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An Examination of Effective Length in Moment Frames

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Honors Thesis
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I. Column Buckling

When loaded axially in compression columns experience a failure mode in compression that axially loaded members don’t experience in tension. This failure mode, elastic column buckling, doesn’t involve yielding or rupture; the column changes shape and deforms to the side.

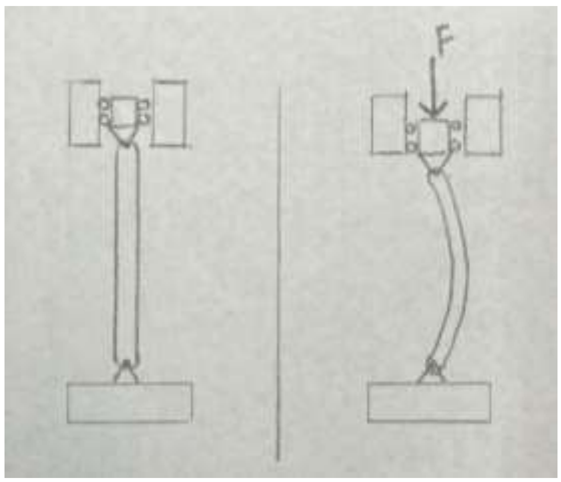


Figure 1: Column Buckling

To find the axial load that would cause a buckling failure Leonhard Euler, figure 2, developed a mathematical solution. The solution (Less Boring Lectures 2021) was developed for a column with pinned ends from the second derivative of a beam deflection equation because a column is just a vertical beam experiencing axial load.

Beam Deflection Equation: $\frac{d^2y}{dx^2} = \frac{M}{EI}$

$$\frac{d^2y}{dx^2} = \frac{-Py}{EI}$$

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = 0$$

Allow, $\propto^2 = \frac{P}{EI}$ and $y" = \frac{d^2y}{dx^2}$

Substitute, $y" + \propto y = 0$

The derivation for a column with two pinned ends begins with the assumption that the deflected shape corresponds to that of a sine or cosine curve as shown.

General Solution: $$y = Asin(\propto x) + Bcos(\propto x)$$

Where, $$\propto = \sqrt{\frac{P}{EI}}$$

Using two boundary conditions of a pinned-pinned column the two coefficients of A and B can be found, and a critical buckling load can be determined.

Boundary Conditions:

1. $x = 0, y = 0$
2. $x = L, y = 0$

Boundary Condition 1: $$0 = Asin(\propto * 0) + Bcos(\propto * 0)$$

$$0 = 0 + B * 1$$

$$B = 0$$

Boundary Condition 2: $$0 = Asin(\propto * L) \qquad A \neq 0$$

$$\sin(\propto L) = 0$$

When $x = \pi$, $$\sin(x) = 0$$

$$\propto L = \pi$$

$$\propto = \frac{\pi}{L}$$

Plug back in, $$\frac{\pi}{L} = \sqrt{\frac{P}{EI}}$$

Thus, Euler's Buckling Equation for critical load:

$$P_{cr} = \frac{\pi^2 * E * I}{L^2} \qquad [1]$$

Where,

E = modulus of elasticity
I = moment of inertia
L = Length of column

Figure 2: Leonhard Euler, born in 1707 was a mathematician, physicist, and engineer who played a crucial role in the development of mathematics. He was born in Basel, Switzerland where he was a friend of the Bernoulli family, a very influential family in mathematics. He eventually attended University of Basel where he received his Master of Philosophy. Euler is credited for many common expressions used in math today such as the use of “π,” the letter “i” for an imaginary number, and the term f(x). Euler developed many foundational formulas used today in physics, astronomy, and engineering, but in 1757 is when he developed his critical buckling load equation for structural design.

1. The Julian & Lawrence Nomograph

For a pinned-pinned column, the effective length is equal to the length of the column. When, however, a column is part of a frame with rigid connections, the column ends are not free to rotate, and Euler's solution must be modified by replacing the actual column length with an effective column length. The effective length is the distance between two points of zero moment, (inflection) points. Different end condition solutions for the general solution of the buckling derivation result in different effective lengths, as shown in Figure 3 from the AISC code manual.

