修 士 論 文 の 和 文 要 旨

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論 文 題 目	Multifold tiling with polyominoes and convex lattice polygons（ポリオミノと格子凸多角形による多層タイル張り）		

要　　旨

平面図形の集合族$\mathcal{T}$が隙間や重なりの無いように平面を充填するとき, $\mathcal{T}$をタイル張りと呼び, $\mathcal{T}$に属する全ての平面図形が互いに合同であるとき，その平面図形をタイルと呼ぶ．本研究ではこれらを拡張した，平面を層数kで充填するk層タイル張りと，それに属するk層タイルを考える．k層タイル張りは，簡潔に言えば，平面図形の無限個の複製が平面上のほとんど全ての点においてちょうどk回重なるように充填するということを意味する．また，（1 層）タイルは任意の正整数kに対して自明にk層タイルであることから，「タイルではないが，$k(\geq 2)$層タイルではある」という性質を持つ平面図形が研究対象となる．この性質を持つ平面図形を非自明なk層タイルと呼ぶ.

k層タイル張り$\mathcal{T}=\{T_1,T_2,T_3,\dots\}$に属する全ての$T_i$が$T_1$を平行移動させたものであるとき，$\mathcal{T}$を$k$層平行移動タイル張りと呼び，$T_1$を$k$層平行移動タイルと呼ぶ．多層平行移動タイル張りに関する様々な研究が既に存在するが，回転移動や対称移動も許した上での多層タイル張りに関する研究は存在しないようである．従って本研究では，そのような非自明な多層タイル張りを考える．本論文では，基本的な平面図形として主にポリオミノと格子凸多角形に着目し，いくつかの事実を明らかにする．具体的には以下の通りである．まず，任意の整数$k\geq 2$に対し，「任意の正整数$h<k$に対してh層タイルではないが，k層タイルではある」という性質を持つポリオミノが存在することを示す．また，任意の整数$k\geq 2$に対し，非自明なk層タイルであるようなポリオミノのうちセル数が最小のものを明らかにする．次に，$k=5$と任意の整数$k\geq 7$に対し，非自明なk層タイルであるような面積kの格子凸多角形が存在することと，$k=2$と$k=3$に対し，そのような格子凸多角形が存在しないことを証明する．

Master Thesis

Multifold tiling with polyominoes and convex lattice polygons

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Multifold tiling with polyominoes and convex lattice polygons

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Abstract

A family of plane shapes $\mathcal{T}$ is called a *tiling* if the shapes in $\mathcal{T}$ cover the whole plane without gaps or overlaps, and if all shapes belonging to $\mathcal{T}$ are congruent each other, then the shape is called a *tile*. As extensions of the conventional tilings and tiles, we study *k-fold tilings*, which cover the plane with multiplicity k, and *k-fold tiles* belonging to it. Intuitively k-fold tilings mean that a family of plane shapes (that are all congruent each other) covers the whole plane such that the shapes overlap exactly k times at almost every point in the plane. Since clearly a (1-fold) tile is a k-fold tile for any positive integer k, we are interested in *nontrivial k-fold tiles*, that is, plane shapes with property "not a tile, but a $k(\geq 2)$-fold tile."

If all shapes in a k-fold tiling $\mathcal{T} = \{T_1, T_2, T_3, \ldots\}$ are translates of T_1, then $\mathcal{T}$ is called a *k-fold translative tiling*, and T_1 is called a *k-fold translative tile*. Although there is various research on multiple translative tilings, there seems to be no research on multiple tilings that also allow rotations and reflections. Therefore, the subjects of our research are such nontrivial multiple tilings. In this thesis, we mainly consider polyominoes and convex lattice polygons as basic plane shapes and present some properties as follows: first, we show that for any integer $k \geq 2$, there exists a polyomino with property "not an h-fold tile for any positive integer $h < k$, but a k-fold tile." We also find for any integer $k \geq 2$, polyominoes with the minimum number of cells among ones that are nontrivial k-fold tiles. Next, we prove that for any integer $k = 5$ or $k \geq 7$, there exists a convex lattice polygon that is a nontrivial k-fold tile whose area is k, and for $k = 2$ or 3, there exists no such convex lattice polygon.

Multifold tiling with polyominoes
and convex lattice polygons

Contents

Chapter 1 Introduction

1.1 What are k-fold tilings?

An infinitely large family of plane shapes $\mathcal{T}$ is called a *tiling* if the shapes in $\mathcal{T}$ cover the whole plane without gaps or overlaps. A tiling $\mathcal{T}$ is said to be *monohedral* [8] if any two shapes belonging to $\mathcal{T}$ are congruent. If a tiling $\mathcal{T}$ is monohedral, the unique shape in the tiling $\mathcal{T}$ is called a *tile*. In this thesis, we consider only monohedral tilings. There is a great deal of existing research on monohedral tilings (see Section 1.2), and they still have been actively studied. As extensions of the conventional tilings and tiles, we study *k-fold tilings*, which cover the plane with multiplicity k, and *k-fold tiles* belonging to it. Although the strict definition of a k-fold tile will be described later, intuitively it means that an infinitely large family of copies of the shape (translations, rotations, and reflections are allowed) covers the whole plane such that the shapes overlap exactly k times at almost every point in it. As an example, we now consider the regular hexagon chipped a right triangle as shown in Fig. 1, which is an 11-fold tile. Because by overlapping 12 appropriately rotated or reflexed copies of it, we can obtain the regular hexagon with multiplicity 11, and by arranging them as shown in Fig. 2, we can obtain an 11-fold tiling.

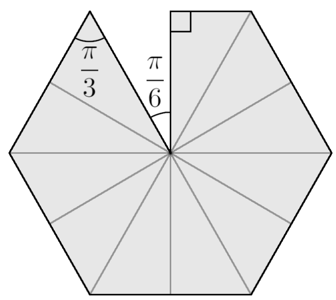


Figure 1: An 11-fold tile of the chipped regular hexagon

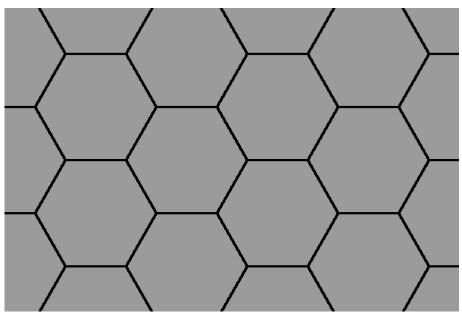


Figure 2: An 11-fold tiling with the regular hexagons with multiplicity 11

Note that this is not a tile (clearly, we can not fill up the chipped $\pi/6$ part in any way). Since we can obtain a k-fold tiling by piling up k sheets of a tiling, it is trivial to consider constructing a k-fold tiling with (1-fold) tiles. Hence we are interested in plane shapes with the property "not a tile, but a $k(\geq 2)$-fold tile." We call a plane shape having this property a *nontrivial k-fold tile*. That is, the chipped regular hexagon in Fig. 1 is a nontrivial 11-fold tile.

1.2 Background

We show some known results on simple ($k = 1$) tiling as follows:

Theorem 1 ([6])**.** *Any triangle and any quadrilateral is a tile.*

Theorem 2 ([20])**.** *For any integer $n \geq 7$, any convex n-gon is not a tile.*

Theorem 3 ([12, 26])**.** *Let the lengths of the sides of a convex hexagon H be denoted by a, b, $\ldots$, f, consecutively, and its angles between a and b, b and c, $\ldots$, f and a by A, B, $\ldots$, F, respectively. H is a tile if and only if it is at least one of the following three types.*

$$\begin{array}{ll} \text{(i)} & A + B + C = 2\pi,\ a = d, \\ \text{(ii)} & A + B + D = 2\pi,\ a = d,\ c = e, \\ \text{(iii)} & A = C = E = \dfrac{2}{3}\pi,\ a = b,\ c = d,\ e = f. \end{array}$$

Theorem 4 ([12, 17, 26, 27, 29])**.** *Let the lengths of the sides of a convex pentagon P be denoted by a, b, $\ldots$, e, consecutively, and its angles between a and b, b and c, $\ldots$, e and a by A, B, $\ldots$, E, respectively. P is a tile if it is at least one of the following 15 types.*

$$\begin{array}{ll} \text{(i)} & A + B + C = 2\pi, \\ \text{(ii)} & A + B + D = 2\pi,\ a = d, \\ \text{(iii)} & A = C = D = \dfrac{2}{3}\pi,\ a = b,\ d = c + e, \\ \text{(iv)} & A = C = \dfrac{\pi}{2},\ a = b,\ c = d, \\ \text{(v)} & A = \dfrac{\pi}{3},\ C = \dfrac{2}{3}\pi,\ a = b,\ c = d, \\ \text{(vi)} & A + B + D = 2\pi,\ A = 2C,\ a = b = e,\ c = d, \end{array}$$

(vii) $2B + C = 2D + A = 2\pi, \ a = b = c = d,$

(viii) $2A + B = 2D + C = 2\pi, \ a = b = c = d,$

(ix) $2E + B = 2D + C = 2\pi, \ a = b = c = d,$

(x) $E = \frac{\pi}{2}, \ A + D = 2B - D = \pi, \ 2C + D = 2\pi, \ a = e = b + d,$

(xi) $A = \frac{\pi}{2}, \ C + E = \pi, \ 2B + C = 2\pi, \ d = e = 2a + c,$

(xii) $A = \frac{\pi}{2}, \ C + E = \pi, \ 2B + C = 2\pi, \ 2a = d = c + e,$

(xiii) $A = C = \frac{\pi}{2}, \ 2B + D = 2E + D = 2\pi, \ e = 2c = 2d,$

(xiv) $A = \frac{\pi}{2}, \ 2B + C = 2\pi, \ C + E = \pi, \ 2a = 2c = d = e,$

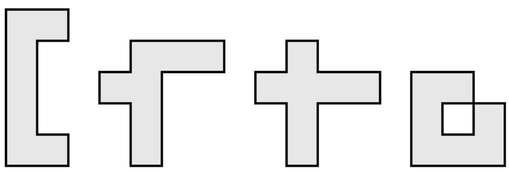
(xv) $A = \frac{\pi}{3}, \ B = \frac{3}{4}\pi, \ C = \frac{7}{12}\pi, \ D = \frac{\pi}{2}, \ E = \frac{5}{6}\pi, \ a = 2b = 2d = 2e.$

In 2017, Rao [25] claimed that Theorem 4 is true even if "if" is replaced with "only if," that is, there are only the 15 types of tiles of convex pentagons mentioned in Theorem 4. This was shown by using a computer, and it seems that it has not been fully verified at this time.

On polyominoes, the following facts are known.

Theorem 5 ([6])**.** *For any positive integer $n \leq 6$, any n-omino is a tile.*

Theorem 6 ([6])**.** *A heptomino is a tile if and only if it is not any of those four listed in Fig. 3.*



C7 F7 X7 G7

Figure 3: The heptominoes that are not tiles

If all shapes in a k-fold tiling $\mathcal{T} = \{T_1, T_2, T_3, \ldots\}$ are translates of T_1, then $\mathcal{T}$ is called a *k-fold translative tiling*, and T_1 is called a *k-fold translative tile*. In particular, if the translative vectors of T_i form a lattice $\Lambda = \{a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 \mid a_1, a_2 \in \mathbb{Z}\}$ in $\mathbb{R}^2$ (where $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^2$ are linearly independent vectors), then $\mathcal{T}$ is called a *k-fold lattice tiling*, and T_1 is called a *k-fold lattice tile*. These terms are defined in [31, 35].

