DYNAMIC ANALYSIS OF CONICAL SHELLS CONTAINING FLOWING FLUID

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RÉSUMÉ

Dans ce mémoire, nous présentons une analyse de l'équilibre statique et dynamique des coques coniques minces anisotropes contenant un fluide en écoulement turbulent. La coque peut être uniforme ou non-uniforme en autant que sa géométrie soit axialement symétrique. Les équations sont solutionnées par la méthode des éléments finis où l'élément fini est conique, mais les fonctions de déplacements sont dérivées des équations des coques.

L'équation du potentiel des vitesses et l'équation de Bernouilli de l'élément fini liquide nous permettent d'exprimer la pression exercée par le fluide, p, comme étant une fonction des déplacements nodaux de l'élément et trois forces (d'inertie, centrifuge et de Coriolis) du fluide en écoulement.

Plusieurs exemples illustrent la théorie et le comportement de la coque pour différentes conditions d'utilisation.

ABSTRACT

The report presents a general theory for the dynamic analysis of thin conical anisotropic shells containing turbulent flowing fluid. The shell may be uniform or non-uniform, provided that its geometry is axisymmetric.

It is finite element theory, using conical finite elements, but the displacement functions are determined by means of classical shell theory.

The velocity potential and Bernoulli equation for a liquid finite element yield an expression for fluid pressure p as a function of the nodal displacements of the element and three forces (inertial, centrifugal and Coriolis) operative in the fluid flow.

A number of examples are given to illustrate the theory and the dynamic behaviour of the shell.

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LIST OF SYMBOLS

a	defined by equation (3.3)
a_c	shell apex half-angle
a_1, a_2	real roots of the characteristic equation
A_1, A_2	Lamé parameters
$A_j (j=1,...,8)$	constant in equation defining u
$B_j (j=1,...,8)$	constant in equation defining v
$C_j (j=1,...,8)$	constant in equation defining w
b/L	liquid level ratio
$\overline{C_j}$	vector C elements
B_{ij}	elements of the elasticity matrix $(i=1,...,6,\ j=1,...,6)$
c	speed of sound in the fluid
D	membrane stiffness, $Et/(1-\nu^2)$
E	Young's modulus
G	shear modulus of elasticity
K	bending stiffness, $Et^3/12(1-\nu^2)$
k	defined by equation (3.17)
ℓ	length defined by equation (2.3)
ℓn	Napierian logarithm
ℓ_i, ℓ_j	value of x coordinate at nodes i and j
m	number of axial half-waves
$M_1, M_2, M_{12}, M_{21}, \overline{M_{12}}$	resultant moments
$M_x, M_\theta, \overline{M_{x\theta}}$	resultant couples for a conical shell
$\overline{\overline{M_1}}$	boundary couple value
n	circumferential mode number

N	number of finite elements
$N_1, N_2, \overline{N_{12}}$	resultant constraints
$N_x, N_\theta, \overline{N_{x\theta}}$	resultant constraints for a conical shell
p	fluid pressure exerted on inner wall
Q	defined by equation (3.2)
Q_1, Q_2	resultant shear constraints
r	average shell radius
R_1, R_2	radii of curvature of surface of reference
R_q, S_q	defined by equation (3.12)
t	wall thickness
$\overline{\overline{T_{12}}}$	resultant boundary shear
U	defined by equation (3.2)
u, v, w	axial, tangential and radial displacements
u_n, v_n, w_n	amplitudes of u, v, w associated with nth circumferential mode number
x	coordinate along cone generator
y	coordinate defined by equation (2.3)
V_x, V_θ, V_α	components of fluid velocity along x, θ and α axes as defined in (3.6)
α	coordinate along cone apex half-angle
α_j, β_j	defined by (2.9)
$\overline{\alpha_j}, \overline{\beta_j}$	defined by (2.10)
δ	defined by (2.20)

$\epsilon_x, \epsilon_\theta, \overline{\epsilon_{x\theta}}$	deformations of reference surface
θ	circumferential coordinate
K_1, K_2	real parts of λ_j
$K_x, K_\theta, K_{x\theta}$	changes in curvatures and torsion of surface of reference
Λ	eigenvalue
λ_j	complex roots of the characteristic equation
μ_1, μ_2	imaginary parts of λ_j
ν	Poisson's ratio
φ	potential fluid velocity
ρ	density of shell material
ρ_ℓ	fluid density inside the shell
ω	vibration frequency (rad/s)
ω_0	defined by equation (3.29)
ζ_1, ζ_2	coordinates at the reference surface

LIST OF MATRICES

$[A]$	defined by relation (2.15)
$[B]$	defined by relation (2.18)
$\{\bar{C}\}$	vector for arbitrary constants
$[c_f]$	damping matrix for a fluid column element
$[C_f]$	global damping matrix for the fluid column
$[C]$	global damping matrix for the system
$[D]$	defined by relation (2.5)
$[D_f]$	defined by relations (3.24, 3.25)
$\{f_i\}$	vector for forces applied at node i
$\{F\}$	vector for external forces
$[G]$	defined by relation (2.24)
$[G_f]$	defined by relations (3.26, 3.27)
$[k_s]$	stiffness matrix for a finite shell element
$[k_f]$	stiffness matrix for a fluid column element
$[K_s]$	global stiffness matrix for the shell
$[K_f]$	global stiffness matrix for the fluid column
$[K]$	global stiffness matrix for the system
$[m_s]$	mass matrix for a finite shell element
$[m_f]$	mass matrix for a fluid column element
$[M_s]$	global mass matrix for the shell
$[M_f]$	global mass matrix for the fluid column

$[M]$	global mass matrix for the system
$[N]$	defined by equation (2.1)
$[P]$	matrix of elasticity
$[Q]$	defined by relation (2.18)
$[R]$	defined by relation (2.12)
$[S]$	defined by relation (2.24)
$[S_4]$	defined by relations (3.22 and 3.23)
$[T]$	defined by relation (2.12)
$[T^i]$	defined by relation (2.25)
$\{\delta\}$	degrees of freedom vector for total shell
$\{\delta_i\}$	degrees of freedom vector at node i
$\{\epsilon\}$	deformation vector
$\{\sigma\}$	stress vector

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15 Variation in the natural frequencies of a liquid-filled conical shell as a function of shell length (X_2-X_1) and n (V=W=0, n=2 to 9, m=1)

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CHAPTER I

INTRODUCTION

1.1 General

Thin shells are used in a wide variety of applications, as for example in the aerospace, aeronautical, petroleum and nuclear industries, to name just a few. The static and dynamic behaviour of the shells has therefore been studied extensively. Most of the investigations deal with empty shells, however, and in a good number of applications, the shells used act as reservoirs or piping and contain a fluid, which alters system behaviour.

Among the researchers who have worked on conical shells in interaction with a fluid, we could mention Dokuchaev [1], who used Galerkin's method, Shkivarchuk [2], who used Ritz and the successive approximation method and Stepanyuk and Babien [3], who used the Bubnov-Galerkin method.

Niordson [4] was one of the first to report a study investigating the effect of a liquid on a shell's natural frequencies. Berry and Reissner [5] studied the case of a simply supported cylindrical shell containing a pressurized gas and

Lindholm, Kana and Abramson [6] conducted an experimental and theoretical investigation of a cylindrical shell completely or partially filled with a non-pressurized liquid.

Analytical solution of the equations of motion for thin shells is generally difficult, and only approximation methods tend to be used. Two such methods are the finite difference and the numerical integration methods where initial values for the frequency are set a priori. These procedures require a great deal of processing time and have been demonstrated [7] to be unable to determine the complete spectrum of vibrations. Similarly, Stodola-type iterations and the Galerkin method lose their accuracy at high shell frequencies. The Rayleigh-Ritz method also has a drawback: the displacement functions chosen must allow for boundary conditions.

The present study is based on the finite element method, which has the following advantages:

a) Arbitrary shell geometry: the method applies equally well to a cylinder as to a cone or any other axisymmetric shell.

b) Thickness discontinuities, variations in material properties, different materials making up the shell are all readily included.

c) Arbitrary boundary conditions: the problem can be solved for a simply supported, clamped-free or clamped-clamped shell without the displacement functions having to be changed for each individual case.

d) The high and low frequency characteristics are obtained immediately.

The finite element we selected was conical and bounded by two circular nodes. This therefore made it possible to use shell theory to determine the displacement functions.

To account for the fluid effect on the shell, we considered a conical finite fluid element bounded by nodes i and j. Solving the equations of motion again for the fluid element, we obtained an expression for fluid pressure as a function of the element's displacements at nodes i and j. Double analytical integration for the pressure distributed along the element gives us three components: the stiffness, the mass and the damping matrices for the fluid.

This thesis is part of a research project being conducted by Dr. Aouni A. Lakis and aimed at using finite elements that are conical, cylindrical, spherical, plates, etc., to determine the resonant frequencies and vibration modes of a thin shell having any shape whatsoever and which may contain a fluid.

Reference [8] reports a case of free vibrations of non-uniform cylindrical shells subjected to different boundary conditions. In reference [9], the dynamic behaviour of cylindrical shells containing flowing fluid was analyzed. The same case was given non-linear treatment in reference [10]. Similarly, the vibration characteristics of a Pickering steam generator were studied in reference [11]. In addition, a dynamic analysis of anisotropic conical shells was performed in reference [12].

1.2 Research Objectives

The purpose of this study was to explore the dynamic behaviour and responses of a conical shell partially filled with liquid or with internal fluid flow.

We consider that the dynamic behaviour of the shell can be expressed by the following equation:

$$\left[[M_o] - [M_f]\right] \{\ddot{\delta}\} + [C_f] \{\dot{\delta}\} + \left[[K_o] - [K_f]\right] \{\delta\} = \{0\} \qquad (1.1)$$

where δ is the displacement vector, $[M_o]$ and $[K_o]$ are the mass and stiffness matrices, respectively, of the in vacuo system and $[M_f]$, $[C_f]$ and $[K_f]$ represent the inertial, Coriolis and centrifugal forces in the fluid flow.

The mass and stiffness matrices for the in vacuo system, $[M_o]$ and $[K_o]$, have been determined in reference [12]. Our objective in this report is therefore to find the mass, damping and stiffness matrices for the fluid component and then to solve equation (1.1). In order for the rationale underlying this research to be valid, we have to accept the following hypotheses:

- The cone is incomplete: the cone's apex and surroundings are not considered;

- The cone is geometrically symmetric and circular: the shell reference surface is produced by rotating a generator around a fixed axis while maintaining <u>constant</u> angle α.

- The cone is right-angled: the bases are circles.

- Cone thickness is variable and assumed to be a linear function of the coordinate given by the generator. The method can, however, be used even if the variation is not linear.

- The shell if made up of one or more layers of isotropic or orthotropic material.

- Displacements are small enough to be geometrically linear.

- The rotary inertia terms are ignored.

- Constraints under static equilibrium are zero.

With regard to the fluid, we assume that:

- The flow is potential.

- The fluid is incompressible.

- The pressure exerted on the wall by the fluid is purely lateral.

- The fluid velocity distribution is assumed constant through a shell section.

- Internal pressure is not unduly high.

- There is no separation between the fluid and the shell wall.

1.3 Contents of the Report

The report is divided into six chapters:

In the next chapter, Chapter two, we choose the displacement functions for a finite element and describe how the mass and stiffness matrices for the solid portion are obtained.

In Chapter three, we develop the apparent mass, damping and stiffness matrices for the fluid and solve the equation of motion for the shell-fluid system to determine the resonant frequencies and eigenvectors of the system.

Chapter four gives an outline of the program logic.

Chapter five contains a summary of results.

Finally, Chapter six presents the general conclusions.

CHAPTER II

DYNAMIC ANALYSIS OF ANISOTROPIC CONICAL SHELLS

2.1 Introduction

In this chapter, we determine the displacement functions for a conical finite element and describe the method that was used to determine the mass and stiffness matrices for the solid component of the system in vacuo, as given in references [8] and [12].

2.2 Displacement Functions

The shell was subdivided into conical finite elements having nodes i and j, which are circles at the boundaries (Fig. 1). The displacement function can then be defined by:

$$[u(x,\theta),\ w(x,\theta),\ v(x,\theta)]^T = [N]\,[\delta_i,\ \delta_j]^T \qquad (2.1)$$

where δ_i and δ_j are the nodal displacements and the elements of [N] are generally a function of shell position and anisotropy.

In order to study equilibrium in the conical shell while taking into account membrane effects and elongation of the reference surface, Sanders' first-order equations were used. These equations are based on Love's first approximation and yield zero deformation for rigid-body motion, which is not the case with other formulations.

The equilibrium equations were expressed as a function of axial, circumferential and radial displacements, u,v and w, respectively. This gives us three ordinary differential equations which, when solved, give the displacement functions.

$$u = u_n(x) \quad \cos n\Theta$$

$$v = v_n(x) \quad \sin n\Theta \qquad (2.2)$$

$$w = w_n(x) \quad \cos n\Theta$$

where n is the circumferential mode number.

$$u_n(x) = A\, x^{\lambda}$$

$$v_n(x) = B\, x^{\lambda}$$

$$w_n(x) = C\, x^{\lambda}$$

λ is a complex root generally.

To avoid getting into impractical magnitudes, we defined a non-dimensional variable:

$$y = \sqrt{\frac{x}{\ell}} \tag{2.3}$$

where ℓ is an arbitrary reference length. We thus obtain:

$$\{u_n(x),\ v_n(x),\ w_n(x)\} = y^{\lambda-1}\ \{A,\ B,\ C\} \tag{2.4}$$

where λ, A, B, and C are complex roots.

2.3 Characteristic Equation

Replacing u, v and w with (2.2) and (2.4) in the equilibrium equations, we obtain a system of algebraic equations we shall write as:

$$[D] \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \{0\} \tag{2.5}$$

So that the solution is non-trivial, the [D] determinant must be zero. That gives us the following characteristic equation:

$$\det([D]) = h_8\lambda^8 + h_6\lambda^6 + h_4\lambda^4 + h_2\lambda^2 + h_0 = 0 \tag{2.6}$$

The value of the h_i coefficients in this eighth-degree polynomial is given in Appendix A-2.

The roots of the equation can be in two forms:

Complex roots:

$$
\begin{aligned}
\lambda_1 &= K_1 + i\,\mu_1 & \lambda_5 &= K_2 + i\,\mu_2 \\
\lambda_2 &= K_1 - i\,\mu_1 & \lambda_6 &= K_2 - i\,\mu_2 \\
\lambda_3 &= -K_1 + i\,\mu_1 & \lambda_7 &= -K_2 + i\,\mu_2 \\
\lambda_4 &= -K_1 - i\,\mu_1 & \lambda_8 &= -K_2 - i\,\mu_2
\end{aligned}
\qquad (2.7)
$$

or Real roots:

$$
\begin{aligned}
\lambda_1 &= K_1 + i\,\mu_1 & \lambda_5 &= a_1 \\
\lambda_2 &= K_1 - i\,\mu_1 & \lambda_6 &= a_2 \\
\lambda_3 &= -K_1 + i\,\mu_1 & \lambda_7 &= -a_1 \\
\lambda_4 &= -K_1 - i\,\mu_1 & \lambda_8 &= -a_2
\end{aligned}
\qquad (2.8)
$$

where K_1, μ_1, K_2, μ_2, a_1 and a_2 are real and positive.

Each λ_j root gives a displacement, $[u_n(x),\ v_n(x),\ w_n(x)]_j$. It is solved by linear combination of these eight displacements with constants A_j, B_j and C_j where $j = 1, 2, \ldots, 8$.

A_j, B_j and C_j are not independant for a given λ_j

A_j and B_j are usually expressed as a function of C_j. Setting:

$$A_j = \alpha_j C_j \qquad B_j = \beta_j C_j \tag{2.9}$$

α_j and β_j depend on the terms in matrix [D] and thus on λ_j. In light of the form of equation (2.5), we only have two pairs of equations to solve because λ_2 is the conjugate root of λ_1, similarly, λ_3 and λ_4 are conjugates as are λ_5 and λ_6, λ_7 and λ_8. Therefore:

$$\begin{array}{ll} \alpha_1 = \bar{\alpha}_1 + i\,\bar{\alpha}_2 & \alpha_5 = \bar{\alpha}_5 + i\,\bar{\alpha}_6 \\ \alpha_2 = \bar{\alpha}_1 - i\,\bar{\alpha}_2 & \alpha_6 = \bar{\alpha}_5 - i\,\bar{\alpha}_6 \\ \alpha_3 = \bar{\alpha}_3 + i\,\bar{\alpha}_4 & \alpha_7 = \bar{\alpha}_7 + i\,\bar{\alpha}_8 \\ \alpha_4 = \bar{\alpha}_3 - i\,\bar{a}_4 & a_8 = \bar{a}_7 - i\,\bar{a}_8 \end{array} \tag{2.10}$$

$$\begin{array}{ll} \beta_1 = \bar{\beta}_1 + i\,\bar{\beta}_2 & \beta_5 = \bar{\beta}_5 + i\,\bar{\beta}_6 \\ \beta_2 = \bar{\beta}_1 - i\,\bar{\beta}_2 & \beta_6 = \bar{\beta}_5 - i\,\bar{\beta}_6 \\ \beta_3 = \bar{\beta}_3 + i\,\bar{\beta}_4 & \beta_7 = \bar{\beta}_7 + i\,\bar{\beta}_8 \\ \beta_4 = \bar{\beta}_3 - i\,\bar{\beta}_4 & \beta_8 = \bar{\beta}_7 - i\,\bar{\beta}_8 \end{array} \tag{2.11}$$

Since we know that the displacements are real functions, the final form of u_n, v_n and w_n will also be real. The displacement corresponding to circumferential mode n is therefore:

$$\begin{Bmatrix} u(x,\theta) \\ w(x,\theta) \\ v(x,\theta) \end{Bmatrix} = [T]\,[R]\,\{\bar{C}\} \qquad (2.12)$$

where matrices [T] and [R] are shown in Appendix A-2 and

$$\{\bar{C}\} = \begin{Bmatrix} \bar{C}_1 \\ \vdots \\ \bar{C}_8 \end{Bmatrix} \qquad (2.13)$$

The C_i terms are the only free constants in the problem and will be determined using the eight boundary conditions, four at each end of the cone.

Let us now express displacement as a function of the degrees of freedom δ_j and δ_i at the finite element's nodes i and j. Displacement at node i can be defined by the vector:

$$\{\delta_i\} = \begin{Bmatrix} u_{ni} \\ w_{ni} \\ (dw_n/dx)_i \\ v_{ni} \end{Bmatrix} \qquad (2.14)$$

These components represent the amplitude of u, v, w and dw/dx associated with the nth circumferental mode. As the element has two nodes and eight degrees of freedom, the displacement will be:

$$\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = \begin{Bmatrix} u_{ni} \\ w_{ni} \\ (dw_n/dx)_i \\ v_{ni} \\ u_{nj} \\ w_{nj} \\ (dw_n/dx)_j \\ v_{nj} \end{Bmatrix} = [A] \quad \{\bar{C}\} \qquad (2.15)$$

The terms in [A] are obtained from the terms in matrix [R]. Multiplying by $[A]^{-1}$, we obtain:

$$\{C\} = [A]^{-1} \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \qquad (2.16)$$

Substituting into equation (2.12), we get:

$$\begin{Bmatrix} u \\ w \\ v \end{Bmatrix} = [T]\ [R]\ [A]^{-1} \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [N] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \qquad (2.17)$$

which is the equation defining the displacement function.

2.4 Stiffness, Mass and Stress Matrices for a Finite Element

The deformation vector can be expressed as a function of displacements u, v and w, as given in Appendix A-1. Using equation (2.17), we derive $\{\epsilon\}$ from the nodal displacements.

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_\Theta \\ 2\epsilon_{x\Theta} \\ K_x \\ K_\Theta \\ 2K_{x\Theta} \end{Bmatrix} = \begin{bmatrix} [T] & [0] \\ [0] & [T] \end{bmatrix} [Q]\ [A]^{-1} \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [B] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{2.18}$$

where matrices [T] and [Q] are as given in Appendix A-2.

The stress vector is obtained from equations (2.18) and the stress-strain relationship.

$$\{\sigma\} = \begin{Bmatrix} N_x \\ N_\Theta \\ N_{x\Theta} \\ M_x \\ M_\Theta \\ M_{x\Theta} \end{Bmatrix} = [P]\ \{\epsilon\} = [P]\ [B] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{2.19}$$

where [P] is the elasticity matrix.

For shells composed of anisotropic and orthotropic material and having a thickness varying along the x axis

$$t = \delta\ x \tag{2.20}$$

The matrix takes the form:

$$[P] = \begin{bmatrix} x\,B_{11} & x\,B_{12} & 0 & x^2\,B_{14} & x^2\,B_{15} & 0 \\ x\,B_{21} & x\,B_{22} & 0 & x^2\,B_{24} & x^2\,B_{25} & 0 \\ 0 & 0 & x\,B_{33} & 0 & 0 & x^2\,B_{36} \\ x^2\,B_{41} & x^2\,B_{42} & 0 & x^3\,B_{44} & x^3\,B_{45} & 0 \\ x^2\,B_{51} & x^2\,B_{52} & 0 & x^3\,B_{54} & x^3\,B_{55} & 0 \\ 0 & 0 & x^2\,B_{63} & 0 & 0 & x^3\,B_{66} \end{bmatrix} \qquad (2.21)$$

The B_{ij} are given in detail in Appendix A-2.

