DYNAMIC ANALYSIS OF ANISOTROPIC THIN CYLINDRICAL SHELLS SUBJECTED TO BOUNDARY - LAYER - INDUCED RANDOM PRESSURE FIELDS

by

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CONTENTS

Summary

A theory is presented for the determination of the free vibration characteristics of uniform or axially non-uniform anisotropic thin cylindrical shells and the response of such shells to boundary-layer pressure fields caused by subsonic internal flow. It is a hybrid of finite-element and classical shell theories. The finite elements are cylindrical frusta and the displacement functions are determined from anisotropic shell equations. The random pressure forces are lumped at the nodes of the finite elements. The mean square response of the displacements of the shell are obtained for a boundary-layer pressure field and some calculations are conducted to illustrate the theory.

1. INTRODUCTION

A careful study of the shells used in practical applications leads to the conclusion that they are most often anisotropic (naturally or structurally) and in many cases are anisotropic and laminar. Although the problem of determining the natural frequencies of isotropic shells has produced many papers, the litterature reveals a very limited number of methods which have been generally developed for special cases of anisotropic cylindrical shells. The need is evident for a theory which can be used for the dynamic analysis of any kind of anisotropic circular cylindrical shell subjected to various boundary conditions. A practical case in point is concerned with the prediction of the natural frequencies of a double-walled steam generator [1-2].

This work attempts to fill these voids by producing a general theory with a minimum of limitations for the free vibration characteristics and the response of anisotropic cylindrical shells subjected to random pressure fields which originate from the turbulent boundary layer of an internal flow.

The analysis is based on a recently developed method for the case of isotropic cylindrical shells [4]. It is a hybrid theory based on the finite element method, with the displacement functions determined by exact solution of the equations of equilibrium of a thin cylindrical shell. The finite elements are cylindrical frusta; thus a given non-uniform shell is first subdivided into its component uniform cylindrical segments and then, generally, each segment is similarly subdivided into a number of cylindrical finite elements.

The theory for predicting the response due to random pressure fields is developed in reference [5]. The continuous pressure field is transformed to a discrete set of forces; then, the cross-correlation spectral density and the mean square values of the displacement of the shell are expressed in terms of correlation functions of the boundary-layer pressure fields.

Here the dynamics of a cylindrical shell and its response will be considered, with the following aims: (i) to extend the theory of [4] to cases where the shells are anisotropic and especially for the case of shells consisting of an arbitrary number of orthotropic layers; (ii) to use the theory of [5] to predict the response of such shells to a pressure field arising from the turbulent boundary-layer of internal flow. This generalized theory will be more directly pertinent to engineering applications, since in nearly all practical cases the shells are often anisotropic; e.g., heat exchangers and liquid metal cooled channels used in the nuclear industry. A number of assumptions are made during the course of the investigation; a compendium of these assumptions and the limitations of the theory will be given in the text.

2. FREE VIBRATION

2.1 General Theory

A given shell is subdivided into a number of finite elements, each being defined by the two nodes, i and j, and the corresponding nodal circle boundaries (Fig. 1). Then, the displacement functions may be defined by

$$[U(x,\varphi),\ W(x,\varphi),\ V(x,\varphi)]^T = [N]\,[\delta_i,\ \delta_j]^T \tag{1}$$

where $\{\delta_i\}$ and $\{\delta_j\}$ represent the nodal displacements, and the elements of [N] are in general functions of position and the shell's anisotropy.

It is noted that the finite-element method yields useful results provided that the displacement functions chosen represent adequately the true displacements; accordingly, the displacement functions should satisfy the convergence criterion of the finite-element method stating that strains within the element should be zero when the nodal displacements are generated by rigid-body motions. To this end, we shall employ the equations of thin cylindrical shells to obtain the displacement functions, instead of using the more common arbitrary polynomial forms.

Sander's theory [7] for thin cylindrical shells is used for the determination of these displacement functions. This shell theory which is based on Love's first approximation was preferred, for the following reason: in Sander's theory all strains vanish for small rigid body motions, which is not true for Love's or Timoshenko's theories, for instance. By using such displacement functions, we automatically satisfy the convergence criterion of the finite-element method previously stated.

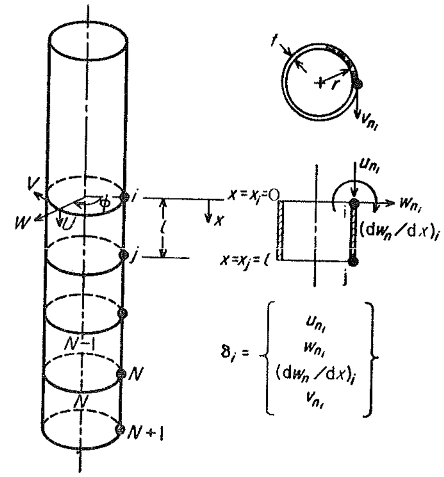

Figure 1. Definition of the finite element used and the displacement vector associated with node i, $\{\delta_i\}$.

2.2 Equations of Motion

Using Love's first approximation, we obtain the following elasticity relationships between the stress-resultant and the deformations of the middle surface for the general case of a multi-layer anisotropic shell

$$\begin{Bmatrix} N_x \\ N_\varphi \\ \bar{N}_{x\varphi} \\ M_x \\ M_\varphi \\ \bar{M}_{x\varphi} \end{Bmatrix} = [P]\{\varepsilon\} , \quad (2) \qquad [P] = \begin{bmatrix} p_{11} & p_{12} & 0 & p_{14} & p_{15} & 0 \\ p_{21} & p_{22} & 0 & p_{24} & p_{25} & 0 \\ 0 & 0 & p_{33} & 0 & 0 & p_{36} \\ p_{41} & p_{42} & 0 & p_{44} & p_{45} & 0 \\ p_{51} & p_{52} & 0 & p_{54} & p_{55} & 0 \\ 0 & 0 & p_{63} & 0 & 0 & p_{66} \end{bmatrix} , \quad (3)$$

the elements p_{ij} of the elasticity matrix [P] characterize the shell's anisotropy which depends on the mechanical properties of the material of the structure.

The strain vector $\{\varepsilon\}$ is the modified strain-displacement relations of Sanders [7] and is given by

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_\varphi \\ 2\epsilon_{x\varphi} \\ \varkappa_x \\ \varkappa_\varphi \\ 2\bar{\varkappa}_{x\varphi} \end{Bmatrix} = \begin{Bmatrix} \partial U/\partial x \\ (1/r)\ (\partial V/\partial \varphi) + (W/r) \\ \partial V/\partial x + (1/r)\ (\partial U/\partial \varphi) \\ -\partial^2 W/\partial x^2 \\ -(1/r^2)\ [(\partial^2 W/\partial \varphi^2) - (\partial V/\partial \varphi)] \\ -(2/r)\ (\partial^2 W/\partial x \partial \varphi) + (3/2r)\ (\partial V/\partial x) - (1/2r^2)\ (\partial U/\partial \varphi). \end{Bmatrix} \tag{4}$$

Upon substituting equations (2) - (4) into Sanders' shell equations of motion [7], the authors obtain the equations of equilibrium in terms of elements p_{ij} of [P] ~~and~~ in terms of U, V and W, namely

$$\begin{aligned}
&(\partial^2 U/\partial x^2)+(1/r)p_{12}(\partial W/\partial x)-p_{14}(\partial^3 W/\partial x^3)+[(1/r)(p_{12}+p_{33})+(1/r^2)(p_{15}+p_{36})-(3/4r^3)p_{66}]. \\
&.(\partial^2 V/\partial\varphi\partial x)+(1/r^2)[p_{33}-(1/r)p_{36}+(1/4r^2)p_{66}]\ (\partial^2 U/\partial\varphi^2)-(1/r^2)\ [p_{15}+2p_{36}- \\
&-(1/r)p_{66}](\partial^3 W/\partial x\partial\varphi^2) = 0\ , \\
&r)[p_{33}+p_{21}+(1/r)p_{36}+(1/r)p_{51}-(3/4r^2)p_{66}](\partial^2 U/\partial\varphi\partial x)+(1/r^2)\ [p_{22}+(1/r^2)p_{55}+ \\
&+(2/r)p_{25}](\partial^2 V/\partial\varphi^2)+[p_{33}+(3/r)p_{36}+(9/4r^2)p_{66}]\ (\partial^2 V/\partial x^2)+[p_{22}+(1/r)p_{52}](1/r^2). \\
&.(\partial W/\partial\varphi)-(1/r^3)[p_{25}+(1/r)p_{55}](\partial^3 W/\partial\varphi^3)-(1/r)\ [2p_{36}+p_{24}+(3/r)p_{66}+(1/r)p_{54}]. \\
&.(\partial^3 W/\partial\varphi\partial x^2) = 0\ , \\
&/r)p_{21}(\partial U/\partial x)-(1/r^2)\ [p_{22}+(1/r)p_{25}](\partial V/\partial\varphi)-(1/r^2)p_{22}W+p_{41}(\partial^3 U/\partial x^3)+(1/r^2)[p_{51}+ \\
&+2p_{63}-(1/r)p_{66}](\partial^3 U/\partial x\partial\varphi^2)+(1/r^3)\ [p_{52}+(1/r)p_{55}](\partial^3 V/\partial\varphi^3)+(1/r)\ [p_{42}+2p_{63}+ \\
&+(1/r)p_{45}+(3/r)p_{66}](\partial^3 V/\partial\varphi\partial x^2)+(2/r^3)p_{25}(\partial^2 W/\partial\varphi^2)-(1/r^4)p_{55}(\partial^4 W/\partial\varphi^4)+(2/r)p_{24}\ . \\
&.(\partial^2 W/\partial x^2)-p_{44}(\partial^4 W/\partial x^4)-(1/r^2)(2p_{45}+4p_{66})(\partial^4 W/\partial x^2\partial\varphi^2) = 0\ .
\end{aligned} \tag{5}$$

Here U, V and W are, respectively, the axial, circumferential and radial displacements of the middle surface of the shell, and r its mean radius (Fig. 1). The solution of these equations will give the displacement functions.

