\big(\frac{\xi}{\sqrt{\varepsilon}}\big)\big]\|_{\mathcal{L}(\overline{\mathrm{Im} B}, L^2)} - o(1) \text{ as } \varepsilon \to 0.$$

From Theorem 2.2.2 we have

$$\|G(Nt) + C\|_{\mathcal{L}(\overline{\mathrm{Im} B}, L^2)} \geq \|G^s_\varepsilon(Nt)\|_{\mathcal{L}(\overline{\mathrm{Im} B}, L^2)} - O(\sqrt{\varepsilon}) - o(1) \text{ as } \varepsilon \to 0.$$

And Proposition 3.2.2 implies

$$\|G(Nt) + C\|_{\mathcal{L}(\overline{\mathrm{Im}B},L^2)} \geq \Theta_I(Nt) - O(\sqrt{\varepsilon}) - o(1) \text{ as } \varepsilon \to 0.$$

Letting $\varepsilon \to 0$,

$$\|G(Nt) + C\|_{\mathcal{L}(\overline{\mathrm{Im}B},L^2)} \geq \Theta_I(Nt). \tag{3.21}$$

Since C was arbitrary, we have

$$\|G(Nt) \mid_{\overline{ImB}} \|_{\mathcal{K}} \geq \Theta_I(Nt),$$

where $\| \cdot \|_{\mathcal{K}}$ denotes Nussbaum's seminorm, introduced in Section 2.2. Take the Nth root of both sides of the equation and use properties of the logarithm to get

$$\|G(Nt) \mid_{\overline{ImB}} \|_{\mathcal{K}}^{1/N} \geq e^{t \frac{1}{Nt} \log(\Theta_I(Nt))}.$$

If we take the limits as $N \to \infty$, for 3-dimensional flows we have

$$r_{ess}(G(t) \mid_{\overline{\mathrm{Im}\ B}}) \geq e^{\mu_{3I} t}.$$

Thus we have the lower bound for $\overline{\mathrm{Im}\ B}$.

To compute a lower bound for the factor space, we assume supp(ω) is a proper subset of the fluid domain, $\mathbb{T}^3$. In this case we may use Proposition 3.2.2.

We remark that for any $T \in \mathcal{L}(L^2_{sol})$ such that T leaves $\overline{\mathrm{Im}B}$ invariant, we may consider $T_F \in \mathcal{L}(F)$ where T_F denotes T acting on the factor space. For any $x \in L^2_{sol}$, we let $[x] \in F$ denote the equivalence class in $F := L^2_{sol}/\overline{\mathrm{Im}B}$

represented by x. Let π denote the orthogonal projection of L^2_{sol} onto Ker$B$. Since $L^2_{sol} = \overline{\text{Im}B} \oplus \text{Ker}B$, we have $[x] = [\pi x]$. It follows that

$$\|T_F\|_{\mathcal{L}(F)} := \sup_{\substack{[x]\in F\\ [x]\neq[0]}} \frac{\|T[x]\|_F}{\|[x]\|_F} = \sup_{\substack{x\in L^2_{sol}\\ \pi x\neq 0}} \frac{\|T_F[\pi x]\|_F}{\|[\pi x]\|_F}.$$

Because T leaves $\overline{\text{Im}B}$ invariant, we may say

$$\|T_F\|_{\mathcal{L}(F)} = \sup_{\substack{x\in L^2_{sol}\\ \pi x\neq 0}} \frac{\|\pi T \pi x\|_{L^2_{sol}}}{\|\pi x\|_{L^2_{sol}}} = \|\pi T \pi\|_{\mathcal{L}(L^2_{sol})}. \tag{3.22}$$

Whenever we have an operator $S \in \mathcal{L}(L^2_{sol})$ that does not leave $\overline{\text{Im}B}$ invariant, the notation $\|S\|_{\mathcal{L}(F)}$ denotes $\|\pi S \pi\|_{\mathcal{L}(L^2_{sol})}$.

We also remark that any operator $K \in \mathfrak{S}_\infty(F)$ can be lifted to an operator $\overline{K} \in \mathfrak{S}_\infty$ as follows: Let $\{\tilde{f}_j\}_{j=1}^\infty$ be a Schauder basis for Ker$B$. In the canonical sense, $\{[\tilde{f}_j]\}_{j=1}^\infty$ is also a Schauder basis for the factor space, F. We may write

$$K = \sum_{j=1}^{\infty} [\tilde{g}_j]([\tilde{f}_j], \cdot),$$

where $\tilde{g}_j \in \text{Ker}B$ for each $j = 1, 2....$ Then we define

$$\overline{K} := \sum_{j=1}^{\infty} \tilde{g}_j(\tilde{f}_j, \cdot). \tag{3.23}$$

Notice that $\overline{K}$ leaves $\overline{\text{Im}B}$ invariant and $\overline{K}_F = K$.

Let $\|\cdot\|_{\mathcal{K}(F)}$ be the Nussbaum seminorm on F. Then

$$\|T_F\|_{\mathcal{K}(F)} := \inf_{K\in\ \mathfrak{S}_\infty(F)} \|T_F + K\|_{\mathcal{L}(F)}.$$

So we have

$$
\begin{aligned}
\|G_F(t)\|_{\mathcal{K}(F)} &= \inf_{K \in \mathfrak{S}_\infty(F)} \|G_F(t) + K\|_{\mathcal{L}(F)} \\
&= \inf_{K \in \mathfrak{S}_\infty(F)} \|\pi\big(G(t) + \overline{K}\big)\pi\|_{\mathcal{L}(L^2_{sol})} \\
&\geq \inf_{C \in \mathfrak{S}_\infty(L^2_{sol})} \|\pi\big(G(t) + C\big)\pi\|_{\mathcal{L}(L^2_{sol})}, \qquad (3.24)
\end{aligned}
$$

where $\overline{K}$ is the lift of an operator $K \in \mathfrak{S}_\infty(F)$ in the sense of (3.23).

Again we begin with equation (3.19) in the factor space norm: For any finite rank $C \in \mathcal{L}(L^2_{sol})$ we have

$$
\|G_F(t) + C\|_{\mathcal{L}(F)} := \|\pi\big(G(t) + C\big)\pi\|_{\mathcal{L}(L^2_{sol})}.
$$

Hence,

$$
\|G_F(t) + C\|_{\mathcal{L}(F)} \geq \|(G(t) + C) \circ \mathrm{op}_\varepsilon\big[1 - \chi\big(\frac{\xi}{\sqrt{\varepsilon}}\big)\big]\|_{\mathcal{L}(F)}.
$$

From equation (3.20) we have

$$
\|C \circ \mathrm{op}_\varepsilon\big[1 - \chi\big(\frac{\xi}{\sqrt{\varepsilon}}\big)\big]\|_{\mathcal{L}(F)} = o(1) \text{ as } \varepsilon \to 0.
$$

And from Theorem 2.2.2 we have

$$
\|G(t) \circ \mathrm{op}_\varepsilon\Big[1 - \chi\big(\frac{\xi}{\sqrt{\varepsilon}}\big)\Big] - G^s_\varepsilon(t)\|_{\mathcal{L}(F)} = O(\sqrt{\varepsilon}). \qquad (3.25)
$$

If we apply Theorem 2.2.2 and Proposition 3.2.2, it follows that for any $C \in \mathcal{L}(L^2_{sol})$ of finite rank

$$
\begin{aligned}
\|G_F(Nt) + C\|_{\mathcal{L}(F)} &\geq \|G^s_\varepsilon(t)\|_{\mathcal{L}(F)} - O(\sqrt{\varepsilon}) - o(1) \\
&\geq \Theta_F(Nt) - O(\sqrt{\varepsilon}) - o(1).
\end{aligned}
$$

Now take the limit as $\varepsilon \to 0$ as above. Since C is an arbitrary finite rank operator, from (3.24) we have

$$\|G_F(Nt)\|_{\mathcal{K}(F)} \geq \Theta_F(Nt).$$

Take the Nth root of both sides of the equation, exponentiate the RHS as we did for the image case and then take the limit as $N \to \infty$. Thus for 3-dimensional flows where supp(ω) is a proper subset of $\mathbb{T}^3$ we have

$$r_{ess}(G_F(t)) \geq e^{\mu_{3F} t}.$$

$\square$

Remark 9. *The proof of Theorem 3.2.1 did not depend on our flow being 3-dimensional. In Section 4.2 we will introduce 2-dimensional propositions similar to Proposition 3.2.2 and reference the proof of 3.2.1 to prove a similar theorem for 2-dimensional flows, Theorem 4.2.3.*

We have the following corollaries to Theorem 3.2.1:

Corollary 3.2.3. *For a 3-dimensional flow with vorticity ω, if supp(ω) is a proper subset of $\mathbb{T}^3$, then*

$$r_{ess}(G(t)) = \max\{r_{ess}(G_F(t)), r_{ess}(G(t) \mid_{\overline{Im\ B}})\}.$$

Corollary 3.2.4. *If the support of ω is the entire fluid domain, T^3, then*

$$r_{ess}(G(t) \mid_{\overline{ImB}}) = r_{ess}(G(t)).$$

Before proving these corollaries, we need the following proposition:

Proposition 3.2.5. *For 2- or 3-dimensional flows and for any $t > 0$,*

$$r_{ess}(G(t)|_{\overline{ImB}}) \leq r_{ess}(G(t)),$$

and

$$r_{ess}(G|_F(t)) \leq r_{ess}(G(t)).$$

Proof. Clearly,

$$\|G(t)|_{\overline{\mathrm{Im}B}}\|_{\mathcal{K}} \leq \|G(t)\|_{\mathcal{K}}.$$

Then it follows that

$$\lim_{N\to\infty} \|G(Nt)|_{\overline{\mathrm{Im}B}}\|_{\mathcal{K}}^{1/N} \leq \lim_{N\to\infty} \|G(Nt)\|_{\mathcal{K}}^{1/N}.$$

Then by semigroup properties for $G(t)$ and $G(t)|_{\overline{\mathrm{Im}B}}$ and Nussbaum's Theorem 2.2.3, we have

$$r_{ess}(G(t)|_{\overline{\mathrm{Im}B}}) \leq r_{ess}(G(t)).$$

Now we prove the second statement. Recall from the proof of Theorem 3.2.1 that for any $T \in \mathfrak{L}(L^2_{sol})$ such that T leaves $\overline{\mathrm{Im}B}$ invariant, we may consider $T_F \in \mathfrak{L}(F)$ where T_F denotes T acting on the factor space and we have

$$\|T_F\|_{\mathfrak{L}(F)} = \|\pi T \pi\|_{\mathfrak{L}(L^2_{sol})}. \tag{3.26}$$

In the proof of Theorem 3.2.1 we also showed that any operator $K \in \mathfrak{S}_\infty(F)$ can be lifted to an operator $\overline{K} \in \mathfrak{S}_\infty$ such that $\overline{K}$ leaves $\overline{\mathrm{Im}B}$ invariant and $\overline{K}_F = K$.

Let $\|\cdot\|_{\mathcal{K}(F)}$ be the Nussbaum seminorm on F. Then

$$\|T_F\|_{\mathcal{K}(F)} := \inf_{K\in\ \mathfrak{S}_\infty(F)} \|T_F + K\|_{\mathfrak{L}(F)}.$$

Then from (3.26) we have

$$\inf_{K\in\ \mathfrak{S}_\infty(F)} \|T_F + K\|_{\mathcal{L}(F)} = \inf_{K\in\ \mathfrak{S}_\infty(F)} \|\pi(T+\overline{K})\pi\|_{\mathcal{L}(L^2_{sol})},$$

where $\overline{K} \in \mathfrak{S}_\infty$ is the lift of $K \in \mathfrak{S}_\infty$ defined by (3.23).

