NON-LINEAR ANALYSIS OF CYLINDRICAL SHELLS

by

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NON-LINEAR ANALYSIS OF THIN CYLINDRICAL SHELLS

ABSTRACT

A theory to predict the influence of geometric non-linearities on the natural frequencies of an empty anisotropic cylindrical shell was presented in this report. It was a hybrid of finite element and classical thin shell theories. Sanders-Koiter's non-linear and strain-displacement relations were used. Displacement functions were evaluated using linearized equations of motion. Modal coefficients were then obtained for these displacement functions. Expressions for the mass, linear and non-linear stiffness matrices were derived through the finite element method. The uncoupled equations were solved with the help of elliptic functions. The period and frequency variations were first determined as a function of shell amplitudes and then compared with the results in the literature.

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LIST OF SYMBOLS

$a_{rs}^{(1)}$, $a_{rs}^{(2)}$	:	coefficients determined by equation (5.13)
$b_{rs}^{(1)}$,	:	coefficient determinde by equation (A-3.2a)
$c_{rs}^{(1)}$, $c_{rs}^{(2)}$	:	coefficients determined by equation (A-3, 2b, c)
A_1, A_2	:	lamé parameters
A_p	:	motion amplitude
A_p, B_p, C_p	:	constants in equations U, V and W, respectively
A_{pq}, B_{pq}, C_{pq}	:	modal coefficients determined by equations (5.1)
A_{prsq}, B_{prsq}, C_{prsq}, D_{prsq}, E_{prsq}	:	modal coefficients determined by the equations (5.2)
cn	:	Jacobi elliptic function
e	:	Exponential function
E	:	Young's modulus of elasticity
f, g, h	:	spatial functions
f_p	:	function determined by equation (6.9)
G	:	shear modulus of elasticity
G (p,q)	:	coefficient determined by equation (5.18)

h_p (p=0,1,4,6,8)	: coefficients of the characteristic equation (4.5)
H_{pl}	: matrix elements H
J	: number of applied constraints
k_p^*	: parameter of the integral elliptic function K
$K\ (k_p^*)$	: elliptic complete integral of the first kind
l	: length of element
L	: total length of shell
m	: axial mode
M_{11}, M_{22}, M_{12}, M_{21}, M_{12}	: resultant moments
M_{xx}, $M_{\theta\theta}$, $M_{x\theta}$, $M_{\theta x}$, $M_{x\theta}$	: resultant moments of a cylindrical shell
$\bar{\bar{M}}_{11}$	: boundary moment couple
n	: circumferential mode
N_{11}, N_{22}, N_{12}, N_{21}, $\bar{N}_{12}$	: resultant constraints
N_{xx}, $N_{\theta\theta}$, $N_{x\theta}$, $N_{\theta x}$, $\bar{N}_{x\theta}$	: resultant constraints for a cylindrical shell
$\bar{\bar{N}}_{11}$	: boundary constraint value
N	: number of finite elements
NDF	: number of degrees of freedom
p_1, p_2, p_A	: loads applied

p_x, p_θ, p_n	:	loads applied to a cylindrical shell
p_{ij}	:	elements of the matrix of elasticity (i,j,1...,6)
Q_1, Q_2	:	shear constraint resultants
Q_x, Q_θ	:	shear constraint resultants for a cylindrical shell
R_1, R_2	:	radius of curvature for the surface of reference
R	:	radius of the shell
t	:	thickness of the shell
T_L	:	linear vibration period
T_{NL}	:	non-linear vibration period
U, V, W	:	axial, tangential and radial displacements, respectively
$\bar{\bar{V}}_{11}$	:	shear constraint resultants at the boundary
x	:	the coordinate generator of the shell
α_p, β_p	:	determined by equation (4.7) and given by equation (4.8)
ε_{xx}, $\varepsilon_{\theta\theta}$, $\varepsilon_{x\theta}$	:	deformations of the surface of reference

ε_{1k}	:	element of matrix E
κ_{xx}, $\kappa_{\theta\theta}$, $\kappa_{x\theta}$	:	rotations at the surface of reference
Λ_p	:	coefficient determined by equation (6.16)
λ_p	:	complex roots of characteristic equation (4.5)
ϕ, ϕ_x, ϕ_θ	:	determined by equation (3.5d)
ν	:	Poisson's ratio
ρ	:	density of the shell
ω_p	:	linear frequency of free vibrations
ω_p^*	:	non-linear frequency of free vibrations
θ	:	circumferential coordinates
τ	:	time related coordinates
ξ_1, ξ_2	:	coordinates for surface of reference

LIST OF MATRICES

$[A]$	:	determined by equation (4.11)
$[B]$, $[BB']$	:	determined by equations (4.14) and (4.18), respectively
$[A^+]$, $[B^+]$, $[C^+]$	:	matrices of modal coefficients determined by equation (5.1)
$[A^*]$, $[B^*]$, $[C^*]$	:	determined by equations (5.4)
$[A']$, $[B']$, $[C']$	:	determined by equations (5.3)
$\|A^{++}\|$, $\|B^{++}\|$, $\|C^{++}\|$, $\|D^{++}\|$, $\|E^{**}\|$	:	matrices of modal coefficients determined by equation (5.2)
$[A^{**}]$, $[B^{**}]$, $[C^{**}]$, $[D^{**}]$, $[E^{**}]$	:	determined by equation (5.9)
$\{C\}$	:	vector of arbitrary constraints
$\{E\}$	:	matrix function of $[A]$
$[F]$	:	vector of external forces
$[H]$	:	determined by equation (4.4)
$[J]$	:	determined by equation (4.14)
$[k_L]$, $[k_{NL}]$	:	linear and non-linear stiffness matrices for a finite element, respectively
$[k^*_{NL}]$	:	determined by equation (5.16)
$[K_L]$, $[K_{NL}]$	:	linear and non-linear stiffness matrices for the entire shell, respectively
$[K_L^{(r)}]$, $[K_{NL}^{(r)}]$	:	reduced linear and non-linear stiffness matrices, for the entire shell, respectively

$[K_L^{(D)}]$	:	global diagonal linear stiffness matrix
$[L]$	:	determined by equation (4.10)
$[m]$	:	mass matrix of a finite element
$[M]$	:	mass matrix for total shell
$[M^{(r)}]$	:	reduced mass matrix for total shell
$[M^{(D)}]$	:	diagonal mass matrix for total shell
$[N]$	:	determined by equation (4.12)
$[P]$	:	matrix of elasticity
$\{q\}$	:	time-related vector coordinates
$[Q]$	:	determined by equation (4.13)
$[R]$, $[R']$	:	determined, respectively, by equations (4.9) and (4.18)
$[T]$	:	determined by equation (4.2)
$[X]$	:	diagonal matrix function of coordinate x
$[\delta_i]$	:	vector of degrees of freedom for node i
$[\delta]$	:	vector of degrees of freedom for total shell
$\{\delta^{(r)}\}$	:	reduced vector of degrees of freedom for total shell
$\{\varepsilon\}$	:	deformation vector
$\{\varepsilon_L\}$, $\{\varepsilon_{NL}\}$	:	linear and non-linear components of the deformation vector, respectively
$[\Phi]$	:	matrix of eigenvectors
$\{\sigma\}$	:	stress vector

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figure

CHAPTER 1

INTRODUCTION

1.1 General

Thin shells are structures that have been widely used in a variety of fields. The diversity of their applications is extensive, from space vehicles to home appliances. Consequently, the analysis of thin shells under static or dynamic load has been the focus of many theories. Most of the research in this field has involved analysis of linear thin shells. The results have proven to be satisfactory in cases where deflections of the shell were very small, especially low-level for bending, even when allowing for the thickness of the shell itself. In several practical experiments, however, the linear analysis was not sufficiently precise for design assurance. In those cases, a non-linear analysis was required.

The first attempt to formulate a theory for thin shells was derived from Aron's general equation of elasticity in 1874. It was followed, in 1888, by an approximation theory known as the "First Approximation" of LOVE [1]. Since then, the linear theory of elastic shells has been re-examined trhoughout the years in the literature ([2] to [7]).

The non-linear theory of thin elastic shells has also been the focus of many studies. Thus, beginning with the tridimensional elasticity equations, there are now several articles available dealing with non-geometric linearities in shells of arbitrary shapes ([8] to [12]).

More specifically, several methods were developed for the analysis of dynamic non-linear thin cylindrical shells. Among these were Galerkin's well-balanced method ([13] to [15]), the small perturbation method ([16] to

[18]), the modal expansion method [19] and most recently the finite element method [20]. All of these methods have their advantages and disadvantages. The best test of any method is probably its general content: i.e. the method should quantify the component displacements and provide for precise characterization of the high and low frequencies of the shell.

These criteria were not met in Galerkin's small perturbation method, and studies [13] to [18] applied only to the particular case where the shell was supported on both edges. The modal expansion and finite element methods, however, were satisfactory on both counts.

In references [13] to [15], only lateral displacement was applied. In [13], the restrictions of tangential displacement continuity were satisfied although to the detriment of actual bending at the edges of the shell. In order to meet the criteria of continuity by including bending at the edges of the shell, Evensen [14] modified the lateral displacement expression by using a symmetric mode to include the coupling. This modification, however, led to actual moments at the edge of the shell such that the boundary conditions lay somewhere between the simply supported and clamped cases. Boundary-condition effects on the other components of displacement were ignored, moreover, in [13] and [14].

Similarly, in [15] coupling with the symmetric mode led to the derivation of motion by assuming:

a) The condition of continuity for the tangential components of displacement.

b) A geometric boundary condition on the axial component.

c) A natural boundary condition.

These three conditions, however, were only satisfied in a general sense.

Alturi [16] also used these three conditions and suggested that a lateral displacement with three modes be indluded. The displacement and axial bending moment were zero at the edges of the shell. Contrary to Dowell and Ventres [15], Alturi [16] solved the problem by using the small perturbation method. The unknowns appearing in the modal equations were expressed by means of an asymptotic series and terms of small parameter.

The formulas in [13] to [16] have serious drawbacks:

a) Having only assumed the form of the lateral displacement, special attention must be given to the conditions of continuity for the other components. Should these not be satisfied automatically, it would be necessary to include other modes and these modes are obtained intuitively. This procedure can hardly be generalized to include other shell geometries.

b) It is extemely difficult to satisfy the geometric boundary conditions on tangential displacement, especially for a circular shell.

c) The analytical solution of the problem requires several manual calculations. These become increasingly difficult so that the inclusion of other means becomes necessary.

d) The formulation is not applicable when the shape of the modes are not simple analytic functions.

e) Generalizations from arbitrary shells are not valid.

Part of the disadvantages were eliminated in Chen and Babcock [17]. The small perturbation method was used to transform the non-linear equations to a linear system, by expanding the unknown variables in a power series with respect to a small parameter. Applying the boundary conditions of circumferential continuity for a simply supported shell, lateral displacement was then obtained. The major advantage of this technique, compared to other methods requiring an initial hypothesis regarding the form of the vibration mode, is that the results are not preconceived.

Other refinements were raised in Ginsberg's article [18]. The equations for a simply supported circular shell were obtained using an energy formulation. All three displacements, U, V and W, were considered and a more exact theory was used. Due to algebraic difficulties encountered during derivation of the general equations, the perturbation technique had to be used. For this reason, therefore, limitations (d) and (e) still apply.

The above mentioned shortcomings restrict use of the methods employed in [13] to [18], (Donnell's simplified non-linear theory), because the theory neglects the plane of inertia effect. By incorporating the nodal expansion technique, Radium and Genin [19] improved upon the methods used in [13] to [18] and eliminated the weaknesses therein by using Sanders-Koiter's [10,11] general non-linear theory.

The authors of the present paper derived and validated the general nodal equation's for analysis of a static and dynamic arbitrary non-linear geometric shell. The three displacement components were considered in these cases.

There are two advantages to these formualtions:

a) Greater simplicity in problem formulation and solution, compared with the other methods.

b) Whatever the shell structure might be, the formultion of the equations retains the same format once the corresponding non-linear nodal equations are derived.

However, this method has a serious disadvantages: the analytical forms for the displacement components apply only to those cases where a cylinder is supported at both ends.

References [13] to [19] adopted the analytical method as their numerical approach to solving the problem. The finite element method likewise suggests a numerical approach. This method offers many advantages, some of which are:

a) Arbitrary shell geometry: the method applies equally well to the cylinder, to the cone or to all other axisymmetric shells with positive or negative shaped curvatures.

b) Simple inclusion of thickness discontinuities, material property variations, differences in materials comprising the shell.

c) Arbitrary boundary conditions: the problem can be resolved for a supported, clamped-free or clamped-clamped shell without changing the displacement functions in each particular case.

d) High and low frequency characteristics are obtained immediately.

After adopting the finite element method, Raju and Rao [20] obtained, for various boundary conditions, frequency variation in conjunction with the maximum normal displacement of a point situated on the average surface of the shell.

The Sanders-Koiter's relationship was derived from strain-displacement non-linear theory. A curved element with two nodes having six degrees of freedom each was used to restrain the shell. The displacement functions were not derived from thin shell theory but were instead described as a cubic polynomial in relation to the orthogonal coordinate. Their algorithm was iterative at each assumed normal displacement value, the approximate vector and frequencies were calculated until the convergence criterion was satisfied.

The research done in [14] to [20] was limited to studies of isotropic shells. Only Nowinski [13] made a generalization concerning orthotropic shells by incorporating Donnel's simplified theory. Ambartsumyan [21] produced an important work involving a number of cases anisotropic shells.

1.2 Research objectives

The present research project presents a general approach to analysis of non-linear thin cylindrical anisotropic shells. The finite element method was employed, but it a hybrid, a combination of the finite element method and classical shell theory. The finite element chosen was a cylindrical one. This choice allowed us to use the complete equilibrium equations to determine the displacement functions and, further, the mass and stiffness matrices.

This theory proved to be more accurate than the usual finite element methods. In addition, if offers the advantages listed in the paragraph below, it can only be used to analyze a cylindrical shell or a straight conical shell with a circular section.

In order to eliminate these weaknesses, Radwan and Genin [19] improved the technique, using more general non-linear theory from Sanders-Koiter's [10-11]. The authors derived a valid general modal equation for analysis of a non-linear static or dynamic load on an arbitrary geometric shell. In this particular case, all three displacement components were considered.

This formulation has two advantages:

a) Definition and solution of the problem are greatly simplified, compared to other methods.

b) Whatever the shape of the shell, the formulation of equations has the same format, once the corresponding non-linear model equations have been derived.

This method does nevertheless, have a serious flaw; it may be applied only in cases where the shell is simply supported at the edges.

The analytical solution involves two steps:

a) The displacement functions are determined by solving the linear system equations. The linear mass and stiffness matrices are then obtained together with the eigenvalues and eigenvectors [22,23].

b) Using strain-displacement relationships from Sanders-Koiter's non-linear theory [10,11], the modal coefficients are obtained from the displacement functions. The non-linear mass and stiffness matrices for a finite element are then calculated with respect to modal coefficients [19].

1.3 Contents of the report

The present study is divided into nine chapters, the contents of which will be described briefly.

Chapter 2 deals with a review of non-linear thin shell theory as well as the basic methodologies employed.

Chapter 3 proposes three non-linear differential equations of motion as a function of the displacement of the shell surface of reference and the components of the elasticity matrix, beginning with the general equations for arbitrary shells and their stress-strain relatiohnships.

The displacement functions are determined in Chapter 4, by solution of the linear systems. With these functions the mass and linear stiffness matrices for finite elements are constructed.

In Chapter 5, the displacement functions defined serve as a basis for interpretation of the modal coefficients as well as the displacement functions determined in chapter 4.

In Chapter 6, the methods for analytical solution of uncoupled non-linear equations are described. The influence of geometric non-linearities of the walls or the frequencies of the shell is recorded.

Chapter 8 presents the algorithm for the mass and stiffness matrices.

The numerical results obtained are reported in Chapter 8 and compared with other methods.

Finally, Chapter 9 contains the general conclusions.

CHAPTER II

BASIC THEORY AND METHOD

2.1 Hypotheses under non-linear elastic thin shell theory

Non-linear elastic thin shell theory is derived by approximation from the tridimensional elasticity equation. Like linear theory, it is also based on LOVE's "first approximation" but the assumption concerning the order of magnitude of the bending has been modified.

The non-linear theory is based upon the following hypotheses:

a) Thickness (t) is infinitesimal in comparison with the minimal radius of curvature (R_{min});

b) The displacement gradients are small and the squares of the rotation do not exceed reference surface deformation in order of magnitude;

c) The normal constraints, normal to the surface of reference are negligible;

d) The normals to the surface of reference remain normal after deformation and are not subject to any elongation.

Hypothesis (a) represents the definition of thin shells ($R/t \geqslant 10$).

Hypothesis (b) corroborates the non-linearities of the equations. Explained by physical bending terminology, these elements have the same thickness as the shell itself.

Hypotheses (c) and (d) allow us to neglect the stresses normal to the surface and the transversel shear deformation.

The theory based on these four hypotheses is known as "SANDERS-KOITER's non linear theory [10,11]"; it has been adopted throughout this paper.

2.2 Method

As mentioned in paragraph 1.2, the analysis was divided into two parts: the first deals with linear behaviour and the second with non-linearities and strain-displacement relationships.

The main steps in the method we propose are as follows:

a) The shell is subdivided into several cylindrical elements (Figure 2). Each shell element is defined by two nodal circles and two nodal points i and j (Figure 4). The displacement functions are defined by:

$$\begin{Bmatrix} U\ (x,\theta) \\ W\ (x,\theta) \\ V\ (x,\theta) \end{Bmatrix} = [N] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}$$

where $\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}$ represent nodal displacements, and the elements of matrix [N] are in general a function of position. These displacement functions must, on the one hand, adequately express real displacements of the shell and, on the other hand, satisfy at least the geometric boundary conditions.

b) The linear component of the procedure is presented in reference [23], where the displacement functions are determined by solving the three differential equations of motion from SANDER's theory [5].

c) For the non-linear component, the modal coefficients [19] are derived from the results obtained in the previous step.

d) The linear and non-linear natural vibration frequency ratio is then obtained for the cases of uncoupled modal equations.

CHAPTER III

EQUATIONS OF MOTION FOR ANISOTROPIC CYLINDRICAL SHELLS

3.1 Strain-displacement and stress-strain relations

Non-linear SANDERS-KOITER's theory for thin shells postulated differences in the first and second fundamental forms between the reference surfaces, deformed and non deformed, as deformation measures in elongation and bending respectively.

Generally, the deformation vector $\{\varepsilon\}$ is written as:

$$\{\varepsilon\} = \{\varepsilon_L\} + \{\varepsilon_{NL}\} = \begin{matrix} \varepsilon_{xx} \\ \varepsilon_{\theta\theta} \\ 2\varepsilon_{x\theta} \\ \kappa_{xx} \\ \kappa_{\theta\theta} \\ 2\kappa_{x\theta} \end{matrix} \qquad (3.1)$$

where subscripts "L" and "NL" mean "linear" and "non-linear", respectively.

For a cylindrical shell, the expressions for $\{\varepsilon_L\}$ and $\{\varepsilon_{NL}\}$ are given by:

$$\{\varepsilon_L\} = \begin{Bmatrix} \dfrac{\partial U}{\partial x} \\ \dfrac{1}{R}\left(\dfrac{\partial V}{\partial \theta} + W\right) \\ \dfrac{\partial V}{\partial x} + \dfrac{1}{R}\dfrac{\partial U}{\partial \theta} \\ -\dfrac{\partial^2 W}{\partial x^2} \\ -\dfrac{1}{R^2}\left(\dfrac{\partial^2 W}{\partial \theta^2} - \dfrac{\partial V}{\partial \theta}\right) \\ -\dfrac{2}{R}\dfrac{\partial^2 W}{\partial x \partial \theta} + \dfrac{3}{2R}\dfrac{\partial V}{\partial x} - \dfrac{1}{2R^2}\dfrac{\partial U}{\partial \theta} \end{Bmatrix} \tag{3.2a}$$

and

$$\{\varepsilon_{NL}\} = \begin{Bmatrix} \dfrac{1}{2}\left(\dfrac{\partial W}{\partial x}\right)^2 + \dfrac{1}{8}\left(\dfrac{\partial V}{\partial x} - \dfrac{1}{R}\dfrac{\partial U}{\partial \theta}\right)^2 \\ \dfrac{1}{2R^2}\left(V - \dfrac{\partial W}{\partial \theta}\right)^2 + \dfrac{1}{8}\left(\dfrac{\partial V}{\partial x} - \dfrac{1}{R}\dfrac{\partial U}{\partial \theta}\right)^2 \\ \dfrac{1}{2R}\left(\dfrac{\partial W}{\partial x}\dfrac{\partial W}{\partial \theta} - V\dfrac{\partial W}{\partial x}\right) \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.2b}$$

where U, V and W are, respectively, the axial, tangential and radial displacements of the shell's surface of reference.

It is evident that in equations (3.2a, b) the expressions for components κ_{xx}, $\kappa_{\theta\theta}$, et $2\kappa_{x\theta}$ are linear. This fits in with hypothesis (b) from paragraph 2.1.

This constituent relations between the stress and deformation vectors of the surface of reference for anisotropic shells are given as follows:

$$\{\sigma\} = \begin{Bmatrix} N_{xx} \\ N_{\theta\theta} \\ N_{x\theta} \\ M_{xx} \\ M_{\theta\theta} \\ M_{x\theta} \end{Bmatrix} = [P]\ \{\varepsilon\} \qquad (3.3)$$

where [P] is the matrix of elasticity.