The origin of the study of multiple tilings is the one by Furtwängler [5] in 1936. He considered trivial multiple lattice tilings in the Euclidean space as a generalization of what is called Minkowski's conjecture (see Zong's survey [33]). As far as we know, Marley [18, 19] first did the study focusing on nontrivial multiple translative tilings in $\mathbb{R}^2$. He discovered nontrivial 5-, 6-, and 35-fold (lattice) tiles of the convex 8-, 10-, and 12-gons, respectively. Recently, Yang and Zong [32] gave a characterization of all convex k-fold translative tiles in $\mathbb{R}^2$ for any $k = 2, 3, 4, 5$. Specifically, for any $k = 2, 3, 4$, they are classified as either parallelograms or centrally symmetric hexagons (this is also true for $k = 1$ [11]), and for $k = 5$, they are classified as any one of parallelograms, centrally symmetric hexagons, two classes of octagons, and one class of decagons.

Multiple coverings and *multiple packings*, which are similar notions to multiple tilings, have already been proposed [8, 21, 22]. Intuitively, a *k-fold covering* is defined as a family of plane shapes that covers the whole plane with overlapping at least k times at any point in it, and a *k-fold packing* is defined as a family of plane shapes that covers the whole plane with overlapping at most k times at almost every point in it. Thus a k-fold tiling considered in this thesis can be regarded as a k-fold covering and a k-fold packing. Pach and Tóth [22], Aloupis et al. [1], and Gibson and Varadarajan [7] showed upper bounds of $k = k(P, c)$ such that any k-fold covering with translates of a plane shape P can be decomposed into c (1-fold) coverings. Their results can be applied to a practical problem, the *sensor cover problem* (see [3]), even though decomposition of multiple coverings was initially proposed by Pach [21] motivated only by theoretical interest. Our motivation to study multiple tilings is also theoretical interest, and we hope their applications will be found later.

Although there is various research on multiple translative tilings in the Euclidean space other than those mentioned above (for example, [2, 9, 10, 13, 15, 16, 31]), there seems to be no research on multiple tilings that also allow rotations and reflections. Therefore, the subjects of our research are such nontrivial multiple tilings. In this thesis, we mainly consider polyominoes and convex unit-lattice polygons as basic plane shapes and present some properties as follows: first, we show that for any integer $k \geq 2$, there is a polyomino whose minimum tile-fold number is k. Second, we find for any integer $k \geq 2$, polyominoes with the minimum number of cells among ones that are nontrivial k-fold tiles. Last, we prove that for any integer $k = 5$ or $k \geq 7$, there is a convex unit-lattice polygon that is a nontrivial k-fold tile whose area is k, and for $k = 2$ or 3, there is no such convex unit-lattice polygon. We also find that for $k = 4$, such a convex unit-lattice polygon must be one of certain pentagons, if any.

Chapter 2 Preliminaries

Let $\mathbb{N}^+$ be the set of positive integers and $\mathbb{N}_0^+$ be the set of nonnegative integers.

2.1 k-fold tiles

We give the strict definition of k-fold tiles.

Definition 1. Let $\mathcal{T}$ be an indexed family of T_i: $\{T_i \mid i \in \mathbb{N}^+\}$ where T_i is a shape constituted by a closed and bounded set on the Euclidean plane $\mathbb{R}^2$, and for any $i, j \in \mathbb{N}^+$, T_i and T_j are congruent. $\mathcal{T}$ is called a *k-fold tiling* if for any point $(x, y) \in \mathbb{R}^2$ that is not included in the boundary of any T_i, there exist exactly $k \in \mathbb{N}^+$ distinct i such that $(x, y) \in T_i$. A 1-fold tiling may be simply called a *tiling*.

Definition 2. The shape belonging to a k-fold tiling is called a *k-fold tile*. A 1-fold tile may be simply called a *tile*.

Hereafter, we simply refer to the Euclidean plane as the plane.

Definition 3. If a plane shape P is a $k(\in \mathbb{N}^+)$-fold tile, then k is a *tile-fold number* of P. The set of tile-fold numbers of P is denoted by $\mathrm{TFN}(P)$. If an integer k satisfies that $k \in \mathrm{TFN}(P)$ and $h \notin \mathrm{TFN}(P)$ for every positive integer $h < k$, then we call k the *minimum tile-fold number* of P, and it is denoted by $\tau^\bullet(P)$ [35].

The following facts are trivial.

Observation 1. *For any plane shape P and any $h, k \in \mathbb{N}^+$, if $h, k \in \mathrm{TFN}(P)$, then $h + k \in \mathrm{TFN}(P)$.*

Observation 2. *For any plane shape P and any $k \in \mathbb{N}^+$, if $k \in \mathrm{TFN}(P)$, then for any $\ell \in \mathbb{N}^+$, $k\ell \in \mathrm{TFN}(P)$.*

The following lemma holds as an extension of the above observations.

Lemma 1. *For any plane shape P and any coprime integers $h, k \geq 2$, if $h, k \in \mathrm{TFN}(P)$, then for any integer $\ell \geq (h-1)(k-1)$, $\ell \in \mathrm{TFN}(P)$.*

Lemma 1 is derived from the following lemma.

Lemma 2 ([28])**.** *For any coprime $a, b \in \mathbb{N}^+$ and any integer $n \geq (a-1)(b-1)$, there exists $x, y \in \mathbb{N}_0^+$ such that $n = ax + by$.*

Proof of Lemma 1. Clear from Observations 1, 2, and Lemma 2. $\square$

2.2 Polyominoes

A polyomino is a plane shape formed by joining one or more congruent squares edge to edge intuitively. The strict definition is shown as follows.

Definition 4 ([8])**.** A closed unit square on the plane with its sides parallel to the coordinate axes and with its center at $(u, v) \in \mathbb{Z}^2$ is called a *cell*. The graph $G = (V, E)$ with a vertex set $V \subset \mathbb{Z}^2$ and an edge set $E = \{\{(u, v), (r, s)\} \mid (u, v), (r, s) \in V,\ |u-r|+|v-s| = 1\}$ is called a *square-grid graph*. If the induced subgraph of the square-grid graph with a finite set $V' \subseteq V$ is connected, then a plane shape that is a union of cells corresponding to all vertices $(u, v) \in V'$ is called a *polyomino*. A polyomino with exactly n cells is called an *n-omino*.

For example, the four shapes listed in Fig. 4 are all polyominoes.

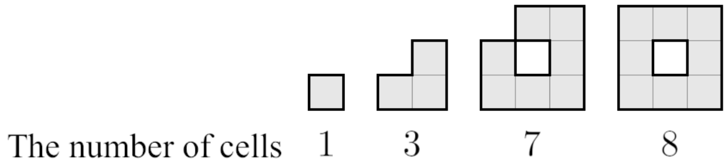


Figure 4: Examples of polyominoes

We can prove the following lemma.

Lemma 3. *For any positive integer k, any k-omino is a k-fold lattice tile.*

Proof. Let us divide the plane into the cells by the unit grid. Along them, we arbitrarily arrange a k-omino P on the plane, and consider a family of k-ominoes $\{P + z \mid z \in \mathbb{Z}^2\}$. Then, each of the k cells composing the k-omino overlaps one of the cells on the plane exactly once. Since this happens in any cell on the plane, k-ominoes cover the plane with multiplicity k. $\square$

2.3 Unit-lattice polygons

Definition 5. A simple polygon whose all vertices lie in $\mathbb{Z}^2$ is called a *unit-lattice polygon*.

For example, the four shapes listed in Fig. 5 are all unit-lattice polygons.

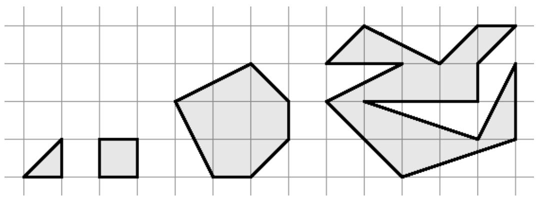


Figure 5: Examples of unit-lattice polygons

2.4 Fold bands

We introduce the following notion of *fold bands*, which will be useful later in some discussions.

Definition 6. Consider $n \in \mathbb{N}^+$ and $m_1, m_2, \ldots, m_n \in \mathbb{N}_0^+$. A linear filling with multiplicity m_1 and the width 1 as shown in Fig. 6 is called an *m_1-fold band*. A series of m_1, m_2, $\ldots$, m_n-fold bands as shown in Fig. 7 is called an *$(m_1, m_2, \ldots, m_n)$-fold band*. If $m_1 = m_2 = \cdots = m_n = m$, then an $(m_1, m_2, \ldots, m_n)$-fold band may be also called an *$n \times m$-fold band*. If n is clear from the context, an $n \times m$-fold band may be called an *m-fold band* for simplicity.

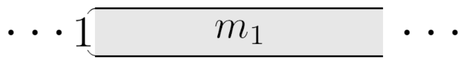


Figure 6: An m_1-fold band

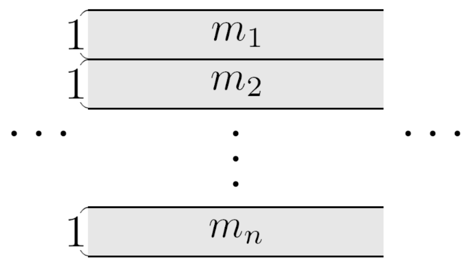


Figure 7: An $(m_1, m_2, \ldots, m_n)$-fold band

Chapter 3 Multifold tiles on polyominoes

In this chapter, we consider nontrivial k-fold tiles on polyominoes.

3.1 The minimum tile-fold number of polyominoes

A plane shape shown in Fig. 8 is called a *holed-p-I* (an analogous shape is presented in [19], although only multiple translative tilings are considered there), where p is an integer greater than or equal to 2. A closed curve composed of a rectilinear polygonal line AB protruding from the left side and a line segment AB (and its interior) is called a *bump part*, and a closed curve composed of a rectilinear polygonal line CD congruent to AB and a line segment CD is called a *hole part*. We assume that the bump part (and the hole part) does not have any line or rotational symmetry. We also assume that the polygonal line AB can be exactly overlapped with the polygonal line CD by translating it in the horizontal direction. As long as these conditions are all satisfied, the shape of the bump part (and the hole part) can be arbitrary. Note that if the length of every edge is rational, we can regard it as a polyomino (by changing the unit of length).

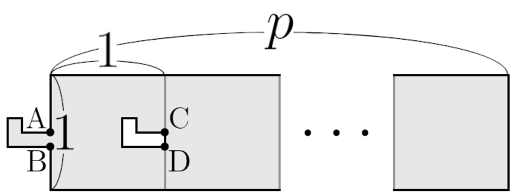


Figure 8: A holed-p-I

Theorem 7. *For any integer $k \geq 2$, there exists a polyomino P that satisfies $\tau^{\bullet}(P) = k$.*

To prove this theorem, we introduce a holed-p-I shown in Fig. 9, which is a polyomino. We prepare some lemmas before showing the proof. Note that $a = 2^{2p+1} - 2$. From here to Fig. 18, let p be an arbitrary integer greater than or equal to 2.