Notice that for $C \in \mathfrak{S}_\infty$, there is some $K_C \in \mathfrak{S}_\infty(F)$ such that $\overline{K_C} = \pi C\pi$, where $\overline{K_C}$ denotes the lift of K_C in the sense of (3.23). Since $\overline{K_C} = \pi\overline{K_C}\pi$, we have

$$\begin{aligned}
\inf_{K\in\ \mathfrak{S}_\infty(F)} \|\pi(T+\overline{K})\pi\|_{\mathcal{L}(L^2_{sol})} &\le \inf_{C\in\mathfrak{S}_\infty} \|\pi(T+\overline{K_C})\pi\|_{\mathcal{L}(L^2_{sol})} \\
&= \inf_{C\in\mathfrak{S}_\infty} \|\pi(T+C)\pi\|_{\mathcal{L}(L^2_{sol})} \\
&\le \inf_{C\in\mathfrak{S}_\infty} \|T+C\|_{\mathcal{L}(L^2_{sol})}.
\end{aligned}$$

Thus, for any $T \in \mathcal{L}(L^2_{sol})$ which leaves $\overline{\mathrm{Im} B}$ invariant,

$$\|T_F\|_{\mathcal{K}(F)} \le \|T\|_{\mathcal{K}(L^2_{sol})}.$$

Then we have

$$\|G_F(Nt)\|_{\mathcal{K}(F)} \le \|G(Nt)\|_{\mathcal{K}},$$

for any $N \in \mathbb{N}$. The mapping $T \mapsto T_F$ is a vector space homomorphism from $\{T \in \mathcal{L}(L^2_{sol}) |\ T \hookrightarrow \overline{\mathrm{Im} B}\}$ to $\mathcal{L}(F)$. Hence, $G_F(Nt) = \big(G(Nt)\big)_F$. Then we may repeat the computations above and apply Nussbaum's Theorem again to get

$$r_{ess}(G|_F(t)) \le r_{ess}(G(t)).$$

□

Proof of Corollary 3.2.3. From the definitions of $\Theta_I(t)$ and $\Theta_F(t)$ we have

$$\sup_{\substack{x_0, |b_0|=|\xi_0|=1 \\ \xi_0 \perp b_0}} |b(x_0, \xi_0, b_0; t)| = \max\{\Theta_I(t), \Theta_F(t)\}.$$

Hence,

$$\mu := \lim_{t\to\infty} \frac{1}{t} \log \sup_{\substack{x_0, |b_0|=|\xi_0|=1 \\ \xi_0 \perp b_0}} |b(x_0, \xi_0, b_0; t)| = \lim_{t\to\infty} \frac{1}{t} \log \max\{\Theta_I(t), \Theta_F(t)\},$$

where μ is the Lyapunov-type exponent defined in Theorem 2.2.1. Thus $\mu = \max\{\mu_{3I}, \mu_{3F}\}$. By Theorem 2.2.1 and Theorem 3.2.1 we have

$$r_{ess}(G(t)) = e^{\mu t} = \max\{e^{\mu_{3I}}, e^{\mu_{3F}}\} \leq \max\{r_{ess}(G_F(t)), r_{ess}(G(t) \mid_{\overline{\text{Im } B}})\}.$$

Then by Proposition 3.2.5

$$r_{ess}(G(t)) = \max\{r_{ess}(G_F(t)), r_{ess}(G(t) \mid_{\overline{\text{Im } B}})\}.$$

□

Proof of Corollary 3.2.4. If we assume $\text{supp}(\omega) = \mathbb{T}^3$, then $\mu = \mu_{3I}$, where μ is the Lyapunov-type exponent from Theorem 2.2.1. Then by Theorem 2.2.1 and Theorem 3.2.1 $r_{ess}(G(t)) = e^{\mu t} \leq r_{ess}(G(t)|_{\overline{\text{Im} B}})$. Hence, by Proposition 3.2.5 we have

$$r_{ess}(G(t)) = r_{ess}(G(t)|_{\overline{\text{Im} B}}).$$

□

3.3 3-dimensional hyperbolic stagnation point example

In this section we give a specific example of a flow with instability from the first class of perturbations. Our instability comes from a hyperbolic stagnation point, so we first prove some facts about hyperbolic stagnation points in fluid flows that are independent of the dimension of the flow.

Consider a 3-dimensional example of a steady flow with a hyperbolic stagnation point. In general, a point x_s is a hyperbolic stagnation point when the spectrum of the matrix $\frac{\partial u}{\partial x}(x_s)$ does not intersect the imaginary axis.

Proposition 3.3.1. *Let u be a steady solution to Euler's equation on $\mathbb{T}^n$, $n = 2, 3$, and x_s a hyperbolic stagnation point of the flow associated with u. Then $\frac{\partial u}{\partial x}(x_s)$ is a symmetric matrix with n real, non-zero eigenvalues corresponding to orthogonal eigenvectors.*

Proof. From Euler's equation, we have:

$$\frac{\partial u_i}{\partial x^j} u_j = -\partial_i p \quad 1 \leq i, j \leq n, \tag{3.27}$$

where p is the scalar pressure. Take the partial derivative with respect to the kth coordinate direction and consider 3.27 evaluated at the stagnation point:

$$\left(\frac{\partial u}{\partial x}\right)^2_{i,k}(x_s) = -\partial_k \partial_i p(x_s). \tag{3.28}$$

This implies $\left(\frac{\partial u}{\partial x}\right)^2(x_s)$ is a symmetric matrix, hence it has n real positive eigenvalues (by the hyperbolic assumption none are zero) and orthogonal eigenvectors. The eigenvalues of $\frac{\partial u}{\partial x}(x_s)$ are the square roots of the eigenvalues of $\left(\frac{\partial u}{\partial x}\right)^2(x_s)$ and they correspond to the same eigenvectors. This proves the proposition. ∎

Corollary 3.3.2. *Let u be a steady solution to Euler's equation on $\mathbb{T}^n$, $n = 2, 3$ and x_s a hyperbolic stagnation point of the flow associated with u. Then $\omega(x_s) = 0$.*

Proof. For any vector $v \in \mathbb{R}^n$, $\omega(x) \times v = \left(\frac{\partial u}{\partial x}(x) - \frac{\partial u}{\partial x}^T(x)\right)v$. Since $\frac{\partial u}{\partial x}(x_s)$ is symmetric, it follows that $\omega(x_s) \times v \equiv 0$ for all $v \in \mathbb{R}^n$. Thus, $\omega(x_s) = 0$. $\square$

Now consider the more specific example:

$$u_1 = \cos x_2 - \sin x_3, \qquad u_2 = \cos x_3 - \sin x_1, \qquad u_3 = \cos x_1 - \sin x_2,$$

at the stagnation point $x_s = (\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$.

$$\left(\frac{\partial u}{\partial x}\right)^T(x_s) = \frac{\partial u}{\partial x}(x_s) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The eigenvalues and associated eigenvectors of $\frac{\partial u}{\partial x}(x_s)$:

$$\begin{aligned} \lambda_1 = -2 \quad &\leftrightarrow \quad v_1 = (0, \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}), \\ \lambda_2 = 1 \quad &\leftrightarrow \quad v_2 = (\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}), \\ \lambda_3 = 1 \quad &\leftrightarrow \quad v_3 = (-\tfrac{2}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}). \end{aligned}$$

Consider the BAS system of ODEs:

$$\text{(BAS)} \begin{cases} \dot{x} = u(x), & x(0) = x_0; \\ \dot{\xi} = -\left(\frac{\partial u}{\partial x}\right)^T \xi, & \xi(0) = \xi_0; \\ \dot{b} = -\left(\frac{\partial u}{\partial x}\right)b + 2\left(\frac{\partial u}{\partial x}b, \xi\right)\frac{\xi}{|\xi|^2}, & b(0) = b_0. \end{cases}$$

At the stagnation point, we get the solution $\xi(t) = v_2 e^{-\lambda_2 t}$, $b(t) = v_1 e^{-\lambda_1 t}$. Since $-\lambda_1 = 2$, $|b(t)|$ grows exponentially. This instability can be caused by a vector field in $\overline{\text{Im}B}$. To see this, we will carefully construct a sequence of vector fields approaching $\overline{\text{Im}B}$ whose linear evolution approches this exponential growth.

We must use the continuous dependence of $b(t) := b(x_0, \xi_0, b_0; t)$ on x_0. We want to choose x_0 such that $(\omega(x_0), v_2) \neq 0$ and

$$|b(x_0, v_2, v_1; t) - b(t)| < \alpha, \tag{3.29}$$

for some $\alpha > 0$ and $b(x_s, v_2, v_1; t) = v_1 e^{-\lambda_1 t}$ Then let $h_0 \in C^\infty(\mathbb{T}^3)$ such that h_0 is supported in the ball of radius 1 centered at 0. Just as in Lemma 3.1.3 for any $0 < \zeta < 1$ define

$$h_\zeta(x) = h_0\big(\frac{x - x_0}{\zeta}\big).$$

Now define $\psi_{\zeta,\delta}$

$$\psi_{\zeta,\delta}(x) = \delta \nabla \times \big((iv_2 \times v_1) h_\zeta(x) e^{ix\cdot\xi/\delta}\big), \tag{3.30}$$

where $\delta^{-1} \in \mathbb{Z}_+$, v_1 and v_2 are defined above. From Lemma 3.1.3 we can adjust the parameters δ and ζ to get $\psi_{\zeta,\delta}$ as close to $\overline{\text{Im}B}$ as we like. From Lemma 3.1.2 we have

$$\lim_{\delta\to 0} \|(G_\varepsilon(t)\psi_{\zeta,\delta})(x)\|_{L^2} = \frac{\|h_\zeta(g^{-t}x) b(g^{-t}x, v_2, v_1; t)\|_{L^2}}{\|h_\zeta\|_{L^2}}.$$

Taking the limit in ζ gives,

$$\lim_{\zeta\to 0} \frac{\|h_\zeta(g^{-t}x) b(g^{-t}x, v_2, v_1; t)\|_{L^2}}{\|h_\zeta\|_{L^2}} = |b(x_0, v_2, v_1; t)|.$$

From (3.29) we have constructed a sequence of fast oscillating vector fields $\psi_{\zeta,\delta}$ whose image under $G(t)$ approaches exponential growth if we take the limit first in δ, then in ζ.

Chapter 4

Lower bounds for growth in 2-dimensions

This chapter parallels Chapter 3, only here we deal with 2-dimensional flows. The main differences in the results stem from the fact that the scalar vorticity of a 2-dimensional inviscid, incompressible fluid flow is constant along flow lines.

4.1 Classifying 2-dimesional fast oscillating vector fields

In this section we introduce our 2-dimensional high frequency perturbations and, in Lemma 4.1.3, estimate their linear evolution. We also establish when such a perturbation is approximately in the factor space and when one is approximately in $\overline{\mathrm{Im}B}$, our first class of perturbations. To begin, we more closely investigate the operator B for a 2-dimensional flow u acting on 2-dimensional perturbations.

