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Vertex-Magic Graphs

Karissa L. Massud

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1 Introduction

Graph theory is a branch of discrete mathematics. It can be defined as the study of graphs, which are structures made of a collection of points and lines. We will begin with some definitions.

Definition 1. *A **graph** is made up of a set of vertices (points) and a set of edges (lines).*

Figure 1 is an example of a graph with 5 vertices and 7 edges. When we count the amount of vertices and edges a graph has, we describe the order and the size of the graph.

Definition 2. *The **order**, v, of a graph is defined as the number of vertices on the graph and the **size**, e, of the graph is defined as the number of edges.*

In Figure 1, the order of the graph would be 5 and the size of the graph would be 7. In this graph, *A, B, C, D,* and *E* are vertices and *f, g, h, i, j, k*, and *l* are edges.

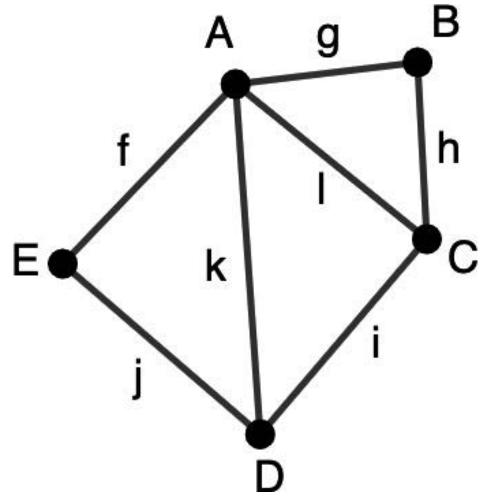

Figure 1: General graph with v = 5 and e = 7

Definition 3. *Vertices joined by an edge are called **adjacent vertices**. The edge between two vertices is called an **incident edge** to the vertices at its endpoints.*

For example, in Figure 1, *A* and *B* are adjacent vertices, and *g* is incidentto vertices *A* and *B*. Vertices *C* and *E* are non-adjacent. Figure 1 is an example of a connected graph. A **connected graph** is a graph where every pair of vertices are joined together by a sequence of vertices and edges.

Graphs are often used to model real world situations by representing connections and relationships between any given number of objects. Graph theory is used in many fields other than mathematics, like chemistry, biol- ogy, and computer science [6]. For example in chemistry, graphs are used to represent molecules and study atoms. In biology, graph theory is used to represent movement between regions where species exist. This can be usefulwhen studying breeding patterns or tracking the spread of diseases. In com- puter science, graphs represent networks and are used for the development ofvarious algorithms, which can analyze graphs and solve important problems.The Internet is also a great example of a graph, where Internet links are the edges and the pages they lead to are the vertices.

2 Graph Labeling

Graph labeling is an important area of graph theory. Graph labeling is the assignment of labels, usually represented by integers, to the vertices or the edges of a graph, and sometimes both. Graph labelings were first introduced in the 1960s as a tool for making progress on proving a conjecture about subgraphs [3]. The first type of graph labeling that was introduced is now known as graceful labeling. Since then, many other graph labelings have been introduced. For example, **graceful labelings** were coined by Solomon Golomb after originally being named β-labelings by Alexander Rosa in 1967[1]. To determine a graceful labeling, distinct labels are assigned to the vertices of the graph from the integer set $\{0, 1, 2, ..., e\}$. Each edge is labeled with the absolute difference of the incident vertex labels. If all of the edges of the graph are labeled with distinct integers from the set $\{1, 2, ..., e\}$, then the labeling is considered graceful. There are many more types of graph labelings that are not discussed here.

Graph labeling is extremely important, not only in the field of math- ematics, but especially in ordinary life. Adding labels to a graph can be beneficial because they give us extra information about a situation, like the distance between two stops on a mail route or the cost of traveling the most efficiently. Some contemporary uses of graph labeling that have substantiallyimpacted the modern world we live in include the global positioning system (GPS), Google searches, and public transportation. The GPS can create the quickest route from one location to another, Google searches sort informa- tion and track site visits, and sensible, efficient routes are devised for public transportation [6]. Other types of labelings are edge-graceful labelings, har- monious labelings, and even graph coloring, which resolved the renowned Four-Color Problem of graph theory.