The elements p_{ij} in [P] determine the anisotropy of the shell, which depends upon the mechanical characteristics of the structure's material.

In general, this implies that:

$$[P] = \begin{bmatrix} p_{11} & p_{12} & 0 & p_{14} & p_{15} & 0 \\ p_{21} & p_{22} & 0 & p_{24} & p_{25} & 0 \\ 0 & 0 & p_{33} & 0 & 0 & p_{36} \\ p_{41} & p_{42} & 0 & p_{44} & p_{45} & 0 \\ p_{51} & p_{52} & 0 & p_{54} & p_{55} & 0 \\ 0 & 0 & p_{63} & 0 & 0 & p_{66} \end{bmatrix} \qquad (3.4)$$

3.2 Equations of equilibrium

By applying the virtual work principle to the infinitesimal element of the deformed surface of reference, the five equations of equilibrium, describing the non-linear behaviour of an arbitrarily formed shell, are then obtained (appendix A-1).

By eliminating shear forces Q_x and Q_e by means of equations (A-1.5d,e), when external loading is non-existent, we obtain:

$$\frac{\partial N_{xx}}{\partial x} + \frac{1}{R}\frac{\partial \bar{N}_{x\theta}}{\partial \theta} - \frac{1}{2R^2}\frac{\partial \bar{M}_{x\theta}}{\partial \theta} - \frac{1}{2R}\frac{\partial}{\partial \theta}\ \phi\ (\ N_{xx} + N_{\theta\theta}\ ) = 0 \qquad (3.5a)$$

$$\frac{1}{R}\frac{\partial N_{\theta\theta}}{\partial \theta} + \frac{\partial \bar{N}_{x\theta}}{\partial x} + \frac{1}{R^2}\frac{\partial M_{\theta\theta}}{\partial \theta} + \frac{3}{2R}\frac{\partial \bar{M}_{x\theta}}{\partial x} - \frac{1}{R}\ (\ \phi_x \bar{N}_{x\theta} + \phi_\theta N_{\theta\theta}\ ) +$$

$$\frac{1}{2}\frac{\partial}{\partial x}\ \phi\ (\ N_{xx} + N_{\theta\theta}\ ) \quad 0 \qquad (3.5b)$$

$$\frac{\partial^2 M_{xx}}{\partial x^2} + \frac{2}{R}\frac{\partial^2 \bar{M}_{x\theta}}{\partial x \partial \theta} + \frac{1}{R^2}\frac{\partial^2 M_{\theta\theta}}{\partial \theta^2} - \frac{1}{R} N_{\theta\theta} - \frac{\partial}{\partial x}\ \phi_x N_{xx} + \phi_\theta \bar{N}_{x\theta} \ -$$

$$\frac{1}{R}\frac{\partial}{\partial \theta}\ \phi_x \bar{N}_{x\theta} + \phi_\theta N_{\theta\theta} = 0 \qquad (3.5c)$$

where

$$\phi = \frac{1}{2}\ (\ \frac{\partial V}{\partial x} - \frac{1}{R}\frac{\partial U}{\partial \theta}\ )\ ,\ \phi_x = -\frac{\partial W}{\partial x} \ \text{ et } \ \phi_\theta = -\frac{1}{R}\ (\ \frac{\partial W}{\partial \theta} - V\ ) \qquad (3.5d)$$

Substituting equations (3.2) to (3.4) for the equilibrium equations (3.5), we obtain new equation (3.6) for functions of the elements p_{ij} in [P] and and the axial, tangential and radial displacements U, V and W of the shell surface of reference:

$$
\begin{aligned}
L_1\ (U,V,W,p_{ij}) + N_1\ (U,V,W,p_{ij}) &= 0 \\
L_2\ (U,V,W,p_{ij}) + N_2\ (U,V,W,p_{ij}) &= 0 \qquad (3.6) \\
L_3\ (U,V,W,p_{ij}) + N_3\ (U,V,W,p_{ij}) &= 0
\end{aligned}
$$

Functions L_i and N_i (i = 1 to 3) represent, respectively, the linear and non-linear equations of equilibrium. These equations are given in Appendix A-2.

3.3 Matrix of elasticity

The matrix of elasticity [P] is generally given by equation (3.4); the present theory can therefore be applied to:

(i) Shells composed of only one layer or of an arbitrary number of isotropic or orthotropic layers;

(ii) Double-walled shells, with slabs or ribs;

(iii) Ring-stiffered shells with grooves of known characteristics;

(iv) Shells where [P] can be experimentally evaluated.

Here we will confine ourselves to shells composed of only one layer or an arbitrary number of symmetric isotropic or orthotropic layers arranged relative to the surface coordinates.

For an arbitrary number of orthotropic layers [2], it is postulated that there is no slippage between the layers and that the principal directions of elasticity on every point of the shell coincide with the directions of the coordinate lines.

(i) For an even number of layers, equal to 2v, the elements p_{ij} of [P] can be written as:

$$p_{ij} = 2 \quad B^{s}_{ij} \left(t_s - t_{s+1} \right) \qquad i = 1 \text{ to } 3 \text{ and } j = 1 \text{ to } 6$$
$$p_{ij} = 2/3 \quad B^{s}_{i-3,j-3} \left(t^3_s - t^3_{s+1} \right) \qquad i = 4 \text{ to } 6 \text{ and } j = 4 \text{ to } 6 \qquad (3.7)$$

(ii) For an odd number 2v + 1, we obtain:

$$p_{ij} = 2 \quad B^{v+1}_{ij} t_{v+1} + \quad B^{s}_{ij} \left(t_s - t_{s+1} \right) \qquad i = 1 \text{ to } 3 \text{ and } j = 1 \text{ to } 6$$
$$p_{ij} = 2/3 \quad B^{v+1}_{i-3,j-3} t^3_{v+1} + \quad B^{s}_{i-3,j-3} \left(t^3_s - t^3_{s+1} \right) \qquad i = 4 \text{ to } 6 \text{ and } j = 4 \text{ to } 6 \qquad (3.8)$$

where

$$B^s_{11} = E^s_1 / (1 - \nu^s_1 \nu^s_2) \qquad B^s_{22} = E^s_2 / (1 - \nu^s_1 \nu^s_2)$$
$$B^s_{12} = B^s_{21} = \nu^s_2 E^s_1 / (1 - \nu^s_1 \nu^s_2) \qquad B^s_{33} = 0.5\, G^s_{12} \qquad (3.9)$$
$$B^s_{ij} = 0 \quad \text{elsewhere}$$

t_s is the x^{th} layer coordinate having the surfce of reference as shown in Figure 5; $(E_1^s \nu_1^s)$ and $(E_2^s \nu_2^s)$ are, respectively, Young's modulus and Poisson's ratio in directions x and θ and G_{12}^s > which is the shear modulus of elasticity.

CHAPTER IV

LINEAR MATRIX CONSTRUCTION

4.1 Choice and justification of the method used

In the preceding chapter, the general equations of motion for elements p_{ij} of the elasticity matrix and axial, tangential and radial displacements, U, V, W, respectively, of the shell's surface of reference were obtained. The solution of these non-linear differential equations was highly complicated.

To circumvent the difficulty, the problem was divided into two parts; the first dealing with the linear system and the second, with the non-linearities in the strain-displacement relations.

In order to obtain the stiffness and mass matrices, the displacement functions were derived from the shell's equations of motion.

4.2 Displacement functions

Following the procedure described in paragraph 2.2, the shell was subdivided into several finite elements defined by two nodes i and j and by components U, V and W, representing axial, tangential and radial displacements, respectively, from a point located on the shell's surface of reference.

The linear equations of motion are given by:

$$\begin{aligned} L_1\ (U,V,W,p_{ij}) &= 0 \\ L_2\ (U,V,W,p_{ij}) &= 0 \\ L_3\ (U,V,W,p_{ij}) &= 0 \end{aligned} \tag{4.1}$$

The displacement functions are then assumed to be:

$$\begin{Bmatrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{Bmatrix} = [T] \begin{Bmatrix} u(x) \\ w(x) \\ v(x) \end{Bmatrix} \tag{4.2}$$

$[T]$ is a (3 x 3) matrix in θ given in Appendix A-3 and $u(x)$, $w(x)$ and $v(x)$ are functions of the x coordinate and the shells characteristics.

Assuming:

$$u(x) = Ae^{\lambda x/R}, \quad v(x) = Be^{\lambda x/R}, \quad w(x) = Ce^{\lambda x/R} \tag{4.3}$$

Substituting (4.2) and (4.3) for the equations of motion (4.1), three homogeneous linear functions of constants A, B and C are obtained:

$$[H] \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \{0\} \tag{4.4}$$

For the solution to be non-trivial, the determinant of matrix [H] must be equal to zero. This brings us to the following polynomial equation [23]:

$$\mathrm{Det}([H]) = h_8\lambda^8 - h_6\lambda^6 + h_4\lambda^4 - h_2\lambda^2 + h_0 = 0 \qquad (4.5)$$

The values of coefficients h_p in this eighth-degree polynomial are given in Appendix A-3.

Each root of this equation yields a solution to the equations of motion (4.1). The complete solution is obtained by adding the eight solutions independently with the constants A_p, B_p and C_p (pl,...,8), so that:

$$u(x) = A_p e^{\lambda_p x/R} \qquad (4.6a)$$

$$v(x) = B_p e^{\lambda_p x/R} \qquad (4.6b)$$

$$w(x) = C_p e^{\lambda_p x/R} \qquad (4.6c)$$

The constants A_p, B_p and C_p are not independent. We can therefore express A_p and B_p as a function of C_p, for example:

$$A_p = \alpha_p C_p \quad \text{and} \quad B_p = \beta_p C_p \quad , \quad p \; 1,\ldots,8 \qquad (4.7)$$

The values of α_p and β_p can be obtained from the following relations:

$$\begin{matrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{matrix} \quad \begin{matrix} \alpha_p \\ \beta_p \end{matrix} = \begin{matrix} -H_{13} \\ -H_{23} \end{matrix} \qquad (4.8)$$

where coefficients $[H_k]$ are as given in Appendix A-3.

Substituting expressions (4.6) and (4.7) into equations (4.2), the displacements $U(x,\theta)$, $V(x,\theta)$ and $W(x,\theta)$ can then be expressed in conjuction with the eight C_p constants only. We then have:

$$\begin{matrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{matrix} = [T]\,[R]\,\{C\} \qquad (4.9)$$

where $[R]$ is a (3 x 8) matrix given in Appendix A-3 and θ is an 8th order vector of the C_p constants:

$$\{C\} = \{\, C_1 \; C_2 \; . \; . \; . \; C_8 \,\}^T$$

Setting $[R]=[L]\,[X]$, equation (4.9) becomes:

$$\begin{matrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{matrix} = [T][L][X]\{C\} \qquad (4.10)$$

where matrixes $[L]$ and $[X]$ are given in Appendix A-3.

To determine the eight C_p constants, it is necessary to formulate eight boundary conditions for the finite elements. The axial, tangential and radial displacements, as well as rotation, will be specified for each node. The degres of freedom at node i can be defined by the vector:

$$\{\delta_i\} = \{\ u_i\ w_i\ (\frac{dw}{dx})_i\ v_i\ \}^T$$

The elements which have two nodes and eight degrees of freedom will have i (x = 0) and j (x = 1) as nodal displacements at the boundaries:

$$\begin{matrix}\delta_i \\ \delta_j\end{matrix} = \{\ u_i\ w_i\ (\frac{dw}{dx})_i\ v_i\ u_j\ w_j\ (\frac{dw}{dx})_j\ v_j\ \}^T = [A]\{C\} \quad (4.11)$$

where the terms of matrix [A], given in appendix, are obtained from matrix [R] by successively setting x = 0 and x = 1.

Multiplying equation (4.11) by $[A^{-1}]$ we obtain:

$$\{C\} \equiv [A^{-1}] \begin{matrix}\delta_i \\ \delta_j\end{matrix}$$

Substituting for equations (4.10) we get:

$$\begin{matrix}U(x,\theta) \\ W(x,\theta) \\ V(x,\theta)\end{matrix} = [T][L][X][A^{-1}] \begin{matrix}\delta_i \\ \delta_j\end{matrix} = [N] \begin{matrix}\delta_i \\ \delta_j\end{matrix} \quad (4.12)$$

These equations determine the displacement functions.

4.3 Linear mass and stiffness matrices for an element

The deformation vector can be obtained from equations (3.2a) and (4.12), therefore:

$$\{\varepsilon_L\} = \begin{bmatrix} [T] & [0] \\ [0] & [T] \end{bmatrix} [Q][A^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \qquad (4.13)$$

Setting $[Q] = [J]\,[X]$, equation (4.13) becomes:

$$\{\varepsilon_L\} = \begin{bmatrix} [T] & [0] \\ [0] & [T] \end{bmatrix} [J][X][A^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [B] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \qquad (4.14)$$

Matrix $[J]$ is given in Appendix A-3.

Combining equations (3.3) and (4.14), the stress-strain relations can be written as:

$$\{\sigma_L\} = [P][B] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \qquad (4.15)$$

The mass and stiffness matrices can then be expressed as:

$$[m] = \rho t \int [N^T][N]\, dA$$

$$[k_L] = \int [B^T][P][B]\, dA \qquad (4.16)$$

where $dA = R dx d\theta$. A quick reminder to the reader: "L" means "linear".

Using equations (4.12) and (4.14), equations (4.16), after integration with respect to θ over the interval, become

$$[m] = \rho t\,[A^{-1}]^T \quad \pi R \quad [X^T][L^T][L][X]\,dx \quad [A^{-1}]$$

(4.17)

$$[k_L] = [A^{-1}]^T \quad \pi R \quad [X^T][J^T][P][J][X]\,dx \quad [A^{-1}]$$

After working out the integration as a function of x, we obtain:

$$[m] = \pi R \rho t\,[A^{-1}]^T[R'][A^{-1}]$$

(4.18)

$$[k_L] = \pi R\,[A^{-1}]^T[BB'][A^{-1}]$$

where the (p,q) term from [R'] is:

$$R'(p,q) = \begin{cases} \dfrac{L'(p,q)}{(\lambda_p + \lambda_q)/R} \quad e^{(\lambda_p + \lambda_q)\ell/R} - 1 & \text{si } \lambda_p + \lambda_q \neq 0 \\ L'(p,q)\,.\,\ell & \text{si } \lambda_p + \lambda_q = 0 \end{cases} \qquad (4.19)$$

and where [BB'] is

$$BB'(p,q) = \begin{cases} \dfrac{J'(p,q)}{(\lambda_p + \lambda_q)/R} \quad e^{(\lambda_p + \lambda_q)\ell/R} - 1 & \text{si } \lambda_p + \lambda_q \neq 0 \\ J'(p,q)\,.\,\ell & \text{si } \lambda_p + \lambda_q = 0 \end{cases} \qquad (4.20)$$

L'(p,q) and J'(p,q) are, respectively, the (p,q) terms of the products of matrices $[L^T]$ $[L]$ and $[J^T]$ $[P]$ $[J]$.

CHAPTER V

NON-LINEAR MATRIX CONSTRUCTION

5.1 Introduction

As mentioned in paragraph 4.1, the problem was solved in two parts. The next chapter deals with the linear mass and stiffness matrices. The objective of the present chapter is to determine the non-linear stiffness matrix.

To that end, the following approach, developed in reference [19], was used with particular attention to geometric non-linearities. The coefficients of the modal equations were obtained through the Lagrange method. Thus, the non-linear stiffness matrix, once calculated, was overlaid onto the linear system. Before we embark on matrix formulation, however, a brief summary of the method is in order.

5.2 Method

This section will be limited to the relevant details of the method used to find the non-linear stiffness matrix. For further information, the interested reader should consult reference article [19].

The main steps of this method are as follow:

a) Shell displacements are expressed as generalized product coordinate sums and spatial functions;

b) The deformation vector is written as a function of the generalized coordinates by separating the linear portion from the non-linear;

c) These expressions are then introduced into the Lagrange equations up to and including the degree corresponding to the deformation energy;

d) Substituting the expressions in a) into the strain-displacement relations in SANDERS-KOITER's [10,11] non-linear method, the generalized coordinate coefficients appearing in the equations derived under c) are determined in terms of spatial functions.

5.3 Coefficients of modal equations

If A_{pq}, B_{pq}, C_{pq}, A_{prsq}, B_{prsq}, C_{prsq}, D_{prsq} and E_{prsq} are designed as coefficients of the modal equations mentioned in step d) above for a cylindrical shell, the following expressions [19] are thus obtained:

$$A_{pq} = \frac{1}{8R^2} \left(R \frac{\partial g_p}{\partial x} - \frac{\partial f_p}{\partial \theta} \right) . \left(R \frac{\partial g_q}{\partial x} - \frac{\partial f_q}{\partial \theta} \right) + \frac{1}{2} \frac{\partial h_p}{\partial x} \frac{\partial h_q}{\partial \theta} \qquad (5.1a)$$

where f, g, h are spatial functions determined by matrix [N] in equation (4.12) and:

$$B_{pq} = \frac{1}{8R^2} \left(R \frac{\partial g_p}{\partial x} - \frac{\partial f_p}{\partial \theta} \right) . \left(R \frac{\partial g_q}{\partial x} - \frac{\partial f_q}{\partial \theta} \right) + \frac{1}{2R^2} \left(\frac{\partial h_p}{\partial \theta} - g_p \right) . \left(\frac{\partial h_q}{\partial \theta} - g_q \right) \qquad (5.1b)$$

$$C_{pq} = \frac{1}{4R} \left(\frac{\partial h_p}{\partial x} \frac{\partial h_q}{\partial \theta} + \frac{\partial h_q}{\partial x} \frac{\partial h_p}{\partial \theta} \right) - \frac{1}{4R} \left(g_p \frac{\partial h_q}{\partial x} + g_q \frac{\partial h_p}{\partial x} \right) \qquad (5.1c)$$

$$
\begin{aligned}
A_{prsq} &= 2A_{pq}A_{rs} \\
B_{prsq} &= 2B_{pq}B_{rs} \\
C_{prsq} &= 2C_{pq}C_{rs} \\
D_{prsq} &= 2B_{pq}A_{rs} \\
E_{prsq} &= 2A_{pq}B_{rs}
\end{aligned}
\qquad (5.2)
$$

In equations (5.1) and (5.2), the subscripts p,q and p, q, r, s represent the coupling between two modes. It is arranged in such a way that equation (5.2) is written, r=p and s=q.

For consistency, equations (5.1) and (5.2) are written in matrix format.

Hence, these different matrices can be expressed in conjunction with matrices [T], [L], [X] and $[A^{-1}]$.

The following notation is adopted: the matrices with the "+" superscript represent equations in (5.1) and the ones with the "++" superscript represent the equations in (5.2)

With the (5.1) equations, we obtain:

$$
\begin{bmatrix} [A^{+}] \\ [B^{+}] \\ [C^{+}] \end{bmatrix} = [A^{-1}]^{T}[X^{T}] \begin{bmatrix} [A'] \\ [B'] \\ [C'] \end{bmatrix} [X][A^{-1}] \qquad (5.3)
$$

where matrices [A'], [B'] and [C'] are a function of n, θ and α_p and β_p from roots λ_p of the specific equatin in (4.5) and of constants defined in equations (4.7).

Setting

$$\begin{bmatrix} [A^*] \\ [B^*] \\ [C^*] \end{bmatrix} = [X^T] \begin{bmatrix} [A'] \\ [B'] \\ [C'] \end{bmatrix} [X] \qquad (5.4)$$

equations (5.3) become:

$$\begin{bmatrix} [A^+] \\ [B^+] \\ [C^+] \end{bmatrix} = [A^{-1}]^T \begin{bmatrix} [A^*] \\ [B^*] \\ [C^*] \end{bmatrix} [A^{-1}] \qquad (5.5)$$

Matrices $[A^+]$, $[B^+]$ and $[C^+]$ are square (8 x 8) matrices.

When r=p and s=q, the equations are written:

$$\begin{bmatrix} [A^{++}] \\ [B^{++}] \\ [C^{++}] \\ [D^{++}] \\ [E^{++}] \end{bmatrix} = 2 \begin{bmatrix} [A^+][A^+] \\ [B^+][B^+] \\ [C^+][C^+] \\ [A^+][B^+] \\ [B^+][A^+] \end{bmatrix} \qquad (5.6)$$

Using equations (5.3) we then get:

$$\begin{matrix}[A^{++}] \\ [B^{++}] \\ [C^{++}] \\ [D^{++}] \\ [E^{++}]\end{matrix} = 2\,[A^{-1}]^T[X] \begin{matrix}[A'] \\ [B'] \\ [C'] \\ [A'] \\ [B']\end{matrix} \; [X][A^{-1}][A^{-1}]^T[X] \begin{matrix}[A'] \\ [B'] \\ [C'] \\ [B'] \\ [A']\end{matrix} \; [X][A^{-1}] \qquad (5.7)$$

Using the symetrical properties of matrices [X], [A'], [B'] and [C'].