The operator B takes on a simplified form in 2-dimensions. The vorticity, ω, of our steady flow u is ususally treated as a scalar function when u is 2-dimensional. However, when we define the operator B for 2-dimensional flows, we treat ω as a 3-dimensional vector field with first two components zero and third componenet the scalar vorticity. Thus

$$Bv := \omega \times v - \nabla\alpha,$$

can be simplified:

$$Bv = \omega \cdot v^{\perp} - \nabla\alpha = \pi_{sol}(\omega \cdot v^{\perp}), \tag{4.1}$$

where the pressure $\nabla\alpha \in (L^2(\mathbb{T}^2))^2$ is determined by the requirement that Bv be divergence free and π_{sol} is the orthogonal projection onto divergence free vector fields.

Let $\phi_\delta \in (L^2_{sol}(\mathbb{T}^2))^2$ be defined by,

$$\phi_\delta(x) := -i\delta\nabla^{\perp}(h_0(x)e^{ix\cdot\xi_0/\delta}), \tag{4.2}$$

where $\xi_0 \in \mathbb{Z}^2, \delta^{-1} \in \mathbb{Z}_+, P \perp \xi_0$ is constant and $h_0 \in C^\infty(\mathbb{T}^2)$ is an arbitrary smooth scalar function. We can expand ϕ_δ as follows:

$$\phi_\delta(x) = h_0(x)\xi_0^{\perp}e^{ix\cdot\xi_0/\delta} - i\delta\big[e^{ix\cdot\xi_0/\delta}\nabla^{\perp}h_0(x)\big]. \tag{4.3}$$

In this next Lemma we establish criteria for ϕ_δ to be near $\overline{\mathrm{Im}B}$. Our criteria is based on $\nabla\omega$, the gradient of the scalar vorticity of our steady solution $u \in C^\infty_{sol}$.

Lemma 4.1.1. *Define ϕ_δ as in (4.2) above. If there is a constant c_0 such that $|(\xi_0^{\perp}, \nabla\omega(x))| > c_0$ on supph_0, then there exists a remainder $r_\delta \in L^2$ such that $\phi_\delta + r_\delta \in \overline{ImB}$ and $\|r_\delta\|_{L^2} = O(\delta)$.*

Proof. Assume there exists a constant c_0 such that $|(\xi_0^{\perp}, \nabla\omega(x))| > c_0$ on supph_0. Then we can define a function $g_0 \in C^\infty(\mathbb{T}^2)$ by

$$g_0(x) := \frac{|\xi_0|^2 h_0(x)}{(\xi_0^{\perp}, \nabla\omega(x))}, \tag{4.4}$$

and define a vector field $v \in C^{\infty}_{sol}(\mathbb{T}^2)$ by,

$$v(x) := \nabla^{\perp}(g_0(x)e^{ix\cdot\xi_0/\delta}).$$

From (4.1) the operator B on $\nabla^{\perp}(g_0(x)e^{ix\cdot\xi_0/\delta})$ takes this simplified form:

$$\begin{aligned} Bv &= \pi_{sol}\big(\omega\nabla(g_0(x)e^{ix\cdot\xi_0/\delta})\big) \\ &= \pi_{sol}\big(\nabla(\omega g_0 e^{ix\cdot\xi_0/\delta})\big) - \pi_{sol}\big(g_0(x)e^{ix\cdot\xi_0/\delta}\nabla\omega\big) \\ &= -\pi_{sol}(g_0(x)e^{ix\cdot\xi_0/\delta}\nabla\omega), \end{aligned}$$

since the gradient of a function is irrotational and, hence, orthogonal to the space of divergence free vector fields. If we apply Lemma 3.1.1 we have

$$Bv = -g_0(x)\pi_{\xi_0^{\perp}}(\nabla\omega)e^{ix\cdot\xi_0/\delta} + \tilde{r}_{\delta},$$

where $\|r_{\delta}\|_{L^2} = O(\delta)$. Substitute our definition for g_0 from (4.4) to get

$$Bv = \xi_0^{\perp}h_0(x)e^{ix\cdot\xi_0/\delta} + \tilde{r}_{\delta}.$$

Then the expansion (4.3) for ϕ_{δ} implies

$$Bv = -i\delta\nabla^{\perp}(h_0(x)e^{ix\cdot\xi_0/\delta}) + r_{\delta} =: \phi_{\delta} + r_{\delta},$$

where $r_{\delta} := \tilde{r}_{\delta} - i\delta\big[e^{ix\cdot\xi_0/\delta}\nabla^{\perp}h_0(x)\big] \in L^2$ and $\|r_{\delta}\|_{L^2} = O(\delta)$. Thus we have $\phi_{\delta} + r_{\delta} \in \overline{\mathrm{Im}B}$. $\square$

This lemma establishes criteria for measuring the factor space norm of a slightly generalized version of our fast oscillating vector fields. Recall that our factor space $F := L^2_{sol}(\mathbb{T}^2)/\overline{\mathrm{Im}B}$, with the canonical factor space norm we denote $\|\cdot\|_F$.

Lemma 4.1.2. *Let $x_0 \in \mathbb{T}^n$, $\xi_0 \in \mathbb{R}^n$ such that $\nabla\omega(x_0) \neq 0$ and $(\xi_0^\perp, \nabla\omega(x_0)) = 0$. Let $h_0 \in C^\infty(\mathbb{T}^n)$ be supported on $B_1(0)$, the ball of radius 1 centered at 0 such that $h_0(0) = 1$. For $0 < \zeta << 1$ define h_ζ by*

$$h_\zeta(x) := h_0\big(\frac{x - x_0}{\zeta}\big),$$

and let $\delta^{-1} \in \mathbb{Z}_+$. For any $x \in [0,1) \times [0,1)$ define

$$\phi_{\zeta,\delta}(x) := -i\delta\nabla^\perp(h_\zeta(x)e^{ix\cdot\xi_0/\delta}),$$

and extend $\phi_{\zeta,\delta}$ periodically. Then we have

$$\|\phi_{\zeta,\delta}\|_F = \|\phi_{\zeta,\delta}\|_{L^2} + O(\zeta) + O(\delta),$$

where $O(\zeta)$ is independent of δ and $O(\delta)$ is independent of ζ.

Remark 10. *The conditions on x_0 and ξ_0 imply that ξ_0 is a scalar multiple of $\nabla\omega(x_0)$, so we cannot require $\xi_0 \in \mathbb{Z}^2$ here. To ensure that $\phi_{\zeta,\delta}$ is periodic, we define the vector field on $B_\zeta(x_0)$ and, since $\zeta << 1$, we may extend it periodically.*

Proof. A key idea in this proof is the fact that if a vector field, w, is divergence free, then $w = 0$ if and only if $\text{curl}w = 0$. This fact follows from the Hodge decomposition of vector fields on the torus: $L^2(\mathbb{T}^n) = L^2_{sol}(\mathbb{T}^n) \oplus L^2_{irr}(\mathbb{T}^n)$ discussed in Section 2.1. It follows that since B maps into L^2_{sol}, we can say $v \in \text{Ker}B$ if and only if $v \in \text{Ker}T$ where $T : (L^2_{sol}(\mathbb{T}^2))^2 \to (L^2_{sol}(\mathbb{T}^2))^2$ is defined by

$$Tv := \text{curl}Bv = v \cdot \nabla\omega.$$

For any $x \in \text{supp}(h_\zeta)$, $|x - x_0| \le \zeta$, so

$$|\nabla\omega(x) - \nabla\omega(x_0)| \le \zeta K,$$

where $K := \|\nabla\omega\|_{Lip}$ is the Lipschitz norm of $\nabla\omega$. We may assume $\zeta << |\nabla\omega(x_0)|$, so for any $x \in \text{supp} h_\zeta$, $|\nabla\omega(x)| \ge |\nabla\omega(x_0)| - \zeta K > 0$ and we may write

$$\xi_0^\perp = \xi_0^\perp - \frac{(\xi_0^\perp, \nabla\omega(x))}{|\nabla\omega(x)|^2}\nabla\omega(x) + \frac{(\xi_0^\perp, \nabla\omega(x))}{|\nabla\omega(x)|^2}\nabla\omega(x).$$

We assume $(\xi_0^\perp, \nabla\omega(x_0)) = 0$, so we have

$$\begin{aligned}\frac{|(\xi_0^\perp, \nabla\omega(x))|}{|\nabla\omega(x)|} &= \frac{|(\xi_0^\perp, \nabla\omega(x)) - (\xi_0^\perp, \nabla\omega(x_0))|}{|\nabla\omega(x)|} \\ &\le \frac{\zeta|\xi_0^\perp|}{|\nabla\omega(x_0)| - \zeta K}.\end{aligned}$$

Hence

$$\xi_0^\perp = \xi_0^\perp - \frac{(\xi_0^\perp, \nabla\omega(x))}{|\nabla\omega(x)|^2}\nabla\omega(x) + O(\zeta)\frac{\nabla\omega(x)}{|\nabla\omega(x)|}.$$

For any $x \in \text{supp} h_\zeta$, let

$$\begin{aligned}\eta(x) &:= \xi_0^\perp - \frac{(\xi_0^\perp, \nabla\omega(x))}{|\nabla\omega(x)|^2}\nabla\omega(x) \\ &= \xi_0^\perp - O(\zeta)\frac{\nabla\omega(x)}{|\nabla\omega(x)|}.\end{aligned} \tag{4.5}$$

We can expand $\phi_{\zeta,\delta}$ as in (4.3) and compute

$$\phi_{\zeta,\delta}(x) = h_\zeta(x)\xi_0^\perp e^{ix\cdot\xi_0/\delta} + r_\delta,$$

where $\|r_\delta\|_{L^2} \le \delta C\|\nabla h_\zeta\|_{L^2}$. Notice that in 2-dimensions, $\|\nabla h_\zeta\|_{L^2} = \|\nabla h_0\|_{L^2}$, so $\|r_\delta\|_{L^2} \le \delta C\|\nabla h_0\|_{L^2}$, which is independent of ζ. We also have from the definition of η in (4.5) that

$$\phi_{\zeta,\delta}(x) = h_\zeta(x)\eta(x)e^{ix\cdot\xi_0/\delta} + r_\zeta + r_\delta,$$

where $\|r_\zeta\|_{L^2} = O(\zeta)$ independent of δ. Since $(T\eta)(x) := \eta(x)\cdot\nabla\omega(x) \equiv 0$, we have $\phi_{\zeta,\delta} - r_\zeta - r_\delta \in \mathrm{Ker}B$. Therefore, $\|\phi_{\zeta,\delta}\|_F = \|\phi_{\zeta,\delta}\|_{L^2} + O(\zeta) + O(\delta)$.

□

Now we prove a slightly generalized 2-dimensional version of Lemma 3.1.2 to approximate the linear evolution of our $\phi_{\zeta,\delta}$ vector fields, where it is no longer assumed that the frequency vector $\xi_0 \in \mathbb{Z}^2$.

