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Estimating the Statistics of Operational Loss Through the Analyzation of a Time Series

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ESTIMATING THE STATISTICS OF OPERATIONAL LOSS THROUGH THE ANALYZATION OF A TIME SERIES

A Dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at Virginia Commonwealth University.

by

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Dedication

I dedicate this work to my family and friends. Particularly to my daughter who has always looked to me for guidance and reassurance of her own ability to achieve her dreams, which has driven me to push harder and be a beacon of light for her. I also dedicate this dissertation to those who have believed in me, when at times I didn't. Thank you for being my cheerleader throughout this process. Your belief in me has never wavered. It would be unbecoming of me not to mention my fraternal brothers who have challenged me, shared ideas and motivated me to complete this doctoral program. I will always appreciate all they have done. My colleagues and friends from Philadelphia, PA, Hopewell Virginia, Virginia Army National Guard, Virginia State University, Virginia Commonwealth University, and those whose paths I've crossed from other walks of life, your support has not been taken for granted.

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Abstract

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In the world of finance, appropriately understanding risk is key to success or failure because it is a fundamental driver for institutional behavior. Here we focus on risk as it relates to the operations of financial institutions, namely operational risk. Quantifying operational risk begins with data in the form of a time series of realized losses, which can occur for a number of reasons, can vary over different time intervals, and can pose a challenge that is exacerbated by having to account for both frequency and severity of losses. We introduce a stochastic point process model for the frequency distribution that has two important parameters (average frequency and time scale). The advantages of this model are that the parameters, which we systematically vary to demonstrate accuracy, can be fit with sufficient data but are also intuitive enough to rely on expert judgement when data is insufficient. Furthermore, we address how to estimate the risk of losses on an arbitrary time scale for a specific frequency model where mathematical techniques can be feasibly applied to analytically calculate the mean, variance, and co-variances that are accurate compared to more time-consuming Monte Carlo simulations. Additionally, the auto- and cross-correlation functions become mathematically tractable, enabling analytic calculations of cumulative loss statistics over larger time horizons that would otherwise be intractable due to temporal correlations of losses for long time windows. Finally, we demonstrate the strengths and shortcomings of our new approach by using combined data from a consortium of institutions, comparing this data to our model and correlation calculations, and showing that different time horizons can lead to a large range of loss statistics that can significantly affect calculations of capital requirements.

CHAPTER 1

INTRODUCTION

1.1 Background Information

Risk for some is just a board game, but in the world of finance, it is the key to either success or failure, as risk is the fundamental element that affects financial behavior. Traditionally, operational risk has been neglected compared to credit and market risk; it is often thought of as "other" despite how it has severely harmed institutions when not properly accounted for [1, 2]. For example, in the last few decades operational losses have been extraordinary, as we will discuss in more detail later. With operational risk we often encounter data that is represented by time series of losses which are used in models to calculate statistics that can then be utilized to make predictions, mitigate, and understand the dynamics of operational losses. The Basel II Committee has suggested several approaches to calculate the regulatory capital for the bank's exposure to operational risk. The approaches are given in a 'continuum' of increasing sophistication and risk sensitivity: (1) Basic Indicator Approach, (2) Standardized Approach and (3) Advanced Measurement Approaches (AMA) [1, 2]. Among the 3 approaches, the Advanced Measurement Approaches (AMA) is known to be the most risk sensitive compared to the other 2 approaches (the Basic Indicator Approach and the Standardized Approach) that are often used for smaller institutions [1, 2]. Here we will only focus on the AMA because it is used for the larger banks that tend to suffer larger operational losses, and requires the most sophisticated mathematical and statistical methods

1.1.1 Defining Risk

While it is imperative to calculate risk as accurately as possible, it is important to understand that there is not a unique or uniform definition of risk [3]. However, we will define risk as it relates to the operations of financial institutions, namely operational risk. Whenever the probability of loss exists, it is referred to as risk. There are numerous actions and activities that can fall under this umbrella definition of risk. Therefore, it is important that we identify how we will refer to risk here. Generally speaking, there are three types of risk that concern financial institutions: business risk, non-business risk and financial risk. Financial Risk is one of the major concerns of every business across fields and geographies [4]. Like the overarching term, “Risk”, as we have defined it here, financial risk is also broken down into categories, as depicted in Figure 1. Our focus will be on operational risk.

Fig. 1.: Topology of financial risks in banks [2].

1.1.2 Operational Risk

Operational risk is defined as the risk of loss resulting from inadequate or failed internal processes, people or systems, or from external events [1]. Loss can occur for a number of reasons, providing a number of sources for operational risk. The major sources of operational risk include operational process reliability, IT security, outsourcing of operations, dependence on key suppliers, implementation of strategic change, integration of acquisitions, fraud, error, customer service quality, regulatory compliance, recruitment, training and retention of staff, and social and environmental impacts [1]. Operational risk is a key risk category that can severely harm a business if not properly accounted for. It is intrinsic to financial institutions and poses a significant threat to their financial solvency [5].

1.1.3 Impact of Operational Risk

Over the last 20 years, risks that financial institutions are being forced to face have increased in complexity and present greater challenges that are not related to market or credit risk. These risks have resulted in bankruptcies, mergers, and stock price declines [2]. There have been more than 100 operational losses each exceeding $100 million, and a number of losses exceeding $1 billion, that have impacted financial firms since the end of the 1980s [6]. For example, the nation's five largest mortgage servicers, Bank of America Corporation, JPMorgan Chase & Co., Wells Fargo & Company, Citigroup Inc., and Ally Financial Inc. collectively agreed to a $25 billion settlement with U.S. Federal government to address past improper mortgage loan servicing and foreclosure fraud [5]. UBS lost $2.3 billion due to unauthorized trades, and Royal Bank of Scotland Group PLC (RBS) lost $275 million in a settlement to resolve allegations of misleading investors in mortgage-backed securities [5]. Furthermore, in the most recent Dodd-Frank Act Stress Test (a routine test banks have to go through with federal regulators to determine solvency), the severely adverse scenario projected operational risk losses for the thirty-five participating Bank Holding Companies of $135 billion, or 23 percent of the $578 billion in aggregate losses projected for these firms over the nine quarters ending in March of 2020 [5].

As mentioned previously, the Advanced Measurement Approach (AMA) for calculating how much capital an institution holds is a government regulatory requirement for larger banks [5]. Figure 2 shows the percentage of capital dedication to operational risk of major U.S. Banks. Also, in Figure 3, data from [7] outlines leading operational losses in 2017. The amount of loss was significant and impactful, particularly within the U.S. This type of data further exemplifies operational risk as an important source of risk for banks. Therefore, understanding the key factors of operational risk as well as quantifying their impacts is an important task in risk measurement and management [5].

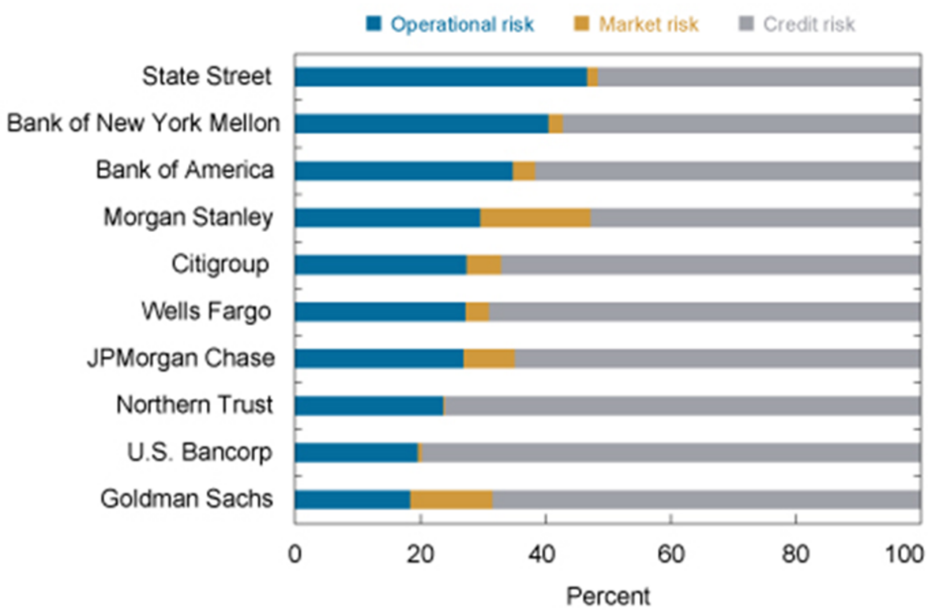

Fig. 2.: Regulatory Capital ratios for the “Advanced Measurement Approach” Banks.