The product $[A^{-1}[[A^{-1}]^T$ represents a matrix of a constant, written as [E]. Substituting equations (5.4) in equations (5.7), we obtain:

$$\begin{matrix}[A^{++}] \\ [B^{++}] \\ [C^{++}] \\ [D^{++}] \\ [E^{++}]\end{matrix} = 2\,[A^{-1}]^T \begin{matrix}[A^{*}] \\ [B^{*}] \\ [C^{*}] \\ [A^{*}] \\ [B^{*}]\end{matrix} \; [E] \begin{matrix}[A^{*}] \\ [B^{*}] \\ [C^{*}] \\ [B^{*}] \\ [A^{*}]\end{matrix} \; [A^{-1}] \qquad (5.8)$$

Setting

$$
\begin{bmatrix} [A^{**}] \\ [B^{**}] \\ [C^{**}] \\ [D^{**}] \\ [E^{**}] \end{bmatrix} = \begin{bmatrix} [A^{*}] \\ [B^{*}] \\ [C^{*}] \\ [A^{*}] \\ [B^{*}] \end{bmatrix} [E] \begin{bmatrix} [A^{*}] \\ [B^{*}] \\ [C^{*}] \\ [B^{*}] \\ [A^{*}] \end{bmatrix} \qquad (5.9)
$$

Equations (5.8) are then written:

$$
\begin{bmatrix} [A^{++}] \\ [B^{++}] \\ [C^{++}] \\ [D^{++}] \\ [E^{++}] \end{bmatrix} = 2\,[A^{-1}]^T \begin{bmatrix} [A^{**}] \\ [B^{**}] \\ [C^{**}] \\ [D^{**}] \\ [E^{**}] \end{bmatrix} [A^{-1}] \qquad (5.10)
$$

Let us now illustrate the development of the expressions for the (p,q) term of matrices $[A^{*}]$ and $[A^{**}]$

For $[A^{*}]$ there is:

$$
A^{*}(p,q) = a_{pq}\, e^{(\lambda_p + \lambda_q)x/R} \qquad (5.11)
$$

and for $[A^{**}]$ there is:

$$A^{**}(p,q) = \quad a_{kq} \qquad a_{pl}\varepsilon_{lk}\, e^{(\lambda_p + \lambda_q + \lambda_k + \lambda_l)x/R} \tag{5.12}$$

the term (l,k) is from matrix $[E]$, and

$$a_{rs} = \frac{1}{2R^2} \quad \frac{1}{4} a_{rs}^{(1)} \sin^2 n\theta + a_{rs}^{(2)} \cos^2 n\theta$$

with

$$a_{rs}^{(1)} = (\beta_r\lambda_r + n\alpha_r)(\beta_s\lambda_s + n\alpha_s)$$

$$a_{rs}^{(2)} = \lambda_r\lambda_s \qquad\qquad r,s \quad 1,\ldots,8 \tag{5.13}$$

Similarly, matrices $[B^{++}]\ldots[E^{++}]$ can be written as a function of α, β, λ, x and θ. The (p,q) terms of these matrices are described in Appendix A-3.

5.4 Non-linear stiffness matrix for an element

The non-linear stiffness matrix for an orthotropic cylindrical shell is written [19,21]:

$$[k_{NL}] = \qquad p_{11}[A^{++}] + p_{22}[B^{++}] + p_{12}\,([D^{++}] + [E^{++}]) + p_{33}[C^{++}] \quad dA \tag{5.14}$$

where $dA = Rdxd\theta$.

Using equations (5.1), equation (5.14) is written:

$$[k_{NL}] = 2[A^{-1}]^T \quad p_{11}[A^{**}] + p_{22}[B^{**}] + p_{12}([D^{**}] + [E^{**}]) + p_{33}[C^{**}] \quad dA \quad [A^{-1}] \tag{5.15}$$

Integrating the expression in parentheses in equation (5.15) for

$0 \leqslant x \leqslant \ell$ and $0 \leqslant \theta \leqslant 2\pi$, and grouping the terms, we find:

$$[k_{NL}] = \frac{1}{\ell R^2}[A^{-1}]^T[k^*_{NL}][A^{-1}] \tag{5.16}$$

The (p,q) term in matrix $[k^*_{NL}]$ is written

$$k^*_{NL}(p,q) = \begin{cases} \dfrac{\varepsilon_{1k}}{(\lambda_p + \lambda_q + \lambda_k + \lambda_1)} G(p,q) \; e^{(\lambda_p + \lambda_q + \lambda_k + \lambda_1)\frac{\ell}{R}} & \text{si } \lambda_p + \lambda_q + \lambda_k + \lambda_1 \neq 0 \\ \varepsilon_{1k} \, G(p,q) \; \ell/R & \text{si } \lambda_p + \lambda_q + \lambda_k + \lambda_1 = 0 \end{cases} \tag{5.17}$$

G(p,q) is a coefficient in conjunction with α, β, λ and elements p_{ij} in matrix [P]. The gneral expression of G(p,q) is:

$$G(p,q) = \frac{3}{16} (p_{11} + p_{22} + 2p_{12}) a^{(1)}_{p1} a^{(1)}_{kq} + 3 (p_{11} a^{(2)}_{p1} a^{(2)}_{kq} +$$

$$p_{22} b^{(1)}_{p1} b^{(1)}_{kq}) + p_{12} (a^{(2)}_{p1} b^{(1)}_{kq} + b^{(1)}_{p1} a^{(2)}_{kq}) +$$

$$\frac{1}{4} (p_{11} + p_{12})(a^{(2)}_{p1} a^{(1)}_{kq} + a^{(1)}_{p1} a^{(2)}_{kq}) + \frac{3}{4} (p_{12} + p_{22}) .$$

$$.(a^{(1)}_{p1} b^{(1)}_{kq} + b^{(1)}_{p1} a^{(1)}_{kq}) + \frac{1}{4} p_{33} (c^{(1)}_{p1} c^{(1)}_{kq} +$$

$$c^{(2)}_{p1} c^{(2)}_{kq} + c^{(2)}_{p1} c^{(1)}_{kq} + c^{(1)}_{p1} c^{(2)}_{kq}) \qquad (5.18)$$

where the terms $a^{(1)}$ and $a^{(2)}$ are given by equations (5.13). Terms $b^{(1)}_{\cdots}$, $c^{(1)}_{\cdots}$ and $c^{(2)}_{\cdots}$ are coefficients appearing in expressions for the elements of matrices $[B^*]$ and $[C^*]$ defined in equations (5.4). These coefficients are given in Appendix A-3.

CHAPTER VI

THE INFLUENCE OF GEOMETRIC NON-LINEARITIES OF THE WALLS ON THE NATURAL FREQUENCIES OF A CYLINDRICAL SHELL

This chapter presents the solution to uncoupled equations of motion after the mass and stiffness matrices for each element have been assembled.

6.1 Global mass and stiffness matrices for the shell

The mass and stiffness matrices obtained in Chapters IV and V apply to only one element. After the shell is subdivided into several cylindrical elements, the global mass and stiffness matrices are determined by assembling the matrices for each element. Assembling is done such that all the equations of motion and the continuity of displacements at each node are satisfied.

Vectors $\{F_i\}$ and $\{F_j\}$ represent the internal forces at each i,j node and $\{\delta_i\}$ and $\{\delta_j\}$ are the displacements associated with $\{F_i\}$ and $\{F_j\}$. The sums of the forces and moments at each node must be equal to the sum of the external forces and the moments applied to the node:

$$\{F\}^e = F_j + F_{i+1}$$

and

$$\delta_j = \delta_{i+1}$$

Using these relations we casn overlay the mass and stiffness matrices for the individual elements in order to obtsain the mass and stiffness matrices for the whole shell. These matrices are designated as [M], $[K_L]$ and $[K_{NL}]$, respectively. They are square matrices of order NDF* (N + L), where N represents the number of finite elements and NDF represents the number of degrees of freedom at each node. This is schematically represented in Figure 6.

6.2 Equations of motion

The dynamic behaviour of an empty cylindrical shell, in the absence of external loads, can be represented by the following system:

$$[M]\{\ddot{\delta}\} + [K_L]\{\delta\} + [K_{NL}]\{\delta^3\} = \{0\} \tag{6.1}$$

where $\{\delta\}$ is the displacement vector; [M], $[K_L]$ and $[K_{NL}]$ are, respectively, the linear and non-linear mass stiffness matrices of the system.

In practice, very specific conditions are applied to the shell boundaries. Thus, matrices [M], $[K_L]$ and $[K_{NL}]$ are reduced to square matrices of order NRDUC NDF*[N + 1] - J, where J represents the number of essential constraints. These reduced matrices are written as $[M^{(r)}]$, $[K_L^{(r)}]$ and $[K_{NL}^{(r)}]$. As noted previously and to apply hereafter, the superscript "r" means "reduced".

The (6.1) system of equations then becomes:

$$[M^{(r)}]\{\ddot{\delta}^{(r)}\} + [K_L^{(r)}]\{\delta^{(r)}\} + [K_{NL}^{(r)}]\{\delta^{(r)^3}\} = \{0\} \quad (6.2)$$

Setting:

$$\{\delta^{(r)}\} = [\Phi]\{q\} \quad (6.3)$$

Where $[\phi]$ represents the square matrix for the eigen vectors of the linear system and $\{q\}$ is a time-related vector.

Substituting equation (6.3) into system (6.2), it becomes:

$$[M^{(r)}][\Phi]\{\ddot{q}\} + [K_L^{(r)}][\Phi]\{q\} + [K_{NL}^{(r)}][\Phi^3]\{q^3\} = \{0\} \quad (6.4)$$

Multiplying equation (6.4) by $[\phi^T]$, we obtain:

$$[\Phi^T][M^{(r)}][\Phi]\{\ddot{q}\} + [\Phi^T][K_L^{(r)}][\Phi]\{q\} + [\Phi^T][K_{NL}^{(r)}][\Phi^3]^{\dagger}\{q^3\}^{\dagger} = \{0\} \quad (6.5)$$

The products of matrix $[\phi^T][M^{(r)}][\phi]$ and $[\phi^T][K_L^{(r)}][\phi]$ represent diagonal matrices, written as $[M^{(D)}]$ and $[K_L^{(D)}]$, respectively.

Finally, the (6.1) system of equations is written:

$$[M^{(D)}]\{\ddot{q}\} + [K_L^{(D)}]\{q\} + [\Phi^T][K_{NL}^{(r)}][\Phi^3]\{q^3\} = \{0\} \quad (6.6)$$

---

† In this development the cancelled products are left out.

6.3 Solution of uncoupled equations

We saw in the preceding paragraph how matrices contained in the linear part of the system (6.1) could be reduced to diagonal matrices. On the other hand, the matrix product $[\phi^T][K_{NL}^{(r)}][\phi^3]$ is not generally described as a diagonal matrix.

A typical equation of the (6.6) system would yield:

$$m_{pp}\, q_p + k_{pp}^{(L)}\, q_p + \quad k_{ps}^{(NL)}\, q_s^3 = 0 \qquad (6.7)$$

where coefficients m_{pp} and $k_{pp}^{(L)}$, represent the p^{th} diagonal terms of matrices $[M^{(D)}]$ and $[K_L^{(D)}]$, respectively, and $k_{ps}^{(NL)}$ is the (p,s) term of the product $[\phi^T][K_{NL}^{(r)}][\phi^3]$.

We have NREDUC simultaneous equations of the form of (6.7). Solution of the equations was extremely difficult. At the first approximation, we were limited to solving these equations by ignoring the coupling between different modes, with the product $[\phi^T][K_{NL}^{(r)}][\phi^3]$ thereby becoming diagonal.

Equation (6.7) would then be written.

$$m_{pp}\, q_p + k_{pp}^{(L)}\, q_p + k_{pp}^{(NL)}\, q_p^3 = 0 \qquad (6.8)$$

Setting

$$q_p(\tau) = A_p\, \delta_p(\tau) \qquad (6.9)$$

which satisfies the conditions:

$$f_p(0)=1 \text{ and } f_p(0)=0$$

Equation (6.8) becomes, after the A_p simplification:

$$m_{pp}\, \delta_p + k_{pp}^{(L)}\, \delta_p + k_{pp}^{(NL)}\, A_p^2\, \delta_p = 0 \qquad (6.11)$$

which is equivalent to:

$$m_{pp}\, \delta_p + k_{pp}^{(L)}\, \delta_p + k_{pp}^{(NL)}\, t^2\, (A_p/t)^2\, \delta_p = 0 \qquad (6.12)$$

where t represents shell thickness.

Dividing this last equation by m_{pp}, it becomes:

$$\delta_p + \frac{k_{pp}^{(L)}}{m_{pp}}\, \delta_p + \frac{k_{pp}^{(NL)}}{m_{pp}}\, t^2\, (A_p/t)^2\, \delta_p^3 = 0 \qquad (6.13)$$

The coefficient $k_{pp}^{(L)}/m_{pp}$ represents the p^{th} linear vibration frequency of the shell. We then obtain:

$$\delta_p + \omega_p^2\, \delta_p + \Lambda_p\, (A_p/t)^2\, \delta_p^3 = 0 \qquad (6.14)$$

where

$$\omega_p^2 = \frac{k_{pp}^{(L)}}{m_{pp}} \quad (6.15)$$

and

$$\Lambda_p = \frac{k_{pp}^{(NL)}}{m_{pp}} t^2 \quad (6.16)$$

The solution $f_p(\tau)$ of this non-linear differential equation which satisfies the conditions in (6.1) is the JACOBI elliptic function $cn(w^* t, k^*)$, given by:

$$cn(\omega_p^* t, k_p^*) = \cos(am\ u) = \cos(\psi) \quad (6.17)$$

where

$$u = \frac{d\theta}{\sqrt{1 - k_p^{*2} \sin^2 \theta}} \quad (6.18)$$

(ϕ am u is called the amplitude of u)

Q = am u:

In this case:

$$\omega_p^{*2} = [\ \omega_p^2 + \Lambda_p\ (\ A_p/t\ )^2\ ]^{1/2} \quad (6.19)$$

$$k_p^{*2} = \frac{\Lambda_p\ (\ A_p/t\ )^2}{2\ [\ \omega_p^2 + \Lambda_p\ (\ A_p/t\ )^2\ ]} \quad (6.20)$$

The ratio of the non-linear to linear period is then determined by:

$$T_{NL} / T_L = 2K / (\pi [1 + (A_p/\omega_p^2)(A_p/t)^2]^{1/2}) \tag{6.21}$$

where $K=K(k_p^*)$ represents the complete integral elliptic of the first kind given by the infinite sum:

$$K(k_p^*) = \frac{1}{2}\pi [1 + (\frac{1}{2})^2 k_p^{*2} + (\frac{1.3}{2.4})^2 k_p^{*4} + ... + (\frac{(2n)!}{2^{2n}(n!)^2})^2 k_p^{*2n} + ...] \tag{6.22}$$

The equation in (6.21) represents the influence of the geometric non-linearity of the walls on the natural frequencies of an empty shell when the equations are uncoupled. The ratio T_{NL}/T_L is expressed in conjunction with non-dimensional ratio (A_p/t) where A_p is the vibration amplitude.

CHAPTER VII

THE ALGORITHM

The irregular cylindrical shell was subdivided into sufficient numbers of finite elements. Calculations for each finite element were performed in two steps: first, the linearity and second, the non-linearity of the strain-displacement relationships.

The computer program is written in FORTRAN IV and performed on a CDC (model CYBER 173). The calculation of algorithm is as follows:

a) The input consists of:

i) the number of finite elements

ii) the radius thickness and length of each element

iii) the mechanical properties of each distinct section of the shell

iv) the harmonic number n

b) The program proceeds as follows for each finite element:

b-1) for the linear component

i) the roots (p=1,..,8) of the characteristic equation (4.5) are determined by the LAGUERRE method with the help of the ZPOLR subroutine from IMSL. The α_p and β_p terms are obtained from equations (4.8)

ii) calculate the intermediate matrices [A], [R'] and [BB'] given by equation (4.18)

iii) the mass [m] and linear stiffness $[k_L]$ matrices are then determined from equations (4.17)

b-2 for the non-linear component

i) calculate coefficients $a_{rs}^{(1)}$, $a_{rs}^{(2)}$, $b_{rs}^{(1)}$, $c_{rs}^{(1)}$ and $c_{rs}^{(2)}$ (r,s=1,..,8), given by equations (5.13)

ii) calculate the terms of the intermediate matrix $[k_{NL}^*]$ defined by equation (5.17)

iii) the non-linear stiffness matrix $[k_{NL}]$ is then obtained with the help of equation (5.16)

c) Assemble the mass and stiffness matrices for the total shell following the procedure described in paragraph 6.1.

d) Application of matrix conditions: [M], $[K_L]$ and $[K_{NL}]$ are now reduced to square matrices of order NDF *(N + 1) - J, where J is the number of applied constraint equations. Only the geometric boundary conditions have been specified. Thus, for a shell with free ends, J=0; for a simply supported shell (with V=W=0) J=4, and for a shell clamped at both ends, J=8.

e) The natural linear frequencies ω_p and the corresponding modes (eigen vectors) of the matrices $[M^{(r)}]^{-1}[K_L^{(r)}]$ are obtained, where p=1, 11, NREDUC. $[M^{(t)}]$ and $[K_L^{(r)}]$ are real symmetric matrices. The calculation is done with the help of IMSL's EIGZF subroutine. The corresponding frequencies and modes are real.

f) Diagonalize the matrices $[M^{(r)}]$ and $[K_1{}^{(r)}]$ according to equation (6.5)

g) Work out the product $[\phi^T]\,[K_{NL}{}^{(r)}]\,[\phi^3]$ of system (6.6), then multiply the result by t^2, where t is the shell thickness.

CHAPTER VIII

CALCULATIONS AND DISCUSSION

This chapter presents the numerical results obtained with the method used. The influence of the wall's geometric non-linearity on the cylindrical shell's free vibrations is expressed by equations (6.19) and (6.21). For a cylindrical shell having the particular physical characteristics given, equations (6.19) and (6.21) have been graphically represented in Figures 7 to 10 with respect to the non-dimensional ratio, A_p/t. The straight horizontal line separating the two types of curvature represents the linear vibration cases, where the frequency is independent of the motion's amplitude. Two types of boundary conditions were studied. The circumferential mode was kept constant, at n=4.

8.1 Non-linear free vibration of an empty cylindrical shell

The first example of calculations to determine the influence of non-linearities in strain-displacement relations on the free vibrations of a cylindrical shell was the analyses in references [13] to [20]. The shell had the following properties:

$E=2.96 \times 10^7$ lb/in^2, $\nu=0.3$, R=1 in, t=0.01 in, $L=\pi/2$ in et $\rho=7.33 \times 10^{-4}$ lb.s^2/in^4.

The boundary conditons were for a shell simply supported at both ends, such that U=V=W=0.

The variation in natural frequencies of this shell was calculated using the method, we propose and compared to the results NOWINSKI [13] and RAJU and RAO [20] obtained for the case of m=1 (Figure 7).

NOWINSKI [13] based his analytical development upon DONNELL's simplified non-linear method. Only lateral displacement was considered. For their part, RAJU and RAO [20], beginning with an energy formulation, used the finite element method.

The shell was subdivided into four equal finite elements and our findings matched results obtained by others, in particular RAJU and RAO [20].

In the case where n=4 and m=1 (Figure 7), we observed that the variation ratio between the linear and non-linear periods decreased as ratio A/t increased. The frequency ratios demonstrated inverse behaviour. A non-linear trend of the strengthening type resulted from the Λ/ω ratio being positive. These variations are minimal for Λ/ω ratio values A/t below 1.0. For values above 1.0, the identified variation was more pronounced than what NOWINSKI [13] and RAJU and RAO [20] obtained.

We are able to ascertain that these differences might be due to Nowinski's [13] neglecting plane inertia. Furthermore, the authors noted a radial displaçement that was not cancelled out at the ends of the shell. As for RAJU and RAO [20], who used SANDERS-KOITER's [10,11] non-linear theory, they expressed the displacements of components along the shell generator in polynomial form.

The present method also accounts for the high frequency characteristics found for a given value of circumferential mode n. Typical curvatures are shown in Figures 8, 9 and 10. Here too, non-linearity had a strengthening effect.

Figure 8 shows the variations in the period and frequency ratios as a function of A/t for m=2 and 3 on the one hand, and for m=4 and 5 on the other, with the more accurate form being closer to the second. The same phenomena can be observed in Figure 9 for m=6 and 9, and for m=7 and 8. However in this Figure, the gaps between each pair of curves are approximately the same.

Finally, for high frequencies, the variation is small in the case of m=11 and m=12 and more pronounced for m=10. The variations in ratios T_{NL}/T_L and ω^*/ω corresponding to the last two modes, m=13 and m=14, are left out. With reference to T_{NL}/T_L, these variations are less than 1.5% and 0.004% for m=13 and m=14, respectively.

One of the great advantages of the finite element method is howm easily it can be applied to whatever the boundary conditions are. Thus, the second calculation example, the one in the RAJU and RAO [20] analysis dealt with a cylindrical shell with circumferential constraints at both ends. We then had V=0 as the boundary condition. The shell had the same physical properties as the preceding one e.g.:

$$E=2.96 \times 10^7 \text{ lb/in}^2, \ \nu=0.3, \ R=1 \text{ in}, \ t=0.01 \text{ in}, \ L=\pi/2 \text{ in et } \rho=7.33 \times 10^{-4} \text{ lb.s}^2/\text{in}^4$$

The shell was divided into four equal finite elements and the results obtained by the present methods are the same as in reference [20] and are shown in Figure 7. As in the first calculation example, the same differences were observed between the two methods. Again, the trends in non-linearities are of the strengthening type, the ω^*/ω ratio increasing as A/t increases.