Lemma 4.1.3. *Let $h_0 \in C^\infty(\mathbb{T}^2)$ be supported on $B_1(0)$, the ball centered at 0 of radius 1. For $0 < \zeta < 1$ and fixed $x_0 \in \mathbb{T}^2$, define h_ζ by*

$$h_\zeta(x) := h_0\big(\frac{x - x_0}{\zeta}\big).$$

Let $\xi_0 \in \mathbb{R}^n$, $\delta^{-1} \in \mathbb{Z}_+$ and define $\phi_{\zeta,\delta}(x) := -i\delta\nabla^\perp(h_\zeta e^{ix\cdot\xi_0/\delta})$. Then for any fixed $t > 0$ we can approximate $G_\varepsilon(t)\phi_{\zeta,\delta}(x) := (op_\varepsilon[a_0]\circ\mathfrak{g}_u(t)\phi_{\zeta,\delta})(x)$ as follows:

$$(op_\varepsilon[a_0]\circ\mathfrak{g}_u(t)\phi_{\zeta,\delta})(x) = h_\zeta(g^{-t}x)b(g^{-t}x,\xi_0,\xi_0^\perp;t)e^{ig^{-t}x\cdot\xi_0/\delta} + r_\delta(x),$$

where $b(g^{-t}x,\xi_0,\xi_0^\perp;t)$ is the solution of (BAS) at time t with initial conditions $(g^{-t}x,\xi_0,\xi_0^\perp)$, $\|r_\delta\|_{L^2} = O(\delta)$.

Before we begin the proof, recall from Section 2.2 that we decompose the symbol a_0 into two parts: A_0 the part homogeneous of degree 0 in ξ and $1 - X$, the evolution of the smooth function $1-\chi$ that cuts out the origin in ξ-space.

$$a_0(x,\xi,t) = A_0(x,\xi,t)\big(1 - X\big(x,\frac{\xi}{\sqrt{\varepsilon}},t\big)\big),$$

where A_0 and X are solutions to the following system:

$$\begin{cases} \partial_t A_0 = -\nabla_u A_0 - \frac{\partial u}{\partial x} A_0 + 2\frac{\xi\otimes\xi}{|\xi|^2}\frac{\partial u}{\partial x} A_0, \\ A_0(x,\xi,0) = 1 - \frac{\xi\otimes\xi}{|\xi|^2}, \\ \partial_t X = -\nabla_u X, \quad \text{where } X(x,\xi,0) = \chi(\xi), \end{cases} \tag{4.6}$$

where

$$\chi(\xi) = \begin{cases} 1 & \text{if } |\xi| \le \frac{1}{2}, \\ 0 & \text{if } |\xi| \ge \frac{2}{3}. \end{cases}$$

Remark 11. *It follows that for* $\tilde{C} := \|\frac{\partial u}{\partial x}\|_{L^\infty(\mathbb{T}^2)}$*, if* $|\xi| \le \frac{1}{2}e^{-\tilde{C}t}\sqrt{\varepsilon}$ *then* $a_0(x,\xi,t) = 0$*. Also, if* $|\xi| \ge \frac{2}{3}e^{\tilde{C}t}\sqrt{\varepsilon}$*, then* $a_0(x,\xi,t) = A_0(x,\xi,t)$ *which is homogeneous of degree* 0 *in* ξ*. Thus by Remark 4 in Section 2.4,* $a_0 \in S^0_{1,0}(\mathbb{T}^2) \subset S^0_{0,0}(\mathbb{T}^2)$*.*

We also have that for any $(x_0,\xi_0,b_0) \in T^*(\mathbb{T}^n)\times\mathbb{R}^n$,

$$b(x_0,\xi_0,b_0;t) = A_0(g^t x_0, (g_*^{-t}(x))^*\xi_0, t)b_0. \tag{4.7}$$

Proof of Lemma 4.1.3. From the expansion of $\phi_{\zeta,\delta}$ as in (4.3) we have

$$\phi_{\zeta,\delta}(x) = h_\zeta(x)\xi_0^\perp e^{ix\cdot\xi_0/\delta} + r_\delta(x),$$

where $\|r_\delta\|_{L^2} = O(\delta)$. Let $t > 0$ and since $\mathfrak{g}_u(t)$ is a unitary operator on $L^2(\mathbb{T}^n)$, we have

$$\big(\mathfrak{g}_u(t)\phi_{\zeta,\delta}\big)(x) = h_\zeta(g^{-t}x)\xi_0^\perp e^{ig^{-t}x\cdot\xi_0/\delta} + \tilde{r}_\delta(x),$$

where $\tilde{r}_\delta := \mathfrak{g}_u(t)r_\delta$, thus $\|\tilde{r}_\delta\|_{L^2} = O(\delta)$. And since $\mathrm{op}_\varepsilon[a_0]$ is bounded on $L^2(\mathbb{T}^2)$, we have

$$(\mathrm{op}_\varepsilon[a_0]\circ\mathfrak{g}_u(t)\phi_{\zeta,\delta})(x) = (\mathrm{op}_\varepsilon[a_0]\circ\mathfrak{g}_u(t)h_\zeta\circ g^{-t}\xi_0^\perp e^{ig^{-t}(\cdot)\cdot\xi_0/\delta})(x) + \overline{r}_\delta(x), \tag{4.8}$$

where $\overline{r}_\delta = \mathrm{op}_\varepsilon[a_0]\tilde{r}_\delta$, hence $\|\overline{r}_\delta\|_{L^2} = O(\delta)$.

We will apply Lemma 2.4.4 and Lemma 2.4.5 to the main order term of $\phi_{\zeta,\delta}$ in (4.8), so we must carefully define our exponent function $S \in C^\infty(\mathbb{R}^2)$. Let $\kappa \in C^\infty\big(g^{-t}(B_{4\zeta}(x_0))\big)$ such that

$$\kappa(x) = \begin{cases} 1 \text{ if } & g^{-t}x \in B_{2\zeta}(x_0), \\ 0 \text{ if } & g^{-t}x \notin B_{3\zeta}(x_0). \end{cases}$$

Now extend κ periodically so $\kappa \in C^\infty(\mathbb{T}^2)$ (this is not a problem since we assume $\zeta << 1$). Define $S \in C^\infty(\mathbb{R}^2)$ by

$$S(x) := \kappa(x) g^{-t}x \cdot \xi_0.$$

Then $h_\zeta(g^{-t}x)\xi_0^\perp e^{ig^{-t}x\cdot\xi_0/\delta} = h_\zeta(g^{-t}x)\xi_0^\perp e^{iS(x)/\delta}$. Also both S and ∇S are well defined in $C^\infty(\mathbb{T}^2)$. Hence S satisfies the hypothesis of Lemma 2.4.4. Now for a $\gamma > 0$ to be specified later, we may define $\sigma \in S^0_{0,0}(\mathbb{T}^n)$ as follows:

$$\sigma(x,\xi) := \big(1 - X(x,\gamma\xi,t)A_0(x,\xi,t)\big).$$

And if we apply Lemma 2.4.4 we have

$$\big(\mathrm{op}_\delta[\sigma]h_\zeta \circ g^{-t}\xi_0^\perp e^{iS/\delta}\big)(x) = \sigma(x,\nabla S(x))h_\zeta(g^{-t}x)\xi_0^\perp e^{iS(x)/\delta} + R_\delta, \tag{4.9}$$

where $\|R_\delta\|_{L^\infty} = O(\delta)$.

Notice that since $\mathrm{op}_\varepsilon[a_0] = \mathrm{op}_1[a_0(x,\varepsilon\xi,t)]$ and $\mathrm{op}_\delta[\sigma] = \mathrm{op}_1[\sigma(x,\delta\xi,t)]$, we have

$$\begin{aligned} \mathrm{op}_\varepsilon[a_0]h_\zeta(g^{-t}x)\xi_0^\perp e^{iS(x)/\delta} &= \big(\mathrm{op}_\delta[\sigma]h_\zeta \circ g^{-t}\xi_0^\perp e^{iS/\delta}\big)(x) \qquad (4.10) \\ &+ \mathrm{op}_1\big[\big(X(x,\gamma\delta\xi,t) - X(x,\sqrt{\varepsilon}\xi,t)\big)A_0(x,\xi,t)\big]h_\zeta(g^{-t}x)\xi_0^\perp e^{iS(x)/\delta}. \end{aligned}$$

We will use Lemma 2.4.5 twice to estimate the second term on the RHS of (4.10). Define

$$c_S := \inf_{x \in supp(h_\zeta \circ g^{-t})} \left|\nabla S(x)\right| = \inf_{x \in supp(h_\zeta \circ g^{-t})} \left|(g_*^{-t}(x))^* \xi_0\right|.$$

The constant $c_S > 0$ since $(g_*^{-t}(x))^* \xi_0$ is a solution to the cotangent flow equation

$$\dot{\xi} = -(\frac{\partial u}{\partial x})^T \xi.$$

Since the cotangent flow is reversible, we may use negative time Gronwall estimates to show that given a fixed t, there is a constant $\tilde{C}(t)$ such that $|\xi(\xi_0, t)| > \tilde{C}(t)|\xi_0|$ for all initial conditions ξ_0.

Since X is the evolution of our cutoff function χ along the cotangent flow, we have that there is another constant $C(t)$ such that $X(x, \xi, t) = 0$ for $|\xi| > C(t)$. Let $\gamma := \frac{2C(t)}{c_S}$, so we have $0 < \frac{1}{\gamma} < \frac{c_S}{C(t)}$. Also the homogeneity of A_0 in ξ implies $\sigma \in S^0_{0,0}(\mathbb{T}^2)$ (see Remark 11), so by Lemma 2.4.5 we have

$$\left\|\mathrm{op}_1\left[X(x, \gamma\delta\xi, t)A_0(x, \xi, t)\right] h_\zeta(g^{-t}x)\xi_0^\perp e^{iS(x)/\delta}\right\|_{L^\infty} = O(\delta).$$

Also, we assume $\delta << \varepsilon$, so $0 < \frac{\delta}{\sqrt{\varepsilon}} < \frac{c_S}{C(t)}$ and by Lemma 2.4.5 we have

$$\left\|\mathrm{op}_1\left[X(x, \sqrt{\varepsilon}\xi, t)A_0(x, \xi, t)\right] h_\zeta(g^{-t}x)\xi_0^\perp e^{iS(x)/\delta}\right\|_{L^\infty} = O(\delta).$$

Hence, from (4.9) and (4.10) we have

$$\left\|\mathrm{op}_\varepsilon[a_0] h_\zeta(g^{-t}x)\xi_0^\perp e^{iS(x)/\delta} - \sigma(x, \nabla S(x)) h_\zeta(g^{-t}x)\xi_0^\perp e^{iS(x)/\delta}\right\|_{L^\infty} = O(\delta). \tag{4.11}$$

For any $x \in supp(h_\zeta \circ g^{-t})$, we have $\nabla S(x) \geq c_S > \frac{C(t)}{\gamma}$. It follows that $X(x, \gamma\nabla S(x), t) = 1$ and

$$\begin{aligned} h_\zeta(g^{-t}x)\sigma(x, \nabla S(x))\xi_0^\perp =& h_\zeta(g^{-t}x)(1 - X(x, \gamma\nabla S(x), t))A_0(x, \nabla S(x), t)\xi_0^\perp \\ =& h_\zeta(g^{-t}x)A_0(x, (g_*^{-t}(x))^*\xi_0, t)\xi_0^\perp. \end{aligned}$$

Since $A_0(x, (g_*^{-t}(x))^*\xi_0, t)\xi_0^\perp$ solves (BAS) with initial conditions $(g^{-t}x, \xi_0, \xi_0^\perp)$, we have

$$h_\zeta(g^{-t}x)\sigma(x, \nabla S(x))\xi_0^\perp = h_\zeta(g^{-t}x)b(g^{-t}x, \xi_0, \xi_0^\perp; t).$$

Then from (4.8) and (4.11) we have

$$\|(\mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}_u(t)\phi_{\zeta,\delta})(x) - h_\zeta(g^{-t}x)b(g^{-t}x, \xi_0, \xi_0^\perp; t)e^{ig^{-t}x\cdot\xi_0}\|_{L^2} = O(\delta),$$

since for any $f \in L^2(\mathbb{T}^2)$, $\|f\|_{L^2} \leq (2\pi)^2\|f\|_{L^\infty}$. This completes the proof. $\square$

4.2 Main theorem for 2-dimensional flows

The approach for finding lower bounds for the essential spectral radius of the linear evolution of 2-dimensional perturbations in each class is completely similar to that taken in Chapter 3. Before stating our main theorem, we prove the following propositions similar to Proposition 3.2.2.

