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Generalizing Binary Operations

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GENERALIZING BINARY OPERATIONS

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Most day to day calculations take place within the field of real numbers with the two binary operations of addition and multiplication. In this field, these two operations are definitionally independent of one another. However, if we approach binary operations from a different point of view, e.g. that of recursive formulae, we can develop multiplication from addition by use of the concept of repeated addition. Along similar lines, we can develop exponentiation from multiplication by repeated multiplication. The next logical step would be to try to develop another binary operation based on repeated exponentiation.

Professor D. F. Borrow of the University of Georgia in the American *Mathematical* Monthly, **43 (1936)**, p. 150, developed some theorems and a notation for repeated exponentiation. As Σ is used for summation and Π is used for products, he used E for repeated exponentiation. The development of a "fourth operation" would depend on all the indexed Terms of E being equal, similar to what is necessary in developing multiplication and exponentiation itself.

In order to clarify relations and notations, let us look at addition, multiplication, exponentiation, and a projected new fourth operation in terms of functions and recursive formulae. Let

$$f_1(n,m) = n + m$$

$$f_2(n,m) = n \cdot m$$

and

$$f_3(n,m) = n^m .$$

We know the following:

$$n \cdot m = n + [n \cdot (m - 1)] = \sum_1^m n_i \qquad \text{(where all } n_i = n\text{)}$$

and

$$n^m = n \cdot [n^{(m-1)}] = \prod_1^m n_i \qquad \text{(where all } n_i = n\text{)} .$$

Using our functional notation, we can write the above equations as recursive formulae:

$$f_2(n,m) = f_1[n,\ f_2(n,\ m-1)]$$

$$f_3(n,m) = f_2[n,\ f_3(n,\ m-1)]\ .$$

By comparing these two formulae, we can easily proceed to the definition of a fourth operation in terms of previous operations. Thus, let

$$f_4(n,m) = f_3[n,\ f_4(n,\ m-1)]\ ,$$

and, in general, for a kth operation, let

$$f_k(n,m) = f_{k-1}[n,\ f_k(n,\ m-1)]\ .$$

The question now arises, how does one define the first term in this recursive formula? In other words, what is $f_4(n,1)$? To answer this question, let us first look at $f_2(n,1)$, and $f_3(n,1)$, which are based on a similar process of recursive formulae and repeated operations. We know that $f_2(n,1) = \sum_1^1 n = n$ and we also know that $f_3(n,1) = \prod_1^1 n = n$. We can thus similarly define $f_4(n,1) = \mathrm{E}_1^1\ n$ to be equal to n by the same line of reasoning, that is, "one n" combined together by the process of [addition/multiplication/exponentiation] is still only "one n."

What about $f_4(n,2) = \mathrm{E}_1^2\ n$? This would be equal to

$$f_3[n,\ f_4(n,\ 2-1)] = f_3[n,\ f_4(n,1)] = f_3(n,n) = n^n\ .$$

Thus we see that our formulation of the recursive formula is consistent with what our initial intuitive feel was for what this new fourth function should be. Similarly, we obtain $f_4(n,3) = n^{(n^n)}$. At this point we might notice that, unlike our definitions of exponentiation and multiplication in terms of multiplication and addition respectively, our definition of f_4 does not allow associativity. In other words, $f_4(n,3) = n^{(n^n)} \neq (n^n)^n$, and, in general,

$$f_4(n,m) = n^{(n^{(n^{(n\cdots)})})} \neq ((((n^n)^n\cdots{}^n)^n)^n)^n = n^{n(m-1)}\ .$$

At this point, two questions may arise: What can one do with f and what about other operations? In particular, does there exist an f_0?

In answer to the first question, it is obvious that tables of f_4 are not readily available, and are not particularly useful, either. The numbers balloon quite rapidly. For example, $f_4(2,4) = 65{,}536$, and

$f_4(2,5) = f_3(2, 65{,}536) = 2^{65{,}536}$, while $f_4(3,3)$ exceeds ten digits. The only easily computable numbers are of the form $f_4(n,2) = n^n$. Even then, the numbers get fairly large, rather rapidly. For example, $f_4(8,2) = 16{,}777{,}216$.

There are other paths which can be taken with f_4 from here. As with an initial development of multiplication or exponentiation, we can develop definitions for $f_4(x,y)$ when y is zero, rational, real, or complex, and then develop definitions when x is zero, rational, real, or complex. For example, in developing exponentiation, one method of developing rational exponents is as follows:

Define $x = y^{(1/n)}$ to be equivalent to

$$y = x^n .$$

If one raises x to the power of m, then one has

$$z = x^m = y^{(m/n)} ,$$

and thus one has defined exponentiation for rational exponents.

Let us do something similar for f_4.

Define $x = f_4(y, 1/n)$ to be equivalent to

$$y = f_4(x,n) .$$

If we then operate on x by m, then we have

$$z = f_4(x,m) = f_4(y, m/n)$$

We can likewise work with negatives. In multiplication, $y = x \cdot (-n) = f_2(x,-n)$. But this is equivalent to saying $y + x \cdot n = 0 = I_1$ (the identity for f_1), or, using our functional notation, $f_1[y, f_2(x,n)] = I_1 = 0$. Likewise for exponentiation, $y = x^{-n} = \frac{1}{X}$ which is equivalent to saying $f_2[y, f_3(x,n)] = I_2 = 1$. Similarly, for our f_4, we can define $y = f_4(x,-n)$ as being equivalent to $f_3[y, f_4(x,n)] = I_3 = 1$.

Now, let us look at our other question--the possibility of f_0, that is, a binary operation more "basic" than addition. If it did exist, it would have to comply with our recursive formulae developed above and also to the general intuitive scheme of the functional notation. Now for any k, we saw that $f_k(n,m) = f_{k-1}[n, f_k(n, m - 1)]$. Let us take a closer look at what happens if k = 1. We would then have

$$f_1(n,m) = f_0[n, f_1(n, m - 1)] .$$

But f_1 is addition. Thus, we have

$$n + m = f_0[n, n + m - 1] .$$

If we now let $m = 1$, then we have

$$n + 1 = f_0[n,n] .$$

From the functional approach we know that $f_3(n,2) = n^2 = n \cdot n = f_2(n,n)$. Similarly, $f_2(n,2) = n \cdot 2 = n + n = f_1(n,n)$. If we are to be consistent, f_1 and f_0 should be similarly related (assuming f_0 exists). Thus, $f_1(n,2) = n + 2 = n \circ n = f_0(n,n)$ [where $f_0(n,n) = n \circ n$]. But above we showed that $f_0(n,n)$ was $n + 1$. From this contradiction resulting from the initial assumption that f_0 exists, we have shown that addition is the "most basic" operation we can have.

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At the suggestion of the Pi Mu Epsilon Council we have had a supply of 10 × 14-inch Fraternity crests printed. One in each color will be sent free to each local chapter on request. Additional posters may be ordered at the following rates:

(1) Purple on goldenrod stock - - - - - - $1.50/dozen,

(2) Purple and lavendar on goldenrod- - - $2.00/dozen.

SPECIAL ANNOUNCEMENT FOR CHAPTER PRESIDENTS

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