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ELEMENTS OF KINEMATICS WITH APPLICATIONS TO PROBLEMS IN NAVIGATION AND ROBOTICS

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SUMMARY

Concepts concerning kinematics are presented that are fundamental for the description of motion of rigid bodies. These concepts are formulated using the mathematical setting provided by rotation matrices, Euler's vectors, Euler's angles and unitary quaternions. Basic algebraic and differential properties of these elements are reviewed with emphasis on their physical interpretation and on application examples relevant to problems encountered in the navigation and robotics fields.

1. MEASURE OF A VECTOR RELATIVE TO A REFERENCE FRAME

A vector, $\underline{v}$, is a mathematical object characterized by a direction and an amplitude. In the 3-dimensional Euclidean space, $\underline{v}$ can represent a variety of physical elements: a force, a velocity (angular or linear velocity), the position of a point, a direction etc.. Independently from its physical interpretation, the measure of $\underline{v}$ relative to a reference frame A, denoted with the symbol $[\underline{v}]_A$, is represented by a triplet of scalars

$$[\underline{v}]_A := \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \tag{1}$$

where v_x, v_y, v_z are such that

$$\underline{v} = \begin{bmatrix} \underline{i} & \underline{j} & \underline{k} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = v_x \underline{i} + v_y \underline{j} + v_z \underline{k}, \tag{2}$$

$\underline{i}$, $\underline{j}$, $\underline{k}$ being the directional vectors of A.

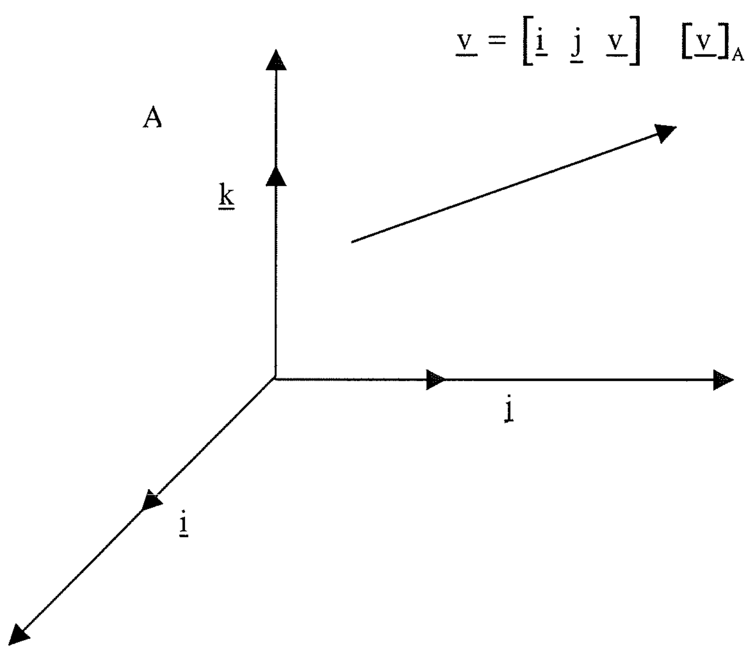

Figure 1: The notion of measure of a vector relative to a frame

Example 1: The velocity of a mobile A with respect to a mobile B is a vector that we denote with the symbol ${}^{B}\underline{v}_{A}$. To mathematically represent this vector we can give its measure relative to a frame attached to A, or a frame attached to B, or any other arbitrarily chosen frame C. These measures denoted with the symbols $\left[{}^{B}\underline{v}_{A}\right]_{A}$, $\left[{}^{B}\underline{v}_{A}\right]_{B}$, $\left[{}^{B}\underline{v}_{A}\right]_{C}$, are usually different one from the other. However, each one of them, together with the triad of directional vectors associated with the reference frame, represents in a bi-univocal fashion the vector ${}^{B}\underline{v}_{A}$.

Example 2: The position of a point P relative to a frame, A, is defined by the measure relative to A, $\left[\overrightarrow{0_A P}\right]_A$, of the vector $\overrightarrow{0_A P}$ obtained by joining the origin of A with P. The position of P with respect to any other frame, B, represents the measure of vector $\overrightarrow{0_B P}$, relative to B.

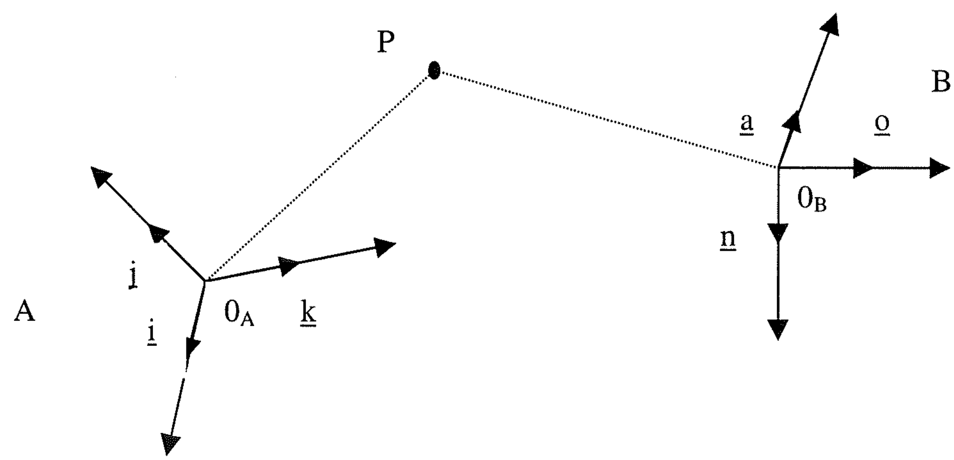

Figure 2: Measure of the position of a point with respect to a frame

2. PHYSICAL ELEMENTS INTERVENING IN THE DESCRIPTION OF THE MOTION OF A RIGID BODY

Given two frames, A and B, the position of A with respect to B is defined by the measure relative to B, $\left[\overrightarrow{0_B 0_A}\right]_B$, of the vector joining the origin of B with the origin of A, $\overrightarrow{0_B 0_A}$. The orientation of A with respect to B is defined by the orientation of the directional vectors of A with respect to the directional vectors of B. This orientation is often represented by the Euler vector $\vartheta\underline{v}$ (or equivalently ($\underline{v}$, ϑ)), a vector defined by the property that by imposing a rotation of an angle ϑ about the unitary vector $\underline{v}$ to a frame C, initially coincident with B, C assumes the same orientation as A.

The linear velocity of A with respect to B is defined by the equation

$$\left[{}^{B}v_{A}\right]_{B} = \frac{d}{dt}\left[\overrightarrow{0_{B}\,0_{A}}\right]_{B} \tag{1}$$

where $\overrightarrow{0_B 0_A}$ is the vector describing the position of A with respect to B. The angular velocity of A with respect to B is defined by the equation

$$\left[{}^{B}\Omega_{A}\right]_{B} = \lim_{\Delta t \to 0}\left[\frac{{}^{B}\Theta_{A}(\Delta t)}{\Delta t}\right]_{B}, \tag{2}$$

where ${}^{B}\Theta_{A}(\Delta t)$ is the Euler vector representing the change of orientation of A with respect to B that has occurred in the time interval (t, t + Δt).

Linear and angular accelerations of A with respect to B are defined by the following equations

$$\left[{}^{B}a_{A}\right]_{B} = \frac{d}{dt}\left[{}^{B}v_{A}\right]_{B} \tag{3}$$

$$\left[{}^{B}a_{\Omega_A}\right]_{B} = \frac{d}{dt}\left[{}^{B}\Omega_{A}\right]_{B}. \tag{4}$$

3. ROTATION MATRICES

A rotation matrix is a 3 x 3 dimensional ortho-normal matrix describing the orientation of a frame B with respect to a reference frame A. More in particular,

$$\mathrm{Rot}(A,B) := \left[[\underline{n}]_A, [\underline{o}]_A, [\underline{a}]_A\right] \tag{1}$$

where $[\underline{n}]_A, [\underline{0}]_A, [\underline{a}]_A$ represent the measures relative to A of the directional vectors of B. Matrix Rot(A,B) is often called the cosinus matrix because its entries represent the cosinuses of the angles that the directional vectors of A form with the directional vectors of B.

Remark 1: In some scientific applications (as for example in computer graphics or in some aeronautical applications), the rotation matrix is rather defined as given by

$$\text{Rot}(A,B) := \left[[\underline{n}]_A, [\underline{o}]_A, [\underline{a}]_A\right]' \tag{2}$$

where $\left[[\underline{n}]_A, [\underline{o}]_A, [\underline{a}]_A\right]$ represent once again the measure relative to A of the directional vectors of B. This alternative definition reflects a preference to work with row vectors rather than with column vectors.

Remark 2: With $[\underline{v}]_A$ and $[\underline{v}]_B$ denoting the measures relative to frames A and B of a given vector $\underline{v}$, one has $[\underline{v}]_B = \text{Rot}(B,A)[\underline{v}]_A$.

Proof: Let $\underline{i}$, $\underline{j}$ and $\underline{k}$ the directional vectors of B; $\underline{n}$, $\underline{o}$ and $\underline{a}$ the directional vectors of A.

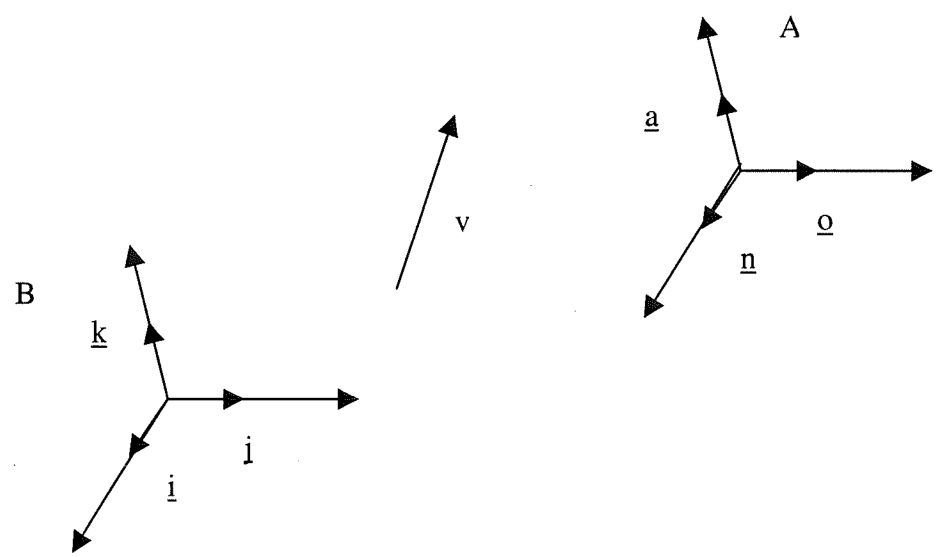

Figure 1: Relation between the measures $[\underline{v}]_A$, $[\underline{v}]_B$ of a given vector.

From the definition of $[\underline{v}]_A$ and $[\underline{v}]_B$ one has

$$\underline{v} = [\underline{i} \quad \underline{j} \quad \underline{k}][\underline{v}]_B \tag{3}$$

$$\underline{v} = [\underline{n} \quad \underline{o} \quad \underline{a}][\underline{v}]_A \cdot \tag{4}$$

Since

$$\text{Rot}(B, A) = [[\underline{n}]_B \ [\underline{o}]_B \ [\underline{a}]_B] \tag{5}$$

with

$$\underline{n} = [\underline{i} \quad \underline{j} \quad \underline{k}][\underline{n}]_B \tag{6}$$

$$\underline{o} = [\underline{i} \quad \underline{j} \quad \underline{k}][\underline{o}]_B \tag{7}$$

$$\underline{a} = [\underline{i} \quad \underline{j} \quad \underline{k}][\underline{a}]_B \tag{8}$$

it follows

$$[\underline{n} \quad \underline{o} \quad \underline{a}] = [\underline{i} \quad \underline{j} \quad \underline{k}]\,\text{Rot}\,(B, A) \tag{9}$$

From (3) and (4) one has

$$\begin{aligned} [\underline{i} \quad \underline{j} \quad \underline{k}][\underline{v}]_B &= [\underline{n} \quad \underline{o} \quad \underline{a}][\underline{v}]_A \\ &= [\underline{i} \quad \underline{j} \quad \underline{k}]\,\text{Rot}\,(B, A)\,[\underline{v}]_A\ , \end{aligned} \tag{10}$$

hence

$$[\underline{v}]_B = \text{Rot}(B, A)[\underline{v}]_A . \tag{11}$$

Remark 3: With $\left[\overrightarrow{0_A P}\right]_A, \left[\overrightarrow{0_B P}\right]_B$ the position relative to A and B of a point P, one has

$$\begin{bmatrix} \left[\overrightarrow{0_B P}\right]_B \\ 1 \end{bmatrix} = T[B, A] \begin{bmatrix} \left[\overrightarrow{0_A P}\right]_A \\ 1 \end{bmatrix}. \tag{12}$$

where T[B,A] is the 4x4 homogeneous transformation matrix defined by the position/orientation of A with respect to B (Craig, chapter 2),

$$T[B,A] := \left[\begin{array}{c|c} \mathrm{Rot}(B,A) & \left[\overrightarrow{0_B\, 0_A}\right]_B \\ \hline 0 \quad 0 \quad 0 & 1 \end{array}\right] . \tag{13}$$

3.1 Product and inversion of rotation matrices

With the orientations of B with respect to A, of C with respect to B, and of A with respect to C, represented by the matrices Rot(A,B), Rot(B,C) et Rot(C,A), the following relations hold

$$\begin{gathered} \mathrm{Rot}(B,A) = \mathrm{Rot}(B,C)\ \mathrm{Rot}(C,A) \\ \mathrm{Rot}(A,B) = \mathrm{Rot}(B,A)^{-1} = \mathrm{Rot}(B,A)\,' \end{gathered} \tag{1}$$

Proof: With $[\underline{v}]_A, [\underline{v}]_B, [\underline{v}]_C$ denoting the measures of a vector $\underline{v}$ relative to A, B, C, one has

$$[\underline{v}]_B = \mathrm{Rot}(B,A)\,[\underline{v}]_A\ ; [\underline{v}]_B = \mathrm{Rot}(B,C)\,[\underline{v}]_C\ ; [\underline{v}]_C = \mathrm{Rot}(C,A)\,[\underline{v}]_A . \tag{2}$$

It follows

$$\mathrm{Rot}(B,A)\,[\underline{v}]_A = \mathrm{Rot}(B,C)\,\mathrm{Rot}(C,A)\,[\underline{v}]_A \text{ for each } [\underline{v}]_A . \tag{3}$$

which implies

$$\begin{gathered} \mathrm{Rot}(B,A) = \mathrm{Rot}(B,C)\ \mathrm{Rot}(C,A), \\ \mathrm{Rot}(B,A)\ \mathrm{Rot}(A,B) = I_3. \end{gathered} \tag{4}$$

Remark 1: It is often useful to represent the relation of orientation among different frames with a directed graph as indicated in Figure 2.