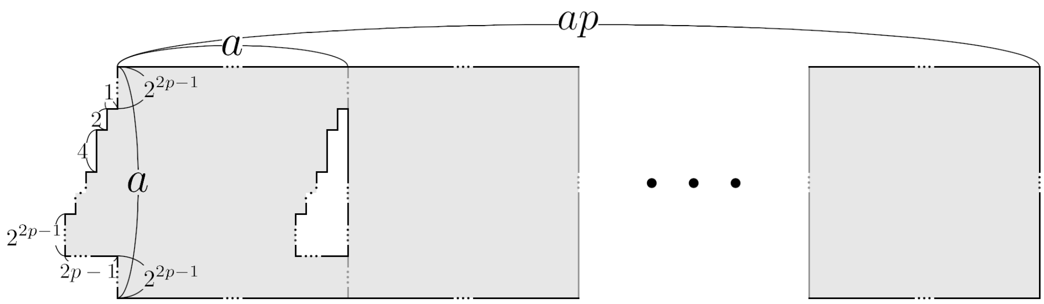


Figure 9: The holed-p-I in the proof of Theorem 7

Fact 1. *Consider any cell edge e on the boundary of the holed-p-I. Then there is a square with the side length of 2^{2p-1} that is contained in the holed-p-I and one of whose side includes e.*

Proof. Clear from Fig. 9. □

We positively orient the boundary of the holed-p-I as shown in Fig. 10. The red-colored edges shown in Fig. 10 are denoted by $e_1, e_2, \ldots, e_{2p-1}$ in order from the top, and we call them *red edges*. For any red edge e_i ($1 \leq i \leq 2p-1$), the holed-p-I contains a cell adjacent to e_i from the left but no cell adjacent to e_i from the right, and hence there is a difference in the multiplicity 1 between the left and right of e_i. We call this a *gap* of (a red edge) e_i.

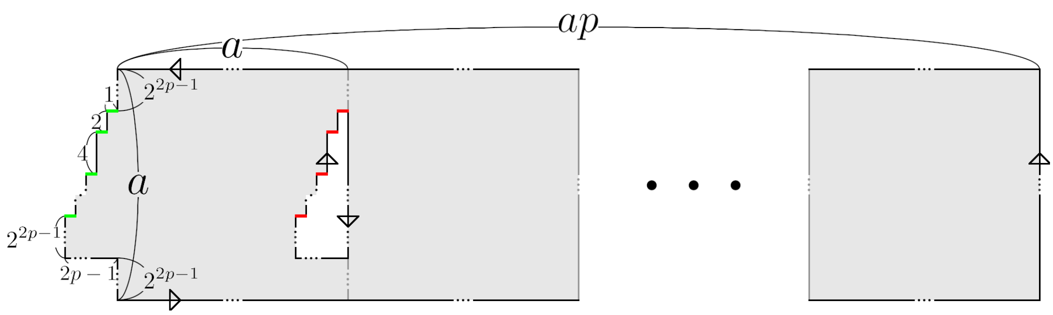


Figure 10: The oriented boundary and gaps of the red edges

Lemma 4. *Consider two arbitrary distinct red edges of the holed-p-I. If both of the two gaps of these edges are eliminated by using only one piece of the holed-*

p-I, then the hole part of the first piece is completely filled with the bump part of the second piece.

Proof. The positional relationship of arbitrarily chosen two distinct red edges e_i and e_j $(1 \leq i < j \leq 2p-1)$ is as shown in Fig. 11, and a pair of edges that eliminates these two gaps must be as shown in Fig. 12. By seeing Figs. 10 and 13 (the latter is the reflection of Fig. 10), we can observe that there is no pair of edges with the positional relationship in Fig. 12 except for the $2p-1$ green-colored edges shown in Fig. 10 (note that $2^j - 2^i$ is even and $2 \leq 2^j - 2^i \leq 2^{2p-1} - 2$).

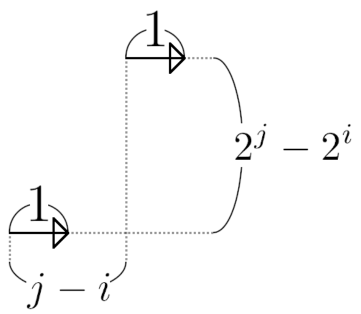


Figure 11: The positional relationship of two red edges

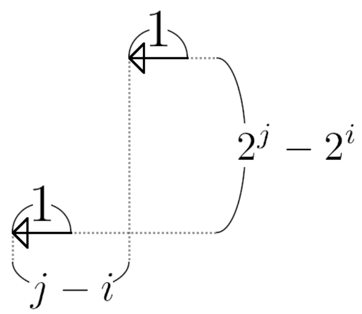


Figure 12: The positional relationship of two edges that eliminate gaps of edges in Fig. 11

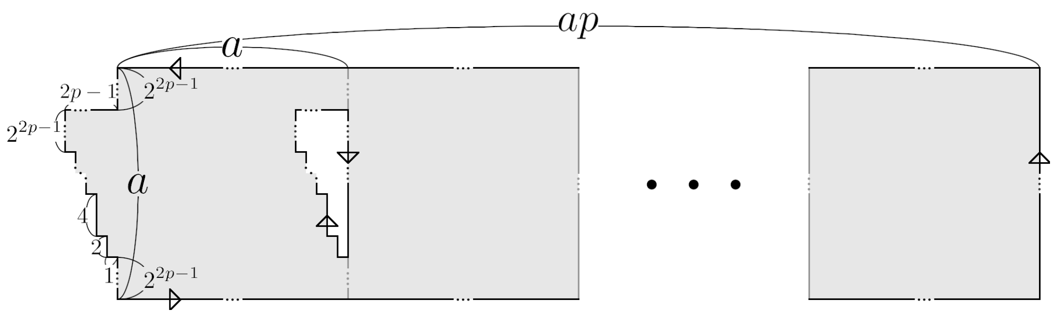


Figure 13: The reflection of Fig. 10

Now only the $2p-1$ green-colored edges in Fig. 10 leave the possibility. Let them be denoted as e'_1, e'_2, ..., e'_{2p-1} in the same way as the red edges, and we call them *green edges*. A pair of edges on a piece that simultaneously eliminates gaps of two red edges e_i and e_j is denoted by $e'_{i'}$ and $e'_{j'}$ ($1 \le i' < j' \le 2p-1$), respectively, and then the following equation holds:

$$2^j - 2^i = 2^{j'} - 2^{i'}. \tag{1}$$

If we assume that $i < i'$, then since $j-i$, $j'-i$, and $i'-i$ are all positive integers, $2^i(2^{j-i} - 2^{j'-i} + 2^{i'-i} - 1) \neq 0$, and this contradicts (1). From symmetry, if we assume that $i' < i$, then it also contradicts (1), and hence $i = i'$. From (1) it follows that $2^j = 2^{j'}$ and clearly $j = j'$. Therefore, e'_i and e'_j must be arranged correspondingly to e_i and e_j, respectively, which proves the statement. □

Lemma 5. *If an $m(\in \mathbb{N}^+)$-fold tiling $\mathcal{T}$ by the holed-p-I includes a piece whose hole part is not completely filled with a bump part of any piece, then there is a region where at least p pieces overlap.*

Proof. Let T_1 be the piece, whose hole part is not completely filled with a bump part of any piece. Assume that T_1 is placed on the plane as shown in Fig. 14. Let c_0 and c_1 be the yellow-colored and blue-colored cells shown in Fig. 14, respectively. We show that at least p pieces overlap at c_0 or c_1.

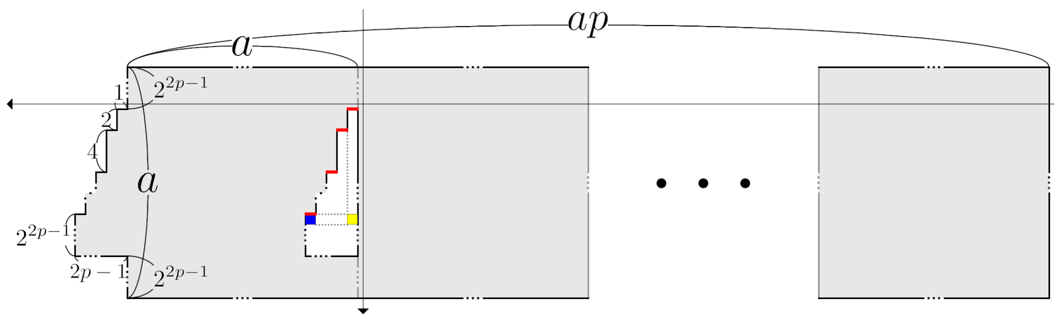


Figure 14: T_1, c_0, and c_1

Based on T_1, we set the 2-dimensional Cartesian coordinate system with the x-axis in the horizontally leftward direction and the y-axis in the vertically downward direction such that the center of the cell with the red edge e_1 as the upper horizontal side is located at $(1, 1)$ as shown in Fig. 14. Let a cell whose center coordinates are $(i, j) \in \mathbb{Z}^2$ be denoted by *cell* (i, j). According to this system, c_0 and c_1 are represented by $(1, 2^{2p-1} - 1)$ and $(2p - 1, 2^{2p-1} - 1)$, respectively. To construct an m-fold tiling that includes T_1, it is necessary to eliminate all gaps of the $2p - 1$ red edges of T_1. However, from the assumption, the hole part is not completely filled with a bump part of any piece. From this and Lemma 4 it follows that at most one of the gaps of the red edges of T_1 can be eliminated by a piece. Suppose that another piece, T_2, is arranged such that it eliminates the gap of any red edge e_i $(1 \leq i \leq 2p - 1)$ of T_1. Then, from Fact 1, it clearly covers at least one of c_0 or c_1. Since T_1 has $2p - 1$ red edges, at least p pieces cover c_0 or c_1. □

Proof of Theorem 7. Consider the holed-p-I shown in Fig. 9 and an $m (\in \mathbb{N}^+)$-fold tiling by it. If there is a piece whose hole part is not completely filled with a bump part of any piece, then from Lemma 5, $m \geq p$. Thus we can assume that the hole part of any piece is completely filled with the bump part of another piece. Then, they are infinitely overlapped in the horizontal direction as shown in Fig. 15, and this is a p-fold band. Hence the holed-p-I is also not an h-fold tile for any positive integer $h < p$ in this case. We can obtain a $(\ldots, p, p, p, \ldots)$-fold band, or a p-fold tiling by arranging p-fold bands as shown in Fig. 16.

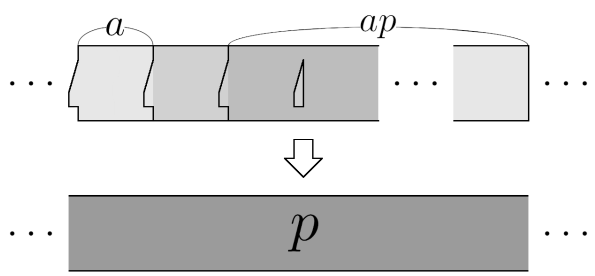


Figure 15: A p-fold band with the holed-p-I (bump parts and hole parts are approximated by the trapezoids)

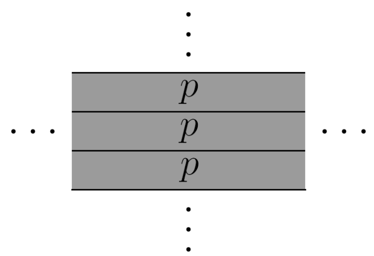


Figure 16: A p-fold tiling with p-fold bands

From the discussions above, for any integer $k \geq 2$, the holed-k-I shown in Fig. 9 satisfies the statement. □

If we cut Fig. 16 with the vertical line, the cross-section of it looks as Fig. 17 from the side. We call this a *cross-section representation of fold bands* (or *a k-fold tiling*).

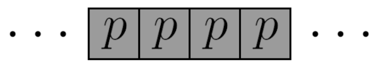


Figure 17: The cross-section representation of a p-fold tiling with p-fold bands

For a plane shape P, let $\tau(P)$ and $\tau^*(P)$ be the minimum integer k such that P is a k-fold translative tile and is a k-fold lattice tile, respectively, as with $\tau^\bullet(P)$. For convenience, we define $\tau^\bullet(P) = \infty$, $\tau(P) = \infty$, and $\tau^*(P) = \infty$ if P is not any multiple, multiple translative, and multiple lattice tile, respectively. For any plane shape P, the following inequality clearly holds:

$$\tau^\bullet(P) \leq \tau(P) \leq \tau^*(P). \tag{2}$$

These definitions and the inequality (2) are given in [35].