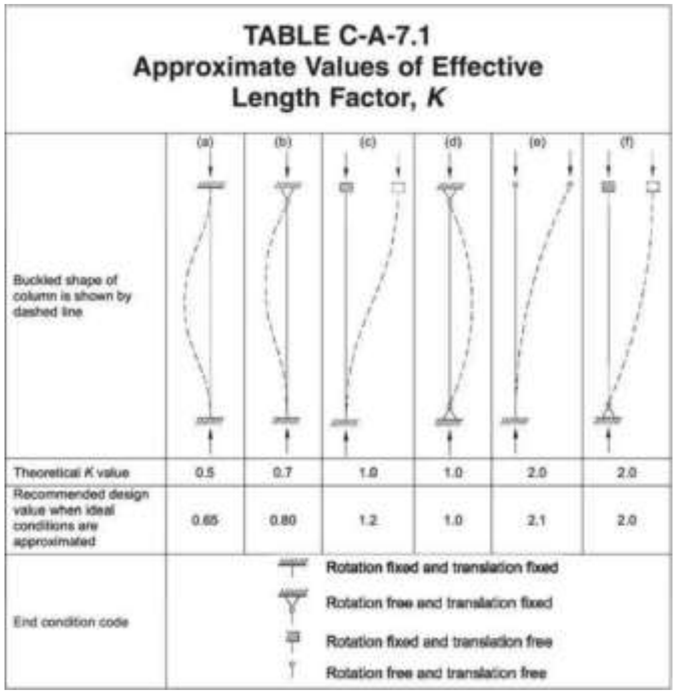


TABLE C-A-7.1
Approximate Values of Effective Length Factor, *K*

	(a)	(b)	(c)	(d)	(e)	(f)
Buckled shape of column is shown by dashed line						
Theoretical *K* value	0.5	0.7	1.0	1.0	2.0	2.0
Recommended design value when ideal conditions are approximated	0.65	0.80	1.2	1.0	2.1	2.0

End condition code:
- Rotation fixed and translation fixed
- Rotation free and translation fixed
- Rotation fixed and translation free
- Rotation free and translation free

Figure 3: AISC Effective Length Factors

If stiff beams prevent column end rotation the corresponding inflection point is forced away from the intersection, resulting in a shorter effective length. If the column is flexurally stiff compared to the beams, the inflection point occurs near the intersection, and the effective length is longer. These effective lengths can be represented by an effective length factor, *K*, times the length of the column, *L*.

$$P_{cr} = \frac{\pi^2 * E * I}{(KL)^2} \qquad [2]$$

There are two types of framed systems, braced and unbraced. A braced frame resists lateral displacement and forces by use of bracing or shear walls. An unbraced frame resist sidesway through moment resisting connections between columns and beams. In this project only braced frames are considered. An effective length factor nomograph was created by Julian and Lawrence to determine the effective length factor of a column in a frame system. One of the first times the Julian & Lawrence nomograph was formally presented was in Thomas C. Kavanagh's *Effective Length of Framed Columns* (Figure 4, Kavanagh 1960, 18).

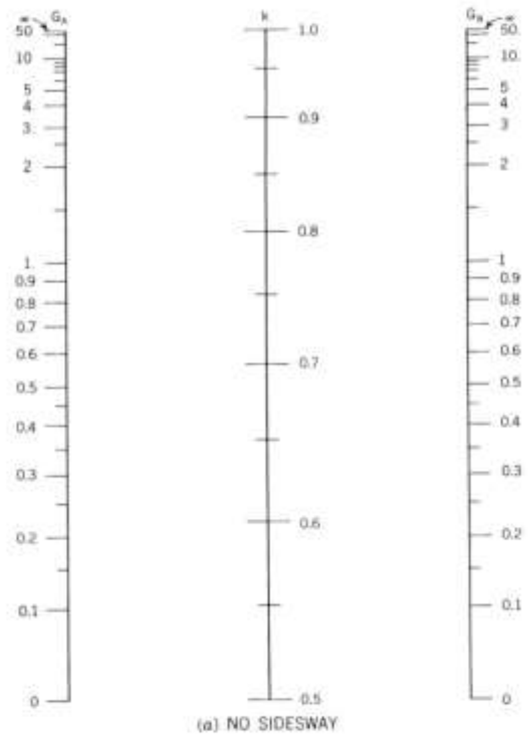


Figure 4: Effective length nomograph developed by Julian and Lawrence

In the nomograph, *G* represents the beams ability to resist joint rotation through a ratio of the column stiffnesses at one joint relative to the beam stiffnesses at that joint. The flexural stiffness of each member rigidly connected at each column end joint is given by