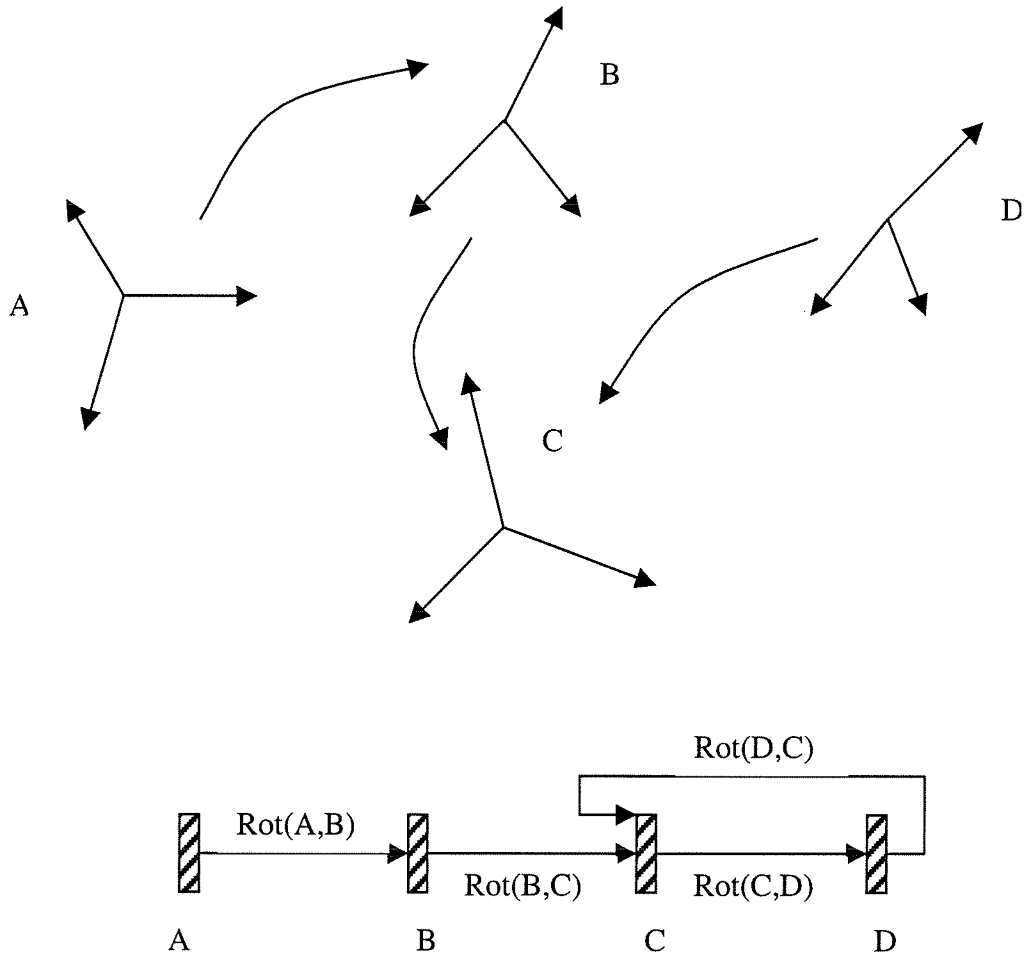

Figure 1: Directed graph representation of the orientations of an ensemble of reference frames.

3.2 Examples

When the orientation of a frame A relative to a frame B corresponds to that obtained by rotating a frame originally coincident with B of an angle γ about the x_ axis, one has

$$\mathrm{Rot}(B,A) = \mathrm{Rot}(x,\gamma) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}. \tag{1}$$

Similarly, if one considers a rotation of an angle β about the y_axis, or a rotation of an α about the z_ axis, then

$$\mathrm{Rot}(y,\beta):=\begin{bmatrix}\cos\beta & 0 & \sin\beta\\ 0 & 1 & 0\\ -\sin\beta & 0 & \cos\beta\end{bmatrix} \tag{2}$$

$$\mathrm{Rot}(z,\alpha):=\begin{bmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}. \tag{3}$$

These matrices describe a rotation about one of the principal axes and are called matrices of simple rotation. Their importance stems from the fact that any arbitrary rotation matrix can be decomposed into the product of matrices of simple rotation, as for example:

$$\mathrm{Rot}(B,A)=\mathrm{Rot}(z,\alpha)\mathrm{Rot}(y,\beta)\mathrm{Rot}(x,\gamma). \tag{4}$$

3.3 Rotation matrices and rotational displacements

When a frame A is submitted to a rotational displacement such that starting from an initial orientation A_0 it ends up into a final orientation A_{fin}, this displacement can be represented by the matrix Rot(A_0, A_{fin}). If A is submitted to a sequence of rotational displacements represented by the matrices R_1, R_2, ... R_n, the final orientation of A (A_{fin}) with respect to its original orientation (A_0) is represented by the matrix

$$\mathrm{Rot}\ (A_0, A_{fin}) = R_1\, R_2 \ldots R_n \tag{1}$$

if the sequence of rotations under consideration is described in a relative fashion; the final orientation of A is described by the matrix

$$\mathrm{Rot}\ (A_0, A_{fin}) = R_n\, R_{n-1} \ldots R_1 \tag{2}$$

if the sequence of rotations is rather described in an absolute fashion.

A sequence of rotations R_1, R_2, ... R_n is described in a relative fashion if for each i = 1, .., n, the i-th rotation matrix R_i describes an orientation relative to a frame obtained by imposing to A the first i-1 rotations. More explicitly, in a relative description one has

$$R_1 = \mathrm{Rot}(A_0, A_1)\ ,\ R_2 = \mathrm{Rot}(A_1, A_2), \ldots \tag{3}$$

where A_i denotes a frame having the orientation that A has after having been submitted to the rotations $R_1, R_2, \ldots R_i$.

To determine the matrix resulting from a sequence of rotations described in this way, let $\left[\overrightarrow{0P_0}\right]_A$ represent the co-ordinates of a point P_0 relative to A_0, a frame having the original orientation of A. After having been submitted to rotation R_1, A_0 will be displaced into A_1, P_0 into P_1 with

$$\left[\overrightarrow{0\,P_1}\right]_{A_0} = R_1\left[\overrightarrow{0\,P_1}\right]_{A_1} = R_1\left[\overrightarrow{0\,P_0}\right]_{A_0} \tag{4}$$

($\left[\overrightarrow{0\,P_1}\right]_{A_1} = \left[\overrightarrow{0\,P_0}\right]_{A_0}$ since P_1 has, relative to A_1, the same position as P_0 relative to A_0). After having been submitted to R_2, A_1 will be displaced into A_2, P_1 into P_2, with

$$\left[\overrightarrow{0\,P_2}\right]_{A_0} = R_1\left[\overrightarrow{0\,P_2}\right]_{A_1} = R_1R_2\left[\overrightarrow{0\,P_2}\right]_{A_2} = R_1R_2\left[\overrightarrow{0\,P_0}\right]_{A_0}. \tag{5}$$

Reiterating the argument, it follows that $\left[\overrightarrow{0\,P_n}\right]_{A_0} = R_1R_2\cdots R_n\left[\overrightarrow{0\,P_0}\right]_{A_0}$. Hence,

$$\mathrm{Rot}(A_0, A_{fin}) = R_1R_2\cdots R_n. \tag{6}$$

A sequence of rotations $R_1, \ldots R_n$ is described in absolute terms if each matrix R_i represents the change in orientation relative to a fixed frame that a frame A is submitted to, after having been submitted to $R_1, \ldots R_{i-1}$. To determine the rotation matrix resulting from the implementation of this sequence of rotations, let $\left[\overrightarrow{0\,P_0}\right]_{A_0}$ represent the co-ordinates of a point P_0 relatives à A_0, a frame with the original orientation of A. After having been submitted to R_1, P_0 will be displaced into P_1 with

$$[0\,P_1]_{A_0} = R_1[0\,P_0]_{A_0}. \tag{7}$$

After having been submitted to R_2, P_1 will be displaced into P_2 with

$$[0\,P_2]_{A_0} = R_2[0\,P_1]_{A_0} = R_2R_1[0\,P_0]_{A_0}. \tag{8}$$

After having been submitted to R_n, P_{n-1} will be displaced into P_n with

$$[0\,P_n]_{A_0} = R_n \cdots R_2R_1[0\,P_0]_{A_0}. \tag{9}$$

It follows that the rotation resulting from the application of the sequence $R_1, \ldots R_n$ is described by the matrix

$$\mathrm{Rot}(A_0, A_n) = R_n \ldots R_1. \tag{10}$$

Example: Let R_1 and R_2 represent two rotations described by the matrices

$$R_1 = \mathrm{Rot}(x, 90^0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{11}$$

$$R_1 = \mathrm{Rot}(z, 90^0) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{12}$$

If these two rotations are applied by viewing the sequence $\{R_1, R_2\}$ as described in a relative fashion, the resultant rotation matrix is

$$R_{rés} = R_1R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{13}$$

If these two rotations are applied by viewing the sequence $\{R_1, R_2\}$ as described in an absolute fashion, then

$$R_{rés} = R_2R_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \neq R_1R_2 . \tag{14}$$

3.4 Euler vector and rotation matrix

The orientation of a frame B relative to a frame A is described by the Euler vector $w := \vartheta\,\underline{k}$, with ϑ a scalar and $\underline{k}$ a unitary vector, if the orientation of B can be thought as obtained by imposing to a frame initially coincident with A a rotation of an angle ϑ about $\underline{k}$.

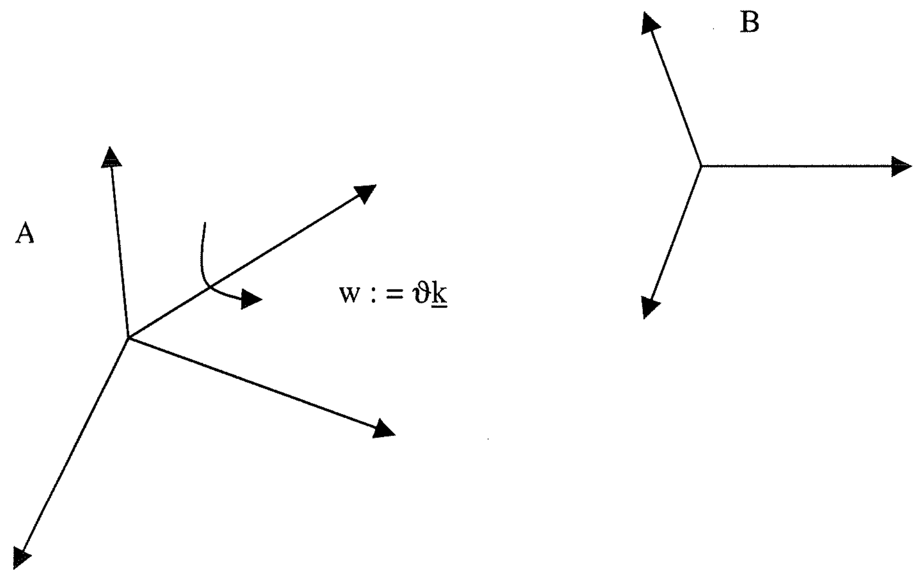

Figure 1: Euler's vector: the orientation of B is obtained by rotating A of an angle ϑ (counter clockwise) about $\underline{k}$

Let the orientation of a frame B relative to a frame A be represented by the Euler's vector

$$\vartheta\,\underline{k}\quad,\quad [\underline{k}]_A = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \tag{1}$$

and by the rotation matrix

$$\mathrm{Rot}(A,B) := [r_{ij}] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}. \tag{2}$$

These two representations are related via the following equations:

$$[r_{ij}] = \begin{bmatrix} k_x k_x v\vartheta + c\vartheta & k_x k_y v\vartheta - k_z s\vartheta & k_x k_z v\vartheta + k_y s\vartheta \\ k_x k_y v\vartheta + k_z s\vartheta & k_y k_y v\vartheta + c\vartheta & k_y k_z v\vartheta - k_x s\vartheta \\ k_x k_z v\vartheta - k_y s\vartheta & k_y k_z v\vartheta + k_x s\vartheta & k_z k_z v\vartheta + c\vartheta \end{bmatrix} \tag{3}$$

where

$$\begin{aligned} v\vartheta &:= 1 - c\vartheta \\ c\vartheta &:= \cos\vartheta \\ s\vartheta &:= \sin\vartheta \cdot \end{aligned} \tag{4}$$

Inversely, we have

$$\vartheta = \operatorname{Arc}\cos\left\{\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right\} \quad 0 \le \vartheta \le 180 \tag{5}$$

$$[\underline{k}]_A = \frac{1}{\sqrt{(r_{32} - r_{23})^2 + (r_{13} - r_{31})^2 + (r_{21} - r_{12})^2}} \begin{bmatrix} r_{32} & - & r_{23} \\ r_{13} & - & r_{31} \\ r_{21} & - & r_{12} \end{bmatrix}. \tag{6}$$

Proof: i) From Euler vector to rotation matrix.

Let rotation matrix $\mathrm{Rot}(A,B) = \mathrm{Rot}\left([\underline{k}]_{R_A}, \vartheta\right)$ represent an orientation obtained by applying to a frame B initially coincident with frame A a rotation of an angle ϑ about unitary vector $\underline{k}$. By applying to a frame initially coincident with an arbitrarily chosen frame R_A, a rotation of ϑ about $\underline{k}$, we have a frame R_B such that

$$\mathrm{Rot}(R_A, R_B) = \mathrm{Rot}(A,B) = \mathrm{Rot}\left([\underline{k}]_{R_A}, \vartheta\right). \tag{7}$$

R_B and B having been obtained by imposing the same rotation to two frames initially coincident respectively with R_A and A, the orientation of R_B relative to B is identical to the orientation of R_A relative to A. It follows,

$$\mathrm{Rot}(A, R_A) = \mathrm{Rot}(B, R_B). \tag{8}$$

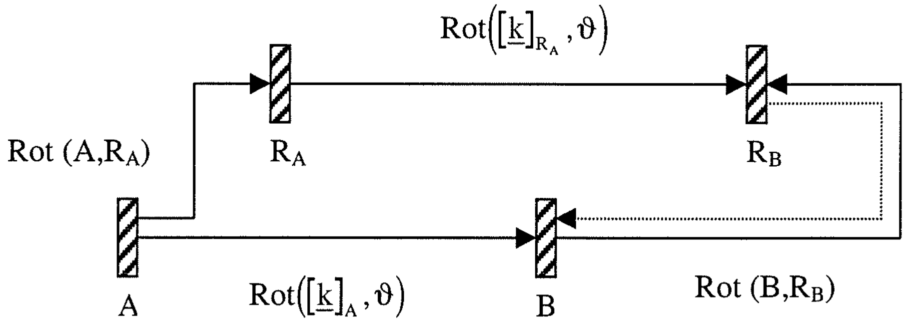

Figure 2 : Rotation matrix and Euler vector

With the help of figure 1, we can then write

$$\begin{aligned}\mathrm{Rot}(A,B)=\mathrm{Rot}([k]_A,\vartheta)&=\mathrm{Rot}(A,R_A)\mathrm{Rot}([\underline{k}]_{R_A},\vartheta)\mathrm{Rot}(R_B,B)\\&=\mathrm{Rot}(A,R_A)\mathrm{Rot}([\underline{k}]_{R_A},\vartheta)\mathrm{Rot}(R_A,A)\\&=\mathrm{Rot}(A,R_A)\mathrm{Rot}([\underline{k}]_{R_A},\vartheta)\mathrm{Rot}'(A,R_A).\end{aligned}\tag{9}$$

By now choosing R_A so that the x_axis of R_A coincides with $\underline{k}$, that is so that $[\underline{k}]_{R_A}=\begin{bmatrix}1\\0\\0\end{bmatrix}$,

we have

$$\mathrm{Rot}(A,R_A)=\mathrm{Rot}(z,\alpha)\mathrm{Rot}(y,\beta)\tag{10}$$

where α and β represent the yaw and pitch angles of $\underline{k}$ relative to A:

$\alpha :=$ angle between the $\underline{z}$ $\underline{x}$ plane of A and the plane $\underline{z}$ $\underline{k}$

$\beta :=$ angle between the $\underline{x}$ $\underline{y}$ plane of A and vector $\underline{k}$.

Using the notation $[\underline{k}]_A=[k_x,k_y,k_z]'$, we have

$$\cos\beta = \sqrt{k_x^2 + k_y^2} \quad , \quad \sin\beta = k_z$$

$$\cos\alpha = k_x \Big/ \sqrt{k_x^2 + k_y^2}$$

$$\sin\alpha = k_y \Big/ \sqrt{k_x^2 + k_y^2} ,$$

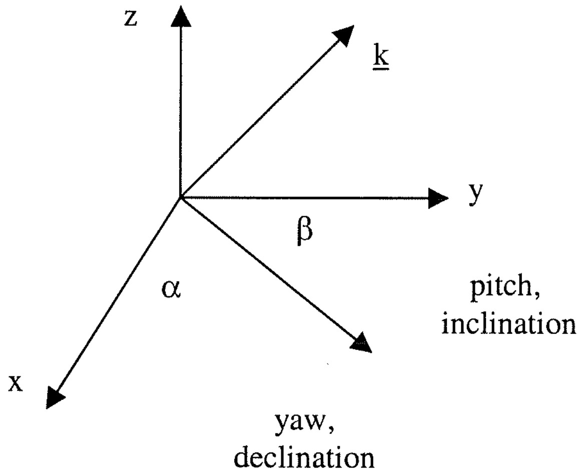

Figure 3 : Pitch and yaw angles of a vector relative to a frame.