	Event	Loss amount	Business line	Event type	Country
1	Former BNDES president under investigation in relation to improper transactions totalling 8.1 billion real	$2.52bn	Commercial banking	Internal theft and fraud	Brazil
2	Shoko Chukin ordered to repay Japanese government following discovery it granted Y265 billion of fraudulent loans	$2.39bn	Commercial banking	Internal theft and fraud	Japan
3	SEC charges Woodbridge Group of Companies with running $1.22 billion Ponzi scheme	$1.22bn	Asset management	Internal theft and fraud	US
4	Societe Generale agrees to pay €963 million to settle dispute with Libyan Investment Authority	$1.18bn	Trading and sales	Improper business or market Practices	Libya
5	Thema International Fund and its affiliates pay $1.06 billion to Madoff trustee	$1.06bn	Asset management	Improper business or market Practices	US
6	Former executives of Catalunya Caixa accused of causing €720 million hole through irregular transactions	$880m	Commercial banking	Internal theft and fraud	Spain
7	Eight Indian banks defrauded of 49.3 billion rupees by Kingfisher Airlines founder Vijay Mallya	$770m	Commercial banking	External theft and fraud	India
8	Western Union pays $586 million for AML and wire fraud failures	$586m	Clearing	Improper business or market Practices	US
9	Deutsche Bank reaches €450 million settlement over Kaupthing CLNs	$550m	Trading and Sales	Improper business or market Practices	Iceland
10	Agricultural Bank of China defrauded of 3.2 billion yuan by employees of billionaire Guo Wengui	$497m	Commercial Banking	External theft and fraud	China

Fig. 3.: Top loss events of 2017, Industry wide [7].

1.2 Analyzing Operational Risk with Time Series, Frequency and Loss Distributions

Embrechts and Hofert [8] informs us that according to the 2008 LDCE, the vast majority of banks model frequency and severity distributions separately. For modeling the frequency distribution, the Poisson (93%) and the Negative Binomial (19%) are usually used. For modeling the severity distribution, 31% of the banks use a single distributional model (with log-normal (33%) and Weibull (17%) the most common choices), 29% use two distributions glued together (for the body, empirical (26%) and log-normal (19%); for the tail, generalized Pareto (31%) and log-normal (14%)), and 19% use two separate distributions for high-frequency/low-severity and low-frequency/highseverity losses (2011). However, here we recognize operational losses modeled as realizations of a continuous stochastic process from combining a counting process (frequency) and a continuous severity distribution [3]. The

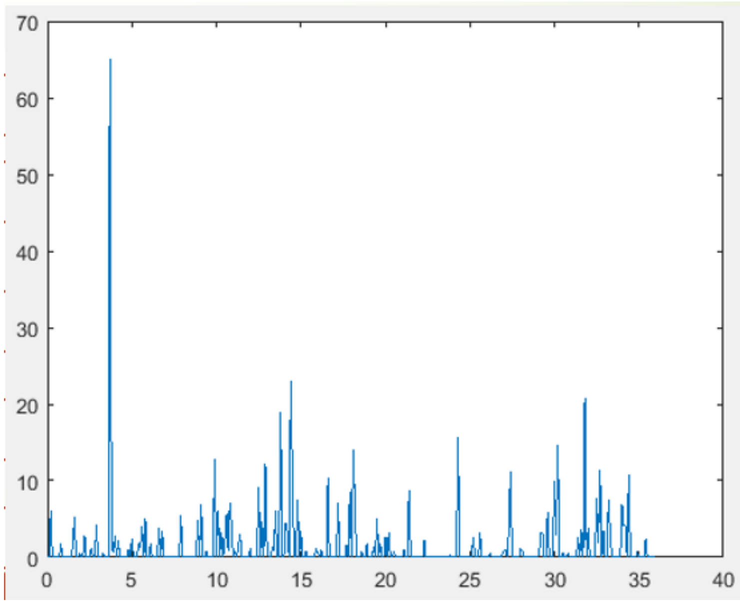

Fig. 4.: Example of Time Series with time represented on the x-axis and the magnitude of loss distribution on the y-axis.

broad loss categories discussed previously naturally have different time scales for frequency of events as well as different (severity) magnitudes. A sequence of numerical data points in successive order is defined as a time series. An examiner of data chooses a convenient time window to analyze statistically. This information can then be used to find associated patterns to predict future activity, to model how risk changes over time, to make comparisons to shifts in variables within the same time window, or to identify seasonality, peaks and troughs. In operational risk one examines the magnitude of losses as well as the frequency of occurrence of loss. Loss events are chaotic in nature with events occurring at irregular instances of time, which makes it imperative to examine frequency distributions to understand the underlying loss arrival process [2]. The most common distributions used when analyzing frequency include binomial, geometric, Poisson, and negative binomial distributions, with the latter two being more common than the others. The magnitude of losses are depicted by severity distributions. Chernobai [2] states that due to the specific nature of operational loss data, the distributions that most often apply to modeling loss are right skewed or heavy-tailed. These distributions include the gamma, lognormal, Weibull, Pareto and the Burr. The severity distribution selected is determined by the type of loss being examined, but heavy tailed distributions, such as the GPD or Weibull commonly describe operational loss magnitudes [2].

For the severity distribution we consider several distribution functions $f_S(x)$:

$$f_S(x) = \frac{1}{\Gamma(\alpha)\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}\exp\Big(-x/\beta\Big),\ \text{for } x>0; \alpha,\beta>0,\ \text{Gamma} \tag{1.1}$$

$$f_S(x) = \frac{1}{x\sqrt{2\pi}\sigma}\exp\Big(-\frac{(\log(x)-\mu)^2}{2\sigma^2}\Big),\ \text{for } x>0; \mu\in\mathbb{R},\sigma>0,\ \text{Lognormal} \tag{1.2}$$

$$f_S(x) = \frac{1}{\sigma}\Big(1+k\frac{x}{\sigma}\Big)^{-1-1/k},\ \text{for } x>0,\ \text{GPD} \tag{1.3}$$

$$f_S(x) = \frac{b}{a}\left(\frac{x}{a}\right)^{b-1}\exp\Big(-\left(\frac{x}{a}\right)^b\Big),\ \text{for } x>0; b>0; a>0,\ \text{Weibull} \tag{1.4}$$

$$f_S(x) = \frac{\frac{kc}{\alpha}\left(\frac{x}{\alpha}\right)^{c-1}}{\Big(1+\left(\frac{x}{\alpha}\right)^c\Big)^{k+1}},\ \text{for } x>0; \alpha>0; c>0; k>0,\ \text{Burr} \tag{1.5}$$

The statistics for these common parametric PDFs are displayed in Table 1.

Table 1.: Severity distribution family and statistics.

Here $\Gamma(x) = \int_0^\infty z^{x-1}e^{-z}\,dz$ and $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$. To insure the mean and variances are finite, in **GPD** we have $0 \le k < \frac{1}{2}$, in **Burr** we have $k > \frac{2}{c}$.