Using relations (2.18) and (2.19) where $\{\delta_i \;\; \delta_j\}^T$ and $[A]^{-1}$ are constant for a finite element, we obtain the element's equations of motion in matrix format:

$$[m_o] \begin{Bmatrix} \ddot{\delta}_i \\ \ddot{\delta}_j \end{Bmatrix} + [k_o] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = \{0\} \qquad (2.22)$$

$[m_o]$ and $[k_o]$ are the mass and stiffness matrices and are given by [12]:

$$\begin{aligned} [k_o] &= [[A]^{-1}]^T \;\; [G]\;[A]^{-1} \\ [m_o] &= [[A]^{-1}]^T \;\; [S]\;[A]^{-1} \end{aligned} \qquad (2.23)$$

where

$$[G] = \int_{A_i} [Q]^T [T^1]^T [F] [T^1] [Q] \, dA$$

$$[S] = \frac{1}{2} \int_{A_i} \rho [R]^T [T]^T [T] [R] \, t \, dA \qquad (2.24)$$

$$[T^1] = \begin{bmatrix} [T] & [0] \\ [0] & [T] \end{bmatrix} \qquad (2.25)$$

A_i : element surface area

2.5 Global Mass and Stiffness Matrices

Equation (2.22) relates to a finite element. If $\{\delta\}$ designates the degrees of freedom vector for all the nodes, then shell motion is governed by an analogous equation which we would write:

$$[M_o] \{\ddot{\delta}\} + [K_o] \{\delta\} = \{F\} \qquad (2.26)$$

$[M_o]$ and $[K_o]$ are the mass and stiffness matrices for the complete structure. They are obtained by assembling elementary matrices, $[m_o]$ and $[k_o]$, so as to satisfy the following two conditions:

- Continuity of nodal displacements from one element to the next, such that:

$$\{\delta_{i+1}\} = \{\delta_j\} \qquad (2.27)$$

- The external forces and moments applied at a given node must be equal to the internal forces and moments, respectively, such that:

$$\left\{f^{(e)}\right\} = \left\{f_j\right\} + \left\{f_{i+1}\right\} \qquad (2.28)$$

The matrices are assembled by overlay procedure, as indicated in Figure 2. If N is the number of finite elements, $[M_o]$ and $[K_o]$ are two symmetric and semi-finite square matrices of order 4(N+1). They are also two banded matrices, having a half-band width equal to 8.

CHAPTER III

NATURAL VIBRATIONS OF CONICAL SHELLS CONTAINING FLOWING FLUID

3.1 Introduction

A shell containing flowing liquid was subjected to inertial, centrifugal and Coriolis forces coupled with elastic deformations of the shell. The mathematical model we chose to represent this system is as follows:

$$[[M_o]-[M_f]]\{\ddot{\delta}\} - [C_f]\{\dot{\delta}\} + [[K_o]-[K_f]]\{\delta\} = \{0\} \qquad (3.1)$$

where $\{\delta\}$ is a displacement vector. $[M_o]$ and $[K_o]$ are the mass matrix and stiffness matrix, respectively, for the empty system and $[M_f]$, $[K_f]$ and $[C_f]$ represent the inertial, Coriolis and centrifugal forces of the flow.

$[M_o]$ and $[K_o]$ were calculated in Chapter 2 and in reference [12].

3.2 Hypotheses

We considered the shell to be subjected to only potential flow, inducing inertial, Coriolis and centrifugal forces contributing to the structure vibration.

The mathematical model which was developed is based on the following hypotheses:

a) Fluid flow is potential
b) The fluid is incompressible.
c) Fluid pressure on the wall is purely lateral.
d) The fluid velocity distribution is assumed constant across a shell section.
e) Internal pressure is not unduly high.
f) The shell apex half-angle may not be equal to 0 degrees (singular point).
g) Linear theory (small deformations).
h) There is no separation between the fluid and the shell wall.

3.3 Definitions

At any given moment, the flow is constant across all sections of the cone. This flow is equal to the fluid velocity multiplied by the flow area. The flow area is proportionate to the square of the cone radius or to the square of the x coordinate multiplied by the sine of the cone apex half-angle. Thus, for a given cone, average fluid velocity, U, multiplied by the square of the x coordinate is equal to a constant at any point inside the shell which we shall call Q (see Figure 3).

$$Q = U \cdot x^2 = cte$$

Thus,
$$U = \frac{Q}{x^2} \qquad (3.2)$$

a is the angle locally defining the inner wall of the shell.

$$a = a_c - \tan^{-1} \left(\frac{t}{2x}\right) \qquad (3.3)$$

where t is the shell thickness and a_c the half-angle at the cone apex.

3.4 Equation for Velocity Potential

Using the Bernoulli equation, the continuity equation and the Euler equation, we obtain the following equation for the velocity potential [13]:

$$\nabla^2 \varphi = \frac{1}{c^2} \left\{ \frac{\partial^2 \varphi}{\partial t^2} + 2\, \nabla\varphi \cdot \nabla \frac{\partial \varphi}{\partial t} + \nabla\varphi \cdot (\nabla\varphi \cdot \nabla)\, \nabla\varphi \right\} \qquad (3.4)$$

where c is the speed of sound in the fluid and φ is the potential function representing potential velocity.

We also have [14]:

$$V = \nabla\varphi \tag{3.5}$$

$$V_x = \frac{Q}{x^2} + \frac{\partial\varphi}{\partial x} \quad ; \quad V_\theta = \frac{1}{x \sin\alpha}\frac{\partial\varphi}{\partial\theta} \quad ; \quad V_\alpha = \frac{1}{x}\frac{\partial\varphi}{\partial\alpha} \tag{3.6}$$

where V_x, V_θ and V_α are the x, θ and α components of the fluid velocity.

If we neglect $\frac{\partial\varphi}{\partial x}$, $\frac{\partial\varphi}{\partial\theta}$ and $\frac{\partial\varphi}{\partial\alpha}$, which are perturbation terms, and substitute (3.5) and (3.6) into (3.4), we obtain:

$$\nabla^2\varphi = \frac{1}{c^2}\left\{\frac{\partial^2\varphi}{\partial t^2} + \frac{2Q}{x^2}\frac{\partial^2\varphi}{\partial x\partial t} + \frac{Q^2}{x^4}\frac{\partial^2\varphi}{\partial x^2}\right\} \tag{3.7}$$

In addition, we know that for a system of conical coordinates, the Laplace is equal to [15]:

$$\nabla^2\varphi = \frac{2}{x}\frac{\partial\varphi}{\partial x} + \frac{\partial^2\varphi}{\partial x^2}\frac{1}{x^2\sin^2\alpha}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{x^2\tan\alpha}\frac{\partial\varphi}{\partial\alpha} + \frac{1}{x^2}\frac{\partial^2\varphi}{\partial x^2} \tag{3.8}$$

3.5 Pressure exerted by the Fluid

From the Bernoulli equation, we obtain the pressure exerted on the inner wall by the fluid:

$$p = - \rho_\ell \left\{ \frac{\partial \varphi}{\partial t} + \frac{Q}{2}_x \frac{\partial \varphi}{\partial x} \right\}_{\alpha=a} \qquad (3.9)$$

where a is the half-angle of the shell's inner wall opening.

Furthermore, the condition:

$$(V_\alpha)_{\alpha=a} = \frac{1}{x} \frac{\partial \varphi}{\partial \alpha} \Bigg|_{\alpha=a} = (\dot{w} + \frac{Q}{2}_x w')_{\alpha=a} \qquad (3.10)$$

must be satisfied for there to be permanent contact between the shell surface and the peripheral fluid layer.

Assuming that the form of the displacement functions is given by equation (2.17), we get:

$$w(x,\theta,t) = \sum_{q=1}^{8} C_q \, y^{\lambda_q - 1} \cos n\theta \, e^{i\omega t} \qquad (3.11)$$

$$y = \sqrt{\frac{x}{\ell}}$$

Setting:

$$\varphi(x,\ \theta,\ \alpha,t) = \sum_{q=1}^{8} R_q(\alpha) \qquad S_q(x,\ \theta,\ t) \qquad (3.12)$$

Substituting (3.11) and (3.12) into (3.10), we obtain:

$$S_q(x,\ \theta,\ t) = \frac{x}{R_q^{'}(a)} \left(\dot{w} + \frac{Q}{x^2} w^{'}\right)_{\alpha=a} \qquad (3.13)$$

and equation (3.12) becomes:

$$\varphi(x,\ \theta,\ \alpha,\ t) = \sum_{q=1}^{8} x \frac{R_q(\alpha)}{R_q^{'}(a)} \left(\dot{w} + \frac{Q}{x^2} w^{'}\right)_{\alpha=a} \qquad (3.14)$$

Using relations (3.7) and (3.8) for incompressible fluid ($c \rightarrow \infty$) and solving for a constant x, we obtain, for each q value:

$$R_q^{''}(\alpha) + \frac{1}{\tan\alpha} R_q^{'}(\alpha) - \frac{n^2}{\sin^2\alpha} R_q(\alpha) = 0 \qquad (3.15)$$

By means of the Frobenius method [15] and knowing that we are dealing with a case of internal flow, we obtain the following truncated solution:

$$R_q(\alpha) = A\,\alpha^n \left\{ 1 + \frac{n}{12}\alpha^2 + \frac{(5n+7)n}{1440}\alpha^4 + \frac{n(n+4)\ (5n+1)}{51840}\alpha^6 \right\} \qquad (3.16)$$

Substituting into (3.14):

$$\varphi(x,\ \theta,\ \alpha,\ t) = \sum_{q=1}^{8} k \quad x(\dot{w} + \frac{Q}{x^2} \ w')_{\alpha=a} \qquad (3.17)$$

where

$$k = \frac{a}{n} \, \frac{1 + \frac{n}{12} a^2 + \frac{(5n+7)}{1440} n \, a^4 + \frac{n(n+4)\ (5n+1)}{51840} a^6}{1 + \frac{(n+2)}{12} a^2 + \frac{(n+4)\ (5n+7)}{1440} a^4 + \frac{(n+4)\ (n+6)\ (5n+1)}{51840} a^6}$$

From (3.9) we obtain:

$$p = -\rho\ell \ k \sum_{q=1}^{8} \left\{ x \ \ddot{w}_q + \frac{Q}{x^2} \lambda_q \ \dot{w}_q + \frac{Q^2}{x^5} \frac{(\lambda_q^2 - 6\lambda_q + 5)}{4} \ w_q \right\} \qquad (3.18)$$

3.6 Inertia, Coriolis and Centrifugal Matrices

Introducing equation (2.17) into equation (3.18), performing the matrix operations required in the finite element method and integrating with respect to x and θ, we obtain:

$$[m_f] = [A_f^{-1}]^T \ [S_f] \ [A_f^{-1}] \qquad (3.19)$$

$$[c_f] = [A_f^{-1}]^T \ [D_f] \ [A_f^{-1}] \qquad (3.20)$$

and

$$[k_f] = [A_f^{-1}]^T [G_f] [A_f^{-1}] \qquad (3.21)$$

which are, respectively, the inertia, Coriolis and centrifugal matrices for an element.

For $\lambda_m + \lambda_n = -4$:

$$S_F(m, n) = -\pi \rho \ell k \sin a \ell^3 \ell n \left(\frac{\ell j}{\ell i}\right) \qquad (3.22)$$

where ℓi and ℓj are the value of the x coordinate at nodes i and j.

For $\lambda_m + \lambda_n \neq -4$:

$$S_F(m,n) = -\frac{2\pi \rho \ell k \sin a}{(\lambda m + \lambda n + 4)} \left\{ e^{\left[\left(\frac{\lambda m+\lambda n+4}{2}\right)\ell n \ell j - \left(\frac{\lambda m+\lambda n-2}{2}\right)\ell n \ell\right]} - e^{\left[\left(\frac{\lambda m+\lambda n+4}{2}\right)\ell n \ell i - \left(\frac{\lambda m+\lambda n-2}{2}\right)\ell n \ell\right]} \right\} \qquad (3.23)$$

For $\lambda_m + \lambda_n = 2$:

$$D_F(m, n) = -\pi \rho \ell k \sin a \Omega \lambda n \ell n \left(\frac{\ell j}{\ell i}\right) \qquad (3.24)$$

For $\lambda_m + \lambda_n \neq 2$:

$$D_F(m,n) = -\frac{2\pi \ \rho\ell \ k \ \sin a \ D \ \lambda n}{(\lambda m + \lambda n - 2)} \left\{ e^{\left[\left(\frac{\lambda m + \lambda n - 2}{2}\right) \ell n \left(\frac{\ell j}{\ell i}\right)\right]} - e^{\left[\left(\frac{\lambda m + \lambda n - 2}{2}\right) \ell n \left(\frac{\ell i}{\ell j}\right)\right]} \right\} \qquad (3.25)$$

For $\lambda_m + \lambda_n = 8$:

$$G_F(m, n) = -\frac{\pi}{4} \rho\ell \ k \ \sin a \ \frac{D^2}{\ell^3} (\lambda m^2 - 6\lambda m + 5) \ \ell n \left(\frac{\ell j}{\ell i}\right) \qquad (3.26)$$

For $\lambda_m + \lambda_n \neq 8$:

$$G_F(m,n) = -\frac{\pi}{2} \rho\ell \ k \ \sin a \ D^2 \left(\frac{\lambda m \quad - 6\lambda m + 5}{\lambda m + \lambda n - 8}\right)$$

$$* \left\{ e^{\left[\left(\frac{\lambda m + \lambda n - 8}{2}\right) \ell n \ (\ell j) - \left(\frac{\lambda m + \lambda n - 2}{2}\right) \ell n \ (\ell)\right]} - e^{\left[\left(\frac{\lambda m + \lambda n - 8}{2}\right) \ell n \ (\ell i) - \left(\frac{\lambda m + \lambda n - 2}{2}\right) \ell n \ (\ell)\right]} \right\} \qquad (3.27)$$

Matrices $[M_F]$, $[C_F]$ and $[K_F]$ are, respectively, the apparent mass, stiffness and damping matrices of the fluid column, which are obtained by superimposing the mass $[m_f]$, stiffness $[k_f]$ and damping $[c_f]$ matrices for each individual fluid column element (see Figure 4).

3.7 Eigenvalue and Eigenvector Problem

The eigenvalue and eigenvector problem is solved by means of the equation reduction technique. Equation (3.1) is therefore rewritten as follows:

$$\begin{bmatrix} [0] & \frac{1}{\omega_o}[M] \\ \frac{1}{\omega_o^2}[M] & \frac{1}{\omega_o}[C] \end{bmatrix} \begin{Bmatrix} \ddot{\delta} \\ \dot{\delta} \end{Bmatrix} + \begin{bmatrix} -\frac{1}{\omega_o}[M] & [0] \\ [0] & [K] \end{bmatrix} \begin{Bmatrix} \dot{\delta} \\ \delta \end{Bmatrix} = 0 \qquad (3.28)$$

where

$$[M] = [M_o] - [M_F]$$

$$[K] = [K_o] - [K_F] \qquad (3.29)$$

$$[C] = [C_F]$$

$$\omega_o = P\ (1,1,1)$$

and the eigenvalue problem will be given by:

$$|[DD] - \Lambda\,[I]| = 0 \qquad (3.30)$$

where

$$[DD] = \begin{bmatrix} [0] & [I] \\ -\dfrac{1}{\omega_o^2}[K]^{-1}[M] & -\dfrac{1}{\omega_o}[K]^{-1}[C] \end{bmatrix} \qquad (3.31)$$

$$\wedge = \frac{1}{\omega_o^2 \, \omega^2}$$

where ω is the natural frequency of the system.

<u>Special Case</u>:

If the velocity of the fluid inside the shell is zero (Q = 0), the eigenvalue and eigenvector problem reduces to:

$$\left| \frac{1}{\omega_o^2} K^{-1} M - \wedge [I] \right| = 0 \qquad (3.32)$$

and $\omega \, (rad/p) = \dfrac{1}{\omega_o \quad \wedge}$

Matrices [K], [M] and [C] are square matrices of order NDF (N+1) where NDF is the number of degrees of freedom at each node and N is the number of finite elements in the shell.

CHAPTER IV

THE ALGORITHM

4.1 Introduction

The non-uniform conical shell was subdivided into a sufficient number of conical finite elements. The calculations for each finite element were performed in two stages, the first dealing with the shell itself and the second with the effect of the fluid contained by the shell.

4.2 Program Organization

4.2.1 The input included:

i) the number of finite elements

ii) the geometry of each finite element: length, apex angle and coefficient.

iii) the mechanical properties of each finite element: Young's modulus, Poisson's ratio and density

iv) harmonic number n

v) boundary conditions

vi) characteristics of the fluid: density, velocity multiplied by (x coordinate)^2.

4.4.2 Processing stages:

a) Compute the shell's mass and stiffness matrices [M_o] and [K_o].

For the liquid component:

i) Compute matrices [S_f], [G_f] and [D_f], which are given by equations (3.22) to (3.27).

ii) Determine matrices [M_f], [K_f] and [C_f], which are given by equations (3.19) to (3.22).

iii) Overlay these matrices onto the mass and stiffness matrices for the empty shell.

iv) The frequencies and principal modes are obtained by solving equation system (3.28), where [K], [M] and [C] are square matrices of order [NDF (N+1)]. The two HSVEC and HESSEIN subroutines from IMSL do the calculations.

CHAPTER V

RESULTS AND DISCUSSION

5.1 Introduction

The articles we found in the literature which deal with conical shells interacting with a fluid [1,2 and 3] provide no results against which we could test our own findings. Our present calculations, though, were based on the same foundations as were used in designing the program for a cylinder partially filled with liquid or containing flowing liquid. Since this program produced results which match existing experience, it stands in firm support of our results.

In this chapter, therefore, we shall strive to set forth the general rules for the "designer" wishing to know the effect of variations in the shell apex angle, shell length and amount of liquid in the shell on the natural frequencies of a conical shell containing fluid.

5.2 Convergence

We started by looking at the behaviour of the solution as a function of the number of finite elements used to model the structure, that is, as a function of mesh fineness.

The characteristics of the shell under study were:

α = 14,2°

x_1 = 0,141 m

$x_2 - x_1$ = 0,174 m

t = $2{,}56 \times 10^{-4}$ m

ρ = 7800 kg/m^3

E = 200×10^9 Pa

ν = 0,3

The shell was filled with a liquid having zero average velocity and a density of:

$$\rho_\ell = 1000 \text{ kg/m}^3$$

and the conditions at the two boundaries were:

$$w = v = 0.$$

For n = 5, m = 1, we obtained the results below:

number of finite elements	2	3	5	8	10
frequency	$0,708 \times 10^3$	$0,683 \times 10^3$	$0,680 \times 10^3$	$0,610 \times 10^3$	$0,610 \times 10^3$

We would later be using 8 finite elements.

5.3 Free Vibrations of Simply Supported Shells

In Figure 5, we see the behaviour of the same shell as a function of the number of circumferential modes for the shell when empty and full.

For an m equal to 1 and 2, the curve drops on the average to a third of its value when the liquid is added but its general shape does not change. This was to be expected since by filling the shell with liquid, its mass is increased accordingly. For this case, the lowest natural frequency corresponds to n = 5 , m = 1.

If we check the relative amplitudes of the shell motion, it will be seen that as in the empty conical shell, it is the radial motion that predominates. For example, when the cone described previously is filled with liquid, the following relative amplitudes are obtained:

	n=5	n=5
	m=1	m=2
U_{max}/W_{max}	0,049325	0,038429
V_{max}/W_{max}	0,196139	0,201568

Figure 6 shows that when the shell is partially filled, the frequency begins by dropping sharply then, as the shell becomes fuller, the frequency drops less quickly.

The effect is quite different depending on what side of the shell the fluid is in. This is understandable if we look at the radial displacement amplitude (see Figures 7 to 10), which is the predominant form. For m = 1, when the shell is filled according to b_1, the liquid mass is nearer the region where radial displacement amplitude is maximum (antinode) than when filling occurs according to b_2. For m = 2, the reverse is true.

Figures 11 and 12 give the normalized shell displacements for

$n = 5$, $m = 1, 2$ and $W = V = 0$.

If we analyze the natural frequencies of the same shell and vary its apex angle α, we see that there is a drop in frequency corresponding to an increase in α angle (see Figure 3). The greater α becomes, the higher the n the lowest frequency corresponds to. By increasing α, we are increasing the amount of liquid inside the shell and the ratio of shell thickness to average radius decreases (see Figure 14). For $\alpha=10°$, the effect of the fluid is maximum at $n=2$. For higher α values, the fluid will lower the natural frequencies further and its effect will reach maximum at a higher n value. For example, for $\alpha=30°$, the fluid effect peaks at $n = 5$.