Corollary 1. *For any integer $k \geq 2$, there exists a polyomino P that satisfies $\tau^\bullet(P) = \tau(P) = \tau^*(P) = k$.*

Proof. Clear from the inequality (2), Theorem 7, and that the way of the multiple tiling shown in the proof of Theorem 7 is a multiple lattice tiling. □

Note that one can show Theorem 7 by using a polyomino with no hole. For example, let us construct the "indented"-p-I by cutting the $(2p - 1) \times 2^{2p-1}$ rectangle under the hole part out of the holed-p-I and attaching it to under the bump part as shown in Fig. 18. This change does not affect the proof of Theorem 7.

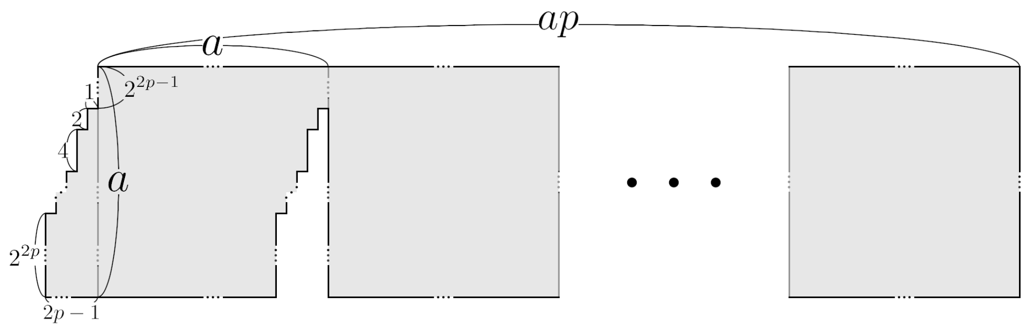


Figure 18: The "indented"-p-I

Theorem 8 can be proven by using another way: it is to use *indented-*(p,q)*-L* defined in the author's graduation thesis [36] and *indented-J* newly defined in this thesis. The definitions of these shapes are shown below.

A plane shape shown in Fig. 19 is called an *indented-*(p,q)*-L*, where both p and q are integers greater than or equal to 2. A closed curve composed of a rectilinear polygonal line AB protruding from the left side and a line segment AB (and its interior) is called a *bump part*, and a closed curve composed of a rectilinear polygonal line CD congruent to AB and a line segment CD is called a *dent part*. As with a holed-p-I, we assume that the bump part (and the dent part) does not have any line or rotational symmetry, and that the polygonal line AB can be exactly overlapped with the polygonal line CD by translating it in the horizontal direction. As long as these conditions are all satisfied, the shape of the bump part (and the dent part) can be arbitrary. Note that if the lengths of every edge is rational, we can regard it as a polyomino (by changing the unit of length). In addition, a plane shape shown in Fig. 20, which is similar to an indented-(p,q)-L, is called an *indented-J*.

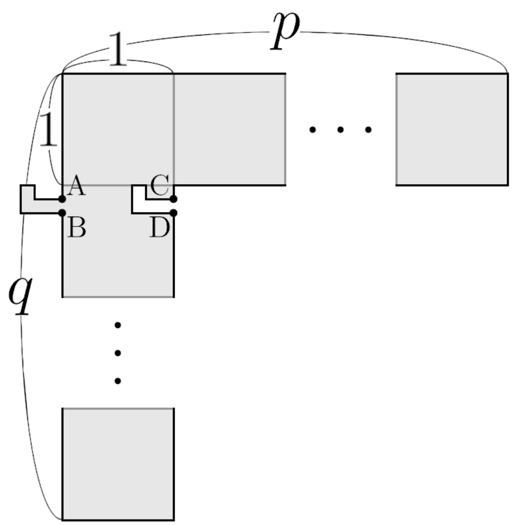


Figure 19: An indented-(p, q)-L

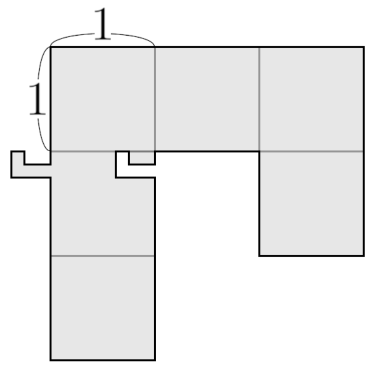


Figure 20: An indented-J

To show another proof of Theorem 7, we introduce an indented-$(p, 2)$-L shown in Fig. 21 and an indented-J shown in Fig. 22 (the bump and dent parts of each of shapes are approximated by the trapezoids), which are both polyominoes. We prepare some lemmas before showing the proof. The bump and dent parts of each of shapes are similar to the bump and hole parts of the holed-p-I shown in Fig. 9, respectively, where $b = 2^{2p+3} - 2$ and $c = 2^7 - 2$. From here to the second proof of Theorem 7, let $p \geq 2$ be an arbitrary positive integer greater than or equal to 2.

Lemma 6. *If an $m(\in \mathbb{N}^+)$-fold tiling $\mathcal{T}$ by the indented-$(p, 2)$-L includes a piece whose dent part is not completely filled with a bump part of any piece, then there is a region where at least $p + 1$ pieces overlap.*

Lemma 7. *If an $m(\in \mathbb{N}^+)$-fold tiling $\mathcal{T}$ by the indented-J includes a piece whose dent part is not completely filled with a bump part of any piece, then there is a region where at least three pieces overlap.*

The proofs of Lemmas 6 and 7 are omitted since they can be done in the same logic of the proof of Lemma 5.

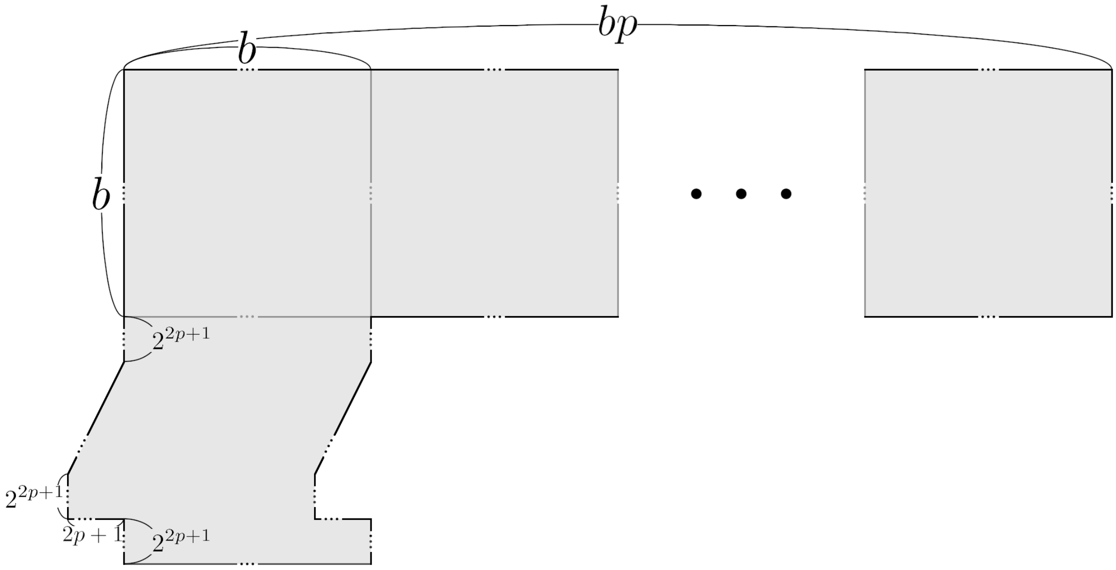


Figure 21: The indented-$(p, 2)$-L in the second proof of Theorem 7

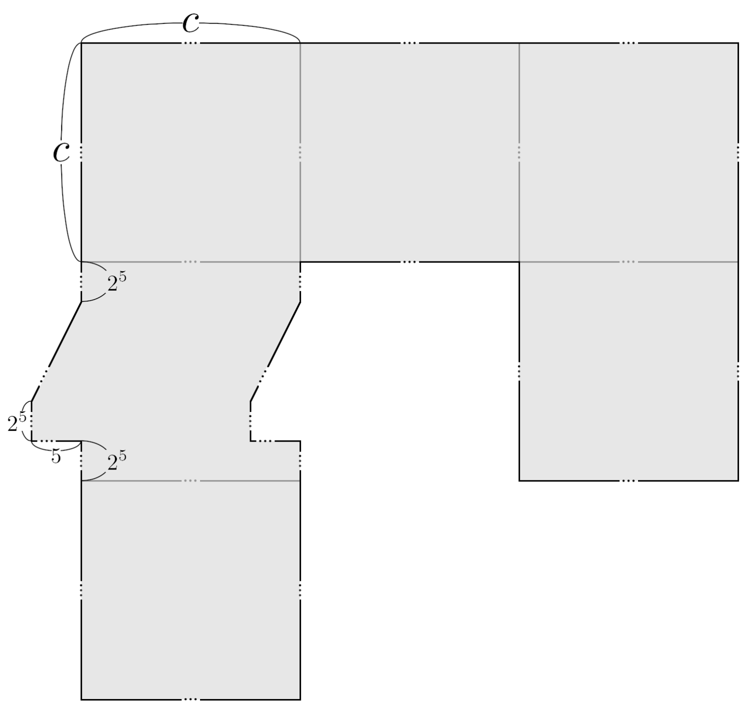


Figure 22: The indented-J in the second proof of Theorem 7

Another Proof of Theorem 7. We show that the indented-J shown in Fig. 22 is the desired polyomino for $k = 3$ and the indented-$(p,2)$-L shown in Fig. 21 is the desired polyomino for $k = p = 2$ and $k = p + 1 \geq 4$ as follows. Let $\mathcal{T}$ and $\mathcal{T}'$ be an $m(\in \mathbb{N}^+)$-fold tiling by the indented-$(p,2)$-L and an $m'(\in \mathbb{N}^+)$-fold tiling by the indented-J, respectively. In the m-fold tiling $\mathcal{T}$, if there is a piece whose dent part is not completely filled with a bump part of any piece, then from Lemma 6, $m \geq p + 1$. Similarly, from Lemma 7, $m' \geq 3$. Thus we can assume that the dent part of any piece is completely filled with the bump part of another piece. We consider following three cases: (i) the indented-$(2,2)$-L, (ii) the indented-J, and (iii) the indented-$(p,2)$-L ($p \geq 3$).

(i) They are infinitely overlapped in the horizontal direction as shown in Fig. 23, and this is a $(2,1)$-fold band.

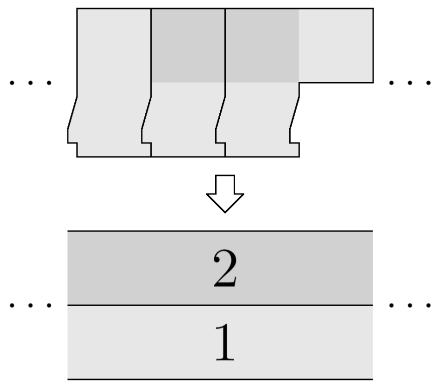


Figure 23: A $(2,1)$-fold band with the indented-$(2,2)$-L

Thus the indented-$(2,2)$-L is also not a tile in this case. We can obtain a $(2,1+1,2)$-fold band by combining two $(2,1)$-fold bands and a 2-fold tiling by arranging an infinite number of copies of it. The cross-section of it is shown in Fig. 24.