More and more we are finding that ratio T_{NL}/T_L decreases more rapidly when V=0 at both ends. So, for A/t=3.0, for example, the present method showed that T_{NL}/T_L goes from 0.64 to 0.56, whereas with RAJU and RAO's method [20], the decrease was from 0.84 to 0.76.

This is probably due to the greater flexibility of the shell, there being a constraint here, only upon circumferential displacement. On the other hand, for both types of boundary conditions considered, the gap between the ω^*/ω vs A/t curves is greater than between T_{NL}/T_L vs A/t.

As in the first example as well, the characteristics of high frequencies were obtained. A few typical curves are shown in Figures 11, 12 and 13. The non-linearity trends are again of the strengthening type.

Thus, in Figure 11, the period and frequency variations in conjunction with the ratio corresponding to modes m=2, 3 and 4 were plotted. The Λ/ω ratio for m=5 scarcely differs from ratio Λ/ω for m=4 (less than 0.02%), and is the reason why the curves for m=5 were not drawn. It should be noted that gaps between the T_{NL}/T_L and ω^*/ω ratios are almost identical when going from m=2 to m=3, and from m=3 to m=4. This remark is equally valid for Figures 12 and 13, which correspond to modes 6, 7, 8, 9 and 10, 11, 13, 15, respectively. However, in these last two cases, the curves are comparatively much closer to each other than in the preceding example, except for m=15 (its behaviour is closer to the linear case).

The variations in modes m=12, 14, 16, 17 and 18 are negligible (≤3%). It was noted that mode m=18 is ;of the weatening type, ratio Λ/ω being negative. In this case the maximum deviation in the linear behaviour is in the order of 0.005%.

On the whole, by comparing the high frequency curves for both types of boundary condition studied, it can be concluded that these curves are closer to each other where V=0 and more spread out where U=V=W=0.

Finally, two points that are common to the two cases of boundary conditions should be emphasized:

a) For m=1 and for all other modes, the variation in the ratio of periods T_{NL}/T_L seems to possess an asymptotic limit when ratio A/t rises above 2.0.

b) The influence of the geometric non-linearity ;of the walls is left out in the last frequencies (m≥15 for V=0).

8.2 Coupling of modes

Although we were limited to solving the equation of motion in the approximation cases, the coupling between different modes was ignored. The fact nevertheless remains that the present theory constitutes a general approach to the dynamic study of non-linear cylindrical shells.

The dynamic behaviour of the shell, however, is ;not adequately described by equation (6.8). When we keep the non-linearities in minds, therefore, the coupling between different modes can no longer be left out. It then becomes necessary to develop a method for solving the system of uncoupled equations (6.7).

CHAPTER IX

CONCLUSION

The methods discussed in this paper demonstrated the influence of geometric non-linearities of the walls on the free vibrations of empty cylindrical shells. It was a hybrid method, based on a combination of thin shell theory and the finite element method.

A cylindrical finite element was used, so that the displacement functions could be derived directly from classical thin shell theory.

Only cases with $n \geq 2$ were dealt with in this report. The solution was divided into two parts. In part one, the displacement functions were obtained from linear shell theory [22,23] and the mass and linear stiffness matrices were determined by the finite element procedure. In part two, the modal coefficients corresponding to non-linearities in strain-displacement relations were obtained for the displacement functions by the method developed in reference [19]. The non-linear stiffness matrix was then calculated using the finite element method.

With the help of a computer program, variations in the free vibration frequencies and periods were determined in conjunction with motion amplitude for a cylindrical shell. Deviations in terms of linear vibrations were observed. The results obtained with this numerical method for the two types of boundary conditions were in agreement with other analytical and numerical methods.

The methods developed in the present research may be applied to the study of forced vibrations of a cylindrical shell under dynamic bads. This theory may also be applicable to problems of normal cones with circular sections.

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APPENDIX A

APPENDIX A-1

SANDERS-KOITER NON-LINEAR THIN SHELL THEORY

a) General Equations of Equilibrium

The fire differential equations of motion form Sanders-Koiter non-linear theory [10,11] for thin shells are (cf. Figure 1):

$$\frac{\partial(A_2N_{11})}{\partial\xi_1} + \frac{\partial(A_1\bar{N}_{12})}{\partial\xi_2} + \bar{N}_{12}\frac{\partial A_1}{\partial\xi_2} - N_{22}\frac{\partial A_2}{\partial\xi_1} + \frac{A_1A_2}{R_1}Q_1 + \frac{A_1}{2}\frac{\partial}{\partial\xi_2}\left[(\frac{1}{R_1} - \frac{1}{R_2})\bar{M}_{12}\right] -$$

$$\frac{A_1A_2}{R_1}(\phi_1N_{11} + \phi_2\bar{N}_{12}) - \frac{A_1}{2}\frac{\partial}{\partial\xi_2}\left[\phi(N_{11} + N_{22})\right] + A_1A_2\,p_1 = 0 \qquad (a)$$

$$\frac{\partial(A_1N_{22})}{\partial\xi_2} + \frac{\partial(A_2\bar{N}_{12})}{\partial\xi_1} + \bar{N}_{12}\frac{\partial A_2}{\partial\xi_1} - N_{11}\frac{\partial A_1}{\partial\xi_2} + \frac{A_1A_2}{R_2}Q_2 + \frac{A_2}{2}\frac{\partial}{\partial\xi_1}\left[(\frac{1}{R_2} - \frac{1}{R_1})\bar{M}_{12}\right] -$$

$$\frac{A_1A_2}{R_2}(\phi_1\bar{N}_{12} + \phi_2N_{22}) + \frac{A_2}{2}\frac{\partial}{\partial\xi_1}\left[\phi(N_{11} + N_{22})\right] + A_1A_2\,p_2 = 0 \qquad (b)$$

(A-1.1)

$$\frac{\partial(A_2Q_1)}{\partial\xi_1} + \frac{\partial(A_1Q_2)}{\partial\xi_2} - A_1A_2(\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2}) - \frac{\partial}{\partial\xi_1}\left[A_2(\phi_1N_{11} + \phi_2\bar{N}_{12})\right] -$$

$$\frac{\partial}{\partial\xi_2}\left[A_1(\phi_1\bar{N}_{12} + \phi_2N_{22})\right] + A_1A_2\,p_n = 0 \qquad (c)$$

$$\frac{\partial(A_2M_{11})}{\partial\xi_1} + \frac{\partial(A_1\bar{M}_{12})}{\partial\xi_2} + \bar{M}_{12}\frac{\partial A_1}{\partial\xi_2} - M_{22}\frac{\partial A_2}{\partial\xi_1} - A_1A_2\,Q_1 = 0 \qquad \text{(d)}$$

$$\frac{\partial(A_1M_{22})}{\partial\xi_2} + \frac{\partial(A_2\bar{M}_{12})}{\partial\xi_1} + \bar{M}_{12}\frac{\partial A_2}{\partial\xi_1} - M_{11}\frac{\partial A_1}{\partial\xi_2} - A_1A_2\,Q_2 = 0 \qquad \text{(e)}$$

with $\bar{N}_{12} = \frac{1}{2}(N_{12} + N_{21})$

$\bar{M}_{12} = \frac{1}{2}(M_{12} + M_{21})$

b) Deformation vector

Side by side with the equilibrium equations, there is a second group of equations determining the state of constraint in the shell, the law of elasticity. To that purpose, we shall be using the deformation vector $\{\varepsilon\}$, given by:

$$\varepsilon_{11} = \frac{1}{A_1}\frac{\partial u_1}{\partial\xi_1} + \frac{u_2}{A_1A_2}\frac{\partial A_1}{\partial\xi_2} + \frac{w}{R_1} + \frac{1}{2}\phi_1^2 + \frac{1}{2}\phi^2$$

$$\varepsilon_{22} = \frac{1}{A_2}\frac{\partial u_2}{\partial\xi_2} + \frac{u_1}{A_1A_2}\frac{\partial A_2}{\partial\xi_1} + \frac{w}{R_2} + \frac{1}{2}\phi_2^2 + \frac{1}{2}\phi^2$$

$$\varepsilon_{12} = \frac{1}{2}\left[\frac{1}{A_1}\frac{\partial u_2}{\partial\xi_1} + \frac{1}{A_2}\frac{\partial u_1}{\partial\xi_2} - \frac{u_1}{A_1A_2}\frac{\partial A_1}{\partial\xi_2} - \frac{u_2}{A_1A_2}\frac{\partial A_2}{\partial\xi_1} + \phi_1\phi_2\right] \qquad \text{(A-1.2)}$$

$$\kappa_{11} = \frac{1}{A_1A_2}\left[A_2 \frac{\partial\phi_1}{\partial\xi_1} + \phi_2 \frac{\partial A_1}{\partial\xi_2}\right]$$

$$\kappa_{22} = \frac{1}{A_1A_2}\left[A_1 \frac{\partial\phi_2}{\partial\xi_2} + \phi_1 \frac{\partial A_2}{\partial\xi_1}\right]$$

$$\kappa_{12} = \frac{1}{2}\left[\frac{1}{A_1}\frac{\partial\phi_2}{\partial\xi_1} + \frac{1}{A_2}\frac{\partial\phi_1}{\partial\xi_2} - \frac{1}{A_1A_2}\left(\phi_2\frac{\partial A_2}{\partial\xi_1} + \phi_1\frac{\partial A_1}{\partial\xi_2}\right) + \left(\frac{1}{R_2} - \frac{1}{R_1}\right)\phi\right]$$

with $\phi_1 = -\frac{1}{A_1}\frac{\partial w}{\partial\xi_1} + \frac{u_1}{R_1}$

$$\phi_2 = -\frac{1}{A_2}\frac{\partial w}{\partial\xi_2} + \frac{u_2}{R_2}$$

$$\phi = \frac{1}{2A_1A_2}\left[\frac{\partial(A_2u_2)}{\partial\xi_1} - \frac{\partial(A_1u_1)}{\partial\xi_2}\right]$$

c) Boundary conditions

The boundary conditions are given by:

$$N_{11} = \bar{\bar{N}}_{11} \qquad \text{or} \quad u_1 = \bar{\bar{u}}_1$$

$$\bar{N}_{12} + \frac{1}{2}\left(\frac{3}{R_2} - \frac{1}{R_1}\right)\bar{M}_{12} + \frac{1}{2}(N_{11} + N_{22})\,\phi = \bar{\bar{T}}_{12} \qquad \text{or} \quad u_2 = \bar{\bar{u}}_2$$

$$Q_1 + \frac{1}{A_2}\frac{\partial\bar{M}_{12}}{\partial\xi_2} - \phi_1N_{11} - \phi_2\bar{N}_{12} = \bar{\bar{V}}_{11} \qquad \text{or} \quad w = \bar{\bar{w}} \qquad \text{(A-1.3)}$$

$$M_{11} = \overline{\overline{M}}_{11} \qquad \text{or} \quad \phi_1 = \overline{\overline{\phi}}_1$$

for a boundary with constant ξ_1, where the double-barred quantities correspond to boundary values.

For boundary with constant ξ_2, we only have to interchange subscripts 1 and 2 and change the sign of the term to ϕ.

Terms ϕ_1, ϕ_2 and ϕ are defined in the previous paragraph.

d) Parameter for a cylindrical shell of revolution (Figs. 2 and 3)

We have:

$$\begin{array}{llll} \xi_1 = x & A_1 = 1 & R_1 = \infty & u_1 = U \\ \xi_2 = \theta & A_2 = R & R_2 = R & u_2 = V \\ & & & w = W \end{array} \qquad (A-1.4)$$

Substituting these parameters into the five equilibrium equations (A-1.1), we obtain:

$$\frac{\partial N_{xx}}{\partial x} + \frac{1}{R}\frac{\partial \overline{N}_{x\theta}}{\partial \theta} - \frac{1}{2R^2}\frac{\partial \overline{M}_{x\theta}}{\partial \theta} - \frac{1}{2R}\frac{\partial}{\partial \theta}\left[\phi(N_{xx} + N_{\theta\theta})\right] + p_x = 0 \qquad (a)$$

$$\frac{1}{R}\frac{\partial N_{\theta\theta}}{\partial\theta} + \frac{\partial \bar{N}_{x\theta}}{\partial x} + \frac{1}{2R}\frac{\partial \bar{M}_{x\theta}}{\partial x} + \frac{1}{R}Q_\theta - \frac{1}{R}(\phi_x\bar{N}_{x\theta} + \phi_\theta N_{\theta\theta}) + \frac{1}{2}\frac{\partial}{\partial x}\left[\phi(N_{xx} + N_{\theta\theta})\right] +$$

$$p_\theta = 0 \qquad \text{(b)}$$

(A-1.5)

$$\frac{\partial Q_x}{\partial x} + \frac{1}{R}\frac{\partial Q_\theta}{\partial\theta} - \frac{1}{R}N_{\theta\theta} - \frac{\partial}{\partial x}\left[\phi_x N_{xx} + \phi_\theta \bar{N}_{x\theta}\right] - \frac{1}{R}\frac{\partial}{\partial\theta}\left[\phi_x\bar{N}_{x\theta} + \phi_\theta N_{\theta\theta}\right] +$$

$$p_n = 0 \qquad \text{(c)}$$

$$\frac{\partial M_{xx}}{\partial x} + \frac{1}{R}\frac{\partial \bar{M}_{x\theta}}{\partial\theta} - Q_x = 0 \qquad \text{(d)}$$

$$\frac{1}{R}\frac{\partial M_{\theta\theta}}{\partial\theta} + \frac{\partial \bar{M}_{x\theta}}{\partial x} - Q_\theta = 0 \qquad \text{(e)}$$

APPENDIX A-2

EQUATIONS OF MOTION

This appendix contains the equations of motion for a thin cylindrical anisotropic shell; which were referenced in the various chapters of this report. The contents are divided into two parts: part one deals with the linear system operators and part two, with the non-linear.

a) Equations of motion for a cylindrical shell: linear system

$$L_1(U,V,W,P_{ij}) = p_{11}\frac{\partial^2 U}{\partial x^2} + \frac{p_{12}}{R}\left(\frac{\partial^2 V}{\partial x\partial\theta} + \frac{\partial W}{\partial x}\right) - p_{14}\frac{\partial^3 W}{\partial x^3} + \frac{p_{15}}{R^2}\left(-\frac{\partial^3 W}{\partial x\partial\theta^2} + \frac{\partial^2 V}{\partial x\partial\theta}\right) +$$

$$\left(\frac{p_{33}}{R} - \frac{p_{63}}{2R^2}\right)\left(\frac{\partial^2 V}{\partial x\partial\theta} + \frac{1}{R}\frac{\partial^2 U}{\partial\theta^2}\right) + \left(\frac{p_{36}}{R^2} - \frac{p_{66}}{2R^3}\right)\left(-\frac{2\partial^3 W}{\partial x\partial\theta^2} + \frac{3}{2}\frac{\partial^2 V}{\partial x\partial\theta} - \frac{1}{2R}\frac{\partial^2 U}{\partial\theta^2}\right)$$

$$L_2(U,V,W,P_{ij}) = \left(\frac{p_{21}}{R} + \frac{p_{51}}{R^2}\right)\frac{\partial^2 U}{\partial x\partial\theta} + \frac{1}{R}\left(\frac{p_{22}}{R} + \frac{p_{52}}{R^2}\right)\left(\frac{\partial^2 V}{\partial\theta^2} + \frac{\partial W}{\partial\theta}\right) - \left(\frac{p_{24}}{R} + \frac{p_{54}}{R^2}\right)$$

$$\left(\frac{\partial^3 W}{\partial x^2\partial\theta}\right) + \frac{1}{R^2}\left(\frac{p_{25}}{R} + \frac{p_{55}}{R^2}\right)\left(-\frac{\partial^3 W}{\partial\theta^3} + \frac{\partial^2 V}{\partial\theta^2}\right) + \left(p_{33} + \frac{3p_{63}}{2R}\right)\left(\frac{\partial^2 V}{\partial x^2} + \frac{1}{R}\frac{\partial^2 U}{\partial x\partial\theta}\right) +$$

$$\frac{1}{R}\left(p_{36} + \frac{3p_{66}}{2R}\right)\left(-2\frac{\partial^3 W}{\partial x^2\partial\theta} + \frac{3}{2}\frac{\partial^2 V}{\partial x^2} - \frac{1}{2R}\frac{\partial^2 U}{\partial x\partial\theta}\right)$$

$$L_3(U,V,W,P_{ij}) = p_{41}\ \frac{\partial^3 U}{\partial x^3} + \frac{p_{42}}{R}\ \left(\frac{\partial^3 V}{\partial x^2\partial\theta} + \frac{\partial^2 W}{\partial x^2}\right) - p_{44}\ \frac{\partial^4 W}{\partial x^4} + \frac{p_{45}}{R^2}$$

$$\left(-\ \frac{\partial^4 W}{\partial x^2\partial\theta^2} + \frac{\partial^3 V}{\partial x^2\partial\theta}\right) + \frac{2p_{63}}{R}.\quad \frac{2p_{63}}{R}\ .\ \frac{\partial^3 V}{\partial x^2\partial\theta} + \frac{1}{R}\ \frac{\partial^3 U}{\partial x\partial\theta^2} + \frac{2p_{66}}{R^2}\ \left(-\ \frac{2\partial^4 W}{\partial x^2\partial\theta^2} +\right.$$

$$\left.\frac{3}{2}\ \frac{\partial^3 V}{\partial x^2\partial\theta} - \frac{1}{2R}\ \frac{\partial^3}{\partial x\partial\theta^2}\right) + \frac{p_{51}}{R^2}\ \frac{\partial^3 U}{\partial x\partial\theta^2} + \frac{p_{52}}{R^3}\ \left(\frac{\partial^3 V}{\partial\theta^3} + \frac{\partial^2 W}{\partial\theta^2}\right) + \frac{p_{55}}{R^4}\ \left(-\ \frac{\partial^4 W}{\partial\theta^4} + \frac{\partial^3 V}{\partial\theta^3}\right.$$

$$-\ \frac{p_{54}}{R^2}\ \frac{\partial^4 W}{\partial x^2\partial\theta^2}$$

b) <u>Equations of motion for a cylindrical shell: non-linear system</u>

$$N_1(U,V,W,P_{ij}) = p_{11}\ \frac{\partial W}{\partial x}\ \frac{\partial^2 W}{\partial x^2} + \frac{p_{12}}{R^2}\ \left(\frac{\partial W}{\partial\theta}\ \frac{\partial^2 W}{\partial x\partial\theta} - V\ \frac{\partial^2 W}{\partial x\partial\theta} - \frac{\partial W}{\partial\theta}\ \frac{\partial V}{\partial x} + V\ \frac{\partial V}{\partial x}\right) +$$

$$\frac{1}{R}\ \left(\frac{p_{33}}{R} - \frac{p_{63}}{2R^2}\right)\ \cdot\ \left(\frac{\partial W}{\partial x}\ \frac{\partial^2 W}{\partial\theta^2} + \frac{\partial W}{\partial\theta}\ \frac{\partial^2 W}{\partial x\partial\theta} - V\ \frac{\partial^2 W}{\partial x\partial\theta} - \frac{\partial W}{\partial x}\ \frac{\partial V}{\partial\theta}\right) + (p_{11} + p_{12})\ \cdot$$

$$\left[\frac{1}{4}\ \frac{\partial V}{\partial x}\ \cdot\ \frac{\partial^2 V}{\partial x^2} + \frac{1}{4R^2}\ \frac{\partial U}{\partial\theta}\ \cdot\ \frac{\partial^2 U}{\partial x\partial\theta} - \frac{1}{4R}\ \frac{\partial U}{\partial\theta}\ \cdot\ \frac{\partial^2 V}{\partial x^2} - \frac{1}{4R}\ \frac{\partial V}{\partial x}\ \cdot\ \frac{\partial^2 U}{\partial x\partial\theta}\right] -$$

$$\left(\frac{p_{11} + p_{21}}{4R}\right)\ \cdot\ \left[\frac{\partial V}{\partial x} - \frac{1}{R}\ \frac{\partial U}{\partial\theta}\right]\ \cdot\ \left[\frac{\partial^2 U}{\partial x\partial\theta} + \frac{\partial W}{\partial x}\ \cdot\ \frac{\partial^2 W}{\partial x\partial\theta}\right] - \left(\frac{p_{12} + p_{22}}{4R}\right)\ \cdot$$

$$\left[\frac{\partial V}{\partial x} - \frac{1}{R}\ \frac{\partial U}{\partial\theta}\right]\ \cdot\ \left[\frac{1}{R}\ \frac{\partial^2 V}{\partial\theta^2} + \frac{1}{R}\ \frac{\partial W}{\partial\theta} + \frac{1}{R^2}\ \frac{\partial W}{\partial\theta}\ \cdot\ \frac{\partial^2 W}{\partial\theta^2} - \frac{V}{R^2}\ \frac{\partial^2 W}{\partial\theta^2} - \frac{1}{R^2}\ \frac{\partial W}{\partial\theta}\ \cdot\ \frac{\partial V}{\partial\theta} +\right.$$

$$\frac{V}{R^2} \cdot \frac{\partial V}{\partial \theta}\Big] + \left(\frac{P_{14} + P_{24}}{4R}\right) \cdot \left[\frac{\partial V}{\partial x} - \frac{1}{R}\frac{\partial U}{\partial \theta}\right] \cdot \left[\frac{\partial^3 W}{\partial x^2 \partial \theta}\right] - \left(\frac{P_{15} + P_{25}}{4R}\right) \cdot$$