If, instead of varying the angle, we increase x_2, that is, if we elongate the shell, the frequency goes down here again (see Figure 15). Figure 16 shows that when x_2 is increased, the fluid has a greater influence on the natural frequencies, and the internal liquid volume and the average shell length-to-radius ratio both increase.

In conclusion, the fluid lowers the shell's resonant frequency and it is therefore unwarranted to ignore this effect.

5.4 Processing Time

The computer program was run on a Cyber 855 in the University of Montreal Computer Centre. This CDC computer makes possible an internal representation of the numbers in 60-bit floating-point mode with single precision (48 bits for the mantissa, 11 for the exponent and 1 for the sign).

Below is a comparison of running times and memory space required to process the case of an empty shell versus a liquid-filled shell (shell having the same characteristics as the one used in the test of convergence).

	CPU time (sec)	memory space (bytes)	cost (Can $)
empty shell	46	144800	19
full shell	67	154500	29

The increase in time and memory needed to handle the fluid-filled case does not seem too serious when all the potential advantages are considered.

We would mention that the program was written for compatibility with other programs dealing with cylindrical and spherical shells. A complete restructuring of these programs would upgrade their performance.

CHAPTER VI

CONCLUSIONS

The method described in this report allows for predicting the effect that a stationary or flowing fluid will have on the dynamic behaviour of an anisotropic conical shell. To solve the equations, we used conical finite elements, enabling us to derive the displacement functions from the equations of motion for the shell.

This method has already been used for cylindrical shells partially filled with liquid or containing flowing fluid ([8], [9]) and for an empty anisotropic conical shell [12]. In both these cases, the results we obtained matched the experimental results. We are justified therefore in considering our method to be appropriate for predicting the vibration characteristics of anisotropic conical shells partially filled with liquid or containing flowing fluid.

We have, in the foregoing pages, demonstrated the consequences of varying the cone apex angle, elongating the shell and partially filling the shell with liquid on the dynamic behaviour of the shell.

There still remains a great deal to be done in the investigation of shells. An interesting follow-up to this work would be to place conical, cylindrical, hemispherical, flat, etc., finite elements end to end, in order to simulate the behaviour of any arbitrary shape of shell.

REFERENCES

[1] DOKUCHAEV, L.V., "On the equation of elastic vibrations of a cavity partially filled with fluid", Mekhanika, No. 3, 1965, pp. 149-153.

[2] SHKLYARCHUK, F.N., "About an approximate method of calculating axially symmetrical oscillations of shells of rotation with liquid filling", Mekhanika, No. 6, 1965, pp. 123-129.

[3] STEPANYUK, V.V. and BABIEN, D.V., "Vibrations and stability of a conical triple-layered shell with fluid flow in the middle layer", Investigations of Applied Hydrodynamics, Izd-Vo Naukova Dumka, Kiev, 1965, pp. 15-19.

[4] NIORDSON, F.I.N., "Vibration of a cylindrical tube containing flowing fluid", Transactions of the Royal Institute of Technology, Stockholm, Sweden, No. 73, 1953.

[5] BERRY, J.G. and REISSNER, E., "The effect of an internal compressible fluid column on the breathing vibrations of thin pressurized cylindrical shells", Journal of the Aerospace Sciences, 25, 288, 1958.

[6] LINDHOLM, U.S., KANA, D.D. and ABRAMSON, H.N., "Breathing vibrations of a circular cylindrical shell with an internal liquid", Journal of the Aerospace Sciences, 29, 1052, 1962.

[7] COHEN, GERALD, A., "Computer analysis of asymmetric free vibrations of ring stiffened orthotropic shells of revolution", AIAAJ, Vol. 3, No. 12, Dec. 1965, pp. 2305-2312.

[8] LAKIS, A.A. and PAIDOUSSIS, M.P., "Dynamic analysis of axially non-uniform thin cylindrical shells", J. Mech. Eng. Sci., Vol. 14, No. 1, pp. 49-71, 1972.

[9] LAKIS, A.A., "Effects of internal and external flow on the vibration characteristics of anisotropic cylindrical shells", rapport technique No EP77-R-11, École Polytechnique, Montréal, Février 1977.

[10] LAVEAU, A., "Analyse dynamique non linéaire des coques cylindriques anisotropes contenant un fluide", Mémoire de M.Sc.A., École Polytechnique de Montréal, Département de génie mécanique, avril 1985.

[11] LAKIS, A.A. and PAIDOUSSIS, M.P., "Shell natural frequencies of the pickering steam generator", Atomic Energy of Canada Ltd., AECL Report No 4362, 1973.

[12] OURICHE, H., "Analyse dynamique des coques coniques anisotropes", Mémoire de M.Sc.A., École Polytechnique de Montréal, Département de génie mécanique, Avril 1984.

[13] OWCZAREK, JERZY A., "Fundamentals of Gas Dynamics", International Textbook Company, Scranton, Pennsylvania, 1964.

[14] LIEPMANN, H.W. and ROSHKO, A., "Elements of gas dynamics", Wiley, New York, 1957.

[15] KORN, G.A. and KORN, T.M., "Mathematical Handbook for Scientists and Engineers", second edition, McGraw-Hill Book Company, New York, 1968.

[16] MEIROVITCH, LEONARD, "Analytical Methods in Vibrations", The MacMillan Company, Collier-MacMillan Limited, London, 1967.

[17] STREETER, VICTOR L., WYLIE, E. BENJAMIN, "Fluid Mechanics, seventh edition, McGraw-Hill Book Company, New York, 1979.

[18] LAKIS, A.A., "Effects of fluid pressures on the Vibration Characteristic of Cylindrical Vessels", Second International Conference on pressure Surge organized by BHRA Fluid Engineering, Cranfield, Bedford, England, Sept. 22-24, 1976.

[19] ABRAMSON, NORMAN H., "The dynamic behavior of Liquids in moving containers", NASA SP-106, 1966.

[20] ZILLIACUS, S., "Fluid-structure interaction and ADINA", Computer & Structures, vol. 17, no 5-6, pp. 763-773, 1983.

[21] SUNDQVIST, Jan, "An application of ADINA to the solution of fluid-structure interaction problems, vol. 17, no 5-6, pp. 793-807, 1983.

APPENDIX A

APPENDIX A-1

Sanders' Thin Shell Equations

a) General equations of equilibrium

There is an abundant literature on developments of equations dealing with the static equilibrium of thin shells. Here we shall limit ourselves strictly to the five final equations of motion given by Sanders in the form (Figure 1).

$$\frac{\partial A_2 N_1}{\partial \zeta_1} + \frac{\partial A_1 \bar{N}_{12}}{\partial \zeta_2} + \bar{N}_{12} \frac{\partial A_1}{\partial \zeta_2} - N_2 \frac{\partial A_2}{\partial \zeta_1} + \frac{A_1 A_2}{R_1} Q_1 + \frac{A_1}{2} \frac{\partial}{\partial \zeta_2} \left[\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \bar{M}_{12}\right] = 0 \qquad \text{(a)}$$

$$\frac{\partial A_2 \bar{N}_{12}}{\partial \zeta_1} + \frac{\partial A_1 N_2}{\partial \zeta_2} + \frac{\partial A_2}{\partial \zeta_1} \bar{N}_{12} - \frac{\partial A_1}{\partial \zeta_2} N_1 + \frac{A_1 A_2}{R_2} Q_2 + \frac{A_2}{2} \frac{\partial}{\partial \zeta_1} \left[\left(\frac{1}{R_2} - \frac{1}{R_1}\right) \bar{M}_{12}\right] = 0 \qquad \text{(b)}$$

$$\frac{\partial A_2 Q_1}{\partial \zeta_1} + \frac{\partial A_1 Q_2}{\partial \zeta_2} - \left(\frac{N_1}{R_1} + \frac{N_2}{R_2}\right) A_1 A_2 = 0 \qquad \text{(c)} \quad \text{(A-1.1)}$$

$$\frac{\partial A_2 M_1}{\partial \zeta_1} + \frac{\partial A_1 \bar{M}_{12}}{\partial \zeta_2} + \bar{M}_{12} \frac{\partial A_1}{\partial \zeta_2} - M_2 \frac{\partial A_2}{\partial \zeta_1} - A_1 A_2 Q_1 = 0 \qquad \text{(d)}$$

$$\frac{\partial A_2 \bar{M}_{12}}{\partial \zeta_1} + \frac{\partial A_1 M_2}{\partial \zeta_2} + \bar{M}_{12} \frac{\partial A_2}{\partial \zeta_1} - M_1 \frac{\partial A_1}{\partial \zeta_2} - A_1 A_2 Q_2 = 0 \qquad \text{(e)}$$

with
$$\begin{cases} \bar{N}_{12} = 1/2\,(N_{12} + N_{21}) \\ \bar{M}_{12} = 1/2\,(M_{12} + M_{21}) \end{cases}$$

b) Deformation vector

Beside the equilibrium equations, there is a second group of equations determining the state of constraints in the shell, the law of elasticity. To that purpose we shall be using deformation vector $\{\epsilon\}$ which is given by:

$$\epsilon_1 = \frac{1}{A_1} \frac{\partial U_1}{\partial \zeta_1} + \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \zeta_2} U_2 + \frac{W}{R_1}$$

$$\varepsilon_2 = \frac{1}{A_2}\frac{\partial U_2}{\partial \zeta_2} + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \zeta_1} U_1 + \frac{W}{R_2}$$

$$\bar{\varepsilon}_{12} = \frac{1}{2A_1A_2}\left(A_2 \frac{\partial U_2}{\partial \zeta_1} + A_1 \frac{\partial U_1}{\partial \zeta_2} - \frac{\partial A_1}{\partial \zeta_2} U_1 - \frac{\partial A_2}{\partial \zeta_1} U_2\right) \tag{A-1.2}$$

$$\kappa_1 = \frac{1}{A_1}\frac{\partial \beta_1}{\partial \zeta_1} + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \zeta_2} \beta_2$$

$$\kappa_2 = \frac{1}{A_2}\frac{\partial \beta_2}{\partial \zeta_2} + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \zeta_1} \beta_1$$

$$\bar{\kappa}_{12} = \frac{1}{2A_1A_2}\left[A_2 \frac{\partial \beta_2}{\partial \zeta_1} + A_1 \frac{\partial \beta_1}{\partial \zeta_2} - \frac{\partial A_1}{\partial \zeta_2} \beta_1 - \frac{\partial A_2}{\partial \zeta 1} \beta_2 + \frac{1}{2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\left(\frac{\partial A_2 U_2}{\partial \zeta_1} - \frac{\partial A_1 U_1}{\partial \zeta_2}\right)\right]$$

with
$$\begin{cases} \beta_1 = \dfrac{U_1}{R_1} - \dfrac{1}{A_1}\dfrac{\partial W}{\partial \zeta_1} \\[2ex] \beta_2 = \dfrac{U_2}{R_2} - \dfrac{1}{A_2}\dfrac{\partial W}{\partial \zeta_2} \end{cases}$$

c) Boundary conditions

The boundary conditions are given by:

$$\frac{M_1}{R_1} + N_1 = \bar{\bar{N}}_1 \qquad \text{or} \qquad U_1 = \bar{\bar{U}}_1$$

$$\bar{N}_{12} + \left(\frac{3}{R_2} - \frac{1}{R_1}\right)\frac{\bar{M}_{12}}{2} = \bar{\bar{T}}_{12} \qquad \text{or} \qquad U_2 = \bar{\bar{U}}_2 \tag{A-1.3}$$

$$Q_1 + \frac{1}{A_2}\frac{\partial \bar{M}_{12}}{\partial \zeta_2} = \bar{\bar{V}}_1 \qquad \text{or} \qquad W = \bar{\bar{W}}$$

$$M_1 = \bar{\bar{M}}_1 \qquad \text{or} \qquad \frac{\partial W}{\partial \zeta_1} = \frac{\partial \bar{\bar{W}}}{\partial \zeta_1}$$

for a boundary with constant ζ_1 where the double-barred quantities correspond to the boundary values.

d) Parameters for a conical shell of revolution (Figs. 2 and 3)

We have quantities:

$$\zeta_1 = x \qquad A_1 = r_\phi = \infty \qquad R_1 = r_\phi = \infty \qquad U_1 = U$$

$$\zeta_2 = \theta \qquad A_2 = r \qquad R_2 = r_\theta \qquad U_2 = V \qquad W = W \tag{A-1.4}$$

with $\lim_{r_\phi \to \infty} (r_\phi \, d\phi) = dx$

and $\dfrac{dr}{d\phi} = r_\phi \cos \phi$

Carrying these parameters over into the five equilibrium equations (A-1.1), we obtain:

$$\frac{\partial(r\,Nx)}{\partial x} + \frac{\partial \overline{N}x\theta}{\partial \theta} - N\theta \frac{\partial r}{\partial x} - \frac{1}{2}\frac{\partial}{\partial \theta}\left(\frac{\overline{M}_{x\theta}}{r_\theta}\right) = 0 \tag{a}$$

$$\frac{\partial(r\,\overline{N}_{x\theta})}{\partial x} + \frac{\partial N\theta}{\partial \theta} + \overline{N}x\theta \frac{\partial r}{\partial x} + \frac{r}{r_\theta} Q_\theta + \frac{r}{2}\frac{\partial}{\partial x}\left(\frac{\overline{M}_{x\theta}}{r_\theta}\right) = 0 \tag{b}$$

$$\frac{\partial(r\,Q_x)}{\partial x} + \frac{\partial Q_\theta}{\partial \theta} - \frac{N_\theta}{r_\theta} r = 0 \tag{c}$$

(A-1.5)

$$\frac{\partial(r\,M_x)}{\partial x} + \frac{\partial \overline{M}_{x\theta}}{\partial x} - M_\theta \frac{\partial r}{\partial x} - r\,Q_x = 0 \tag{d}$$

$$\frac{\partial(r\,\overline{M}_{x\theta})}{\partial x} + \frac{\partial M_\theta}{\partial \theta} + \overline{M}_{x\theta} \frac{\partial r}{\partial x} - r\,Q_\theta = 0 \tag{e}$$

APPENDIX A-2

MATRICES

This appendix contains the matrices referenced in Chapter 2 which relate to empty shells.

These matrices are classified as follows:

[D]	(table 1)
[R]; [R_R]; [T]	(table 2)
[Q]; [Q_R]	(table 3)
[P]	(table 4)

TABLE 1

Matrix $\underset{3x3}{[D]}$

$$\underset{3x3}{[D]} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \{0\} \qquad ; [D] = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}$$

with:

$$d_{11} = A\lambda^2 + C$$

$$d_{12} = R\lambda + T$$

$$d_{13} = F\lambda^3 + G\lambda^2 + H\lambda + J$$

$$d_{21} = -R\lambda + T$$

$$d_{22} = Y\lambda^2 + L$$

$$d_{23} = M\lambda^2 + N\lambda + P$$

$$d_{31} = -F\lambda^3 + G\lambda^2 - H\lambda + J$$

$$d_{32} = M\lambda^2 - N\lambda + P$$

$$d_{33} = Q\lambda^4 + S\lambda^2 + Z$$

and:

$$A = \frac{B_{11}}{4} \sin\alpha$$

$$C = B_{12} \sin\alpha + n^2 B_{36} \frac{\cot\alpha}{\sin\alpha} - \frac{n^2 B_{33}}{\sin\alpha} - B_{22} \sin\alpha - n^2 B_{66} \frac{\cot^2\alpha}{4\sin\alpha} - \frac{B_{11}}{4} \sin\alpha$$

$$R = \frac{n}{2} B_{12} + \frac{n}{2} B_{15} \cot\alpha + \frac{n}{2} B_{33} + \frac{n}{2} B_{36} \cot\alpha - \frac{3n}{8} B_{66} \cot^2\alpha$$

$$T = \frac{n}{2} B_{12} + \frac{n}{2} B_{15} \cot \alpha - n B_{22} - n B_{25} \cot \alpha - \frac{3n}{2} B_{33} - \frac{3n}{2} B_{36} \cot \alpha$$

$$+ \frac{9n}{8} B_{66} \cot^2 \alpha$$

$$F = -\frac{B_{41}}{8} \sin \alpha$$

$$G = \frac{3}{8} B_{41} \sin \alpha - \frac{B_{51}}{4} \sin \alpha + \frac{B_{42}}{4} \sin \alpha$$

$$H = \frac{B_{21}}{2} \cos \alpha + \frac{n^2 B_{51}}{2 \sin \alpha} + \frac{n^2 B_{63}}{\sin \alpha} + \frac{B_{52}}{2} \sin \alpha - \frac{n^2 B_{66} \cot \alpha}{2 \sin \alpha} - B_{42} \sin \alpha$$

$$+ \frac{B_{41}}{8} \sin \alpha$$

$$J = \frac{B_{12}}{2} \cos \alpha + \frac{n^2 B_{51}}{2 \sin \alpha} + B_{51} \frac{\sin \alpha}{4} - B_{22} \cos \alpha - \frac{n^2 B_{52}}{\sin \alpha} - B_{52} \frac{\sin \alpha}{2}$$

$$- \frac{3}{8} B_{41} \sin \alpha + \frac{3}{4} B_{42} \sin \alpha - \frac{3 n^2}{\sin \alpha} B_{63} + 3n^2 B_{66} \frac{\cot \alpha}{2 \sin \alpha}$$

$$Y = \frac{B_{33}}{4} \sin \alpha + \frac{3}{4} B_{36} \sin \alpha + \frac{9}{16} B_{66} \cos \alpha \cot \alpha$$

$$L = \frac{-9}{4} B_{33} \sin \alpha - \frac{27}{4} B_{36} \cos \alpha - \frac{81}{16} B_{66} \cos \alpha \cot \alpha - 2n^2 B_{52} \frac{\cot \alpha}{\sin \alpha}$$

$$- n^2 B_{55} \frac{\cot^2 \alpha}{\sin \alpha} - \frac{n^2 B_{22}}{\sin \alpha}$$

$$M = \frac{n}{4} B_{24} + \frac{n}{2} B_{36} + \frac{n}{4} B_{54} \cot \alpha + \frac{3n}{4} B_{66} \cot \alpha$$

$$N = -n B_{24} + \frac{n}{2} B_{25} - n B_{54} \cot \alpha + \frac{n}{2} B_{55} \cot \alpha$$

$$P = -n B_{22} \cot \alpha - \frac{n^3 B_{25}}{\sin^2 \alpha} - \frac{n}{2} B_{25} - n B_{52} \cot^2 \alpha + \frac{3n}{4} B_{24} - \frac{9n}{2} B_{36}$$

$$+ \frac{3n}{4} B_{54} \cot \alpha - n^3 B_{55} \frac{\cot \alpha}{\sin^2 \alpha} - \frac{n}{2} B_{55} \cot \alpha - \frac{27n}{4} B_{66} \cot \alpha$$

$$Q = -B_{44} \frac{\sin \alpha}{16}$$

$$S = B_{42}\frac{\cos\alpha}{2} - \frac{3}{4}B_{45}\sin\alpha + \frac{n^2 B_{66}}{\sin\alpha} + \frac{n^2 B_{54}}{\sin\alpha} + B_{55}\frac{\sin\alpha}{4} + \frac{5}{8}B_{44}\sin\alpha$$

$$Z = -\frac{9n^2}{\sin\alpha}B_{66} - 2n^2 B_{25}\frac{\cot\alpha}{\sin\alpha} - B_{25}\cos\alpha - \frac{n^2 B_{55}}{\sin\alpha} - \frac{n^4 B_{55}}{\sin^3\alpha} - \frac{B_{55}}{4}\sin\alpha$$

$$- B_{22}\cos\alpha\cot\alpha + \frac{3}{2}B_{42}\cos\alpha + \frac{3n^2}{2\sin\alpha}B_{54} - \frac{9}{16}B_{44}\sin\alpha + \frac{3}{4}B_{45}\sin\alpha$$

The characteristic equation (3.5) is of the form:

$$h_8\lambda^8 + h_6\lambda^6 + h_4\lambda^4 + h_2\lambda^2 + h_0 = 0$$

where

$$h_8 = AYQ + F^2Y$$

$$h_6 = ALQ + CYQ + AYS - AM^2 + R^2Q - 2RFM - G^2Y + 2HFY + F^2L$$

$$h_4 = CLQ + ALS + CYS + AYZ - 2AMP + AN^2 - M^2C - T^2Q + R^2S - 2RFP - 2TFN$$
$$+ 2RGN + 2TGM - 2RHM - 2GJY + H^2Y - G^2L + 2HFL$$

$$h_2 = CLS + ALZ + CYZ - AP^2 - 2CMP + CN^2 - T^2S + R^2Z + 2RJN + 2TJM - 2RHP$$
$$- 2THN + 2TGP - J^2Y - 2GJL + H^2L$$

$$h_0 = CLZ - CP^2 - T^2Z - J^2L + 2TJP$$

Special case of isotropic material:

