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# Almost periodic functions and almost periodic equidistributed functions

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Almost periodic functions	Equidistributed sequences	Almost periodic equidistribution	The Weyl Criterion	References	Q & A

#### Almost Periodic Functions and Equidistribution

#### Yihan Zhu

joint work with

Dr. Mehdi S. Monfared

University of Windsor

March, 2022



Almost periodic functions	Equidistributed sequences	Almost periodic equidistribution	The Weyl Criterion	Q & A 0
Outline				



- 2 Equidistributed sequences
- 3 Almost periodic equidistribution
- 4 The Weyl Criterion



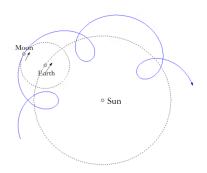
#### Periodic phenomena in our lives:

- The recurrences of days and nights repeat every 24 hours.
- Seasons occur regularly roughly every  $365\frac{1}{4}$  days.



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### Figure 1: Motion of the Earth around the Sun and motion of the Moon around the Earth $\langle \Box \rangle \langle \overline{z} \rangle \langle \overline{z} \rangle \langle \overline{z} \rangle$

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### Question: Is the linear combination of two periodic motions still be periodic?



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If f and g are two periodic functions, f+g need not be periodic. For example, the function

$$h(t) = \cos t + \cos \sqrt{2}t$$

is not periodic since the equation

$$h(t)=2,$$

has a single solution t = 0.

This phenomenon leads us to the notion of "almost periodicity".

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#### An Almost Periodic Motion

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In 1926 and 1927, mathematicians Bohr and Bochner gave equivalent version of definitions of almost periodic functions on real line  $\mathbb{R}$  respectively. In this research we will consider almost periodicity of functions on topological group G [1]. We denote the set of all continuous almost periodic functions on topological group G by AP(G).



In 1926 and 1927, mathematicians Bohr and Bochner gave equivalent version of definitions of almost periodic functions on real line  $\mathbb{R}$  respectively. In this research we will consider almost periodicity of functions on topological group G [1]. We denote the set of all continuous almost periodic functions on topological group G by AP(G).

#### Some advantages of almost periodic functions:

(i) Every periodic function is almost periodic.

(ii) If f, g are in AP(G), then f + g and fg are in AP(G).

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## Almost periodic functions equidistributed sequences of the sequences of th

Equidistribution is also known as uniform distribution. The formal definition of equidistribution mod 1 of sequences was given initially by Weyl [4] in 1916. A sequence  $\{x_n\}_n$  in [0, 1] is **equidistributed** if for every interval  $[a, b] \subset [0, 1]$ ,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{[a,b]}(x_n) = b - a.$$
 (1)

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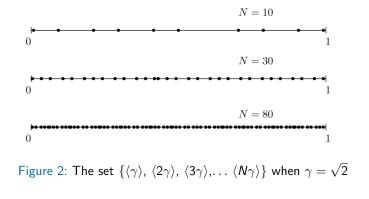
For example, the sequence

$$0, \frac{1}{2}, 0, \frac{1}{3}, \frac{2}{3}, 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 0, \frac{1}{5}, \frac{2}{5}, \cdots$$

is equidistributed along [0, 1].

	Equidistributed sequences	Almost periodic equidistribution	The Weyl Criterion	Q & A 0
Example 1				

If  $\gamma$  is irrational, then the sequence of fractional parts  $\langle \gamma \rangle$ ,  $\langle 2\gamma \rangle$ ,  $\langle 3\gamma \rangle$ ,... is equidistributed in [0, 1). (Bohl, Sierpiński, Weyl, 1909-1910)



	Almost periodic equidistribution	The Weyl Criterion	Q & A 0
Example 2			

In particular, an equidistributed sequence must be dense in [0,1], and finite sequences are not equidistributed in this sense.

For example, the sequence  $\langle \frac{np}{q} \rangle$  is not equidistributed since it is finite.

Almost periodic functions	Equidistributed sequences	Almost periodic equidistribution •0	The Weyl Criterion	Q & A 0
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2 Equidistributed sequences

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#### Almost periodic equidistribution

In our research, the notion of equidistribution was extended from sequences to the more general case of functions.

#### Definition 1 (a.p.-equidistribution)

Let *S* be a locally compact space equipped with a regular Borel measure  $\lambda$ , and a monotone compact cover  $\{K_{\alpha}\}_{\alpha \in I}$ . Let *H* be a topological group and *M* the invariant mean on AP(H). A continuous mapping  $\varphi : S \longrightarrow H$  is called **a.p-equidistributed** along  $\{K_{\alpha}\}_{\alpha \in I}$  if for all  $f \in AP(G)$ ,

$$\langle M, f \rangle = \lim_{\alpha \in I} \frac{1}{\lambda(K_{\alpha})} \int_{K_{\alpha}} f \circ \varphi(s) \, d\lambda(s).$$
 (2)

OO	OOOO	Almost periodic equidistribution	●00	

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An important tool for verification of equidistribution of sequences is the Weyl criterion, which is given by Weyl [4] in 1916.

The sequence  $(x_n)_n$  of real numbers is equidistributed mod 1 if and only if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e^{2\pi ihx_n}=0$$
(3)

for all integers  $h \neq 0$ .

The Weyl Criterion

#### Our Generalization on the Weyl Criterion

In our research [3], this criterion has been extended so that it becomes a useful tool for verification of equidistribution of functions.

#### Theorem 2 (the generalized Weyl criterion)

Let S be a locally compact space equipped with a regular Borel measure  $\lambda$  and a monotone compact cover  $\{K_{\alpha}\}_{\alpha \in I}$ . Then a continuous mapping  $\varphi : S \longrightarrow H$  is a.p.-equidistributed along  $\{K_{\alpha}\}_{\alpha \in I}$  if and only if

$$\lim_{\alpha \in I} \frac{1}{\lambda(K_{\alpha})} \int_{K_{\alpha}} (\sigma_{ij} \circ \varphi)(s) \, d\lambda(s) = 0, \tag{4}$$

for all  $\sigma \in \mathscr{R}_H$ ,  $\sigma \not\cong 1_H$ ,  $1 \leqslant i, j \leqslant d_{\sigma}$ .

Almost periodic functions	Equidistributed sequences	Almost periodic equidistribution	The Weyl Criterion	References •	Q & A 0
References I					

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- [4] H. Weyl, Über die gleichverteilung von zahlen mod. eins, Math. Ann. 77 (1916), 313–352.

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## Thank you!

Yihan Zhu

Almost Periodic Functions and Equidistribution

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