Distribution	**Mean** $\mathbb{E}[S] = \mu_S$	**Variance** $\mathbb{E}[S^2] - \mathbb{E}[S]^2 = \sigma_S^2$	**Allowable Parameters**
Gamma	$\alpha\beta$	$\alpha\beta^2$	$\alpha, \beta > 0$
Lognormal	$\exp(\mu + \sigma^2/2)$	$\Big(e^{\sigma^2} - 1\Big)e^{2\mu+\sigma^2}$	$\sigma > 0$
GPD	$\frac{\sigma}{(1-k)}$	$\frac{\sigma^2}{(1-k)^2(1-2k)}$	$\sigma > 0$ and $0 \le k < \frac{1}{2}$
Weibull	$a\Gamma\Big(1+\frac{1}{b}\Big)$	$a^2\Gamma\left(1+\frac{2}{b}\right) - a^2\Big(\Gamma\left(1+\frac{1}{b}\right)\Big)^2$	$a, b > 0$
Burr	$k\alpha B\left(k-\frac{1}{c}, 1+\frac{1}{c}\right)$	$k\alpha^2 B\left(k-\frac{2}{c}, 1+\frac{2}{c}\right) - (k\alpha)^2 B\left(k-\frac{1}{c}, 1+\frac{1}{c}\right)^2$	$\alpha, c, k > 0$ and $k > \frac{2}{c}$

1.2.1 Issues in Analyzing Operational Risk

To control operational risk it is necessary to understand both advantages as well as limits of a mathematical prediction [3]. Due to the fact that losses can occur with different time-scales, ranging from a daily, monthly, or yearly basis, one must consider frequency distribution models carefully, as well as the manner in which data is collected. For instance, one must determine if time will be based on the date of occurrence of a risk loss event, the date of discovery, or the date of invoice of loss, as each can result in different outputs.[8] states that Operational Risk losses arising from legal events are often not identified until months after the date of their occurrence. They further note that, the question arises about the date that a bank should assign such losses within its internal loss database (e.g., date of occurrence, date of discovery, date of accounting, date of agreement or settlement). Choices of this date are not specified by Basel II. However, this can have a significant impact on the assessment of a bank's OR profile [8]. For our purposes, we will only address how the frequency distribution is modeled. Another consideration is how a single event contributes to other types of loss, which then implies dependency. Not only does the dependency create an issue in calculating predictions and statistics, but the ability to even determine such dependency in reporting data is a major focus for modelers. In order to instigate the selection of proper measures for operational risk loss mitigation, financial institutions must be able to consider multiple factors and assumptions, but still maintain confidence in their predictions of loss. Developments in information systems and computing technology have facilitated improvements in the measurement and management of operational risk. However, the same technological advancements that are reshaping today's business environment are also likely to expose banks to even greater operational risks in the future. Thus, the desire to develop these methods stem from the inability in current common practices to assess operational risk losses in certain categories and time scales, and as mentioned by [3], it is therefore imperative that banks and supervisors continue to strengthen their approach to managing and mitigating operational risk.

There are risk categories where the data is scarce (e.g., Employee Practices & Workplace Safety, Technology and Infrastructure Failure, see Fig. 3) and commonly used methods to fit a loss distribution model to data would suffer from insufficient sample size. Similarly, estimating the correlation between loss categories with scarce data is challenging because the status quo is to fit a heavy-tailed copula (e.g., t-copula) to the data [9]. This challenge is further exacerbated by the common practice of fitting the copula to the raw data that has more samples rather than on cumulative data summed over the desired time window, i.e., yearly capital is a common desired output [2] but correlations might be calculated with monthly/daily data that has more samples.

1.3 Plans and Assumptions

Here we introduce a stochastic point process model for the frequency distribution that has two main parameters: average frequency and a time scale parameter. The advantages of this model are as follows: i) the parameters could be fit with sufficient data but are also intuitive enough to rely on expert judgement when data is insufficient, ii) when coupled with an independent severity distribution model, the auto- and cross-correlation functions are mathematically tractable, so we can analytically calculate the cumulative loss statistics over varying time windows using tools from signal processing [10] and computational neuroscience [11, 12, 13, 14, 15, 16]. Finally, we will demonstrate the strengths and shortcomings of our new approach by using freely available loss data from several operational risk categories. Our loss distribution model and correlation calculations will be compared to common industry practices [9], that in particular heavily rely on data for fitting a heavy-tailed copula.Using combined data from a consortium of institutions, we fit our model to loss statistics of an average institution, and show that different time windows of (cumulative) losses can lead to a large range of loss statistics, which in turn can significantly effect calculations of capital requirements.

In general, the objective is to estimate a cumulative loss distribution and to derive entities of interest from it [3], i.e., the 99.9 percentile of the cumulative loss distribution is a common measure of capital, or Value-at-Risk (VaR). Additionally, our proposed methods could potentially address the shortcomings we have previously highlighted in the AMA.

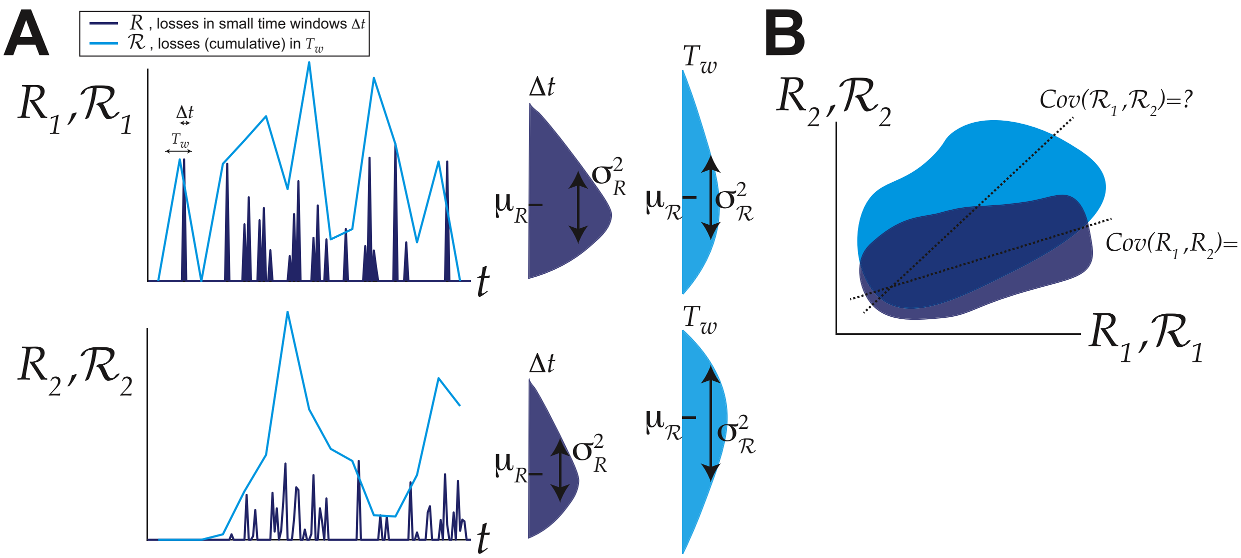

Fig. 5.: Varying operational risk loss statistics.

Figure 5 displays how operational risk loss statistics can vary depending on risk category and on time window operations. **A** gives examples of two different loss time series (top and bottom). For each loss category, two time series from the same loss data are shown: losses in small windows (Δt dark blue) which are denoted by $R_j(t)$ and cumulative losses in time window T_w denoted by $\mathcal{R}_j(t)$. Right panels show aggregate loss distributions averaged over time, marked with corresponding statistics of interest. **B** depicts the range of the joint distributions of (R_1, R_2) (dark blue) and $(\mathcal{R}_1, \mathcal{R}_2)$ (light blue). The covariance and correlation are important for calculating aggregate capital different risk categories, but can depend on time window of observations.