In this case, coefficients d_{ij} of matrix [D] and coefficients h_j of the characteristic equation are the same form as previously (anisotropic material). The coefficients that change form are the following:

$$A = \frac{\sin\alpha}{4}$$

$$C = -\frac{\sin\alpha}{4}(5 - 4\nu) - \frac{n^2(1-\nu)}{2\sin\alpha}\left[1 + \frac{k}{4}\cot^2\alpha\right]$$

$$R = \frac{n}{4}\left[(1+\nu) - \frac{3}{4}(1-\nu)\, k \cot^2\alpha\right]$$

$$T = \frac{n}{4}\left[(5\nu - 7) + \frac{9}{4}(1-\nu)\, k \cot^2\alpha\right]$$

$$F = G = 0$$

$$H = \frac{\cos\alpha}{2}\left[\nu - \frac{n^2 k(1-\nu)}{2\sin^2\alpha}\right]$$

$$J = \frac{\cos\alpha}{2}\left[(\nu - 2) + \frac{3\, n^2 k\,(1-\nu)}{2\sin^2\alpha}\right]$$

$$Y = \frac{(1-\nu)}{8}\sin\alpha\left[1 + \frac{9k}{4}\cot^2\alpha\right]$$

$$L = \frac{9}{8}(\nu - 1)\sin\alpha\left[1 + \frac{9k}{4}\cot^2\alpha\right] - \frac{n^2}{\sin\alpha}(1 + k\cot^2\alpha)$$

$$M = \frac{(3-\nu)}{8}\, n\, k \cot\alpha$$

$$N = \frac{(1-2\nu)}{2}\, n\, k\cot\alpha$$

$$P = -n\cot\alpha\left[\frac{k}{8}(31 - 33\nu) + \frac{n^2 k}{\sin^2\alpha} + 1\right]$$

$$Q = \frac{-k\sin\alpha}{16}$$

$$S = k\left[\frac{(7-6\nu)}{8}\sin\alpha + \frac{n^2}{2\sin\alpha}\right]$$

$$Z = \frac{k}{\sin\alpha}\left[\frac{(12\nu - 13)}{16}\sin^2\alpha + \frac{(12\nu - 11)}{2} n^2 - \frac{n^4}{\sin^2\alpha}\right] - \frac{\cos^2\alpha}{\sin\alpha}$$

Stiffness:

Since we set $t = \delta x$ for an isotropic shell, we obtain:

$$k = \frac{\delta^2}{12} = \frac{K}{D}$$

with $D = \dfrac{E\delta}{(1-\nu^2)}\, x$ membrane stiffness

$$K = \frac{E\delta^3}{4(1-\nu^2)} x^2 \qquad \text{bending stiffness}$$

where E is Young's modulus

ν is Poisson's ratio

TABLE 2

Matrix $\underset{3x8}{[R]}$

$$\begin{Bmatrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{Bmatrix}_C = \sum_{n=0}^{\infty} \underset{3x3}{[T]} \; \underset{3x8}{[R]} \; \underset{8x1}{\begin{Bmatrix} \bar{C}_1 \\ \bar{C}_2 \\ \vdots \\ \bar{C}_8 \end{Bmatrix}} \quad \text{with } [T] = \begin{bmatrix} \cos n\theta & 0 & 0 \\ 0 & \cos n\theta & 0 \\ 0 & 0 & \sin n\theta \end{bmatrix}$$

$$R(1,1) = y^{\kappa_1 - 1}\,[\bar{\alpha}_1 \cos(\mu_1 \ln y) - \bar{\alpha}_2 \sin(\mu_1 \ln y)]$$

$$R(1,2) = y^{\kappa_1 - 1}\,[\bar{\alpha}_2 \cos(\mu_1 \ln y) + \bar{\alpha}_1 \sin(\mu_1 \ln y)]$$

$$R(1,3) = y^{-\kappa_1 - 1}\,[\bar{\alpha}_3 \cos(\mu_1 \ln y) - \bar{\alpha}_4 \sin(\mu_1 \ln y)]$$

$$R(1,4) = y^{-\kappa_1 - 1}\,[\bar{\alpha}_4 \cos(\mu_1 \ln y) + \bar{\alpha}_3 \sin(\mu_1 \ln y)]$$

$$R(1,5) = y^{\kappa_2 - 1}\,[\bar{\alpha}_5 \cos(\mu_2 \ln y) - \bar{\alpha}_6 \sin(\mu_2 \ln y)]$$

$$R(1,6) = y^{\kappa_2 - 1}\,[\bar{\alpha}_6 \cos(\mu_2 \ln y) + \bar{\alpha}_5 \sin(\mu_2 \ln y)]$$

$$R(1,7) = y^{-\kappa_2 - 1}\,[\bar{\alpha}_7 \cos(\mu_2 \ln y) - \bar{\alpha}_8 \sin(\mu_2 \ln y)]$$

$$R(1,8) = y^{-\kappa_2 - 1}\,[\bar{\alpha}_8 \cos(\mu_2 \ln y) + \bar{\alpha}_7 \sin(\mu_2 \ln y)]$$

$$R(2,1) = y^{\kappa_1 - 1} \cos(\mu_1 \ln y) \quad ; \quad R(2,5) = y^{\kappa_2 - 1} \cos(\mu_2 \ln y)$$

$$R(2,2) = y^{\kappa_1 - 1} \sin(\mu_1 \ln y) \quad ; \quad R(2,6) = y^{\kappa_2 - 1} \sin(\mu_2 \ln y)$$

$$R(2,3) = y^{-\kappa_1 - 1} \cos(\mu_1 \ln y) \quad ; \quad R(2,7) = y^{-\kappa_2 - 1} \cos(\mu_2 \ln y)$$

$$R(2,4) = y^{-\kappa_1 - 1} \sin(\mu_1 \ln y) \quad ; \quad R(2,8) = y^{-\kappa_2 - 1} \sin(\mu_2 \ln y)$$

$$R(3,1) = y^{\kappa_1 - 1}\,[\bar{\beta}_1 \cos(\mu_1 \ln y) - \bar{\beta}_2 \sin(\mu_1 \ln y)]$$

$$R(3,2) = y^{\kappa_1 - 1}\,[\bar{\beta}_2 \cos(\mu_1 \ln y) + \bar{\beta}_1 \sin(\mu_1 \ln y)]$$

$$R\ (3,\ 3) = y^{-\kappa_1 - 1}\ [\bar{\beta}_3 \cos\ (\mu_1 \ell n y) - \bar{\beta}_4 \sin\ (\mu_1 \ell n y)]$$

$$R\ (3,\ 4) = y^{-\kappa_1 - 1}\ [\bar{\beta}_4 \cos\ (\mu_1 \ell n y) + \bar{\beta}_3 \sin\ (\mu_1 \ell n y)]$$

$$R\ (3,\ 5) = y^{\kappa_2 - 1}\ [\bar{\beta}_5 \cos\ (\mu_2 \ell n y) - \bar{\beta}_6 \sin\ (\mu_2 \ell n y)]$$

$$R\ (3,\ 6) = y^{\kappa_2 - 1}\ [\bar{\beta}_6 \cos\ (\mu_2 \ell n y) + \bar{\beta}_5 \sin\ (\mu_2 \ell n y)]$$

$$R\ (3,\ 7) = y^{-\kappa_2 - 1}\ [\bar{\beta}_7 \cos\ (\mu_2 \ell n y) - \bar{\beta}_8 \sin\ (\mu_2 \ell n y)]$$

$$R\ (3,\ 8) = y^{-\kappa_2 - 1}\ [\bar{\beta}_8 \cos\ (\mu_2 \ell n y) + \bar{\beta}_7 \sin\ (\mu_2 \ell n y)]$$

Matrix $[R_r]_{3\times 8}$

$$\begin{Bmatrix} U\ (x,\ \theta) \\ W\ (x,\ \theta) \\ V\ (x,\ \theta) \end{Bmatrix}_C = \sum_{n=0}^{\infty} \underset{3\times 3}{[T]}\ \underset{3\times 8}{[R_r]}\ \underset{8\times 1}{\{\bar{C}\}}$$

$$R_r\ (i,\ j) = R\ (i,\ j) \qquad \text{for} \begin{cases} i = 1,\ 2,\ 3 \\ j = 1,\ 2,\ 3,\ 4 \end{cases}$$

$$R_r\ (1,\ 5) = \alpha_5\ y^{a_1-1}$$

$$R_r\ (1,\ 6) = \alpha_6\ y^{a_2-1}$$

$$R_r\ (1,\ 7) = \alpha_7\ y^{-a_1-1}$$

$$R_r\ (1,\ 8) = \alpha_8\ y^{-a_2-1}$$

$$R_r\ (2,\ 5) = y^{a_1-1}$$

$$R_r\ (2,\ 6) = y^{a_2-1}$$

$$R_r\ (2,\ 7) = y^{-a_1-1}$$

$$R_r\ (2,\ 8) = y^{-a_2-1}$$

$$R_r\ (3,\ 5) = \beta_5\ y^{a_1-1}$$

$$R_r\ (3,\ 6) = \beta_6\ y^{a_2-1}$$

$$R_r\ (3,\ 7) = \beta_7\ y^{-a_1-1}$$

$$R_r\ (3,\ 8) = \beta_8\ y^{-a_2-1}$$

TABLE 3

Matrix $\underset{6x8}{[Q]}$

$$\underset{6x1}{\{\varepsilon\}} = \sum_{n=0}^{\infty} \underset{6x6}{\begin{bmatrix} [T] & [0] \\ [0] & [T] \end{bmatrix}} \underset{6x8}{[Q]} \underset{8x1}{\{\overline{C}\}} \quad ; \quad [T] = \begin{bmatrix} \cos n\theta & 0 & 0 \\ 0 & \cos n\theta & 0 \\ 0 & 0 & \sin n\theta \end{bmatrix}$$

$$Q(1, 1) = \frac{y^{\kappa_1 - 3}}{2\ell} \{a_{11} \cos \mu_1 \ell n y - a_{12} \sin \mu_1 \ell n y\}$$

$$Q(1, 2) = \frac{y^{\kappa_1 - 3}}{2\ell} \{a_{12} \cos \mu_1 \ell n y + a_{11} \sin \mu_1 \ell n y\}$$

$$Q(1, 3) = \frac{y^{-\kappa_1 - 3}}{2\ell} \{a_{13} \cos \mu_1 \ell n y - a_{14} \sin \mu_1 \ell n y\}$$

$$Q(1, 4) = \frac{y^{-\kappa_1 - 3}}{2\ell} \{a_{14} \cos \mu_1 \ell n y + a_{13} \sin \mu_1 \ell n y\}$$

$$Q(1, 5) = \frac{y^{\kappa_2 - 3}}{2\ell} \{a_{15} \cos \mu_2 \ell n y - a_{16} \sin \mu_2 \ell n y\}$$

$$Q(1, 6) = \frac{y^{\kappa_2 - 3}}{2\ell} \{a_{16} \cos \mu_2 \ell n y + a_{15} \sin \mu_2 \ell n y\}$$

$$Q(1, 7) = \frac{y^{-\kappa_2 - 3}}{2\ell} \{a_{17} \cos \mu_2 \ell n y - a_{18} \sin \mu_2 \ell n y\}$$

$$Q(1, 8) = \frac{y^{-\kappa_2 - 3}}{2\ell} \{a_{18} \cos \mu_2 \ell n y + a_{17} \sin \mu_2 \ell n y\}$$

$$Q(2, 1) = \frac{y^{\kappa_1 - 3}}{\ell} \{a_{21} \cos \mu_1 \ell n y - a_{22} \sin \mu_1 \ell n y\}$$

$$Q(2, 2) = \frac{y^{\kappa_1 - 3}}{\ell} \{a_{22} \cos \mu_1 \ell n y + a_{21} \sin \mu_1 \ell n y\}$$

$$Q(2, 3) = \frac{y^{-\kappa_1 - 3}}{\ell} \{a_{23} \cos \mu_1 \ell n y - a_{24} \sin \mu_1 \ell n y\}$$

$$Q\ (2,\ 4) = \frac{y^{-\kappa_1-3}}{\ell}\ \{a_{24} \cos \mu_1 \ell n\, y + a_{23} \sin \mu_1 \ell n\, y\}$$

$$Q\ (2,\ 5) = \frac{y^{\kappa_2-3}}{\ell}\ \{a_{25} \cos \mu_2 \ell n\, y - a_{26} \sin \mu_2 \ell n\, y\}$$

$$Q\ (2,\ 6) = \frac{y^{\kappa_2-3}}{\ell}\ \{a_{26} \cos \mu_2 \ell n\, y + a_{25} \sin \mu_2 \ell n\, y\}$$

$$Q\ (2,\ 7) = \frac{y^{-\kappa_2-3}}{\ell}\ \{a_{27} \cos \mu_2 \ell n\, y - a_{28} \sin \mu_2 \ell n\, y\}$$

$$Q\ (2,\ 8) = \frac{y^{-\kappa_2-3}}{\ell}\ \{a_{28} \cos \mu_2 \ell n\, y + a_{27} \sin \mu_2 \ell n\, y\}$$

$$Q\ (3,\ 1) = \frac{y^{\kappa_1-3}}{\ell}\ \{a_{31} \cos \mu_1 \ell n\, y - a_{32} \sin \mu_1 \ell n\, y\}$$

$$Q\ (3,\ 2) = \frac{y^{\kappa_1-3}}{\ell}\ \{a_{32} \cos \mu_1 \ell n\, y + a_{31} \sin \mu_1 \ell n\, y\}$$

$$Q\ (3,\ 3) = \frac{y^{-\kappa_1-3}}{\ell}\ \{a_{33} \cos \mu_1 \ell n\, y - a_{34} \sin \mu_1 \ell n\, y\}$$

$$Q\ (3,\ 4) = \frac{y^{-\kappa_1-3}}{\ell}\ \{a_{34} \cos \mu_1 \ell n\, y + a_{33} \sin \mu_1 \ell n\, y\}$$

$$Q\ (3,\ 5) = \frac{y^{\kappa_2-3}}{\ell}\ \{a_{35} \cos \mu_2 \ell n\, y - a_{36} \sin \mu_2 \ell n\, y\}$$

$$Q\ (3,\ 6) = \frac{y^{\kappa_2-3}}{\ell}\ \{a_{36} \cos \mu_2 \ell n\, y + a_{35} \sin \mu_2 \ell n\, y\}$$

$$Q\ (3,\ 7) = \frac{y^{-\kappa_2-3}}{\ell}\ \{a_{37} \cos \mu_2 \ell n\, y - a_{38} \sin \mu_2 \ell n\, y\}$$

$$Q\ (3,\ 8) = \frac{y^{-\kappa_2-3}}{\ell}\ \{a_{38} \cos \mu_2 \ell n\, y + a_{37} \sin \mu_2 \ell n\, y\}$$

$$Q\ (4,\ 1) = \frac{y^{\kappa_1-5}}{4\ell^2}\ \{a_{41} \cos \mu_1 \ell n\, y - a_{42} \sin \mu_1 \ell n\, y\}$$

$$Q\ (4,\ 2) = \frac{y^{\kappa_1-5}}{4\ell^2}\ \{a_{42} \cos \mu_1 \ell n\, y + a_{41} \sin \mu_1 \ell n\, y\}$$

$$Q\ (4,\ 3) = \frac{y^{-\kappa_1-5}}{4\ell^2}\ \{a_{43} \cos \mu_1 \ell n\, y - a_{44} \sin \mu_1 \ell n\, y\}$$

$$Q\ (4,\ 4) = \frac{y^{-\kappa_1-5}}{4\ell^2}\ \{a_{44} \cos \mu_1 \ell n y + a_{43} \sin \mu_1 \ell n y\}$$

$$Q\ (4,\ 5) = \frac{y^{\kappa_2-5}}{4\ell^2}\ \{a_{45} \cos \mu_2 \ell n y - a_{46} \sin \mu_2 \ell n y\}$$

$$Q\ (4,\ 6) = \frac{y^{\kappa_2-5}}{4\ell^2}\ \{a_{46} \cos \mu_2 \ell n y + a_{45} \sin \mu_2 \ell n y\}$$

$$Q\ (4,\ 7) = \frac{y^{-\kappa_2-5}}{4\ell^2}\ \{a_{47} \cos \mu_2 \ell n y - a_{48} \sin \mu_2 \ell n y\}$$

$$Q\ (4,\ 8) = \frac{y^{-\kappa_2-5}}{4\ell^2}\ \{a_{48} \cos \mu_2 \ell n y + a_{47} \sin \mu_2 \ell n y\}$$

$$Q\ (5,\ 1) = \frac{y^{\kappa_1-5}}{\ell^2}\ \{a_{51} \cos \mu_1 \ell n y - a_{52} \sin \mu_1 \ell n y\}$$

$$Q\ (5,\ 2) = \frac{y^{\kappa_1-5}}{\ell^2}\ \{a_{52} \cos \mu_1 \ell n y + a_{51} \sin \mu_1 \ell n y\}$$

$$Q\ (5,\ 3) = \frac{y^{-\kappa_1-5}}{\ell^2}\ \{a_{53} \cos \mu_1 \ell n y - a_{54} \sin \mu_1 \ell n y\}$$

$$Q\ (5,\ 4) = \frac{y^{-\kappa_1-5}}{\ell^2}\ \{a_{54} \cos \mu_1 \ell n y + a_{53} \sin \mu_1 \ell n y\}$$

$$Q\ (5,\ 5) = \frac{y^{\kappa_2-5}}{\ell^2}\ \{a_{55} \cos \mu_2 \ell n y - a_{56} \sin \mu_2 \ell n y\}$$

$$Q\ (5,\ 6) = \frac{y^{\kappa_2-5}}{\ell^2}\ \{a_{56} \cos \mu_2 \ell n y + a_{55} \sin \mu_2 \ell n y\}$$

$$Q\ (5,\ 7) = \frac{y^{-\kappa_2-5}}{\ell^2}\ \{a_{57} \cos \mu_2 \ell n y - a_{58} \sin \mu_2 \ell n y\}$$

$$Q\ (5,\ 8) = \frac{y^{-\kappa_2-5}}{\ell^2}\ \{a_{58} \cos \mu_2 \ell n y + a_{57} \sin \mu_2 \ell n y\}$$

$$Q\ (6,\ 1) = \frac{y^{\kappa_1-5}}{\ell^2}\ \{a_{61} \cos \mu_1 \ell n y - a_{62} \sin \mu_1 \ell n y\}$$

$$Q\ (6,\ 2) = \frac{y^{\kappa_1-5}}{\ell^2}\ \{a_{62} \cos \mu_1 \ell n y + a_{61} \sin \mu_1 \ell n y\}$$

$$Q\ (6,\ 3) = \frac{y^{-\kappa_1-5}}{\ell^2} \{a_{63} \cos \mu_1 \ell n y - a_{62} \sin \mu_1 \ell n y\}$$

$$Q\ (6,\ 4) = \frac{y^{-\kappa_1-5}}{\ell^2} \{a_{64} \cos \mu_1 \ell n y + a_{63} \sin \mu_1 \ell n y\}$$

$$Q\ (6,\ 5) = \frac{y^{\kappa_2-5}}{\ell^2} \{a_{65} \cos \mu_2 \ell n y - a_{66} \sin \mu_2 \ell n y\}$$

$$Q\ (6,\ 6) = \frac{y^{\kappa_2-5}}{\ell^2} \{a_{66} \cos \mu_2 \ell n y + a_{65} \sin \mu_2 \ell n y\}$$

$$Q\ (6,\ 7) = \frac{y^{-\kappa_2-5}}{\ell^2} \{a_{67} \cos \mu_2 \ell n y - a_{68} \sin \mu_2 \ell n y\}$$

$$Q\ (6,\ 8) = \frac{y^{-\kappa_2-5}}{\ell^2} \{a_{68} \cos \mu_2 \ell n y + a_{67} \sin \mu_2 \ell n y\}$$