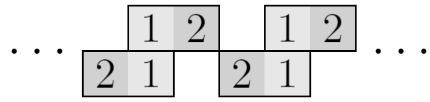


Figure 24: The cross-section representation of a 2-fold tiling with $(2, 1)$-fold bands

(ii) They are infinitely overlapped in the horizontal direction as shown in Fig. 25, and this is a $(3, 2, 1)$-fold band.

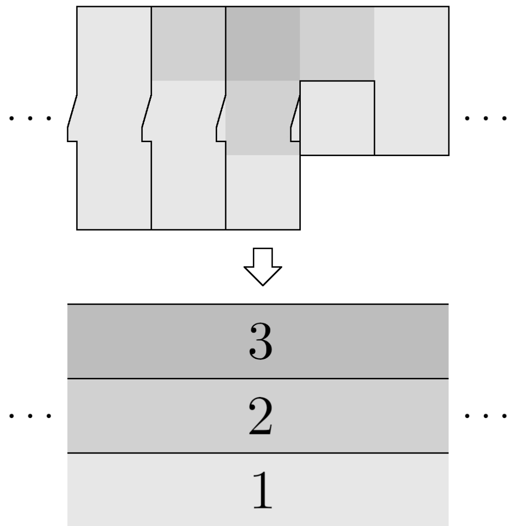


Figure 25: A $(3, 2, 1)$-fold band with the indented-J

Hence the indented-J is also not an h-fold tile for any positive integer $h < 3$ in this case. We can obtain a $(3, 2 + 1, 1 + 2, 3)$-fold band by combining two $(3, 2, 1)$-fold bands and a 3-fold tiling by arranging an infinite number of copies of it. The cross-section of it is shown in Fig. 26.

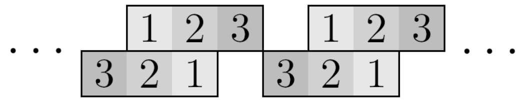


Figure 26: The cross-section representation of a 3-fold tiling with $(3, 2, 1)$-fold bands

(iii) They are infinitely overlapped in the horizontal direction as shown in Fig. 27, and this is a $(p, 1)$-fold band.

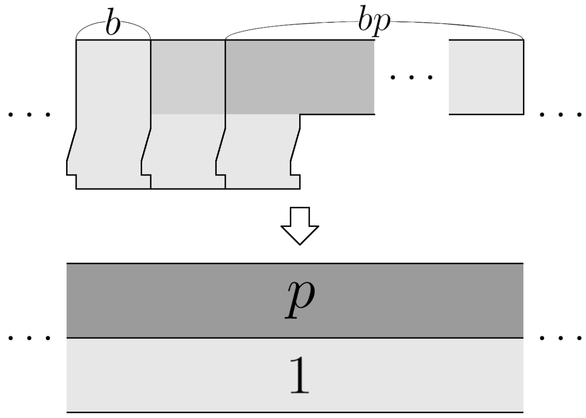


Figure 27: A $(p, 1)$-fold band with the indented-$(p, 2)$-L

From $p \geq 3$, it is clearly impossible to construct a p-fold tiling from $(p, 1)$-fold bands. Hence the indented-(p,2)-L is also not a h-fold tile for any positive integer $h < p + 1$ in this case. We can obtain a $(p + 1, 1 + p)$-fold band by combining two $(p, 1)$-fold bands and a $(p+1)$-fold tiling by arranging an infinite number of copies of it. The cross-section of it is shown in Fig. 28.

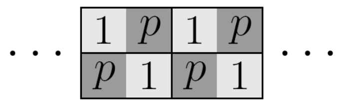


Figure 28: The cross-section representation of a $(p+1)$-fold tiling with $(p, 1)$-fold bands

From the above discussions, the indented-$(2, 2)$-L shown in Fig. 21, the indented-J shown in Fig. 22, and the indented-$(k - 1, 2)$-L shown in Fig. 21 satisfy the statement for $k = 2$, $k = 3$, and $k \geq 4$, respectively. $\square$

As a generalization of Theorem 7, we can consider the following problem.

Open Problem 1. For any integer $k \geq 2$, is there a polygon P that satisfies $\mathrm{TFN}(P) = \{k\ell \mid \ell \in \mathbb{N}^+\}$?

A solution to Open Problem 1 has not been found for any $k \geq 2$. However, we found that for any integer $p \geq 2$ and any indented-$(p, 2)$-L P, P is not a solution to Open Problem 1 for any $k \geq 2$. This is directly deduced from the following proposition.

Proposition 1. *For any integer $p \geq 2$ and any indented-$(p, 2)$-L P, $p^2 - 1 \in \mathrm{TFN}(P)$ and $p^2 \in \mathrm{TFN}(P)$.*

We show some lemmas to prove Proposition 1.

Lemma 8. *For any integer k, $\lfloor \frac{k}{2} \rfloor - \lceil \frac{k-1}{2} \rceil = 0$.*

Proof. If k is odd, that is, if there is an integer m such that $k = 2m + 1$, then $\lfloor \frac{k}{2} \rfloor - \lceil \frac{k-1}{2} \rceil = \lfloor \frac{2m+1}{2} \rfloor - \lceil \frac{2m}{2} \rceil = m - m = 0$. If k is even, that is, if there is an integer m such that $k = 2m$, then $\lfloor \frac{k}{2} \rfloor - \lceil \frac{k-1}{2} \rceil = \lfloor \frac{2m}{2} \rfloor - \lceil \frac{2m-1}{2} \rceil = m - m = 0$. $\square$

Lemma 9. *For any integer k, $\lfloor \frac{k-1}{2} \rfloor - \lceil \frac{k}{2} \rceil = -1$.*

Proof. If k is odd, that is, if there is an integer m such that $k = 2m + 1$, then $\lfloor \frac{k-1}{2} \rfloor - \lceil \frac{k}{2} \rceil = \lfloor \frac{2m}{2} \rfloor - \lceil \frac{2m+1}{2} \rceil = m - (m + 1) = -1$. If k is even, that is, if there is an integer m such that $k = 2m$, then $\lfloor \frac{k-1}{2} \rfloor - \lceil \frac{k}{2} \rceil = \lfloor \frac{2m-1}{2} \rfloor - \lceil \frac{2m}{2} \rceil = (m - 1) - m = -1$. $\square$

Proof of Proposition 1. We can obtain a $(p, 1)$-fold band or a $(p+1)$-fold tiling by arranging an infinite number of copies of P appropriately as mentioned in the second proof of Theorem 7. From this and Observation 2, $(p + 1)(p - 1) = p^2 - 1 \in \mathrm{TFN}(P)$. Next, we show a way of constructing a p^2-fold band with

$(p,1)$-fold bands. Let us set the 2-dimensional Cartesian coordinate system with the x-axis in the horizontally rightward direction and the y-axis in the vertically downward direction on the plane. For any intergers a and b, we call the region (not including the boundary) between the lines $y=a$ and $y=b$ *region* (a,b). For every nonnegative integer $i \leq 2p$, we consider arranging $p-\lceil \frac{i}{2} \rceil$ and $\lfloor \frac{i}{2} \rfloor$ pieces of $(p,1)$-fold bands in region $(i,i+2)$ such that p-fold bands and 1-fold bands included in $(p,1)$-fold bands are on the upper side, respectively. Then, for any nonnegative integer $j \leq 2p+1$, the multiplicity of region $(j,j+1)$, which will be denoted by $\ell(j)$, is as follows:

$$\ell(j) = \begin{cases} p(p-\lceil \frac{j}{2} \rceil) + \lfloor \frac{j}{2} \rfloor, & j=0, \\ p(p-\lceil \frac{j}{2} \rceil) + (p-\lceil \frac{j-1}{2} \rceil) + \lfloor \frac{j}{2} \rfloor + p\lfloor \frac{j-1}{2} \rfloor, & 1 \leq j \leq 2p, \\ (p-\lceil \frac{j-1}{2} \rceil) + p\lfloor \frac{j-1}{2} \rfloor, & j=2p+1. \end{cases}$$

One can easily check that $\ell(0)=\ell(2p+1)=p^2$. Moreover, from Lemmas 8 and 9, for any positive integer $j \leq 2p$,

$$\begin{aligned} \ell(j) &= p(p-\lceil \tfrac{j}{2} \rceil) + (p-\lceil \tfrac{j-1}{2} \rceil) + \lfloor \tfrac{j}{2} \rfloor + p\lfloor \tfrac{j-1}{2} \rfloor \\ &= p^2 + p(\lfloor \tfrac{j-1}{2} \rfloor - \lceil \tfrac{j}{2} \rceil) + p + (\lfloor \tfrac{j}{2} \rfloor - \lceil \tfrac{j-1}{2} \rceil) \\ &= p^2 - p + p = p^2. \end{aligned}$$

Therefore, we can construct a $2(p+1)\times p^2$-fold band with $2p^2$ pieces of $(p,1)$-fold bands, and it follows that $p^2 \in \mathrm{TFN}(P)$. □

A statement similar to Proposition 1 may also hold for indented-(p,q)-L ($q \geq 3$). Moreover, the shape of a holed-p-I is simpler than that of an indented-(p,q)-L. Thus using a holed-p-I rather than an indented-(p,q)-L for a proof can be expected to simplify it. In particular, if we use a holed-p-I for solving Open Problem 1, we may be able to take advantage of the fact that it is impossible to construct any multiple tiling with multiplicity other than multiples of p by using p-fold bands. From these facts, we think that a holed-p-I, especially the holed-p-I shown in Fig. 9, is promising as a solution to Open Problem 1.

3.2 The lower bound of the number of cells

Next, we focus our attention on the number of cells of a polyomino.

Definition 7. If an $n(\in \mathbb{N}^+)$-omino P is a nontrivial $k(\in \mathbb{N}^+)$-fold tile and there is no n'-omino that is a nontrivial k-fold tile for any positive integer $n' < n$, then n is called the *minimum size of nontrivial k-fold-tile polyomino* and P is called a *minimum-sized nontrivial k-fold-tile polyomino.*

We show the following theorem.

Theorem 8. *For any integer $k \geq 2$, the minimum size of nontrivial k-fold-tile polyomino is 7, and the heptominoes C7, F7, and X7 listed in Fig. 3 are all minimum-sized nontrivial k-fold-tile polyominoes. Furthermore, the heptomino G7 in Fig. 3 is also a minimum-sized nontrivial k-fold-tile polyomino for every $k \geq 2$ except for $k = 3, 5$.*

Note that it is only open whether or not G7 is a k-fold tile for $k = 3, 5$. We show some lemmas used for proving this theorem. First, we consider overlapping a heptomino in a diagonal direction. We consider an operation of repeating the process to translate a heptomino in the rightward and downward (leftward and upward) directions by 1 and to overlap them (see Fig. 29). We call this operation *Operation I.*

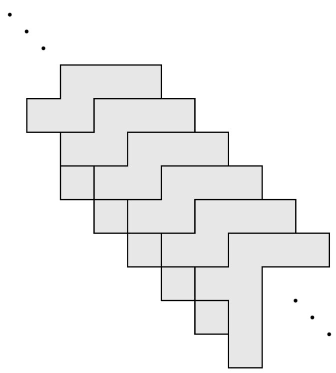


Figure 29: Operation I of F7

The multiplicity of each cell of Fig. 29 is shown in Fig. 30. Moreover, by rotating it 180 degrees, we obtain the arrangement shown in Fig. 31. We call each of these arrangements a *diagonal* $(1, 2, 1, 1, 1, 1)$*-fold band*.