2.3 The Displacement Functions

In the continuum, we express U, V and W of the middle surface of the shell by

$$\begin{Bmatrix} U(x,\varphi) \\ W(x,\varphi) \\ V(x,\varphi) \end{Bmatrix} = \begin{bmatrix} \cos n\varphi & o & o \\ o & \cos n\varphi & o \\ o & o & \sin n\varphi \end{bmatrix} \begin{Bmatrix} u_n(x) \\ w_n(x) \\ v_n(x) \end{Bmatrix} = [T] \begin{Bmatrix} u_n(x) \\ w_n(x) \\ v_n(x) \end{Bmatrix} , \tag{6}$$

where n is the circumferential wave-number. By substituting equation (6) into equation (5) and letting

$$u_n(x) = A\, e^{\lambda x/r} , \quad v_n(x) = B\, e^{\lambda x/r} , \quad w_n(x) = C\, e^{\lambda x/r} , \tag{7}$$

we obtain three simultaneous ordinary linear equations in A, B, C of the form

$$[H] \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \{0\}. \tag{8}$$

For non-trivial solution, the determinant of [H] must vanish, leading to the following characteristic equation

$$h_8\lambda^8 - h_6\lambda^6 + h_4\lambda^4 - h_2\lambda^2 + h_o = 0 , \tag{9}$$

where

$$= (h_9/r^2)(p_{11}p_{44} - p_{14}^2) ,$$

$$= (n^2/r^2)\,[h_9(h_1p_{44} + 2p_{11}p_{45} + 4p_{11}p_{66} - 2h_5rp_{14}) + h_7(p_{11}p_{44} - p_{14}^2) - r^2h_{11}^2p_{11} - -h_3^2p_{44} + 2rh_3h_{11}p_{14}] + (2/r)h_9(p_{11}p_{24} - p_{14}p_{12}) ,$$

$$(n^4/r^2)\,[h_1h_7p_{44} + h_9p_{11}p_{55} + (2p_{45} + 4p_{66})(h_1h_9 + h_7p_{11} - h_3^2) + (p_{25} + (1/r)p_{55}).$$
$$.(2h_3p_{14} - 2h_{11}p_{11}r) + h_{11}r^2(2h_3h_5 - h_1h_{11}) - rh_5(2h_7p_{14} + rh_5h_9)] + (n^2/r).$$
$$.[2(p_{25} + rp_{22})\,((h_3/r)p_{14} - h_{11}p_{11}) - 2p_{12}(h_5h_9r + h_7p_{14} - h_3h_{11}r) - 2p_{24}(h_3^2 - h_1h_9 - h_7p_{11})$$
$$+ 2h_9p_{11}p_{25}] + h_9\,(p_{11}p_{22} - p_{12}^2) ,$$

$$= (n^6/r^2)\,[h_1h_7(2p_{45} + 4p_{66}) + p_{55}(h_1h_9 + h_7p_{11} - h_3^2) - r^2h_5^2h_7 + (p_{25} + (1/r)p_{55}).$$
$$.(-2rh_1h_{11} + 2rh_3h_5 - p_{11}p_{25} - (1/r)\,p_{11}p_{55})] + (n^4/r)\,[2h_1h_7p_{24} + 2p_{25}(h_1h_9 + h_7p_{11} -$$
$$-h_3^2) - 2p_{12}(rh_5h_7 - h_3p_{25} - (h_3/r)p_{55}) - 2(p_{25} + rp_{22})(h_1h_{11} + (1/r)p_{11}p_{25} + (1/r^2)p_{11}p_{55} -$$
$$-h_3h_5)] + n^2[p_{22}(h_1h_9 + h_7p_{11} - h_3^2) - (1/r)(p_{25} + rp_{22})((1/r)p_{11}p_{25} + p_{11}p_{22} - 2h_3p_{12}) -$$
$$-h_7p_{12}^2]\,,$$
$$= n^4h_1h_7[p_{22} + (2/r)n^2p_{25} + (n^4/r^2)p_{55}] - n^2h_1[(n^3/r)(p_{25} + (1/r)p_{55} + (n/r)(p_{25} + rp_{22})]^2$$

and the parameters h_i , i = 1, 3, 5, 7, 9, 11 are given by

$$h_1 = p_{33} - (1/r)p_{36} + (1/4r^2)p_{66}\,, \quad h_3 = p_{12} + p_{33} + (1/r)(p_{15} + p_{36}) - (3/4r^2)p_{66}\,,$$
$$h_5 = (1/r)(p_{15} + 2p_{36} - (1/r)p_{66})\,, \quad h_7 = p_{22} + (1/r^2)p_{55} + (2/r)p_{25} \qquad ,(10)$$
$$h_9 = p_{33} + (3/r)p_{36} + (9/4r^2)p_{66}\,, \quad h_{11} = (1/r)[2p_{36} + p_{24} + (3/r)p_{66} + (1/r)p_{54}].$$

This characteristic equation for anisotropic cylindrical shells which is a quartic in λ^2, has the same general form as equation (5) of [4] for isotropic one. The eight roots λ_i may therefore be written as follows

$$\lambda_1 = -\varkappa_1 + i\,\mu_1\,,\quad \lambda_2 = -\varkappa_1 - i\,\mu_1\,,\quad \lambda_3 = -\varkappa_2 + i\,\mu_2\,,\quad \lambda_4 = -\varkappa_2 - i\,\mu_2\,,$$
$$\lambda_5 = \varkappa_1 + i\,\mu_1\,,\quad \lambda_6 = \varkappa_1 - i\,\mu_1\,,\quad \lambda_7 = \varkappa_2 + i\,\mu_2\,,\quad \lambda_8 = \varkappa_2 - i\,\mu_2\,. \qquad (11)$$

where $\varkappa_i$ and μ_i are real. Each root, λ_j, yields a solution of equation (5), the complete solution being obtained by the sum of all eight with the constants Aj, Bj and Cj, j = 1, 2,... 8.

For every j, the three constants Aj, Bj and Cj are related among each other by the linear equations (8), so that u_n, v_n and w_n may be expressed in terms of only eight constants. To this end, we let

$$A_j = \alpha_j C_j, \qquad B_j = \beta_j C_j , \tag{12}$$

where α_j and β_j, for $j = 1$ and 3, may be expressed as follows

$$\alpha_1 = \bar{\alpha}_1 + i\bar{\alpha}_2 , \quad \alpha_3 = \bar{\alpha}_3 + i\bar{\alpha}_4 , \quad \bar{\beta}_1 + i\bar{\beta}_2 , \quad \beta_3 = \bar{\beta}_3 + i\bar{\beta}_4 \tag{13}$$

The real and imaginary parts of α_j, β_j, $j = 1$ and 3, may be obtained from the following relationships

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} \alpha_j \\ \beta_j \end{Bmatrix} = \begin{Bmatrix} -a_{13} \\ -a_{23} \end{Bmatrix} , \tag{14}$$

where

$$a_{11} = n^2 h_{11} - \lambda_j^2 p_{11} , \quad a_{12} = -n\lambda_j h_3 , \quad a_{13} = -\lambda_j (n^2 h_5 + p_{12}) + (1/r)\lambda_j^3 p_{14} ,$$

$$a_{21} = a_{12} , \quad a_{22} = -n^2 h_7 + \lambda_j^2 h_9 , \quad a_{23} = -(n/r)\ (1+n^2) p_{25} - np_{22}$$

$$- (n^3/r^2) p_{55} + n\lambda_j^2 h_{11} .$$

By inspecting the coefficients of equations (8), the other α_j, β_j can be given by

$$\begin{array}{lll} \alpha_2 = \bar{\alpha}_1 - i\ \bar{\alpha}_2 & \alpha_5 = \bar{\alpha}_5 + i\ \bar{\alpha}_6 = -\alpha_2 & \beta_5 = \bar{\beta}_5 + i\ \bar{\beta}_6 = \beta_2 \\ \alpha_4 = \bar{\alpha}_3 - i\ \bar{\alpha}_4 & \alpha_6 = \bar{\alpha}_5 - i\ \bar{\alpha}_6 = -\alpha_1 & \beta_6 = \bar{\beta}_5 - i\ \bar{\beta}_6 = \beta_1 \\ \beta_2 = \bar{\beta}_1 - i\ \bar{\beta}_2 & \alpha_7 = \bar{\alpha}_7 + i\ \bar{\alpha}_8 = -\alpha_4 & \beta_7 = \bar{\beta}_7 + i\ \bar{\beta}_8 = \beta_4 \\ \beta_4 = \bar{\beta}_3 - i\ \bar{\beta}_4 & \alpha_8 = \bar{\alpha}_7 - i\ \bar{\alpha}_8 = -\alpha_3 & \beta_8 = \bar{\beta}_7 - i\ \bar{\beta}_8 = \beta_3 . \end{array} \tag{15}$$

Upon substituting the relations (12)-(15) into equation (7) and thence into equations (6) we obtain expressions for the displacement functions in terms of eight constants $\bar{C}_j$. These expressions may be written as

$$\begin{Bmatrix} U(x,\varphi) \\ W(x,\varphi) \\ V(x,\varphi) \end{Bmatrix} = [T]\ [R]\ \{\bar{C}\} , \tag{16}$$

where [R] is given in appendix I and $\{\bar{C}\} = [\bar{C}_1 \ldots \bar{C}_8]^T$. The eight $\bar{C}_j$ are the only free constant which must be determined from eight boundary conditions, four at each edge of the finite element. The nodal displacements(Fig. 1) at nodes i, (x = 0) and j, (x = ℓ) are defined by

$$\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = \{u_{ni}, w_{ni}, (dw_n/dx)_i, v_{ni}, u_{nj}, w_{nj}, (dw_n/dx)_j, v_{nj}\}^T = [A]\{\bar{C}\} , \qquad (17)$$

where [A] is given in appendix I, its element being determined from those of [R]. Finally, combining equations (16) and (17), we obtain

$$\begin{Bmatrix} U(x,\varphi) \\ W(x,\varphi) \\ V(x,\varphi) \end{Bmatrix} = [T]\,[R]\,[A]^{-1} \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [N] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \qquad (18)$$

This equation defines the displacement functions in terms of $n\varphi$, x, the elements p_{ij} of [P] and the nodal displacements $\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}$.

2.4 Determination of the Mass and Stiffness Matrices.

Substituting equations (18) into equations (4) we obtain the strain vector $\{\varepsilon\}$ in terms of $\{\delta_i\}$ and $\{\delta_j\}$ as follows:

$$\{\varepsilon\} = \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} [Q]\,[A]^{-1} \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [B] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} , \qquad (19)$$

where [Q] is given in Appendix I. The corresponding stress-resultant matrix may be found from equation (2), i.e.,

$$\{\sigma\} = [P]\ \{\varepsilon\} = [P]\,[B] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} , \qquad (20)$$

where [P] is the elasticity matrix for anisotropic shells.

The stiffness and mass matrices for one finite element [3] are expressed as

$$[k] = \iint [B]^T [P] [B] \, dA, \quad [m] = \rho t \iint [N]^T [N] \, dA, \tag{21}$$

where $dA = r d\varphi dx$, ρ is the density of the shell ant t its thickness. Integrating over φ and using equations (18)-(20) we obtain

$$[k] = [\,[A]^{-1}]^T \{\pi r \int_0^{\ell} [Q]^T [P] [Q] dx\} [A]^{-1} = [\,[A]^{-1}]^T [G] [A]^{-1} \tag{22}$$

$$[m] = \rho t \, [\,[A]^{-1}]^T \{\pi r \int_0^{\ell} [R]^T [R] dx\} [A]^{-1} = \rho t \, [\,[A]^{-1}]^T [S] [A]^{-1} \tag{23}$$

where [G] and [S] are defined by the above equations.

[G] and [S] were obtained analytically for the case of isotropic shell in reference [4] by carrying out the necessary matrix operations and integrating over x in equations (22) and (23). To do this it was found necessary to introduce several intermediate matrices, eventually obtaining expressions for the general terms k_{ij} and m_{ij} of [k] and [m], respectively.

For the case of anisotropic shells, the elements of [G] and [S] are similar to those of reference [4], for the following reason: in [4], the (i,j)th terms of [G] and [S] are determined functions of the elements of [P] and of the general terms, κ and μ, of the roots λ's which have the same general form as those of equation (11). Because of the complexity of the manipulations, only the final result will be given here. The interested reader is referred to reference [4] for details.