The overall loss distribution model is often based on assumptions about the frequency and severity of operational risk loss categories [1, 2]. To enable mathematical analysis, one of our key assumptions is to assume that the time series is stationary (i.e., the statistics do not vary over time) and that frequency is independent of severity. That is:

$$\mu(t_0) = \mu(t_1) = \mu_X \tag{1.6}$$

Another common assumption in mathematical calculations of time series is that the system is ergodic, so that the average over time is equal to the average over trials:

$$\mu_X = \frac{1}{T}\int_0^T x(t)\,dt,\;\; T = \textit{time interval} \tag{1.7}$$

CHAPTER 2

SINGLE LOSS TIME SERIES WITH HOMOGENEOUS POISSON PROCESS

We first consider a simple time series where the frequency distribution is given by a homogeneous Poisson Process with rate ν events/time, and independent severity distribution f_S. We present the calculation of second order statistics of the whole time series for a given loss category. For each risk type and a given period, operational losses are represented by a time series R_j, where j indexes a time interval of size Δt (assumed to be small enough so that at most one operational loss event can occur).

In a homogeneous Poisson process, the probability of a loss event in a time bin is $\nu_r \Delta t$ and each time bin is independent. To derive the first and second order statistics, we use the PDF of R_j:

$$f_R = I_{t_j} * S_j,$$

where

$$I_{t_j} = \begin{cases} 0, & \text{no loss event at } t_j \\ 1, & \text{have loss event at } t_j \end{cases} \tag{2.1}$$

We have the mean is:

$$\begin{aligned} \mathbb{E}[R] &= 0 * (1 - \nu_r \Delta t) + \int_0^T s(\nu_r \Delta t) f(s)\, ds && (2.2) \\ &= \nu_r \Delta t \int_0^T s f(s)\, ds = (\nu_r \Delta t) \mu_s && (2.3) \end{aligned}$$

where μ_S is the mean of the chosen severity distribution. The variance is:

$$\sigma_R^2 = \mathbb{E}[R^2] - \mathbb{E}[R]^2 = \mathbb{E}[R^2] - (\nu_r \Delta t \mu_s)^2$$

where

$$\mathbb{E}[R^2] = 0^2 * (1 - \nu_r \Delta t) + \int_0^T s^2 (\nu_r \Delta t) f(s)\, ds = (\nu_r \Delta t)\mu_{s^2}$$

therefore,

$$\sigma_R^2 = (\nu_r \Delta t)\mu_{s^2} - (\nu_r \Delta t \mu_s)^2 \tag{2.4}$$

2.1 Cumulative Loss Statistics in Arbitrary Time Windows

For the purposes of aggregating capital over different time horizons (i.e., yearly capital assessment is regulatory requirement [1, 2]), we consider cumulative losses over different time windows to understand how this practice of using different time windows might effect the statistics. Development of methods to capture cumulative losses in arbitrary time windows may also help institutions handle certain operational loss categories that occur infrequently and other categories where data are collected infrequently. Recall that R_j is the loss in small time bin Δt, the cumulative losses in a time window of length T_w that contains $n := T_w/\Delta t$ time bins is:

$$\mathcal{R}_l = \sum_{j=1}^{n} R_{j+(l-1)*n}$$

where the subscript l denotes the l^{th} window of length T_w. The mean is:

$$\mu_{\mathcal{R}} = \sum \mathbb{E}[R_j] = n\Delta t \Big(\nu_r \mu_S \Big) = T_w \nu_r \mu_S \tag{2.5}$$

With a homogeneous Poisson Process model, each of the R_j are independent, making the calculation of the variance tractable because the covariance terms are 0:

$$\sigma_{\mathcal{R}}^2 = \sum Var(R_j) + 2 \sum_{j<k} Cov(R_j, R_k) = n\Delta t \left[(\nu_r \Delta t)\mu_{s^2} - (\nu_r \Delta t \mu_s)^2 \right] \tag{2.6}$$

2.2 Relationship Between Autocovariance Function and Cumulative Loss Statistics

The autocovariance function is a common tool used to quantify the temporal dynamics of a time series; here we define it as:

$$A(t) = \mathbb{E}_\tau \left[R_{t+\tau} R_\tau \right] - \mathbb{E}_\tau \left[R_\tau \right]^2 \tag{2.7}$$

$A(t)$ specifies how correlated (or degree of co-variability) two points separated by time t are, on average. Note that $A(t=0) = \sigma_T^2$ is the (point-wise) variance in time. The autocovariance has a nice relationship with the cumulative loss in a window of length T_w whenever the time series satisfies stationarity [10]:

$$\sigma_{\mathcal{R}}^2 = \int_{-T_w}^{T_w} A(t) \Big(T_w - |t| \Big) \, dt \tag{2.8}$$

Indeed, we can use this equation to derive equation (2.6); note that the Autocovariance function of R_j with a Poisson Process frequency distribution and independent severity distribution is:

$$A(t) = \sigma_R^2 \delta(t) = \sigma_R^2 \frac{1}{\Delta t}. \tag{2.9}$$

Substituting this into equation (2.8), we get:

$$\sigma_{\mathcal{R}}^2 = \sigma_R^2 \Big(T_w - |t| \Big) \Big|_{t=0} = \sigma_R^2 T_w = \Big(\nu_r \mu_{s^2} - (\nu_r \mu_s)^2 \Delta t \Big) T_w,$$

noting that $n := T_w / \Delta t$ we get:

$$\sigma_{\mathcal{R}}^2 = \sigma_R^2 T_w = n \Delta t [(\nu \Delta t) \mu_{s^2} - (\nu \Delta t \mu_s)^2]$$

We discover that this is the same as equation (2.6).This nice property will become useful as a method to determine variance in more complex cases, such as when we consider dependent events.