Proposition 4.2.1. *Let $u \in (C^\infty(\mathbb{T}^2))^2$ be a solution to steady Euler's equation (SE) with scalar vorticity $\omega := curlu$ and fix $T > 0$. Let*

$$\mathcal{A} := \{(x_0, \xi_0, b_0) \in \mathbb{T}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 |\ \xi_0 \perp b_0,\ |\xi_0| = |b_0| = 1\}.$$

Define $\Theta_I(t)$ by

$$\Theta_I(t) = \sup_{\substack{(x_0,\xi_0,b_0)\in\mathcal{A} \\ x_0 \in supp(\nabla\omega)}} |b(x_0, \xi_0, b_0; t)|,$$

where $b(x_0, \xi_0, b_0; t)$ *is a solution to (BAS) with initial conditions* (x_0, ξ_0, b_0)*. Then for any* $\varepsilon > 0$ *and* $t \in [0, T]$ *we have*

$$\|G^s_\varepsilon(t)\|_{\mathcal{L}(\overline{ImB}, L^2_{sol})} + O(\sqrt{\varepsilon}) \geq \Theta_I(t),$$

where the constant in O *is uniform for* $t \in [0, T]$*.*

Proposition 4.2.2. *Let* $u \in (C^\infty(\mathbb{T}^2))^2$ *be a solution to steady Euler's equation (SE) with scalar vorticity* $\omega := curlu$ *and fix* $T > 0$*. Let*

$$\mathcal{A} := \{(x_0, \xi_0, b_0) \in \mathbb{T}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 |\ \xi_0 \perp b_0,\ |\xi_0| = |b_0| = 1\}.$$

(i) If we define $\tilde{\Theta}_F(t)$ *by*

$$\tilde{\Theta}_F(t) := \sup_{\substack{(x_0,\xi_0,b_0)\in\mathcal{A} \\ x_0 \notin supp \nabla\omega}} |b(x_0, \xi_0, b_0; t)|,$$

where $b(x_0, \xi_0, b_0; t)$ *is a solution to (BAS) with initial conditions* (x_0, ξ_0, b_0)*. Then for any* $\varepsilon > 0$ *and* $t \in [0, T]$ *we have*

$$\|G^s_\varepsilon(t)\|_{\mathcal{L}(F)} + O(\sqrt{\varepsilon}) \geq \tilde{\Theta}_F(t),$$

where the constant in O *is uniform for* $t \in [0, T]$*.*

(ii) If we define $\overline{\Theta}_F(t)$ *by*

$$\overline{\Theta}_F(t) := \sup_{\substack{\{x_0\in\mathbb{T}^2 |\ |\nabla\omega(x_0)|>0\} \\ |b_0|=1 \\ b_0 \perp \nabla\omega(x_0)}} |b(x_0, \nabla\omega(x_0), b_0; t)|,$$

where $b(x_0, \nabla\omega(x_0), b_0; t)$ *is a solution to (BAS) with initial conditions* $(x_0, \nabla\omega(x_0), b_0)$*. Then for any* $\varepsilon > 0$ *and* $t \in [0, T]$ *we have*

$$\|G^s_\varepsilon(t)\|_{\mathcal{L}(F)} + O(\sqrt{\varepsilon}) \geq \overline{\Theta}_F(t),$$

where the constant in O *is uniform for* $t \in [0, T]$*.*

The proofs of these propositions are very similar to the proof of Proposition 3.2.2. We begin by looking at sequences of fast oscillating vector fields and showing that the appropriate norms of their images under $G^s_\varepsilon(t)$ approach the appropriate Θ-function. First we approximate the evolution of our general 2-dimensional fast oscillating perturbations. Consider the vector field $\phi_\delta \in C^\infty(\mathbb{T}^2)$ defined by

$$\phi_\delta(x) := \delta\nabla^\perp(h_0(x)e^{i\xi_0\cdot x/\delta}), \tag{4.12}$$

where $\delta^{-1} \in \mathbb{Z}_+, \delta < 1, \xi_0 \in \mathbb{Z}^2$ and $h_0 \in C^\infty(\mathbb{T}^2)$ is an arbitrary smooth scalar function. If we consider ϕ_δ as a 3-dimensional planar vector field on $\mathbb{T}^3$, then

$$\phi_\delta = \delta\nabla\times\left(\frac{i\xi_0\times\xi_0^\perp}{|\xi_0|^2}h_0(x)e^{ix\cdot\xi_0/\delta}\right). \tag{4.13}$$

Thus, by Lemma 3.1.2 and Remark 7 from Section 3.1 we have

$$\begin{aligned}\|\mathrm{op}_\varepsilon[a_0]\circ\mathfrak{g}_u(t)\phi_\delta\|_{L^2} &= \|h_0(g^{-t}\cdot)A_0(\cdot,(g_*^{-t}(\cdot))^*\xi_0,t)\xi_0^\perp e^{ig^{-t}(\cdot)\cdot\xi_0/\delta}\|_{L^2} + O(\delta)\\ &= \|h_0(g^{-t}\cdot)b(\cdot,(g_*^{-t}(\cdot))^*\xi_0,\xi_0^\perp,t)\|_{L^2} + O(\delta). \end{aligned}\tag{4.14}$$

We also remark that in the proof of Proposition 3.2.2 we showed

$$\|G^s_\varepsilon(t) - \mathrm{op}_\varepsilon[a_0]\circ\mathfrak{g}_u(t)\|_{\mathcal{L}(L^2_{sol}(\mathbb{T}^3))} = O(\sqrt{\varepsilon}).$$

It follows that we have the same estimate in 2-dimensions:

$$\|G^s_\varepsilon(t) - \mathrm{op}_\varepsilon[a_0]\circ\mathfrak{g}_u(t)\|_{\mathcal{L}(L^2_{sol}(\mathbb{T}^2))} = O(\sqrt{\varepsilon}). \tag{4.15}$$

Proof of Proposition 4.2.1. Let $x_0 \in \mathbb{T}^2$, $\xi_0 \in \mathbb{Z}^2$ such that $(\xi_0^\perp, \nabla\omega(x_0)) \neq 0$. We can choose $h_0 \in C^\infty(\mathbb{T}^2)$ supported such that there is some constant c_0

where $|(\xi_0^\perp, \nabla\omega(x))| > c_0$ for all $x \in \mathrm{supp} h_0$. We will call any function h_0 that satisfies these properties, *localized at* x_0. Then for $\delta^{-1} \in \mathbb{Z}_+$, let $\phi_\delta := -i\delta\nabla^\perp(h_0 e^{ix\cdot\xi_0/\delta})$. Then from Lemma 4.1.1, ϕ_δ is approximately in the image of B. More specifically, there is some remainder r_δ such that $\|r_\delta\|_{L^2_{sol}} = O(\delta)$ and $\phi_\delta + r_\delta \in \overline{\mathrm{Im} B}$. Take the limit of the estimate 4.14 as $\delta \to 0$, to get

$$\|\mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}_u(t)\|_{\mathcal{L}(\overline{\mathrm{Im}B}, L^2)} \geq \sup_{\substack{x_0\in\mathbb{T}^2,\xi_0\in\mathbb{Z}^2 \\ (\xi_0^\perp,\nabla\omega(x_0))\neq 0 \\ h_0 \text{ localized at } x_0}} \frac{\|h_0(g^{-t}x)b(g^{-t}x,\xi_0,\xi_0^\perp;t)\|_{L^2_{sol}}}{\|h_0\xi_0^\perp\|_{L^2_{sol}}}.$$

The flow map g^{-t} is measure preserving, so as a change of variables it will not affect the L^2-norm. We are taking the supremum over all functions h_0 *localized at* x_0 for some x_0 such that $\nabla\omega(x_0) \neq 0$. This is the same as taking the supremum over all functions $h_0 \in C^\infty(\mathbb{T}^2)$ with $\mathrm{supp}(h_0) \subset \{x : \nabla\omega(x) \neq 0\}$. Also, $b(x_0, \xi_0, b_0; t)$ depends linearly on the initial condition b_0. Thus if we take into account that b is also homogeneous of degree 0 in ξ_0, depends continuously on ξ_0 and any $\xi \in \mathbb{R}^2$ can be approximated by a vector $\xi_0 \in \mathbb{Q}^2$, we have

$$\|\mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}_u(t)\|_{\mathcal{L}(\overline{\mathrm{Im}B}, L^2)} \geq \sup_{\substack{x_0\in\mathbb{T}^2,\xi_0\in\mathbb{R}^2 \\ |\xi_0|=1 \\ (\xi_0^\perp,\nabla\omega(x_0))\neq 0}} |b(x_0,\xi_0,b_0;t)|. \tag{4.16}$$

Take the closure of the condition $(\xi_0^\perp, \nabla\omega(x_0)) \neq 0$ on the supremum in line 4.16 and, since $b(x_0, \xi_0, \xi_0^\perp; t)$ depends continuously on the initial conditions, we have

$$\|\mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}_u(t)\|_{\mathcal{L}(\overline{\mathrm{Im}B}, L^2_{sol})} \geq \sup_{\substack{|\xi_0|=|b_0|=1 \\ b_0\perp\xi_0 \\ x_0\in\mathrm{supp}\nabla\omega}} |b(x_0,\xi_0,b_0;t)| =: \Theta_I(t).$$

Hence, from (4.15), we have $\|G^s_\varepsilon(t)\|_{\mathcal{L}(\overline{\mathrm{Im}B}, L^2_{sol})} + O(\sqrt{\varepsilon}) \geq \Theta_I(t)$. This concludes the proof of Proposition 4.2.1. □

To prove Proposition 4.2.2, it will be important to examine the evolution of $\nabla\omega$ along a path of flow. Let $u \in C^\infty(\mathbb{T}^2)$ be our steady solution to Euler's equation and let the scalar function $\omega := \text{curl} u$ in the 2-dimensional sense. Consider the vorticity equation for steady flows in 2-dimensions:

$$u^i \partial_i \omega = 0. \tag{4.17}$$

Take the j^{th} partial derivative of 4.17 to get

$$\partial_j u^i \partial_i \omega + u^i \partial_j \partial_i \omega = 0.$$

Which implies

$$(\frac{\partial u}{\partial x})^T \nabla\omega + (u \cdot \nabla)\nabla\omega = 0.$$

It follows that

$$\frac{d}{dt}\nabla\omega(g^t x) = -(\frac{\partial u}{\partial x})^T \nabla\omega.$$