The (i,j)th term of [G] is given by

$$\frac{2}{\pi r}G(i,j) = e^{A_1 l}\left\{\frac{[(D_1-D_4)A_1-(D_2+D_3)\times(B_1+C_1)]\cos[(B_1+C_1)l]}{[A_1^2+(B_1+C_1)^2]} + \frac{[(D_1-D_4)(B_1+C_1)+(D_2+D_3)A_1]\sin[(B_1+C_1)l]}{[A_1^2+(B_1+C_1)^2]} + \frac{[(D_1+D_4)A_1-(D_2-D_3)\times(B_1-C_1)]\cos[(B_1-C_1)l]}{[A_1^2+(B_1-C_1)^2]} + \frac{[(D_1+D_4)(B_1-C_1)+(D_2-D_3)A_1]\sin[(B_1-C_1)l]}{[A_1^2+(B_1-C_1)^2]}\right\} + \frac{(B_1+C_1)(D_2+D_3)-A_1(D_1-D_4)}{[A_1^2+(B_1+C_1)^2]} + \frac{(B_1-C_1)(D_2-D_3)-A_1(D_1+D_4)}{[A_1^2+(B_1-C_1)^2]} \tag{24}$$

for all i,j = 1, 2, ..., 8, except for the following elements:

G(1,5), G(1,6), G(2,5), G(2,6), G(3,7), G(3,8), G(4,7), G(4,8),

G(5,1), G(6,1), G(5,2), G(6,2), G(7,3), G(8,3), G(7,4), G(8,4)

which can be written as follows:

$$G(i,j) = \frac{\pi r}{2}\left[\frac{(D_1-D_4)\sin(2B_1 l)+2(D_2+D_3)\sin^2(B_1 l)}{2B_1} + (D_1+D_4)l\right] \tag{25}$$

In equations (24) and (25), A_1, B_1, C_1, D_1, D_2, D_3, D_4 represent the (i,j)th elements, correspondingly, of matrices $[A_1]$, $[B_1]$, $[C_1]$, $[D_1]$, $[D_2]$, $[D_3]$ and $[D_4]$ which are given in appendix I.

Similarly, the (i,j)th term of [S] is given by

$$\frac{2}{\pi r} S(i,j) = e^{A_1 l} \left\{ \frac{[(E_1-E_4)A_1-(E_2+E_3)\times(B_1+C_1)]\cos[(B_1+C_1)l]}{[A_1^2+(B_1+C_1)^2]} + \frac{[(E_1-E_4)(B_1+C_1)+(E_2+E_3)A_1]\sin[(B_1+C_1)l]}{[A_1^2+(B_1+C_1)^2]} + \frac{[(E_1+E_4)A_1-(E_2-E_3)\times(B_1-C_1)]\cos[(B_1-C_1)l]}{[A_1^2+(B_1-C_1)^2]} + \frac{[(E_1+E_4)(B_1-C_1)+(E_2-E_3)A_1]\sin[(B_1-C_1)l]}{[A_1^2+(B_1-C_1)^2]} \right\} + \frac{(B_1+C_1)(E_2+E_3)-A_1(E_1-E_4)}{[A_1^2+(B_1+C_1)^2]} + \frac{(B_1-C_1)(E_2-E_3)-A_1(E_1+E_4)}{[A_1^2+(B_1-C_1)^2]} \tag{26}$$

for all i,j = 1, 2, ..., 8 except for the following elements:

S(1,5), S(1,6), S(2,5), S(2,6), S(3,7), S(3,8), S(4,7), S(4,8),

S(5,1), S(6,1), S(5,2), S(6,2), S(7,3), S(8,3), S(7,4), S(8,4)

which can be written as follows:

$$S(i,j) = \frac{\pi r}{2}\left[\frac{(E_1-E_4)\sin(2B_1 l)+2(E_2+E_3)\sin^2(B_1 l)}{2B_1} + (E_1+E_4)l\right] \tag{27}$$

Here again, E_1, E_2, E_3, E_4, B_1 and C_1, in equations (26) and (27), represent the (i,j)th elements of the corresponding matrices given in appendix I.

With [m] and [k] determined, the global mass and stiffness matrices for the whole shell, [M] and [K], respectively, may be constructed by superposition in the normal manner as describe in [4]. Each of these matrices is of order $4(N+1)$, where N is the total number of finite element.

2.5 Elasticity Matrix

The elasticity matrix [P] given by equation (3) is quite general, so this theory may be applied to: (i) shells consisting of single or an arbitrary number of isotropic or orthotropic layers, (ii) double-walled, gridwork or folded shells and (iii) shells with rings and stringers provided their characteristics are known. Here we limit ourselves to shells consisting of single or an arbitrary number of isotropic or orthotropic layers symmetrically arranged relative to the coordinate surface.

For isotropic shells, the elements p_{ij} of [P] are listed in reference [4]. In the case of an arbitrary number of orthotropic layers [8], we assume that these layers function concurrently without slippage and as previously stated that the principal directions of elasticity at each point of the shell coincide with the directions of coordinate lines; (i) for an even number of layers, 2v, the elements p_{ij} of [P] may be written in the form

$$p_{ij} = 2 \sum_{s=1}^{v} B_{ij}^{s} (t_s - t_{s+1}), \quad i = 1 \text{ to } 3, \text{ and } j = 1 \text{ to } 6,$$

$$p_{ij} = (2/3) \sum_{s=1}^{v} B_{i-3,\, j-3}^{s} (t_s^3 - t_{s+1}^3), \quad i = 4 \text{ to } 6, \text{ and } j = 4 \text{ to } 6. \tag{28}$$

(ii) for an odd number, 2v+1, we obtain

$$p_{ij} = 2[B_{ij}^{v+1} t_{v+1} + \sum_{s=1}^{v} B_{ij}^{s} (t_s - t_{s+1})], \quad i = 1 \text{ to } 3 \text{ and } j = 1 \text{ to } 6,$$

$$p_{ij} = (2/3)[B_{i-3,j-3}^{v+1} t_{v+1}^3 + \sum_{s=1}^{v} B_{i-3,j-3}^{s} (t_s^3 - t_{s+1}^3)], \quad i = 4 \text{ to } 6 \text{ and } j = 4 \text{ to } 6, \tag{29}$$

where

$$B_{11}^S = [E_1^S/(1-\nu_1^S\ \nu_2^S)]\ , \quad B_{22}^S = [E_2^S/(1-\nu_1^S\ \nu_2^S)]\ ,$$

$$B_{12}^S = B_{21}^S = [\nu_2^S\ E_1^S/(1-\nu_1^S\ \nu_2^S)]\ , \quad B_{23}^S = 0.5G_{12}^S\ ,$$

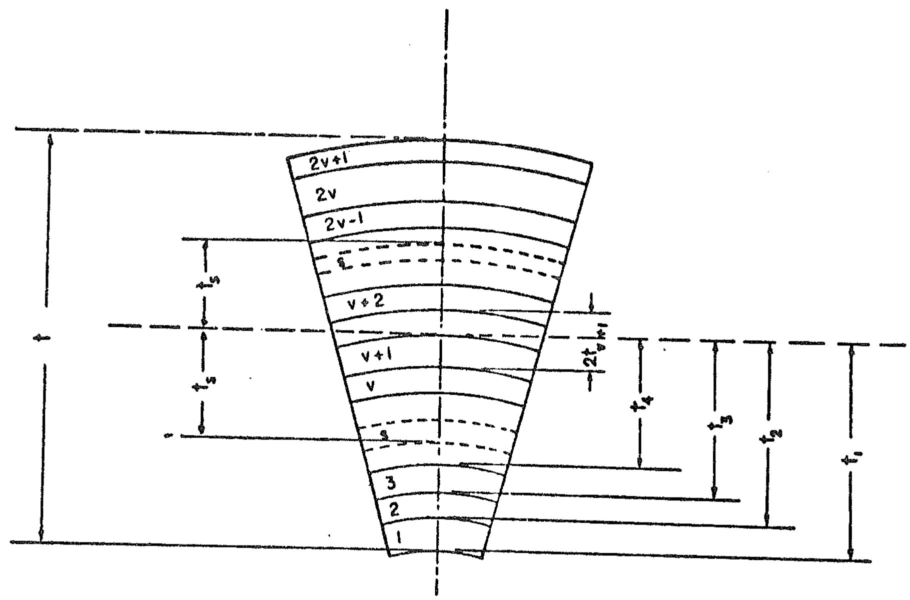

Fig. 2 Shells consisting of an odd number $(2\nu+1)$ of anisotropic layers.

t_s is the coordinate of the S^{th} layer with respect to the middle surface as shown in Figure 2, (E_1^S, ν_1^S) and $E_2^S, \nu_2^S)$ are its Young's modulus and Poisson's ratio in the x and φ directions, respectively, and G_{12}^S is the shear modulus. All other terms of B_{ij}^S are zero.

2.6 Free Vibration

For free vibration, the equation of motion may be written in the form

$$[M]\ \{\ddot{\Delta}\} + [K]\ \{\Delta\} = \{0\}, \tag{30}$$

where $\{\Delta\} = \{\delta_1, \delta_2, \ldots, \delta_{N+1}\}^T$, N is the number of finite elements, [M] and [K] are real, symmetric matrices of order 4(N+1), and $\{\delta_{N+1}\}$ being the displacement vector associated with the lower edge of the last finite element.

In cases where the shell has rigid edge constraints, the kinematic boundary conditions must be taken into consideration. Accordingly, [K] and [M] are reduced to square matrices of order 4(N+1)-J, where J is the number of constraint equations imposed. Thus, for a shell with two edges supported, we must have $v_n = w_n = 0$ in the displacement vectors $\{\delta_1\}$ and $\{\delta_{N+1}\}$, and J = 4; for a free shell, J = 0; and for one with two clamped edges J = 8. The solution of equation (30) now follows by standard matrix techniques, yielding the natural frequencies, ω_i, i = 1, 2, ..., 4(N+1)-J, and the corresponding eigenvectors.

3. RESPONSE TO BOUNDARY-LAYER PRESSURE FIELD

3.1 General Theory

In this section we are concerned with the vibration of thin anisotropic cylindrical shells due to a pressure field arising from the turbulent boundary layer of an internal subsonic flow. It is based on a recently developped theory [5] by the author for the case of isotropic cylindrical shells. Only an outline of the theory is given here; for a detailed account the reader is referred to reference [5].

The equations of motion of the shell subjected to arbitrary load is given by

$$[M]\{\ddot{y}\} + [C]\{\dot{y}\} + [K]\{y\} = \{F\}, \qquad (31)$$

where $\{y\}$ is a nodal displacement vector, $\{F\}$ is a vector of the external forces, and $[M]$, $[C]$ and $[K]$ are the mass, damping and stiffness matrices, respectively.

Whereas equation (31) is quite general, the particular form of its constituent terms depends on the particular theory used. In this theory $[M]$ and $[K]$ are determined by equations (22) and (23), $[C]$ is assumed to be linearly related to $[M]$ and $[K]$, or to either one, and the external forces $\{F\}$ represent the internal random pressure field.

3.2 Assumptions

In reference [6] we have indicated how the intertial effects of a stationnary fluid contained by the shell may be taken into account. However, when the fluid is flowing, the shell is also subjected to "centrifugal" and Coriolis-type pressure forces. The former have the effect of diminishing the natural frequencies of the system, while the latter have a damping effect on vibrations in cases where one end of the shell is free. Unless we are dealing with very flexible shells, very heavy fluids, or very high velocities, the effects of these forces will be relatively small. Accordingly, for metal shells conveying fluid with flow velocity in the normal engineering range, these effects are negligible and are not taken into account.

The displacements are assumed small enough for the resultant forces to be normal to the shell's surface. It is also assumed that the pressure field is spatially continuous and that it has the properties of a weakly stationary, ergodic process. We further assume that the pressure drop in the length of the shell is sufficiently small for the mean pressure to be considered constant over the length of the shell. Finally, the continuous random pressure field of the deformable body is approximated by a finite set fo discrete forces and moments acting at the nodal points [11].

3.3 Representation of Pressure Field at Nodal Points.

As previously mentioned, the shell is divided into N finite elements, each of which is a cylindrical frustum. The position of the N+1 nodal points may be chosen arbitrarily (Fig. 1).

Any pressure field is considered to be acting on an area S_e surrounding the node e of coordinate ℓ_e as shown in Figure 3 (a). We define the pressure distribution acting over this area S_e by two mutually perpendicular forces per unit length. We may write, for the actual resultant force per unit length,

$$F(x,\varphi,t) = \sum_n f_{Rn}(x,t).\cos n\varphi + \sum_n f_{Cn}(x,t).\sin n\varphi, \qquad (32)$$

where f_{Rn} and f_{Cn} are at a distance x_o from the origin of the shell as shown in Figure 3 (a).