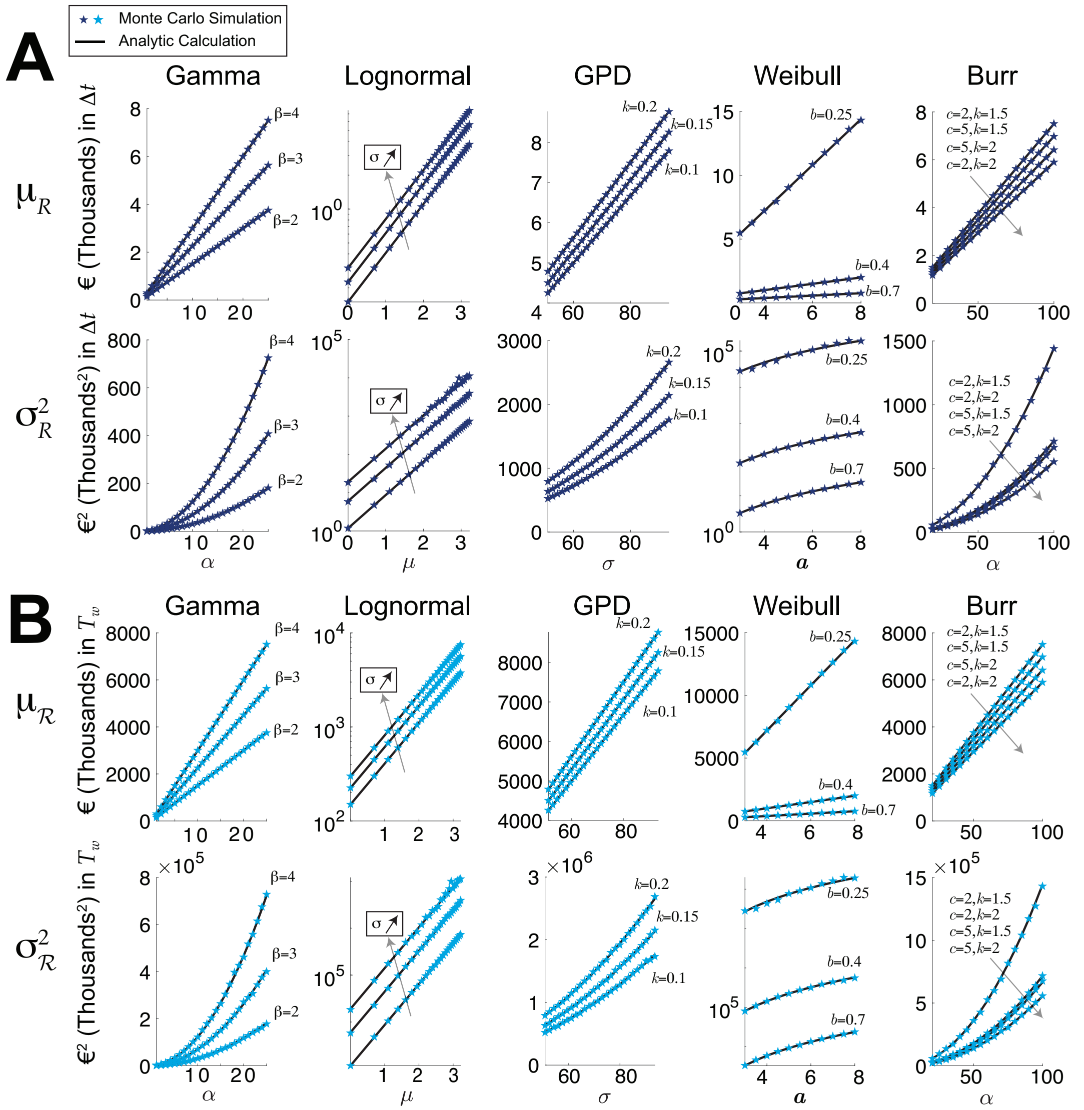

Fig. 6.: Validation of results for homogeneous Poisson Process frequency model.

2.3 Summary of Homogeneous Poisson Process

Results from Figure 6 demonstrates the accuracy of our analytic calculations for a wide variety of severity distribution parameters. The dark blue stars are the loss statistics in small time windows Δt calculated by Monte Carlo, while the analytic formulas are solid black curves. The light blue stars are the loss statistics in a larger time window ($T_w = 1$ year here), following the dark/light blue coloring convention in Fig 5. Overall we see that our formulas are very accurate. The analytic calculations for the mean and variance of aggregate loss distributions with frequency distribution as a homogeneous Poisson process with rate $\nu_r = 75$ (events/year) for commonly used severity distributions (see Table 1); see Table 7 to compare to industry-wide frequency averages. **A** shows the mean μ_R and variance σ_R^2 of aggregate loss distribution in small windows $\Delta t = 0.001$ in time units of years (≈ 0.365 day), with Monte Carlo simulations in stars and calculations with solid curves (Eq. (2.3) and (2.4)). For Lognormal, $\sigma = \sqrt{2\log(2)}, \sqrt{2\log(3)}, \sqrt{2\log(4)}$; ; vertical axis is log-scale for Lognormal, and Weibull variance. **B** Shows the mean $\mu_{\mathcal{R}}$ and variance $\sigma_{\mathcal{R}}^2$ of cumulative losses in large time windows $T_w = 1$ year, with Monte Carlo simulations in stars and calculations with solid curves (Eq. (2.5) and (2.6)).

The formulas we derived for the mean and variance of aggregate loss distributions, assuming a homogeneous Poisson process frequency distribution and independent severity distribution (Equation (2.3), (2.4), (2.5), (2.6)), are straightforward calculations but have scarcely been reported. This could be due to a number of factors, including: focus on tailend loss distribution, desire to keep methods proprietary, reliance on Monte Carlo simulations, lack of temporal dependence, etc. Nevertheless, our work shows analytic calculations are possible for this common frequency distribution model and very accurate.

CHAPTER 3

TIME SERIES WITH INHOMOGENEOUS POISSON PROCESS

Realistic models of loss frequencies must account for dependence between loss events; operations of institutions are not simply memory-less. Operational risk models, such as the loss distribution approach, frequently use past internal losses to forecast operational loss exposure. However, the ability of past losses to predict exposure, particularly tail exposure, has not been thoroughly examined in the literature [17]. [18] notes that while making advances in some areas, banks still rely on many highly subjective operational-risk detection tools, centered on self-assessment and control reviews. Such tools have been ineffective in detecting cyberrisk, fraud, aspects of conduct risk, and other critical operational-risk categories. Additionally, they miss low-frequency, high-severity events, such as misconduct among a small group of frontline employees (2020). Thus, it is imperative that we investigate a time series utilizing an inhomogeneous Poisson Process model where each of the R_j are dependent, and the probability of $R_t > 0$ depends on time. This makes the calculation of the statistics, the variance in particular, difficult to analyze. We revisit the definition of variance:

$$\sigma_{\mathcal{R}}^2 = \sum Var(R_j) + 2\sum_{j<k} Cov(R_j, R_k)$$

In this case the covariance terms will result in combinatorial blowup. That is, the covariance of cumulative losses in larger windows T_w are unwieldy because of the possible correlation of a large number, $\binom{n}{2}$, of different R_jR_k terms (here $n = T_w/(\Delta t)$). In addition, the non-Markovian nature of R_j and R_k pose calculation hardships. However, we are not left without a method to determine the variance, as we can use equation (2.8). This provides us with the unique situation because the autocovariance $A(t)$ is tractable. Therefore, with an inhomogeneous Poisson Process, the autocovariance not only serves as a statistical analysis tool, but also as a method to determine the variance in a time series. While the pdf of R will not change, that is: $R = S * \nu_t \Delta t$. What does change is our calculation of ν. We model ν depending on time by a stochastic differential equation.

$$\tau \nu_t' = -\nu_t + \tau a \sum_k \delta(t - t_k) \tag{3.1}$$

Here t_k is a random point in time given by a homogeneous point process with rate γ, a is the jump size of ν_t at times t_k, and τ is the time-scale that determines how fast ν_t decays to 0 in the absence of random jumps at t_k. This frequency distribution model has three parameters that essentially control two main factors: (γ, a) that control the magnitude of the frequency of events, and τ that is a measure the memory of a loss category. In contrast to other time series models, this stochastic differential equation formulation consists of parameters that are intuitive and easy to understand for decision makers: memory or time-scale and frequency. We let $D(t) = \sum_k \delta(t - t_k)$, $\mathbb{E}[D(t)] = \gamma$, and recognize that the only source of randomness on the rhs is D, to acquire the mean of ν_t:

$$\nu_t = a \int_0^\infty D(t') e^{\frac{-(t-t')}{\tau}} dt$$

$$\mathbb{E}\left[\nu_t\right] = a \int_0^\infty \mathbb{E}\left[D(t')\right] e^{\frac{-(t-t')}{\tau}} dt = a\gamma\tau$$

This yields

$$\mathbb{E}\left[R\right] = \mathbb{E}\left[S\right] * \mathbb{E}\left[\nu_t\right] = \mu_s(a\gamma\tau) \tag{3.2}$$

For the variance we will require the 2^{nd} moment of ν_t.

$$\nu_t^2 = a^2 \int_0^\infty \int_0^\infty D(t-u) D(t-v) e^{\frac{-u}{\tau}} e^{\frac{-v}{\tau}} dudv \tag{3.3}$$

Note that $\mathbb{E}\left[D(t-u)D(t-v)\right] = \gamma^2 + \gamma\delta(u-v)$.

$$\mathbb{E}\left[\nu_t^2\right] = a^2 \int_0^\infty \int_0^\infty \left[\gamma^2 + \gamma\delta(u-v)\right] e^{\frac{-u}{\tau}} e^{\frac{-v}{\tau}} dudv$$

$$= \gamma^2(a\tau)^2 + a^2\gamma \int_0^\infty e^{\frac{-2v}{\tau}} dv = \gamma^2(a\tau)^2 + \frac{a^2\gamma\tau}{2}$$

The variance of ν_t is:

$$Var(\nu_t) = \frac{a^2\gamma\tau}{2} \tag{3.4}$$

The autocovariance $\mathbb{E}[\nu_{t'}\nu_{t'+t}] - \mathbb{E}[\nu_t]^2$ is similarly calculated by replacing $D(t-u)$ with $D(t'+t-u)$ in equation (3.3):

$$A_{\nu_t}(t) = \frac{a^2\gamma\tau}{2} e^{-|t|/\tau} \tag{3.5}$$

Since the second moment of R is:

$$\mathbb{E}_\nu\Big[\mathbb{E}[R^2|\nu_t]\Big] = \mu_{S^2}\Delta t \mathbb{E}_{\nu_t}[\nu_t] = \mu_{S^2}(a\tau\gamma\Delta t),$$

the variance of R is:

$$\sigma_R^2 = \mu_{S^2}(a\tau\gamma\Delta t) - (\mu_S a\tau\gamma\Delta t)^2 \tag{3.6}$$

For cumulative losses in larger time windows, the statistics are calculated similarly, with:

$$\mu_{\mathcal{R}} = \sum \mathbb{E}[R_j] = n\Delta t\Big(a\gamma\tau\mu_S\Big) = T_w a\gamma\tau\mu_S \tag{3.7}$$

We have determined our mean for cumulative losses in large time windows of an inhomogeneous Poisson process, but the variance will require the use of the calculation of our auto covariance. We can then substitute into equation (2.8), and get the variance of cumulative losses in a time window T_w.