Hence, $\nabla\omega$ evolves like a covector along the flow g^t and we have

$$\nabla\omega(g^t x_0) = (g_*^{-t}(x_0))^* \nabla\omega(x_0). \tag{4.18}$$

Proof of Proposition 4.2.2 (i). Let $h_0 \in C^\infty$ such that $\nabla\omega(x) = 0$ for any $x \in \text{supp} h_0$. Now let $\delta^{-1} \in \mathbb{Z}_+$ and choose any $\xi_0 \in \mathbb{Z}^2$ and consider the resulting fast oscillating vector field, $\phi_\delta := -i\delta\nabla^\perp(h_0 e^{ix\cdot\xi_0/\delta})$. Just as in the proof of Lemma 4.1.2 we consider the operator $T = \text{curl} B$ defined by

$$Tv := v \cdot \nabla\omega \quad v \in (C^\infty(\mathbb{T}^2))^2.$$

Since Bv is divergence free, $Bv = 0$ if and only if $Tv = 0$. Since $\nabla\omega \equiv 0$ on $\text{supp}(\phi_\delta)$, it is clear that $\phi_\delta \in \text{Ker} T = \text{Ker} B$. Hence, if recall the expansion

(4.3) we have

$$\|\phi_\delta\|_F = \|\phi_\delta\|_{L^2} = \|h_0\xi_0^\perp\|_{L^2} + O(\delta). \tag{4.19}$$

The evolution of $\nabla\omega$ (4.18) along the flow implies $\nabla\omega(g^t x) \equiv 0$ for all $x \in \mathrm{supp} h_0$. It follows that $\nabla\omega \equiv 0$ on $\mathrm{supp}(h_0 \circ g^{-t})$ and

$$h_0(g^{-t}x)b(g^{-t}x, \xi_0, \xi_0^\perp; t)e^{ig^{-t}x\cdot\xi_0/\delta} \in \mathrm{Ker}T.$$

Hence, from the estimate (4.14) we have

$$\begin{aligned}\|\mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}_u^t\phi_\delta\|_F =& \|h_0(g^{-t}x)b(g^{-t}x, \xi_0, \xi_0^\perp; t)e^{ig^{-t}x\cdot\xi_0/\delta}\|_F + O(\delta) \\ &= \|h_0(g^{-t}x)b(g^{-t}x, \xi_0, \xi_0^\perp; t)e^{ig^{-t}x\cdot\xi_0/\delta}\|_{L^2} + O(\delta). \end{aligned} \tag{4.20}$$

Consider (4.19) and (4.20) and take the limit as $\delta \to 0$ to get

$$\|\mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}_u^t\|_{\mathcal{L}(F)} \geq \sup_{\substack{\xi\in\mathbb{Z}^2, x\in\mathbb{T}^2 \\ \mathrm{supp}(h_0)\subset\{x:\nabla\omega(x)=0\}}} \frac{\|h_0(g^{-t}x)b(g^{-t}x, \xi_0, \xi_0^\perp; t)e^{ig^{-t}x\cdot\xi_0/\delta}\|_{L^2}}{\|h_0\xi_0^\perp\|_{L^2}}.$$

We simplify the supremum on the RHS using that (i) g^{-t} is measure preserving so the corresponding coordinate change does not affect the L^2 norm, (ii) $b(g^{-t}x, \xi_0, \xi_0^\perp; t)$ homogeneous of degree zero in ξ_0 and any vector in $\mathbb{R}^2$ can be approximated by a vector in $\mathbb{Q}^2$, so we can take our supremum over $\xi_0 \in \mathbb{R}^n$ such that $|\xi_0| = 1$, and (iii) $b(g^{-t}x, \xi_0, \xi_0^\perp; t)$ depends linearly on $\xi_0^\perp$ and ξ_0 is a 2-dimensional vector, so it is equivalent to consider all $b_0 \perp \xi_0$ such that $|b_0| = 1$ in our supremum. We also note that (iv) if $\mathrm{supp}(h_0) \subset \{x : \nabla\omega(x) = 0\}$, then

$$\mathrm{supp}(h_0 \circ g^{-t}) \subset \{x : \nabla\omega(x) = 0\},$$

so we may take the supremum over $x_0 = g^{-t}x \in \mathbb{T}^2 \setminus \mathrm{supp}\nabla\omega$:

$$\|\mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}_u^t\|_{\mathcal{L}(F)} \geq \sup_{\substack{|b_0|=|\xi_0|= \\ \xi_0\perp b_0 1 \\ x_0\notin\mathrm{supp}\nabla\omega(x_0)}} |b(x_0, \xi_0, b_0; t)| =: \tilde{\Theta}_F(t).$$

Therefore, from the estimate (4.15) we have

$$\|G^s_\varepsilon(t)\|_{\mathcal{L}(F)} + O(\sqrt{\varepsilon}) \geq \tilde{\Theta}_F(t).$$

□

Proof of Proposition 4.2.2 (ii). Let $x_0 \in \mathbb{T}^n$ such that $\nabla\omega(x_0) \neq 0$ and define $\xi_0 := \frac{\nabla\omega(x_0)}{|\nabla\omega(x_0)|}$. Let $h_0 \in C^\infty(\mathbb{T}^n)$ be supported on $B_1(0)$, the ball of radius 1 centered at 0 such that $h_0(0) = 1$. For $0 < \zeta << 1$ define h_ζ by

$$h_\zeta(x) := h_0\big(\frac{x - x_0}{\zeta}\big)$$

and let $\delta^{-1} \in \mathbb{Z}_+$. For any $x \in [0,1) \times [0,1)$ define

$$\phi_{\zeta,\delta}(x) := -i\delta\nabla^\perp(h_\zeta(x)e^{ix\cdot\xi_0/\delta}),$$

and extend $\phi_{\zeta,\delta}$ periodically. It follows from Lemma 4.1.2 and the expansion (4.3) of $\phi_{\zeta,\delta}$ that

$$\|\phi_{\zeta,\delta}\|_F = \|\phi_{\zeta,\delta}\|_{L^2} + O(\zeta) + O(\delta) = \|h_\zeta\xi_0^\perp\|_{L^2} + O(\zeta) + O(\delta). \tag{4.21}$$

Now we must estimate $\|\mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}^t_u\phi_{\zeta,\delta}\|_F$ for our fixed time $t > 0$. The approach is similar to the proof of Lemma 4.1.2. Here we will also need the operator $T : L^2_{sol}(\mathbb{T}^2) \to L^2_{sol}(\mathbb{T}^2)$ defined by

$$Tv = \mathrm{curl}Bv = v \cdot \nabla\omega. \tag{4.22}$$

For any $v \in L^2_{sol}(\mathbb{T}^2)$, the image Bv is divergence free. Then from Remark 1 in Section 2.1 $Bv = 0$ if and only if $Tv = 0$ and $\mathrm{Ker}B = \mathrm{Ker}T$. Lemma 4.1.3 gives that

$$\big(\mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}^t_u\phi_{\zeta,\delta}\big)(x) = h_\zeta(g^{-t}x)b(g^{-t}x, \xi_0, \xi_0^\perp; t)e^{ig^{-t}x\cdot\xi_0/\delta} + r_\delta(x), \tag{4.23}$$

where $\|r_\delta\|_{L^2} = O(\delta)$. Let $y \in \mathrm{supp}(h_\zeta \circ g^{-t})$. Then $g^{-t}y \in \mathrm{supp}(h_\zeta) \subset B_\zeta(x_0)$. We wish to estimate $|\nabla\omega(y) - \nabla\omega(g^t x_0)|$. From the evolution of $\nabla\omega$ along flow lines, described by (4.18), we have

$$\begin{aligned}\nabla\omega(y) - \nabla\omega(g^t x_0) =& (g_*^{-t}(y_0))^*\nabla\omega(y_0) - (g_*^{-t}(x_0))^*\nabla\omega(x_0)\\ =& \ (g_*^{-t}(y_0))^*\nabla\omega(y_0) - (g_*^{-t}(y_0))^*\nabla\omega(x_0)\\ &+ \ (g_*^{-t}(y_0))^*\nabla\omega(x_0) - (g_*^{-t}(x_0))^*\nabla\omega(x_0),\end{aligned}$$

where $y_0 = g^{-t}y$. If we let

$$c(t) := \sup_{(x,\xi)\in T^*(\mathbb{T}^2), |\xi|=1} |(g_*^{-t}(x))^*\xi|.$$

Then we have

$$\begin{aligned}|(g_*^{-t}(y_0))^*\nabla\omega(y_0) - (g_*^{-t}(y_0))^*\nabla\omega(x_0)| &\le c(t)|\nabla\omega(y_0) - \nabla\omega(x_0)|\\ &\le \zeta c(t) K_1,\end{aligned}$$

where $K_1 := \|\nabla\omega\|_{Lip}$, the Lipschitz norm of $\nabla\omega$. The matrix valued function $x \mapsto (g_*^{-t}(x))^*$ is also Lipschitz, so if we let K_2 denote its Lipschitz norm, we have

$$|(g_*^{-t}(y_0))^*\nabla\omega(x_0) - (g_*^{-t}(x_0))^*\nabla\omega(x_0)| \le \zeta|\nabla\omega(x_0)|K_2.$$

Thus for any $y \in \mathrm{supp}(h_\zeta \circ g^{-t})$ we have

$$|\nabla\omega(y) - \nabla\omega(g^t x_0)| \le \zeta c(t) K_1 + \zeta|\nabla\omega(x_0)|K_2 = \zeta K, \tag{4.24}$$

where $K := c(t)K_1 + |\nabla\omega(x_0)|K_2$.

Let $y \in \text{supp}(h_\zeta \circ g^{-t}) \subset B_\zeta(x_0)$. We may assume $\zeta << 1$, which implies $|\nabla\omega(y)| \geq |\nabla\omega(g^t x_0)| - \zeta K > 0$. Let $b(y) := b(g^{-t}y, \xi_0, \xi_0^\perp; t)$ (Note: the parameters ξ_0 and t are fixed). Then we have

$$b(y) = b(y) - \frac{(b(y), \nabla\omega(y))}{|\nabla\omega(y)|^2}\nabla\omega(y) + \frac{(b(y), \nabla\omega(y))}{|\nabla\omega(y)|^2}\nabla\omega(y).$$

From the estimate (4.24) we have

$$\begin{aligned} |(b(y), \nabla\omega(y))| &\leq |(b(y), \nabla\omega(y)) - (b(y), \nabla\omega(g^t x_0))| + |(b(y), \nabla\omega(g^t x_0))| \\ &\leq \zeta K \|b(\cdot)\|_{L^\infty(\mathbb{T}^2)} + |(b(y), \nabla\omega(g^t x_0))|, \end{aligned} \tag{4.25}$$

Let L denote the Lipschitz norm of the function $x \mapsto b(g^t x)$. From our choice of y it follows that

$$|b(y) - b(g^t x_0)| = |b(g^t(g^{-t}y)) - b(g^t x_0)| \leq \zeta L. \tag{4.26}$$

Since $\xi_0 := \frac{\nabla\omega(x_0)}{|\nabla\omega(x_0)|}$ we have

$$(b(g^t x_0), \nabla\omega(g^t x_0)) = (b(x_0, \xi_0, \xi_0^\perp; t), (g_*^{-t}(x_0))^* \xi_0) = 0,$$

which follows from the construction of (BAS), see equation (2.10). Then from estimate (4.26) we have

$$\begin{aligned} |(b(y), \nabla\omega(g^t x_0))| =& |(b(y), \nabla\omega(g^t x_0)) - (b(g^t x_0), \nabla\omega(g^t x_0))| \\ \leq& \zeta L \|\nabla\omega\|_{L^\infty(\mathbb{T}^2)}. \end{aligned} \tag{4.27}$$