These two forces acting at point A are transformed to two forces and one moment, M_r, acting at the node e, as shown in Figure 3 (b).

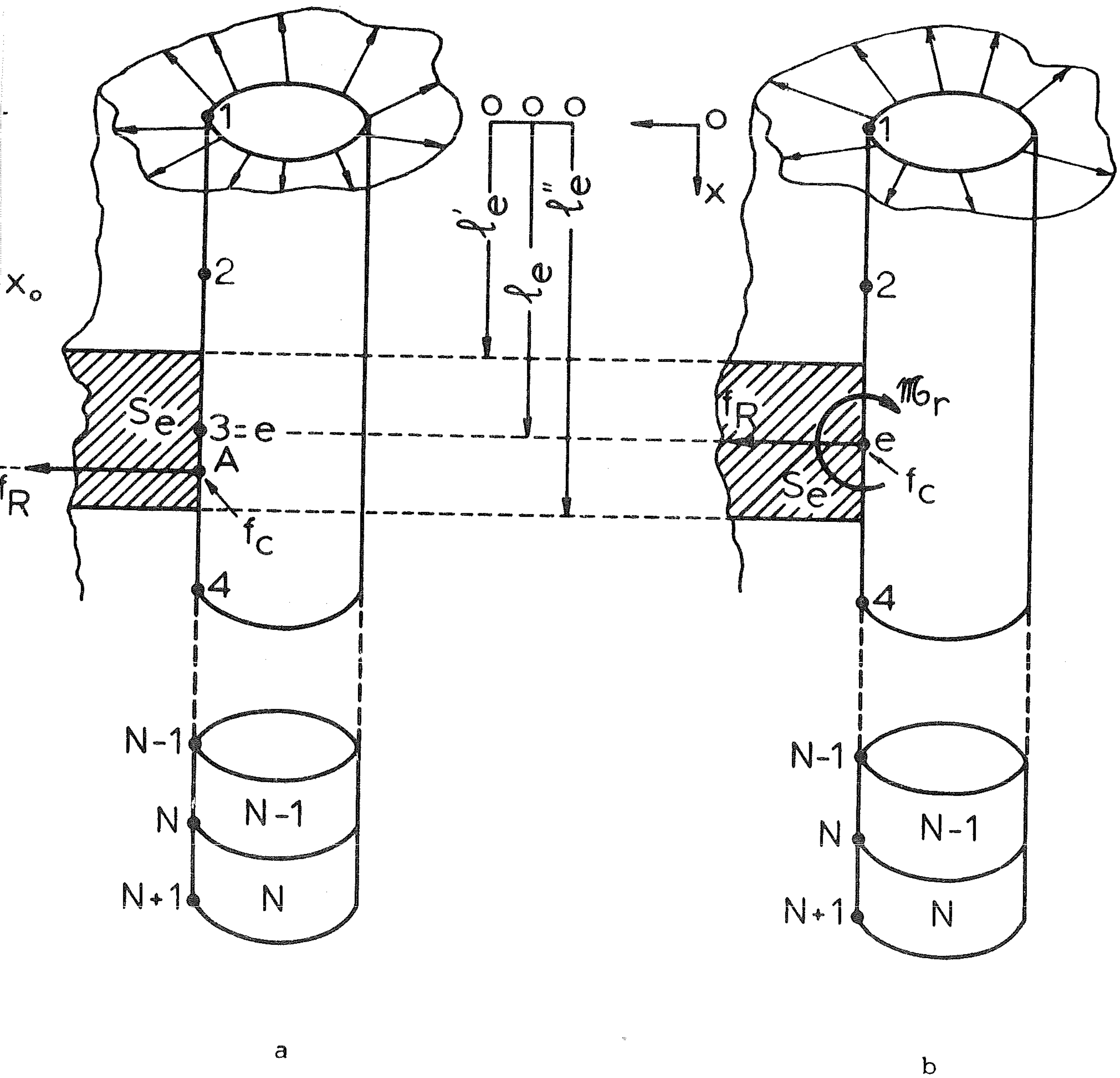

Figure 3 (a) Representation of the pressure field by a discrete force field. (b) The equivalent discrete force field acting at the node e, involving f_{Rn}, f_{Cn} and M_r.

The external force vector associated with the n^{th} circumferential wave number at a typicalnode e can now be written in the following form:

$$\{F(t)\}_e = \left\{ \begin{array}{l} 0 \\ \int_{\ell_i'}^{\ell_i''} f_R(x_i,t)\, dx_i \\ \int_{\ell_j'}^{\ell_j''} (x_j-\ell_j)\, f_R(x_j,t)\, dx_j \\ \int_{\ell_p'}^{\ell_p''} f_C(x_p,t)\, dx_p \end{array} \right\}_e \qquad (33)$$

where f_{Rn} and f_{Cn} are expressed in terms of the instantaneous pressure on the surface, $p(x, \varphi, t)$.

3.4 Mean Square Response

We proceed by first considering the free vibration of the conservative system (30) and determining the natural frequencies ω_i and the eigenvectors $\{\Phi_i\}$, $i = 1, 2, \ldots, 4(N+1)-J$, where J is the number of kinematic boundaries.

We next form the modal matrix

$$[\Phi] = [\Phi_1, \Phi_2, \ldots, \Phi_{4(N+1)-J}], \text{ and define } \{y\} = [\Phi]\{Z\}. \qquad (34),(35)$$

Finally the equations of motion (31) are decoupled and the mean square values of the displacements of the shell are expressed in terms of the axial and circumferential correlation functions of the pressure field, $\psi_p(\xi, o, o)$ and $\psi_p(o, \eta, o)$, respectively; see equations (10)-(25) of reference [5].

In the case of subsonic boundary-layer pressure fluctuations, the streamwise and lateral spatial correlation functions have been examined theoretically and experimentally by Bakewell at al. [9] and Clinch [10].

Bakewell measured and derived expressions for the axial and circumferential correlation functions in experiments with air flowing in a cylindrical pipe. He found the following approximate expressions for the (real) spatial correlations:

$$\psi_{p\omega}(\xi, 0, 0) \approx e^{-b|S_\xi|} \cos a S_\xi, \tag{36}$$

$$\psi_{p\omega}(0, \eta, 0) \approx (1 + c S_\eta^2)^{-1} [2 - e^{-dS_\eta^2}]^{-1} \tag{37}$$

where $S_\xi = \xi\omega/U$ conv. and $S_\eta = \eta\omega/U_{℄}$ are the axial and circumferential Strouhal number, $\xi = |x_i - x_j|$, $\eta = |r(\varphi_i - \varphi_j)|$, ω is the center frequency, and a, b, c, d are constants to be specified; U_{conv} and $U_{℄}$ are, respectively, the convection and the centerline velocities.

The values of the constants used in these two expressions for axial and circumferential correlations depend on the fluid. For turbulent flow in air, the values of a, b, c and d are given in [9]

$$\begin{aligned} &a = 8.7266, b = 1.0, \text{ for } S_\xi = \xi\omega/U_{℄} \\ &c = 20, d = 100, \text{ for } S_\eta = \eta\omega/U_{℄} \end{aligned} \tag{38}$$

Clinch measurements in water proved that these constants are approximately the same for different fluids at the same Strouhal number, at least for sufficiently high Reynolds number.

Upon usign the experimentally based relations (36)-(38), we obtain the following expression for the mean square response of the shell [5]:

$$\overline{y_q^2(t)} = \sum_{r=1}^{4(N+1)-J} \Phi_{qr}^2 \frac{r^2}{16\,\pi^2\omega_r^4\mathcal{M}_r^2} \times$$
$$\times \left[\sum_{i=1}^{N+1}\sum_{u=1}^{N+1} \Phi_{ir}\Phi_{ur}|\Gamma_{iu}^F| + 2\sum_{i=1}^{N+1}\sum_{k=1}^{N+1} \Phi_{ir}\Phi_{kr}|\Gamma_{ki}^M| + \right.$$
$$\left. + \sum_{j=1}^{N+1}\sum_{k=1}^{N+1} \Phi_{jr}\Phi_{kr}|\Gamma_{jk}^{MM}| + \sum_{p=1}^{N+1}\sum_{v=1}^{N+1} \Phi_{pr}\Phi_{vr}|\Gamma_{pv}^F|\right], \tag{39}$$

where Φ_{qr} is the $(qr)^{th}$ element of the modal matrix $[\Phi]$, M_r is the element of the generalized mass matrix, ω_r, the r^{th} natural frequency and r is the mean radius of the shell; Γ_{iu}^F, Γ_{ki}^M and Γ_{jk}^{MM} are derived analytically in reference [5].

Equation (39) is then the response of the shell to a subsonic boundary-layer pressure field at the nodal points $q(x,\varphi)$. This response is associated with a specific n, where n is the circumferential wave number (section 2.3). By repeating the analysis for a sufficient number of n, the total response for any point on the nodal circles may be obtained by superposition, in accordance with the assumption that there is no coupling between the circumferential wavenumbers.

4. CALCULATION AND DISCUSSION

The computer program of reference [5] has been modified to determine the eigenvalues, eigenvectors and the response of a given uniform or non-uniform anisotropic cylindrical shell subjected to a boundary-layer pressure field. It is written in FORTRAN V language for the IBM 360/70 computer, using double precision arithmetic throughout all the overlays.

The necessary step of the computational method may be outlined as follows: a) We first specify the imposed boundary conditions, their number, J, and the values of n (≥ 2) for which calculations should be done; b) The shell is then subdivided into a sufficient number, N, of finite elements (sufficiency in this context is related to the complexity of the structure); c) And finally the computer program, for given input data, calculated the mass and stiffness matrices for each element, assembles the global mass and stiffness matrices for the whole shell, calculates the natural frequencies and the eigenvectors, determines the damping matrix, and executes the necessary steps to obtain the response.

The necessary input data for each finite element are the mean radius, r, wall thickness, t, length of the individual element, ℓ, material density, ρ, and the elements p_{ij} of [P].

For given r, t, ℓ, ρ and p_{ij}, the computer program executes the following steps for each element: i) the eight complex roots, λ_j, of the characteristic equation (9), are calculated by Newton-Raphson iterative technique, and hence, we obtain $\varkappa_1$, $\varkappa_2$, μ_1, μ_2, α_j, β_j $(j = 1, 2, \ldots, 8)$, and $\bar{\alpha}_j$, $\bar{\beta}_j$; ii) the intermediate matrices are determined; iii) the displacement functions, mass and stiffness matrices, [N], [m] and [k], respectively, are computed by the relationshpis given by equations (18), (22) and (23).

When the stiffness and mass matrices have thus been computed for each element, the global [M] and [K] are constructed and reduced appropriately to take account of the boundary conditions.

For free vibration, the computer program proceeds to find the natural frequencies, ωi, where $i = 1, 2, \ldots, 4(N+1)-J$ for each n, and the corresponding eigenvectors of a real square non-symmetric matrix of the special form $[M]^{-1}[K]$, where both [M] and [K] are real, symmetric matrices and [M] is positive definite.

Knowing the damping factor, the fluid velocity and its density at each node of the structure, equation (39) is finally executed to obtain the response to a boundary-layer pressure field.

Calculations have already been conducted to test the theory in the case of isotropic shells [5]. The free vibration characteristics of uniform and axially non-uniform shells were obtained for a variety of boundary conditions [4]. The computed natural frequencies and the response were compared with those obtained by other theories and from experiments; agreement was found to be good and, in the majority of cases, was even better with the experiments.

Here we repeat only one calculation to test the computer program and the modified theory. More results concerning the anisotropic shells will be presented and discussed at the conference.