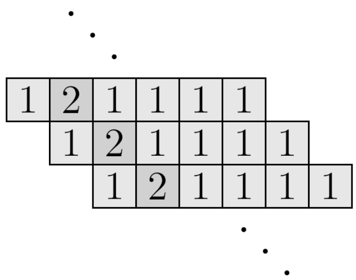


Figure 30: The multiplicity of each cell of Fig. 29

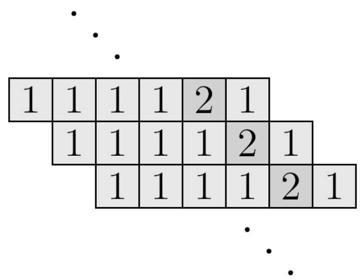


Figure 31: The 180 degree rotation of Fig. 30

Such diagonal fold bands similarly construct a k-fold tiling as normal fold bands. Therefore, unless otherwise required, hereafter we also simply call them *fold bands*. In addition, by either or both of rotating F7 90 degrees clockwise and reflecting it in a vertical line before applying Operation I to it, we obtain a $(1, 2, 1, 1, 1, 1)$-fold band again or a $(1, 2, 2, 2)$-fold band. Similarly, by applying Operation I to C7, F7, X7, and G7, we obtain the (diagonal) fold bands as shown in Table 1. Note that we do not have to consider reflecting each of C7, X7, and G7 before applying the operation since they all have line symmetry.

Table 1: The fold bands obtained by Operation I

Heptomino	Obtained fold bands
C7	$(1, 2, 1, 1, 1, 1)$
F7	$(1, 2, 1, 1, 1, 1), (1, 2, 2, 2)$
X7	$(1, 2, 1, 2, 1), (2, 2, 1, 2)$
G7	$(1, 2, 2, 2), (1, 2, 1, 2, 1)$

In Operation I, we considered overlapping a heptomino with the slope -1.

We next consider overlapping it with the slope $-1/2$, that is, an operation of repeating the process to translate a heptomino in the rightward and downward (leftward and upward) directions by 2 and 1, respectively, and to overlap them. We call this operation *Operation II*. We also call an arrangement obtained by this operation a *diagonal fold band* or simply a *fold band* as above. As with Operation I, by applying Operation II to C7, F7, X7, and G7, we obtain the (diagonal) fold bands as shown in Table 2.

Table 2: The fold bands obtained by Operation II

Heptomino	Obtained fold bands
C7	$(1,1,1,0,1,0,1,0,1,1), (1,0,1,1,2,1,1)$
F7	$(1,0,1,1,1,0,1,1,1), (1,1,2,1,1,1), (1,0,1,0,2,2,1), (1,0,1,0,1,1,1,2)$
X7	$(1,0,1,1,1,1,2), (1,0,2,1,1,1,1)$
G7	$(1,1,2,0,2,1), (1,1,1,0,2,1,1)$

Lemma 10. *The four heptominoes listed in Fig. 3 are all 2-fold tiles.*

Proof. By applying Operation I to each of F7 and G7, we obtain a $(1,2,2,2)$-fold band, and by combining two pieces of it, we obtain a $(2,2,2,1+1,2,2,2)$-fold band. By applying Operation I to C7, we obtain a $(1,2,1,1,1,1)$-fold band, and by combining two pieces of it, we obtain a $(1,2,1+1,1+1,1+1,1+1,2,1)$-fold band, or a $(1,2,2,2,2,2,2,1)$-fold band. By applying Operation II to X7, we obtain a $(1,0,1,1,1,1,2)$-fold band, and by combining two pieces of it, we obtain a $(1,0+2,1+1,1+1,1+1,1+1,2+0,1)$-fold band, or a $(1,2,2,2,2,2,2,1)$-fold band. Clearly, for each of the four heptominoes listed in Fig. 3, a 2-fold tiling can be obtained from these fold bands. □

Lemma 11. *The heptominoes C7, F7, and X7 listed in Fig. 3 are all 3-fold tiles.*

Proof. By applying Operation I to each of C7 and F7, we obtain a $(1,2,1,1,1,1)$-fold band, and by combining six pieces of it, we obtain a $(1,1,1+1,1+1,2+1,1+$

$1+1, 2+1, 1+1+1, 1+2, 2+1, 1+1+1, 1+2, 1+1+1, 1+2, 1+1, 1+1, 1, 1)$-fold band, or a $(1, 1, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2, 1, 1)$-fold band. A 3-fold tiling can be obtained by combining an infinite number of copies of it. The cross-section of it is shown in Fig. 32.

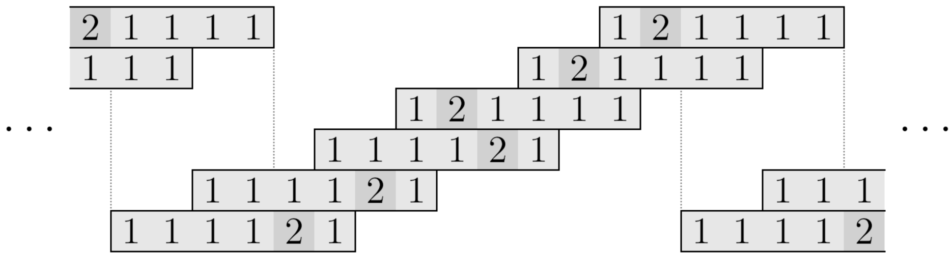


Figure 32: The cross-section representation of a 3-fold tiling with $(1, 2, 1, 1, 1, 1)$-fold bands

By applying Operation I to X7, we obtain both a $(2, 2, 1, 2)$-fold band and a $(1, 2, 1, 2, 1)$-fold band, and by combining two pieces of the former and four pieces of the latter, we obtain a $(1, 2+1, 1+2, 2+1, 1+2, 1+2, 2+1, 1+2, 2+1, 2+1, 1+2, 1+2, 2+1, 1+2, 2)$-fold band, or a $(1, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2)$-fold band. A 3-fold tiling can be obtained by combining an infinite number of copies of it. The cross-section of it is shown in Fig. 33. □

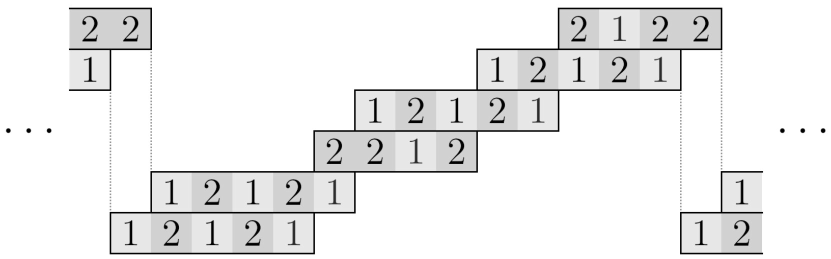


Figure 33: The cross-section representation of a 3-fold tiling with $(2, 2, 1, 2)$-fold bands and $(1, 2, 1, 2, 1)$-fold bands

Proof of Theorem 8. From Lemmas 10 and 11, it follows that C7, F7, and X7 are all 2-fold tiles and 3-fold tiles. Therefore, from Lemma 1, these heptominoes are all k-fold tiles for any integer $k \geq (2-1)(3-1) = 2$. From this and Theorems 5 and 6, they are minimum-sized nontrivial k-fold-tile polyominoes for every $k \geq 2$. Similarly, from Lemmas 1, 3, and 10 and Theorems 5 and 6, it is also proven that G7 is a minimum-sized nontrivial k-fold-tile polyomino for every $k \geq 2$ except for $k = 3, 5$. $\square$

Chapter 4 Multifold tiles on convex unit-lattice polygons

In this chapter, we consider nontrivial k-fold tiles on convex unit-lattice polygons.

In 2012, Gravin et al. [10] presented a convex unit-lattice octagon O_7 shown in Fig. 34 as a simple example of a nontrivial 7-fold (lattice) tile. This can be confirmed by considering $\{O_7 + z \mid z \in \mathbb{Z}^2\}$ as in the proof of Lemma 3. Since O_7 is centrally symmetric, each triangle that occurs from the division of O_7 by the unit grid can combine with another triangle by a translation to constitute a cell. From this and the fact that the area of O_7 is 7, they cover the plane with multiplicity 7. However, it is not a tile from Theorem 2. We show the following theorems generalizing that.

Theorem 9. *For any integer $k = 5$ or $k \geq 7$, there exists a convex unit-lattice polygon that is*

(i) *a nontrivial k-fold tile,*

(ii) *of area is k, and*

(iii) *is a hexagon if $k = 5$ or 8; an octagon otherwise.*

Definition 8 ([4])**.** A map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2 : \boldsymbol{p} \mapsto A\boldsymbol{p} + \boldsymbol{b}$ with $A \in \mathrm{GL}_2(\mathbb{Z})$[1)] and $\boldsymbol{b} \in \mathbb{Z}^2$ is called a $\mathbb{Z}$-*affine transformation.* Two convex unit-lattice polygons P and P' are said to be *equivalent* if there exists a $\mathbb{Z}$-affine transformation φ such that $\varphi(P) = P'$.

Theorem 10. *For $k = 2$ or 3, there exists no convex unit-lattice polygon that satisfies both the conditions (i) and (ii) in Theorem 9. Furthermore, for $k = 4$, if there exists such a convex unit-lattice polygon, then it is equivalent to the pentagon (5,2,6)-b shown in Fig 46.*

Note that it is open whether or not there is a convex unit-lattice polygon

[1)] $\mathrm{GL}_2(\mathbb{Z})$ is the general linear group of degree 2 over $\mathbb{Z}$.

that satisfies both the conditions (i) and (ii) in Theorem 9 for $k = 4, 6$. First, we show some lemmas and prove Theorem 9.

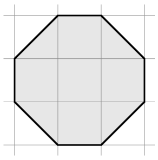


Figure 34: O_7

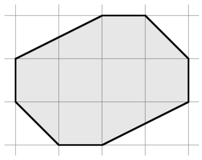


Figure 35: O_9

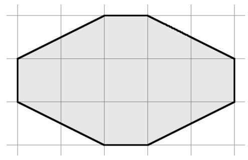


Figure 36: O_{11}

Lemma 12. *For any integer $k = 7$ or $k \geq 9$, there exists a convex unit-lattice octagon of area k that is a nontrivial k-fold (lattice) tile.*

Proof. From Theorem 2, any convex octagon is not a tile. As with O_7, convex unit-lattice octagons O_9 and O_{11} shown in Figs. 35 and 36 are 9- and 11-fold lattice tiles, respectively. We now consider adding three cells to each of O_7, O_9, and O_{11} repeatedly while keeping the property of being a centrally symmetric convex unit-lattice octagon. Then it is clear that an octagon that was originally a k-fold lattice tile becomes a $(k+3)$-fold lattice tile after the addition as shown in Fig. 37. $\square$

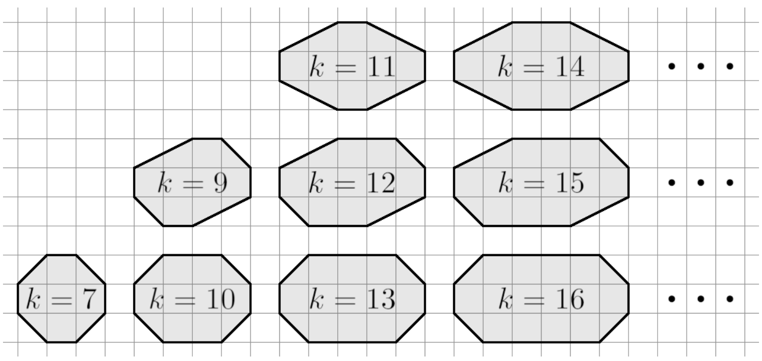


Figure 37: A nontrivial k-fold (lattice) tile of convex unit-lattice octagon for any integer $k = 7$ or $k \geq 9$

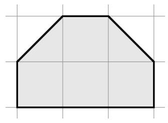


Figure 38: H_5

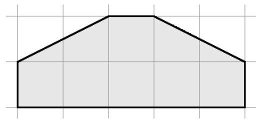


Figure 39: H_8

Lemma 13. *For $k = 5$ or 8, there exists a convex unit-lattice hexagon of area k that is a nontrivial k-fold tile.*

Proof. It is clear from Theorem 3 that hexagons H_5 and H_8 are not tiles. Let us make the pair of H_5 by overlapping them as shown in Fig. 40. We also make the pair of H_8 in a similar way as shown in Fig. 41. Each number in Figs. 40 and 41 indicates the multiplicity.