Graph labeling is an interesting technique used by mathematicians, com- puter scientists, chemists and biologists in important fields like graph the- ory, coding theory, astronomy, circuit design, missile guidance, communica-tion network addressing, data base management, compound chemistry, and biotelemetry. It is so interesting, in fact, that in the past 50 years, over 200 graph labeling techniques have been studied in over 2500 papers [3]. At ap- proximately one paper per week, these numbers alone show how influential graph labeling has been, and continues to be.

3 Vertex-Magic Labeling

In this paper, we will study magic labelings. Magic labelings were first intro-duced by Sedláček in 1963 [3]. At this time, the labels on the graph were onlyassigned to the edges. In 1970, Kotzig and Rosa defined what are now known as edge-magic total labelings, where both the vertices and the edges of the graph are labeled. Following this in 1999, MacDougall, Miller, Slamin, and Wallis introduced the idea of vertex-magic total labelings. There are many different types of magic labelings. In this paper will focus on vertex-magic total labelings.

3.1 Definitions and History

We will begin with a few definitions.

Definition 4. *A **vertex-magic total labeling (VMTL)** is a labeling such that the vertices and edges are assigned consecutive integers between* 1 *and* $v+e$*, where* v *is the order of the graph (number of vertices) and* e *is the sizeof the graph (number of edges).*

Labels are assigned to graphs with the purpose of summing to the magic number of the graph.

Definition 5. *When the sum of the labels of a vertex and its incident edges results in the same integer for each vertex, we have a **vertex-magic graph**. This sum is called the **magic number, k,** of the graph. A raph is magic if at least one labeling exists.*

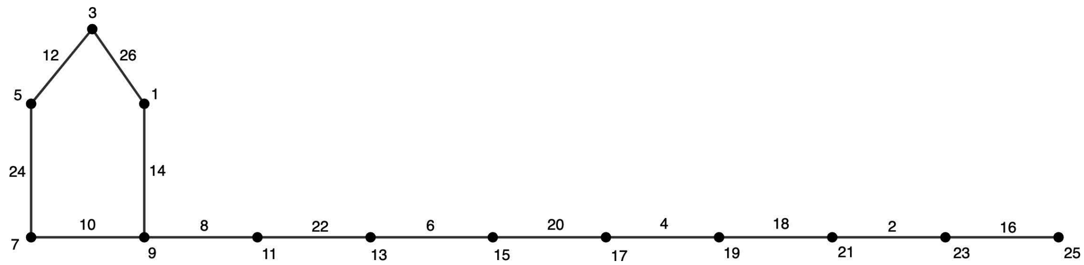

Figure 2: Vertex-Magic Total Labeling on a Kite Graph

In Figure 2, we see that v = 13 and e = 13 because the graph has 13 vertices and 13 edges. The integer labels that produce a VMTL of this graphare $\{1, 2, 3, ..., v + e\}$. In other words, the labels on this graph are all integers $\{1, 2, 3, ..., 26\}$. Since the sum of the labels of each vertex with the labels of its incident edges is 41, we see that the magic number, k, of this graph is 41.

Graphs can have multiple magic numbers. In this case, any of the magic numbers can have vertex-magic labelings. As we see in Figure 3 below, we have the same graph with labelings for four different magic numbers, namely 9, 10, 11 and 12.