Equation (9) becomes

$$\mathrm{Rot}([\underline{k}]_A, \vartheta) = \mathrm{Rot}(z,\alpha)\mathrm{Rot}(y,\beta)\mathrm{Rot}(x,\vartheta)\mathrm{Rot}(y,\beta)'\,\mathrm{Rot}(z,\alpha)'. \tag{11}$$

Considering that

$$\mathrm{Rot}(x,\vartheta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\vartheta & -s\vartheta \\ 0 & s\vartheta & c\vartheta \end{bmatrix} \tag{12}$$

$$\mathrm{Rot}(y,\beta) = \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \tag{13}$$

$$\mathrm{Rot}(x,\alpha) = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{14}$$

and implementing the products specified by equations (9,10,11), it follows

$$[r_{ij}] = \begin{bmatrix} k_x k_x v\vartheta + c\vartheta & k_x k_y v\vartheta - k_z s\vartheta & k_x k_z v\vartheta + k_y s\vartheta \\ k_x k_y v\vartheta + k_z s\vartheta & k_y k_y v\vartheta + c\vartheta & k_y k_z v\vartheta - k_x s\vartheta \\ k_x k_z v\vartheta - k_y s\vartheta & k_y k_z v\vartheta + k_x s\vartheta & k_z k_z v\vartheta + c\vartheta \end{bmatrix} \tag{15}$$

ii) From rotation matrix to Euler vector.

Letting

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} k_x k_x v\vartheta + c\vartheta & k_y k_x v\vartheta - k_z s\vartheta & k_z k_x v\vartheta + k_y s\vartheta \\ k_x k_y v\vartheta + k_z s\vartheta & k_y k_y v\vartheta + c\vartheta & k_z k_y v\vartheta - k_x s\vartheta \\ k_x k_z v\vartheta - k_y s\vartheta & k_y k_z v\vartheta + k_x s\vartheta & k_z k_z v\vartheta + c\vartheta \end{bmatrix}, \qquad (16)$$

observing that

$$r_{11} + r_{22} + r_{33} = k_x^2 + k_y^2 + k_z^2 - c\vartheta + 3c\vartheta = 1 + 2c\vartheta, \qquad (17)$$

we deduce

$$\vartheta = \text{arc}\cos^{-1}\left\{\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right\}, \qquad \vartheta \in [0, \pi]. \qquad (18)$$

In a similar fashion, from

$$r_{32} - r_{23} = -k_z k_y v\vartheta + k_x s\vartheta + k_y k_z v\vartheta + k_x s\vartheta = 2k_x s\vartheta \qquad (19)$$

it follows

$$k_x = \frac{r_{32} - r_{23}}{2s\vartheta}; \qquad (20)$$

from

$$r_{13} - r_{31} = k_z k_x v\vartheta + k_y s\vartheta - k_x k_z v\vartheta + k_y c\vartheta = 2k_y s\vartheta \qquad (21)$$

it follows

$$k_y = \frac{r_{13} - r_{31}}{2s\vartheta}; \qquad (22)$$

from

$$r_{21} - r_{12} = k_x k_y v\vartheta + k_z s\vartheta - k_x k_y v\vartheta + k_z s\vartheta = 2k_z s\vartheta \qquad (23)$$

it follows

$$k_z = \frac{r_{21} - r_{12}}{2s\vartheta}. \qquad (24)$$

we can then conclude

$$[\underline{k}]_A = \begin{bmatrix} k_x \\ k_y \\ k_v \end{bmatrix} = \frac{1}{2s\vartheta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \qquad (25)$$

or, equivalently,

$$[\underline{k}]_A = \frac{1}{\sqrt{(r_{32}-r_{23})^2+(r_{13}-r_{31})^2+(r_{21}-r_{12})^2}}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix} . \tag{26}$$

3.5 Angular velocity matrix

Given Euler vector $\vartheta\underline{v}$ with $\vartheta \cong \Delta\vartheta << 1$, one can use the approximation

$$\text{rot}([\underline{v}]_A, \vartheta) \cong \begin{bmatrix} 1 & -\Delta\vartheta_z & \Delta\vartheta_y \\ \Delta\vartheta_z & 1 & -\Delta\vartheta_x \\ -\Delta\vartheta_y & \Delta\vartheta_x & 1 \end{bmatrix} \tag{1}$$

where

$$\begin{aligned} \Delta\vartheta_x &:= \Delta\vartheta\, v_x \\ \Delta\vartheta_y &:= \Delta\vartheta\, v_y \\ \Delta\vartheta_z &:= \Delta\vartheta\, v_z \; . \end{aligned} \tag{2}$$

Introducing the matrix $S\left\{\begin{bmatrix} \Delta\vartheta_x \\ \Delta\vartheta_y \\ \Delta\vartheta_z \end{bmatrix}\right\}$ defined by the equation

$$S\left\{\begin{bmatrix} \Delta\vartheta_x \\ \Delta\vartheta_y \\ \Delta\vartheta_z \end{bmatrix}\right\} := \begin{bmatrix} 0 & -\Delta\vartheta_z & \Delta\vartheta_y \\ \Delta\vartheta_z & 0 & -\Delta\vartheta_x \\ -\Delta\vartheta_y & \Delta\vartheta_x & 0 \end{bmatrix} \tag{3}$$

we have

$$\text{Rot}([\underline{k}]_A, \Delta\vartheta) \cong I_3 + S\left\{\begin{bmatrix} \Delta\vartheta_x \\ \Delta\vartheta_y \\ \Delta\vartheta_z \end{bmatrix}\right\} . \tag{4}$$

Remark 1: The matrix, S(a), associated to a triplet of scalars $a = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$ by the relation

$$S(a) := \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \tag{5}$$

is called the angular velocity matrix. It is a skew symmetric matrix ($S + S' = 0$) enjoying the property that for any other triplet of scalars $b := \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$, one has

$$S(a)b = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} -a_z b_y & + & a_y b_z \\ a_z b_x & - & a_x b_z \\ -a_y b_x & + & a_x b_y \end{bmatrix} = a \wedge b . \tag{6}$$

Remark 2: For any pair of frames A and B and for any vector $\underline{v}$ one has

$$S\{[\underline{v}]_B\} = Rot(B,A)\, S\{[\underline{v}]_A\}\, Rot(A,B) \cdot \tag{7}$$

Proof: Given a pair of vectors $\underline{v}$ and $\underline{w}$ we have

$$\begin{aligned} [\underline{v} \wedge \underline{w}]_B &= S\{[\underline{v}]_B\}[w]_B = Rot(B,A)\,[\underline{v} \wedge \underline{w}]_A \\ &= Rot(B,A)\, S\{[\underline{v}]_A\}[w]_A \\ &= Rot(B,A)\, S\{[\underline{v}]_A\}\, Rot(A,B)[w]_B \end{aligned} \tag{8}$$

Equation (7) follows from the fact that this equality has to be satisfied for any $[w]_B$.

3.6 Propagation of rotation matrices

Let the orientation of B relative to A be described by the matrix Rot(A,B)(t); let ${}^A\Omega_B$ denote the angular velocity of B with respect to A, and let $[{}^A\Omega_B]_A$, $[{}^A\Omega_B]_B$ be the measures relative to A and B of ${}^A\Omega_B$. The relation among Rot(A,B)(t), ${}^A\Omega_B$ and $\dot{Rot}(A,B)(t)$ is described by the following law of propagation

$$\dot{\mathrm{Rot}}(A,B)\,(t) = S\left\{\left[{}^{A}\Omega_{B}\right]_{A}\right\} \mathrm{Rot}(A,B)\,(t)$$

$$= \mathrm{Rot}(A,B)(t)\;\; S\left\{\left[{}^{A}\Omega_{B}\right]_{B}\right\}. \tag{1}$$

This equation allows one to use measurements of ${}^{A}\Omega_{B}$ to compute the orientation of B with respect to A. Inversely, from

$$S\left\{\left[{}^{A}\Omega_{B}\right]_{A}\right\} = \dot{\mathrm{Rot}}(A,B)\,\mathrm{Rot}'(A,B)\,(t)$$

$$S\left\{\left[{}^{A}\Omega_{B}\right]_{B}\right\} = \mathrm{Rot}(B,A)\,\dot{\mathrm{Rot}}(A,B) \tag{2}$$

it also allows one to compute ${}^{A}\Omega_{B}$ from Rot(A,B) (t) and its derivative.

Proof: Law of propagation of rotation matrices.

Denoting with $\left[\Delta\vartheta_x \Delta\vartheta_y \Delta\vartheta_z\right]$ the measure relative to A of the Euler vector representing the rotation required for frame B to pass from the orientation described by Rot (A,B) (t) : = R(t) to that described by Rot (A,B) (t + Δt) : = R (t + Δt), we have

$$R(t+\Delta t) = R(\Delta t)R(t) = \mathrm{Rot}\left(\Delta\vartheta_x, \Delta\vartheta_y, \Delta\vartheta_z\right)R(t). \tag{3}$$

By now applying the relation between a rotation matrix and an Euler vector, it follows

$$R(t+\Delta t) = \left[I_3 + S\left(\left[\Delta\vartheta_x\;\Delta\vartheta_y\;\Delta\vartheta_z\right]\right)\right] R(t), \tag{4}$$

hence

$$\lim_{\Delta t \to 0} \frac{R(t+\Delta t) - R(t)}{\Delta t} = \lim_{\Delta t \to 0} S\left(\left[\frac{\Delta\vartheta_x}{\Delta t}\;\; \frac{\Delta\vartheta_y}{\Delta t}\;\; \frac{\Delta\vartheta_z}{\Delta t}\right]'\right) R(t) \tag{5}$$

$$= S\left(\left[{}^{A}\Omega_{B}\right]_{A}\right) \mathrm{Rot}(A,B), \tag{6}$$

and therefore

$$\dot{Rot}(A,B)(t) = S\{[{}^{A}\Omega_{B}]_{A}\}\,Rot(A,B). \tag{7}$$

On the other hand, from

$$\dot{Rot}(A,B) = S\{[{}^{A}\Omega_{B}]_{A}\}\,Rot(A,B) \tag{8}$$

one can write

$$\begin{aligned} Rot'(A,B)\,\dot{Rot}(A,B) &= Rot'(A,B)\,S\{[{}^{A}\Omega_{B}]_{A}\}\,Rot(A,B) \\ &= Rot(B,A)\,S\{[{}^{A}\Omega_{B}]_{A}\}\,Rot(A,B) \\ &= S\{[{}^{A}\Omega_{B}]_{B}\} \end{aligned} \tag{9}$$

hence

$$S\{[{}^{A}\Omega_{B}]_{B}\} = Rot(B,A)\,\dot{Rot}(A,B). \tag{10}$$

Remark 1: When matrix $R(\Delta t)$ is described in terms of $[{}^{A}\Omega_{B}]_{A}$, then

$$R(t+\Delta t) = R(\Delta t)\,R(t)\,; \tag{11}$$

when described in terms of $[{}^{A}\Omega_{B}]_{B}$ then $R(t + \Delta t) = R(t)\ R(\Delta t)$.

Remark 2: If frame C has angular velocity ${}^{B}\Omega_{C}$ with respect to B, then

$${}^{A}\Omega_{C} = {}^{B}\Omega_{C} + {}^{A}\Omega_{B}. \tag{12}$$

Proof: From Rot(A,C) = Rot(A,B) Rot(B,C), the law of propagation of rotation matrices gives

$$\dot{Rot}(A,C) = S\{[{}^{A}\Omega_{C}]_{A}\}\,Rot(A,C)\ . \tag{13}$$

We also have

$$\dot{Rot}(A,C) = \dot{Rot}(A,B)\,Rot(B,C) + Rot(A,B)\,\dot{Rot}(B,C)\ ,$$

hence, using once again the law of propagation of rotation matrices,

$$\dot{Rot}(A,C) = S\left\{\left[{}^{A}\Omega_{B}\right]_{A}\right\} Rot(A,B)\,Rot(B,C) + Rot(A,B)\,S\left\{\left[{}^{B}\Omega_{C}\right]_{B}\right\} Rot(B,C)$$

$$= S\left\{\left[{}^{A}\Omega_{B}\right]_{A}\right\} Rot(A,C) + S\left\{\left[{}^{B}\Omega_{C}\right]_{A}\right\} Rot(A,C) \tag{14}$$

$$= S\left\{\left[{}^{A}\Omega_{B}\right]_{A} + \left[{}^{B}\Omega_{C}\right]_{A}\right\} Rot(A,C).$$

A comparison of (13) with (14) gives

$${}^{A}\Omega_{C} = {}^{A}\Omega_{B} + {}^{B}\Omega_{C}\ . \tag{15}$$

With $A \equiv C$, this equation implies

$$\left[{}^{A}\Omega_{A}\right]_{A} = \left[{}^{A}\Omega_{B}\right]_{A} + \left[{}^{B}\Omega_{A}\right]_{A}\ , \tag{16}$$

and, since $\left[{}^{A}\Omega_{A}\right]_{A} = 0$,

$$\left[{}^{A}\Omega_{B}\right]_{A} = -\left[{}^{B}\Omega_{A}\right]_{A}\ . \tag{17}$$

3.7 Best ortho-normal approximant of a non ortho-normal matrix

When matrix Rot (A,B) is obtained as a result of numerical computations, it may happen that it does not enjoy the ortho-normality property required for it to be a rotation matrix. A matrix $X := [x_{ij}]$ is referred to as the best ortho-normal approximant of a given not necessarily ortho-normal matrix $R := [r_{ij}]$, if X is ortho-normal and if it minimizes the function

$$\sum_{ij} \left(x_{ij} - r_{ij}\right)^2 = Trace(X-R)'(X-R)\ . \tag{1}$$

X can be computed from R using the formula $X = (R\,R')^{1/2}\,R'^{-1}$.

Proof: One first observes the equivalence of the following conditions:

i Trace $(X-R)'(X-R)$ minimal with $X X' = I_3$

ii Trace $(R X')$ maximal with $X X' = I_3$

iii $R X'$ symmetric with $X X' = I_3$.

By imposing $R X' = X R'$, it follows

$$R X' X R' = X R' X R' \tag{2}$$

$$R R' = (X R')^2$$

$$X R' = (R R')^{1/2} \quad \rightarrow X = (R R')^{1/2} R'^{-1} . \tag{3}$$

3.8 Relation between vector measures relative to distinct frames

Let $[\underline{v}]_A$, $[\underline{v}]_B$ be the measures relative to A and B of a vector $\underline{v}$; $\frac{d}{dt}[\underline{v}]_A, \frac{d}{d}[\underline{v}]_B$ the derivatives of these measures.

From

$$[\underline{v}]_A = \text{Rot}(A,B)\,[\underline{v}]_B \tag{1}$$

it follows

$$\frac{d}{dt}[\underline{v}]_A = \dot{\text{Rot}}(A,B)\,[\underline{v}]_B + \text{Rot}(A,B)\frac{d}{dt}[\underline{v}]_B$$

$$= S\left(\left[{}^A\Omega_B\right]_A\right)\text{Rot}(A,B)[\underline{v}]_B + \text{Rot}(A,B)\frac{d}{dt}[\underline{v}]_B \tag{2}$$

$$= S\left[{}^A\Omega_B\right]_A [\underline{v}]_A + \text{Rot}(A,B)\frac{d}{dt}[\underline{v}]_B$$

and therefore

$$\frac{d}{dt}[\underline{v}]_A = \text{Rot}(A,B)\left[\left[{}^A\Omega_B\right]_B \wedge [v]_B + \frac{d}{dt}[\underline{v}]_B\right]. \tag{3}$$

4. EULER ANGLES (RPY)

The Euler angles are a triplet of angles (roll, pitch and yaw) describing the orientation of a frame B relative to a frame A. They are defined as follows.