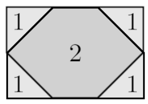


Figure 40: The pair of H_5

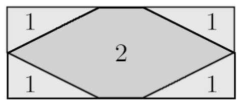


Figure 41: The pair of H_8

By arranging pairs of H_5 on the plane in a manner of a tiling with hexagons, each of which is constituted by the 2-fold part of it, we obtain a $(\ldots, 3, 2, 3, 2, \ldots)$-fold band as shown in Fig. 42. By combining two $(\ldots, 3, 2, 3, 2, \ldots)$-fold bands, we obtain a $(\ldots, 3+2, 2+3, 3+2, 2+3, \ldots)$-fold band, or a 5-fold tiling. Similarly, by arranging pairs of H_8, we obtain a $(\ldots, 3, 3, 2, 3, 3, 2, \ldots)$-fold band as shown in Fig. 43. By combining three $(\ldots, 3, 3, 2, 3, 3, 2, \ldots)$-fold bands, we obtain a $(\ldots, 3+2+3, 3+3+2, 2+3+3, 3+2+3, 3+3+2, 2+3+3, \ldots)$-fold band, or an 8-fold tiling. The cross-section of it is shown in Fig. 44.

Therefore, H_5 and H_8 are convex unit-lattice hexagons of area k that are nontrivial k-fold tiles for $k = 5$ and $k = 8$, respectively. $\square$

Proof of Theorem 9. Clear from Lemmas 12 and 13. $\square$

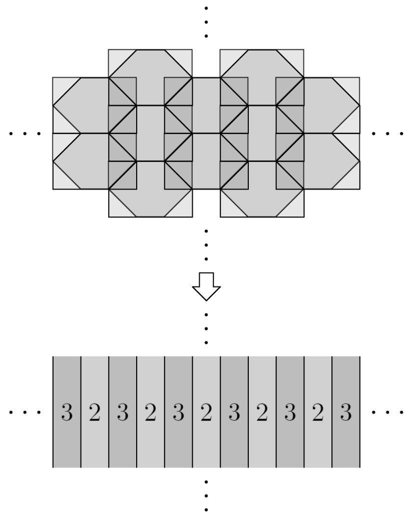


Figure 42: The arrangement of H_5

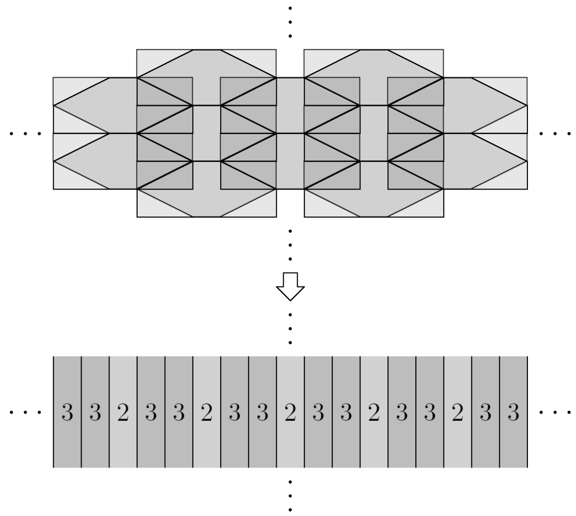


Figure 43: The arrangement of H_8

	3	2	3	3	2	3	3	2	3	
$\cdots$	2	3	3	2	3	3	2	3	3	$\cdots$
	3	3	2	3	3	2	3	3	2	

Figure 44: The cross-section representation of an 8-fold tiling with $(\dots, 3, 3, 2, 3, 3, 2, \dots)$-fold bands

Next, we prove Theorem 10. Any $\mathbb{Z}$-affine transformation maps $\mathbb{Z}^2$ bijectively onto itself. We call points lying in $\mathbb{Z}^2$ *unit-lattice points*. For a unit-lattice polygon P, let $v = v(P)$, $b = b(P)$, $i = i(P)$, and $a = a(P)$ be the number of vertices of P, the number of unit-lattice points on the boundary of P, the number of interior unit-lattice points of P, and the area of P, respectively. It is known that $\mathbb{Z}$-affine transformations map convex unit-lattice polygons to convex unit-lattice polygons and preserve values v, b, i, and a. We show some known theorems on unit-lattice polygons and prove a lemma as a preparation.

Theorem 11 ([23])**.** *For any unit-lattice polygon,*

$$a = i + b/2 - 1. \tag{3}$$

Theorem 12 ([14, 24])**.** *Every convex unit-lattice polygon satisfying* $i = 0$ *is equivalent to one of the following polygons:*

(i) *A triangle with vertices* $(0, 0)$, $(n, 0)$, *and* $(0, 1)$, *where* n *is any positive integer.*

(ii) *A trapezoid with vertices* $(0, 0)$, $(n, 0)$, $(m, 1)$, *and* $(0, 1)$, *where* n *and* m *are any positive integers that satisfy* $n \geq m$.

(iii) *The triangle with vertices* $(0, 0)$, $(2, 0)$, *and* $(0, 2)$.

Theorem 13 ([14, 24])**.** *Every convex unit-lattice polygon satisfying* $i = 1$ *is equivalent to exactly one of the 16 polygons shown in Fig 45.*

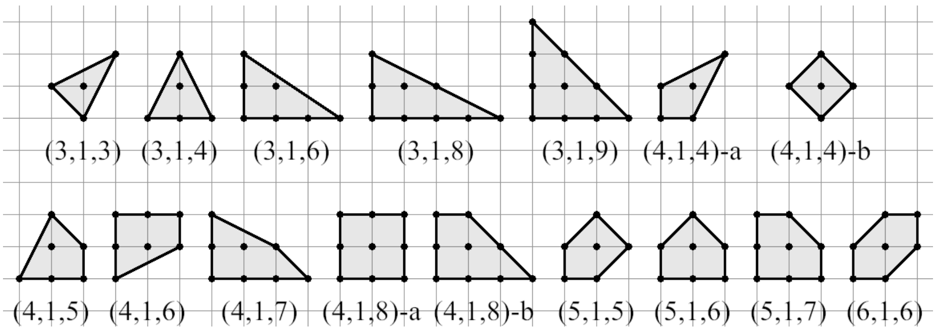


Figure 45: The 16 polygons in Theorem 13 (each polygon is named after (v, i, b))

Theorem 14 ([30])**.** *Every convex unit-lattice polygon satisfying* $i = 2$ *is equivalent to exactly one of the 45 polygons shown in Fig 46.*

Lemma 14. *If* P *is a convex unit-lattice pentagon of Type (i) in Theorem 4, then for any* $\mathbb{Z}$*-affine transformation* φ, $\varphi(P)$ *is also a convex unit-lattice pentagon of Type (i) in Theorem 4. Additionally, if* H *is a convex unit-lattice hexagon of Type (i) in Theorem 3, then for any* $\mathbb{Z}$*-affine transformation* φ, $\varphi(H)$ *is also a convex unit-lattice hexagon of Type (i) in Theorem 3.*

Proof. A convex pentagon is Type (i) in Theorem 4 if and only if it has at least one pair of parallel sides. A convex hexagon is Type (i) in Theorem 3 if and

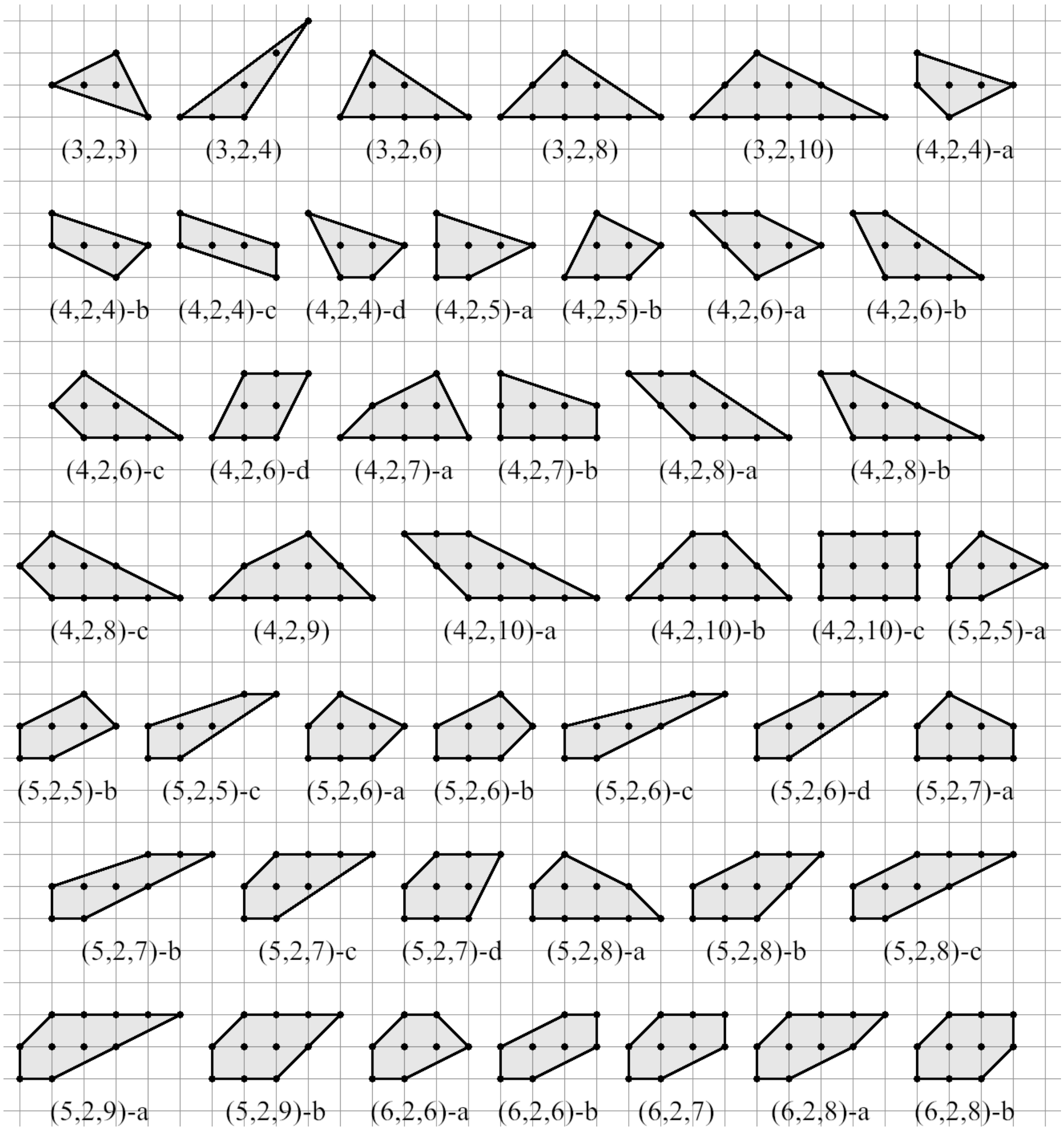


Figure 46: The 45 polygons in Theorem 14 (each polygon is named after (v, i, b))

only if it has at least one pair of parallel opposite sides of equal length. Since any non-singular affine transformation in $\mathbb{R}^2$ preserves both parallelism and the ratio of lengths of two parallel line segments, the statement is proved. $\square$