The set of calculations undertaken here was first studied experimentally and theoretically by Clinch [12]. It is a long, simply supported cylindrical shell conveying water with flow velocities in the range 248-520 in/sec. The pertinent data are as follows: $r = 3$ in(.0762 m), $L = 240$ in(6.096 m), $t = 0.025$ in($.63 \times 10^{-3}$ m), $E = 28.5 \times 10^6$ lb/in^2(1.995×10^{11} N/m^2), $\nu = 0.305$, $\rho = .749 \times 10^{-3}$ lb-sec^2/in^4(8.0048×10^3 kg/m^3). Clinch obtained the response in the frequency range of 100-1,000 Hz, approximately.

This shell was also analysed by the theory of reference [5] by subdividing the shell into 8 elements and calculating the response for $n = 2$ to 6. Here we repeat the same calculations for $n = 2$ to 12 from which the approximate "total" and the high-frequency responses of this theory are shown in Figure 4; also shown are Clinch's experimental and theoretical results.

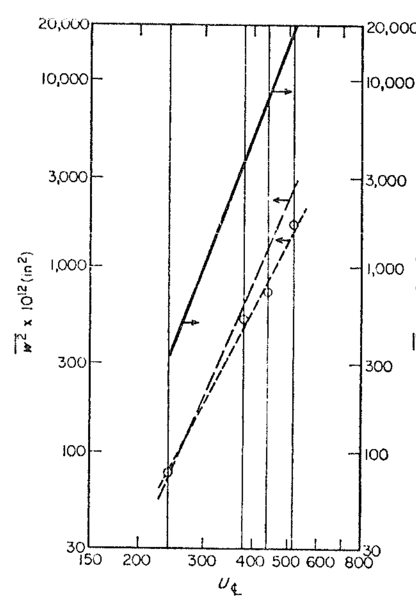

Figure 4. The mean square response of the maximum radial displacement of a shell first studied by Clinch, as a function of the centerline velocity. --- 0 ---, Clinch's experimental and theoretical results for high-frequency response; ---, theoretical results obtained by this theory ($n = 2$ to 12) for high frequency response (98-1,000 Hz); ——, "total" response obtained by this theory ($n = 2$ to 12) considering all frequency components.

It is evident from Figure 4 that the response at the high frequency range is but a small part of the total. This observation demonstrates the limitations of Clinch theory if one is interested in the total response rather than only the high-frequency range. On the other hand, the agreement between this theory and experiment, in the frequency range of 100-1,000 Hz, approximately, is quite good. This is the first and, so far, only experimental verification of this theory, as experimental data are very scarce; the results lend confidence that the values of the overall response of the shell are also reliable.

5. CONCLUSION

The hybrid finite-element, classical theory developed in this paper is used to obtain the free vibration characteristics and to predict the response, to boundary-layer pressure field of an axially non-uniform, anisotropic thin cylindrical shell. To this end the shell is subdivided into a number of cylindrical finite elements, each with two nodes, the nodal displacements being the axial, circumferential and radial displacements and a rotation. The shell equations employed which are solved for the determination of the displacement functions, are such that the convergence criteria of the finite-element method are satisfied. The pressure field is similarly rendered discrete and is represented by two forces and a moment at each node. Finally, the pressure correlation functions used in this analysis are applicable only for flow velocities corresponding to Mach number 0.3 or less; there is no assurance that such correlation functions can be applied at higher Mach numbers when compressibility effects become important.

This theory was computerized so that if the dimensions and material properties of each finite element, and the properties and flow velocity of the fluid, are given as input, the program gives as output the natural frequencies and eigenvectors of the shell and the r.m.s. values of the nodal displacements. The analysis proceeds separately for each circumferential wavenumber, n; the total response may then be found by summing over n.

The effort involved in producing such complex theory is deemed to be justified. In this connection, it is noted that accurate knowledge of some of the high and the low frequencies is essential for the accurate determination of the response of shells to random pressure field, such as those generated by internal or external flow. Accordingly, the present method, because of its usage of classical theory for the displacement functions, may lead to the determination of the high as well as the low frequencies with high accuracy [4]. Apart from this, the main advantage of this theory is that it may be used, without modification, to obtain the free vibration characteristics and the response of any anisotropic cylindrical shell which is geometrically axially symmetric, no matter how many property discontinuities may be present, and for whatever boundary conditions.

The extension of this theory to the more general case of curved-shell finite elements is envisaged, with which shells of any shape could be analysed with enhanced precision. Another extension to this work will be to consider the effects of all the components arising from the presence of flowing or stationary fluids, on the natural frequencies for the cases of completely or partially-filled shells.

ACKNOWLEDGEMENTS

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APPENDIX I

List of Matrices

Appendix I contains the matrices referred to in the text which were too large to be included therein.

These matrices are listed as follows.

[R], [A] (see table 1)

[Q] (see table 2)

[Γ] (see table 3)

[Δ], [A_1] (see table 4)

[B_1], [C_1], [D_1] to [D_4] (see table 5)

[E_1], [E_2], [E_3], [E_4] are obtained, respectively, from matrices [D_1], [D_2], [D_3], [D_4] by substituting in these matrices the elements of matrix $[\gamma] = [\Gamma]^T [P] [\Gamma]$ by the elements of matrix $[RJ] = [\Delta]^T [\Delta]$.

Table 1. Matrices [R] and [A].

Matrix [R]

$e^{-\psi_1}[\bar{\alpha}_1 \cos\zeta_1 - \bar{\alpha}_2 \sin\zeta_1]$	$e^{-\psi_1}[\bar{\alpha}_2 \cos\zeta_1 + \bar{\alpha}_1 \sin\zeta_1]$	$e^{-\psi_2}[\bar{\alpha}_3 \cos\zeta_2 - \bar{\alpha}_4 \sin\zeta_2]$	$e^{-\psi_2}[\bar{\alpha}_4 \cos\zeta_2 + \bar{\alpha}_3 \sin\zeta_2]$
$e^{-\psi_1}\cos\zeta_1$	$e^{-\psi_1}\sin\zeta_1$	$e^{-\psi_2}\cos\zeta_2$	$e^{-\psi_2}\sin\zeta_2$
$e^{-\psi_1}[\bar{\beta}_1 \cos\zeta_1 - \bar{\beta}_2 \sin\zeta_1]$	$e^{-\psi_1}[\bar{\beta}_2 \cos\zeta_1 + \bar{\beta}_1 \sin\zeta_1]$	$e^{-\psi_2}[\bar{\beta}_3 \cos\zeta_2 - \bar{\beta}_4 \sin\zeta_2]$	$e^{-\psi_2}[\bar{\beta}_4 \cos\zeta_2 + \bar{\beta}_3 \sin\zeta_2]$

$e^{\psi_1}[\bar{\alpha}_5 \cos\zeta_1 - \bar{\alpha}_6 \sin\zeta_1]$	$e^{\psi_1}[\bar{\alpha}_6 \cos\zeta_1 + \bar{\alpha}_5 \sin\zeta_1]$	$e^{\psi_2}[\bar{\alpha}_7 \cos\zeta_2 - \bar{\alpha}_8 \sin\zeta_2]$	$e^{\psi_2}[\bar{\alpha}_8 \cos\zeta_2 + \bar{\alpha}_7 \sin\zeta_2]$
$e^{\psi_1}\cos\zeta_1$	$e^{\psi_1}\sin\zeta_1$	$e^{\psi_2}\cos\zeta_2$	$e^{\psi_2}\sin\zeta_2$
$e^{\psi_1}[\bar{\beta}_5 \cos\zeta_1 - \bar{\beta}_6 \sin\zeta_1]$	$e^{\psi_1}[\bar{\beta}_6 \cos\zeta_1 + \bar{\beta}_5 \sin\zeta_1]$	$e^{\psi_2}[\bar{\beta}_7 \cos\zeta_2 - \bar{\beta}_8 \sin\zeta_2]$	$e^{\psi_2}[\bar{\beta}_8 \cos\zeta_2 + \bar{\beta}_7 \sin\zeta_2]$

$$\omega_j = \kappa_j l/r, \quad \eta_j = \mu_j l/r, \quad \psi_j = \kappa_j x/r, \quad \zeta_j = \mu_j x/r; \qquad j = 1, 2,$$

Matrix [A]

$\bar{\alpha}_1$	$\bar{\alpha}_2$	$\bar{\alpha}_3$	$\bar{\alpha}_4$	$\bar{\alpha}_5$	$\bar{\alpha}_6$	$\bar{\alpha}_7$	$\bar{\alpha}_8$
1	0	1	0	1	0	1	0
$-\kappa_1/r$	μ_1/r	$-\kappa_2/r$	μ_2/r	κ_1/r	μ_1/r	κ_2/r	μ_2/r
$\bar{\beta}_1$	$\bar{\beta}_2$	$\bar{\beta}_3$	$\bar{\beta}_4$	$\bar{\beta}_5$	$\bar{\beta}_6$	$\bar{\beta}_7$	$\bar{\beta}_8$
$e^{-\omega_1}[\bar{\alpha}_1 \cos\eta_1 - \bar{\alpha}_2 \sin\eta_1]$	$e^{-\omega_1}[\bar{\alpha}_2 \cos\eta_1 + \bar{\alpha}_1 \sin\eta_1]$	$e^{-\omega_2}[\bar{\alpha}_3 \cos\eta_2 - \bar{\alpha}_4 \sin\eta_2]$	$e^{-\omega_2}[\bar{\alpha}_4 \cos\eta_2 + \bar{\alpha}_3 \sin\eta_2]$	$e^{\omega_1}[\bar{\alpha}_5 \cos\eta_1 - \bar{\alpha}_6 \sin\eta_1]$	$e^{\omega_1}[\bar{\alpha}_6 \cos\eta_1 + \bar{\alpha}_5 \sin\eta_1]$	$e^{\omega_2}[\bar{\alpha}_7 \cos\eta_2 - \bar{\alpha}_8 \sin\eta_2]$	$e^{\omega_2}[\bar{\alpha}_8 \cos\eta_2 + \bar{\alpha}_7 \sin\eta_2]$
$e^{-\omega_1}\cos\eta_1$	$e^{-\omega_1}\sin\eta_1$	$e^{-\omega_2}\cos\eta_2$	$e^{-\omega_2}\sin\eta_2$	$e^{\omega_1}\cos\eta_1$	$e^{\omega_1}\sin\eta_1$	$e^{\omega_2}\cos\eta_2$	$e^{\omega_2}\sin\eta_2$
$\frac{e^{-\omega_1}}{r}[-\kappa_1 \cos\eta_1 - \mu_1 \sin\eta_1]$	$\frac{e^{-\omega_1}}{r}[\mu_1 \cos\eta_1 - \kappa_1 \sin\eta_1]$	$\frac{e^{-\omega_2}}{r}[-\kappa_2 \cos\eta_2 - \mu_2 \sin\eta_2]$	$\frac{e^{-\omega_2}}{r}[\mu_2 \cos\eta_2 - \kappa_2 \sin\eta_2]$	$\frac{e^{\omega_1}}{r}[\kappa_1 \cos\eta_1 - \mu_1 \sin\eta_1]$	$\frac{e^{\omega_1}}{r}[\mu_1 \cos\eta_1 + \kappa_1 \sin\eta_1]$	$\frac{e^{\omega_2}}{r}[\kappa_2 \cos\eta_2 - \mu_2 \sin\eta_2]$	$\frac{e^{\omega_2}}{r}[\mu_2 \cos\eta_2 + \kappa_2 \sin\eta_2]$
$e^{-\omega_1}[\bar{\beta}_1 \cos\eta_1 - \bar{\beta}_2 \sin\eta_1]$	$e^{-\omega_1}[\bar{\beta}_2 \cos\eta_1 + \bar{\beta}_1 \sin\eta_1]$	$e^{-\omega_2}[\bar{\beta}_3 \cos\eta_2 - \bar{\beta}_4 \sin\eta_2]$	$e^{-\omega_2}[\bar{\beta}_4 \cos\eta_2 + \bar{\beta}_3 \sin\eta_2]$	$e^{\omega_1}[\bar{\beta}_5 \cos\eta_1 - \bar{\beta}_6 \sin\eta_1]$	$e^{\omega_1}[\bar{\beta}_6 \cos\eta_1 + \bar{\beta}_5 \sin\eta_1]$	$e^{\omega_2}[\bar{\beta}_7 \cos\eta_2 - \bar{\beta}_8 \sin\eta_2]$	$e^{\omega_2}[\bar{\beta}_8 \cos\eta_2 + \bar{\beta}_7 \sin\eta_2]$