$$\frac{E*I}{L} \quad [3]$$

where,

E = modulus of elasticity
I = moment of inertia
L = length of column or beam

Then,

$$G = \frac{\sum(E*I/L)_{Columns}}{\sum(E*I/L)_{Beams}} \quad [4]$$

G_A is the ratio of flexural stiffnesses at one end of the column while G_B is the ratio of flexural stiffnesses at the other end of the column. Once a *G* value is found for end of the column, the nomograph is used by plotting the two values on each side and then connecting them with a straight line. The value in the middle that the connecting line crosses is the effective length factor, *K*, of the column.

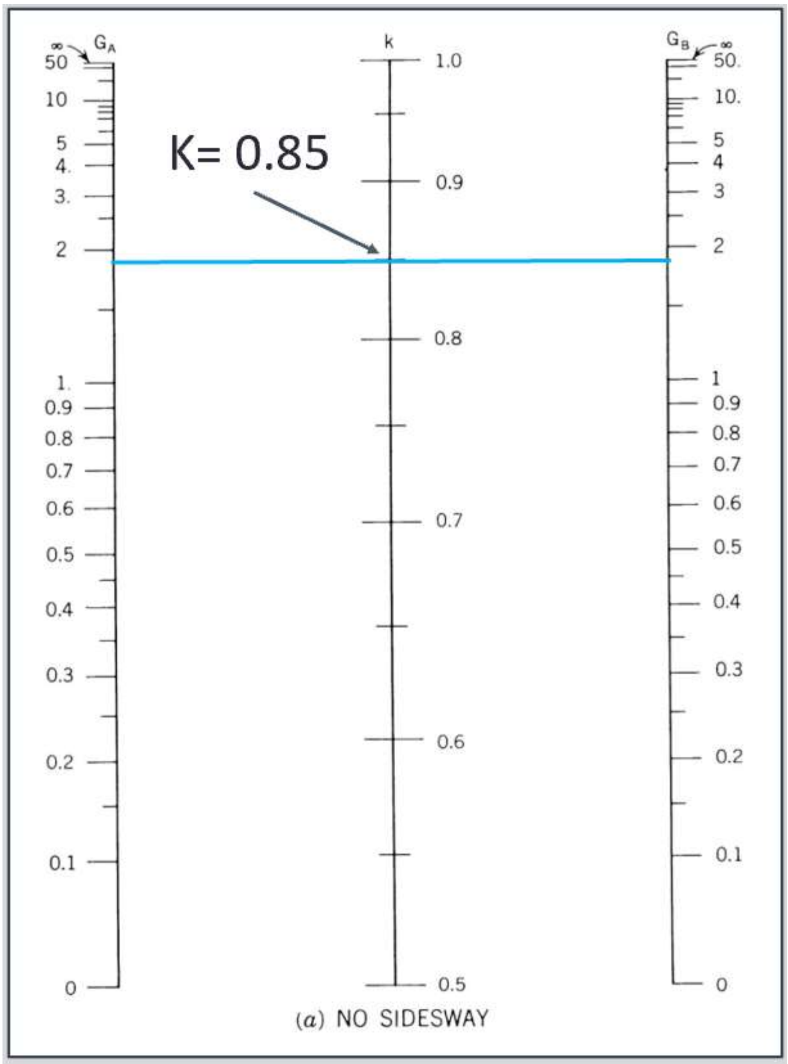


Figure 5: Finding K-factor from Nomograph

Different supports will have different *G* values, resulting in different *K* values. An end that is pinned has no beams tying into it and, therefore a *G* factor which is theoretically infinity. In practice, however, a value of 10 is recommended in such cases. Similarly, a fixed end support would be represented by beams with, effectively, infinite stiffness and a corresponding G factor of 0, though a design value of 1 is generally recommended for use in practice by the AISC Steel Construction Manual (AISC 16.1-573).

2. Matrix Analysis

This thesis explores the effect of join rotations caused by asymmetric beam loads, which the nomograph does not account for, using matrix structural analysis software. Matrix structural analysis, also known as the Direct Stiffness Method, is a type of analysis that solves problems of trusses, beams, and frames. It was developed by William McGuire and Richard H. Gallagher. Many solution methods, like Euler's, apply forces and from the applied forces, will find deflections. What makes matrix analysis unique is that it solves a set of deflection equations for compatible displacements and then use those to find forces and moments.