$$\left[\frac{\partial V}{\partial x} - \frac{1}{R}\frac{\partial U}{\partial \theta}\right] \cdot \left[- \frac{1}{R^2}\frac{\partial^3 W}{\partial \theta^3} + \frac{1}{R^2}\frac{\partial^2 V}{\partial \theta^2}\right] - \left(\frac{P_{11} + P_{21} + P_{12} + P_{22}}{4R}\right) \cdot$$

$$\left[\frac{\partial V}{\partial x} - \frac{1}{R}\frac{\partial U}{\partial \theta}\right] \cdot \Big[\frac{1}{4}\frac{\partial V}{\partial x} \cdot \frac{\partial^2 V}{\partial x \partial \theta} + \frac{1}{4R^2}\frac{\partial U}{\partial \theta} \cdot \frac{\partial^2 U}{\partial \theta^2} - \frac{1}{4R}\frac{\partial U}{\partial \theta} \cdot \frac{\partial^2 V}{\partial x \partial \theta} -$$

$$\frac{1}{4R}\frac{\partial V}{\partial x}\frac{\partial^2 U}{\partial \theta^2}\Big] - \left(\frac{P_{11} + P_{21}}{4R}\right) \cdot \left[\frac{\partial^2 V}{\partial x \partial \theta} - \frac{1}{R}\frac{\partial^2 U}{\partial \theta^2}\right] \cdot \left[\frac{\partial U}{\partial x} + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2\right] -$$

$$\left(\frac{P_{12} + P_{22}}{4R}\right) \cdot \left[\frac{\partial^2 V}{\partial x \partial \theta} - \frac{1}{R}\frac{\partial^2 U}{\partial \theta^2}\right] \cdot \Big[\frac{1}{R}\frac{\partial V}{\partial \theta} + \frac{W}{R} + \frac{1}{2R^2}\left(\frac{\partial W}{\partial \theta}\right)^2 - \frac{V}{R^2}\frac{\partial W}{\partial \theta} +$$

$$\frac{V^2}{2R^2}\Big] + \left(\frac{P_{14} + P_{24}}{4R}\right) \cdot \left[\frac{\partial^2 V}{\partial x \partial \theta} - \frac{1}{R}\frac{\partial^2 U}{\partial \theta^2}\right] \cdot \left[\frac{\partial^2 W}{\partial x^2}\right] - \left(\frac{P_{15} + P_{25}}{4R}\right) \cdot$$

$$\left[\frac{\partial^2 V}{\partial x \partial \theta} - \frac{1}{R}\frac{\partial^2 U}{\partial \theta^2}\right] \cdot \left[- \frac{1}{R^2}\frac{\partial^2 W}{\partial \theta^2} + \frac{1}{R^2}\frac{\partial V}{\partial \theta}\right] - \left(\frac{P_{11} + P_{21} + P_{12} + P_{22}}{4R}\right) \cdot$$

$$\left[\frac{\partial^2 V}{\partial x \partial \theta} - \frac{1}{R}\frac{\partial^2 U}{\partial \theta^2}\right] \cdot \left[\frac{1}{8}\left(\frac{\partial V}{\partial x}\right)^2 + \frac{1}{8R^2}\left(\frac{\partial U}{\partial \theta}\right)^2 - \frac{1}{4R}\frac{\partial V}{\partial x} \cdot \frac{\partial U}{\partial \theta}\right]$$

$$N_2(U,V,W,P_{ij}) = (\frac{p_{21}}{R} + \frac{p_{51}}{R^2}) \frac{\partial W}{\partial x} \frac{\partial^2 W}{\partial x \partial \theta} + \frac{1}{R^2} (\frac{p_{22}}{R} + \frac{p_{52}}{R^2}) (\frac{\partial W}{\partial \theta} \frac{\partial^2 W}{\partial \theta^2} - V \frac{\partial^2 W}{\partial \theta^2} -$$

$$\frac{\partial W}{\partial \theta} \frac{\partial V}{\partial \theta} + V \frac{\partial V}{\partial \theta}) + \frac{1}{R} (p_{33} + \frac{3p_{63}}{2R}) (\frac{\partial W}{\partial x} \frac{\partial^2 W}{\partial x \partial \theta} + \frac{\partial W}{\partial \theta} \frac{\partial^2 W}{\partial x^2} - V \frac{\partial^2 W}{\partial x^2} - \frac{\partial W}{\partial x} \frac{\partial V}{\partial x}) +$$

$$(\frac{p_{21} + p_{22}}{R} + \frac{p_{51} + p_{52}}{R^2}) \cdot \left[\frac{1}{4} \frac{\partial V}{\partial x} \cdot \frac{\partial^2 V}{\partial x \partial \theta} + \frac{1}{4R^2} \frac{\partial U}{\partial \theta} \cdot \frac{\partial^2 U}{\partial \theta^2} - \frac{1}{4R} \frac{\partial U}{\partial \theta} \cdot \frac{\partial^2 V}{\partial x \partial \theta} - \frac{1}{4R} \frac{\partial V}{\partial x} \cdot \right.$$

$$\left. \frac{\partial^2 U}{\partial \theta^2} \right] + \frac{p_{33}}{R} \cdot \left[\frac{\partial W}{\partial x}\right] \cdot \left[\frac{\partial V}{\partial x} + \frac{1}{R} \frac{\partial U}{\partial \theta} + \frac{1}{R} \frac{\partial W}{\partial x} \cdot \frac{\partial W}{\partial \theta} - \frac{V}{R} \cdot \frac{\partial W}{\partial x}\right] + \frac{p_{36}}{R} \cdot$$

$$\left[\frac{\partial W}{\partial x}\right] \cdot \left[- \frac{2}{R} \frac{\partial^2 W}{\partial x \partial \theta} + \frac{3}{2R} \frac{\partial V}{\partial x} - \frac{1}{2R^2} \frac{\partial U}{\partial \theta}\right] + \left[(\frac{p_{11} + p_{21}}{4}) \cdot (\frac{\partial^2 V}{\partial x^2} - \frac{1}{R} \frac{\partial^2 U}{\partial x \partial \theta}) - \right.$$

$$\left. \frac{p_{21}}{R} \cdot (- \frac{1}{R} \frac{\partial W}{\partial \theta} + \frac{V}{R})\right] \cdot \left[\frac{\partial U}{\partial x} + \frac{1}{2} (\frac{\partial W}{\partial x})^2\right] + \left[(\frac{p_{12} + p_{22}}{4}) \cdot (\frac{\partial^2 V}{\partial x^2} - \frac{1}{R} \frac{\partial^2 U}{\partial x \partial \theta}) - \right.$$

$$\left. \frac{p_{22}}{R} \cdot (- \frac{1}{R} \frac{\partial W}{\partial \theta} + \frac{V}{R})\right] \cdot \left[\frac{1}{R} \cdot \frac{\partial V}{\partial \theta} + \frac{W}{R} + \frac{1}{2R^2} (\frac{\partial W}{\partial \theta})^2 - \frac{V}{R^2} \frac{\partial W}{\partial \theta} + \frac{V^2}{2R^2}\right] +$$

$$\left[\frac{p_{24}}{R} \cdot (- \frac{1}{R} \frac{\partial W}{\partial \theta} + \frac{V}{R}) - (\frac{p_{14} + p_{24}}{4}) \cdot (\frac{\partial^2 V}{\partial x^2} - \frac{1}{R} \frac{\partial^2 U}{\partial x \partial \theta})\right] \cdot \left[\frac{\partial^2 W}{\partial x^2}\right] +$$

$$\left[(\frac{p_{15} + p_{25}}{4}) \cdot (\frac{\partial^2 V}{\partial x^2} - \frac{1}{R} \frac{\partial^2 U}{\partial x \partial \theta}) - \frac{p_{25}}{R} (- \frac{1}{R} \frac{\partial W}{\partial \theta} + \frac{V}{R})\right] \cdot \left[\frac{-1}{R^2} \frac{\partial^2 W}{\partial \theta^2} + \right.$$

$$\left. \frac{1}{R^2} \frac{\partial V}{\partial \theta}\right] + \left[(\frac{p_{11} + p_{21} + p_{12} + p_{22}}{4}) \cdot (\frac{\partial^2 V}{\partial x^2} - \frac{1}{R} \frac{\partial^2 U}{\partial x \partial \theta}) - (\frac{p_{21} + p_{22}}{R}) \cdot \right.$$

$$\left(-\frac{1}{R}\frac{\partial W}{\partial \theta}+\frac{V}{R}\right)\right] \cdot \left[\frac{1}{8}\left(\frac{\partial V}{\partial x}\right)^2+\frac{1}{8R^2}\left(\frac{\partial U}{\partial \theta}\right)^2-\frac{1}{4R}\frac{\partial V}{\partial x}\cdot\frac{\partial U}{\partial \theta}\right]+\left(\frac{p_{11}+p_{21}}{4}\right)\cdot$$

$$\left[\frac{\partial V}{\partial x}-\frac{1}{R}\frac{\partial U}{\partial \theta}\right]\cdot\left[\frac{\partial^2 U}{\partial x^2}+\frac{\partial W}{\partial x}\cdot\frac{\partial^2 W}{\partial x^2}\right]+\left(\frac{p_{12}+p_{22}}{4}\right)\cdot\left[\frac{\partial V}{\partial x}-\frac{1}{R}\frac{\partial U}{\partial \theta}\right]\cdot$$

$$\left[\frac{1}{R}\frac{\partial^2 V}{\partial x\partial \theta}+\frac{1}{R}\frac{\partial W}{\partial x}+\frac{1}{R^2}\frac{\partial W}{\partial \theta}\cdot\frac{\partial^2 W}{\partial x\partial \theta}-\frac{V}{R^2}\frac{\partial^2 W}{\partial x\partial \theta}-\frac{1}{R^2}\cdot\frac{\partial W}{\partial \theta}\cdot\frac{\partial V}{\partial x}+\frac{V}{R^2}\cdot\frac{\partial V}{\partial x}\right]-$$

$$\left(\frac{p_{24}+p_{14}}{4}\right)\cdot\left[\frac{\partial V}{\partial x}-\frac{1}{R}\frac{\partial U}{\partial \theta}\right]\cdot\left[\frac{\partial^3 W}{\partial x^3}\right]+\left(\frac{p_{15}+p_{25}}{4}\right)\cdot\left[\frac{\partial V}{\partial x}-\frac{1}{R}\frac{\partial U}{\partial \theta}\right]\cdot$$

$$\left[-\frac{1}{R^2}\frac{\partial^3 W}{\partial x\partial \theta^2}+\frac{1}{R^2}\frac{\partial^2 V}{\partial x\partial \theta}\right]+\left(\frac{p_{11}+p_{21}+p_{12}+p_{22}}{4}\right)\cdot\left[\frac{\partial V}{\partial x}-\frac{1}{R}\frac{\partial U}{\partial \theta}\right]\cdot$$

$$\left[\frac{1}{4}\frac{\partial V}{\partial x}\cdot\frac{\partial^2 V}{\partial x^2}+\frac{1}{4R^2}\frac{\partial U}{\partial \theta}\cdot\frac{\partial^2 U}{\partial x\partial \theta}-\frac{1}{4R}\frac{\partial U}{\partial \theta}\cdot\frac{\partial^2 V}{\partial x^2}-\frac{1}{4R}\frac{\partial V}{\partial x}\cdot\frac{\partial^2 U}{\partial x\partial \theta}\right]$$

$$N_3(U,V,W,P_{ij}) = p_{41}\left[\left(\frac{\partial^2 W}{\partial x^2}\right)^2+\frac{\partial W}{\partial x}\frac{\partial^3 W}{\partial x^3}\right]+\frac{p_{42}}{R^2}\left[\left(\frac{\partial^2 W}{\partial x\partial \theta}\right)^2+\frac{\partial W}{\partial \theta}\frac{\partial^3 W}{\partial x^2\partial \theta}-\right.$$

$$\left.\frac{\partial V}{\partial x}\frac{\partial^2 W}{\partial x\partial \theta}-V\frac{\partial^3 W}{\partial x^2\partial \theta}-\frac{\partial^2 W}{\partial x\partial \theta}\frac{\partial V}{\partial x}-\frac{\partial W}{\partial \theta}\frac{\partial^2 V}{\partial x^2}+\left(\frac{\partial V}{\partial x}\right)^2+V\frac{\partial^2 V}{\partial x^2}\right]+\frac{2p_{63}}{R^2}$$

$$\left[\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial \theta^2}+\frac{\partial W}{\partial x}\frac{\partial^3 W}{\partial x\partial \theta^2}+\left(\frac{\partial^2 W}{\partial x\partial \theta}\right)^2+\frac{\partial W}{\partial \theta}\frac{\partial^3 W}{\partial x^2\partial \theta}-\frac{\partial V}{\partial x}\frac{\partial^2 W}{\partial x\partial \theta}-V\frac{\partial^3 W}{\partial x^2\partial \theta}-\right.$$

$$\frac{\partial^2 W}{\partial x^2}\frac{\partial V}{\partial \theta} - \frac{\partial W}{\partial x}\frac{\partial^2 V}{\partial x \partial \theta}\Bigg] + \frac{p_{51}}{R^2}\Bigg[\left(\frac{\partial^2 W}{\partial x \partial \theta}\right)^2 + \frac{\partial W}{\partial x}\frac{\partial^3 W}{\partial x \partial \theta^2}\Bigg] + (p_{41} + p_{42}) \cdot$$

$$\Bigg[\frac{1}{4}\left(\frac{\partial^2 V}{\partial x^2}\right)^2 + \frac{1}{4}\frac{\partial V}{\partial x} \cdot \frac{\partial^3 V}{\partial x^3} + \frac{1}{4R^2}\left(\frac{\partial^2 U}{\partial x \partial \theta}\right)^2 + \frac{1}{4R^2}\frac{\partial U}{\partial \theta} \cdot \frac{\partial^3 U}{\partial x^2 \partial \theta} - \frac{1}{2R}\frac{\partial^2 U}{\partial x \partial \theta} \cdot$$

$$\frac{\partial^2 V}{\partial x^2} - \frac{1}{4R} \cdot \frac{\partial U}{\partial \theta} \cdot \frac{\partial^3 V}{\partial x^3} - \frac{1}{4R}\frac{\partial V}{\partial x} \cdot \frac{\partial^3 U}{\partial x^2 \partial \theta}\Bigg] + \frac{p_{52}}{R^2}\Bigg[\frac{1}{R^2}\left(\frac{\partial^2 W}{\partial \theta^2}\right)^2 + \frac{1}{R^2} \cdot \frac{\partial W}{\partial \theta} \cdot$$

$$\frac{\partial^3 W}{\partial \theta^3} - \frac{1}{R^2}\frac{\partial V}{\partial \theta} \cdot \frac{\partial^2 W}{\partial \theta^2} - \frac{V}{R^2}\frac{\partial^3 W}{\partial \theta^3} - \frac{1}{R^2} \cdot \frac{\partial^2 W}{\partial \theta^2} \cdot \frac{\partial V}{\partial \theta} - \frac{1}{R^2}\frac{\partial W}{\partial \theta} \cdot \frac{\partial^2 V}{\partial \theta^2} + \frac{1}{R^2}\left(\frac{\partial V}{\partial \theta}\right)^2 +$$

$$\frac{V}{R^2}\frac{\partial^2 V}{\partial \theta^2}\Bigg] + \left(\frac{p_{51} + p_{52}}{R^2}\right) \cdot \Bigg[\frac{1}{4}\left(\frac{\partial^2 V}{\partial x \partial \theta}\right)^2 + \frac{1}{4}\frac{\partial V}{\partial x} \cdot \frac{\partial^3 V}{\partial x \partial \theta^2} + \frac{1}{4R^2}\left(\frac{\partial^2 U}{\partial \theta^2}\right)^2 +$$

$$\frac{1}{4R^2} \cdot \frac{\partial U}{\partial \theta} \cdot \frac{\partial^3 U}{\partial \theta^3} - \frac{1}{2R}\frac{\partial^2 U}{\partial \theta^2} \cdot \frac{\partial^2 V}{\partial x \partial \theta} - \frac{1}{4R}\frac{\partial U}{\partial \theta} \cdot \frac{\partial^3 V}{\partial x \partial \theta^2} - \frac{1}{4R}\frac{\partial V}{\partial x} \cdot \frac{\partial^3 U}{\partial \theta^3}\Bigg] +$$

$$p_{11}\left[\frac{\partial W}{\partial x}\right] \cdot \left[\frac{\partial^2 U}{\partial x^2} + \frac{\partial W}{\partial x} \cdot \frac{\partial^2 W}{\partial x^2}\right] + p_{12} \cdot \left[\frac{\partial W}{\partial x}\right] \cdot \Bigg[\frac{1}{R}\frac{\partial^2 V}{\partial x \partial \theta} + \frac{1}{R}\frac{\partial W}{\partial x} +$$

$$\frac{1}{R^2}\frac{\partial W}{\partial \theta} \cdot \frac{\partial^2 W}{\partial x \partial \theta} - \frac{V}{R^2} \cdot \frac{\partial^2 W}{\partial x \partial \theta} - \frac{1}{R^2}\frac{\partial W}{\partial \theta} \cdot \frac{\partial V}{\partial x} + \frac{V}{R^2} \cdot \frac{\partial V}{\partial x}\Bigg] - p_{14}\left[\frac{\partial W}{\partial x}\right] \cdot$$

$$\left[\frac{\partial^3 W}{\partial x^3}\right] + p_{15}\left[\frac{\partial W}{\partial x}\right] \cdot \left[-\frac{1}{R^2}\frac{\partial^3 W}{\partial x \partial \theta^2} + \frac{1}{R^2}\frac{\partial^2 V}{\partial x \partial \theta}\right] + (p_{11} + p_{12}) \cdot \left[\frac{\partial W}{\partial x}\right] \cdot$$

$$\left[\frac{1}{4}\frac{\partial V}{\partial x}\cdot\frac{\partial^2 V}{\partial x^2}+\frac{1}{4R^2}\frac{\partial U}{\partial \theta}\cdot\frac{\partial^2 U}{\partial x\partial \theta}-\frac{1}{4R}\frac{\partial U}{\partial \theta}\cdot\frac{\partial^2 V}{\partial x^2}-\frac{1}{4R}\frac{\partial V}{\partial x}\cdot\frac{\partial^2 U}{\partial x\partial \theta}\right]$$

$$\left[\frac{\partial U}{\partial x}+\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2\right]\cdot\left[-p_{11}\frac{\partial^2 W}{\partial x^2}+\frac{p_{21}}{R}\left(-\frac{1}{R}\frac{\partial^2 W}{\partial \theta^2}+\frac{1}{R}\frac{\partial V}{\partial \theta}+1\right)\right]-\left[\frac{1}{R}\frac{\partial V}{\partial \theta}+\right.$$

$$\left.\frac{W}{R}+\frac{1}{2R^2}\left(\frac{\partial W}{\partial \theta}\right)^2-\frac{V}{R^2}\frac{\partial W}{\partial \theta}+\frac{V^2}{2R^2}\right]\cdot\left[-p_{12}\frac{\partial^2 W}{\partial x^2}+\frac{p_{22}}{R}\left(-\frac{1}{R}\cdot\frac{\partial^2 W}{\partial \theta^2}+\frac{1}{R}\frac{\partial V}{\partial \theta}+\right.\right.$$

$$\left.\left.1\right)\right]-\left[\frac{1}{8}\left(\frac{\partial V}{\partial x}\right)^2+\frac{1}{8R^2}\left(\frac{\partial U}{\partial \theta}\right)^2-\frac{1}{4R}\frac{\partial V}{\partial x}\cdot\frac{\partial U}{\partial \theta}\right]\cdot\left[-(p_{11}+p_{12})\cdot\right.$$

$$\left.\left(\frac{\partial^2 W}{\partial x^2}\right)+\left(\frac{p_{21}+p_{22}}{R}\right)\cdot\left(-\frac{1}{R}\frac{\partial^2 W}{\partial \theta^2}+\frac{1}{R}\frac{\partial V}{\partial \theta}+1\right)\right]+\frac{\partial^2 W}{\partial x^2}\cdot\left[-p_{14}\frac{\partial^2 W}{\partial x^2}+\right.$$

$$\left.\frac{p_{24}}{R}\left(-\frac{1}{R}\frac{\partial^2 W}{\partial \theta^2}+\frac{1}{R}\frac{\partial V}{\partial \theta}\right)\right]-\left[-\frac{1}{R^2}\frac{\partial^2 W}{\partial \theta^2}+\frac{1}{R^2}\frac{\partial V}{\partial \theta}\right]\cdot\left[-p_{15}\frac{\partial^2 W}{\partial x^2}+\right.$$

$$\left.\frac{p_{25}}{R}\left(-\frac{1}{R}\frac{\partial^2 W}{\partial \theta^2}+\frac{1}{R}\frac{\partial V}{\partial \theta}\right)\right]-p_{33}\cdot\left[-\frac{1}{R}\frac{\partial W}{\partial \theta}+\frac{V}{R}\right]\cdot\left[\frac{\partial^2 V}{\partial x^2}+\frac{1}{R}\frac{\partial^2 U}{\partial x\partial \theta}+\right.$$