Table 2. Matrix [Q].

$$
[Q] = \begin{bmatrix}
\frac{e^{-\psi_1}}{r}[(-\kappa_1\bar{\alpha}_1-\mu_1\bar{\alpha}_2)\cos\zeta_1+(\kappa_1\bar{\alpha}_2-\mu_1\bar{\alpha}_1)\sin\zeta_1] &
\frac{e^{-\psi_1}}{r}[(-\kappa_1\bar{\alpha}_2+\mu_1\bar{\alpha}_1)\cos\zeta_1-(\kappa_1\bar{\alpha}_1+\mu_1\bar{\alpha}_2)\sin\zeta_1] &
\frac{e^{-\psi_2}}{r}[(-\kappa_2\bar{\alpha}_3-\mu_2\bar{\alpha}_4)\cos\zeta_2+(\kappa_2\bar{\alpha}_4-\mu_2\bar{\alpha}_3)\sin\zeta_2] &
\frac{e^{-\psi_2}}{r}[(-\kappa_2\bar{\alpha}_4+\mu_2\bar{\alpha}_3)\cos\zeta_2-(\kappa_2\bar{\alpha}_3+\mu_2\bar{\alpha}_4)\sin\zeta_2] &
\frac{e^{\psi_1}}{r}[(\kappa_1\bar{\alpha}_5-\mu_1\bar{\alpha}_6)\cos\zeta_1-(\kappa_1\bar{\alpha}_6+\mu_1\bar{\alpha}_5)\sin\zeta_1] &
\frac{e^{\psi_1}}{r}[(\kappa_1\bar{\alpha}_6+\mu_1\bar{\alpha}_5)\cos\zeta_1+(\kappa_1\bar{\alpha}_5-\mu_1\bar{\alpha}_6)\sin\zeta_1] &
\frac{e^{\psi_2}}{r}[(\kappa_2\bar{\alpha}_7-\mu_2\bar{\alpha}_8)\cos\zeta_2-(\kappa_2\bar{\alpha}_8+\mu_2\bar{\alpha}_7)\sin\zeta_2] &
\frac{e^{\psi_2}}{r}[(\kappa_2\bar{\alpha}_8+\mu_2\bar{\alpha}_7)\cos\zeta_2+(\kappa_2\bar{\alpha}_7-\mu_2\bar{\alpha}_8)\sin\zeta_2] \\
\frac{e^{-\psi_1}}{r}[(n\beta_1+1)\cos\zeta_1-n\beta_2\sin\zeta_1] &
\frac{e^{-\psi_1}}{r}[n\beta_2\cos\zeta_1+(n\beta_1+1)\sin\zeta_1] &
\frac{e^{-\psi_2}}{r}[(n\beta_3+1)\cos\zeta_2-n\beta_4\sin\zeta_2] &
\frac{e^{-\psi_2}}{r}[n\beta_4\cos\zeta_2+(n\beta_3+1)\sin\zeta_2] &
\frac{e^{\psi_1}}{r}[(n\beta_5+1)\cos\zeta_1-n\beta_6\sin\zeta_1] &
\frac{e^{\psi_1}}{r}[n\beta_6\cos\zeta_1+(n\beta_5+1)\sin\zeta_1] &
\frac{e^{\psi_2}}{r}[(n\beta_7+1)\cos\zeta_2-n\beta_8\sin\zeta_2] &
\frac{e^{\psi_2}}{r}[n\beta_8\cos\zeta_2+(n\beta_7+1)\sin\zeta_2] \\
\frac{e^{-\psi_1}}{r}[(-\kappa_1\beta_1-\mu_1\beta_2-n\bar{\alpha}_1)\times\cos\zeta_1+(\kappa_1\beta_2-\mu_1\beta_1+n\bar{\alpha}_2)\sin\zeta_1] &
\frac{e^{-\psi_1}}{r}[(-\kappa_1\beta_2+\mu_1\beta_1-n\bar{\alpha}_2)\times\cos\zeta_1-(\kappa_1\beta_1+\mu_1\beta_2+n\bar{\alpha}_1)\sin\zeta_1] &
\frac{e^{-\psi_2}}{r}[(-\kappa_2\beta_3-\mu_2\beta_4-n\bar{\alpha}_3)\times\cos\zeta_2+(\kappa_2\beta_4-\mu_2\beta_3+n\bar{\alpha}_4)\sin\zeta_2] &
\frac{e^{-\psi_2}}{r}[(-\kappa_2\beta_4+\mu_2\beta_3-n\bar{\alpha}_4)\times\cos\zeta_2-(\kappa_2\beta_3+\mu_2\beta_4+n\bar{\alpha}_3)\sin\zeta_2] &
\frac{e^{\psi_1}}{r}[(\kappa_1\beta_5-\mu_1\beta_6-n\bar{\alpha}_5)\times\cos\zeta_1-(\kappa_1\beta_6+\mu_1\beta_5-n\bar{\alpha}_6)\sin\zeta_1] &
\frac{e^{\psi_1}}{r}[(\kappa_1\beta_6+\mu_1\beta_5-n\bar{\alpha}_6)\times\cos\zeta_1+(\kappa_1\beta_5-\mu_1\beta_6-n\bar{\alpha}_5)\sin\zeta_1] &
\frac{e^{\psi_2}}{r}[(\kappa_2\beta_7-\mu_2\beta_8-n\bar{\alpha}_7)\times\cos\zeta_2-(\kappa_2\beta_8+\mu_2\beta_7-n\bar{\alpha}_8)\sin\zeta_2] &
\frac{e^{\psi_2}}{r}[(\kappa_2\beta_8+\mu_2\beta_7-n\bar{\alpha}_8)\times\cos\zeta_2+(\kappa_2\beta_7-\mu_2\beta_8-n\bar{\alpha}_7)\sin\zeta_2] \\
\frac{-e^{-\psi_1}}{r^2}[(\kappa_1^2-\mu_1^2)\cos\zeta_1+2\kappa_1\mu_1\sin\zeta_1] &
\frac{-e^{-\psi_1}}{r^2}[-2\kappa_1\mu_1\cos\zeta_1+(\kappa_1^2-\mu_1^2)\sin\zeta_1 &
\frac{-e^{-\psi_2}}{r^2}[(\kappa_2^2-\mu_2^2)\cos\zeta_2+2\kappa_2\mu_2\sin\zeta_2] &
\frac{-e^{-\psi_2}}{r^2}[-2\kappa_2\mu_2\cos\zeta_2+(\kappa_2^2-\mu_2^2)\sin\zeta_2] &
\frac{-e^{\psi_1}}{r^2}[(\kappa_1^2-\mu_1^2)\cos\zeta_1-2\kappa_1\mu_1\sin\zeta_1] &
\frac{-e^{\psi_1}}{r^2}[2\kappa_1\mu_1\cos\zeta_1+(\kappa_1^2-\mu_1^2)\sin\zeta_1] &
\frac{-e^{\psi_2}}{r^2}[(\kappa_2^2-\mu_2^2)\cos\zeta_2-2\kappa_2\mu_2\sin\zeta_2] &
\frac{-e^{\psi_2}}{r^2}[2\kappa_2\mu_2\cos\zeta_2+(\kappa_2^2-\mu_2^2)\sin\zeta_2] \\
\frac{e^{-\psi_1}}{r^2}[(n^2+n\beta_1)\cos\zeta_1-n\beta_2\sin\zeta_1] &
\frac{e^{-\psi_1}}{r^2}[n\beta_2\cos\zeta_1+(n^2+n\beta_1)\sin\zeta_1] &
\frac{e^{-\psi_2}}{r^2}[(n^2+n\beta_3)\cos\zeta_2-n\beta_4\sin\zeta_2] &
\frac{e^{-\psi_2}}{r^2}[n\beta_4\cos\zeta_2+(n^2+n\beta_3)\sin\zeta_2] &
\frac{e^{\psi_1}}{r^2}[(n^2+n\beta_5)\cos\zeta_1-n\beta_6\sin\zeta_1] &
\frac{e^{\psi_1}}{r^2}[n\beta_6\cos\zeta_1+(n^2+n\beta_5)\sin\zeta_1] &
\frac{e^{\psi_2}}{r^2}[(n^2+n\beta_7)\cos\zeta_2-n\beta_8\sin\zeta_2] &
\frac{e^{\psi_2}}{r^2}[n\beta_8\cos\zeta_2+(n^2+n\beta_7)\sin\zeta_2] \\
\frac{e^{-\psi_1}}{r^2}\left[\left(-2n\kappa_1-\frac{3}{2}\kappa_1\beta_1-\frac{3}{2}\mu_1\beta_2+\frac{n\bar{\alpha}_1}{2}\right)\cos\zeta_1+\left(-2n\mu_1+\frac{3}{2}\kappa_1\beta_2-\frac{3}{2}\mu_1\beta_1-\frac{n\bar{\alpha}_2}{2}\right)\sin\zeta_1\right] &
\frac{e^{-\psi_1}}{r^2}\left[\left(2n\mu_1-\frac{3}{2}\kappa_1\beta_2+\frac{3}{2}\mu_1\beta_1+\frac{n\bar{\alpha}_2}{2}\right)\cos\zeta_1+\left(-2n\kappa_1-\frac{3}{2}\kappa_1\beta_1-\frac{3}{2}\mu_1\beta_2+\frac{n\bar{\alpha}_1}{2}\right)\sin\zeta_1\right] &
\frac{e^{-\psi_2}}{r^2}\left[\left(-2n\kappa_2-\frac{3}{2}\kappa_2\beta_3-\frac{3}{2}\mu_2\beta_4+\frac{n\bar{\alpha}_3}{2}\right)\cos\zeta_2+\left(-2n\mu_2+\frac{3}{2}\kappa_2\beta_4-\frac{3}{2}\mu_2\beta_3-\frac{n\bar{\alpha}_4}{2}\right)\sin\zeta_2\right] &
\frac{e^{-\psi_2}}{r^2}\left[\left(2n\mu_2-\frac{3}{2}\kappa_2\beta_4+\frac{3}{2}\mu_2\beta_3+\frac{n\bar{\alpha}_4}{2}\right)\cos\zeta_2+\left(-2n\kappa_2-\frac{3}{2}\kappa_2\beta_3-\frac{3}{2}\mu_2\beta_4+\frac{n\bar{\alpha}_3}{2}\right)\sin\zeta_2\right] &
\frac{e^{\psi_1}}{r^2}\left[\left(2n\kappa_1+\frac{3}{2}\kappa_1\beta_5-\frac{3}{2}\mu_1\beta_6+\frac{n\bar{\alpha}_5}{2}\right)\cos\zeta_1+\left(-2n\mu_1-\frac{3}{2}\kappa_1\beta_6-\frac{3}{2}\mu_1\beta_5-\frac{n\bar{\alpha}_6}{2}\right)\sin\zeta_1\right] &
\frac{e^{\psi_1}}{r^2}\left[\left(2n\mu_1+\frac{3}{2}\kappa_1\beta_6+\frac{3}{2}\mu_1\beta_5+\frac{n\bar{\alpha}_6}{2}\right)\cos\zeta_1+\left(2n\kappa_1+\frac{3}{2}\kappa_1\beta_5-\frac{3}{2}\mu_1\beta_6+\frac{n\bar{\alpha}_5}{2}\right)\sin\zeta_1\right] &
\frac{e^{\psi_2}}{r^2}\left[\left(2n\kappa_2+\frac{3}{2}\kappa_2\beta_7-\frac{3}{2}\mu_2\beta_8+\frac{n\bar{\alpha}_7}{2}\right)\cos\zeta_2+\left(-2n\mu_2-\frac{3}{2}\kappa_2\beta_8-\frac{3}{2}\mu_2\beta_7-\frac{n\bar{\alpha}_8}{2}\right)\sin\zeta_2\right] &
\frac{e^{\psi_2}}{r^2}\left[\left(2n\mu_2+\frac{3}{2}\kappa_2\beta_8+\frac{3}{2}\mu_2\beta_7+\frac{n\bar{\alpha}_8}{2}\right)\cos\zeta_2+\left(2n\kappa_2+\frac{3}{2}\kappa_2\beta_7-\frac{3}{2}\mu_2\beta_8+\frac{n\bar{\alpha}_7}{2}\right)\sin\zeta_2\right]
\end{bmatrix}
$$

$$
\{\epsilon\} = \begin{bmatrix}[T][0]\\ [0][T]\end{bmatrix}[Q]\{C\} = \begin{bmatrix}[T][0]\\ [0][T]\end{bmatrix}[Q][A]^{-1}\{\delta\}
$$

Table 3. Matrix [Γ].