3.1 Autcovariance with Inhomogeneous Poisson Process

As mentioned above, the autocovariance (**ACF**) serves dual purposes in regards to investigating the statistics of the inhomogeneous Poisson process. We have utilized some of the properties of autocovariance to determine the variance of ν and R. More importantly however, the autocovariance will help us determine the dependency of our random variables at different time points. Simply put, it will give us an indication of whether or not one occurrence of loss is dependent upon a loss from a previous time step. To determine the autocovariance of ν_t we utilize $\mathbb{E}[\nu_{t'}\nu_{t'+t}] - \mathbb{E}[\nu_t]^2$ and use similar calculation but simply replacing $D(t-u)$ with $D(t'+t-u)$ in equation (3.3) to get:

$$A_{\nu_t}(t) = \frac{a^2\gamma\tau}{2}e^{-|t|/\tau} \tag{3.8}$$

When determining the ACF of R_t it is first important to note that the autocovariance contains the variance when $\tau = 0$. We argue that due to the temporal structure of R_t, which is solely based upon ν_t, the auto covariance of our time series is approximately:

$$A_R(t) = A_{\nu_t}(t) * var(R_t). \tag{3.9}$$

By substituting $t = 0$, we readily determine the ACF of R_t because it is just $A_{\nu_t}(t)$ scaled by a factor. Therefore,

$$A_R(t) = \sigma_R^2 e^{-|t|/\tau}. \tag{3.10}$$

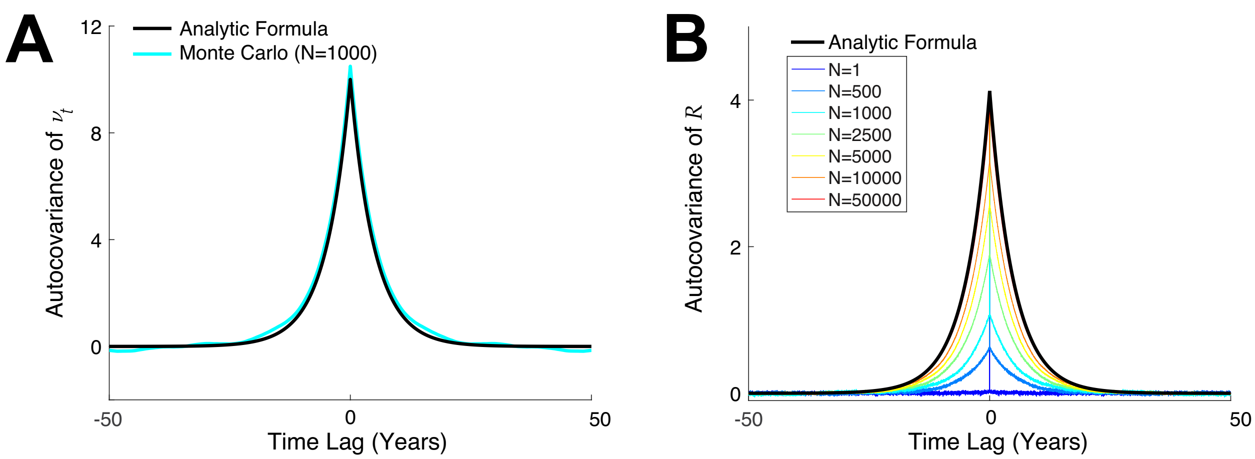

Fig. 7.: The effect of the number of realizations on $A_R(t)$.

From Figure 7, it initially appears as if our analytic calculation of $A_R(t)$ has large errors. However, when we increase the number of realizations we can see our analytical calculation begin to converge to the Monte Carlo simulation, which convinces us that our calculations are accurate.

Specifically, the number of realizations effects $A_R(t)$ significantly. Consider the inhomogeneous Poisson process model for the frequency distribution in (3.8) ($\nu_i = 4$ events/year, $a = 1$ event/year, $\tau = 5$ years), with a gamma distribution for the severity $S \sim$Gamma$(\alpha = 1.5, \beta = 2.5)$. **A** is a comparison of $A_{\nu_t}(t)$ calculated from Monte Carlo simulations (cyan) of Eq. (3.1) with theoretical calculation (black, from Eq. (3.8)) shows good agreement with relatively few realizations ($N = 1000$, 1000 years for each realization). **B** is a comparison of $A_R(t)\Delta t$ calculated from Monte Carlo simulations (rainbow) with theoretical calculation (black, from Eq. (3.10)) shows that many realizations are required to achieve accuracy.

Just as we have accomplished previously, we can confidently use the calculation of our autocovariance, equation (3.10), substitute into (2.8), and get the variance of cumulative losses in a time window T_w:

$$\sigma_{\mathcal{R}}^2 = \frac{1}{\Delta t}\int_{-T_w}^{T_w} A_R(t)\Big(T_w - |t|\Big)\,dt =$$

because of symmetry

$$2\frac{1}{\Delta t}\int_0^{T_w} \sigma_R^2 e^{-t/\tau}\Big(T_w - t\Big)\,dt = 2\frac{\sigma_R^2}{\Delta t}\left[\int_0^{T_w} e^{-t/\tau} T_w\,dt - \int_0^{T_w} \sigma_R^2 e^{-t/\tau} t\,dt\right] =$$

Now, using integration by parts,

$$
\begin{aligned}
\sigma_{\mathcal{R}}^2 &= 2\frac{\sigma_R^2}{\Delta t}\left[-T_w\tau e^{-t/\tau}\Big|_0^{T_w} - \left(t(-\tau e^{-t/\tau}) - \int_0^{T_w} -\tau e^{-t/\tau}\,dt\right)\right] \\
&= 2\frac{\sigma_R^2}{\Delta t}\left[-T_w\tau e^{-t/\tau}\Big|_0^{T_w} - \left(-t\tau e^{-t/\tau} + \tau^2 e^{-t/\tau}\right)\Big|_0^{T_w}\right] \\
&= 2\frac{\sigma_R^2}{\Delta t}\left[\left(-T_w\tau e^{-t/\tau} + T_w\tau\right) - \left((-T_w\tau e^{-t/\tau} + \tau^2 e^{-t/\tau}) - \left(0+\tau^2\right)\right)\right] \\
&= 2\frac{\sigma_R^2}{\Delta t}\tau\left(T_w + \tau(e^{-T_w/\tau} - 1)\right)
\end{aligned}
\tag{3.11}
$$