The program, Visual Analysis, by IES Software out of Bozeman, Montana was selected. To verify the ability of the software to model column buckling the first task done on Visual Analysis was creating a 14-foot long, square column broken up into ten smaller parts. The column was pinned at the bottom connection and x-direction movement was restricted in the top connection. This results in a column with pinned ends which corresponds to Euler's derivation and thus, resulting in an effective length factor of K=1.0. The section properties were chosen to make sure the column was slender. The 8"x8"x14' column had a radius of gyration of .289 resulting in a slenderness ratio of 48.5. Using the section properties, the critical buckling load was calculated using Equation 2. Giving a P_{cr} of 3461 kips. Visual Analysis validated this by showing the column failed when loaded to 3462 kips.

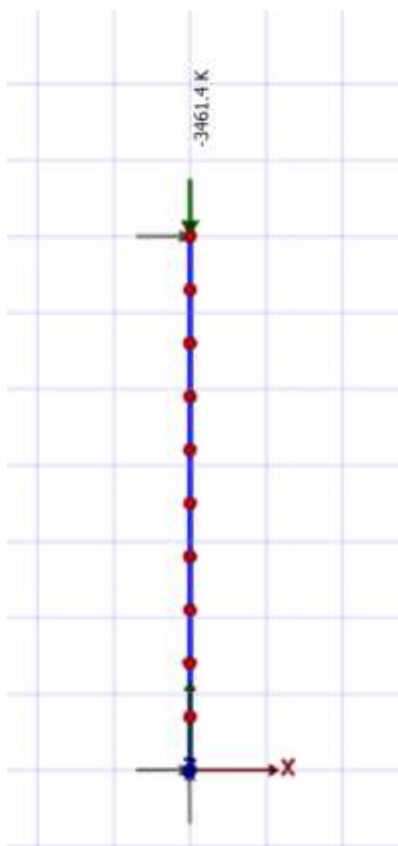


Figure 6: Square Column Analysis in VA

The next step was creating a moment frame with four beams and three columns, where the middle column was also split into ten smaller sections. This step was done to verify the classic frame solution of the nomograph, that just deals with axial loads. The far ends of the beams were fixed, and the columns were pinned at the bottom and restricted in the x-direction on top.

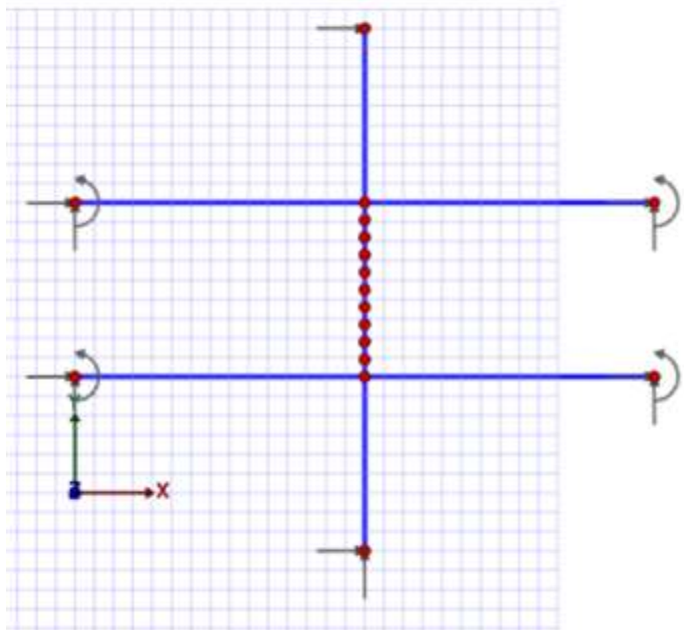


Figure 7: Moment Frame Analysis in VA

Initially, W18x50 beams and W12x40 columns were used. This would give beams with a span to depth ratio of about 20 and columns with a slenderness ratio of about 50. The beams had 30' spans, while the columns were 12' in height. Using Equation 4, the G values of the two end connections were found to be 1.83 and then using the nomograph in figure 1, a K factor of .85 was found.