Matrix $\underset{6\times 8}{[Q_r]}$

$$\underset{6\times 1}{\{\varepsilon\}} = \sum_{n=0}^{\infty} \underset{6\times 6}{\begin{bmatrix} [T] & 0 \\ 0 & [T] \end{bmatrix}} \underset{6\times 8}{[Q_r]} \underset{8\times 1}{\{\bar{C}\}}$$

$$Q_r\,(i,j) = Q\,(i,j) \qquad \text{for} \begin{cases} i = 1, \ldots, 6 \\ j = 1, \ldots, 4 \end{cases}$$

$$Q_r\,(1,5) = \frac{y^{a_1-3}}{2\ell}\, a_{15}^{*} \qquad Q_r\,(3,5) = \frac{y^{a_1-3}}{\ell}\, a_{35}^{*}$$

$$Q_r\,(1,6) = \frac{y^{a_2-3}}{2\ell}\, a_{16}^{*} \qquad Q_r\,(3,6) = \frac{y^{a_2-3}}{\ell}\, a_{36}^{*}$$

$$Q_r\,(1,7) = \frac{y^{-a_1-3}}{2\ell}\, a_{17}^{*} \qquad Q_r\,(3,7) = \frac{y^{-a_1-3}}{\ell}\, a_{37}^{*}$$

$$Q_r\,(1,8) = \frac{y^{-a_2-3}}{2\ell}\, a_{18}^{*} \qquad Q_r\,(3,8) = \frac{y^{-a_2-3}}{\ell}\, a_{38}^{*}$$

$$Q_r\,(2,5) = \frac{y^{a_1-3}}{\ell}\, a_{25}^{*} \qquad Q_r\,(4,5) = \frac{y^{a_1-5}}{4\ell^2}\, a_{45}^{*}$$

$$Q_r\,(2,6) = \frac{y^{a_2-3}}{\ell}\, a_{26}^{*} \qquad Q_r\,(4,6) = \frac{y^{a_2-5}}{4\ell^2}\, a_{46}^{*}$$

$$Q_r\,(2,7) = \frac{y^{-a_1-3}}{\ell}\, a_{27}^{*} \qquad Q_r\,(4,7) = \frac{y^{-a_1-5}}{4\ell^2}\, a_{47}^{*}$$

$$Q_r\,(2,8) = \frac{y^{-a_2-3}}{\ell}\, a_{28}^{*} \qquad Q_r\,(4,8) = \frac{y^{-a_2-5}}{4\ell^2}\, a_{48}^{*}$$

$$a_{51} = \frac{n^2}{\sin^2 \alpha} + \frac{n\,\bar{\beta}_1 \cot \alpha}{\sin \alpha} + \frac{(1-\kappa_1)}{2}$$

$$a_{52} = \frac{n\,\bar{\beta}_2 \cot \alpha}{\sin \alpha} - \frac{\mu_1}{2}$$

$$a_{53} = \frac{n^2}{\sin^2 \alpha} + \frac{n\,\bar{\beta}_3 \cot \alpha}{\sin \alpha} + \frac{(\kappa_1+1)}{2}$$

$$a_{54} = \frac{n\,\bar{\beta}_4 \cot \alpha}{\sin \alpha} - \frac{\mu_1}{2}$$

$$a_{55} = \frac{n^2}{\sin^2 \alpha} + \frac{n\,\bar{\beta}_5 \cot \alpha}{\sin \alpha} + \frac{(1-\kappa_2)}{2}$$

$$a_{56} = \frac{n\,\bar{\beta}_6 \cot \alpha}{\sin \alpha} - \frac{\mu_2}{2}$$

$$a_{57} = \frac{n^2}{\sin^2 \alpha} + \frac{n\,\bar{\beta}_7 \cot \alpha}{\sin \alpha} + \frac{(\kappa_2+1)}{2}$$

$$a_{58} = \frac{n\,\bar{\beta}_8 \cot \alpha}{\sin \alpha} - \frac{\mu_2}{2}$$

$$a_{61} = \frac{n\,\bar{\alpha}_1 \cot \alpha}{2 \sin \alpha} + \frac{3}{4} \cot \alpha \left[\bar{\beta}_1 (\kappa_1-3) - \bar{\beta}_2 \mu_1\right] + \frac{n\,(\kappa_1-3)}{\sin \alpha}$$

$$a_{62} = \frac{n\,\bar{\alpha}_2 \cot \alpha}{2 \sin \alpha} + \frac{3}{4} \cot \alpha \left[\bar{\beta}_2 (\kappa_1-3) + \bar{\beta}_1 \mu_1\right] + \frac{n\,\mu_1}{\sin \alpha}$$

$$a_{63} = \frac{n\,\bar{\alpha}_3 \cot \alpha}{2 \sin \alpha} - \frac{3}{4} \cot \alpha \left[\bar{\beta}_3 (\kappa_1+3) + \bar{\beta}_4 \mu_1\right] - \frac{n\,(\kappa_1+3)}{\sin \alpha}$$

$$a_{64} = \frac{n\,\bar{\alpha}_4 \cot \alpha}{2 \sin \alpha} - \frac{3}{4} \cot \alpha \left[\bar{\beta}_4 (\kappa_1+3) - \bar{\beta}_3 \mu_1\right] + \frac{n\,\mu_1}{\sin \alpha}$$

$$a_{65} = \frac{n\,\bar{\alpha}_5 \cot \alpha}{2 \sin \alpha} + \frac{3}{4} \cot \alpha \left[\bar{\beta}_5 (\kappa_2-3) - \bar{\beta}_6 \mu_2\right] + \frac{n\,(\kappa_2-3)}{\sin \alpha}$$

$$a_{66} = \frac{n\,\bar{\alpha}_6 \cot \alpha}{2 \sin \alpha} + \frac{3}{4} \cot \alpha \left[\bar{\beta}_6 (\kappa_2-3) + \bar{\beta}_5 \mu_2\right] + \frac{n\,\mu_2}{\sin \alpha}$$

$$a_{67} = \frac{n\,\bar{\alpha}_7 \cot \alpha}{2 \sin \alpha} - \frac{3}{4} \cot \alpha \left[\bar{\beta}_7 (\kappa_2+3) + \bar{\beta}_8 \mu_2\right] - \frac{n\,(\kappa_2+3)}{\sin \alpha}$$

$$a_{68} = \frac{n\,\bar{\alpha}_8 \cot \alpha}{2 \sin \alpha} - \frac{3}{4} \cot \alpha \left[\bar{\beta}_8 (\kappa_2+3) - \bar{\beta}_7 \mu_2\right] + \frac{n\,\mu_2}{\sin \alpha}$$

$$Q_r\,(5,5) = \frac{y^{a_1-5}}{\ell^2}\,a_{55}^* \qquad Q_r\,(6,5) = \frac{y^{a_1-5}}{\ell^2}\,a_{65}^*$$

$$Q_r\,(5,6) = \frac{y^{a_2-5}}{\ell^2}\,a_{56}^* \qquad Q_r\,(6,6) = \frac{y^{a_2-5}}{\ell^2}\,a_{66}^*$$

$$Q_r\,(5,7) = \frac{y^{-a_1-5}}{\ell^2}\,a_{57}^* \qquad Q_r\,(6,7) = \frac{y^{-a_1-5}}{\ell^2}\,a_{67}^*$$

$$Q_r\,(5,8) = \frac{y^{-a_2-5}}{\ell^2}\,a_{58}^* \qquad Q_r\,(6,8) = \frac{y^{-a_2-5}}{\ell^2}\,a_{68}^*$$

$$a_{15}^* = \alpha_5\,(a_1-1)$$

$$a_{16}^* = \alpha_6\,(a_2-1)$$

$$a_{17}^* = -\alpha_7\,(a_1+1)$$

$$a_{18}^* = -\alpha_8\,(a_2+1)$$

$$a_{25}^* = \alpha_5 + \cot\alpha + \frac{n\,\beta_5}{\sin\alpha}$$

$$a_{26}^* = \alpha_6 + \cot\alpha + \frac{n\,\beta_6}{\sin\alpha}$$

$$a_{27}^* = \alpha_7 + \cot\alpha + \frac{n\,\beta_7}{\sin\alpha}$$

$$a_{28}^* = \alpha_8 + \cot\alpha + \frac{n\,\beta_8}{\sin\alpha}$$

$$a_{35}^* = \frac{-n\,\alpha_5}{\sin\alpha} + \frac{\beta_5}{2}\,(a_1-3)$$

$$a_{36}^* = \frac{-n\,\alpha_6}{\sin\alpha} + \frac{\beta_6}{2}\,(a_2-3)$$

$$a_{37}^* = \frac{-n\,\alpha_7}{\sin\alpha} - \frac{\beta_7}{2}\,(a_1+3)$$

$$a_{38}^* = \frac{-n\,\alpha_8}{\sin\alpha} - \frac{\beta_8}{2}\,(a_2+3)$$

$$a_{45}^* = -(a_1-1)\,(a_1-3)$$

$$a_{46}^* = -(a_2-1)\,(a_2-3)$$

$$a_{47}^* = -(a_1+1)\,(a_1+3)$$

$$a_{48}^* = -(a_2+1)\,(a_2+3)$$

$$a_{55}^* = \frac{n^2}{\sin^2\alpha} + \frac{n\,\beta_5\cos\alpha}{\sin^2\alpha} - \frac{(a_1-1)}{2}$$

$$a_{56}^* = \frac{n^2}{\sin^2\alpha} + \frac{n\,\beta_6\cos\alpha}{\sin^2\alpha} - \frac{(a_2-1)}{2}$$

$$a_{57}^{*} = \frac{n^2}{\sin^2 \alpha} + \frac{n\, \beta_7 \cos \alpha}{\sin^2 \alpha} + \frac{(a_1+1)}{2}$$

$$a_{58}^{*} = \frac{n^2}{\sin^2 \alpha} + \frac{n\, \beta_8 \cos \alpha}{\sin^2 \alpha} + \frac{(a_2+1)}{2}$$

$$a_{65}^{*} = \frac{n\, \alpha_5 \cos \alpha}{2 \sin^2 \alpha} + \frac{(a_1-3)}{\sin \alpha} \left(n + \frac{3}{4} \beta_5 \cos \alpha\right)$$

$$a_{66}^{*} = \frac{n\, \alpha_6 \cos \alpha}{2 \sin^2 \alpha} + \frac{(a_2-3)}{\sin \alpha} \left(n + \frac{3}{4} \beta_6 \cos \alpha\right)$$

$$a_{67}^{*} = \frac{n\, \alpha_7 \cos \alpha}{2 \sin^2 \alpha} - \frac{(a_1+3)}{\sin \alpha} \left(n + \frac{3}{4} \beta_7 \cos \alpha\right)$$

$$a_{68}^{*} = \frac{n\, \alpha_8 \cos \alpha}{2 \sin^2 \alpha} - \frac{(a_2+3)}{\sin \alpha} \left(n + \frac{3}{4} \beta_8 \cos \alpha\right)$$

$$a_{11} = \bar{\alpha}_1 (\kappa_1 - 1) - \bar{\alpha}_2 \mu_1$$

$$a_{12} = \bar{\alpha}_2 (\kappa_1 - 1) + \bar{\alpha}_1 \mu_1$$

$$a_{13} = -\bar{\alpha}_3 (\kappa_1 + 1) - \bar{\alpha}_4 \mu_1$$

$$a_{14} = -\bar{\alpha}_4 (\kappa_1 + 1) + \bar{\alpha}_3 \mu_1$$

$$a_{15} = \bar{\alpha}_5 (\kappa_2 - 1) - \bar{\alpha}_6 \mu_2$$

$$a_{16} = \bar{\alpha}_6 (\kappa_2 - 1) + \bar{\alpha}_5 \mu_2$$

$$a_{17} = -\bar{\alpha}_7 (\kappa_2 + 1) - \bar{\alpha}_8 \mu_2$$

$$a_{18} = -\bar{\alpha}_8 (\kappa_2 + 1) + \bar{\alpha}_7 \mu_2$$

$$a_{21} = \bar{\alpha}_1 + \cot \alpha + \frac{n \bar{\beta}_1}{\sin \alpha}$$

$$a_{22} = \bar{\alpha}_2 + \frac{n \bar{\beta}_2}{\sin \alpha}$$

$$a_{23} = \bar{\alpha}_3 + \cot \alpha + \frac{n \bar{\beta}_3}{\sin \alpha}$$

$$a_{24} = \bar{\alpha}_4 + \frac{n \bar{\beta}_4}{\sin \alpha}$$

$$a_{25} = \bar{\alpha}_5 + \cot \alpha + \frac{n \bar{\beta}_5}{\sin \alpha}$$

$$a_{26} = \bar{\alpha}_6 + \frac{n \bar{\beta}_6}{\sin \alpha}$$

$$a_{27} = \bar{\alpha}_7 + \cot \alpha + \frac{n \bar{\beta}_7}{\sin \alpha}$$

$$a_{28} = \bar{\alpha}_8 + \frac{n \bar{\beta}_8}{\sin \alpha}$$

$$a_{31} = \frac{\bar{\beta}_1}{2} (\kappa_1 - 3) - \frac{\bar{\beta}_2}{2} \mu_1 - \frac{n \bar{\alpha}_1}{\sin \alpha}$$

$$a_{32} = \frac{\bar{\beta}_2}{2} (\kappa_1 - 3) + \frac{\bar{\beta}_1}{2} \mu_1 - \frac{n \bar{\alpha}_2}{\sin \alpha}$$

$$a_{33} = -\frac{\bar{\beta}_3}{2} (\kappa_1 + 3) - \frac{\bar{\beta}_4}{-2} \mu_1 - \frac{n \bar{\alpha}_3}{\sin \alpha}$$

$$a_{34} = -\frac{\bar{\beta}_4}{2} (\kappa_1 + 3) + \frac{\bar{\beta}_3}{2} \mu_1 - \frac{n \bar{\alpha}_4}{\sin \alpha}$$

$$a_{35} = \frac{\bar{\beta}_5}{2} (\kappa_2 - 3) - \frac{\bar{\beta}_6}{2} \mu_2 - \frac{n \bar{\alpha}_5}{\sin \alpha}$$

$$a_{36} = \frac{\bar{\beta}_6}{2} (\kappa_2 - 3) + \frac{\bar{\beta}_5}{2} \mu_2 - \frac{n \bar{\alpha}_6}{\sin \alpha}$$

$$a_{37} = -\frac{\bar{\beta}_7}{2} (\kappa_2 + 3) - \frac{\bar{\beta}_8}{2} \mu_2 - \frac{n \bar{\alpha}_7}{\sin \alpha}$$

$$a_{38} = -\frac{\bar{\beta}_8}{2} (\kappa_2 + 3) + \frac{\bar{\beta}_7}{2} \mu_2 - \frac{n \bar{\alpha}_8}{\sin \alpha}$$

$$a_{41} = \mu_1^2 - (\kappa_1 - 1)(\kappa_1 - 3)$$

$$a_{42} = -2\mu_1 (\kappa_1 - 2)$$

$$a_{43} = \mu_1^2 - (\kappa_1 + 1)(\kappa_1 + 3)$$

$$a_{44} = 2\mu_1 (\kappa_1 + 2)$$

$$a_{45} = \mu_2^2 - (\kappa_2 - 1)(\kappa_2 - 3)$$

$$a_{46} = -2\mu_2 (\kappa_2 - 2)$$

$$a_{47} = \mu_2^2 - (\kappa_2 + 1)(\kappa_2 + 3)$$

$$a_{48} = 2\mu_2 (\kappa_2 + 2)$$

TABLE 4

Matrix [P]

6x6

$$[P]=\begin{bmatrix} x B_{11} & x B_{12} & 0 & x^2 B_{14} & x^2 B_{15} & 0 \\ x B_{21} & x B_{22} & 0 & x^2 B_{24} & x^2 B_{25} & 0 \\ 0 & 0 & x B_{23} & 0 & 0 & x^2 B_{36} \\ x^2 B_{41} & x^2 B_{42} & 0 & x^3 B_{44} & x^3 B_{45} & 0 \\ x^2 B_{51} & x^2 B_{52} & 0 & x^3 B_{54} & x^3 B_{46} & 0 \\ 0 & 0 & x^2 B_{63} & 0 & 0 & x^3 B_{66} \end{bmatrix}$$

Below are the B_{ij} expressions for a shell composed of a number of symmetric iso- or orthotropic layers.

- For even number of layers 2n we have:

$$B_{ij} = 2 \sum_{s=1}^{n} Z_{ij}^{s} (\delta_s - \delta_{s+1}) \; ; \; i = 1 \text{ to } 3 \qquad j = 1 \text{ to } 3$$

$$B_{ij} = \frac{2}{3} \sum_{s=1}^{n} Z_{i-3,j-3}^{s} (\delta_s^3 - \delta_{s+1}^3) ; i = 4 \text{ to } 6 \qquad j = 4 \text{ to } 6$$

$$B_{ij} = 0 \; (i = 1 \text{ to } 3 \text{ and } j = 4 \text{ to } 6) \text{ or } (i = 4 \text{ to } 6 \text{ and } j = 1 \text{ to } 3)$$

- For odd number of layers 2n + 1:

$$B_{ij} = 2 \quad Z_{ij}^{n+1} \quad \delta_{n+1} + \sum_{s=1}^{n} Z_{ij}^{s} (\delta_s - \delta_{s+1})$$

$i = 1$ to 3 and $j = 1$ to 3

$$B_{ij} = \frac{2}{3} \quad Z_{i-3,j-3}^{n+1} \quad \delta_{n+1}^{3} + \sum_{s=1}^{n} Z_{i-3,j-3}^{s} (\delta_s^{3} - \delta_{s+1}^{3})$$

$i = 4$ to 6 and $j = 4$ to 6

$B_{ij} = 0$ ($i = 1$ to 3 and $j = 4$ to 6) or ($i = 4$ to 6 and $j = 1$ to 3)

with:

$$Z_{11}^{s} = E_1^{s}/(1 - \nu_1^{s} \nu_2^{s})$$

$$Z_{22}^{s} = E_1^{s}/(1 - \nu_1^{s} \nu_2^{s})$$

$$Z_{12}^{s} = Z_{21}^{s} = E_1^{s} \nu_2^{s}/(1 - \nu_1^{s} \nu_2^{s})$$

$$Z_{33}^{s} = \frac{1}{2} \quad G_{12}^{s}$$

E_1^{s}, ν_1^{s} (resp. E_2^{s}, ν_2^{s}) : Young's modulus and Poisson's ratio with respect to axis x (resp. θ).

G_{12}^{s} : shear modulus of elasticity

δ_5 is the proportionality coefficient for the thickness t_5 of the 5th layer to x.

$$t_5 = \delta_5 x$$

t_5 is measured relative to the surface of the medium.

APPENDIX B

TABLES

n	m = 1 EMPTY	m = 1 FULL	m = 2 EMPTY	m = 2 FULL
2	$0{,}15000 \times 10^{5}$	$0{,}51606 \times 10^{4}$	$0{,}19895 \times 10^{5}$	$0{,}58925 \times 10^{4}$
3	$0{,}27269 \times 10^{4}$	$0{,}65407 \times 10^{3}$	$0{,}86671 \times 10^{4}$	$0{,}20988 \times 10^{4}$
4	$0{,}25567 \times 10^{4}$	$0{,}68703 \times 10^{3}$	$0{,}72106 \times 10^{4}$	$0{,}19734 \times 10^{4}$
5	$0{,}21085 \times 10^{4}$	$0{,}60994 \times 10^{3}$	$0{,}35559 \times 10^{4}$	$0{,}11028 \times 10^{4}$
6	$0{,}27529 \times 10^{4}$	$0{,}83739 \times 10^{3}$	$0{,}49254 \times 10^{4}$	$0{,}16061 \times 10^{4}$
7	$0{,}34735 \times 10^{4}$	$0{,}11035 \times 10^{4}$	$0{,}53979 \times 10^{4}$	$0{,}18154 \times 10^{4}$
8	$0{,}14609 \times 10^{4}$	$0{,}46744 \times 10^{3}$	$0{,}46862 \times 10^{4}$	$0{,}15918 \times 10^{4}$
9	$0{,}50620 \times 10^{4}$	$0{,}17519 \times 10^{4}$	$0{,}70161 \times 10^{4}$	$0{,}25383 \times 10^{4}$
10	$0{,}60021 \times 10^{4}$	$0{,}21577 \times 10^{4}$	$0{,}80378 \times 10^{4}$	$0{,}30061 \times 10^{4}$
11	$0{,}70549 \times 10^{4}$	$0{,}26268 \times 10^{4}$	$0{,}92025 \times 10^{4}$	$0{,}35509 \times 10^{4}$
12	$0{,}82271 \times 10^{4}$	$0{,}31646 \times 10^{4}$	$0{,}10516 \times 10^{5}$	$0{,}41791 \times 10^{4}$
13	$0{,}95273 \times 10^{4}$	$0{,}37772 \times 10^{4}$	$0{,}11988 \times 10^{5}$	$0{,}48985 \times 10^{4}$
14	$0{,}10966 \times 10^{5}$	$0{,}44716 \times 10^{4}$	$0{,}13629 \times 10^{5}$	$0{,}57179 \times 10^{4}$
15	$0{,}12554 \times 10^{5}$	$0{,}52553 \times 10^{4}$	$0{,}15449 \times 10^{5}$	$0{,}66469 \times 10^{4}$
16	$0{,}14302 \times 10^{5}$	$0{,}61359 \times 10^{4}$	$0{,}17453 \times 10^{5}$	$0{,}76945 \times 10^{4}$
17	$0{,}16166 \times 10^{5}$	$0{,}70965 \times 10^{4}$	$0{,}19653 \times 10^{5}$	$0{,}88641 \times 10^{4}$
18	$0{,}18304 \times 10^{5}$	$0{,}82135 \times 10^{4}$	$0{,}22066 \times 10^{5}$	$0{,}10173 \times 10^{5}$
19	$0{,}20561 \times 10^{5}$	$0{,}94183 \times 10^{4}$	$0{,}24662 \times 10^{5}$	$0{,}11611 \times 10^{5}$
20	$0{,}22982 \times 10^{5}$	$0{,}10735 \times 10^{5}$	$0{,}27440 \times 10^{5}$	$0{,}13178 \times 10^{5}$

TABLE 1 - Natural frequencies of an empty and liquid-filled conical shell
($\alpha = 14{,}2°$, $x_1 = 0{,}141$ m, $x_2 - x_1 = 0{,}174$ m, $t = 2{,}56 \times 10^{-4}$ m, $\rho = 7800$ kg/m^3,
$E = 200$ GPa, $\nu = 0.3$, $\rho_\ell = 1000$ kg/m^3, $W = V = 0$).