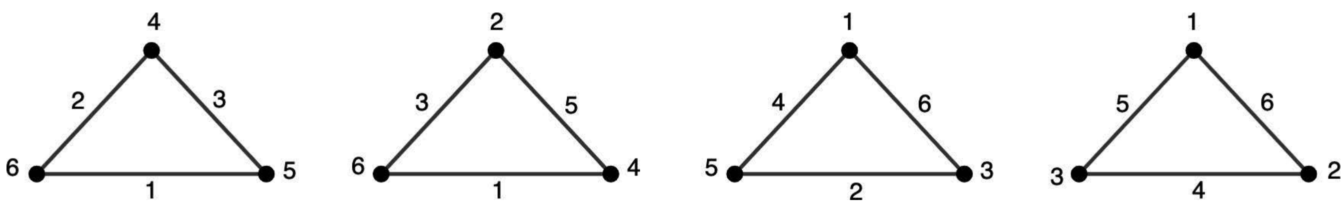

Figure 3: Graphs with Vertex-Magic Total Labelings of Four DifferentMagic Numbers

Some graphs have different labelings that result in the same magic num- ber. The graph in Figure 4 has magic number k = 13. We can see that the labels on each of the graphs are arranged differently. For example, on the graph on the left of the figure, 4 and 6 are integers labeled on the edges of the graph and 2 and 8 are integers labeled on the vertices of the graph. However, on the graph on the right, 4 and 6 are integers labeled on the vertices, and 2 and 8 are integers labeled on the edges. Nevertheless, on both graphs in Figure 4, the labels of each of the vertices and their incident edges sum to 13.

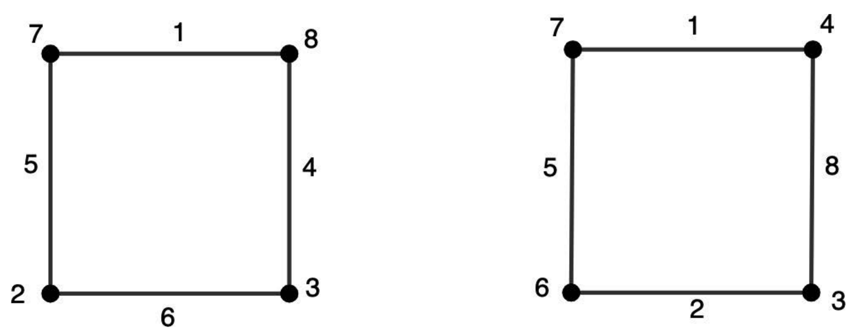

Figure 4: Two Different Vertex- Magic Total Labelings of the Same Magic Number 13

3.2 Important Results

In order to find a magic labeling of a connected graph, we need to know whatinteger to try as a magic number. There is no way that we can try every possible integer as a magic number of each graph. So, we can eliminate somemagic number possibilities by finding the range of magic numbers for a givengraph. For example, a magic graph cannot have a magic number that is less than $v + e$. Suppose we have a magic number less than $v + e$. If we add the labels of a vertex with its incident edges where one of these labels is $v + e$, the sum of the labels will exceed $v + e$. In this instance, we would no longer have a VMTL, and so a vertex-magic graph cannot have a magic number that is less than $v + e$. This gives a **lower bound of the magic number**. On the contrary, a magic graph cannot have a magic number that is greater than the sum of the set of all possible labels. This is impossible because we would not be labeling the vertices on the opposite sides of all of the edges ofthe graph. We would be ignoring the fact that each edge must be incident to two vertices. This gives an **upper bound of the magic number**.

Assume some graph G has a VMTL. Define $\mathbf{V_{sum}}$ to be the sum of all vertex labels and $\mathbf{E_{sum}}$ the sum of all edge labels of any graph G. The follow-ing lemma gives us a useful formula for the magic number of vertex-magic graph.

Lemma 1. *([2, 5]) If G is a vertex-magic graph with v vertices and e edges, then*

$$\frac{(v+e)(v+e+1)}{2v} + \frac{E_{sum}}{v} = k$$

Proof. Let u_i be a vertex of G and $k(u_i)$ be the sum of the labels of u_i andall of its incident edges. We want to find the sum of $k(u_i)$ for all i. To dothis, we need to add the edge and vertex labels at each vertex of the graph. Now, we know that each edge is incident to two vertices on any graph G. Hence, as we add the labels of each vertex and its incident edges, we will count each edge twice. So, when we sum each vertex with its incident edges we get $V_{sum} + E_{sum} + E_{sum}$. On the other hand, since we know that G is a vertex-magic graph, we have that $k(u_i)$ is k. So, we multiply k by the numberof vertices, v. We obtain vk, the sum of all edge labels. Thus, we have shown