However, with the second-order, or P-Delta, analysis on Visual Analysis the correspondence of the calculated critical buckling load and the actual failure load was very low. Meaning the calculated buckling load of the system was a lot lower than what the frame failed at on Visual Analysis. The height of the columns was increased to 18 ft to make the columns slenderer, which made the correspondence closer, but not close enough. The next change made was increasing the size of the beams. After several iterations, a W18x211 gave an acceptable correspondence of 99.1%, with a critical buckling load of 826 Kips.

After going through the process of finding the closest correspondence and not getting the results that were expected, a realization was made that it may be possible that visual analysis was not calculating the P-Delta analysis correctly. To check this a simple test was done. A single column with a pinned end and an x-axis restricted end was modeled. An axially load was placed on the model and using the second order analysis the failure load was found. Then, a lateral load at the midspan of the column was added, which should lower the critical buckling load. However, it did not change the critical failure load at all which proved that the P-Delta analysis on Visual Analysis was not working properly.

With Visual Analysis not working properly a new matrix analysis program was tried. Structural Analysis Program 2000, or SAP2000, was recommended. There was, however, a large learning curve getting used to the new software. There was never any real progress made with SAP2000, while a lot of time was spent learning how to set up a model and how to use the program. While learning about the program, a new idea came to light.

3. Asymmetric Loading Influence on Buckling

Using Visual Analysis, a manual P-Delta analysis could be done. This was done by using the same moment frame that gave 99% correspondence with the failure load. It was then loaded up axially to 99% of the critical failure load, 825 K. It was also loaded in a checkerboard pattern to represent the asymmetric beam loading. A 3 klf distributed load was chosen to give a larger joint rotation.

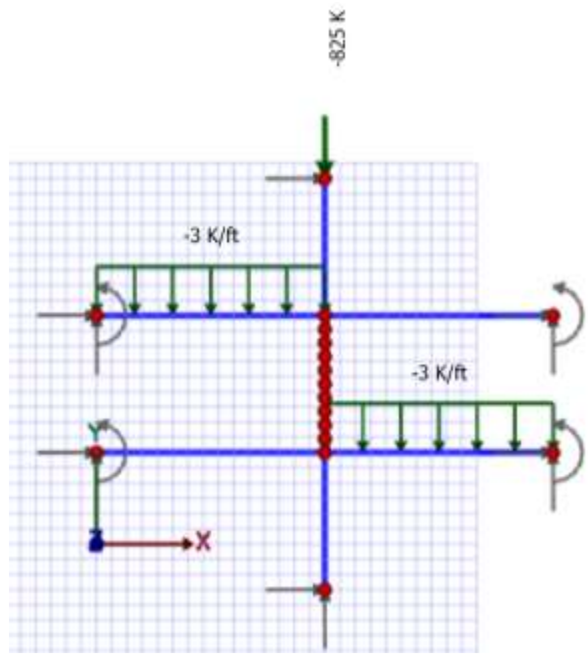


Figure 8: Moment Frame Loading

This was done to give the deflection of the column nodes just before failure.

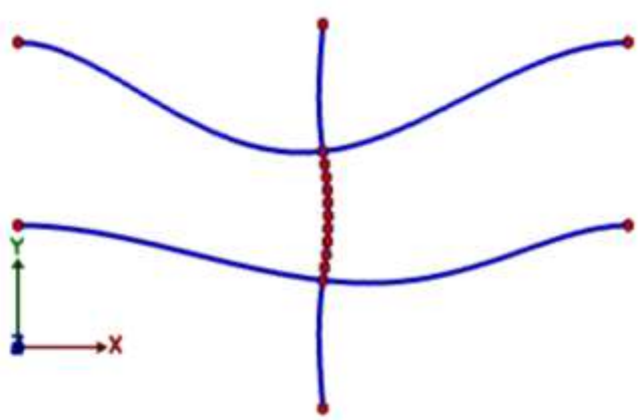
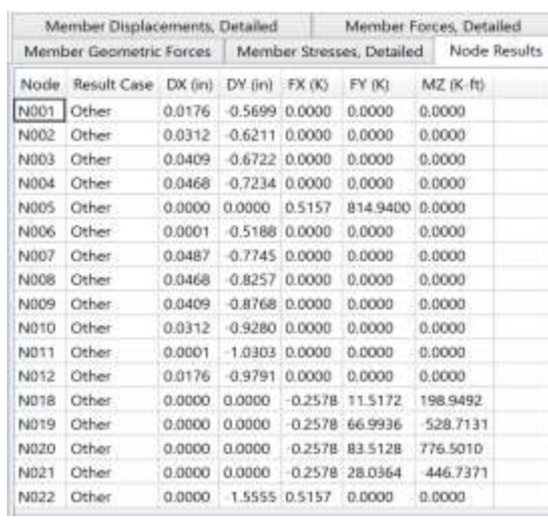