LIQUID LEVEL	$m = 1$	$m = 2$
EMPTY	2,1085	3,5559
FULL	0,6099	1,1028
b_1/L		
1/4	1,2457	2,3991
1/2	0,8431	1,6197
3/4	0,6496	1,1869
b_2/L		
1/4	1,1765	2,8002
1/2	0,7083	2,0485
3/4	0,62119	1,3525

TABLE 2 - Natural frequencies ($\Omega \times 10^{-3}$) of a conical shell partially filled with liquid

($n = 5$, $\alpha = 14,2°$, $x = 0,141$ m, $x_2 - x_1 = 0,174$ m, $t = 2,56 \times 10^{-4}$ m, $\rho = 7800$ kg/m^3, $E = 200$ GPa, $\nu = 0,3$; $\rho_\ell = 1000$ kg/m^3, $W = V = 0$).

n	m	α = cone apex half-angle 10°	14,2°	20°	25°	30°
2	1	0,27952 x 10^4	0,51606 x 10^4	0,29590 x 10^4	0,20498 x 10^4	0,14340 x 10^4
	2	0,89873 x 10^4	0,58925 x 10^4	0,33978 x 10^4	0,23060 x 10^4	0,16517 x 10^4
3	1	0,72584 x 10^3	0,65407 x 10^3	0,36389 x 10^4	0,24910 x 10^4	0,18178 x 10^4
	2	0,22305 x 10^4	0,20988 x 10^4	0,42215 x 10^4	0,29276 x 10^4	0,20340 x 10^4
4	1	0,75204 x 10^3	0,68703 x 10^3	0,49589 x 10^3	0,58968 x 10^3	0,16901 x 10^4
	2	0,16042 x 10^4	0,19734 x 10^4	0,15339 x 10^4	0,18535 x 10^4	0,20432 x 10^4
5	1	0,11948 x 10^4	0,60994 x 10^3	0,48030 x 10^3	0,44173 x 10^3	0,38227 x 10^3
	2	0,20009 x 10^4	0,11028 x 10^4	0,13614 x 10^4	0,13146 x 10^4	0,11139 x 10^4
6	1	0,55125 x 10^{42}	0,83739 x 10^3	0,49029 x 10^3	0,43881 x 10^3	0,31295 x 10^3
	2	0,19958 x 10^4	0,16061 x 10^4	0,10804 x 10^4	0,11498 x 10^4	0,95830 x 10^3
7	1	0,21669 x 10^4	0,11035 x 10^4	0,60902 x 10^3	0,42559 x 10^3	0,24065 x 10^{40}
	2	0,31357 x 10^4	0,18154 x 10^4	0,10756 x 10^4	0,93315 x 10^3	0,45240 x 10^3
8	1	0,28244 x 10^4	0,46744 x 10^3	0,75554 x 10^3	0,48347 x 10^3	0,37498 x 10^3
	2	0,38853 x 10^4	0,15918 x 10^4	0,12794 x 10^4	0,96425 x 10^3	0,78010 x 10^3
9	1	0,33461 x 10^4	0,17519 x 10^4	0,89473 x 10^3	0,62161 x 10^3	0,42503 x 10^3
	2	0,46213 x 10^4	0,25383 x 10^4	0,14287 x 10^4	0,10842 x 10^4	0,83980 x 10^3

TABLE 3 – Natural frequencies of a liquid-filled conical shell as a function of shell apex half-angle

(x_1 = 0,141 m, $x_2 - x_1$ = 0,174 m, t = 2,56 x 10^{-4} m, ρ = 7800 kg/m^3, E = 200 GPa, ν = 0,3, ρ_ℓ = 1000 kg/m^3, W = V = 0).

n	m	$(x_2 - x_1)$ (m) 0,087	0,174	0,435	1,392	3,480
2	1	$0,84000 \times 10^4$	$0,51606 \times 10^4$	$0,26050 \times 10^4$	$0,62950 \times 10^3$	$0,17702 \times 10^3$
	2	$0,97556 \times 10^4$	$0,58925 \times 10^4$	$0,34972 \times 10^4$	$0,90910 \times 10^3$	$0,26252 \times 10^3$
3	1	$0,20492 \times 10^4$	$0,65407 \times 10^3$	$0,20716 \times 10^3$	$0,11678 \times 10^3$	$0,21012 \times 10^3$
	2	$0,55519 \times 10^4$	$0,20988 \times 10^4$	$0,79822 \times 10^3$	$0,76389 \times 10^3$	$0,27165 \times 10^3$
4	1	$0,17357 \times 10^4$	$0,68703 \times 10^3$	$0,15363 \times 10^3$	$0,26439 \times 10^2$	$0,12459 \times 10^2$
	2	$0,52721 \times 10^4$	$0,19734 \times 10^4$	$0,41944 \times 10^3$	$0,85899 \times 10^2$	$0,47121 \times 10^2$
5	1	$0,12800 \times 10^4$	$0,60994 \times 10^3$	$0,18876 \times 10^3$	$0,23902 \times 10^2$	$0,54020 \times 10^1$
	2	$0,36980 \times 10^4$	$0,11028 \times 10^4$	$0,43323 \times 10^3$	$0,65030 \times 10^2$	$0,17745 \times 10^2$
6	1	$0,17929 \times 10^4$	$0,83739 \times 10^3$	$0,23407 \times 10^3$	$0,26532 \times 10^2$	$0,47480 \times 10^1$
	2	$0,35552 \times 10^4$	$0,16061 \times 10^4$	$0,47337 \times 10^3$	$0,68493 \times 10^2$	$0,13275 \times 10^2$
7	1	$0,18015 \times 10^4$	$0,11035 \times 10^4$	$0,20960 \times 10^3$	$0,34334 \times 10^2$	$0,52901 \times 10^1$
	2	$0,28419 \times 10^4$	$0,18154 \times 10^4$	$0,53209 \times 10^3$	$0,75138 \times 10^2$	$0,14094 \times 10^2$
8	1	$0,29568 \times 10^4$	$0,46744 \times 10^3$	$0,35924 \times 10^3$	$0,40934 \times 10^2$	$0,63371 \times 10^1$
	2	$0,44248 \times 10^4$	$0,15918 \times 10^4$	$0,61101 \times 10^3$	$0,81197 \times 10^2$	$0,14832 \times 10^2$
9	1	$0,37407 \times 10^4$	$0,17519 \times 10^4$	$0,39989 \times 10^3$	$0,45685 \times 10^2$	$0,72179 \times 10^1$
	2	$0,52719 \times 10^4$	$0,25383 \times 10^4$	$0,67741 \times 10^3$	$0,87151 \times 10^2$	$0,15447 \times 10^2$

TABLE 4 - Natural frequencies of a liquid-filled conical shell as a function of shell apex half-angle
($\alpha = 14,2°$, $x_1 = 0,141$ m, $t = 2,56 \times 10^{-4}$ m, $\rho = 7800$ kg/m^3,
$E = 200$ GPa, $\nu = 0,3$, $\rho_l = 1000$ kg/m^3, $W = V = 0$).

APPENDIX C

FIGURES

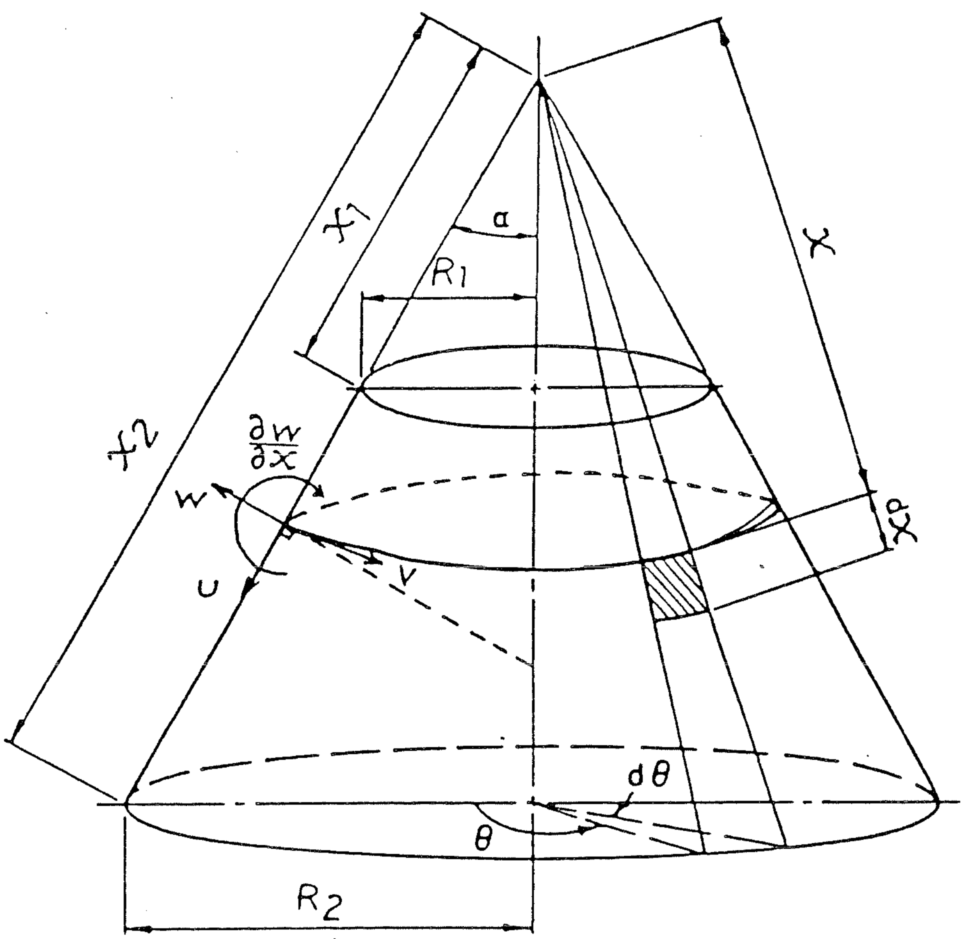

Figure 1 : Reference surface geometry for a conical shell

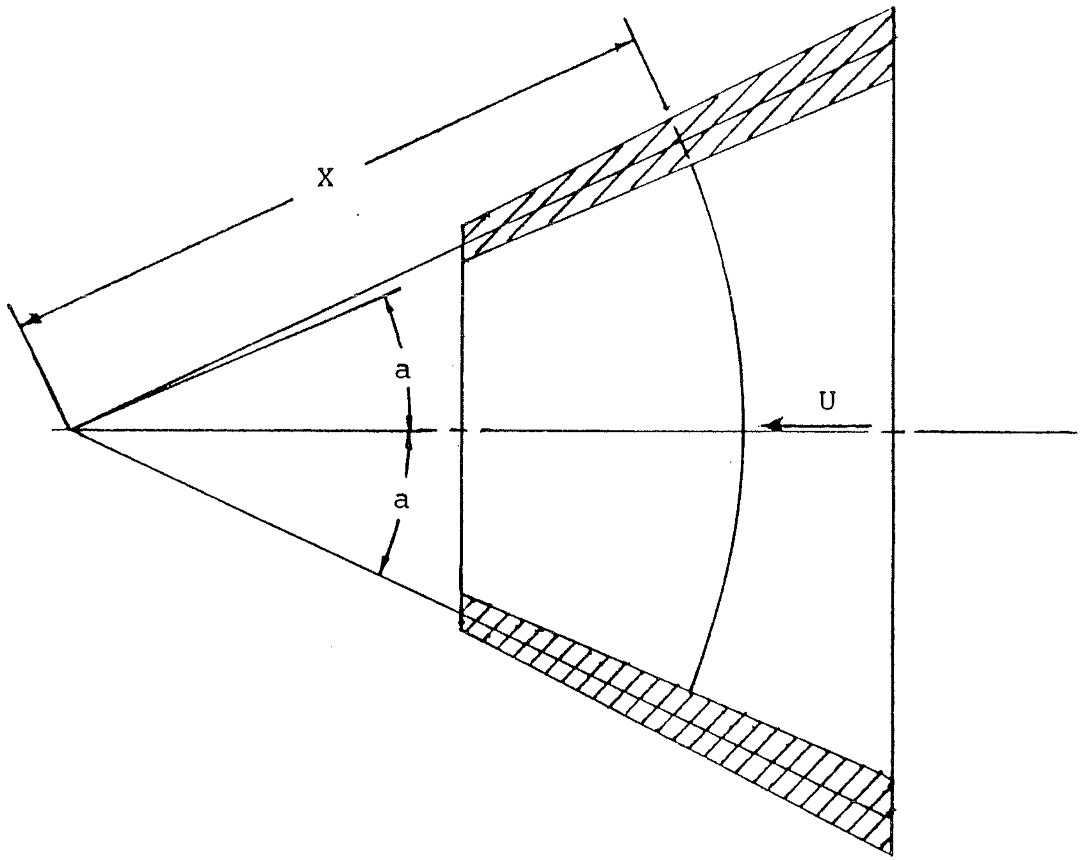

Figure 2 : Conical shell geometry

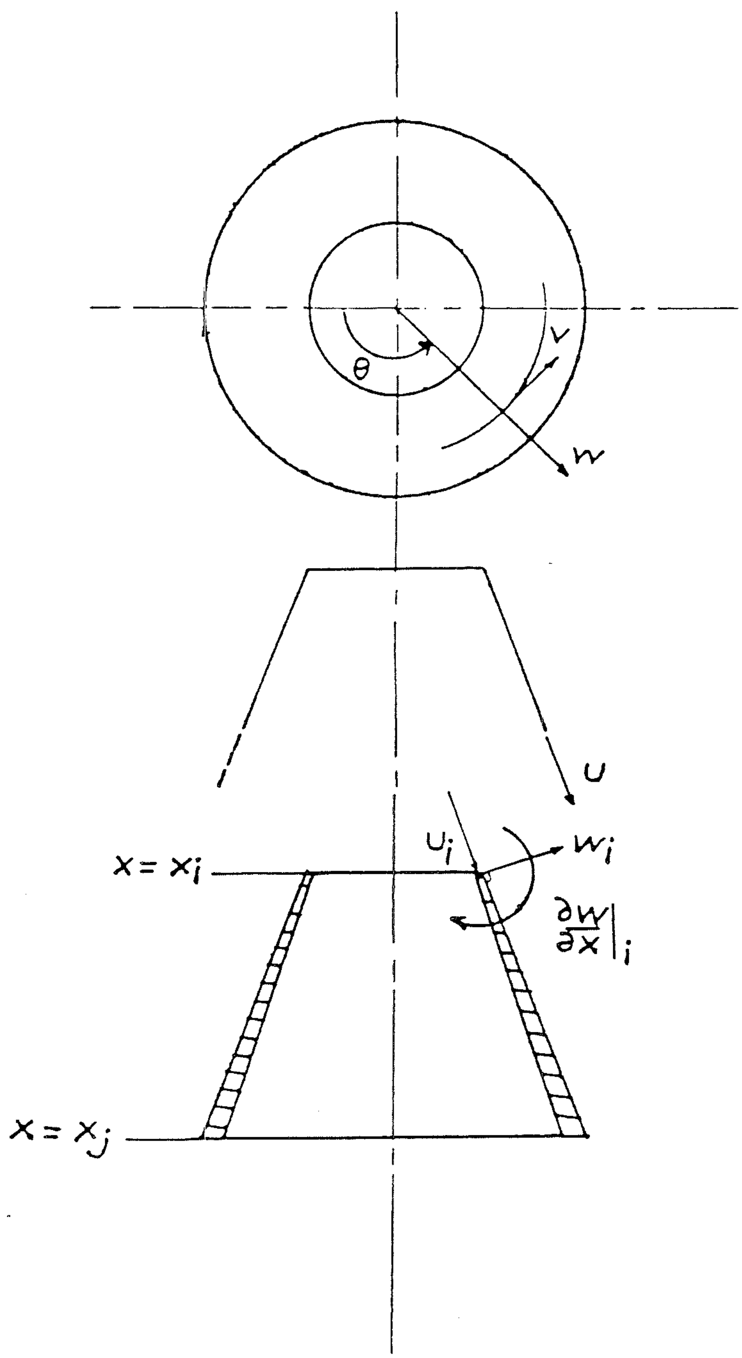

Figure 3 : Displacements and degrees of freedom at a node

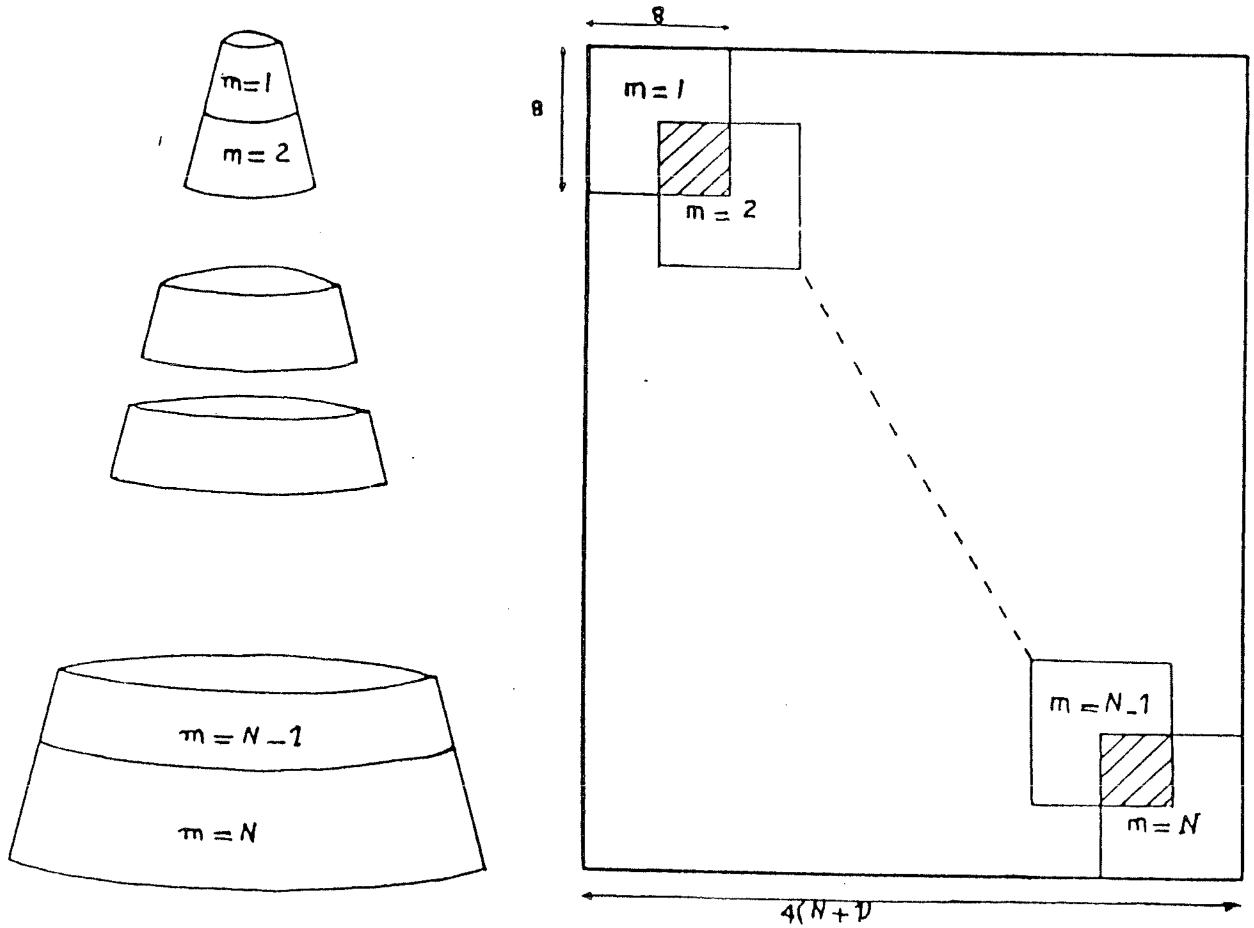

Figure 4 : Assembly diagram for mass, damping and stiffness matrices

FIGURE 5: Natural frequencies of an empty and liquid-filled conical shell with W = V = 0 at both boundaries