Thus from (4.25) and (4.27) we have

$$\begin{aligned} \frac{|(b(y), \nabla\omega(y))|}{|\nabla\omega(y)|} &\leq \frac{\zeta K \|b(\cdot)\|_{L^\infty(\mathbb{T}^2)} + \zeta L \|\nabla\omega\|_{L^\infty(\mathbb{T}^2)}}{|\nabla\omega(y)|} \\ &\leq \frac{\zeta K \|b(\cdot)\|_{L^\infty(\mathbb{T}^2)} + \zeta L \|\nabla\omega\|_{L^\infty(\mathbb{T}^2)}}{|\nabla\omega(g^t x_0)| - \zeta K}. \end{aligned}$$

For any $x \in \mathrm{supp}(h_\zeta \circ g^{-t})$ we define $\eta(x)$ by

$$\eta(y) := b(x) - \frac{(b(x), \nabla\omega(x))}{|\nabla\omega(x)|}, \tag{4.28}$$

Then $h_\zeta(g^{-t}\cdot)\eta(\cdot) \in C^\infty(\mathbb{T}^n)$ and

$$h_\zeta(g^{-t}x)b(x) = h_\zeta \circ g^{-t}\eta(x) + O(\zeta)\frac{h_\zeta(g^{-t}x)\nabla\omega(x)}{|\nabla\omega(x)|},$$

where $O(\zeta)$ is uniform in x and is independent of δ. Since $\|h_\zeta\|_{L^2} = \zeta\|h_0\|_{L^2}$ on $\mathbb{T}^2$, we have

$$\|h_\zeta(g^{-t}\cdot)b(g^{-t}\cdot, \xi_0, \xi_0^\perp; t)e^{ig^{-t}(\cdot)\cdot\xi_0/\delta} - h_\zeta(g^{-t}\cdot)\eta e^{ig^{-t}(\cdot)\cdot\xi_0/\delta}\|_{L^2} = O(\zeta^2). \tag{4.29}$$

Which implies

$$\|h_\zeta(g^{-t}\cdot)b(g^{-t}\cdot, \xi_0, \xi_0^\perp; t)e^{ig^{-t}(\cdot)\cdot\xi_0/\delta} - h_\zeta(g^{-t}\cdot)\eta e^{ig^{-t}(\cdot)\cdot\xi_0/\delta}\|_{F} = O(\zeta^2). \tag{4.30}$$

From the definition of η, it is clear that

$$T\big(h_\zeta(g^{-t}\cdot)\eta e^{ig^{-t}(\cdot)\cdot\xi_0}\big)(x) = h_\zeta(g^{-t}x)e^{ig^{-t}x\cdot\xi_0}\big(\eta\cdot\nabla\omega\big)(x) \equiv 0. \tag{4.31}$$

Hence $h_\zeta(g^{-t}\cdot)\eta e^{ig^{-t}(\cdot)\cdot\xi_0} \in \mathrm{Ker}B$ and we have

$$\|h_\zeta(g^{-t}\cdot)\eta e^{ig^{-t}(\cdot)\cdot\xi_0}\|_F = \|h_\zeta(g^{-t}\cdot)\eta e^{ig^{-t}(\cdot)\cdot\xi_0}\|_{L^2}.$$

Then (4.29) and (4.30) imply that

$$\begin{aligned}&\|h_\zeta(g^{-t}\cdot)b(g^{-t}\cdot, \xi_0, \xi_0^\perp; t)e^{ig^{-t}(\cdot)\cdot\xi_0/\delta}\|_F \qquad (4.32)\\ &\qquad = \|h_\zeta(g^{-t}\cdot)b(g^{-t}\cdot, \xi_0, \xi_0^\perp; t)e^{ig^{-t}(\cdot)\cdot\xi_0/\delta}\|_{L^2} + O(\zeta^2)\end{aligned}$$

Then from (4.23) we have

$$\begin{aligned}\|\mathrm{op}_\varepsilon[a_0]\circ\mathfrak{g}_u^t\phi_{\zeta,\delta}\|_F =& \|h_\zeta(g^{-t}x)b(g^{-t}x,\xi_0,\xi_0^\perp;t)e^{ig^{-t}x\cdot\xi_0/\delta}\|_F+O(\delta)\\ &=\|h_\zeta(g^{-t}x)b(g^{-t}x,\xi_0,\xi_0^\perp;t)e^{ig^{-t}x\cdot\xi_0/\delta}\|_{L^2}+O(\zeta^2)+O(\delta)\\ &=\|\mathrm{op}_\varepsilon[a_0]\circ\mathfrak{g}_u^t\phi_{\zeta,\delta}\|_{L^2}+O(\zeta^2)+O(\delta),\end{aligned}\tag{4.33}$$

where the $O(\zeta^2)$ does not depend on δ. Consider the quotient (4.33) over (4.21) and take the limit as $\delta\to 0$ to get,

$$\|\mathrm{op}_\varepsilon[a_0]\circ\mathfrak{g}_u(t)\|_{\mathcal{L}(F)}+O(\zeta^2)\geq \sup_{\substack{x,\ x_0\in\mathbb{T}^2\\ |\nabla\omega(x_0)|>0\\ \xi_0=\nabla\omega(x_0)/|\nabla\omega(x_0)|}}\frac{\|h_\zeta(g^{-t}x)b(g^{-t}x,\xi_0,\xi_0^\perp;t)\|_{L^2}}{\|h_\zeta\xi_0^\perp\|_{L^2}}.\tag{4.34}$$

For any value of $0<\zeta<1$, $h_\zeta(x_0)=1$, so for fixed $|\xi_0|=1$ we have

$$\lim_{\zeta\to 0}\frac{\|h_\zeta(g^{-t}x)b(g^{-t}x,\xi_0,\xi_0^\perp;t)\|_{L^2}}{\|h_\zeta\xi_0^\perp\|_{L^2}}=|b(x_0,\xi_0,\xi_0^\perp;t)|.$$

Hence, we can take the limit as $\zeta\to 0$ of (4.34) (and use the fact that b is homogeneous of degree 0 in ξ_0) to get

$$\|\mathrm{op}_\varepsilon[a_0]\circ\mathfrak{g}_u(t)\|_{\mathcal{L}(F)}\geq \sup_{\substack{|\nabla\omega(x_0)|>0,\ |b_0|=1\\ b_0\perp\nabla\omega(x_0)}}|b(x_0,\nabla\omega(x_0),b_0;t)|=:\overline{\Theta}_F(t).$$

From (4.15) we have

$$\|G_\varepsilon^s(t)-\mathrm{op}_\varepsilon[a_0]\circ\mathfrak{g}_u(t)\|_{\mathcal{L}(F)}=O(\sqrt{\varepsilon}).$$

Therefore, $\|G_\varepsilon^s(t)\|_{\mathcal{L}(F)}+O(\sqrt{\varepsilon})\geq\overline{\Theta}_F(t)$. □

Definition 8. *Let* $\Theta_F(t):=\max\{\tilde{\Theta}_F(t),\overline{\Theta}_F(t)\}$ *and define* $\mu_{2I},\mu_{2F}\in\mathbb{R}$ *by*

$$\mu_{2I}=lim_{t\to\infty}\frac{1}{t}\log\Theta_I(t),$$
$$\mu_{2F}=lim_{t\to\infty}\frac{1}{t}\log\Theta_F(t).$$

Remark 12. *Because* $\{A_0(x,\xi,t) : (x,\xi) \in T^*(\mathbb{T}^n), t \geq 0\}$ *is a strongly continuous cocyle over the flow* $\{g^t\}_{t\in\mathbb{R}}$*, we have that* $\log \Theta_I(t)$*and* $\log \Theta_F(t)$ *are subadditive, which implies that both limits exists.*

Theorem 4.2.3. *For 2-dimensional flows, we have the following lower bound for the essential spectral radius of our evolution operator restricted to* $\overline{ImB}$*:*

$$e^{\mu_{2I}t} \leq r_{ess}(G(t)|_{\overline{ImB}}).$$

And for 2-dimensional flows we have another lower bound for the essential spectral radius of the evolution operator acting on the factor space:

$$e^{\mu_{2F}t} \leq r_{ess}(G_F(t)),$$

where $G_F(t)$ *denotes* $G(t)$ *on the factor space.*

Proof. The proof for Theorem 4.2.3 is the same as that for Theorem 3.2.1 except that we will use the 2-dimensional propositions from the current section instead of Proposition 3.2.2 (see Remark 9 following the proof of Theorem 3.2.1). To prove $e^{\mu_{2I}t} \leq r_{ess}(G(t)|_{\overline{\mathrm{Im}B}})$ replace Proposition 3.2.2 with Proposition 4.2.1 in the proof of Theorem 3.2.1 for $\overline{\mathrm{Im}B}$. For the factor space estimate, notice that Proposition 4.2.2 implies

$$\|G^s_\varepsilon(t)\|_{\mathcal{L}(F)} + O(\sqrt{\varepsilon}) \geq \Theta_F(t), \tag{4.35}$$

where $\Theta_F(t) := \max\{\tilde{\Theta}_F(t), \overline{\Theta}_F(t)\}$. To prove $e^{\mu_{2F}t} \leq r_{ess}(G_F(t))$, replace Proposition 3.2.2 with the estimate (4.35) above in the proof of Theorem 3.2.1 for the factor space. ∎

Corollary 4.2.4. *For flows in 2D*

$$r_{ess}(G(t)) = \max\{r_{ess}(G_F(t)), r_{ess}(G(t) \mid_{\overline{Im\ B}})\}.$$

Proof. By Proposition 3.2.5 we have

$$\max\{r_{ess}(G_F(t)), r_{ess}(G(t) \mid_{\overline{\text{Im } B}})\} \leq r_{ess}(G(t)).$$

For the other inequality, notice

$$\sup_{\substack{x_0,\ |b_0|=|\xi_0|=1 \\ \xi_0 \perp b_0}} |b(x_0, \xi_0, b_0; t)| = \max\{\Theta_I(t), \tilde{\Theta}_F(t)\}.$$

Hence

$$\lim_{t\to\infty} \frac{1}{t} \log \sup_{\substack{x_0,\ |b_0|=|\xi_0|=1 \\ \xi_0 \perp b_0}} |b(x_0, \xi_0, b_0; t)| \leq \lim_{t\to\infty} \frac{1}{t} \log \max\{\Theta_I(t), \tilde{\Theta}_F(t)\}. \quad (4.36)$$

But the LHS of (4.36) is the Lyapunov-type exponent μ from Theorem 2.2.1, thus we have

$$r_{ess}(G(t)) = e^{\mu t} \leq \max\{e^{\mu_{2I} t}, e^{\mu_{2F} t}\} \leq \max\{r_{ess}(G_F(t)), r_{ess}(G(t) \mid_{\overline{\text{Im } B}})\}.$$

□

4.3 Hyperbolic stagnation points in 2-dimensions

In [6] Friedlander and Vishik demonstrate that for any flow with a hyperbolic stagnation point, there is instability in the essential spectrum. In fact, any instability in the essential spectrum for a 2-dimensional flow is caused by a hyperbolic stagnation point, see [13]. Here we see that for two-dimensional flows where the hyperbolic stagnation point x_s is in the support of the gradient of vorticity, this instability is caused by perturbations in $\overline{\text{Im } B}$ as well as by perturbations in the factor space. At the end of this section we use this idea to demonstrate through an example that our lower bound for $r_{ess}(G_F(t))$ in 3-dimensions may not be sharp.