Figure 9: Pattern Loaded Beam Result



Member Displacements, Detailed | Member Forces, Detailed | Member Geometric Forces | Member Stresses, Detailed | Node Results

Node	Result Case	DX (in)	DY (in)	FX (K)	FY (K)	MZ (K-ft)
N001	Other	0.0176	-0.5699	0.0000	0.0000	0.0000
N002	Other	0.0312	-0.6211	0.0000	0.0000	0.0000
N003	Other	0.0409	-0.6722	0.0000	0.0000	0.0000
N004	Other	0.0468	-0.7234	0.0000	0.0000	0.0000
N005	Other	0.0000	0.0000	0.5157	814.9400	0.0000
N006	Other	0.0001	-0.5188	0.0000	0.0000	0.0000
N007	Other	0.0487	-0.7745	0.0000	0.0000	0.0000
N008	Other	0.0468	-0.8257	0.0000	0.0000	0.0000
N009	Other	0.0409	-0.8768	0.0000	0.0000	0.0000
N010	Other	0.0312	-0.9280	0.0000	0.0000	0.0000
N011	Other	0.0001	-1.0303	0.0000	0.0000	0.0000
N012	Other	0.0176	-0.9791	0.0000	0.0000	0.0000
N018	Other	0.0000	0.0000	-0.2578	11.5172	198.9492
N019	Other	0.0000	0.0000	-0.2578	66.9936	-528.7131
N020	Other	0.0000	0.0000	-0.2578	83.5128	776.5010
N021	Other	0.0000	0.0000	-0.2578	28.0364	-446.7371
N022	Other	0.0000	-1.5555	0.5157	0.0000	0.0000

Figure 10: Visual Analysis Nodal Deflection Results

Using the result view to get the precise deflection values of the column nodes, a new column could be created. A separate model was created of just the middle column; however, the column was set up in it’s already deflected shape. The bottom node was fixed in place while the top node was restricted rotationally and in the x-direction. It was then loaded axially again using the same 99% critical failure load.

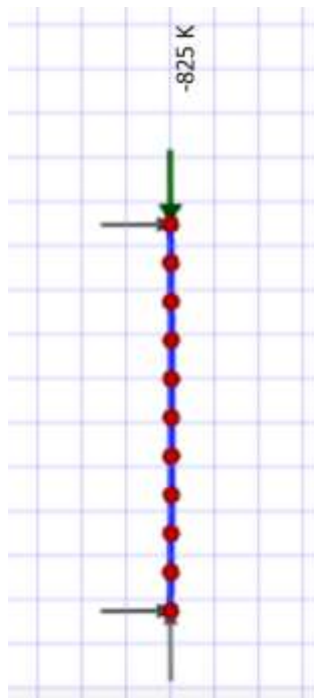


Figure 11: Deflected Column Model

After loading, the new node deflections were found and recorded. Using the new deflections, a third model was created with the new nodal positions. This was set up the same way as the previous iteration, however it was deflected more than the previous. When loaded at the same 825 kips, the nodal deflections were yet again recorded. After three iterations of the deflected column were modeled, the total deflection of the column for each was recorded.

Single Curvature Loading			1				2				3	
99% Pcr	First Deflection				Second Deflection				Third Deflection			
Nodes Top to Bottom	Position x	Position y	dx (in)	dy (in)	Position x	Position y	dx (in)	dy (in)	Position x	Position y	dx (in)	dy (in)
1	30	30	0.0001	-1.0303	30.0001	28.9697	0	-0.5137	30.0001	28.456	0	
2	30	28.2	0.0176	-0.9791	30.0176	27.2209	0.0766	-0.4619	30.0942	26.759	0.4213	
3	30	26.4	0.0312	-0.928	30.0312	25.472	0.145	-0.4104	30.1762	25.0616	0.8005	
4	30	24.6	0.0409	-0.8768	30.0409	23.7232	0.1985	-0.359	30.2394	23.3642	1.1006	
5	30	22.8	0.0468	-0.8257	30.0468	21.9743	0.2323	-0.3079	30.2791	21.6664	1.2932	
6	30	21	0.0487	-0.7745	30.0487	20.2255	0.2438	-0.2569	30.2925	19.9686	1.3588	
7	30	19.2	0.0468	-0.7234	30.0468	18.4766	0.2323	-0.2059	30.2791	18.2707	1.2967	
8	30	17.4	0.0409	-0.6722	30.0409	16.7278	0.1985	-0.1547	30.2394	16.5731	1.1006	
9	30	15.6	0.0312	-0.6211	30.0312	14.9789	0.145	-0.1034	30.1762	14.8755	0.8007	
10	30	13.8	0.0176	-0.5699	30.0176	13.2301	0.0766	-0.0518	30.0942	13.1783	0.4213	
11	30	12	0.0001	-0.5188	30.0001	11.4812	0	0	30.0001	11.4812	0	
Total Deflection			0.0487				0.2438				1.3588	

Figure 12: Recorded Deflections for each Loading Iteration

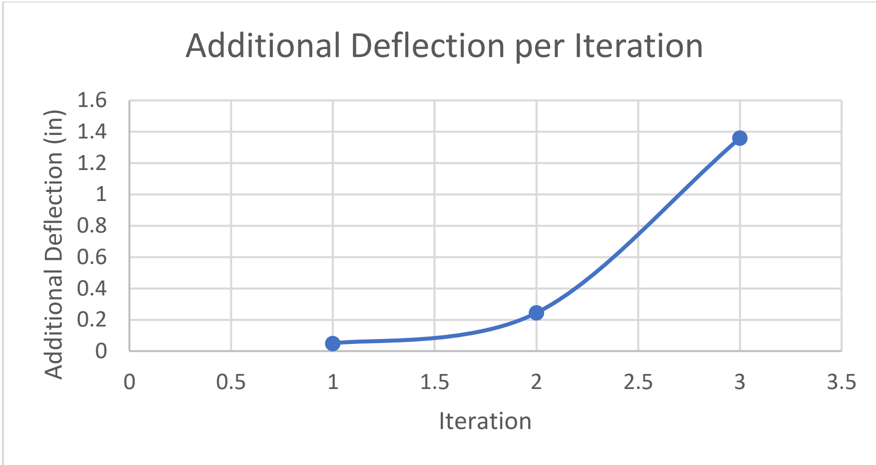


Figure 13: Total Deflection for Each Column Iteration

Even though the loading on the column is less than the calculated critical buckling load, the column continues to deflect. This shows that the column is in fact buckling. Showing, a frame system with asymmetric beam loading that causes joint rotation does influence the critical buckling load. Showing that the current nomograph is not a perfectly viable solution for the critical buckling load of a column in a framed system.

4. Conclusion

Throughout the semester, several tasks were completed to create a finalized thesis. Initially, research was done on column buckling, Leonhard Euler, and the derivation of Euler's buckling equation. Once it was understood where the critical buckling load equation came from and how it works, the Julian & Lawrence nomograph on effective length factors was examined. The nomograph is the current method of determining K values for framed systems where only axial load is taken into consideration. Matrix Analysis was then used to test the theory of if the effective length of a column is affected by asymmetric beam loading. It took several weeks of testing on Visual Analysis and SAP2000, modeling columns and framed systems with different types of loading patterns to test the hypothesis. Through a manual P-Delta analysis, it was determined that asymmetric beam loading causes excess joint rotation that is not accounted for in the nomograph. Thus, when loaded in a checkered pattern the effective column length is longer than expected which would give a lower critical buckling load. Meaning, when framed systems are asymmetrically loaded the nomograph is not correct and the critical column buckling load is not calculated correctly.

For the next student who begins research on effective column lengths and how loading affects it, it is very important to know what the goal of the thesis is. That way there is no confusion along the way of what to do. Another important step is to familiarize oneself with the column buckling, Euler's solution, and the nomograph, as was done early on in this paper. Once all the research is done, it is important to familiarize oneself with a matrix analysis software. I would recommend from the beginning, trying out a new software with a P-delta analysis to see if results can be obtained that way as well. It can always be done manually on Visual Analysis; however it may prove better if done on another software to get results in a different way.

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