Proof of Theorem 10. From Theorem 1, we can assume that $b \geq v \geq 5$. From Theorem 12, every convex unit-lattice polygon without interior unit-lattice points has three or four vertices. Hence we can also assume that $i \geq 1$. Then we can observe that there is no pair of integers (i, b) that satisfies the equation (3) with $a = 2$. From the equation (3) and $a = 3$ or 4, it follows that

$$(i, b) = \begin{cases} (1, 6), & (a = 3) \\ (1, 8), (2, 6). & (a = 4) \end{cases} \tag{4}$$

From Theorem 13, every convex unit-lattice polygon satisfying both $v \geq 5$ and $(i, b) = (1, 6)$ is equivalent to either of the polygons (5,1,6) and (6,1,6) in Fig. 45, and there is no convex unit-lattice polygon satisfying both $v \geq 5$ and $(i, b) = (1, 8)$. Moreover, from Theorem 14, every convex unit-lattice polygon satisfying both $v \geq 5$ and $(i, b) = (2, 6)$ is equivalent to any one of the polygons (5,2,6)-a, (5,2,6)-b, (5,2,6)-c, (5,2,6)-d, (6,2,6)-a, and (6,2,6)-b in Fig. 46. The pentagons (5,1,6), (5,2,6)-a, (5,2,6)-c, and (5,2,6)-d are all Type (i) in Theorem 4, and the hexagons (6,1,6), (6,2,6)-a, and (6,2,6)-b are all Type (i) in Theorem 3. Hence from Lemma 14, any polygon excepting (5,2,6)-b appeared above is a tile. □

The pentagon (5,2,6)-b, which left the possibility in the above proof, is not any type of (i)–(xv) in Theorem 4. In fact, we can show that it is not a tile by examining local tilings with them thoroughly. However, we do not know whether or not any pentagon equivalent to (5,2,6)-b is not a tile. We also do not know whether or not the pentagon (5,2,6)-b or each pentagon equivalent to (5,2,6)-b is a 4-fold tile.

In a similar way, pairs of integers (i, b) are determined for $k = a = 6$ as follows:

$$(i, b) = (1, 12), (2, 10), (3, 8), (4, 6). \tag{5}$$

The assumption $v \geq 5$ and Theorems 13 and 14 eliminate the possibilities of $(i, b) = (1, 12), (2, 10)$. For $(i, b) = (3, 8), (4, 6)$, we can use Castryck's result [4]. For every $1 \leq g \leq 30$, he performed a computer calculation of all lattice polygons satisfying $i = g$ up to equivalence as a generalization of Theorems 13 and 14. The resulting data is available on his website. According to this data, up to equivalence, there are a total of 14 convex unit-lattice polygons satisfying both $v \geq 5$ and $(i, b) = (3, 8)$ as shown in Fig. 47 and 21 convex unit-lattice polygons

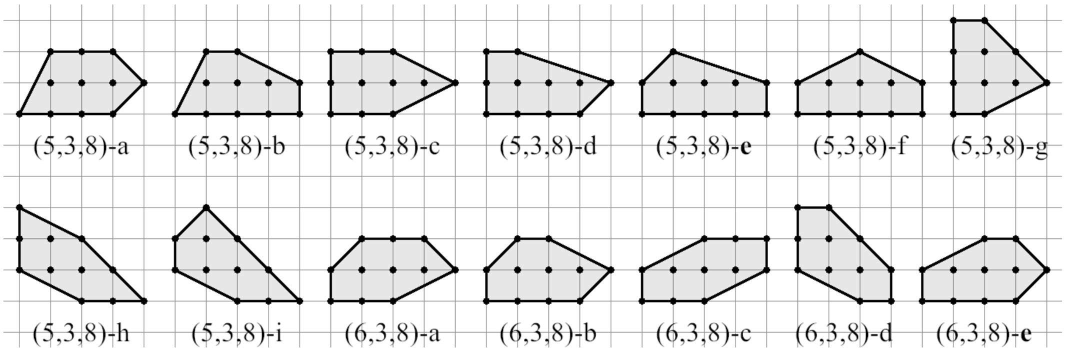


Figure 47: The 14 equivalence classes of convex unit-lattice polygons satisfying both $v \geq 5$ and $(i, b) = (3, 8)$ (each polygon is named after (v, i, b))

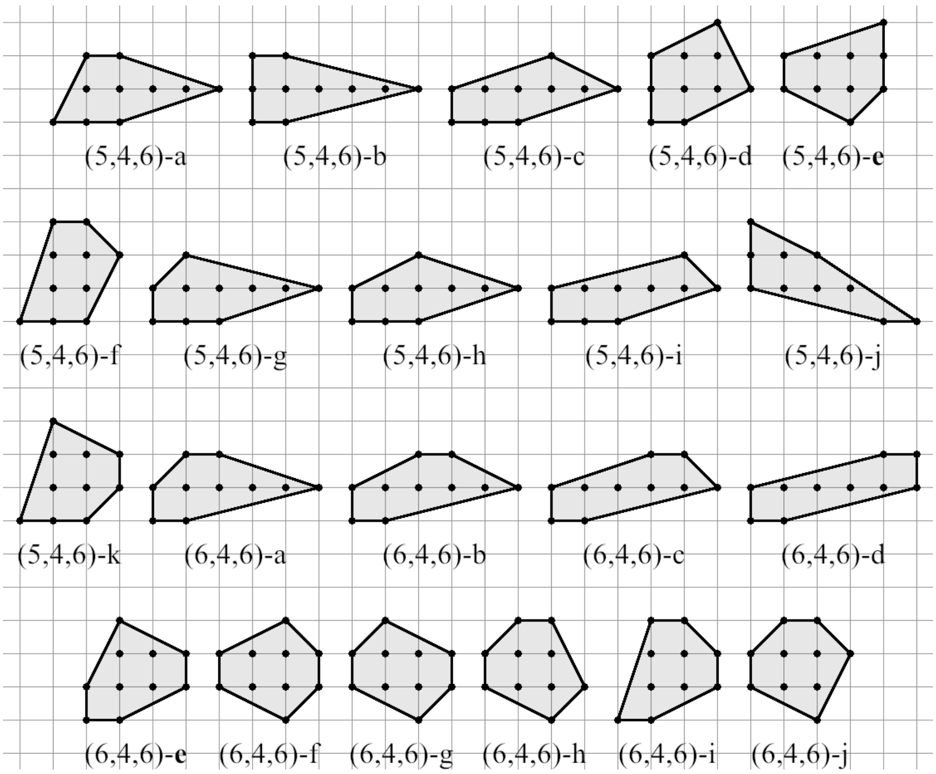


Figure 48: The 21 equivalence classes of convex unit-lattice polygons satisfying both $v \geq 5$ and $(i, b) = (4, 6)$ (each polygon is named after (v, i, b))

satisfying both $v \geq 5$ and $(i, b) = (4, 6)$ as shown in Fig. 48. Moreover, we can use Lemma 14 for the twelve of the former: (5,3,8)-a–h and (6,3,8)-a–d, and 14 of the latter: (5,4,6)-a–f and (6,4,6)-a–h. By examining the remaining nine polygons: (5,3,8)-i, (6,3,8)-e, (5,4,6)-g–k, (6,4,6)-i, and (6,4,6)-j and polygons equivalent to them, it may be able to prove or disprove that there is a convex unit-lattice polygon of area 6 that is a nontrivial 6-fold tile.

Chapter 5 Summary and Future Work

In this research, we studied nontrivial multiple tilings with polyominoes or convex unit-lattice polygons allowing translations, rotations, and reflections, and obtained four theorems. The first one (Theorem 7) claims that for any integer $k \geq 2$, there is a polyomino whose minimum tile-fold number is k. The second one (Theorem 8) claims that for any integer $k \geq 2$, the heptominoes C7, F7, and X7 listed in Fig. 3 are all minimum-sized nontrivial k-fold-tile polyominoes, and for any integer $k \geq 2$ except for $k = 3, 5$, the heptomino G7 listed in Fig. 3 is also minimum-sized nontrivial k-fold-tile polyomino. The third one (Theorem 9) claims that for any integer $k = 5$ or $k \geq 7$, there is a convex unit-lattice polygon that is a nontrivial k-fold tile whose area is k. The last one (Theorem 10) claims that for $k = 2$ or 3, there is no convex unit-lattice polygon that is a nontrivial k-fold tile whose area is k, and for $k = 4$, such a convex unit-lattice polygon must be equivalent to a certain pentagon, if any.

As future work, we have some unsolved parts for minimum-sized nontrivial k-fold-tile polyominoes and nontrivial k-fold tiles of convex unit-lattice polygons whose area is k. For the former, although Theorem 8 presented such polyominoes for any integer $k \geq 2$, it is still open whether G7 is a 3- or 5-fold tile. For the latter, although Theorems 9 and 10 clarified whether or not there is such a polygon for any integer $k = 2, 3, 5$ or $k \geq 7$, it is still open for $k = 4$ and 6; the pentagon and the nine polygons (and shapes equivalent to them) leave the possibility, respectively. In particular, although one can show that the pentagon is not a tile, we do not know whether or not any pentagon equivalent to it is not a tile and whether or not each pentagon equivalent to it is a 4-fold tile. In addition, we can also consider the following problems.

Open Problem 1 (Reshown)**.** For any integer $k \geq 2$, is there a polygon P that satisfies $\mathrm{TFN}(P) = \{k\ell \mid \ell \in \mathbb{N}^+\}$?

Open Problem 2. For any integer $k \geq 2$, is there a polyomino whose minimum tile-fold number is k and whose size (the number of cells) is bounded by a polynomial function of k?

Open Problem 3. For any $k = 2, 3$, or 4, is there a convex polygon that is a nontrivial k-fold tile?

Note that the sizes of the holed-k-I and the indented-$(k, 2)$-L in the first and second proofs of Theorem 7, respectively, are both $O(k \cdot 2^{4k})$, which is exponential. The former may be a solution to Open Problem 1 as mentioned in Section 3.1. Even if this is true, we should confirm that its tile-fold numbers are only multiples of k. Also note that a nontrivial 6-fold (lattice) tile of a convex polygon is already independently discovered by Marley [18, 19] (as mentioned in Section 1.2) and Zong [34].

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List of Publications

1. Kota Chida, Erik D. Demaine, Martin L. Demaine, David Eppstein, Adam Hesterberg, Takashi Horiyama, John Iacono, Hiro Ito, Stefan Langerman, Ryuhei Uehara, and Yushi Uno, Multifold tiles of polyominoes and convex lattice polygons, The 23rd Thailand-Japan Conference on Discrete and Computational Geometry, Graphs, and Games (TJCDCG^3 2020+1), Chiang Mai University (online format), Chiang Mai, Sep. 3–5, 2021. (No proceedings were published.)
2. Kota Chida, Erik D. Demaine, Martin L. Demaine, David Eppstein, Adam Hesterberg, Takashi Horiyama, John Iacono, Hiro Ito, Stefan Langerman, Ryuhei Uehara, and Yushi Uno, Multifold tiles of polyominoes and convex lattice polygons, Thai Journal of Mathematics, submitted.