$$\left.\frac{1}{R}\frac{\partial W}{\partial x}\cdot\frac{\partial^2 W}{\partial x\partial \theta}+\frac{1}{R}\frac{\partial W}{\partial \theta}\cdot\frac{\partial^2 W}{\partial x^2}-\frac{V}{R}\cdot\frac{\partial^2 W}{\partial x^2}-\frac{1}{R}\frac{\partial W}{\partial x}\cdot\frac{\partial V}{\partial x}\right]-p_{36}\left[-\frac{1}{R}\frac{\partial W}{\partial \theta}+\right.$$

$$\left.\frac{V}{R}\right]\cdot\left[-\frac{2}{R}\frac{\partial^3 W}{\partial x^2\partial \theta}+\frac{3}{2R}\frac{\partial^2 V}{\partial x^2}-\frac{1}{2R^2}\frac{\partial^2 U}{\partial x\partial \theta}\right]-\left[-\frac{2}{R}\frac{\partial^2 W}{\partial x\partial \theta}+\frac{1}{R}\frac{\partial V}{\partial x}\right]\cdot$$

$$\left[p_{33}\left(\frac{\partial V}{\partial x}+\frac{1}{R}\frac{\partial U}{\partial \theta}+\frac{1}{R}\frac{\partial W}{\partial x}\cdot\frac{\partial W}{\partial \theta}-\frac{V}{R}\cdot\frac{\partial W}{\partial x}\right)+p_{36}\left(-\frac{2}{R}\frac{\partial^2 W}{\partial x\partial \theta}+\frac{3}{2R}\frac{\partial V}{\partial x}-\right.\right.$$

$$\frac{1}{2R^2}\frac{\partial U}{\partial \theta})\Bigg] + \frac{P_{33}}{R}\left[\frac{\partial W}{\partial x}\right] \cdot \left[\frac{\partial^2 V}{\partial x \partial \theta} + \frac{1}{R}\frac{\partial^2 U}{\partial \theta^2} + \frac{1}{R}\frac{\partial W}{\partial x} \cdot \frac{\partial^2 W}{\partial \theta^2} + \frac{1}{R}\frac{\partial W}{\partial \theta} \cdot \frac{\partial^2 W}{\partial x \partial \theta} - \right.$$

$$\left. \frac{V}{R} \cdot \frac{\partial^2 W}{\partial x \partial \theta} - \frac{1}{R}\frac{\partial W}{\partial x} \cdot \frac{\partial V}{\partial \theta}\right] + \frac{P_{36}}{R}\left[\frac{\partial W}{\partial x}\right] \cdot \left[-\frac{2}{R}\frac{\partial^3 W}{\partial x \partial \theta^2} + \frac{3}{2R}\frac{\partial^2 V}{\partial x \partial \theta} - \frac{1}{2R^2}\frac{\partial^2 U}{\partial \theta^2}\right] -$$

$$\frac{P_{21}}{R}\left[-\frac{1}{R}\frac{\partial W}{\partial \theta} + \frac{V}{R}\right] \cdot \left[\frac{\partial^2 U}{\partial x \partial \theta} + \frac{\partial W}{\partial x} \cdot \frac{\partial^2 W}{\partial x \partial \theta}\right] - \frac{P_{22}}{R} \cdot \left[-\frac{1}{R}\frac{\partial W}{\partial \theta} + \frac{V}{R}\right] \cdot$$

$$\left[\frac{1}{R}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{R}\frac{\partial W}{\partial \theta} + \frac{1}{R^2}\frac{\partial W}{\partial \theta} \cdot \frac{\partial^2 W}{\partial \theta^2} - \frac{V}{R^2} \cdot \frac{\partial^2 W}{\partial \theta^2} - \frac{1}{R^2}\frac{\partial W}{\partial \theta} \cdot \frac{\partial V}{\partial \theta} + \frac{V}{R^2}\frac{\partial V}{\partial \theta}\right] +$$

$$\frac{P_{24}}{R}\left[-\frac{1}{R}\frac{\partial W}{\partial \theta} + \frac{V}{R}\right] \cdot \left[\frac{\partial^3 W}{\partial x^2 \partial \theta}\right] - \frac{P_{25}}{R}\left[-\frac{1}{R}\frac{\partial W}{\partial \theta} + \frac{V}{R}\right] \cdot \left[-\frac{1}{R^2}\frac{\partial^3 W}{\partial \theta^3} + \right.$$

$$\left. \frac{1}{R^2}\frac{\partial^2 V}{\partial \theta^2}\right] - \left(\frac{P_{21} + P_{22}}{R}\right)\left[-\frac{1}{R}\frac{\partial W}{\partial \theta} + \frac{V}{R}\right] \cdot \left[\frac{1}{4}\frac{\partial V}{\partial x} \cdot \frac{\partial^2 V}{\partial x \partial \theta} + \frac{1}{4R^2}\frac{\partial U}{\partial \theta} \cdot \frac{\partial^2 U}{\partial \theta^2} - \right.$$

$$\left. \frac{1}{4R}\frac{\partial U}{\partial \theta} \cdot \frac{\partial^2 V}{\partial x \partial \theta} - \frac{1}{4R}\frac{\partial V}{\partial x}\frac{\partial^2 U}{\partial \theta^2}\right]$$

APPENDIX A-3

The matrices referenced in the course of our analytical developments are given in this appendix.

The matrices are classified as follows:

[H]	(Table 1)
[A]	(Table 2)
[T], [L], [X]	(Table 3)
[J]	(Table 4)
$[B^*]$, $[C^*]$	(Table 5)
$[B^{**}]$, $[C^{**}]$, $[D^{**}]$, $[E^{**}]$	(Table 6)

The eight roots of the characteristic equation (4.5) are represented by λ_p (p = 1,...8). The values for α_p and β_p are defined by ;equations (4.8).

Quantities ℓ and R are the length and radius, respectively, of each finite element.

TABLE 1

MATRIX $\underset{(3,3)}{[H]}$

$$\underset{(3,3)}{[H]} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \{0\} \qquad [H] = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix}$$

Here we shall only be presenting the coefficients appearing in equation (4.8).

$$H_{11} = n^2 h_1 - \lambda^2 p_{11}$$

$$H_{12} = -n\lambda h_3$$

$$H_{21} = H_{12}$$

$$H_{22} = -n^2 h_7 + \lambda^2 h_9$$

$$H_{13} = -\lambda(n^2 h_5 + p_{12}) + \lambda^3 p_{14}/R$$

$$H_{23} = -n(1 + n^2)\ p_{25}/R - np_{22} - n^3 p_{55}/R^2 + n\lambda^2 h_{11}$$

with $h_1 = p_{33} - p_{36}/R + p_{66}/4R^2$

$$h_3 = p_{12} + p_{33} + (p_{15} + p_{36})/R - 3p_{66}/4R^2$$

$$h_5 = (p_{15} + 2p_{36} - p_{66}/R)/R$$

$$h_7 = p_{22} + p_{55}/R^2 + 2p_{25}/R$$

$$h_9 = p_{33} + 3p_{36}/R + 9p_{66}/4R^2$$

$$h_{11} = (2p_{36} + p_{24} + 3p_{66}/R + p_{54}/R)/R$$

The characteristic equation (4.5) is:

$$h_8\lambda^8 - h_6\lambda^6 + h_4\lambda^4 - h_2\lambda^2 + h_0 = 0$$

where $h_8 = (h_9/r^2)(p_{11}p_{44} - p_{14}^2)$

$$h_6 = (n^2/r^2)\,[\,h_9(h_1p_{44} + 2p_{11}p_{45} + 4p_{11}p_{66} - 2h_5rp_{14}) + h_7(p_{11}p_{44} - p_{14}^2) - r^2h_{11}^2p_{11} - h_3^2p_{44} + 2rh_3h_{11}p_{14}\,] + (2/r)\,h_9(p_{11}p_{24} - p_{14}p_{12})$$

$$h_4 = (n^4/r^2)\,[\,h_1h_7p_{44} + h_9p_{11}p_{55} + (2p_{45} + 4p_{66})(h_1h_9 + h_7p_{11} - h_3^2) + (p_{25} + (1/r)\,p_{55}) \cdot (2h_3p_{14} - 2h_{11}p_{11}r) + h_{11}r^2\,(2h_3h_5 - h_1h_{11}) - rh_5\,(2h_7p_{14} + rh_5h_9)\,] +$$

$$(n^2/r) \cdot [2 (p_{25} + rp_{22})((h_3/r)\ p_{14} - h_{11}p_{11}) -$$

$$2p_{12} (h_5h_9r + h_7p_{14} - h_3h_{11}r) - 2p_{24} (h_3^2 - h_1h_9 - h_7p_{11}) +$$

$$2h_9p_{11}p_{25}] + h_9 (p_{11}p_{22} - p_{12}^2)$$

$$h_2 = (n^6/r^2) [h_1h_7 (2p_{45} + 4p_{66}) + p_{55} (h_1h_9 + h_7p_{11} - h_3^2) -$$

$$r^2h_5^2h_7 + (p_{25} + (1/r)\ p_{55}) \cdot (-2rh_1h_{11} + 2rh_3h_5 - p_{11}p_{25} -$$

$$(1/r)\ p_{11}p_{55})] + (n^4/r) [2h_1h_7p_{24} + 2p_{25} (h_1h_9 + h_7p_{11} -$$

$$h_3^2) - 2p_{12} (rh_5h_7 - h_3p_{25} - (h_3/r)\ p_{55}) - 2 (p_{25} + rp_{22})$$

$$(h_1h_{11} + (1/r)\ p_{11}p_{25} + (1/r^2)\ p_{11}p_{55} - h_3h_5)] +$$

$$n^2 [p_{22} (h_1h_9 + h_7p_{11} - h_3^2) - (1/r)(p_{25} + rp_{22})((1/r)$$

$$p_{11}p_{25} + p_{11}p_{22} - 2h_3p_{12}) - h_7p_{12}^2]$$

$$h_0 = n^4h_1h_7 [p_{22} + (2/r)\ n^2p_{25} + (n^4/r^2)\ p_{55}] - n^2h_1$$

$$[(n^3/r)(p_{25} + (1/r)\ p_{55}) + (n/r)(p_{25} + rp_{22})]^2$$

TABLE 2

MATRIX $\underset{(8,8)}{[A]}$

$$\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = \underset{(8,8)}{[A]} \; \underset{(8,1)}{\{C\}}$$

with $\{C\} = \{C_1, C_2, \ldots, C_8\}^T$

$$\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = \left\{ u_i \;\; w_i \;\; (\frac{dw}{dx})_i \;\; v_i \;\; u_j \;\; w_j \;\; (\frac{dw}{dx})_j \;\; v_j \right\}^T$$

$$\begin{aligned} A(1,q) &= \alpha_q \\ A(2,q) &= 1 \\ A(3,q) &= \frac{\lambda_q}{R} \\ A(4,q) &= \beta_q \\ A(5,q) &= A(1,q)\ a_q \\ A(6,q) &= a_q \\ A(7,q) &= A(3,q)\ a_q \\ A(8,q) &= A(4,q)\ a_q \end{aligned} \qquad (A-3.1)$$

$a_q = e^{\lambda_q \ell / R}$ and $q = 1, \ldots, 8$

TABLE 3

MATRICES $\underset{(8,8)}{[T]}$ and $\underset{(3,8)}{[R]}$

$$\begin{Bmatrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{Bmatrix} = \underset{(3,3)}{[T]}\,\underset{(3,8)}{[R]}\,\underset{(8,1)}{\{C\}}$$

with $\{C\} = \{C_1, C_2, \ldots, C_8\}^T$

$$[T] = \begin{bmatrix} \cos n\theta & 0 & 0 \\ 0 & \cos n\theta & \\ 0 & 0 & \sin n\theta \end{bmatrix}$$

$$\underset{(3,8)}{[R]} = \underset{(3,8)}{[L]}\,\underset{(8,8)}{[X]}$$

$$L(1,q) = \alpha_q$$

$$L(2,q) = 1 \qquad q = 1, \ldots, 8$$

$$L(3,q) = \beta_q$$

$$X(p,q) = e^{\lambda_p x/R} \quad \text{si } p = q$$

$$X(p,q) = 0 \quad \text{si } p \neq q \qquad p,q = 1, \ldots, 8$$

TABLE 4

MATRICE $\underset{(6,8)}{[Q]}$

$$\{\varepsilon_L\} = \underset{(6,6)}{\begin{bmatrix} [T] & [0] \\ [0] & [T] \end{bmatrix}} \underset{(6,8)}{[Q]} \underset{(8,8)}{[A^{-1}]} \underset{(8,1)}{\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}}$$

with $\underset{(6,8)}{[Q]} = \underset{(6,8)}{[J]} \underset{(8,8)}{[X]}$

$$J(1,q) = \alpha_q \frac{\lambda_q}{R}$$

$$J(2,q) = \frac{1}{R} (n\beta_q + 1)$$

$$J(3,q) = \frac{1}{R} (\beta_q \lambda_q - n\alpha_q)$$

$$J(4,q) = -\left(\frac{\lambda_q}{R}\right)^2 \qquad q = 1, \ldots, 8$$

$$J(5,q) = \frac{1}{R^2} (n^2 + \beta_q n)$$

$$J(6,q) = \frac{1}{R^2} \left(2n\lambda_q + \frac{3}{2} \beta_q \lambda_q + \frac{1}{2} n\alpha_q\right)$$

TABLE 5

MATRICES $\underset{(8,8)}{[B^*]}$ and $\underset{(8,8)}{[C^*]}$

$$\underset{(8,8)}{\begin{Bmatrix} [B^+] \\ [C^+] \end{Bmatrix}} = \underset{(8,8)}{[A^{-1}]}^T \underset{(8,8)}{\begin{Bmatrix} [B^*] \\ [C^*] \end{Bmatrix}} \underset{(8,8)}{[A^{-1}]}$$

$$B^*(p,q) = b_{pq} \; e^{(\lambda_p + \lambda_q)\, x/R}$$

$$p,q = 1, \ldots, 8$$

$$C^*(p,q) = c_{pq} \; e^{(\lambda_p + \lambda_q)\, x/R}$$

where $$b_{pq} = \frac{1}{2R^2} \left[\frac{1}{4} a_{pq}^{(1)} + b_{pq}^{(1)} \right] \sin^2 n\theta$$

$$c_{pq} = - \frac{1}{4R^2} \left[c_{pq}^{(1)} + c_{pq}^{(2)} \right] \sin n\theta \cos n\theta$$

with $$a_{pq}^{(1)} = (\beta_p \lambda_p + n\alpha_p)(\beta_q \lambda_q + n\alpha_q)$$

$$b_{pq}^{(1)} = (n + \beta_p)(n + \beta_q) \qquad \text{(A-3.2a)}$$

$$p,q = 1, \ldots, 8$$

$$c_{pq}^{(1)} = n\,(\lambda_p + \lambda_q) \qquad \text{(A-3.2b)}$$

$$c_{pq}^{(2)} = (\beta_p \lambda_q + \beta_q \lambda_p) \qquad \text{(A-3.2c)}$$

TABLE 6

MATRICES $\underset{(8,8)}{[B^{**}]}$, $\underset{(8,8)}{[C^{**}]}$, $\underset{(8,8)}{[D^{**}]}$ and $\underset{(8,8)}{[E^{**}]}$

$$\underset{(8,8)}{\begin{Bmatrix} [B^{++}] \\ [C^{++}] \\ [D^{++}] \\ [E^{++}] \end{Bmatrix}} = 2 \underset{(8,8)}{[A^{-1}]^T} \underset{(8,8)}{\begin{Bmatrix} [B^{**}] \\ [C^{**}] \\ [D^{**}] \\ [E^{**}] \end{Bmatrix}} \underset{(8,8)}{[A^{-1}]}$$

$$B^{**}(p,q) = \sum_{k=1}^{8} b_{kq} \left[\sum_{l=1}^{8} b_{pl}\, \varepsilon_{lk}\, e^{(\lambda_p+\lambda_q+\lambda_k+\lambda_l)\, x/R} \right]$$

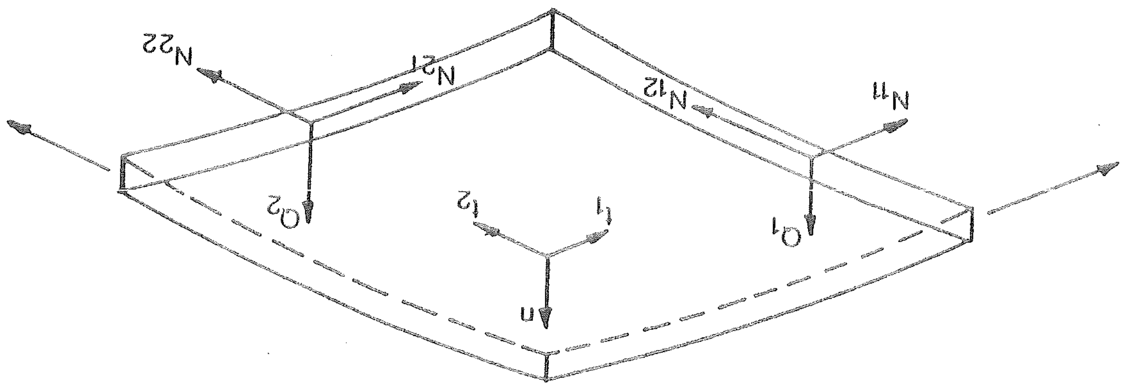
$$C^{**}(p,q) = \sum_{k=1}^{8} c_{kq} \left[\sum_{l=1}^{8} c_{pl}\, \varepsilon_{lk}\, e^{(\lambda_p+\lambda_q+\lambda_k+\lambda_l)\, x/R} \right]$$

$$D^{**}(p,q) = \sum_{k=1}^{8} b_{kq} \left[\sum_{l=1}^{8} a_{pl}\, \varepsilon_{lk}\, e^{(\lambda_p+\lambda_q+\lambda_k+\lambda_l)\, x/R} \right]$$

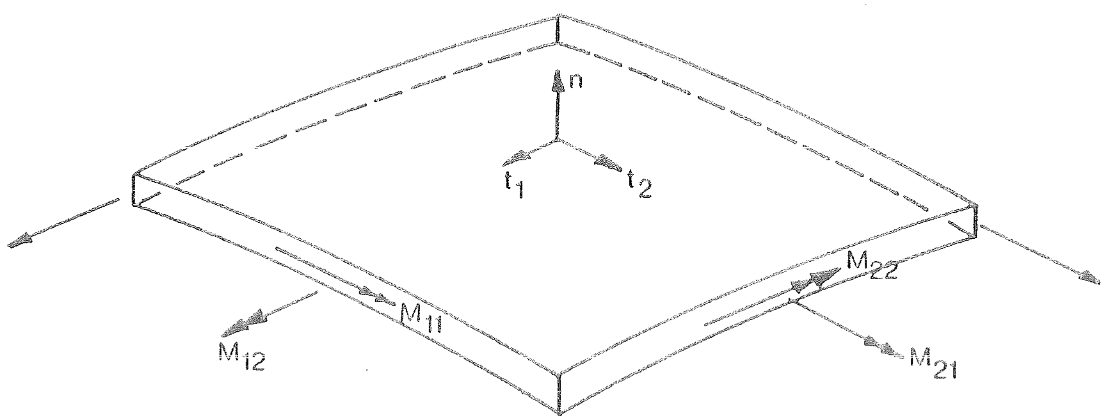
$$E^{**}(p,q) = \sum_{k=1}^{8} a_{kq} \left[\sum_{l=1}^{8} b_{pl}\, \varepsilon_{lk}\, e^{(\lambda_p+\lambda_q+\lambda_k+\lambda_l)\, x/R} \right]$$