Pitch (β): angle that the x_axis of B forms with the xy plane of A;

Roll (γ): angle of the clockwise rotation about its x_axis that B must undergo for its y_axis to become parallel to the xy plane of A;

Yaw (α): angle of the counter clockwise rotation about its z_axis that A must be submitted to for its x_axis to become parallel to the plane defined by the z_axis of A and the x_axis of B.

4.1 Relation between Euler angles and rotation matrix

With the orientation of B relative to A described by the Euler angles γ, β, α, the matrix Rot(A,B) can be determined by considering the rotation necessary for a frame initially coincident with A to assume the orientation of B defined by γ, β and α.

This rotation can be determined by considering the following steps:

i) Let the orientation of a frame R^1 coincide with that of A. By submitting R^1 to a rotation Rot(z,α), (i.e.: a rotation about the z_axis of angle α), the z_axis of R^1 will coincide with the z_axis of A; its x_axis will be parallel to the plane defined by the z_axis of A and the x_axis of B;

ii) Let R^2 be a frame with initial orientation coincident with R^1. By submitting R^2 to a rotation Rot (y, β), the x_axis of R^2 will be parallel to the x_axis of B, the y_axis of R^2 parallel to the y_axis of R^1;

iii) Let R^3 be a frame with initial orientation coincident with R^2. Let us submit R^3 to a rotation Rot(x, γ). The x_axis of R^3 will then be parallel to the x_axis of B, and the y_axis of R^3 parallel to the y_axis of B.

In short, R^3 has the same orientation as B.

The displacement characterizing the orientation of B relative to A can therefore be viewed as the result of the implementation of the sequence of three rotations Rot(z, α), Rot(y, β), Rot(x, γ). Each of these rotations being described in a relative fashion, it follows

$$\text{Rot(A,B)} = \text{Rot}(z, \alpha)\ \text{Rot}(y, \beta)\ \text{Rot}(x, \gamma)\,. \tag{1}$$

By implementing the computation one obtains

$$\text{Rot}(z,\alpha)\,\text{Rot}(y,\beta)\,\text{Rot}(x,\gamma) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} \tag{2}$$

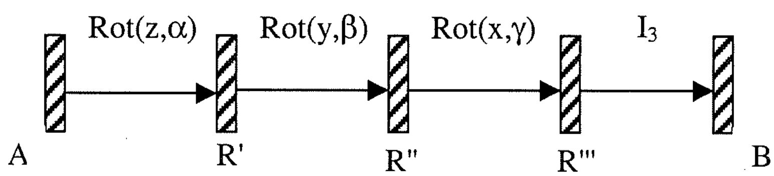

Figure 1 : RPY rotations

Inversely, by setting

$$\text{Rot}(A,B) = \left[r_{ij}\right] \tag{3}$$

one has

$$\beta = \text{A}\tan 2\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right) : \in \left(-90^0, +90^0\right)$$

$$\alpha = \text{A}\tan 2\left(r_{21}/c\beta, r_{11}/c\beta\right) \tag{4}$$

$$\gamma = \text{A}\tan 2\left(r_{32}/c\beta, r_{33}/c\beta\right).$$

Remark 1: The function ϕ = A tan2(y,x) coincides with $\tan^{-1}$ (y/x) except that the value of ϕ is determined by using the quadrant to which a point with co-ordinates (x,y) belongs ($-180^0 \leq \phi \leq 180^0$).

Remark 2: If α, β, $\gamma << 1$, then

$$\mathrm{Rot}(A,B) = \mathrm{Rot}(z,\alpha)\,\mathrm{Rot}(y,\beta)\,\mathrm{Rot}(x,\gamma) \cong \begin{bmatrix} 1 & -\alpha & \beta \\ \alpha & 1 & -\gamma \\ -\beta & \gamma & 1 \end{bmatrix} = I_3 + S\left\{\begin{bmatrix} \gamma \\ \beta \\ \alpha \end{bmatrix}\right\} \tag{5}$$

hence, using the notation $\left[{}^{A}\Omega_B\right]_A' = \left[\Omega_x \Omega_y \Omega_z\right]$,

$$\dot{\alpha} = \Omega_z$$

$$\dot{\beta} = \Omega_y \tag{6}$$

$$\dot{\gamma} = \Omega_x .$$

4.2 Euler angles propagation law

The relation between angular velocity and the derivative of the Euler's angles is obtained by computing the derivative of Rot(A,B) = Rot(z,α) Rot(y,β) Rot(x,γ) (a matrix function of $\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \alpha, \beta, \text{et}\ \gamma$), and by imposing that this derivative be equal to $S\left\{\begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}_A\right\} \mathrm{Rot}(A,B)$ or, equivalently, equal to $\mathrm{Rot}(A,B)\ \ S\left\{\begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}_B\right\}$, according to whether ($\Omega_x$ Ω_y Ω_z) represents a measure relative to A or to B.

When (Ω_x Ω_y Ω_z) represents a measure relative to B, one can write

$$\frac{d}{dt}\left[\mathrm{Rot}(z,\alpha)\mathrm{Rot}(y,\beta)\mathrm{Rot}(x,\gamma)\right] = \mathrm{Rot}(z,\alpha)\mathrm{Rot}(y,\beta)\mathrm{Rot}(x,\gamma)\, S\left\{\begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}_B\right\} \tag{1}$$

that is,

$$
\begin{bmatrix}
-\dot{\alpha}\, s\alpha c\beta - \dot{\beta}\, c\alpha s\beta & \begin{array}{l} -\dot{\alpha}\, s\alpha s\beta s\gamma + \dot{\beta}\, c\alpha c\beta s\gamma + \dot{\gamma}\, c\alpha s\beta c\gamma \\ -\dot{\alpha}\, c\alpha c\gamma \qquad\qquad + \dot{\gamma}\, s\alpha s\gamma \end{array} & \begin{array}{l} -\dot{\alpha}\, s\alpha s\beta c\gamma + \dot{\beta}\, c\alpha c\beta c\gamma \\ +\dot{\alpha}\, c\alpha c\gamma \end{array} \\
\dot{\alpha}\, c\alpha c\beta - \dot{\beta}\, s\beta s\alpha & \begin{array}{l} +\dot{\alpha}\, c\alpha s\beta s\gamma + \dot{\beta}\, c\alpha c\beta s\gamma + \dot{\gamma}\, s\alpha s\beta c\gamma \\ -\dot{\alpha}\, s\alpha s\gamma - \dot{\gamma}\, s\gamma c\alpha \end{array} & \begin{array}{l} \dot{\alpha}\, c\alpha s\beta c\gamma + \dot{\beta}\, s\alpha c\beta c\gamma - \dot{\gamma}\, s\alpha s\beta s\gamma \\ +\dot{\alpha}\, s\alpha s\gamma \qquad\qquad - \dot{\gamma}\, c\alpha c \end{array} \\
-\dot{\beta}\, c\beta & -\dot{\beta}\, s\beta s\gamma + \dot{\gamma}\, c\gamma c\beta & -\dot{\beta}\, s\beta c\gamma \qquad\qquad - \dot{\gamma}\, c\beta s\gamma
\end{bmatrix}
$$

$$
= \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}. \tag{2}
$$

By comparing the 3-1 entry of the left matrix with the correspondent entry of that on the right, one has

$$
c\beta s\gamma \Omega_z - \Omega_y c\beta c\gamma = -\dot{\beta}\, c\beta \tag{3}
$$

hence,

$$
\dot{\beta} = \Omega_y c\gamma - \Omega_z s\gamma . \tag{4}
$$

By now considering entry 3-2, one has

$$
\dot{\gamma}\, c\gamma c\beta - \dot{\beta}\, s\beta s\gamma = \Omega_z s\beta + \Omega_x c\beta c\gamma
$$

whence

$$
\dot{\gamma} = \Omega_z \tan\beta c\gamma + \Omega_x + \Omega_y \tan\beta s\gamma . \tag{5}
$$

From entry 1-1 one has

$$
-\dot{\alpha}\, s\alpha c\beta - \dot{\beta}\, c\alpha s\beta = \Omega_z (c\alpha s\beta s\gamma - s\alpha s\gamma) - \Omega_y (c\alpha s\beta c\gamma + s\alpha s\gamma) \tag{6}
$$

whence

$$
\begin{aligned} -\dot{\alpha}\, s\alpha c\beta = (\Omega_y c\gamma - \Omega_z s\gamma) c\alpha s\beta + \Omega_z (c\alpha s\beta s\gamma - s\alpha s\gamma) \\ - \Omega_z (c\alpha s\beta c\gamma + s\alpha s\gamma) \end{aligned} \tag{7}
$$

and therefore,

$$\dot{\alpha} = \frac{\Omega_z c\gamma}{c\beta} + \frac{\Omega_y s\gamma}{c\beta}. \tag{8}$$

By rewriting equations (4,5,8) under a matrix form, we obtain

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{s\gamma}{c\beta} & \dfrac{c\gamma}{c\beta} \\ 0 & c\gamma & -s\gamma \\ 1 & \tan\beta s\gamma & \tan\beta c\gamma \end{bmatrix} \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}. \tag{9}$$

Note that, in general, $\left(\dot{\alpha}, \dot{\beta}, \dot{\gamma}\right) \neq \left(\Omega_x \Omega_y \Omega_z\right)$.

5. EULER PARAMETERS (UNITARY QUATERNIONS)

With the orientation of frame B relative to frame A represented by the Euler vector $\phi\, \underline{r}$, the representation of this same orientation in terms of Euler parameters is given by the pair (scalar, vector) $p := (\eta, \underline{q})$ where

$$\eta = \cos \phi/2 \qquad \underline{q} := \sin \phi/2\ \underline{r}. \tag{1}$$

In an equivalent fashion, this pair may be viewed as a 4 dimensional vector (unitary quaternion)

$\underline{q} := (\eta, q_1, q_2, q_3)$

$\eta := \cos \phi/2$

$$q_1 := \sin \phi/2\ r_x \qquad q_2 := \sin \phi/2\ r_y \qquad q_3 := \sin \phi/2\ r_z\ . \tag{2}$$

where $[r_x\ r_y\ r_z]'$denotes the measure of $\underline{r}$ relative to B.

A necessary and sufficient condition for a unitary quaternion to represent an ensemble of Euler paramaters is

$$\|\underline{p}\|^2 = \eta^2 + q'q = \eta^2 + q_1^2 + q_2^2 + q_3^2 = 1. \tag{3}$$

5.1 Relation with a rotation matrix

Denoting with Rot(A,B) the orientation of frame B relative to frame A, one has

$$\mathrm{Rot}(A,B)=\begin{bmatrix} 1-2q_2^2-2q_3^2 & 2(q_1q_2-\eta q_3) & 2(q_1q_3+\eta q_2) \\ 2(q_1q_2+\eta q_3) & 1-2q_1^2-2q_3^2 & 2(q_3q_2-\eta q_1) \\ 2(q_1q_3-\eta q_2) & 2(q_2q_3+\eta q_1) & 1-2q_1^2-2q_2^2 \end{bmatrix}. \quad (1)$$

Inversely, using the notation

$$\mathrm{Rot}(A,B):=[r_{ij}] \quad (2)$$

one has

$$q_1=\frac{r_{32}-r_{23}}{4\eta}$$

$$q_2=\frac{r_{13}-r_{31}}{4\eta}$$

$$q_3=\frac{r_{21}-r_{12}}{4\eta} \quad (3)$$

$$\eta=\frac{1}{2}\sqrt{1+r_{11}+r_{22}+r_{33}}\,.$$

Remark 1: These formulas are obtained by considering the relation between the rotation matrix and the Euler vector, by expressing sinφ, cosφ in terms of sinφ/2, cosφ/2, and by identifying the quaternion components.

5.2 Algebraic properties

The Euler parameters enjoy the following properties:

i) Rot (A,B) = $I_3 \Longleftrightarrow (\eta,\underline{q}) = (1, 0_3)$

ii) If $(\eta,\underline{q})$ is the quaternion associated with Rot(B,A) then $(\eta, -q)$ is the quaternion associated with Rot(A,B)'.

iii) By associating to a quaternion the complex notation:

$$(\eta,\underline{q}):=\eta+q_1 i+q_2 j+q_3 k \tag{1}$$

and by defining the products of imaginary numbers i, j, k with the relations :

$$i * i = k * k = j * j = -1$$

$$i * j = k \;\; ; \;\; k * i = j \;\; ; \;\; j * k = i \tag{2}$$

one can define the product of two quaternions

$$(\eta_3,\underline{q}_3)=(\eta_1,\underline{q}_1)(\eta_2,\underline{q}_2), \tag{3}$$

as given by

$$\eta_3+q_{31}i+q_{32}j+q_{33}k=(\eta_1+q_{11}i+q_{12}j+q_{13}k)(\eta_2+q_{21}i+q_{22}j+q_{23}k). \tag{4}$$

or equivalently

$$(\eta_3,\underline{q}_3)=(\eta_1\eta_2-\underline{q}'_1\underline{q}_2,\eta_1\underline{q}_2+\eta_2\underline{q}_1+S(\underline{q}_1)\underline{q}_2). \tag{5}$$

If $(\eta_i, \underline{q}_i)$, i = 1, 2, 3 are quaternions associated with Rot_i, and if $Rot_3 = Rot_1\, Rot_2$, then,

$$(\eta_3,\underline{q}_3)=(\eta_1,\underline{q}_1)(\eta_2,\underline{q}_2). \tag{6}$$

5.3 Euler parameters propagation law

With Rot(A,B) a function of time, i.e. Rot(A,B) = Rot(t), let $(\eta,\underline{q})$ (t) denote the quaternion associated with Rot(t). The law of propagation establishes the relation between $(\eta,\underline{q})$ (t), its derivative $(\dot{\eta},\dot{\underline{q}})(t)$ and the angular velocity ${}^{A}\Omega_{B}$. This relation is described by

$$\begin{bmatrix} \dot{\eta} \\ \dot{\underline{q}} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -\Omega_1 \quad -\Omega_2 \quad -\Omega_3 \\ \Omega_1 & \\ \Omega_2 & \boxed{-S(\Omega)} \\ \Omega_3 & \end{bmatrix} \begin{bmatrix} \eta \\ \underline{q} \end{bmatrix} \tag{1}$$

where $\Omega \triangleq \left[{}^{A}\Omega_{B}\right]_{B}$ represents the angular velocity of B with respect to A, measured with respect to B. From equation (1) we obtain

$$\begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix} = 2 \begin{bmatrix} -q_1 & \eta & q_3 & -q_2 \\ -q_2 & -q_3 & \eta & q_1 \\ -q_3 & q_2 & -q_1 & \eta \end{bmatrix} \begin{bmatrix} \dot{\eta} \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} . \tag{2}$$

To verify the validity of equation (1), observe that

$$\begin{pmatrix} \dot{\eta} \\ \dot{q} \end{pmatrix} = \lim_{\Delta t \to 0} \left\{ \begin{pmatrix} \eta \\ q \end{pmatrix} (t + \Delta t) - \begin{pmatrix} \eta \\ q \end{pmatrix} (t) \right\} \tag{3}$$

where

$$\begin{pmatrix} \eta \\ \underline{q} \end{pmatrix} (t + \Delta t) = \begin{pmatrix} \eta \\ \underline{q} \end{pmatrix} (t) \begin{pmatrix} \eta \\ \underline{q} \end{pmatrix} (\Delta t) . \tag{4}$$

Note that $\begin{pmatrix} \eta \\ \underline{q} \end{pmatrix} (\Delta t)$ represents the change in orientation of B with respect to A, measured relative to B, that has occurred in the interval (t, t + Δt). For a sufficiently small Δt, this change in orientation can be described in terms of the Euler vector ($\Delta \vartheta \, \underline{r}$) with

$$[\underline{r}]_B := \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} := \frac{1}{\sqrt{\Omega_1^2 + \Omega_2^2 + \Omega_3^2}} \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix} ; \Delta\vartheta = \sqrt{\Omega_1^2 + \Omega_2^2 + \Omega_3^2} \, \Delta t \tag{5}$$

where $[\Omega_1 \Omega_2 \Omega_3]'$ represents the measure relative to B of the angular velocity of B with respect to A. It follows

$$\begin{pmatrix} \eta \\ \\ \underline{q} \end{pmatrix}(\Delta t) = \left(\cos\frac{\Delta\vartheta}{2}, \sin\frac{\Delta\vartheta}{2} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \right) \cong \left(1 \;, \; \frac{\Delta t}{2} \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix} \right) \tag{6}$$

whence

$$\begin{pmatrix} \eta \\ \\ q \end{pmatrix}(t+\Delta t) = \begin{pmatrix} \eta \\ \\ q \end{pmatrix}(t) \begin{pmatrix} \eta \\ \\ q \end{pmatrix}(\Delta t) = \begin{bmatrix} \eta - \frac{\Delta t}{2}[\Omega_1 \Omega_2 \Omega_3]\,\underline{q} \\ \frac{\Delta t}{2}\eta \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix} + \underline{q} - \frac{\Delta t}{2} S(\Omega)\,\underline{q} \end{bmatrix}$$

$$\begin{pmatrix} \eta \\ \\ \underline{q} \end{pmatrix}(t+\Delta t) - \begin{pmatrix} \eta \\ \\ \underline{q} \end{pmatrix}(t) = \begin{bmatrix} -\frac{\Delta t}{2}[\Omega_1 \Omega_2 \Omega_3]\,\underline{q} \\ \frac{\Delta t}{2}\eta \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix} - \frac{\Delta t}{2} S(\Omega)\underline{q} \end{bmatrix}. \tag{7}$$

By dividing the last expression by Δt and considering the limit for Δt → 0, one obtains

$$\begin{bmatrix} \dot{\eta} \\ \dot{\underline{q}} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -\Omega_1 \quad -\Omega_2 \quad -\Omega_3 \\ \Omega_1 & \\ \Omega_2 & -S(\Omega) \\ \Omega_3 & \end{bmatrix} \begin{bmatrix} \eta \\ \\ \underline{q} \end{bmatrix}. \tag{8}$$

Remark 1: Equation(1) allows one to determine the orientation of a vehicle starting from measurements of its angular velocity and its initial orientation.