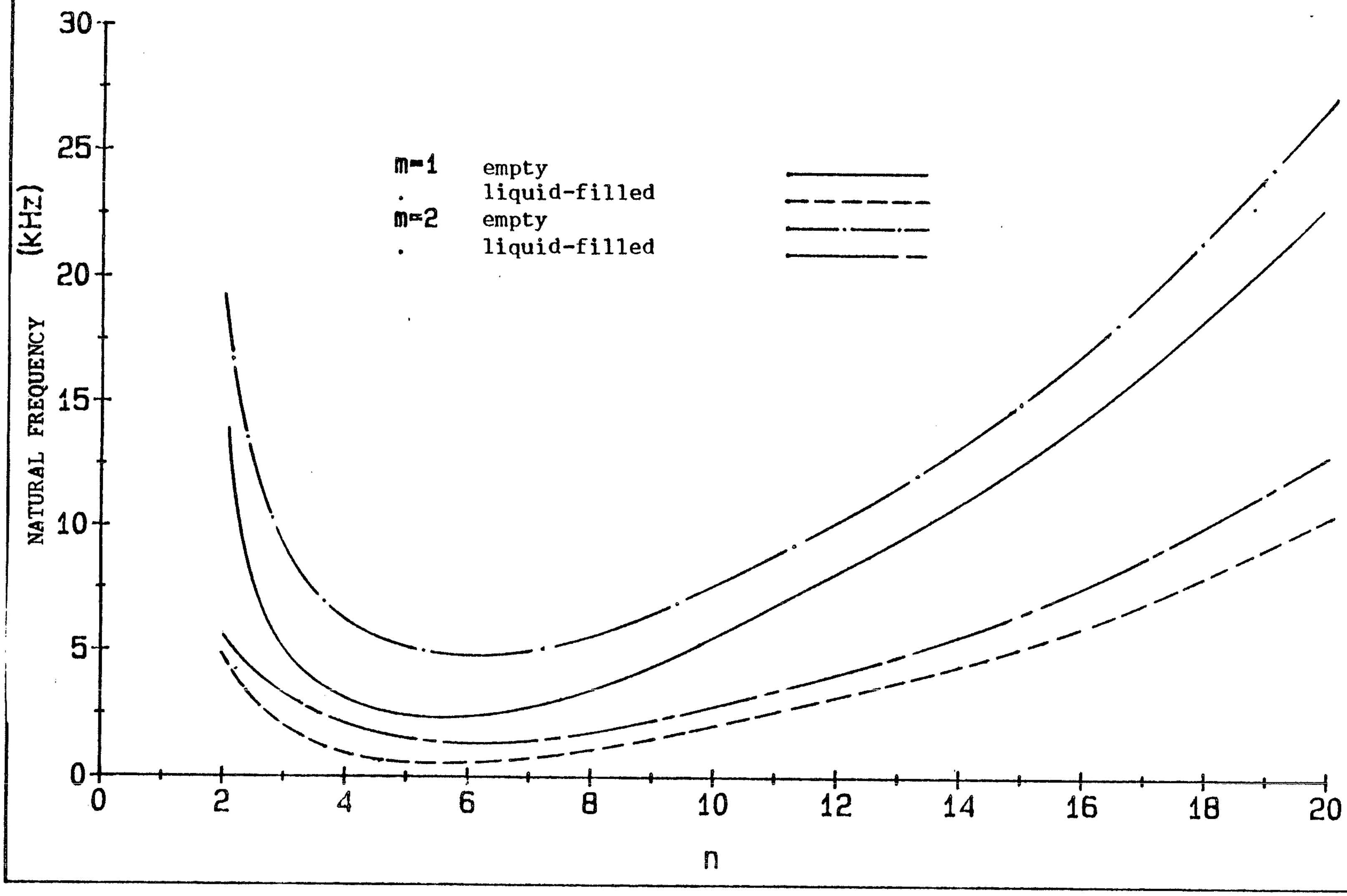

FIGURE 6 Natural frequencies of a conical shell as a function of liquid level with W = V = 0 at both boundaries

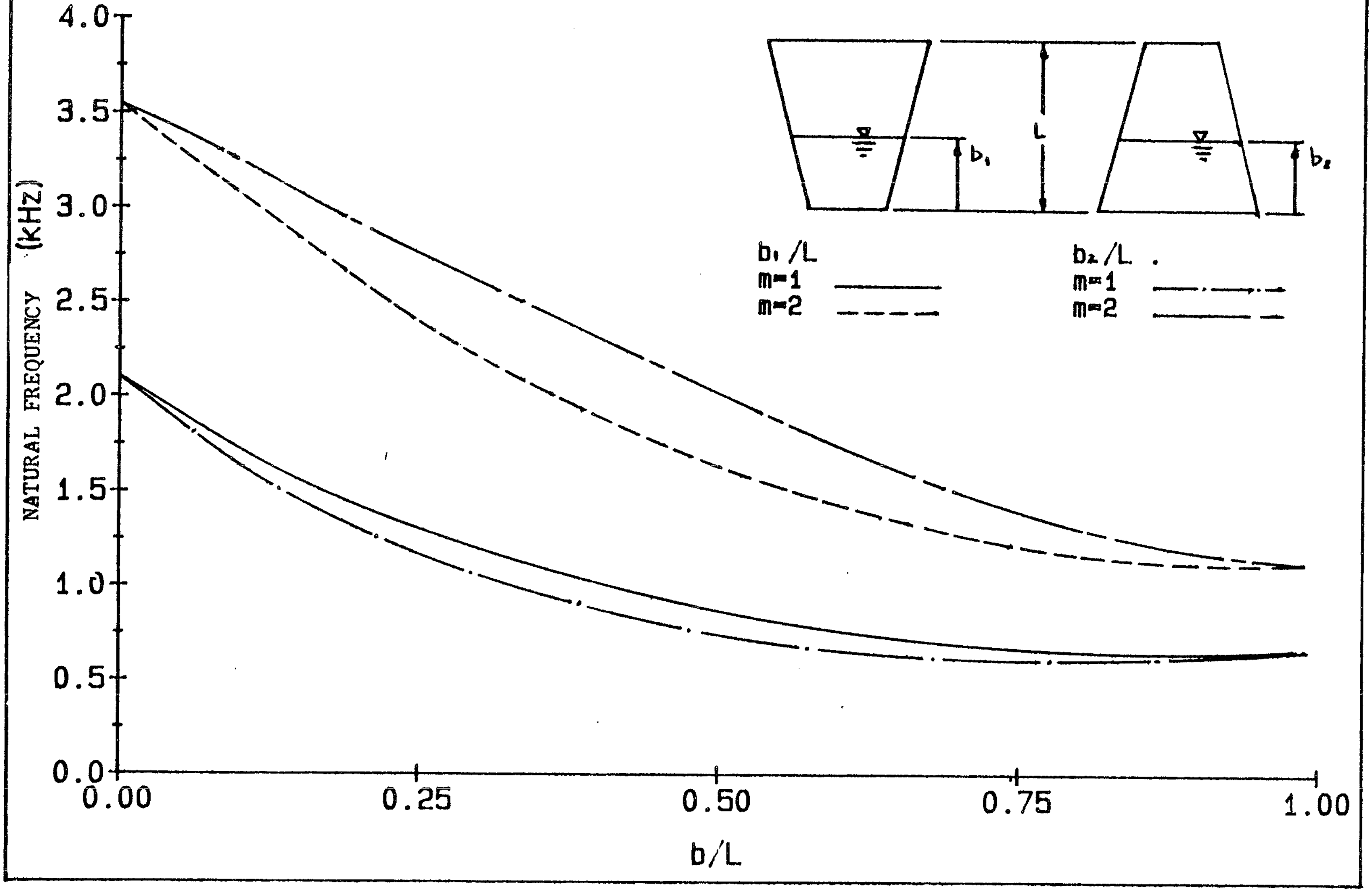

FIGURE 7 Normalized radial displacement mode of a conical shell partially filled with liquid, W = V = 0 at both boundaries. (n=5, m=1)

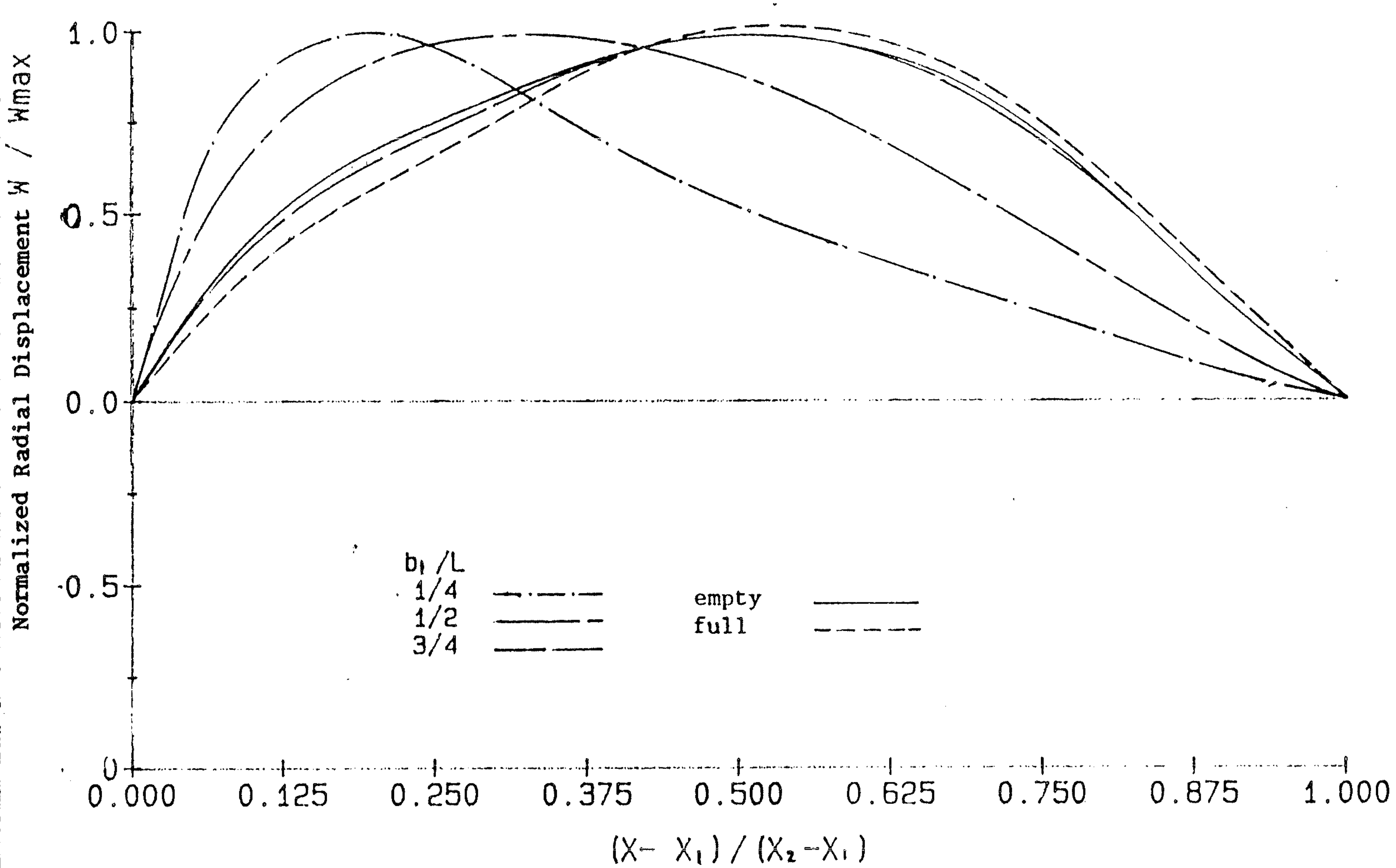

FIGURE 8 Normalized radial displacement mode of a conical shell partially filled with liquid W = V = 0 at both boundaries (n=5, m=2)

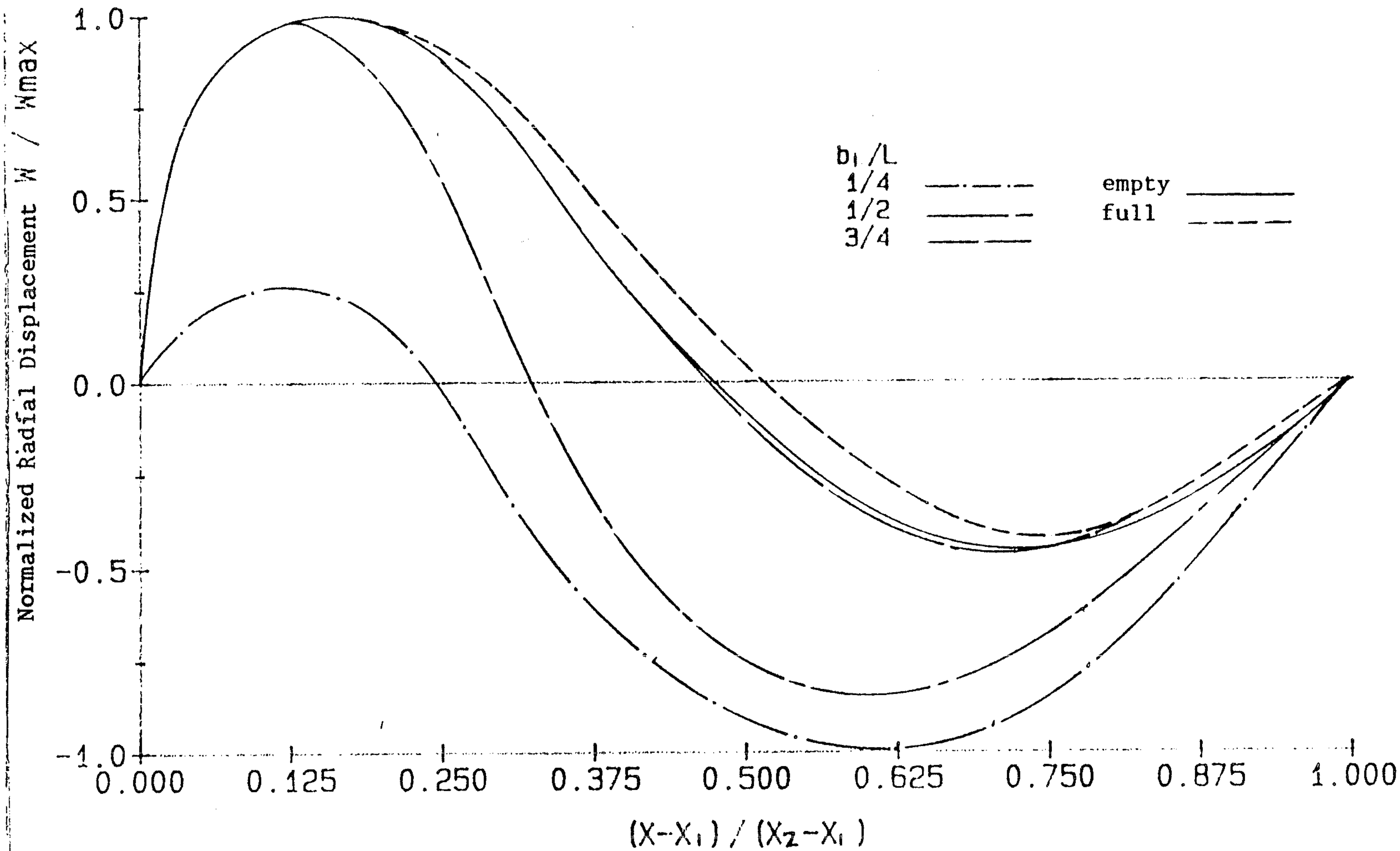

FIGURE 9 Normalized radial displacement mode of a conical shell partially filled with liquid. W = V = 0 at both boundaries (n=5, m=1)

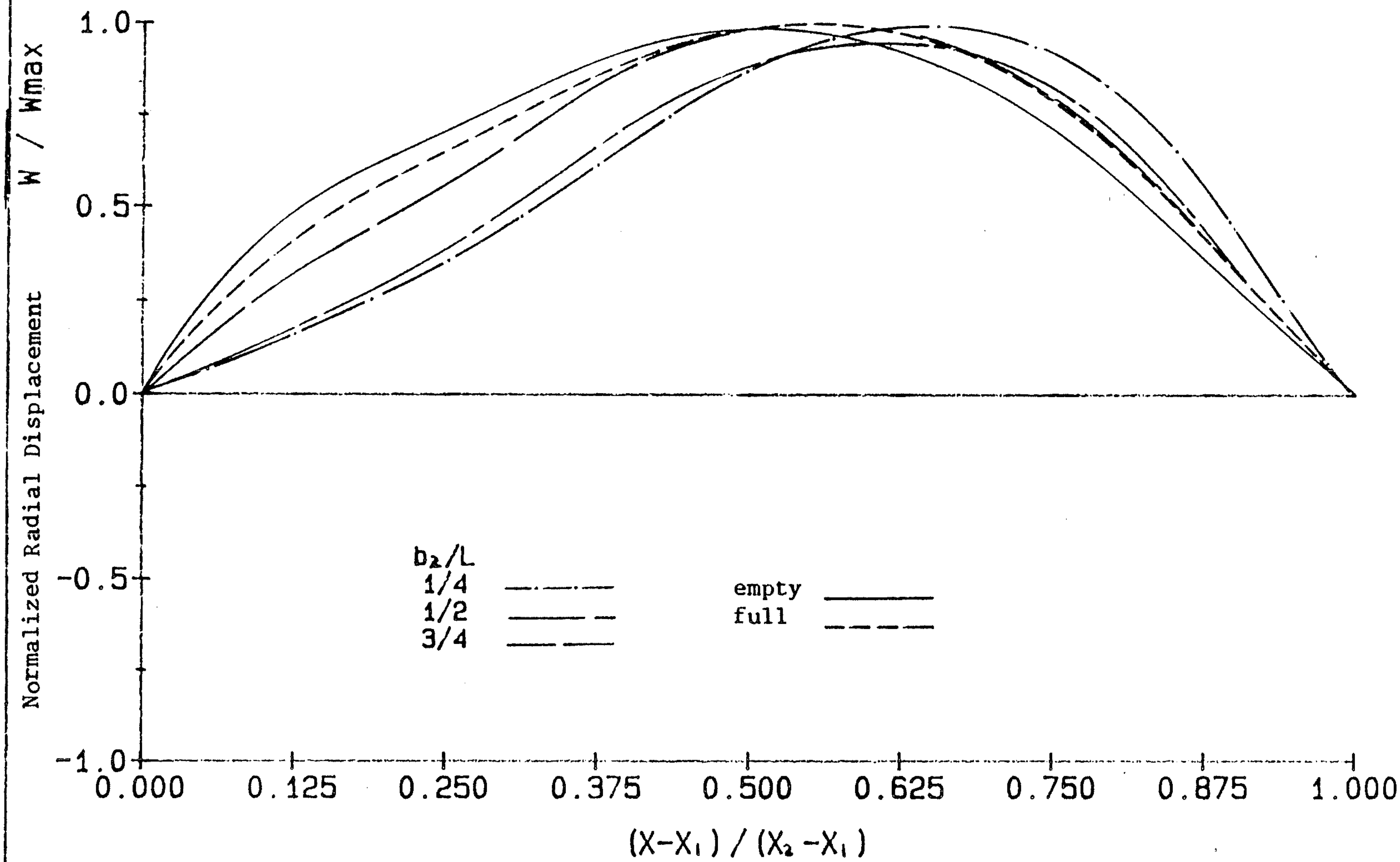

FIGURE 10 Normalized radial displacement mode of a conical shell partially filled with liquid. W = V = 0 at both boundaries (n=5, m=2)

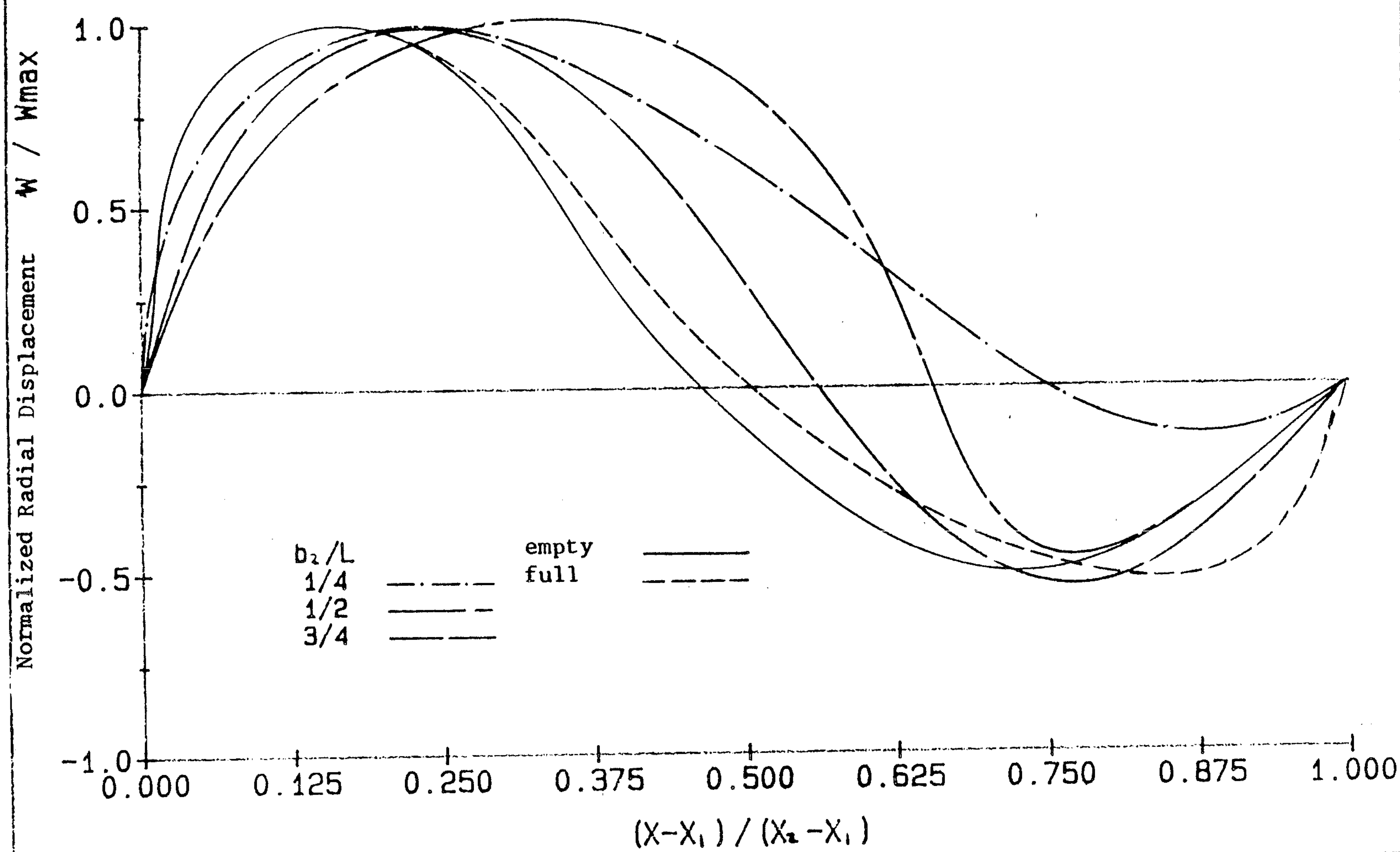

FIGURE 11 Normalized eigenvectors of a liquid-filled conical shell.
W = V = 0 at both boundaries
(n=5, m=1)