$p,q = 1, \ldots, 8$

APPENDIX B

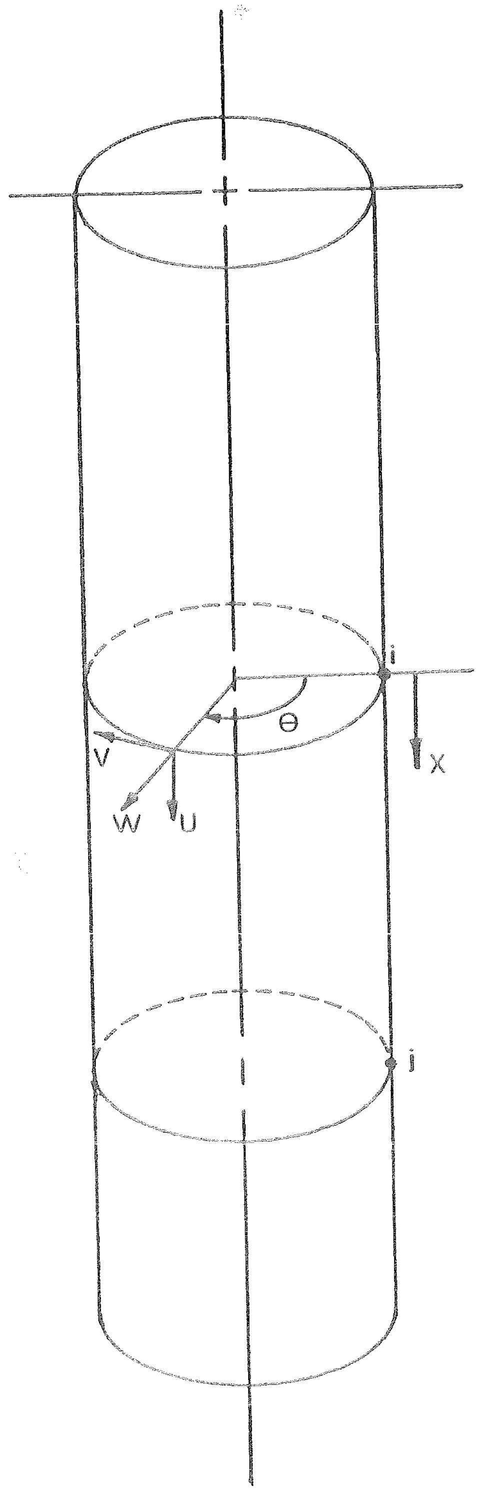


Direction of resultant constraints



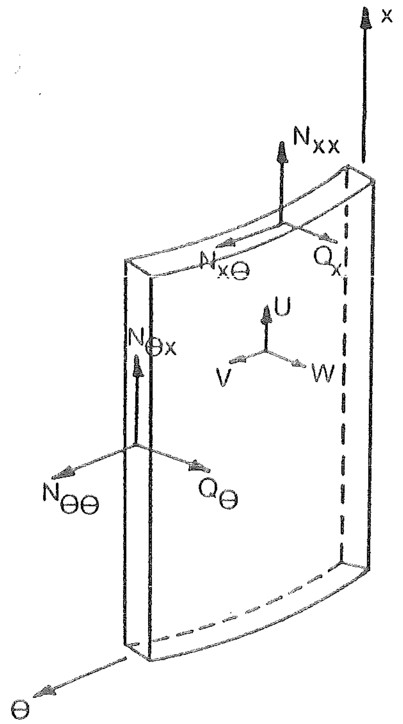
Direction of resultant moments

FIGURE 1: Differential elements for thin shells

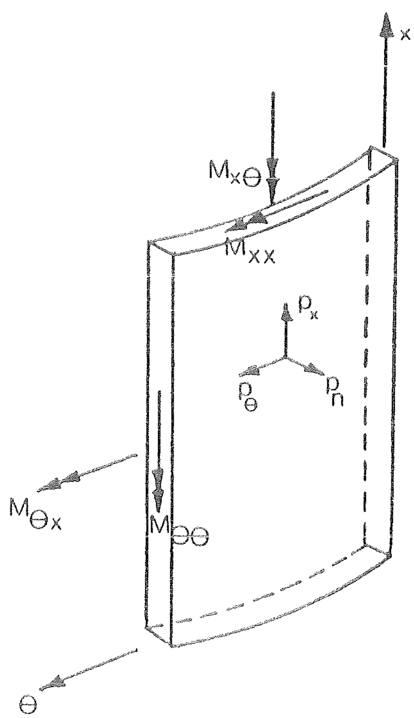


(b) Resultant couples and external loads

FIGURE 2: Differential elements for cylindrical shells



(a) Resultant constraints and displacements



(b) Resultant couples and external loads

FIGURE 3: Differential elements for cylindrical shells

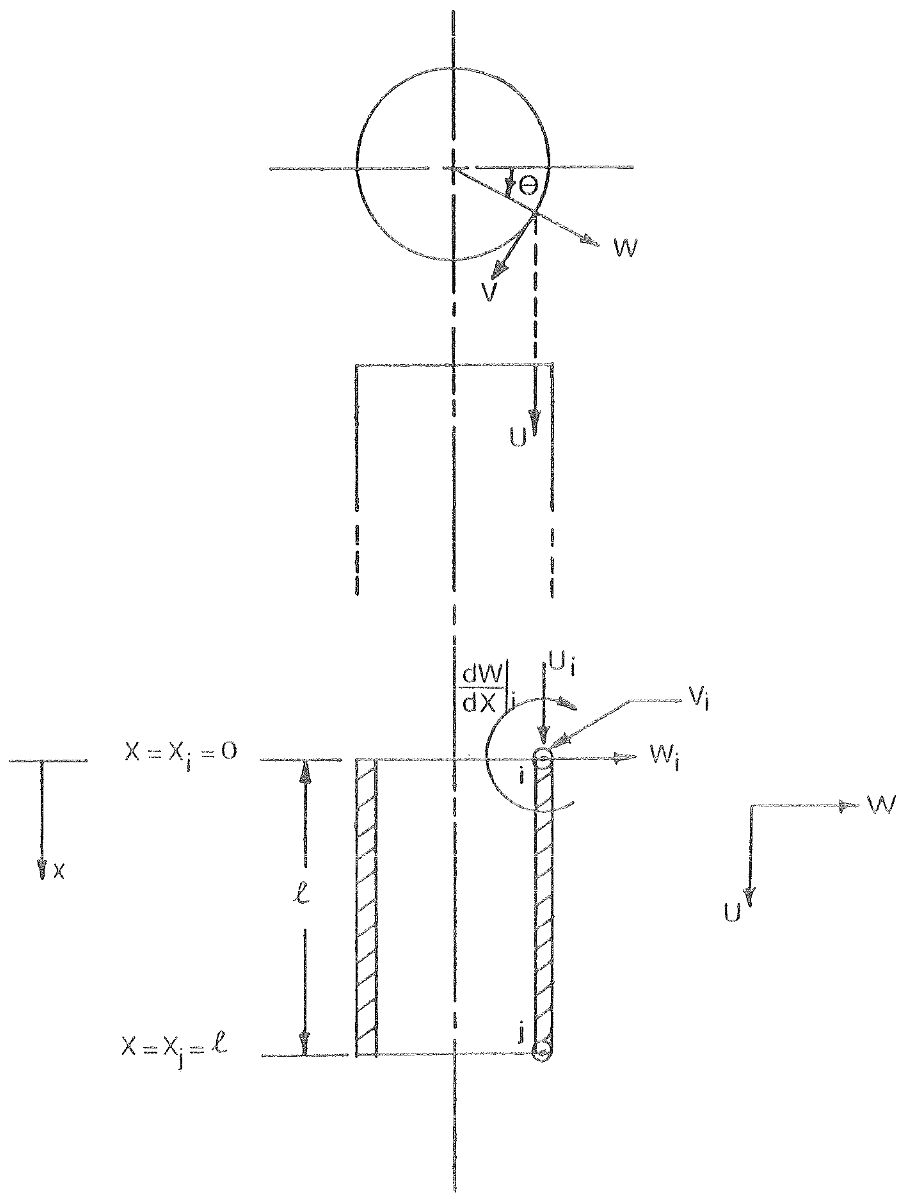


FIGURE 4: Nodal displacements at nodes i and j of a cylindrical element

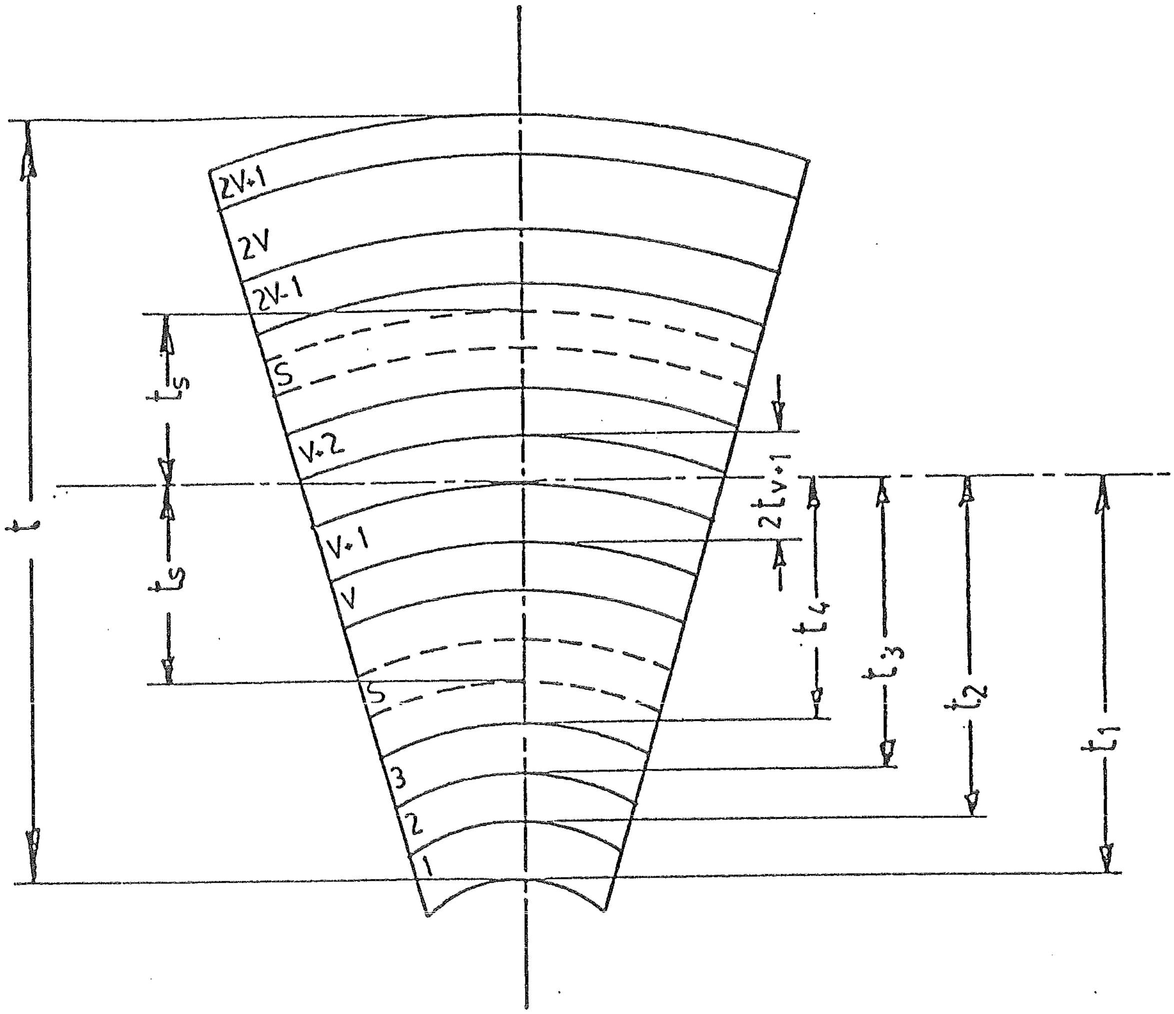


FIGURE 5: Shell composed of an odd number of anisotropic layers

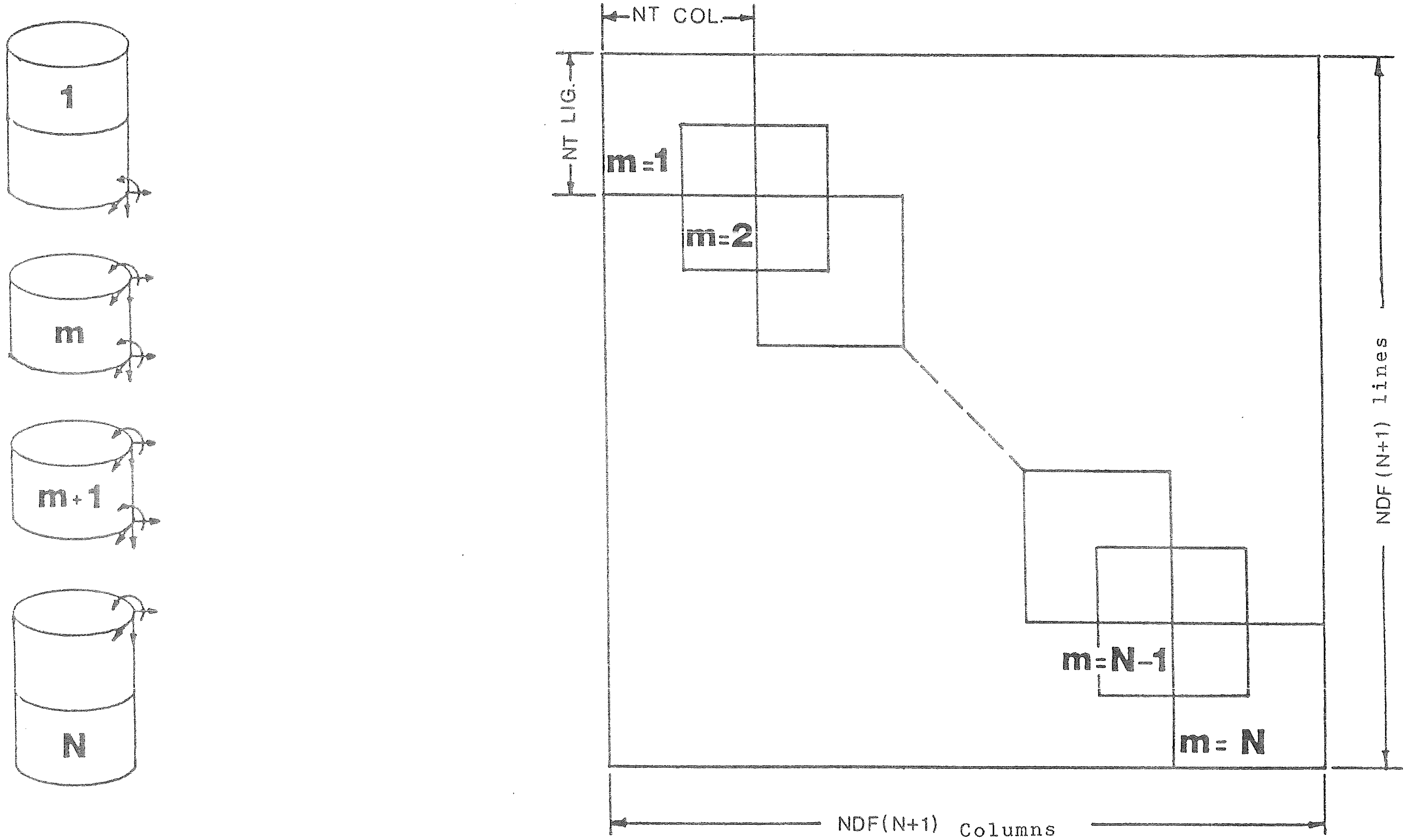


FIGURE 6: Assembly diagram of the mass and stiffness matrices for complete system

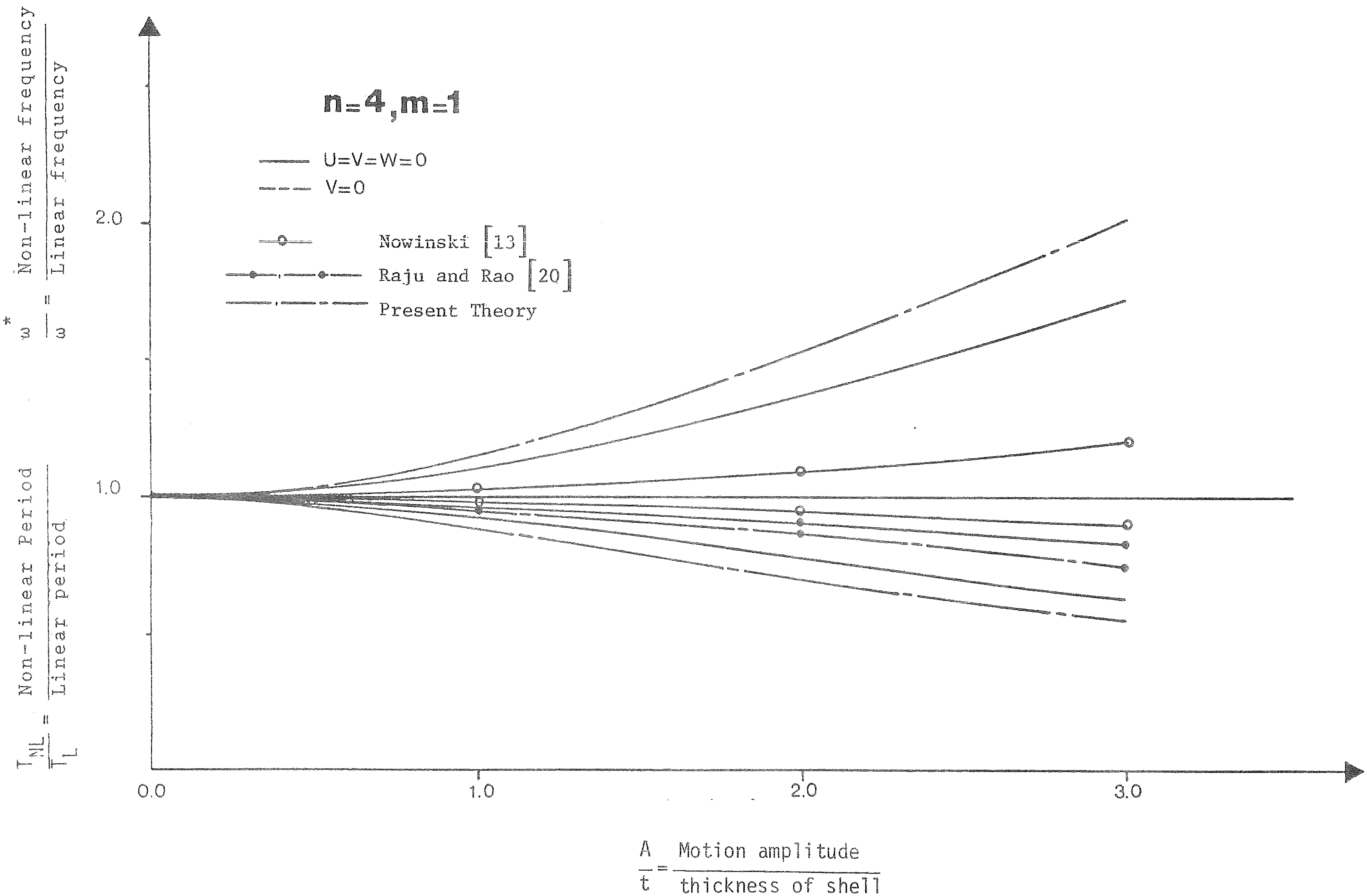


FIGURE 7: Free vibration variations in period and frequency as a function of motion amplitude.
n 4; m 1, U V W 0.