$$V_{sum} + E_{sum} + E_{sum} = vk.$$

Now, we know that we label the vertices and edges of the graph by $1, 2, 3, ..., (v+e)$ by definition. Therefore, we have that the sum of all of the labels of the graph are equal to

$$\begin{aligned} V_{sum} + E_{sum} &= 1 + 2 + 3 + ... + (v+e) \\ &= \sum_{i=1}^{v+e} i \\ &= \frac{(v+e)(v+e+1)}{2}. \end{aligned}$$

Thus, we can substitute this into $V_{sum} + E_{sum} + E_{sum} = vk$ to give

$$\frac{(v+e)(v+e+1)}{2} + E_{sum} = vk.$$

Finally we have

$$\frac{(v+e)(v+e+1)}{2v} + \frac{E_{sum}}{v} = k$$

as we divide by *v*

Solving the formula from Lemma 1 for E_{sum} gives

$$E_{sum} = vk - \frac{(v+e+1)(v+e)}{2}.$$

It is shown to be useful throughout the next few sections, as we use the for- mula to deduce which labels can be placed on the edges of a graph.

It is shown to be useful throughout the next few sections, as we use the for- mula to deduce which labels can be placed on the edges of a graph.

The conclusion of Lemma 1 above can help us find better upper and lower bounds for the magic number of a graph. The next theorem gives us upperand lower bounds for possible magic numbers of graphs.

Theorem 1. *([2, 5]) Let G be a graph with v vertices and e edges. If G is a vertex-magic graph, then the magic number, k, is bounded such that*

$$\frac{e(e+1)+(v+e+1)(v+e)}{2v} \leq k \leq e + \frac{e(e+1)+(v+e+1)(v+e)}{2v}.$$

The following proof is from [2].

Proof. From Lemma 1, we have that

$$E_{sum} = vk - \frac{(v+e+1)(v+e)}{2}.$$

The minimum E_{sum} occurs when the labels 1 through e are assigned to theedges of the graph. So we have that,

$$\sum_{i=1}^{e} i = \frac{e(e+1)}{2} \leq E_{sum}.$$

The maximum E_{sum} occurs when the labels $v+1$ through $v+e$ are assignedto the edges of the graph. Thus,

$$\sum_{i=1}^{e} v+i = \sum_{i=1}^{e} v + \sum_{i=1}^{e} i = ve + \frac{e(e+1)}{2} \geq E_{sum}.$$

It follows that,

$$\frac{e(e+1)}{2} \leq E_{sum} \leq ve + \frac{e(e+1)}{2}.$$

Then we substitute in for E_{sum} which gives

$$\frac{e(e+1)}{2} \leq vk - \frac{(v+e+1)(v+e)}{2} \leq ve + \frac{e(e+1)}{2}.$$

Then we should perform the appropriate operations to isolate k between theinequality to obtain its bounds. Hence,

$$\frac{e(e+1)+(v+e+1)(v+e)}{2v} \leq k \leq e + \frac{e(e+1)+(v+e+1)(v+e)}{2v}.$$

This inequality is important because it shows that a vertex-magic graph with v vertices and e edges has a bounded magic number, k. In Figure 3, the graph with $v = 3$ vertices and $e = 3$ edges has the magic number bounds

$$\frac{3(4)+(7)(6)}{6} \leq k \leq 3 + \frac{3(4)+(7)(6)}{6}$$
$$\frac{54}{6} \leq k \leq \frac{72}{6}$$
$$9 \leq k \leq 12.$$

Each magic number possibility in the bounds turns out to be a magic numberof this graph. As you can see in Figure 3, there is a VMTL for each of the magic numbers.

The aim of this project is to explore vertex-magic total labelings on dif- ferent graph classes. We will discuss path graphs and crown graphs and showsome of their labelings. We arrive at a result for a specialized vertex-magic total labeling of crown graphs.