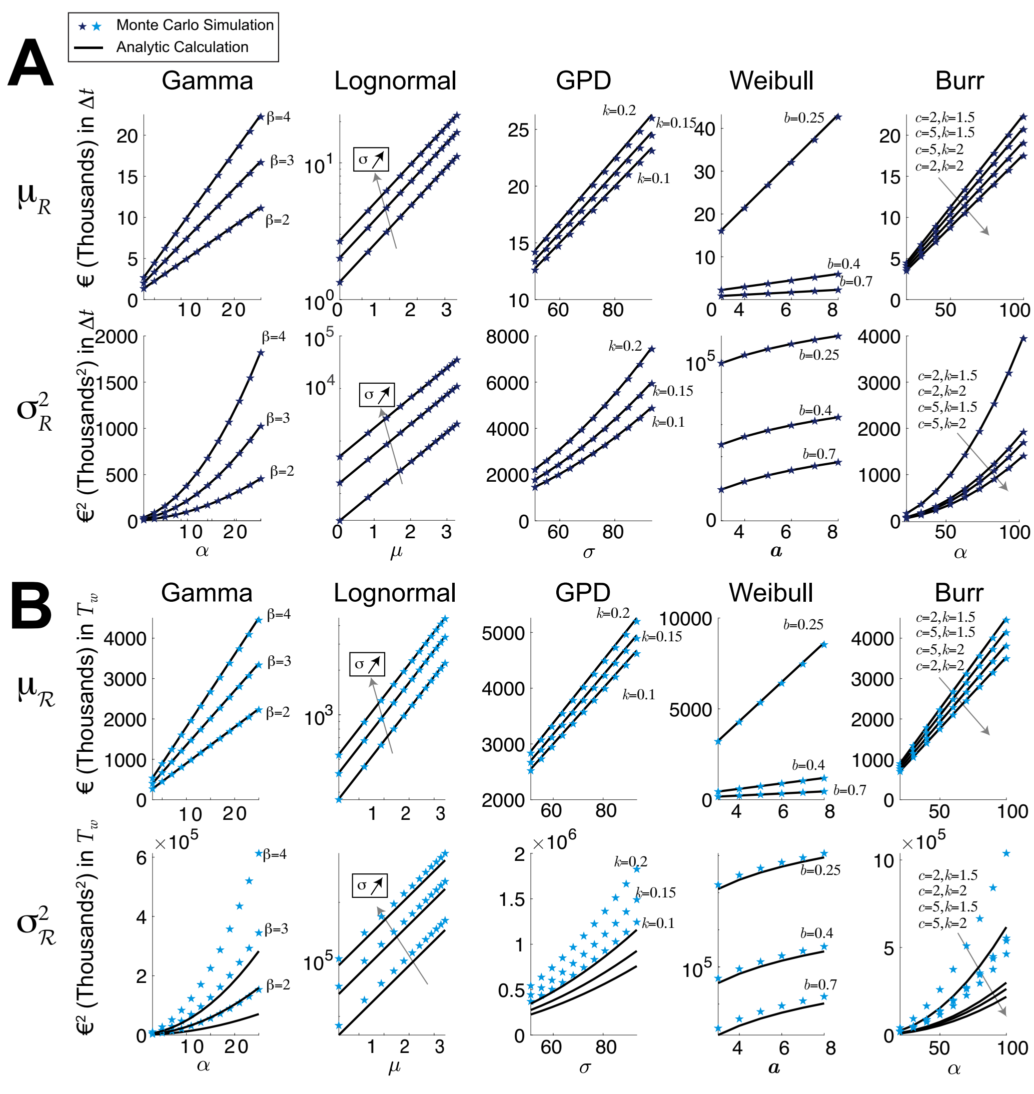

Fig. 8.: Validation of results for inhomogeneous Poisson Process frequency model (3.1).

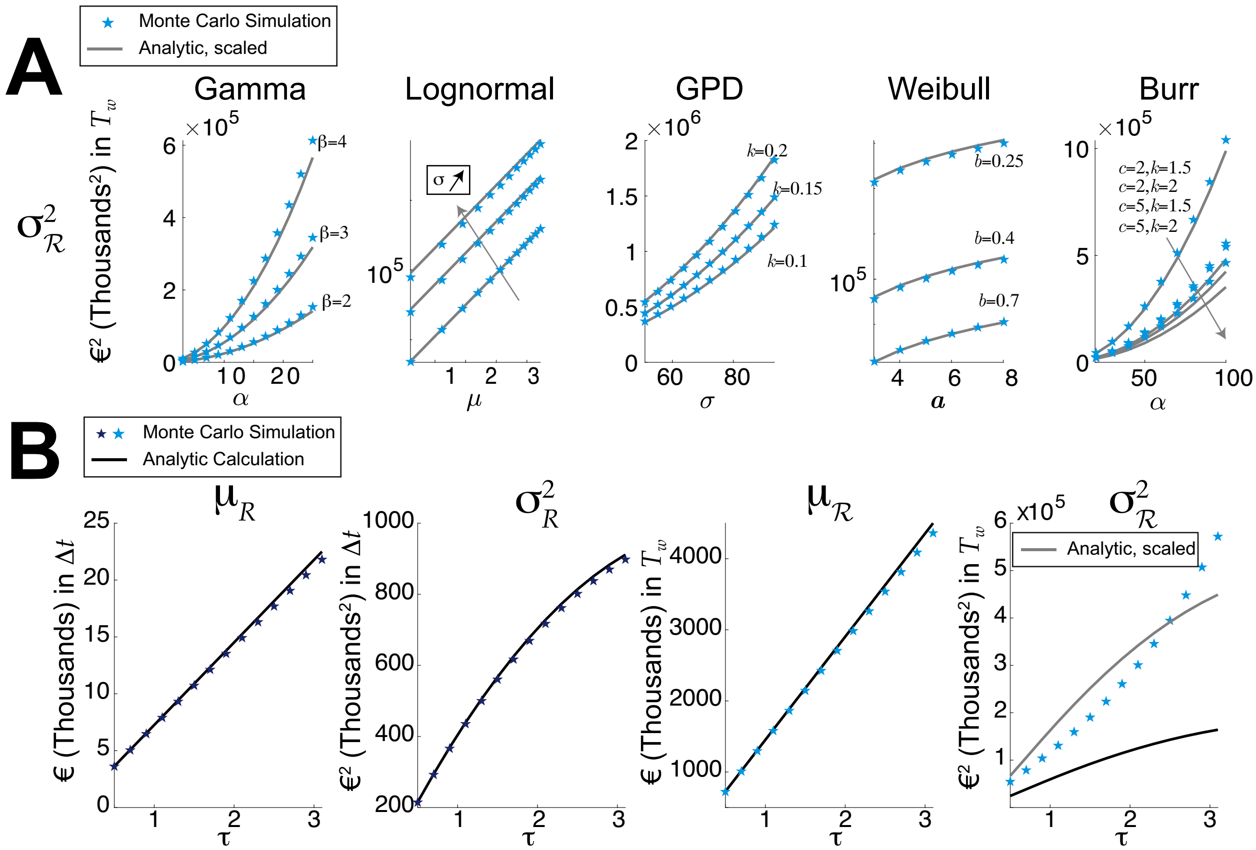

Fig. 9.: Inhomogeneous Poisson process with scalar adjustments to analytical formulas.

3.2 Summary of Time Series with Inhomogeneous Poisson Process

Figure 8 displays our results from an inhomogeneous Poisson process by comparing the mean and var of R and $\mathcal{R}$ following format of Figure 6; here we set $\tau = 1.2, a = 1, \gamma = 75/2$. **A**) Comparing the analytic formula for μ_R (Eq.(3.2), σ_R^2 (Eq.(3.6)) (solid lines) with the Monte Carlo simulations (stars), and similarly for $\mathcal{R}$ in (**B**): $\mu_{\mathcal{R}}$ (Eq.(3.7)), and $\sigma_{\mathcal{R}}^2$ (Eq.(3.11)). The general trends for $\sigma_{\mathcal{R}}^2$ are captured (see Fig. 9A) The Monte Carlo simulations are for 50,000 realizations with each being 100 years.