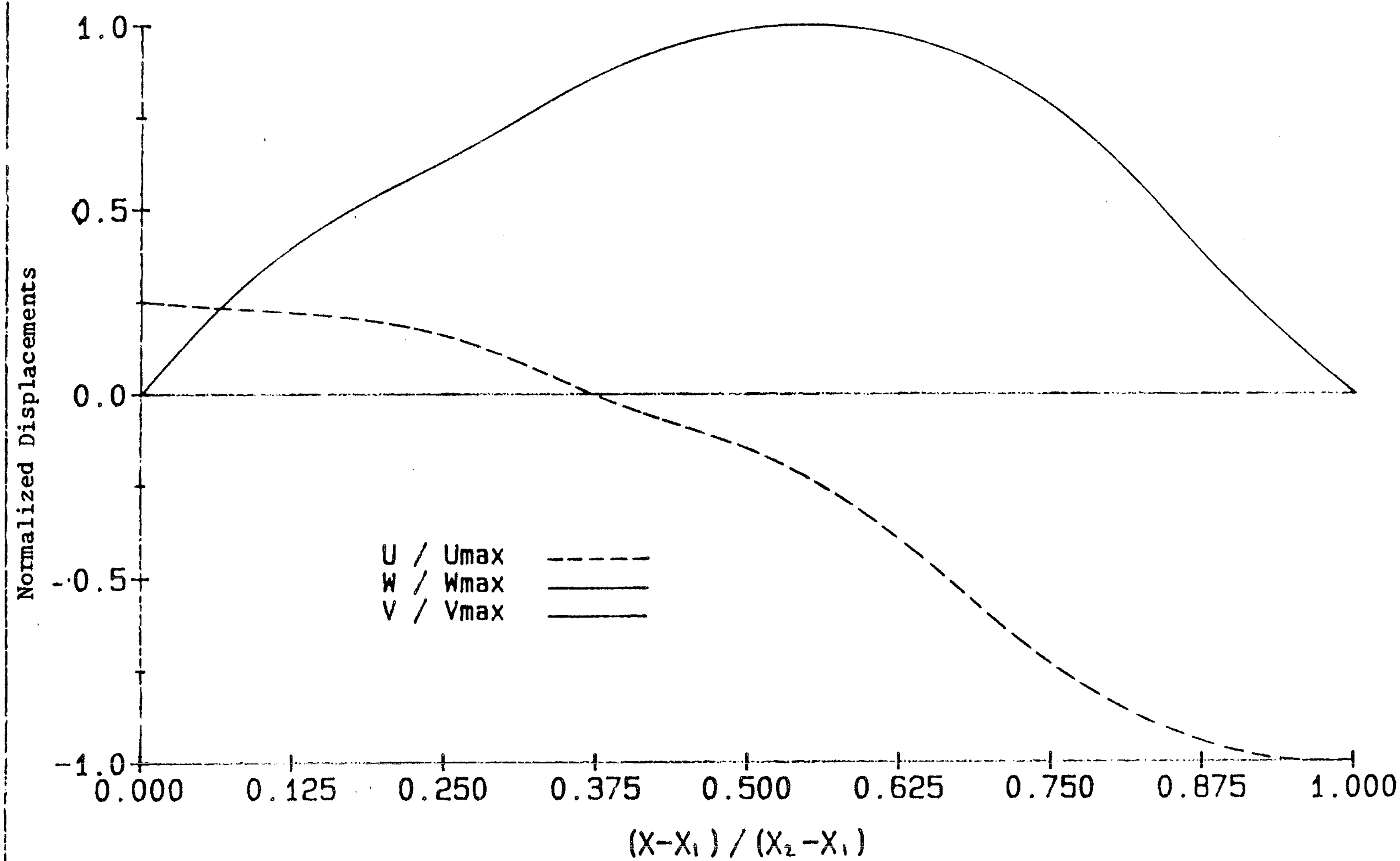

FIGURE 12 Normalized eigenvectors of a liquid-filled conical shell. W = V = 0 at both boundaries. (n=5, m=2)

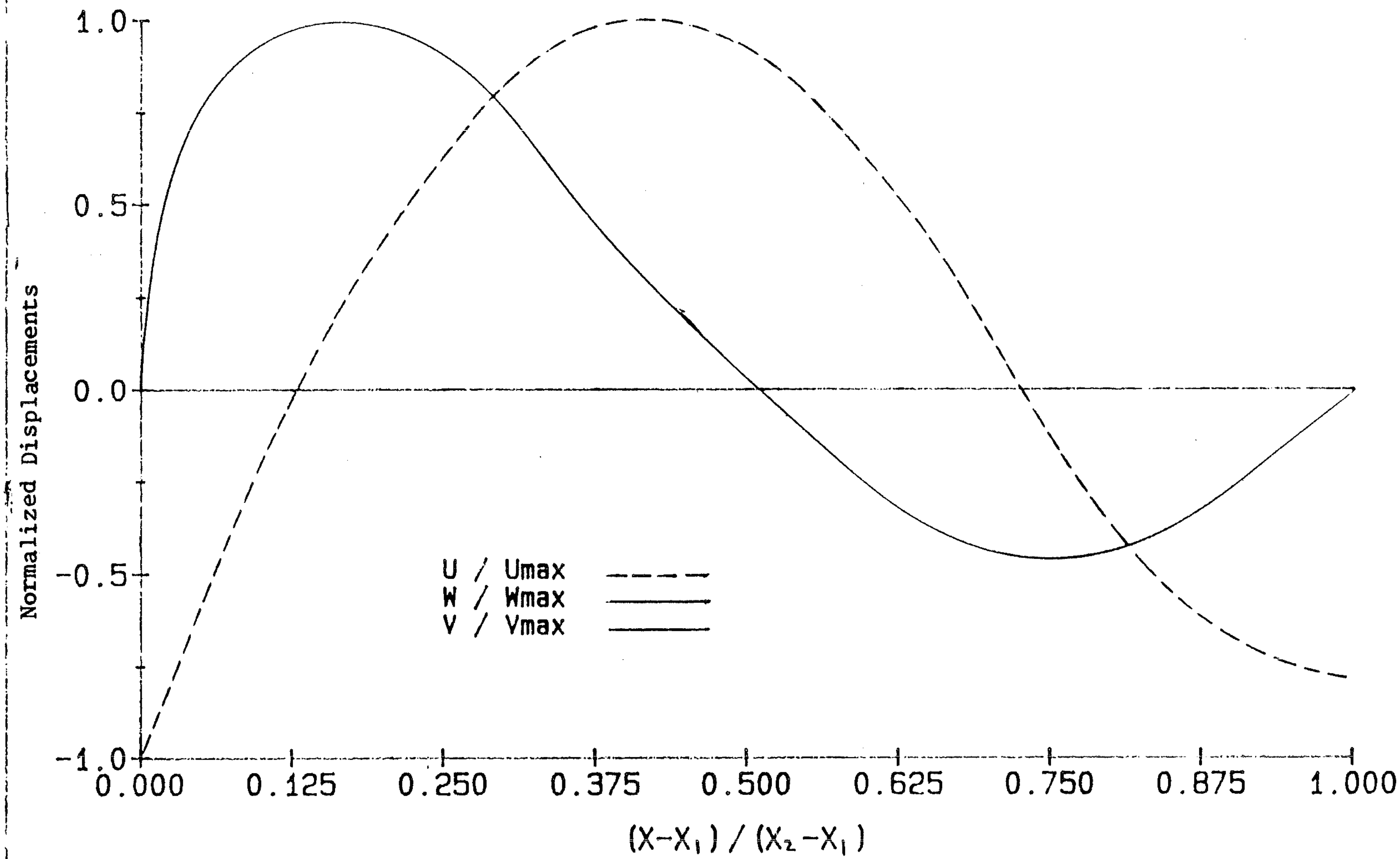

FIGURE 13 Natural frequencies of a liquid-filled conical shell as a function of shell apex half-angle. (m=1)

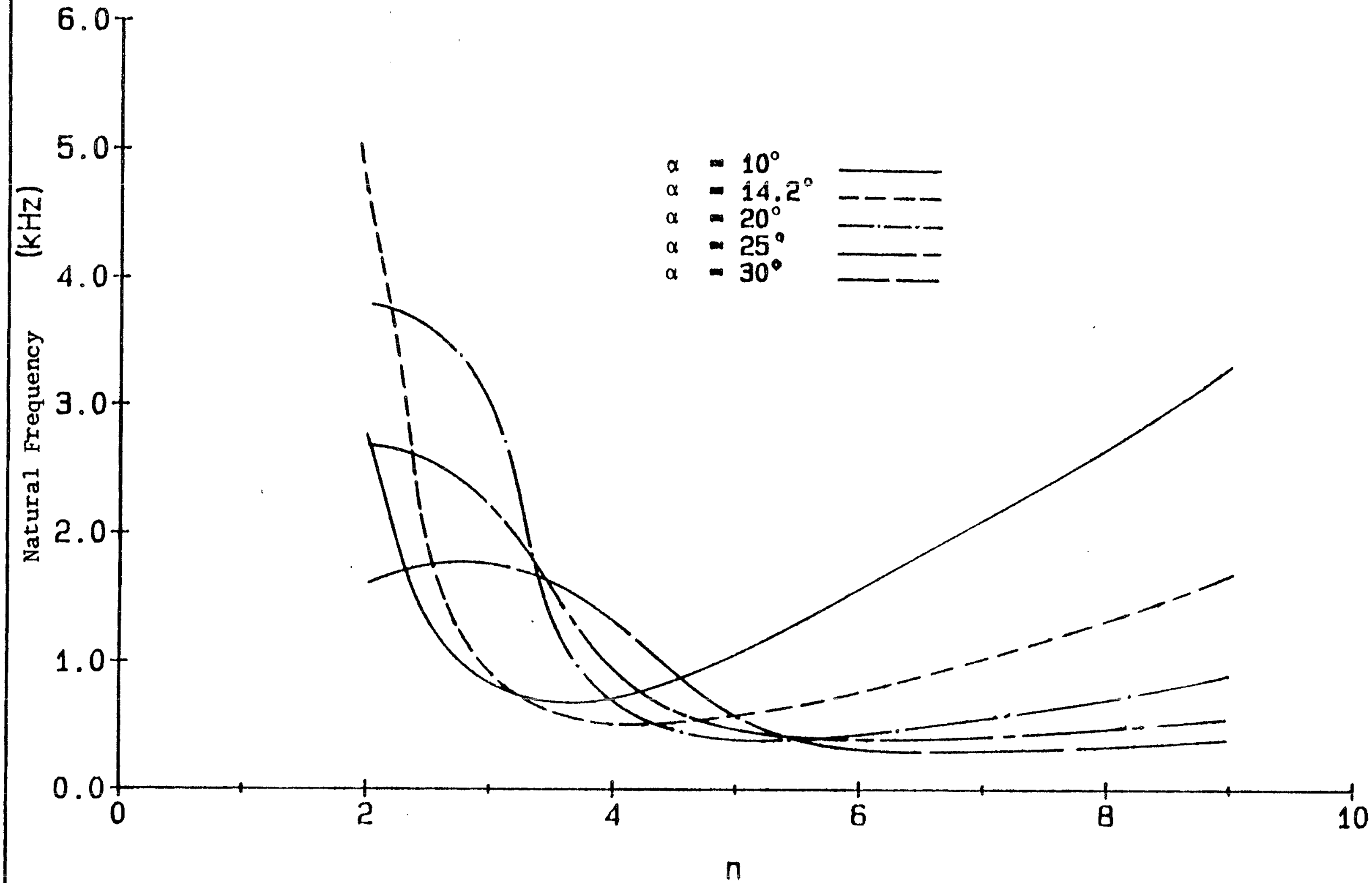

FIGURE 14 Natural frequencies of a liquid-filled conical shell over the natural frequency of an empty conical shell as a function of shell apex half-angle. (m=1)

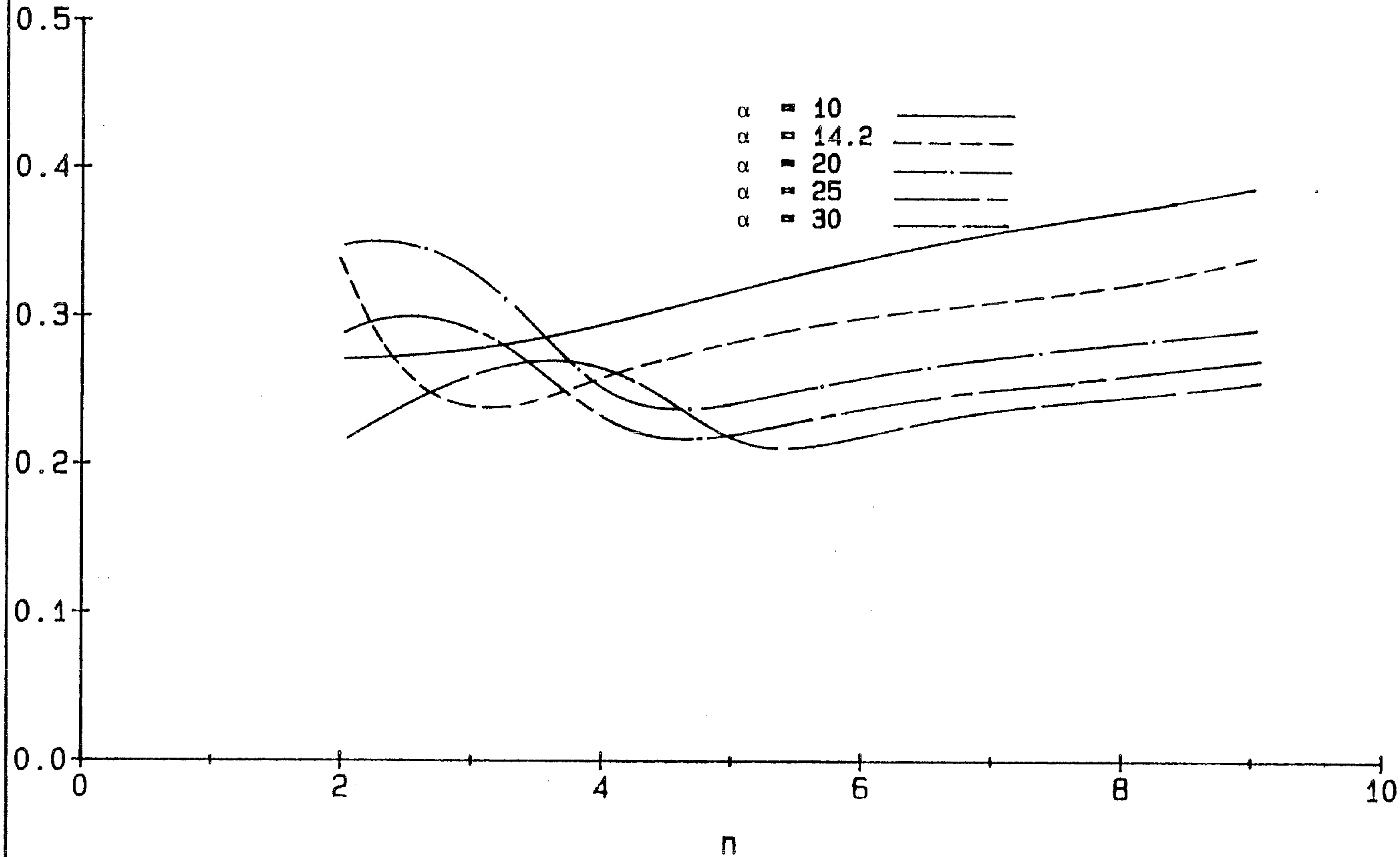

FIGURE 15 Natural frequencies of a liquid-filled conical shell as a function of shell length ($X_2 - X_1$). (m=1)

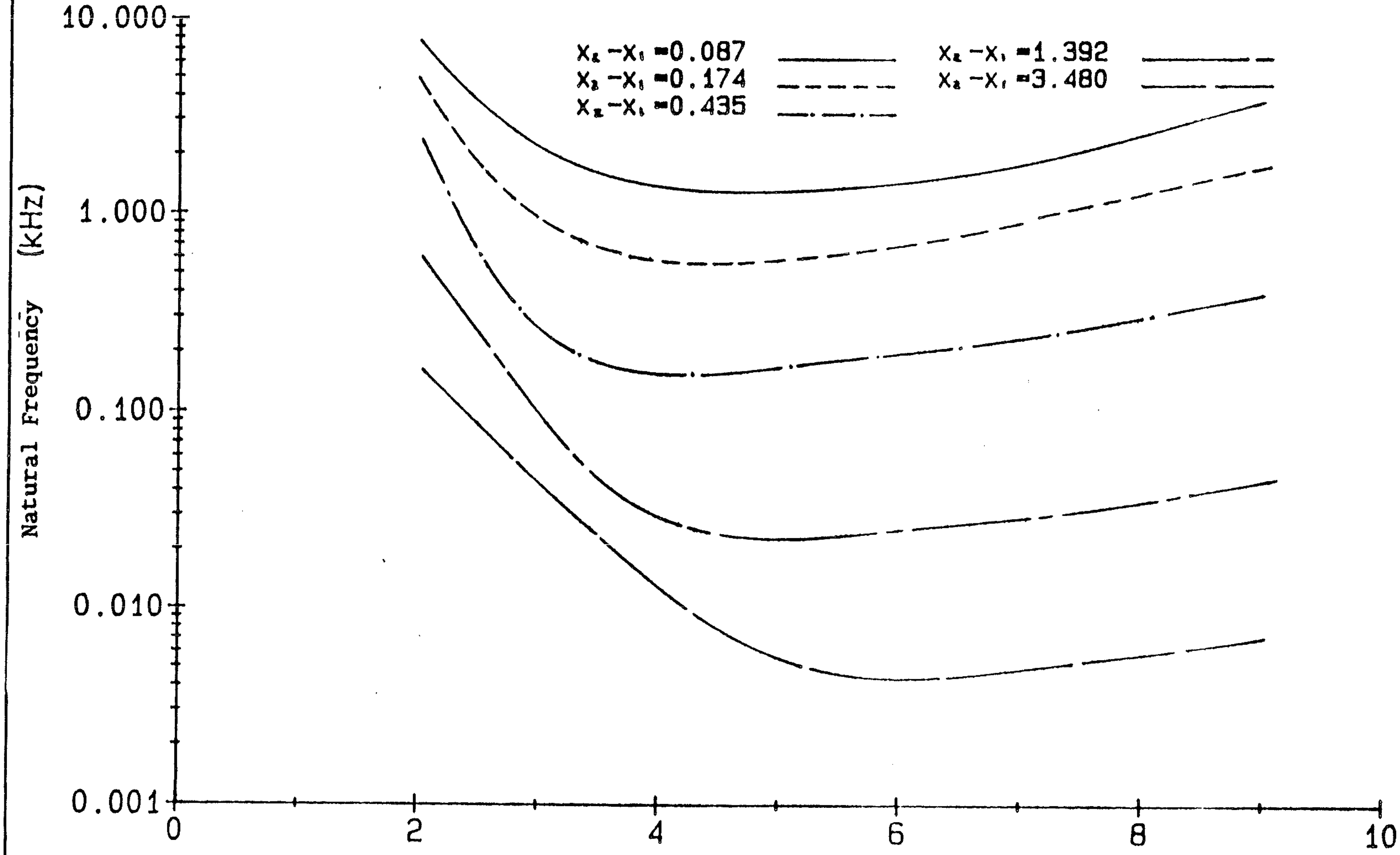

FIGURE 16 Natural frequencies of a liquid-filled conical shell over the natural frequency of an empty conical shell as a function of shell length. (m=1)