4 Labeling Vertex-Magic Graphs

To construct vertex-magic total labelings on different graph classes, we go through a series of steps. Generally, we find the graph's upper and lower bounds for a magic number, k, and calculate our E_{sum} for a chosen k. Wecan then form partitions of E_{sum} depending on how many edges the graph has. We then work to find a VMTL with the possible edge labels. We focus the labeling methods of this paper primarily on path graphs and crown graphs, more specifically.

4.1 Path Graphs

In this section, we will discuss path graphs, their properties, and some detailsthat allow us to find magic labelings for them.

Definition 6. *The **path graph**, P_n, is defined as a graph with n verticesthat are listed in an order such that consecutive vertices are adjacent.*

A path graph P_n has n vertices and n - 1 edges. The figure below is anexample of a labeled vertex-magic path graph with order 7 and size 6.

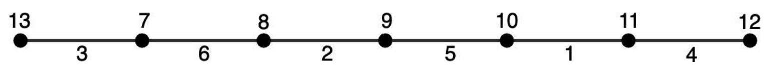

Figure 5: Vertex-Magic Total Labeling of P_7 with Magic Number 16

We will find a vertex-magic labeling for P_5. We have that $v = 5$ and $e = 4$.

Using Theorem 1, we have

$$11 \leq k \leq 15.$$

So the magic number bounds for P_5 are $11 \leq k \leq 15$. . We should choose a magic number to focus on for a magic labeling. Let k be one of the magic numbers in our bounds. We will choose $k = 11$ in this example. The formula for E_{sum} following Lemma 1 gives

$$E_{sum} = 10.$$

This implies that we need combinations of integers that sum to 10 for the edge labels of the graph. Thus, the only possible edge labels are the smallest labels, $\{1, 2, 3, 4\}$. We obtain the vertex-magic labeling shown in Figure 6 below.

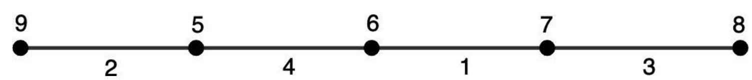

Figure 6: Vertex-Magic Total Labeling of P_5 with Magic Number 11

Notice the connection between the labelings for P_5 and P_7. The vertices havea distinct pattern which help us find the labelings of odd path graphs.

Theorem 2. *Consider P_n where n is odd. There exists a VMTL with vertexl abels assigned by*

$$v_i = \begin{cases} i + (n-2), & i \geq 2 \\ 2n-1, & i = 1 \end{cases}$$

and edge labels assigned by

$$e_i = \begin{cases} \frac{n}{2} - \frac{i}{2}, & i \text{ is odd} \\ n - \frac{i}{2}, & i \text{ is even.} \end{cases}$$

Proof. Let P_n be a path graph with n defined as an odd integer. Now, consider the four cases for labeling the vertices of a path graph using the definitions given for v_i and e_i. When $i = 1$, the sum of the label of the vertex and its incident edges is

$$(2n-1) + (\frac{n}{2} - \frac{1}{2}) = \frac{5n}{2} - \frac{3}{2}.$$

When i is even, the sum is

$$(i + (n-2)) + (\frac{n}{2} - \frac{i-1}{2}) + (n - \frac{i}{2}) = \frac{5n}{2} - \frac{3}{2}.$$

Then when i is odd and $i \neq 1, i \neq n,$ the sum is

$$(i + n - 2) + (n - \frac{i-1}{2}) + (\frac{n}{2} - \frac{i}{2}) = \frac{5n}{2} - \frac{3}{2}.$$

Finally, when $i = n,$ the sum is

$$(i + (n-2)) + (n - \frac{i-1}{2}) = \frac{5n}{2} - \frac{3}{2}.$$

Recall that the sum of each vertex and its incident edges must be equal to have a vertex-magic labeling, by definition. Thus, since all of the possible cases of vertex sums arrive at the same result, we have shown that the two functions that label the vertices and edges of odd path graphs give a vertex- magic labeling.