Suppose the two-dimensional steady flow u has a hyperbolic stagnation point, x_s and that $x_s \in \text{supp}\nabla\omega$. We proved in Proposition 3.3.1 that $\frac{\partial u}{\partial x}(x_s)$ is symmetric. Since we are in 2-dimensional space, it follow that $\frac{\partial u}{\partial x}(x_s)$ has two real eigenvalues and the divergence free condition gives us that the sum of these eigenvalues is 0. Let λ and $-\lambda$ be the eigenvalues of $\frac{\partial u}{\partial x}(x_s)$ associated with the eigenvectors a_+ and a_-, respectively. In [6], the authors demonstrate that (BAS) has a simple solution at the hyperbolic stagnation point:

$$x(t) = x_s$$

$$\xi(t) = a_+ e^{-\lambda t}$$

$$b(t) = a_- e^{\lambda t}.$$

Hence, the Lyapunov-type exponent from Theorem 2.2.1, μ, is positive. Thus from Definitions 1 and 2, we have linear instability.

First we demonstrate that there is a perturbation in $\overline{\text{Im} B}$ that grows exponentially under the linear evolution. Solutions to (BAS), $b(x_0, \xi_0, b_0; t)$ are continuous functions of initial conditions x_0, ξ_0 and b_0, and the continuity is uniform in t on $[0, T]$. So for any $\varepsilon > 0$ there is an $\alpha > 0$ such that if $|x_0 - x_s|, |\xi_0 - a_+|, |b_0 - a_-| \leq \alpha$, then

$$\|b(x_0, \xi_0, b_0; \cdot) - b(x_s, a_+, a_-; \cdot)\|_{L^\infty(0,T)} \leq \varepsilon. \tag{4.37}$$

Since we know $x_s \in \text{supp}\nabla\omega$, we can choose (x_0, ξ_0, b_0) within $\frac{\alpha}{2}$ of (x_s, a_+, a_-) such that $(b_0, \nabla\omega(x_0)) \neq 0$. Now define $h_0 \in C^\infty(\mathbb{T}^2)$ so that $h_0(x_0) = 1$, $\text{supp}h_0$ is contained in a ball of radius $\frac{\alpha}{2}$ centered at x_0 and there is a constant $c_0 > 0$ such that $|(b_0, \nabla\omega(x))| > c_0$ on $\text{supp}h_0$. Then by Lemma 4.1.1 we have that for

$$\phi_\delta := \delta\nabla^\perp(h_0 e^{x\cdot\xi_0/\delta}),$$

$\phi_\delta + r_\delta \in \overline{\text{Im } B}$ where $\|r_\delta\|_{L^2_{sol}} = O(\delta)$. Proposition 3.1.2 along with estimate (4.15) implies

$$G^s_\varepsilon(t)\phi_\delta(x) = h_0(g^{-t}x)b(g^{-t}x, \xi_0, b_0; t)e^{ig^{-t}x\cdot\xi_0} + r_\varepsilon + r_\delta,$$

where $\|r_(\varepsilon)\|_{L^2} = O(\sqrt{\varepsilon})$ and $\|r_(\delta)\|_{L^2} = O(\delta)$. Hence, by the inequality 4.37 above we have that ϕ_δ corresponds to exponential stretching.

The approach to finding exponential growth in the factor space is a bit more delicate because we have to deal with the canonical factor space norm, $\|\cdot\|_F$. We first examine the dynamics of the flow near a stagnation point more closely. A detailed discussion of the dynamics of nonlinear systems near hyperbolic stagnation points (along with the following theorem) can be found in Guckenheimer and Holmes [9].

Theorem 4.3.1 (Stable Manifold Theorem). *Suppose that $\dot{x} = u(x)$ has a hyperbolic fixed point x_s. Then there exists and neighborhood U of x_s with local stable and unstable manifolds,*

$$W^s_{loc}(x_s) := \{x \in U |\ g^t x \to x_s \text{ as } t \to \infty, \text{ and } g^t x \in U \text{ for all } t \geq 0\}$$
$$W^u_{loc}(x_s) := \{x \in U |\ g^t x \to x_s \text{ as } t \to -\infty, \text{ and } g^t x \in U \text{ for all } t \leq 0\}.$$

$W^s_{loc}(x_s), W^u_{loc}(x_s)$ are of the same dimensions as the eigenspaces E^s, E^u of the linearized system and tangent to E^s, E^u at the stagnation point x_s. $W^s_{loc}(x_s), W^u_{loc}(x_s)$ are as smooth as the function u.

Notice that $W^s_{loc}(x_s)$ and $W^u_{loc}(x_s)$ coincide with the paths of the flow through x_s.

First we introduce a notation convention: for any vector v, let $\overline{v} := \frac{v}{|v|}$. Using the continuity of $b(x_0, \xi_0, b_0; t)$ (uniform in t on $(0, T)$) as above, we will choose an point $y_0 \in W^s_{loc}(x_s)$ sufficiently close to x_s to give that $b(y_0, \overline{\nabla\omega}(y_0), \overline{\nabla^\perp\omega}(y_0); t)$ grows exponentially. To justify the existence of such a y_0 we need to begin with $x_0 \in W^s_{loc}(x_s)$ and show that as $s \to \infty$,

$$g^s x_0 \to x_s, \quad \overline{\nabla\omega}(g^s x_0) \to a_+ \text{ and } \overline{\nabla^\perp\omega}(g^s x_0) \to a_-.$$

The first convergence follows from the assumption that $x_0 \in W^s_{loc}(x_s)$. To see the second convergence notice that since vorticity is constant along flow lines, $\overline{\nabla\omega}(g^t x_0)$ is a unit vector perpendicular to $W^s_{loc}(x_s)$ at the point x_0 . Since $W^s_{loc}(x_s)$ is tangent to the eigenspace E^s at the stagnation point, which is spanned by a_-, it follows that a_+ is perpendicular to $W^s_{loc}(x_s)$ at the point x_s. This give us the convergence $\overline{\nabla\omega}(g^s x) \to a_+$. The last convergence follows from a similar argument. See Figure 4.1

Next we must demonstrate that this exponential growth corresponds to growth in the factor space norm. Let $\xi_0 := \overline{\nabla\omega}(x_0)$. Define $h_0 \in C^\infty(\mathbb{T}^2)$ be supported in $B_1(0)$ with $h_0(x_0) = 1$. For $0 < \zeta < 1$ let

$$h_\zeta(x) = h_0(\frac{x - x_0}{\zeta}).$$

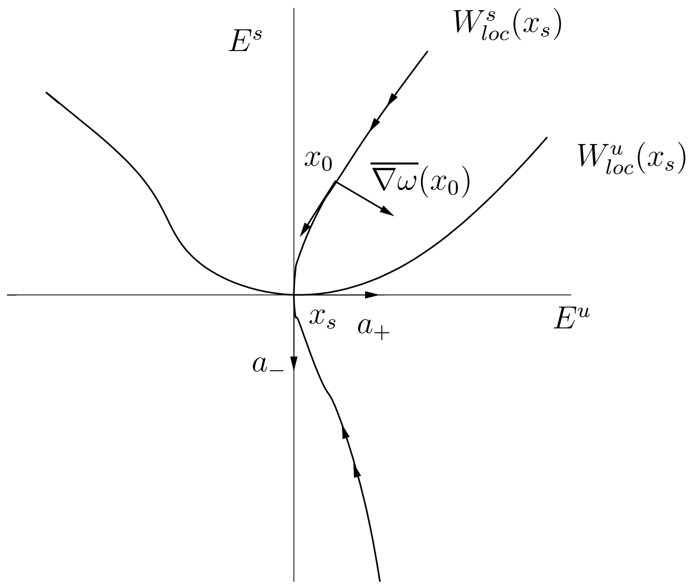

Figure 4.1: Flow dynamics near a 2-dimensional hyperbolic stagnation point.

It follows from Lemma 4.1.2 that for

$$\phi_{\zeta,\delta}(x) := \delta\nabla^\perp(h_\zeta e^{x\cdot\xi_0/\delta}),$$

we have

$$\|\phi_{\zeta,\delta}\|_F = \|\phi_{\zeta,\delta}\|_{L^2_{sol}} + O(\zeta) + O(\delta).$$

We also showed in the proof of Proposition 4.2.2 that

$$\|\mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}^t_u \phi_{\zeta,\delta}\|_F = \|\mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}^t_u \phi_{\zeta,\delta}\|_{L^2} + O(\delta) + O(\zeta).$$

We demonstrated in the proof of Proposition 4.2.2 (ii) that

$$\lim_{\zeta\to 0}\lim_{\delta\to 0} \|\mathrm{op}_\varepsilon[a_0] \circ \mathfrak{g}^t_u \phi_{\zeta,\delta}\|_{L^2} = |b(x_0, \overline{\nabla\omega}(x_0), \overline{\nabla^\perp\omega}(x_0); t)|.$$

Thus we have that the evolution of $\phi_{\zeta,\delta}$ under the linearized flow grows exponentially in the factor space norm.

Finally, we look at an example of a flow that indicates our 3-dimensional lower bound for $r_{ess}(G_F(t))$ is not sharp. Consider the planar 3-dimensional steady flow given by

$$u_1(x) := \sin x_1 \cos x_2 \quad u_2(x) := -\cos x_1 \sin x_2 \quad u_3(x) = 0.$$

The point $x_s = (0,0,0)$ is a hyperbolic stagnation point since

$$\frac{\partial u}{\partial x}(x_s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Clearly, the eigenvalues of $\frac{\partial u}{\partial x}(x_s)$ are ± 1 with corresponding eigenvectors in the first two coordinate directions. This implies that we have some linear instability in the essential spectrum. Theorem 3.2.1 (and Corollary 3.2.4) gives us that this instability corresponds to a perturbation in $\overline{\text{Im}B}$, since $\text{supp}(\omega) = \mathbb{T}^3$ in this example. We remark that if we consider any planar vector field in 3-dimensions, its factor space norm is the same as if we considered the vector field to be in 2-dimensional space. To see this, notice that $\overline{\text{Im}B}$ for the 3-dimensional planar flow is the same as $\overline{\text{Im}B}$ for the 2-dimensional flow: if a is a 3-dimensional vector field, then we have

$$Ba := \pi_{sol}\big(\omega \times a\big).$$

Since u is planar, ω is zero in the first two components and is constant in the 3rd coordinate direction. Hence, Ba is a planar vector field with zero 3rd component and $\overline{\text{Im}B}$ consists only of planar flows corresponding to elements of 2-dimensional $\overline{\text{Im}B}$. Recall

$$\|v\|_F := \inf_{w \in \overline{\text{Im}B}} \|v + w\|_{L^2_{sol}}.$$

Thus, the factor space norm does not depend on the dimension of our vector fields. If we construct a sequence of planar 3-dimensional perturbations just as we constructed $\phi_{\zeta,\delta}$ above, then we can demonstrate the same exponential growth in the factor space subject to the linear evolution associated with u as we would in 2-dimensions. Thus, it is possible for a flow to have vorticity supported in all of $\mathbb{T}^3$ and still have instability in the factor space.

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Vita

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