$[\Gamma] =$

$\frac{1}{r^2}\left[-2n\kappa_1-\frac{3}{2}\kappa_1\bar\beta_1-\frac{3}{2}\mu_1\bar\beta_2+\frac{n\bar\alpha_1}{2}\right]$	$\frac{1}{r^2}[n^2+n\bar\beta_1]$	$-\frac{1}{r^2}[\kappa_1{}^2-\mu_1{}^2]$	$\frac{1}{r}[-\kappa_1\bar\beta_1-\mu_1\bar\beta_2-n\bar\alpha_1]$	$\frac{1}{r}[n\bar\beta_1+1]$	$\frac{1}{r}[-\kappa_1\bar\alpha_1-\mu_1\bar\alpha_2]$
$\frac{1}{r^2}\left[-2n\mu_1+\frac{3}{2}\kappa_1\bar\beta_2-\frac{3}{2}\mu_1\bar\beta_1-\frac{n\bar\alpha_2}{2}\right]$	$\frac{1}{r^2}[-n\bar\beta_2]$	$-\frac{1}{r^2}[2\kappa_1\mu_1]$	$\frac{1}{r}[\kappa_1\bar\beta_2-\mu_1\bar\beta_1+n\bar\alpha_2]$	$\frac{1}{r}[-n\bar\beta_2]$	$\frac{1}{r}[\kappa_1\bar\alpha_2-\mu_1\bar\alpha_1]$
$\frac{1}{r^2}\left[2n\mu_1-\frac{3}{2}\kappa_1\bar\beta_2+\frac{3}{2}\mu_1\bar\beta_1+\frac{n\bar\alpha_2}{2}\right]$	$\frac{1}{r^2}[n\bar\beta_2]$	$\frac{1}{r^2}[2\mu_1\kappa_1]$	$\frac{1}{r}[-\kappa_1\bar\beta_2+\mu_1\bar\beta_1-n\bar\alpha_2]$	$\frac{1}{r}[n\bar\beta_2]$	$\frac{1}{r}[-\kappa_1\bar\alpha_2+\mu_1\bar\alpha_1]$
$\frac{1}{r^2}\left[-2n\kappa_1-\frac{3}{2}\kappa_1\bar\beta_1-\frac{3}{2}\mu_1\bar\beta_2+\frac{n\bar\alpha_1}{2}\right]$	$\frac{1}{r^2}[n^2+n\bar\beta_1]$	$-\frac{1}{r^2}[\kappa_1{}^2-\mu_1{}^2]$	$\frac{1}{r}[-\kappa_1\bar\beta_1-\mu_1\bar\beta_2-n\bar\alpha_1]$	$\frac{1}{r}[n\bar\beta_1+1]$	$\frac{1}{r}[-\kappa_1\bar\alpha_1-\mu_1\bar\alpha_2]$
$\frac{1}{r^2}\left[-2n\kappa_2-\frac{3}{2}\kappa_2\bar\beta_3-\frac{3}{2}\mu_2\bar\beta_4+\frac{n\bar\alpha_3}{2}\right]$	$\frac{1}{r^2}[n^2+n\bar\beta_3]$	$-\frac{1}{r^2}[\kappa_2{}^2-\mu_2{}^2]$	$\frac{1}{r}[-\kappa_2\bar\beta_3-\mu_2\bar\beta_4-n\bar\alpha_3]$	$\frac{1}{r}[n\bar\beta_3+1]$	$\frac{1}{r}[-\kappa_2\bar\alpha_3-\mu_2\bar\alpha_4]$
$\frac{1}{r^2}\left[-2n\mu_2+\frac{3}{2}\kappa_2\bar\beta_4-\frac{3}{2}\mu_2\bar\beta_3-\frac{n\bar\alpha_4}{2}\right]$	$-\frac{1}{r^2}[n\bar\beta_4]$	$-\frac{1}{r^2}[2\kappa_2\mu_2]$	$\frac{1}{r}[\kappa_2\bar\beta_4-\mu_2\bar\beta_3+n\bar\alpha_4]$	$\frac{1}{r}[-n\bar\beta_4]$	$\frac{1}{r}[\kappa_2\bar\alpha_4-\mu_2\bar\alpha_3]$
$\frac{1}{r^2}\left[2n\mu_2-\frac{3}{2}\kappa_2\bar\beta_4+\frac{3}{2}\mu_2\bar\beta_3+\frac{n\bar\alpha_4}{2}\right]$	$\frac{1}{r^2}[n\bar\beta_4]$	$\frac{1}{r^2}[2\kappa_2\mu_2]$	$\frac{1}{r}[-\kappa_2\bar\beta_4+\mu_2\bar\beta_3-n\bar\alpha_4]$	$\frac{1}{r}[n\bar\beta_4]$	$\frac{1}{r}[-\kappa_2\bar\alpha_4+\mu_2\bar\alpha_3]$
$\frac{1}{r^2}\left[-2n\kappa_2-\frac{3}{2}\kappa_2\bar\beta_3-\frac{3}{2}\mu_2\bar\beta_4+\frac{n\bar\alpha_3}{2}\right]$	$\frac{1}{r^2}[n^2+n\bar\beta_3]$	$-\frac{1}{r^2}[\kappa_2{}^2-\mu_2{}^2]$	$\frac{1}{r}[-\kappa_2\bar\beta_3-\mu_2\bar\beta_4-n\bar\alpha_3]$	$\frac{1}{r}[n\bar\beta_3+1]$	$\frac{1}{r}[-\kappa_2\bar\alpha_3-\mu_2\bar\alpha_4]$
$\frac{1}{r^2}\left[2n\kappa_1+\frac{3}{2}\kappa_1\bar\beta_5-\frac{3}{2}\mu_1\bar\beta_6+\frac{n\bar\alpha_5}{2}\right]$	$\frac{1}{r^2}[n^2+n\bar\beta_5]$	$-\frac{1}{r^2}[\kappa_1{}^2-\mu_1{}^2]$	$\frac{1}{r}[\kappa_1\bar\beta_5-\mu_1\bar\beta_6-n\bar\alpha_5]$	$\frac{1}{r}[n\bar\beta_5+1]$	$\frac{1}{r}[\kappa_1\bar\alpha_5-\mu_1\bar\alpha_6]$
$\frac{1}{r^2}\left[-2n\mu_1-\frac{3}{2}\kappa_1\bar\beta_6-\frac{3}{2}\mu_1\bar\beta_5-\frac{n\bar\alpha_6}{2}\right]$	$-\frac{1}{r^2}[n\bar\beta_6]$	$\frac{1}{r^2}[2\kappa_1\mu_1]$	$\frac{1}{r}[-\kappa_1\bar\beta_6-\mu_1\bar\beta_6+n\bar\alpha_6]$	$\frac{1}{r}[-n\bar\beta_6]$	$\frac{1}{r}[-\kappa_1\bar\alpha_6-\mu_1\bar\alpha_5]$
$\frac{1}{r^2}\left[2n\mu_1+\frac{3}{2}\kappa_1\bar\beta_6+\frac{3}{2}\mu_1\bar\beta_5+\frac{n\bar\alpha_6}{2}\right]$	$\frac{1}{r^2}[n\bar\beta_6]$	$-\frac{1}{r^2}[2\kappa_1\mu_1]$	$\frac{1}{r}[\kappa_1\bar\beta_6+\mu_1\bar\beta_5-n\bar\alpha_6]$	$\frac{1}{r}[n\bar\beta_6]$	$\frac{1}{r}[\kappa_1\bar\alpha_6+\mu_1\bar\alpha_5]$
$\frac{1}{r^2}\left[2n\kappa_1+\frac{3}{2}\kappa_1\bar\beta_5-\frac{3}{2}\mu_1\bar\beta_6+\frac{n\bar\alpha_5}{2}\right]$	$\frac{1}{r^2}[n^2+n\bar\beta_5]$	$-\frac{1}{r^2}[\kappa_1{}^2-\mu_1{}^2]$	$\frac{1}{r}[\kappa_1\bar\beta_5-\mu_1\bar\beta_6-n\bar\alpha_5]$	$\frac{1}{r}[n\bar\beta_5+1]$	$\frac{1}{r}[\kappa_1\bar\alpha_5-\mu_1\bar\alpha_6]$
$\frac{1}{r^2}\left[2n\kappa_2+\frac{3}{2}\kappa_2\bar\beta_7-\frac{3}{2}\mu_2\bar\beta_8+\frac{n\bar\alpha_7}{2}\right]$	$\frac{1}{r^2}[n^2+n\bar\beta_7]$	$-\frac{1}{r^2}[\kappa_2{}^2-\mu_2{}^2]$	$\frac{1}{r}[\kappa_2\bar\beta_7-\mu_2\bar\beta_8-n\bar\alpha_7]$	$\frac{1}{r}[n\bar\beta_7+1]$	$\frac{1}{r}[\kappa_2\bar\alpha_7-\mu_2\bar\alpha_8]$
$\frac{1}{r^2}\left[-2n\mu_2-\frac{3}{2}\kappa_2\bar\beta_8-\frac{3}{2}\mu_2\bar\beta_7-\frac{n\bar\alpha_8}{2}\right]$	$-\frac{1}{r^2}[n\bar\beta_8]$	$\frac{1}{r^2}[2\kappa_2\mu_2]$	$\frac{1}{r}[-\kappa_2\bar\beta_8-\mu_2\bar\beta_7+n\bar\alpha_8]$	$\frac{1}{r}[-n\bar\beta_8]$	$\frac{1}{r}[-\kappa_2\bar\alpha_8-\mu_2\bar\alpha_7]$
$\frac{1}{r^2}\left[2n\mu_2+\frac{3}{2}\kappa_2\bar\beta_8+\frac{3}{2}\mu_2\bar\beta_7+\frac{n\bar\alpha_8}{2}\right]$	$\frac{1}{r^2}[n\bar\beta_8]$	$-\frac{1}{r^2}[2\kappa_2\mu_2]$	$\frac{1}{r}[\kappa_2\bar\beta_8+\mu_2\bar\beta_7-n\bar\alpha_8]$	$\frac{1}{r}[n\bar\beta_8]$	$\frac{1}{r}[\kappa_2\bar\alpha_8+\mu_2\bar\alpha_7]$
$\frac{1}{r^2}\left[2n\kappa_3+\frac{3}{2}\kappa_2\bar\beta_7-\frac{3}{2}\mu_2\bar\beta_8+\frac{n\bar\alpha_7}{2}\right]$	$\frac{1}{r^2}[n^2+n\bar\beta_7]$	$-\frac{1}{r^2}[\kappa_2{}^2-\mu_2{}^2]$	$\frac{1}{r}[\kappa_2\bar\beta_7-\mu_2\bar\beta_8-n\bar\alpha_7]$	$\frac{1}{r}[n\bar\beta_7+1]$	$\frac{1}{r}[\kappa_2\bar\alpha_7-\mu_2\bar\alpha_8]$

$$[G] = \pi r\int_0^l [Q]^T[P][Q]\,dx = \pi r\int_0^l [Z]^T[\Gamma]^T[P][\Gamma][Z]\,dx = \pi r\int_0^l [Z]^T[Y][Z]\,dx$$

Table 4. Matrices $[\Delta]$ and $[A_1]$

$[\Delta] =$

$\bar{\alpha}_1$	$-\bar{\alpha}_2$	$\bar{\alpha}_2$	$\bar{\alpha}_1$	$\bar{\alpha}_3$	$-\bar{\alpha}_4$	$\bar{\alpha}_4$	$\bar{\alpha}_3$	$\bar{\alpha}_5$	$-\bar{\alpha}_6$	$\bar{\alpha}_6$	$\bar{\alpha}_5$	$\bar{\alpha}_7$	$-\bar{\alpha}_8$	$\bar{\alpha}_8$	$\bar{\alpha}_7$
1	0	0	1	1	0	0	1	1	0	0	1	1	1	0	1
$\bar{\beta}_1$	$-\bar{\beta}_2$	$\bar{\beta}_2$	$\bar{\beta}_1$	$\bar{\beta}_3$	$-\bar{\beta}_4$	$\bar{\beta}_4$	$\bar{\beta}_3$	$\bar{\beta}_5$	$-\bar{\beta}_6$	$\bar{\beta}_6$	$\bar{\beta}_5$	$\bar{\beta}_7$	$-\bar{\beta}_8$	$\bar{\beta}_8$	$\bar{\beta}_7$

$[A_1] =$

$-2\frac{\kappa_1}{r}$	$-2\frac{\kappa_1}{r}$	$-\frac{\kappa_1}{r}-\frac{\kappa_2}{r}$	$-\frac{\kappa_1}{r}-\frac{\kappa_2}{r}$	0	0	$-\frac{\kappa_1}{r}+\frac{\kappa_2}{r}$	$-\frac{\kappa_1}{r}+\frac{\kappa_2}{r}$
$-2\frac{\kappa_1}{r}$	$-2\frac{\kappa_1}{r}$	$-\frac{\kappa_1}{r}-\frac{\kappa_2}{r}$	$-\frac{\kappa_1}{r}-\frac{\kappa_2}{r}$	0	0	$-\frac{\kappa_1}{r}+\frac{\kappa_2}{r}$	$-\frac{\kappa_1}{r}+\frac{\kappa_2}{r}$
$-\frac{\kappa_1}{r}-\frac{\kappa_2}{r}$	$-\frac{\kappa_1}{r}-\frac{\kappa_2}{r}$	$-2\frac{\kappa_2}{r}$	$-2\frac{\kappa_2}{r}$	$\frac{\kappa_1}{r}-\frac{\kappa_2}{r}$	$\frac{\kappa_1}{r}-\frac{\kappa_2}{r}$	0	0
$-\frac{\kappa_1}{r}-\frac{\kappa_2}{r}$	$-\frac{\kappa_1}{r}-\frac{\kappa_2}{r}$	$-2\frac{\kappa_2}{r}$	$-2\frac{\kappa_2}{r}$	$\frac{\kappa_1}{r}-\frac{\kappa_2}{r}$	$\frac{\kappa_1}{r}-\frac{\kappa_2}{r}$	0	0
0	0	$\frac{\kappa_1}{r}-\frac{\kappa_2}{r}$	$\frac{\kappa_1}{r}-\frac{\kappa_2}{r}$	$2\frac{\kappa_1}{r}$	$2\frac{\kappa_1}{r}$	$\frac{\kappa_1}{r}+\frac{\kappa_2}{r}$	$\frac{\kappa_1}{r}+\frac{\kappa_2}{r}$
0	0	$\frac{\kappa_1}{r}-\frac{\kappa_2}{r}$	$\frac{\kappa_1}{r}-\frac{\kappa_2}{r}$	$2\frac{\kappa_1}{r}$	$2\frac{\kappa_1}{r}$	$\frac{\kappa_1}{r}+\frac{\kappa_2}{r}$	$\frac{\kappa_1}{r}+\frac{\kappa_2}{r}$
$\frac{\kappa_2}{r}-\frac{\kappa_1}{r}$	$\frac{\kappa_2}{r}-\frac{\kappa_1}{r}$	0	0	$\frac{\kappa_2}{r}+\frac{\kappa_1}{r}$	$\frac{\kappa_2}{r}+\frac{\kappa_1}{r}$	$2\frac{\kappa_2}{r}$	$2\frac{\kappa_2}{r}$
$\frac{\kappa_2}{r}-\frac{\kappa_1}{r}$	$\frac{\kappa_2}{r}-\frac{\kappa_1}{r}$	0	0	$\frac{\kappa_2}{r}+\frac{\kappa_1}{r}$	$\frac{\kappa_2}{r}+\frac{\kappa_1}{r}$	$2\frac{\kappa_2}{r}$	$2\frac{\kappa_2}{r}$

Table 5. Matrices $[B_1]$, $[C_1]$ and $[D_1]$ to $[D_4]$ *

$[B_1] =$

$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$
$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$
$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$
$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$
$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$
$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$
$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$
$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$

$[C_1] =$

$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$
$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$
$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$
$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$
$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$
$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_1}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$
$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$
$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$	$\frac{\mu_2}{r}$

$[D_1] =$

y_{11}	y_{13}	y_{15}	y_{17}	y_{19}	$y_{1,11}$	$y_{1,13}$	$y_{1,15}$
y_{31}	y_{33}	y_{35}	y_{37}	y_{39}	$y_{3,11}$	$y_{3,13}$	$y_{3,15}$
y_{51}	y_{53}	y_{55}	y_{57}	y_{59}	$y_{5,11}$	$y_{5,13}$	$y_{5,15}$
y_{71}	y_{73}	y_{75}	y_{77}	y_{79}	$y_{7,11}$	$y_{7,13}$	$y_{7,15}$
y_{91}	y_{93}	y_{95}	y_{97}	y_{99}	$y_{9,11}$	$y_{9,13}$	$y_{9,15}$
$y_{11,1}$	$y_{11,3}$	$y_{11,5}$	$y_{11,7}$	$y_{11,9}$	$y_{11,11}$	$y_{11,13}$	$y_{11,15}$
$y_{13,1}$	$y_{13,3}$	$y_{13,5}$	$y_{13,7}$	$y_{13,9}$	$y_{13,11}$	$y_{13,13}$	$y_{13,15}$
$y_{15,1}$	$y_{15,3}$	$y_{15,5}$	$y_{15,5}$	$y_{15,9}$	$y_{15,11}$	$y_{15,13}$	$y_{15,15}$

$[D_2] =$

y_{21}	y_{23}	y_{25}	y_{27}	y_{29}	$y_{2,11}$	$y_{2,13}$	$y_{2,15}$
y_{32}	y_{43}	y_{45}	y_{47}	y_{49}	$y_{4,11}$	$y_{4,13}$	$y_{4,15}$
y_{52}	y_{54}	y_{65}	y_{67}	y_{69}	$y_{6,11}$	$y_{6,13}$	$y_{6,15}$
y_{72}	y_{74}	y_{76}	y_{87}	y_{89}	$y_{8,11}$	$y_{8,13}$	$y_{8,15}$
y_{92}	y_{94}	y_{96}	y_{98}	$y_{10,9}$	$y_{10,11}$	$y_{10,13}$	$y_{10,15}$
$y_{11,2}$	$y_{11,4}$	$y_{11,6}$	$y_{11,8}$	$y_{11,10}$	$y_{12,11}$	$y_{12,13}$	$y_{12,15}$
$y_{13,2}$	$y_{13,4}$	$y_{13,6}$	$y_{13,8}$	$y_{13,10}$	$y_{13,12}$	$y_{14,13}$	$y_{14,15}$
$y_{16,2}$	$y_{16,4}$	$y_{15,6}$	$y_{16,8}$	$y_{15,10}$	$y_{15,12}$	$y_{15,14}$	$y_{16,15}$

$[D_3] =$

y_{12}	y_{14}	y_{16}	y_{18}	$y_{1,10}$	$y_{1,12}$	$y_{1,14}$	$y_{1,16}$
y_{41}	y_{34}	y_{36}	y_{38}	$y_{3,10}$	$y_{3,12}$	$y_{3,14}$	$y_{3,16}$
y_{61}	y_{63}	y_{56}	y_{58}	$y_{5,10}$	$y_{5,12}$	$y_{5,14}$	$y_{5,16}$
y_{81}	y_{83}	y_{86}	y_{78}	$y_{7,10}$	$y_{7,12}$	$y_{7,14}$	$y_{7,16}$
$y_{10,1}$	$y_{10,3}$	$y_{10,5}$	$y_{10,7}$	$y_{9,10}$	$y_{9,12}$	$y_{9,14}$	$y_{9,16}$
$y_{12,1}$	$y_{12,3}$	$y_{12,5}$	$y_{12,7}$	$y_{12,9}$	$y_{11,12}$	$y_{11,14}$	$y_{11,16}$
$y_{14,1}$	$y_{14,3}$	$y_{14,5}$	$y_{14,7}$	$y_{14,9}$	$y_{14,11}$	$y_{13,14}$	$y_{13,16}$
$y_{16,1}$	$y_{16,3}$	$y_{16,5}$	$y_{16,7}$	$y_{16,9}$	$y_{16,11}$	$y_{16,13}$	$y_{15,16}$

$[D_4] =$

y_{22}	y_{24}	y_{26}	y_{28}	$y_{2,10}$	$y_{2,12}$	$y_{2,14}$	$y_{2,16}$
y_{42}	y_{44}	y_{46}	y_{48}	$y_{4,10}$	$y_{4,12}$	$y_{4,14}$	$y_{4,16}$
y_{62}	y_{64}	y_{66}	y_{68}	$y_{6,10}$	$y_{6,12}$	$y_{6,14}$	$y_{6,16}$
y_{82}	y_{84}	y_{86}	y_{88}	$y_{8,10}$	$y_{8,12}$	$y_{8,14}$	$y_{8,16}$
$y_{10,2}$	$y_{10,4}$	$y_{10,6}$	$y_{10,8}$	$y_{10,10}$	$y_{10,12}$	$y_{10,14}$	$y_{10,16}$
$y_{12,2}$	$y_{12,4}$	$y_{12,6}$	$y_{12,8}$	$y_{12,10}$	$y_{12,12}$	$y_{12,14}$	$y_{12,16}$
$y_{14,2}$	$y_{14,4}$	$y_{14,6}$	$y_{14,8}$	$y_{14,10}$	$y_{14,12}$	$y_{14,14}$	$y_{14,16}$
$y_{16,2}$	$y_{16,4}$	$y_{16,6}$	$y_{16,8}$	$y_{16,10}$	$y_{16,12}$	$y_{16,14}$	$y_{16,16}$

* y_{ij} in theses matrices are elements of the $[\gamma]$ matrix, where

$$[\gamma] = [\Gamma]^T [P] [\Gamma].$$

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