There is more to be said about the vertex-magic labeling shown in Figure

6. By definition, this labeling is a VMTL, but there is something even more special about it. If you look to the edges of the path, you see that the smallest labels are used as edge labels. This introduces a special type of VMTL for P_5 in Figure 6, namely, an **E-super vertex-magic total labeling**.

Definition 7. *An* ***E-super vertex-magic total labeling*** *is a VMTL where the smallest possible labels,* $1, 2, \ldots, e$, *are used as labels for the graph's edges.*

Unlike odd path graphs, it is impossible to find an E-super VMTL whena path is even. This is because the sum of the smallest possible labels we assign to the edges of an even path will never equate to a possible E_{sum}. Ithas been shown that the path graph P_n has a VMTL when $n > 2$ [4], andP_n has an E-super VMTL if and only if n is odd and $n \geq 3$ [7].

4.2 Crown Graphs

Here, we will discuss a new graph class. Then we will find a vertex-magic labeling for the class.

Definition 8. *A* ***crown graph****, denoted* H_n, *is a graph with* $2n$ *vertices, where the vertices are partitioned into two sets of size n,* $\{u_1, u_2, \ldots, u_n\}$ *and*

$\{v_1, v_2, \ldots, v_n\}$. *The vertices* u_i *and* v_j *are adjacent when* $i \neq j$.

To determine the size of H_n, we notice that the graph has n - 1 edgesthat are incident to each of its vertices. When we multiply n - 1 by thetotal number of vertices, $2n$, we obtain the number of edges incident to all vertices, $2n(n\ 1)$. Then we must divide by 2 because the edges of the graph are counted twice. This gives us

$$n(n-1),$$

the count of all of the edges, or the *size* of some crown graph H_n. Figure 7is an example of a crown graph H_5.

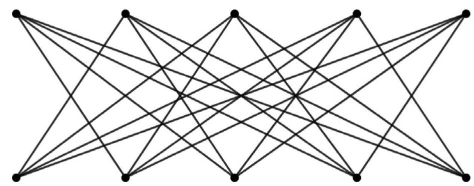

Figure 7: Crown Graph H_5

We will find a vertex-magic labeling for crown graph H_3. We have that $v = 6$

and $e = 6$. Using Theorem 1, we have

$16.5 \leq k \leq 22.5.$

Since k must be an integer, we round up to the next integer on the lower bound and down on the upper bound. Thus, the graph's magic number bounds are

$$17 \leq k \leq 22.$$

Let $k = 17$. The E_{sum} formula following Lemma 1 gives

$$E_{sum} = 24.$$

We consider combinations of edge labels that sum to 24. We should choose 6 edge labels and 6 vertex labels, and their positions on the graph so thatthe graph has a VMTL. We use labels 1, 2, ..., $(v + e)$ for any VMTL. To determine the possible edge labels, we can create partitions of the E_{sum} with 6 numbers each, for the edge labels. We can start by picking the small-est possible labels on the edges, integers $\{1, 2, ..., 6\}$. We find that we cannot use the smallest labels because these edge labels do not sum to 24. In- stead, we can use one of the following three possible sets of integers as edge labels: $\{1, 2, 3, 5, 6, 7\}$, $\{1, 2, 3, 4, 5, 9\}$, or $\{1, 2, 3, 4, 6, 8\}$. If we use the set $\{1, 2, 3, 5, 6, 7\}$, we obtain the labeling in Figure 8 below. This is the next smallest possible E_{sum} of the crown graph H_3.

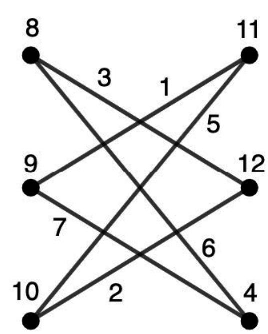

Figure 8: Vertex-Magic Labeling of H_3 with Magic Number 17

Now we will look at another crown graph labeling. In Figure 9, we have a VMTL of H_4, which we see that we can label to be an E-super VMTL. There are many other E-super VMTLs of H_4 that we will see in support of Conjecture 1 towards the end of the paper. In the next section, we will givea result about E-super VMTLs for crown graphs.