Remark 2: Equation (2) allows one to determine the measurement provided by a strap down gyroscope when the vehicle follows a given trajectory.

5.4 Vector measure and Euler parameters

With $[v]_A$, $[v]_B$ representing the measure of a vector $\underline{v}$ relative to frames A and B, and with the orientation of B with respect to A described by the quaternion $p = \cos\frac{\vartheta}{2} + \sin\frac{\vartheta}{2}\underline{q}$, one has

$$Q([v]_A) = \left(\cos\frac{\vartheta}{2} + \sin\frac{\vartheta}{2}\underline{q}\right) Q([v]_B) \left(\cos\frac{\vartheta}{2} - \sin\frac{\vartheta}{2}\underline{q}\right). \tag{9}$$

where $Q([v]_A), Q([v]_B)$, quaternions associated with $[v]_A$ and $[v]_B$ are defined as follows

$$\begin{aligned} Q([v]_A) &:= (0,[v]_A) \\ Q([v]_B) &:= (0,[v]_B). \end{aligned} \tag{10}$$

5.5 Advantages offered by the Euler parameters

The Euler parameters offer the following points of interest :

i. They are routinely used in the treatment of guidance problems in aeronautics and aerospace;

ii. Fundamental problems related to the control of robotic systems (in particular, automatic guidance of vehicles) are often solved using quaternion algebraic and differential properties;

iii. The use of a rotation matrix to compute the change in orientation of a frame submitted to a rotation requires a number of operations (27 multiplications, 18 additions) that is considerably greater than the number of operations required by the use of quaternions (16 multiplications, 12 additions);

iv. The use of a rotation matrix to simulate the dynamics of a rigid body described by the Euler equation $N = {}^C I \dot{\Omega} + S(\Omega) {}^C I \Omega$ also requires a number of operations that is considerably greater than the number of operations required by the use of quaternions.

5.6 Rodriguez-Hamilton parameters

The Rodriguez-Hamilton parameters offer a representation of orientation similar to that offered by the Euler parameters. In particular the Rodriguez-Hamilton parameters associated to an orientation described by the Euler vector $\vartheta\underline{r}$ is given by

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \tan(\vartheta/2)[\underline{r}]\,. \tag{1}$$

The relation between Rodriguez-Hamilton parameters, rotation matrix and Euler parameters is the following

$$Rot = \left(1+p_1^2+p_2^2+p_3^2\right)\begin{bmatrix} 1+p_1^2-p_2^2-p_3^2 & 2(p_1p_2-p_3) & 2(p_1p_3+p_2) \\ 2(p_1p_2+p_3) & 1-p_1^2+p_2^2-p_3^2 & 2(p_1p_3-p_2) \\ 2(p_1p_3-p_2) & 2(p_2p_3+p_1) & 1-p_1^2-p_2^2+p_3^2 \end{bmatrix} \tag{2}$$

$$\eta = \left(1+p_1^2+p_2^2+p_3^2\right)^{-1/2} \qquad q_1 = \eta p_1 \quad q_2 = \eta p_2 \quad q_3 = \eta p_3 \tag{3}$$

The propagation law for the Rodriquez-Hamilton parameters is described by the equation

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+p_1^2 & p_1p_2-p_3 & p_1p_3+p_2 \\ p_1p_2+p_3 & 1+p_2^2 & p_2p_3-p_1 \\ p_1p_3-p_2 & p_1+p_2p_3 & 1+p_3^2 \end{bmatrix}\begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}. \tag{4}$$

6. APPLICATION EXAMPLES

The presentation of the following examples pursues a triple objective:

i) A familiarization with the physical implications of the mathematical elements introduced in the previous sections;

ii) A familiarization with the algebraic manipulations that are routinely carried in the course of the study of navigation and robotics problems;

iii) The establishment of formal relationships that play a fundamental role in the treatment of these problems.

6.1 Reference frames of interest in navigation

When dealing with navigation systems, the following frames are of interest:

eci: earth-centered inertial frame;

ecef : earth-centered earth fixed;

ned: north, east and down local frame;

vehi: a frame attached to the vehicle.

The definition of these frames is illustrated in figures 1-4.

Origin at earth center ;
z axis := earth axis of rotation ;
x axis: =on the plane containing the Grenwich meridian

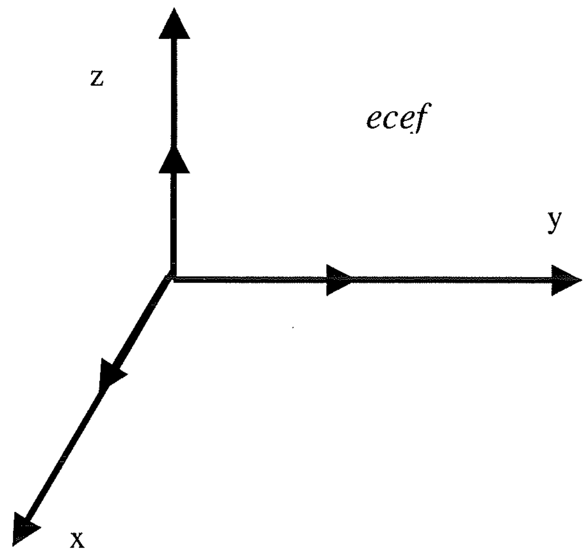

Figure 1: Earth-centered earth fixed Reference frame

Origin at center of the earth ;
z axis : = earth axis of rotation ;
x axis := from origin to vernal point (springtime equinox ; intersection of equatorial plane with the apparent trajectory of the sun around earth, directed from south to north).

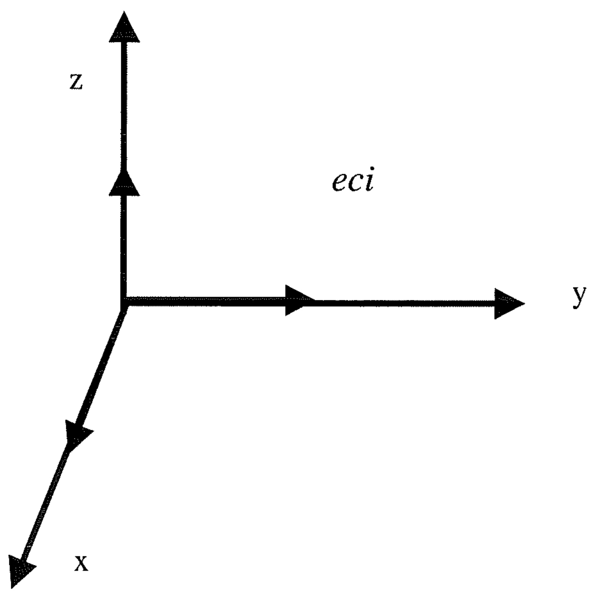

Figure 2: eci: Earth-centered inertial frame Reference frame

Same origin as vehicle origin;
*z axis:=directed toward (**down**) the center of the earth;*
*x axis:=on the Greenwich meridian, directed toward the **n**orth;*
*y axis:= directed toward **e**ast*

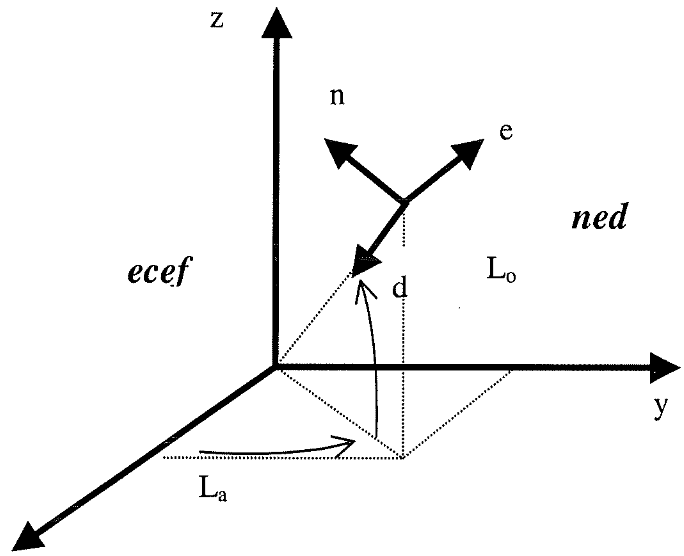

Figure 3: North, east and down local reference frame

Origin:= center of mass of the vehicle ;
x axis:= vehicle's longitudinal axis directed from the back to the front ;
z axis:= on the plane of vertical symmetry of the vehicle, directed from the ceiling to the floor.

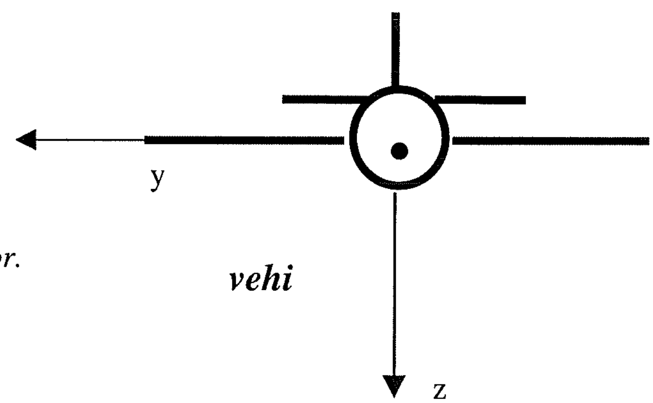

Figure 4: Reference frame attached to the vehicle

6.2 Angular velocity and orientation of earth relative to an inertial frame

Earth turns about its rotation axis with a speed Ω_T = 2π/jour = 20/24x60x60 = $7.27.10^{-5}$ rad/sec. This implies

$$\left[{}^{eci}\Omega_{ecef}\right]_{eci} = \begin{bmatrix} 0 \\ 0 \\ \Omega_T \end{bmatrix} \tag{1}$$

Because of this angular velocity, the earth's orientation relative to the inertial frame, Rot (ecef, eci), varies with time (ecef := the conventional earth-centered-earth-fixed frame; eci : = earth centered-inertial frame). Taking t = 0, the time when the ecef x_axis coincides with the eci x_axis, it follows that, at $t \neq 0$, the earth orientation relative to the inertial frame corresponds to a rotation about the z_axis of an angle $\vartheta = \Omega_T t$. One can then write,

$$Rot(eci, ecef)(t) = Rot(z, \Omega_T t) = \begin{bmatrix} \cos\Omega_T t & -\sin\Omega_T t & 0 \\ \sin\Omega_T t & \cos\Omega_T t & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

It is a useful exercise to see how this formula relates to the law of propagation

$$\dot{\mathrm{Rot}}(\mathrm{A},\mathrm{B}) = \mathrm{S}\left\{\left[{}^{\mathrm{A}}\Omega_{\mathrm{B}}\right]_{\mathrm{A}}\right\} \mathrm{Rot}(\mathrm{A},\mathrm{B}). \tag{3}$$

Using the notation $Rot(eci, ecef) = \left[r_{ij}\right]$, ij = 1, 2, 3, consider the system of differential equations

$$\begin{bmatrix} \dot{r}_{11} & \dot{r}_{12} & \dot{r}_{13} \\ \dot{r}_{21} & \dot{r}_{22} & \dot{r}_{23} \\ \dot{r}_{31} & \dot{r}_{32} & \dot{r}_{33} \end{bmatrix} = \begin{bmatrix} 0 & -\Omega_T & 0 \\ \Omega_T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{4}$$

with initial conditions

$$r_{ij}(0) = 1 \quad \text{si} \quad i = j, \quad = 0 \quad \text{si} \quad i \neq j. \tag{5}$$

It follows

$$\begin{matrix} \dot{r}_{11} = -\Omega_T \;\; r_{21} & \dot{r}_{12} = -\Omega_T \;\; r_{22} & \dot{r}_{13} = -\Omega_T \;\; r_{23} \\ \dot{r}_{21} = \Omega_T \;\; r_{11} & \dot{r}_{22} = \Omega_T \;\; r_{12} & \dot{r}_{23} = \Omega_T \;\; r_{13} \\ \dot{r}_{31} = 0 & \dot{r}_{32} = 0 & \dot{r}_{33} = 0 \end{matrix} \tag{6}$$

whence

$$
\begin{array}{ll}
\ddot{r}_{11} = -\Omega_T^2 \quad r_{11} & r_{21} = -\dot{r}_{11} / \Omega_T \\
\ddot{r}_{12} = -\Omega_T^2 \quad r_{12} & r_{22} = -\dot{r}_{12} / \Omega_T \\
\ddot{r}_{13} = -\Omega_T^2 \quad r_{13} & r_{32} = -\dot{r}_{13} / \Omega_T
\end{array}
\tag{7}
$$

By integrating one obtains (2).

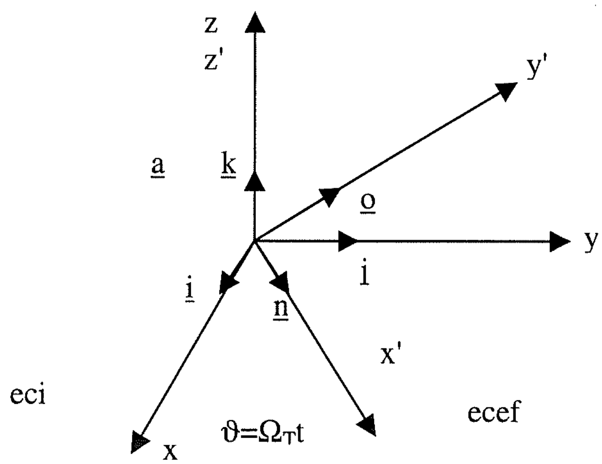

Figure 1 : Rotation of ecef relative to eci

6.3 Latitude, longitude and orientation of the ned frame relative to ecef

Let us determine the rotation matrix describing the orientation of ned relative to ecef when in correspondence with an assigned latitude and longitude L_a and L_o.