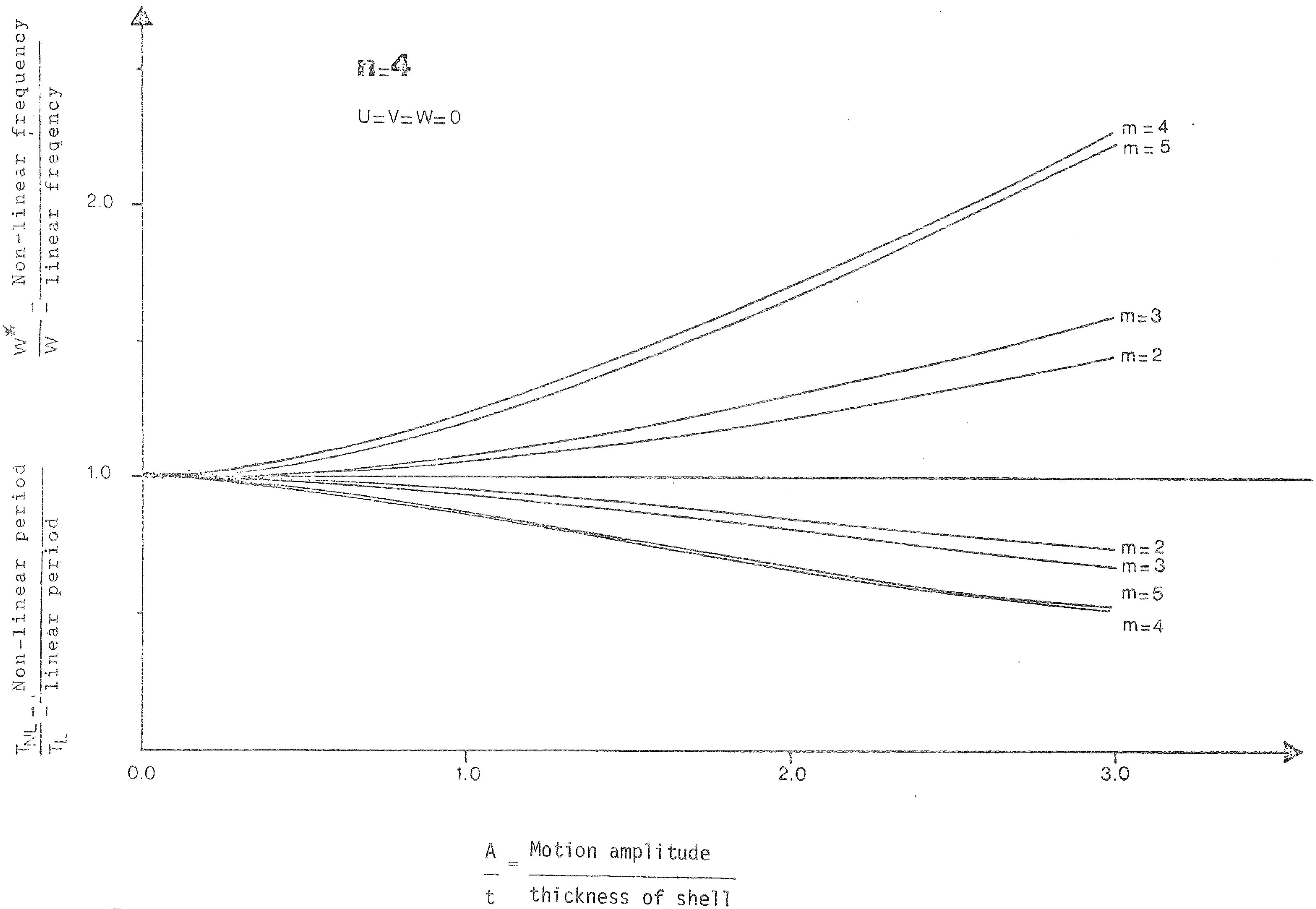
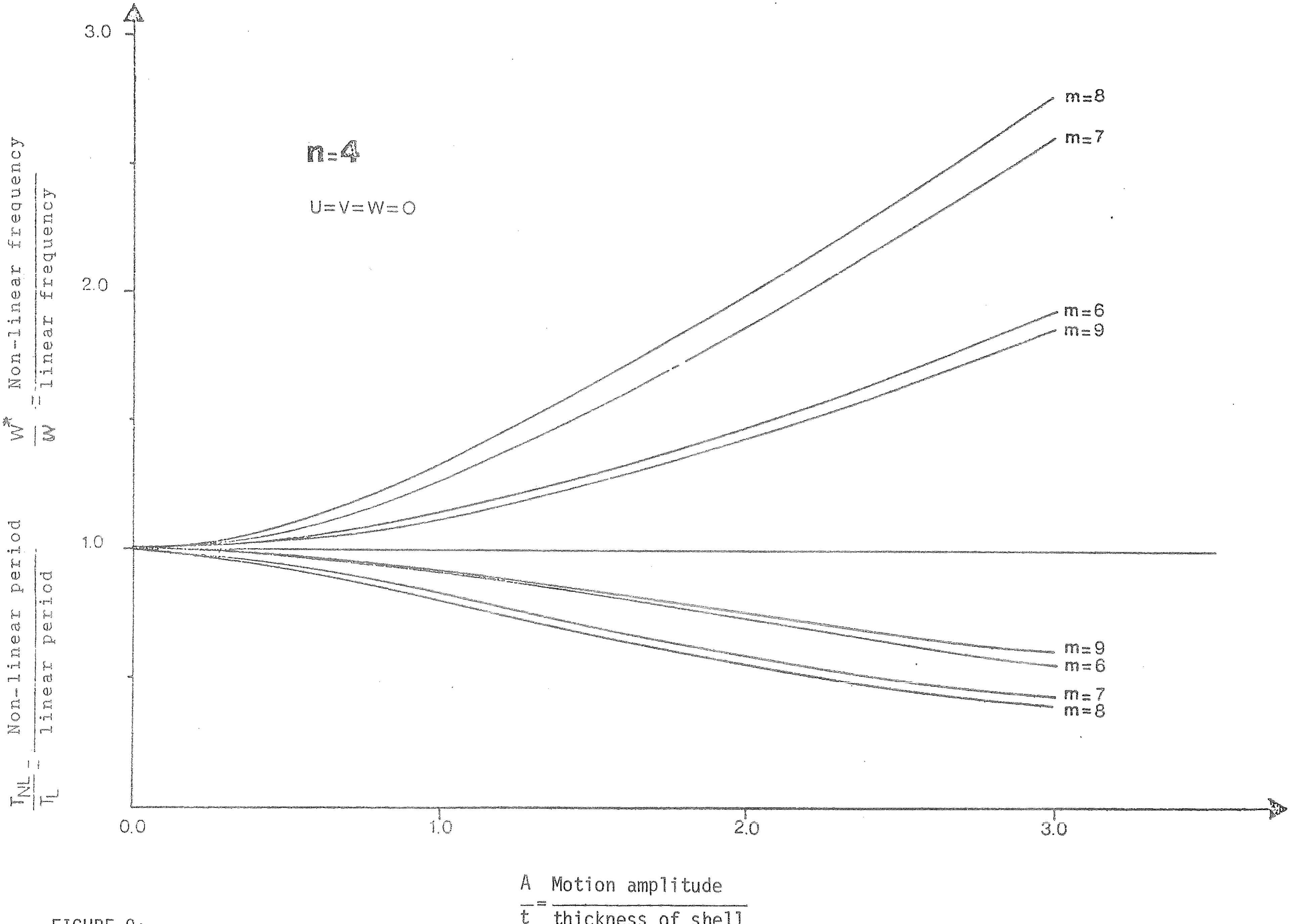

n=4
U=V=W=0
W*/W = Non-linear frequency / linear frequency
T_NL/T_L = Non-linear period / linear period
2.0
1.0
0.0
1.0
2.0
3.0
m=4
m=5
m=3
m=2
m=2
m=3
m=5
m=4
A/t = Motion amplitude / thickness of shell

FIGURE 8:



3.0
2.0
1.0
0.0
1.0
2.0
3.0
n=4
U=V=W=O
m=8
m=7
m=6
m=9
m=9
m=6
m=7
m=8
W*/w = Non-linear frequency / linear frequency
T_NL/T_L = Non-linear period / linear period
A/t = Motion amplitude / thickness of shell

FIGURE 9:

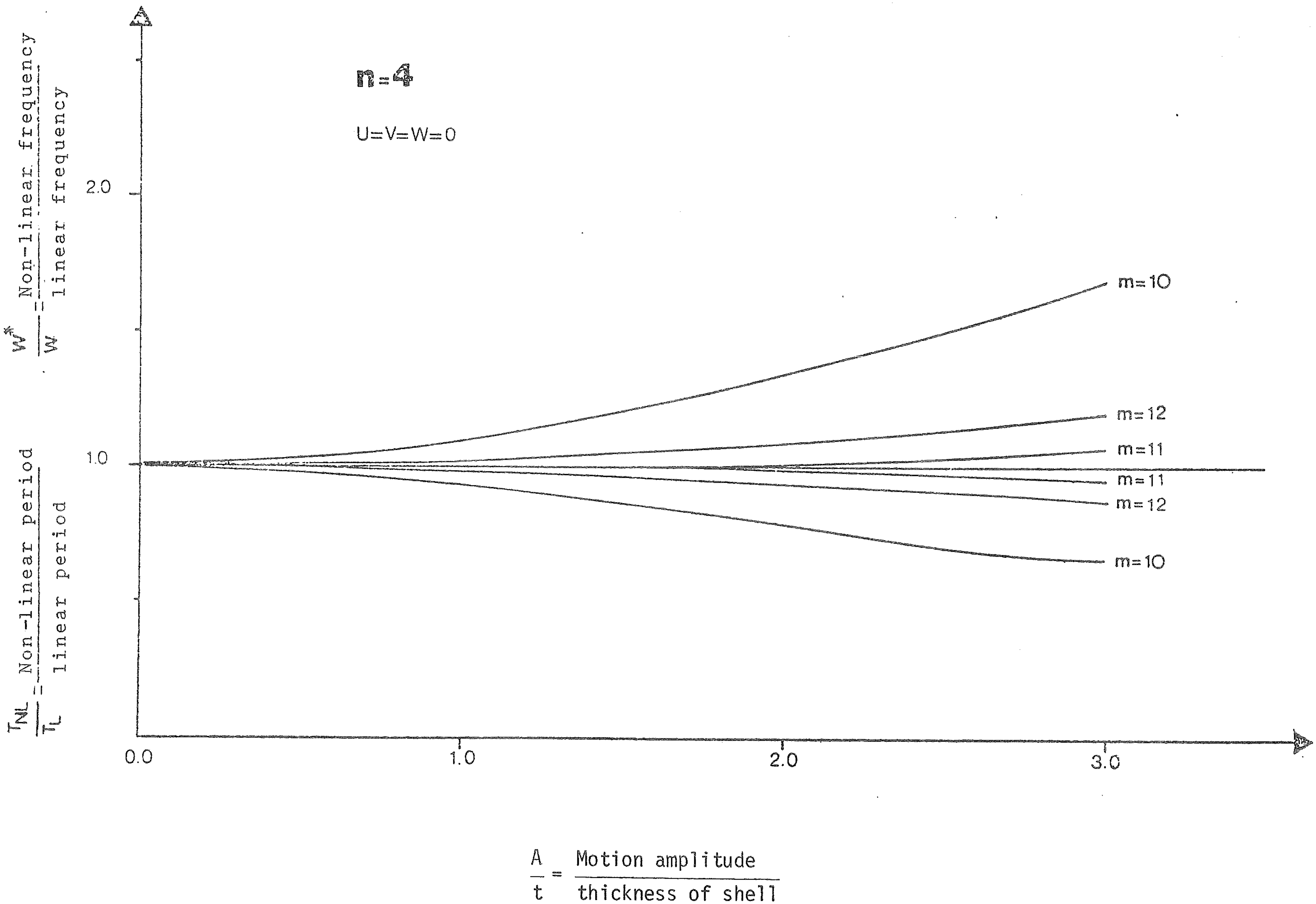

n=4
U=V=W=0
$\frac{w^*}{w}$ = $\frac{\text{Non-linear frequency}}{\text{linear frequency}}$
$\frac{T_{NL}}{T_L}$ = $\frac{\text{Non-linear period}}{\text{linear period}}$
2.0
1.0
0.0
1.0
2.0
3.0
m=10
m=12
m=11
m=11
m=12
m=10
$\frac{A}{t}$ = $\frac{\text{Motion amplitude}}{\text{thickness of shell}}$

FIGURE 10:

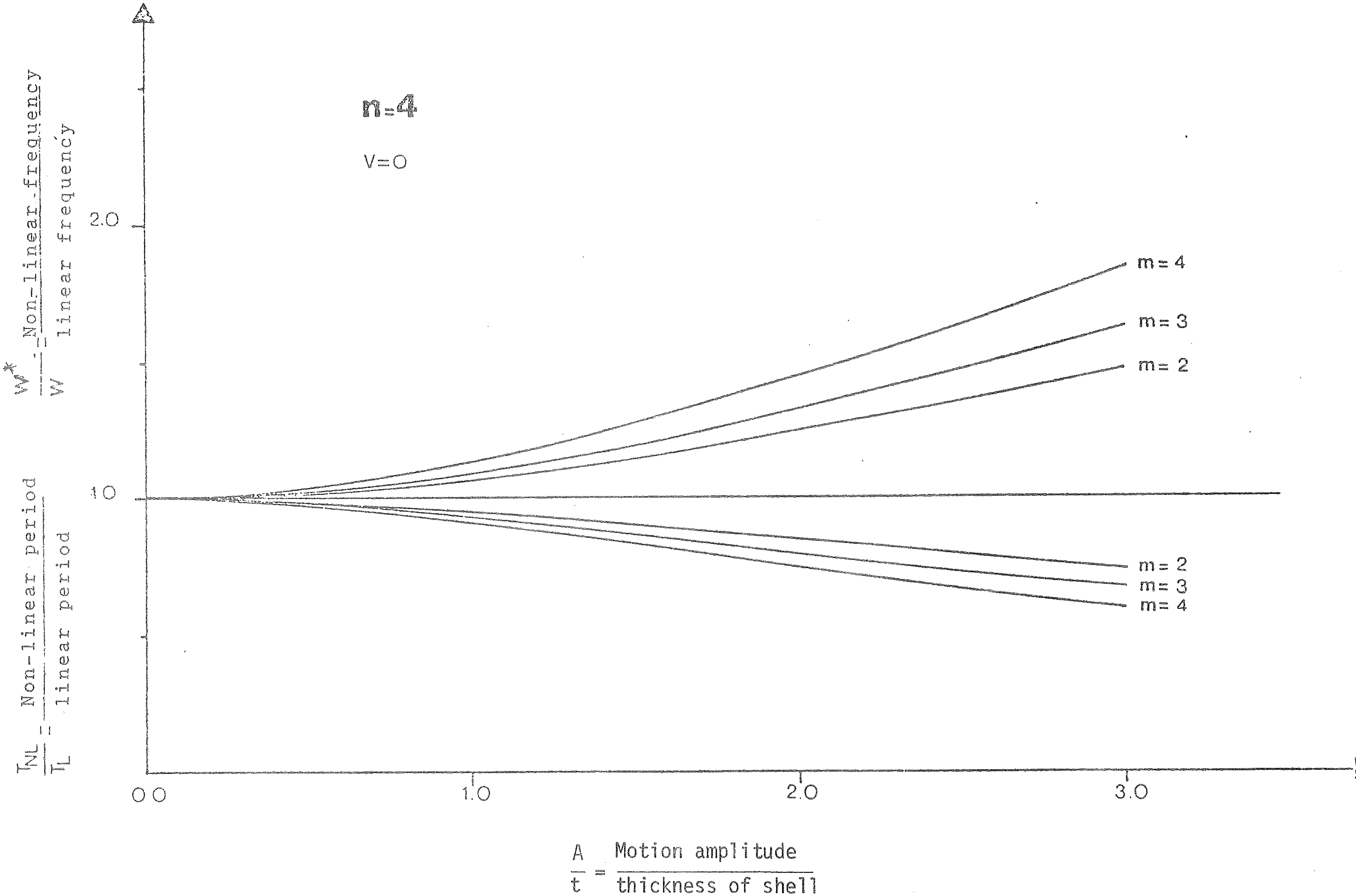

n=4
V=O
$\frac{W^*}{W}$ = $\frac{\text{Non-linear frequency}}{\text{linear frequency}}$
$\frac{T_{NL}}{T_L}$ = $\frac{\text{Non-linear period}}{\text{linear period}}$
2.0
1.0
0.0
1.0
2.0
3.0
m= 4
m= 3
m= 2
m= 2
m= 3
m= 4
$\frac{A}{t}$ = $\frac{\text{Motion amplitude}}{\text{thickness of shell}}$

FIGURE 11:

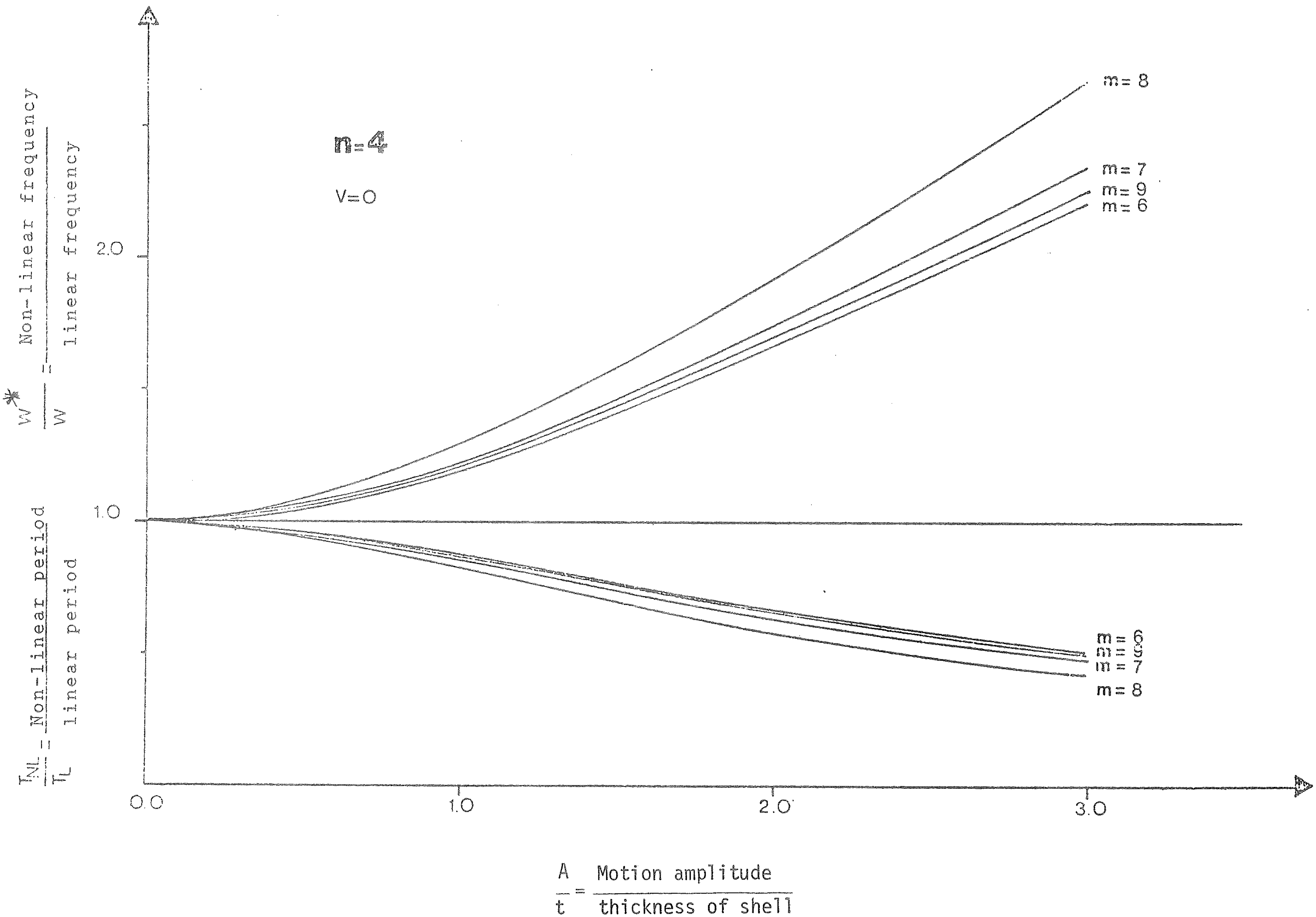
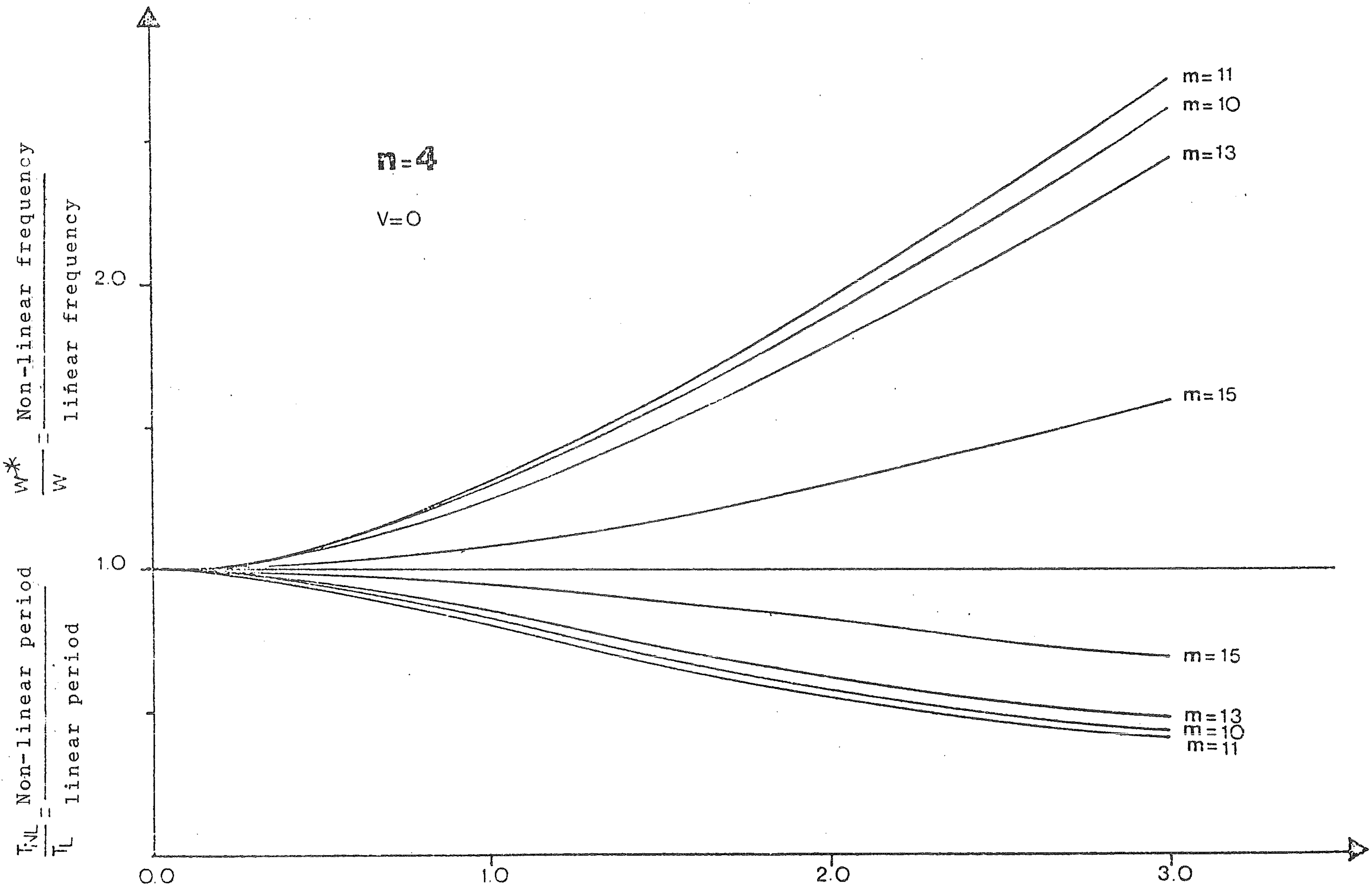

n=4
V=0
W*/W = Non-linear frequency / linear frequency
T_NL/T_L = Non-linear period / linear period
2.0
1.0
0.0
1.0
2.0
3.0
m= 8
m= 7
m= 9
m= 6
m= 6
m= 9
m= 7
m= 8
A/t = Motion amplitude / thickness of shell

FIGURE 12:



$\frac{A}{t} = \frac{\text{Motion amplitude}}{\text{thickness of shell}}$

FIGURE 13: