Almost periodic functions and almost periodic equidistributed functions

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Almost Periodic Functions and Equidistribution

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joint work with

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Outline

Periodic phenomena in our lives:

- The recurrences of days and nights repeat every 24 hours.
- Seasons occur regularly roughly every $365\frac{1}{4}$ days.

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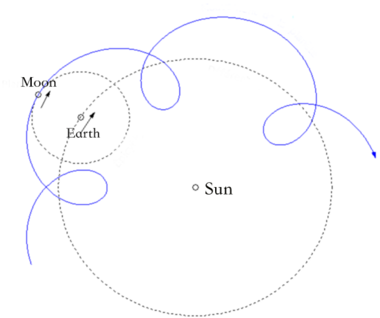

Figure 1: Motion of the Earth around the Sun and motion of the Moon around the Earth

Periodic functions (on real line $\mathbb{R}$)

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If f and g are two periodic functions, $f+g$ need not be periodic. For example, the function

$$h(t) = \cos t + \cos \sqrt{2}t$$

is not periodic since the equation

$$h(t) = 2,$$

has a single solution $t = 0$.

This phenomenon leads us to the notion of "almost periodicity".

An Almost Periodic Motion

Outline

Almost periodic functions on topological groups

In 1926 and 1927, mathematicians Bohr and Bochner gave equivalent version of definitions of almost periodic functions on real line $\mathbb{R}$ respectively. In this research we will consider almost periodicity of functions on topological group G [1]. We denote the set of all continuous almost periodic functions on topological group G by $AP(G)$.

Almost periodic functions on topological groups

In 1926 and 1927, mathematicians Bohr and Bochner gave equivalent version of definitions of almost periodic functions on real line $\mathbb{R}$ respectively. In this research we will consider almost periodicity of functions on topological group G [1]. We denote the set of all continuous almost periodic functions on topological group G by $AP(G)$.

Some advantages of almost periodic functions:

(i) Every periodic function is almost periodic.

(ii) If f, g are in $AP(G)$, then $f + g$ and fg are in $AP(G)$.

Outline

Equidistributed sequences

Equidistribution is also known as uniform distribution. The formal definition of equidistribution mod 1 of sequences was given initially by Weyl [4] in 1916. A sequence $\{x_n\}_n$ in $[0, 1]$ is **equidistributed** if for every interval $[a, b] \subset [0, 1]$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} 1_{[a,b]}(x_n) = b - a. \tag{1}$$

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For example, the sequence

$$0, \frac{1}{2}, 0, \frac{1}{3}, \frac{2}{3}, 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 0, \frac{1}{5}, \frac{2}{5}, \cdots$$

is equidistributed along $[0,1]$.

Example 1

If γ is irrational, then the sequence of fractional parts $\langle\gamma\rangle$, $\langle 2\gamma\rangle$, $\langle 3\gamma\rangle$,... is equidistributed in $[0,1)$. (Bohl, Sierpiński, Weyl, 1909-1910)

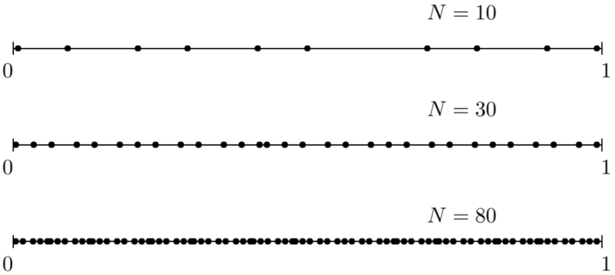

Figure 2: The set $\{\langle\gamma\rangle, \langle 2\gamma\rangle, \langle 3\gamma\rangle, \ldots \langle N\gamma\rangle\}$ when $\gamma = \sqrt{2}$

Example 2

In particular, an equidistributed sequence must be dense in $[0, 1]$, and finite sequences are not equidistributed in this sense.

For example, the sequence $\langle \frac{np}{q} \rangle$ is not equidistributed since it is finite.

Outline

Almost periodic equidistribution

In our research, the notion of equidistribution was extended from sequences to the more general case of functions.

Definition 1 (a.p.-equidistribution)

Let S be a locally compact space equipped with a regular Borel measure λ, and a monotone compact cover $\{K_\alpha\}_{\alpha \in I}$. Let H be a topological group and M the invariant mean on $AP(H)$. A continuous mapping $\varphi : S \longrightarrow H$ is called **a.p-equidistributed** along $\{K_\alpha\}_{\alpha \in I}$ if for all $f \in AP(G)$,

$$\langle M, f \rangle = \lim_{\alpha \in I} \frac{1}{\lambda(K_\alpha)} \int_{K_\alpha} f \circ \varphi(s)\, d\lambda(s). \tag{2}$$

Outline

The Weyl Criterion

An important tool for verification of equidistribution of sequences is the Weyl criterion, which is given by Weyl [4] in 1916.

The sequence $(x_n)_n$ of real numbers is equidistributed mod 1 if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} = 0 \tag{3}$$

for all integers $h \neq 0$.

Our Generalization on the Weyl Criterion

In our research [3], this criterion has been extended so that it becomes a useful tool for verification of equidistribution of functions.

Theorem 2 (the generalized Weyl criterion)

Let S be a locally compact space equipped with a regular Borel measure λ and a monotone compact cover $\{K_\alpha\}_{\alpha\in I}$. Then a continuous mapping $\varphi : S \longrightarrow H$ is a.p.-equidistributed along $\{K_\alpha\}_{\alpha\in I}$ if and only if

$$\lim_{\alpha\in I} \frac{1}{\lambda(K_\alpha)} \int_{K_\alpha} (\sigma_{ij} \circ \varphi)(s)\, d\lambda(s) = 0, \qquad (4)$$

for all $\sigma \in \mathscr{R}_H$, $\sigma \ncong 1_H$, $1 \leqslant i, j \leqslant d_\sigma$.

References I

[1] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, Vol. 1, Second Edition, Springer-Verlag, Berlin, 1979.

[2] V. Limic and N. Limić, *Equidistribution and uniform distribution: a probabilist's perspective*, Probability Surveys **15** (2018), 131–155.

[3] M. S. Monfared and Y. Zhu, *Equidistribution of continuous functions along monotone compact covers*, submitted.

[4] H. Weyl, *Über die gleichverteilung von zahlen mod. eins*, Math. Ann. **77** (1916), 313–352.

Q & A

Thank you!