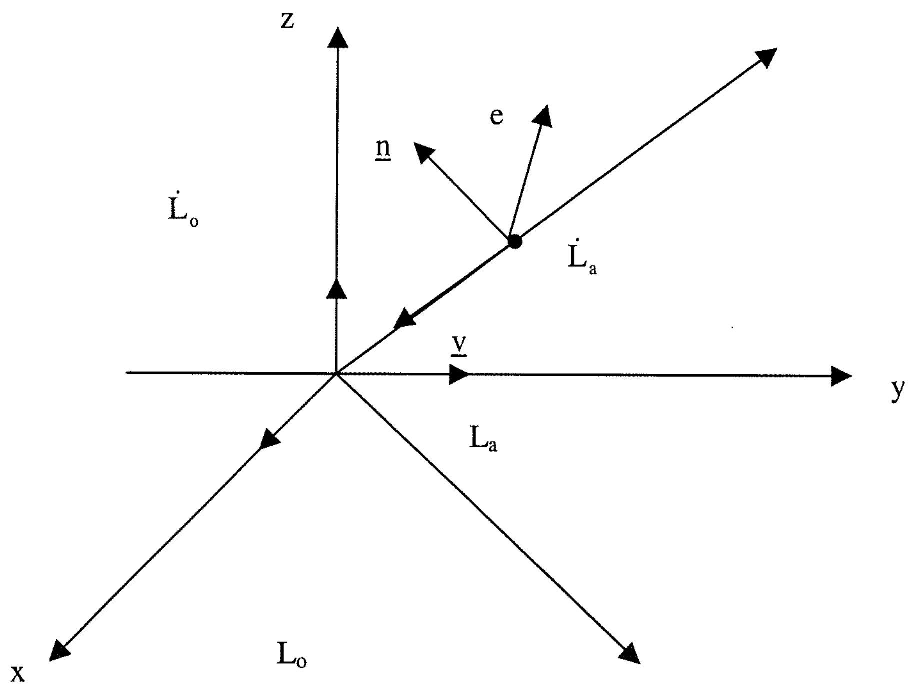

Figure 1 : Orientation of ned relative to ecef

Solving this problem is equivalent to determining the rotation that a frame initially coincident with ecef must be submitted to so as to attain the same orientation as ned.

To this end, note that:

i) If we submit ecef to a rotation about its z_axis of an angle L_o , then

ecef → ecef' | y_axis of ecef' // y_axis of ned.

ii) If we submit ecef' to a clockwise rotation of an angle $L_a + 90^o$ about its own y_axis, then

ecef' → ecef" | y_axis de ecef" // y_axis of ned
x_axis of ecef" // x_axis of ned

iii) The rotation necessary for ecef to attain the orientation of ned is then described by the composition of a first rotation of an angle L_o about the z_axis, followed by a second rotation of an angle $-(L_a + 90^o)$ about the y_axis.

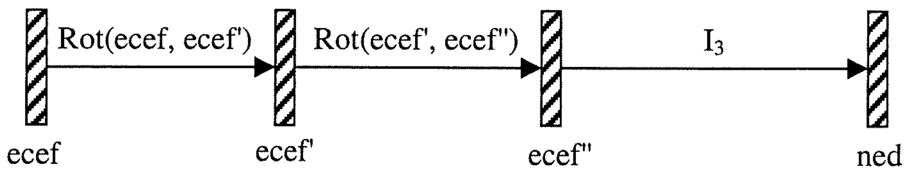

Figure 2 : Rotations required for ecef to be oriented as ned

$$\text{Rot(ecef, ecef')} = \text{Rot}(z, L_o) \tag{1}$$

$$\text{Rot (ecef', ecef")} = \text{Rot}(y, -(L_a + 90^o)) \tag{2}$$

$$\text{Rot (ecef", ned)} = I_3 \tag{3}$$

$$\text{Rot (ecef, ned)} = \text{Rot (ecef, ecef') Rot (ecef', ecef")} \tag{4}$$

From

$$\text{Rot}(z, L_o) = \begin{bmatrix} \cos L_o & -\sin L_o & 0 \\ \sin L_o & \cos L_o & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

$$\begin{aligned} \text{Rot}(y, -(L_a + 90^o)) &= \begin{bmatrix} \cos(-L_a - 90^o) & 0 & \sin(-L_a - 90^o) \\ 0 & 1 & 0 \\ -\sin(-L_a - 90^o) & 0 & \cos(-L_a - 90^o) \end{bmatrix} \\ &= \begin{bmatrix} -\sin L_a & 0 & -\cos L_a \\ 0 & 1 & 0 \\ \cos L_a & 0 & -\sin L_a \end{bmatrix}, \end{aligned} \tag{6}$$

it follows

$$Rot(ecef, ned) = \begin{bmatrix} -\cos L_o \sin L_a & -\sin L_o & -\cos L_o \cos L_a \\ -\sin L_o \sin L_a & \cos L_o & -\sin L_o \cos L_a \\ \cos L_a & 0 & -\sin L_a \end{bmatrix}. \tag{7}$$

6.4 Linear velocity of a vehicle and angular velocity of ned

Let the measurement relative to ned of the linear velocity of a vehicle with respect to earth be represented by

$$\left[{}^{ecef}V_{Vehi}\right]_{ned} = \begin{bmatrix} V_N \\ V_E \\ V_D \end{bmatrix}. \tag{1}$$

Clearly, the vehicle's latitude is not influenced by V_E and V_D and $\dot{L}_a = V_N/(R+h)$, where R is the earth's radius, h the vehicle's altitude. In a similar spirit, the longitude is not influenced by V_N and V_E and

$$\dot{L}_o = V_E/((R+h)\cos L_a) . \tag{2}$$

From knowledge of $\dot{L}_a, \dot{L}_0$ we can compute $\left[{}^{ecef}\Omega_{ned}\right]_{ecef}$ using the relation

$$S\left\{\left[{}^{ecef}\Omega_{ned}\right]_{ecef}\right\} = \dot{Rot}(ecef,ned)Rot'(ecef,ned). \tag{3}$$

Since

$$Rot(ecef,ned) = \begin{bmatrix} -\cos L_o \sin L_a & -\sin L_o & -\cos L_o \cos L_a \\ -\sin L_o \sin L_a & \cos L_o & -\sin L_o \cos L_a \\ \cos L_a & 0 & -\sin L_a \end{bmatrix} \tag{4}$$

and therefore

$$\dot{Rot}(Ter,nev) = \begin{bmatrix} \sin L_o \sin L_a \dot{L}_o - \cos L_o \cos L_a \dot{L}_a & -\cos L_o \dot{L}_o & \sin L_o \cos L_a \dot{L}_o + \cos L_o \sin L_a \dot{L}_a \\ -\cos L_o \sin L_a \dot{L}_o - \sin L_o \cos L_a \dot{L}_a & -\sin L_o \dot{L}_o & -\cos L_o \cos L_a \dot{L}_o + \sin L_o \sin L_a \dot{L}_a \\ -\sin L_a \dot{L}_a & 0 & -\cos L_a \dot{L}_a \end{bmatrix}. \tag{5}$$

By implementing the product required by equation (3) one obtains

$$\dot{Rot}(Ter,nev)Rot'(ecef,ned) = S\{[{}^{ecef}\Omega_{ned}]_{ecef}\} = \begin{bmatrix} 0 & -\dot{L}_o & -\dot{L}_a \cos L_o \\ \dot{L}_o & 0 & -\dot{L}_a \sin L_o \\ L_a \cos L_o & \dot{L}_a \sin L_o & 0 \end{bmatrix} \quad (6)$$

whence

$$[{}^{ecef}\Omega_{ned}]_{ecef} = \begin{bmatrix} \dot{L}_a \sin L_o \\ -\dot{L}_a \cos L_o \\ \dot{L}_o \end{bmatrix}. \quad (7)$$

An alternative and perhaps more direct way to compute ${}^{ecef}\Omega_{ned}$ is to observe that this velocity is given by the sum of angular velocity $\dot{L}_o$ about the z_axis of ecef, plus angular velocity $-\dot{L}_a$ about the y_axis of ned.

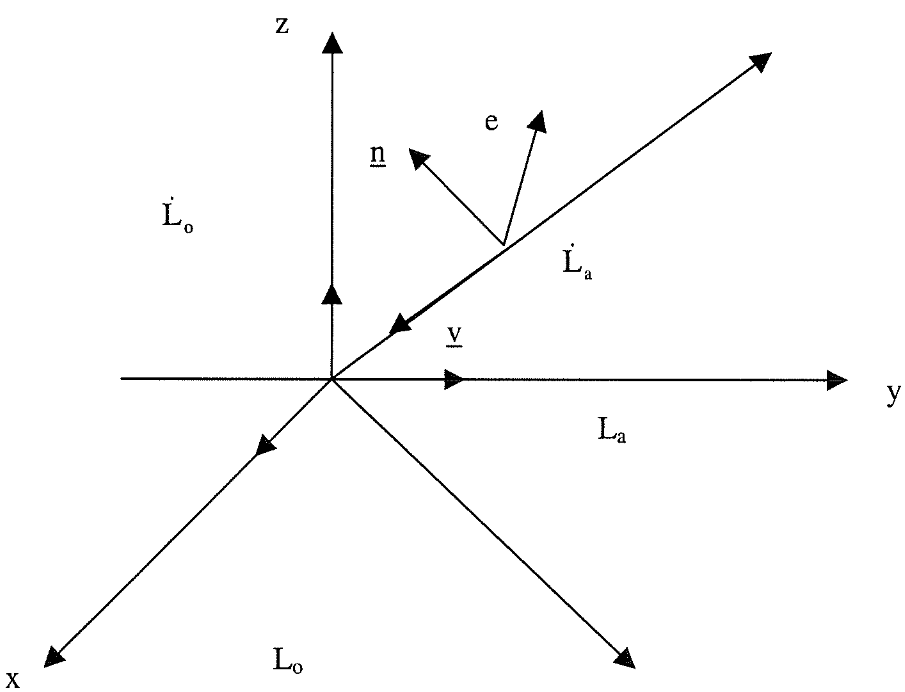

Figure 1 : Linear and angular velocity of ned

It follows

$$[{}^{ecef}\Omega_{ned}]_{ecef} = \begin{bmatrix} 0 \\ 0 \\ \dot{L}_o \end{bmatrix} + Rot(ecef,ned)\begin{bmatrix} 0 \\ -\dot{L}_a \\ 0 \end{bmatrix}. \quad (8)$$

$$= \begin{bmatrix} 0 \\ 0 \\ \dot{L}_o \end{bmatrix} + \begin{bmatrix} -\cos L_o \sin L_a & -\sin L_o & -\cos L_o \cos L_a \\ -\sin L_o \sin L_a & \cos L_o & -\sin L_o \cos L_a \\ \cos L_a & 0 & -\sin L_a \end{bmatrix} \begin{bmatrix} 0 \\ -\dot{L}_a \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{L}_a \sin L_o \\ -\dot{L}_a \cos L_o \\ \dot{L}_o \end{bmatrix}. \quad (9)$$

Let now

$$\left[{}^{ned}\Omega_{Vehi}\right]_{ned} = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} \quad (10)$$

then

$$\left[{}^{ecef}\Omega_{Vehi}\right]_{ecef} = \left[{}^{ecef}\Omega_{ned}\right]_{ecef} + \left[{}^{ned}\Omega_{Vehi}\right]_{ecef} \quad (11)$$

$$= \begin{bmatrix} \dot{L}_a \sin L_o \\ -\dot{L}_a \cos L_o \\ \dot{L}_o \end{bmatrix} + \mathrm{Rot}(ecef, ned) \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} \quad (12)$$

In general $\left[{}^{ned}\Omega_{Vehi}\right]$ is measured relative to body frame vehi, and the orientation of vehi is described in terms of its Euler angles relative to ned.

In this case, one can use the relation

$$\left[{}^{ned}\Omega_{Vehi}\right]_{ned} = \mathrm{Rot}(ned, Vehi)\left[{}^{ned}\Omega_{Vehi}\right]_{Vehi}$$

$$= \mathrm{Rot}(z,\alpha)\,\mathrm{Rot}(y,\beta)\,\mathrm{Rot}(x,\gamma)\left[{}^{ned}\Omega_{Vehi}\right]_{Vehi} \quad (13)$$

with α, β and γ angles of yaw, pitch and roll of the vehicule relative to ned.

6.5 Absolute, relative and transport velocity

Let P be a point representative of the position occupied by a vehicle.

The position of the vehicle relative to earth is described by the co-ordinates $\left[\overrightarrow{0P}\right]_{ecef}$, where 0 denotes the origin of ecef; its velocity relative to earth is given by

$$\left[{}^{ecef}v_{Vehi}\right]_{ecef} = \frac{d}{dt}\left[\overrightarrow{0P}\right]_{ecef}. \tag{1}$$

The vehicle's position relative to eci is given by

$$\left[\overrightarrow{0P}\right]_{eci} = \mathrm{Rot}(I_n, ecef)\left[\overrightarrow{0P}\right]_{ecef}. \tag{2}$$

its velocity relative to eci (absolute velocity of the vehicle) is given by

$$\left[{}^{eci}v_{Vehi}\right]_{eci} = \frac{d}{dt}\left[\overrightarrow{0P}\right]_{eci} = \frac{d}{dt}\left\{\mathrm{Rot}(eci, ecef)\left[\overrightarrow{0P}\right]_{ecef}\right\} \tag{3}$$

$$= \dot{\mathrm{Rot}}(eci, ecef)\left[\overrightarrow{0P}\right]_{ecef} + \mathrm{Rot}(eci, ecef)\frac{d}{dt}\left[\overrightarrow{0P}\right]_{ecef} \tag{4}$$

$$= S\left\{\left[eci\,\Omega_{ecef}\right]_{ecef}\right\}\mathrm{Rot}(eci, ecef)\frac{d}{dt}\left[\overrightarrow{0P}\right]_{ecef} + \mathrm{Rot}(eci, ecef)\frac{d}{dt}\left[\overrightarrow{0P}\right]_{ecef}. \tag{5}$$

It follows

$$\left[{}^{eci}v_{Vehi}\right]_{eci} = \left[{}^{eci}\Omega_{ecef}\right]_{eci} \wedge \left[\overrightarrow{0P}\right]_{eci} + \left[{}^{ecef}v_{Vehi}\right]_{eci} \tag{6}$$

that is

$${}^{eci}v_{Vehi} = {}^{eci}\Omega_{ecef} \wedge \overrightarrow{0P} + {}^{ecef}v_{Vehi} \tag{7}$$

(absolute velocity = transport velocity + relative velocity).

Remark 1: Consider a point P on the surface of the earth, and $\frac{d}{dt}\left[\overrightarrow{0P}\right]_{ecef} = 0$.

Since $\left\|{}^{eci}\Omega_{ecef}\right\|$ = 15 degrees/hour, it follows that at the equator,

$$\left\|\frac{d}{dt}\left[\overrightarrow{0P}\right]_{eci}\right\| = \left\|\left[{}^{eci}\Omega_{ecef}\right]_{eci} \wedge \left[\overrightarrow{0P}\right]_{eci}\right\|$$

$$= 15 \times \frac{2\pi}{360} \times 6400 \cong 1600 \text{ km / h}$$

(the earth radius has been taken equal to 6400 Km).

6.6 Measure of the acceleration and derivative of the measure of velocity

Let

$$\left[{}^{eci}a_{Vehi}\right]_{Vehi} \quad \left[{}^{eci}v_{Vehi}\right]_{Vehi} \quad \left[{}^{eci}\Omega_{Vehi}\right]_{Vehi}$$

represent the measures relative to the vehicle's frame of the linear acceleration and of linear and angular velocities.