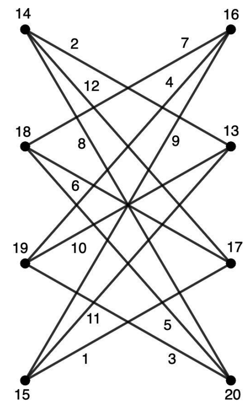

Figure 9: E-Super VMTL of H_4 with Magic Number 36

4.2.1 Crown Graph Results

As seen above in Figure 9, we have that it is possible to generate E-super VMTLs for even crown graphs H_n with $n \geq 4$, where a vertex-magic labelingexists. E-super VMTLs do not exist for any crown graph H_n with odd n.

Theorem 3. *For the crown graph H_n, $n \geq 4$, if there exists an E-super vertex-magic total labeling, then n is even.*

Proof. For any crown graph H_n, we have that $v = 2n$ and $e = n(n-1)$.

Assume there exists a VMTL for H_n. By Lemma 1 we have

$$k = \frac{(2n + n^2 - n)(2n + n^2 - n + 1)}{4n} + \frac{E_{sum}}{2n}.$$

By definition of E-super VMTL,

$$E_{sum} = \sum_{i=1}^{e} i = \frac{(n^2 - n)(n^2 - n + 1)}{2}.$$

So it follows by substitution that

$$k = \frac{(n+1)(n^2+n+1)}{4} + \frac{(n-1)(n^2-n+1)}{4} = \frac{n^3}{2} + n.$$

The magic number, *k,* must be an integer. When *n* is even, *k* is an integer.

We have evidence that there are many VMTLs of H_4. A few of theselabelings are shown in Figure 10. We expect that this will happen for othereven crown graphs as well.

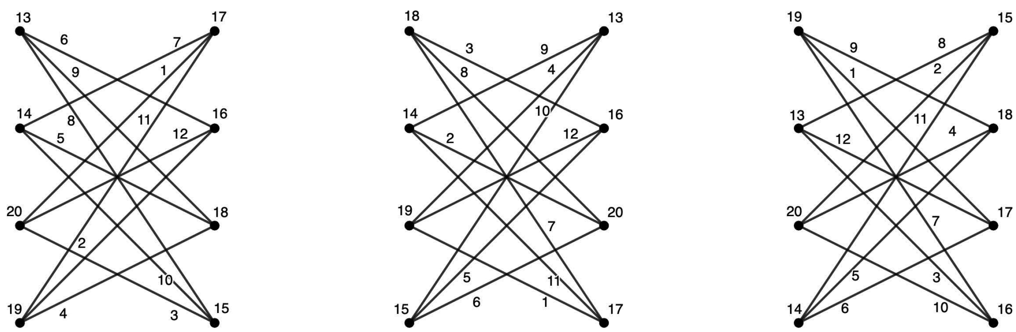

Figure 10: Three additional E-Super VMTL of H_4 with Magic Number 36

Conjecture 1. *If $n \geq 4$ is even, then there exists an E-super vertex-magictotal labeling of crown graph H_n.*

We have shown in Theorem 3 that for any even $n \geq 4$, it is possible for there to exist an E-super VMTL of crown graph H_n, but one will never exist for any odd *n*. This theorem does not guarantee that an E-super VMTL exists, but we conjecture that every even crown graph H_n with $n \geq 4$ will have a VMTL, based on computational evidence.

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About the Author

Karissa Massud is a graduating senior double majoring in Mathematics and Secondary Education and double minoring in Statistics and Spanish. Her research project was completed in the summer of 2021, which in turn supported the completion of her thesis in the fall, all under the mentorship of **Dr. Shannon Lockard** (Mathematics). Karissa's research was made possible with funding provided by the Adrian Tinsley Program (ATP) grant for undergraduate research. Karissa will present this work at the 2022 Mathematical Association of America's (MAA) MathFest and the 2022 National Conference on Undergraduate Research (NCUR). She plans to pursue a Ph.D. in Pure Mathematics and begin teaching after graduation.