By virtue of definition,

$$\left[{}^{eci}a_{Vehi}\right]_{eci} = \frac{d}{dt}\left\{\left[{}^{eci}v_{Vehi}\right]_{eci}\right\} \tag{1}$$

and therefore,

$$\begin{aligned} \left[{}^{eci}a_{Vehi}\right]_{eci} &= \frac{d}{dt}\left\{\mathrm{Rot}(eci, Vehi)\left[{}^{eci}v_{Vehi}\right]_{Vehi}\right\} \\ &= \dot{\mathrm{Rot}}(eci, Vehi)\left[{}^{eci}v_{Vehi}\right]_{Vehi} + \mathrm{Rot}(eci, Vehi)\frac{d}{dt}\left[{}^{eci}v_{Vehi}\right]_{Vehi}. \end{aligned} \tag{2}$$

By applying the propagation law for rotation matrices, one has

$$\dot{\mathrm{Rot}}(eci, Vehi) = S\left\{\left[{}^{eci}\Omega_{Vehi}\right]_{eci}\right\}\mathrm{Rot}(eci, Vehi) \tag{3}$$

whence

$$\left[{}^{eci}a_{Vehi}\right]_{eci} = \left[{}^{eci}\Omega_{Vehi}\right]_{eci} \wedge \left[{}^{eci}v_{Vehi}\right]_{eci} + \mathrm{Rot}(eci, Vehi)\frac{d}{dt}\left[{}^{eci}v_{Vehi}\right]_{Vehi}. \tag{4}$$

By pre-multiplying by Rot(vehi, eci), it follows

$$\left[{}^{eci}a_{Vehi}\right]_{Vehi} = \left[{}^{eci}\Omega_{Vehi}\right]_{Vehi} \wedge \left[{}^{eci}v_{Vehi}\right]_{Vehi} + \frac{d}{dt}\left[{}^{eci}v_{Vehi}\right]_{Vehi}. \tag{5}$$

The importance of this equation is that it allows one to compute $\left[{}^{eci}v_{Vehi}\right]_{Vehi}$ from the measure of $\left[{}^{eci}a_{Vehi}\right]_{Vehi}$. This computation can be implemented by integrating the differential equation

$$\frac{d}{dt}\left[{}^{eci}v_{Vehi}\right]_{Vehi} = \left[{}^{eci}a_{Vehi}\right]_{Vehi} - \left[{}^{eci}\Omega_{Vehi}\right]_{Vehi} \wedge \left[{}^{eci}v_{Vehi}\right]_{Vehi} . \qquad (6)$$

6.7 Absolute, relative, centripetal and Coriolis accelerations

Let ${}^{eci}\Omega_{ecef}$ represent the angular velocity of earth relative to eci; $\left[\overrightarrow{0P}\right]_{ecef}$, ${}^{ecef}v_{Vehi}$, ${}^{ecef}a_{Vehi}$ the position, velocity and acceleration of the vehicle relative to ecef; ${}^{eci}a_{Vehi}$ the vehicle's acceleration relative to eci.

From the relations

$$\left[{}^{ecef}v_{Vehi}\right]_{ecef} = \frac{d}{dt}\left[\overrightarrow{0P}\right]_{ecef} \qquad (1)$$

$$\left[{}^{ecef}a_{Vehi}\right]_{ecef} = \frac{d}{dt}\left[{}^{ecef}v_{Vehi}\right]_{ecef} \qquad (2)$$

$$\left[{}^{eci}a_{Vehi}\right]_{eci} = \frac{d}{dt}\left\{\left[{}^{eci}v_{Vehi}\right]_{eci}\right\} \qquad (3)$$

$$\left[{}^{eci}v_{Vehi}\right]_{eci} = \left[{}^{eci}\Omega_{ecef}\right]_{eci} \wedge \left[\overrightarrow{0P}\right]_{eci} + \left[{}^{ecef}v_{Vehi}\right]_{eci}$$

one obtains

$$\left[{}^{eci}a_{Vehi}\right]_{eci} = \frac{d}{dt}\left\{\left[{}^{eci}\Omega_{ecef}\right]_{eci} \wedge [0P]_{eci} + \left[{}^{ecef}v_{Vehi}\right]_{eci}\right\}. \qquad (4)$$

It follows,

$$\left[{}^{eci}a_{Vehi}\right]_{eci} = \frac{d}{dt}\left\{\left[{}^{eci}\Omega_{ecef}\right]_{eci} \wedge \left(\mathrm{Rot}(eci,ecef)\left[\overrightarrow{0P}\right]_{ecef}\right) + \mathrm{Rot}(eci,ecef)\left[{}^{ecef}v_{Vehi}\right]_{ecef}\right\}$$

$$= \left[{}^{eci}\Omega_{ecef}\right]_{eci} \wedge \left\{\dot{\mathrm{Rot}}(eci,ecef)\,[0P]_{ecef} + \mathrm{Rot}(eci,ecef)\frac{d}{dt}\left[\overrightarrow{0P}\right]_{ecef}\right\}$$

$$+ \dot{\mathrm{Rot}}(eci,ecef)\left[{}^{ecef}v_{Vehi}\right]_{ecef} + \mathrm{Rot}(ci,ecef)\frac{d}{dt}\left[{}^{ecef}v_{Vehi}\right]_{ecef} \qquad (5)$$

$$= \left[{}^{eci}\Omega_{ecef}\right]_{eci} \wedge \left\{\left[{}^{eci}\Omega_{ecef}\right]_{eci} \wedge \mathrm{Rot}(eci,ecef)\left[\overrightarrow{0P}\right]_{ecef} + \mathrm{Rot}(eci,ecef)\frac{d}{dt}\left[\overrightarrow{0P}\right]_{ecef}\right\}$$

$$+ \left[{}^{eci}\Omega_{ecef}\right]_{eci} \wedge \mathrm{Rot}(eci,ecef)\left[{}^{ecef}v_{Vehi}\right]_{ecef} + \mathrm{Rot}(eci,ecef)\frac{d}{dt}\left[{}^{ecef}v_{Vehi}\right]_{ecef}.$$

By pre-multiplying left and right members of this equation by Rot(ecef, eci), it follows

$$= \left[{}^{eci}\Omega_{ecef}\right]_{ecef} \wedge \left\{\left[{}^{eci}\Omega_{ecef}\right]_{ecef} \wedge \left[\overrightarrow{0P}\right]_{ecef}\right\} + \left[{}^{eci}\Omega_{ecef}\right]_{ecef} \wedge \frac{d}{dt}\left[\overrightarrow{0P}\right]_{ecef}$$

$$+ \left[{}^{eci}\Omega_{ecef}\right]_{ecef} \wedge \left[{}^{ecef}v_{Vehi}\right]_{ecef} + \frac{d}{dt}\left[{}^{ecef}v_{Vehi}\right]_{ecef}. \qquad (6)$$

Finally,

$$\left[{}^{eci}a_{Vehi}\right] = {}^{eci}\Omega_{ecef} \wedge \left\{{}^{eci}\Omega_{ecef} \wedge \left[\overrightarrow{0P}\right]_{ecef}\right\} + 2\,{}^{eci}\Omega_{ecef} \wedge {}^{ecef}v_{Vehi} + {}^{ecef}a_{Vehi} \qquad (7)$$

↑ absolute acceleration ↑ centripetal ↑ Coriolis ↑ Relative

Remark 1: To get a feeling about the order of magnitude of the Coriolis and centripetal accelerations, consider a vehicle at the equator, with a linear velocity of 1000 Km/h in the East direction. We have

$$\begin{matrix}\text{Coriolis} \\ \text{acceleration}\end{matrix} = 2\left\| {}^{eci}\Omega_{ecef} \right\| \cdot \left\| {}^{ecef}\mathrm{v}_{Vehi} \right\| = 2\left(\frac{15\times 2\pi}{360}\right)\frac{10^6}{(3.600)^2} = \left(\sim 4.10^{-3}\,\mathrm{g}\right)$$

$$\begin{matrix}\text{centripetal} \\ \text{acceleration}\end{matrix} = \left\| {}^{eci}\Omega_{ecef} \right\|^2 \left\| \overrightarrow{0P} \right\| = \left(\frac{15\times 2\pi}{360}\right)\frac{6.4\,10^6}{(3.600)^2} = \left(\sim 2.10^{-3}\,\mathrm{g}\right)$$

(g: gravitational acceleration).

6.8 Position and orientation of a stationary vehicle relative to ned from inertial data

Let

$$\left[{}^{eci}\Omega_{plat}\right]_{plat} \text{ and } \left[{}^{eci}\mathrm{a}_{a\,plat}\right]_{plat}$$

be the measures of angular velocity and of apparent acceleration provided by the gyros and the accelerometers of the platform. Assuming the platform stationary relative to earth, we have

$$\left[{}^{eci}\Omega_{Plat}\right]_{Plat} = \left[{}^{eci}\Omega_{ecef}\right]_{Plat} \qquad (1)$$

$$\left[{}^{eci}\mathrm{a}_{a\,Plat}\right]_{Plat} = \left[-\mathrm{g}\right]_{Plat}, \qquad (2)$$

where g denotes the vector vulgar weight (acceleration due to gravity + centripetal acceleration).

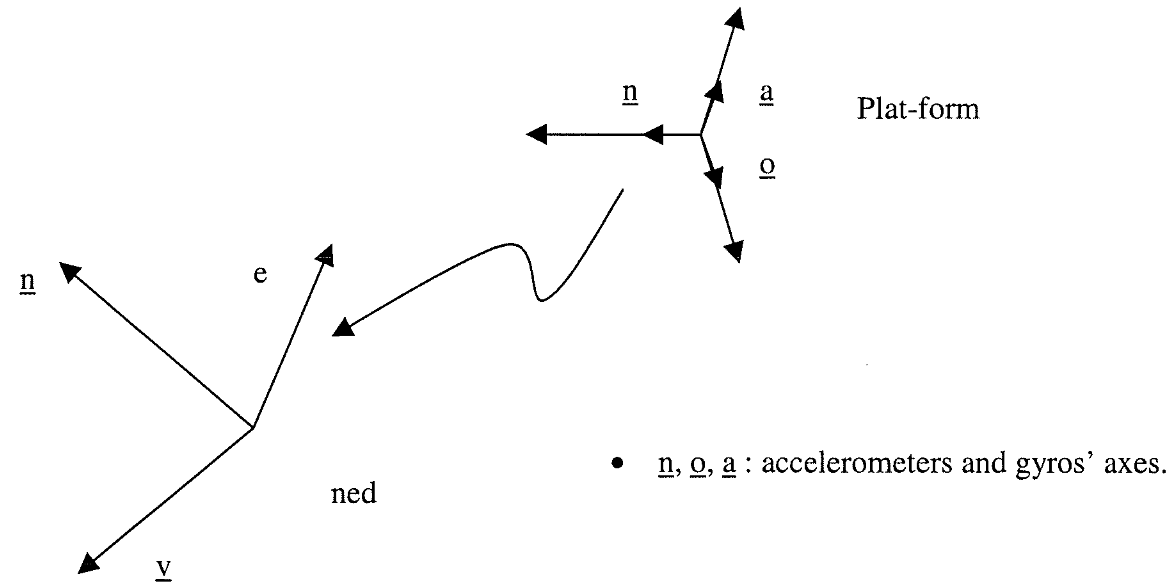

Figure 1 : Ned, and platform frames .

Denoting with n e and d ned's directional vectors, one has

$$[\underline{v}]_{Platform} = \frac{[g]_{Platform}}{\|[g]\|_{Platform}} = -\left[{}^{eci}a_{a\,plat}\right]_{plat} \Big/ \left\|\left[{}^{eci}a_{a\,plat}\right]_{plat}\right\| \tag{3}$$

$$[\underline{e}]_{Platform} = -\frac{\left[{}^{eci}\Omega_{ecef}\right]_{plat} \wedge [\underline{v}]_{plat}}{\left\|\left[{}^{eci}\Omega_{ecef}\right]_{plat} \wedge [\underline{v}]_{plat}\right\|} \tag{4}$$

$$[\underline{n}]_{Platform} = [\underline{e}]_{Platform} \wedge [\underline{v}]_{Platform}. \tag{5}$$

It follows

$$Rot(Plat, ned) = \left[[\underline{n}]_{plat}\ [\underline{e}]_{plat}\ [\underline{v}]_{plat}\right] \tag{6}$$

whence

$$Rot(ned, plat) = Rot'(plat, ned). \tag{7}$$

To determine latitude, L_a, from

$$\left[{}^{eci}\Omega_{ecef}\right]_{ned} = \mathrm{Rot}(ned, plat)\left[{}^{eci}\Omega_{ecef}\right]_{plat} \tag{8}$$

one can use the equation

$$\begin{bmatrix} \Omega_x \\ 0 \\ \Omega_y \end{bmatrix} := \left[{}^{eci}\Omega_{ecef}\right]_{ned} = \begin{bmatrix} \Omega_T \cos L_a \\ 0 \\ -\Omega_T \sin L_a \end{bmatrix}, \tag{9}$$

where $L_a = -\operatorname{atan} 2(\Omega_y, \Omega_x)$.

6.9 Orientation and angular velocity from astral measurements

Let A and B be frames attached to two space vehicles; let 0_A and 0_B be their origins; P and Q two stars. Denote with $v_A(P)$, $v_B(P)$, $v_A(Q)$, $v_B(Q)$ the measures relative to A and B of the directional vectors associated to $\overrightarrow{0_A P}, \overrightarrow{0_B P}, \overrightarrow{0_A Q}, \overrightarrow{0_B Q}$; $\dot{v}_A(p), \dot{v}_B(p), \dot{v}_A(q), \dot{v}_B(q)$ the time derivative of these measures. Consider the problem of obtaining from these measures: i) the orientation and ii) the angular velocity of B with respect to A.

i) Given the great distances separating vehicles and stars, the vectors $\left\{\overrightarrow{0_A P}\right\}$ *and* $\left\{\overrightarrow{0_B P}\right\}$ can be considered to be parallel. From the relation $[v]_A = \mathrm{Rot}(A,B)[v]_B$, one then has

$$v_A(P) = \mathrm{Rot}(A,B)\, v_B(P) \tag{1}$$

$$v_A(Q) = \mathrm{Rot}(A,B)\, v_B(Q)\,. \tag{2}$$

Similarly, from

$$v_A(P) \wedge v_A(Q) = \mathrm{Rot}(A,B)\left(v_B(P) \wedge v_B(Q)\right) \tag{3}$$

it follows

$$\left[v_A(P) \middle| v_A(Q) \middle| v_A(P) \wedge v_A(Q)\right] = \mathrm{Rot}(A,B)\left[v_B(P) \middle| v_B(Q) \middle| v_B(P) \wedge v_B(Q)\right] \tag{4}$$

whence

$$\mathrm{Rot}(A,B) = \left[v_A(P) \middle| v_A(Q) \middle| v_A(P) \wedge v_A(Q)\right]\left[v_B(P) \middle| v_B(Q) \middle| v_B(P) \wedge v_B(Q)\right]^{-1}. \tag{5}$$

ii) From the formula

$$[\dot{\underline{v}}]_A = [\overset{\bullet}{\underline{v}}]_A + \left[{}^N\Omega_A\right]_A \wedge [\underline{v}]_A \tag{6}$$

with eci the usual Newtonian frame and $\underline{v}$ an arbitrary vector, we have (by taking $\underline{v}$: = vect_dir $\left(\overrightarrow{0_A P}\right)$ and $\underline{v}$ = :vect_dir $\left(\overrightarrow{0_A Q}\right)$)

$$\dot{v}_A(P) = -\left[{}^N\Omega_A\right]_A \wedge v_A(P) \tag{7}$$

$$\dot{v}_A(Q) = -\left[{}^N\Omega_A\right]_A \wedge v_A(Q). \tag{8}$$

It follows

$$\dot{v}_A(P) \wedge \dot{v}_A(Q) = \left(\left[{}^N\Omega_A\right]_A \wedge v_A(P)\right) \wedge \left(\left[{}^N\Omega_A\right]_A \wedge v_A(Q)\right). \tag{9}$$

By now invoking the easily verifiable relation

$$\underline{a} \wedge (\underline{b} \wedge \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c} \tag{10}$$

we infer

$$\begin{aligned} \dot{v}_A(P) \wedge \dot{v}_A(Q) = &\left(\left(\left[{}^N\Omega_A\right]_A \wedge v_A(P)\right) \cdot v_A(Q)\right)\left[{}^N\Omega_A\right]_A \\ &-\left(\left(\left[{}^N\Omega_A\right]_A \wedge v_A(P)\right) \cdot \left[{}^N\Omega_A\right]_A\right) v_A(Q) \end{aligned} \tag{11}$$

and therefore (the second term of the right hand side being nul)

$$\left[{}^N\Omega_A\right]_A = \frac{\dot{v}_A(P) \wedge \dot{v}_A(Q)}{\dot{v}_A(P) \cdot v_A(Q)}. \tag{12}$$

By proceeding in an identical fashion,

$$\left[{}^{N}\Omega_{B}\right]_{B} = \frac{\dot{v}_{B}(P) \wedge \dot{v}_{B}(Q)}{\dot{v}_{B}(P) \cdot v_{B}(Q)}. \tag{13}$$

Finally, from

$${}^{A}\Omega_{B} = {}^{N}\Omega_{B} + {}^{A}\Omega_{N} = {}^{N}\Omega_{B} - {}^{N}\Omega_{A}, \tag{14}$$

it follows

$$\left[{}^{A}\Omega_{B}\right]_{A} = \mathrm{Rot}(A,B)\left[{}^{N}\Omega_{B}\right]_{B} - \left[{}^{N}\Omega_{A}\right]_{A}. \tag{15}$$

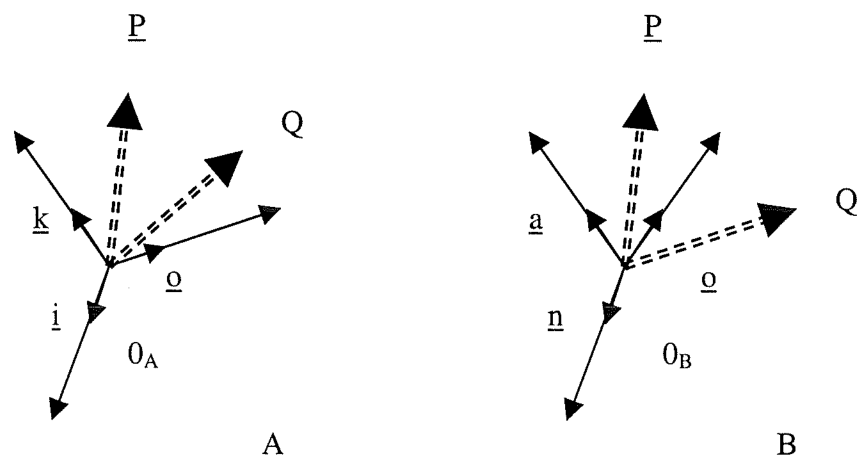

Figure 1: From the position of frames A and B relative to the stars to the orientation and angular velocity of B relative to A.

6.10 Position/orientation and linear/angular velocity of a frame B relative to a frame A from point co-ordinates measurements

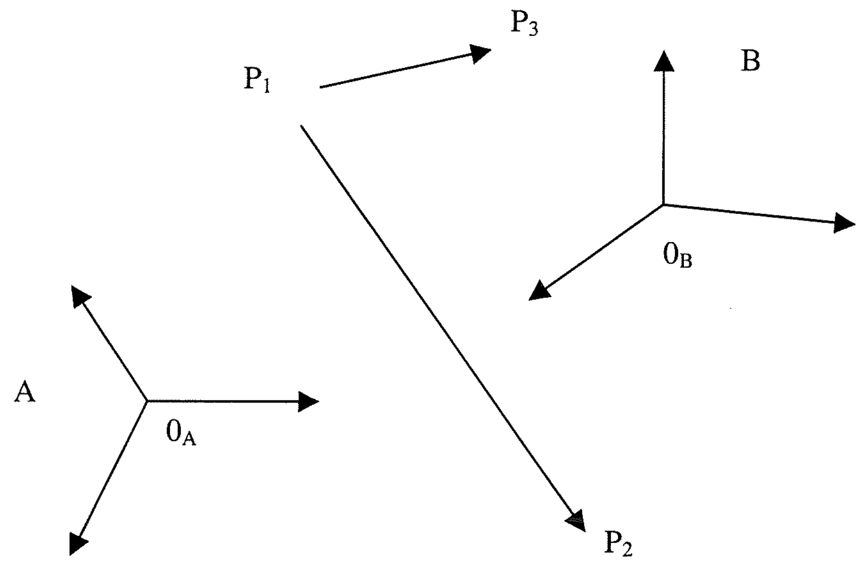

Figure 1 : Point coordinates relative to A and B

With reference with figure 1, consider

$$[0_\alpha P_i]_\alpha \quad \text{pour} \quad i \in \{1,2,3\}, \quad \alpha \in \{A,B\}$$

and therefore

$$[0_\alpha P_i]_\alpha - [0_\alpha P_1]_\alpha = [P_1 P_i]_\alpha \quad , \quad i \in \{1,2,3\}, \alpha \in \{A,B\} \ . \tag{1}$$

Note that

$$\left[[P_1P_2]_A \middle| [P_1P_3]_A \middle| [P_1P_2]_A \wedge [P_1P_3]_A\right] = \mathrm{Rot}(A,B)\left[[P_1P_2]_B \middle| [P_1P_3]_B \middle| [P_1P_2]_B \wedge [P_1P_3]_B\right] \tag{2}$$

hence

$$\mathrm{Rot}(A,B) = \left[[P_1P_2]_A \middle| [P_1P_3]_A \middle| [P_1P_2]_A \wedge [P_1P_3]_A\right]\left[[P_1P_2]_B \middle| [P_1P_3]_B \middle| [P_1P_2]_B \wedge [P_1P_3]_B\right]^{-1} . \tag{3}$$

Furthermore from the relation

$$[0P_1]_A = \mathrm{Rot}(A,B)\,[0P_1]_B + \left[\overrightarrow{0_A 0_B}\right]_A \tag{4}$$

one also obtains

$$\left[\overrightarrow{0_A 0_B}\right]_A = [0P_1]_A - \mathrm{Rot}(A,B)\,[0P_1]_B\,. \tag{5}$$

Remark 1: In regard to the determination of velocities, by introducing the notations

$$\Box_A := \left[[P_1P_2]_A \quad [P_1P_3]_A \quad [P_1P_2]_A \wedge [P_1P_3]_A\right]$$

$$\Box_B := \left[[P_1P_2]_B \quad [P_1P_3]_B \quad [P_1P_2]_B \wedge [P_1P_3]_B\right], \tag{6}$$

one has

$$\dot{\Box}_A = S\left\{\left[{}^A\Omega_B\right]_A\right\} \mathrm{Rot}(A,B)\,\Box_B + \mathrm{Rot}(A,B)\,\dot{\Box}_B\,. \tag{7}$$

It follows

$$S\left\{\left[{}^A\Omega_B\right]_A\right\} = \left\{\dot{\Box}_A - \mathrm{Rot}(A,B)\,\dot{\Box}_B\right\} \Box_B^{-1}\,\mathrm{Rot}(B,A) \tag{8}$$

whence one gets $\left[{}^A\Omega_B\right]_A$. Furthermore, from

$$[0\dot{P}_1]_A = S\left\{\left[{}^A\Omega_B\right]_A\right\} \mathrm{Rot}(A,B)\,[0P_1]_B + \mathrm{Rot}(A,B)\,[0\dot{P}_1]_B + [0_A\dot{0}_B]_A \tag{9}$$

one obtains

$$\left[{}^A v_B\right]_A = [0_A\dot{0}_B]_A = [0\dot{P}_1]_A - S\left\{\left[{}^A\Omega_B\right]_A\right\} \mathrm{Rot}(A,B)\,[0P_1]_B + \mathrm{Rot}(A,B)\,[0\dot{P}_1]_B\,. \tag{10}$$

Remark 2: A DGPS receiver gives the position of a frame A (ROVER) relative to a frame B (BASE) having z_axis parallel to the axis of rotation of earth, x_axis parallel to the Meridian plane passing by Greenwich (frame B has the same orientation as ecef). I n applications, it may be important to have the position of the ROVER in terms of coordinates relative to an auxiliary local frame C. Denoting with T [C,B] the homogeneous transformation matrix , we have

$$\left[{}^{C}pos_{A}\right]_{C} = T[C,B]\left[{}^{B}pos_{A}\right]_{B} \tag{11}$$

where $\left[{}^{\beta}pos_{\alpha}\right]_{\gamma}$ denote the co-ordinates relative to γ of the position of α with respect to β. To determine T [C, B] we place A into a certain number of positions A_i, i = 1 … i and we measure $\left[{}^{C}pos_{A_i}\right]_{C}, \left[{}^{B}pos_{A_i}\right]_{B}$. Subsequently, we compute T [C,B] using the equation X = T [C,B] Y^{-1} where

$$X := \left[\left[{}^{C}pos_{A_1}\right]_{C} \quad \left[{}^{C}pos_{A_2}\right]_{C} \quad \cdots \quad \left[{}^{C}pos_{A_n}\right]_{C}\right] \tag{12}$$

$$Y := \left[\left[{}^{B}pos_{A_1}\right]_{B} \quad \left[{}^{B}pos_{A_2}\right]_{B} \quad \cdots \quad \left[{}^{B}pos_{A_n}\right]_{B}\right], \tag{13}$$

(the various column vectors are expressed in homogeneous co-ordinates).

6.11 Determination of a satellite orientation offset from gyros data

Consider a satellite in a circular orbit around earth; let ω_0 be the frequency of revolutions. Let us determine the relation between the orientation of the satellite with respect to the orbit and the measures provided by the gyros installed on the satellite.

Let Sat be a frame attached to the satellite; L_{oc} an auxiliary frame with origin coincident with Sat and axes x_0, y_0, z_0 defined as follows:

x_0: tangent to the orbit and directed along the orbit's path;

z_0: normal to the orbit, directed toward the center of the orbit;

y_0: bi-normal to the orbit: $y_0 = z_0 \wedge x_0$.

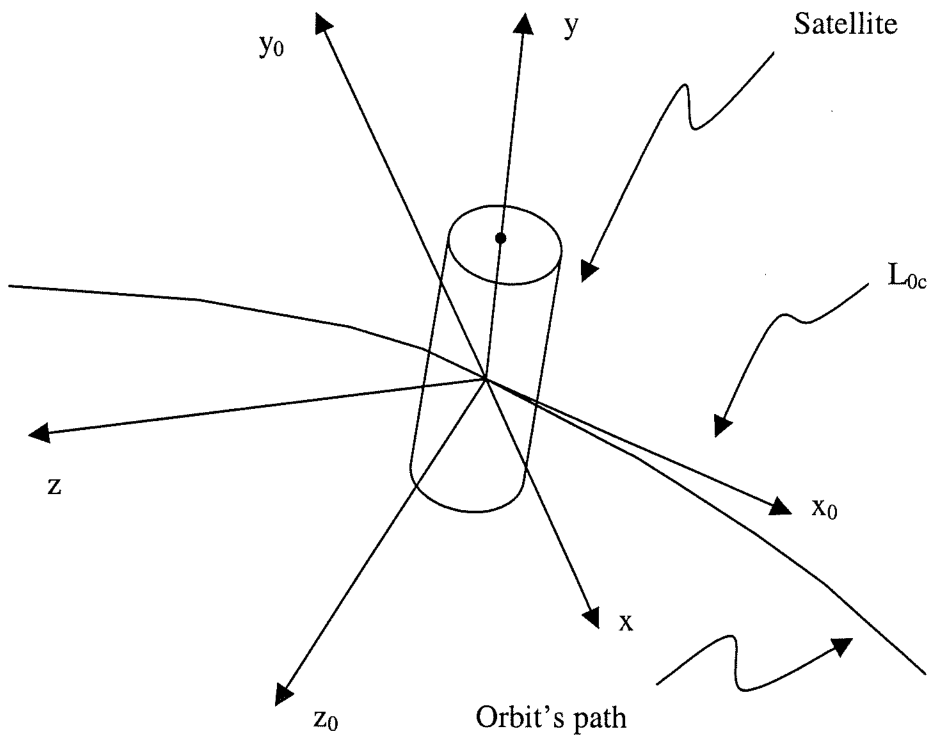

Figure 1: A satellite on a circular earth orbit.

Describe the orientation of the satellite relative to L_{oc} in terms of yaw, pitch and roll angles

α = yaw

β = pitch

γ = roll.

Assuming these angles sufficiently small, one can write

$$\mathrm{Rot}(L_{oc}, \mathrm{Sat}) = I_3 + S\left\{\begin{bmatrix}\gamma \\ \beta \\ \alpha\end{bmatrix}\right\} \tag{1}$$

$$\mathrm{Rot}(\mathrm{Sat}, L_{oc}) = I_3 - S\left\{\begin{bmatrix}\gamma \\ \beta \\ \alpha\end{bmatrix}\right\} \tag{2}$$

$$\left[{}^{L_{oc}}\Omega_{Sat}\right]_{L_{oc}} = \begin{bmatrix} \dot{\gamma} \\ \dot{\beta} \\ \dot{\alpha} \end{bmatrix}. \tag{3}$$

Denoting with eci the usual inertial frame,

$$\left[{}^{eci}\Omega_{Sat}\right]_{L_{oc}} = \left[{}^{eci}\Omega_{L_{oc}}\right]_{L_{oc}} + \left[{}^{L_{oc}}\Omega_{Sat}\right]_{L_{oc}} \tag{4}$$

where

$$\left[{}^{eci}\Omega_{L_{oc}}\right]_{L_{oc}} = \begin{bmatrix} 0 \\ -\omega_o \\ 0 \end{bmatrix}. \tag{5}$$

It follows

$$\left[{}^{eci}\Omega_{Sat}\right]_{Sat} = \mathrm{Rot}(Sat, Loc)\left[{}^{eci}\Omega_{Sat}\right]_{L_{oc}}$$

$$= \left(I_3 - S\left\{\begin{bmatrix} \gamma \\ \beta \\ \alpha \end{bmatrix}\right\}\right)\begin{bmatrix} 0 \\ -\omega_o \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{\gamma} \\ \dot{\beta} \\ \dot{\alpha} \end{bmatrix} \tag{6}$$

$$\cong \begin{bmatrix} 0 \\ -\omega_o \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & -\alpha & \beta \\ \alpha & 0 & -\gamma \\ -\beta & \gamma & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -\omega_o \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{\gamma} \\ \dot{\beta} \\ \dot{\alpha} \end{bmatrix}$$

Introducing the notation $\left[{}^{eci}\Omega_{Sat}\right]_{Sat} := [p\ q\ r]'$, (angular velocity measures provided by the gyros), it follows that the orientation and the angular velocity of S_{at} relative to L_{oc} satisfy the differential equation

$$\begin{bmatrix} \dot{\gamma} \\ \dot{\beta} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} \alpha\omega_o \\ 0 \\ -\gamma\omega_o \end{bmatrix} + \begin{bmatrix} p \\ q + \omega_o \\ r \end{bmatrix}. \tag{7}$$

6.12 Application of the Kalman filtering to the determination of the orientation of a satellite frame gyro's and horizon detector data

With reference to the satellite of example 11, let us consider the problem of computing the best estimate of its orientation, $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$, from the measurements provided by the gyros, modelled by the stochastic processes

$$\begin{bmatrix} \hat{p} \\ \hat{q} \\ \hat{r} \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} + \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \qquad \text{with } \omega_i,\ i = 1, 2, 2, \text{ zero mean white noises,} \tag{1}$$

and from the measurement, $\gamma_m = \gamma + \nu$, ν zero mean white noise, of the roll angle γ provided by a horizon detector.

This problem can be solved by applying the Kalman estimator

$$\begin{bmatrix} \dot{\hat{\gamma}} \\ \dot{\hat{\beta}} \\ \dot{\hat{\alpha}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \omega_o \\ 0 & 0 & 0 \\ -\omega_o & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\gamma} \\ \hat{\beta} \\ \hat{\alpha} \end{bmatrix} + \begin{bmatrix} \hat{p} \\ \hat{q} + \omega_o \\ \hat{r} \end{bmatrix} - \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} (\hat{\gamma} - \gamma_m) \tag{2}$$

The dynamics of the error estimate produced by this estimator is obtained by subtracting (11.7) of (2), which gives

$$\begin{bmatrix} \Delta\dot{\hat{\gamma}} \\ \Delta\dot{\hat{\beta}} \\ \Delta\dot{\hat{\alpha}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \omega_o \\ 0 & 0 & 0 \\ -\omega_o & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\hat{\gamma} \\ \Delta\hat{\beta} \\ \Delta\hat{\alpha} \end{bmatrix} - \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} [1\,0\,0] \begin{bmatrix} \Delta\hat{\gamma} + \nu \\ \Delta\hat{\beta} \\ \Delta\hat{\alpha} \end{bmatrix} + \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

The K_1, K_2 and K_3 minimizing the variance of this error are computed by solving the Riccati differential equation proposed by the Kalman filter theory.

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List of figures

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