The analytic formulas we derived (Eq (3.2), (3.6), (3.7), (3.11); black lines in Fig 8) are generally very accurate in comparison to the Monte Carlo simulations (stars, Fig 8). The glaring exception is the analytic approximation of $\sigma_{\mathcal{R}}^2$ in Fig 8B (bottom row, black lines);

our theory does not accurately capture the Monte Carlo simulations (light blue stars).

This was initially surprising to us, previously believing that there might have been an error in our analytic calculation and/or in the Monte Carlo simulations. After much investigation and careful re-checking, we are certain that this quantitative mismatch is not easily rectifiable in these operational risk models. Indeed, this quantitative mismatch in using this exact same calculation (Eq. (3.11)) to approximate statistics in different time windows has previously been reported in the literature. Although the details of the models are different, we see this in the variance/covariance of (spike) statistics (see Fig 2C, Fig 5B, Fig 6B2 in [14], Fig 3B,C,D in [13], Fig 2 in [15]) where the important trends are captured rather than precise exquisite matching.

Despite quantitative inaccuracies between the Monte Carlo simulation and the analytic formula (Eq.(3.11)) for $\sigma^2_{\mathcal{R}}$, a simple scalar factor can yield accurate results.As such, Our analytic calculation still has value in capturing the qualitative trends of the Monte Carlo simulations. These simple scalar factors can yield relatively accurate approximations (Fig 9), indicating that our formula

$$\sigma^2_{\mathcal{R}} = 2\frac{\mu_{S^2}(a\tau\gamma\Delta t) - (\mu_S a\tau\gamma\Delta t)^2}{\Delta t}\tau\Big(T + \tau(e^{-T/\tau} - 1)\Big) \tag{3.12}$$

reveals the relative trends as parameters are varied. Fig 9A shows the comparisons of the analytic theory but all 3 gray curves in each panel is scaled by a factor. Specifically, **A** is the same as Figure 8B, except analytic formula are scaled by 2, 1.5, 1.6, 1.5, 1.6, respectively. The scaling factors were manually determined.

Fig 9B shows how our analytic calculation approximates the statistics as the time-scale of the inhomogeneous frequency distribution, τ, varies; note that as τ increases there appears to be more discrepancies between the Monte Carlo and the analytic calculations. The last panel on the far-right for $\sigma^2_{\mathcal{R}}$ shows both the original formula (black) and the curve scaled by 2.74 (gray; computed as least-squares fit to the stars). In detail, **B** fixes all parameters

($a = 1, \gamma = 75/3.1$) except τ that varies from 0.5 to 3.1 (unit=years), using a Gamma distributed severity distribution $S \sim$Gamma($\alpha = 20, \beta = 3$) ; last column has $\sigma^2_{\mathcal{R}}$ scaled by 2.74. The accuracy of the theory slightly diminishes as τ increases.

In Figures 8–9, rather than simulating a realization of along time series (as in Fig. 6), we simulated many realizations (50,000) of a moderate length time. This was to insure that the autocovariance of R_t was accurately captured compared to the theory (Eq. (3.10)); see Figure 7B with 50,000 realizations in red. Note that a very small number of realizations can accurately capture the autocovariance of the time-varying inhomogeneous Poisson rate ν_t (Fig. 7A).

CHAPTER 4

MULTIPLE LOSS CATEGORIES

4.1 Calculating Statistics for Two Correlated Loss Time Series

Here we extend the methods for a single loss category to two loss categories, which often suffices for calculating aggregate capital because pairwise correlations are commonly used to determine diversification of risk [2]. It is commonplace in the operational risk literature and elsewhere to describe dependence using correlation [19]. To capture dependencies of potential Operational Risk losses across or within business lines or event types, the notion of correlations, or more general, the notion of copulas, may be used [8]. Although we assume that the frequency distributions for both loss categories have the same form as Eq. (3.1):

$$\tau_1 \nu_1' = -\nu_1 + \tau_1 a_1 \sum_{k_1} \delta(t - t_{k_1}) \tag{4.1}$$

$$\tau_2 \nu_2' = -\nu_2 + \tau_2 a_2 \sum_{k_2} \delta(t - t_{k_2}) \tag{4.2}$$

a key difference is that the frequencies can be correlated. The random times t_{k_1}, t_{k_2} are again governed by an inhomogeneous Poisson Process with rates γ_1 and γ_2, respectively, but the random times (t_{k_1}, t_{k_2}) can be correlated. The parameter $c \in [-1, 1]$ is a measure of the correlation between t_{k_1}, t_{k_2}; letting $\bar{\gamma} := \min(\gamma_1, \gamma_2)$, the value $|c|\bar{\gamma}$ is the probability per unit time that ν_1 and ν_2 instantaneously jump at the same time, both in the positive direction if $c > 0$ and in opposite directions when $c < 0$. The marginal statistics of ν_j, $\mathcal{V}_j$ (the cumulative frequency in T_w), R_j and $\mathcal{R}_j$, for $j \in \{1, 2\}$, are the same as with a single loss time series:

$$\mathbb{E}[\nu_j] = a_j \tau_j \gamma_j \tag{4.3}$$

$$Var(\nu_j) = \frac{a_j^2 \gamma_j \tau_j}{2} \tag{4.4}$$

$$\mathbb{E}[\mathcal{V}_j] = T_w a_j \gamma_j \tau_j \tag{4.5}$$

$$Var(\mathcal{V}_j) = a_j^2 \gamma_j \tau_j^2 \Big(T_w + \tau_j (e^{-T_w/\tau_j} - 1) \Big) \tag{4.6}$$

$$\mu_{R_j} = \mu_{S_j}(a_j \gamma_j \tau_j \Delta t) \tag{4.7}$$

$$\sigma_{R_j}^2 = \mu_{S_j^2}(a_j \tau_j \gamma_j \Delta t) - (\mu_{S_j} a_j \tau_j \gamma_j \Delta t)^2 \tag{4.8}$$

$$\mu_{\mathcal{R}_j} = T_w a_j \gamma_j \tau_j \mu_{S_j} \tag{4.9}$$

$$\sigma_{\mathcal{R}_j}^2 = 2 \frac{\sigma_{R_j}^2}{\Delta t} \tau_j \Big(T_w + \tau_j (e^{-T_w/\tau_j} - 1) \Big). \tag{4.10}$$

To generalize the theory and calculations from the prior section, we consider the cross-covaraince function of $CC_\nu(t) := \mathbb{E}[\nu_1(t'+t)\nu_2(t')] - \mathbb{E}[\nu_1]\mathbb{E}[\nu_2]$; note that $CC_\nu(t) \neq CC_\nu(-t)$, unlike with the autocovariance function where $A(t) = A(-t)$ for both R and ν_t. The analogous equation to Eq. (3.5) is:

$$\nu_1(t')\nu_2(t'+t) = a_1 a_2 \int_0^\infty \int_0^\infty D_1(t'+t-u) D_2(t'-v) e^{\frac{-u}{\tau_1}} e^{\frac{-v}{\tau_2}} \, du \, dv. \tag{4.11}$$

where $D_j(t) = \sum_{k_j} \delta(t - t_{k_j})$. Since

$$\mathbb{E}[D_1(t'+t-u) D_2(t'-v)] - \gamma_1 \gamma_2 = c\bar{\gamma}\delta(t-u+v)$$

by construction, we take the expected value of Eq. (4.11), using similar calculations as before to get:

$$CC_\nu(t) = c\bar{\gamma} a_1 a_2 \frac{\tau_1 \tau_2}{\tau_1 + \tau_2} \begin{cases} e^{-t/\tau_1}, & \text{if } t \geq 0 \\ e^{-|t|/\tau_2}, & \text{if } t < 0 \end{cases} \tag{4.12}$$

We apply the same arguments as before when deriving $A_R(t)$ (Eq. (3.10)): since R_j and D_j have the same temporal support, the cross-covariance functions must be of a similar form.

We first derive the point-wise covariance for the losses in small windows Δt: $Cov(R_1, R_2)$ and set this to $CC_R(t=0)$ to get:

$$CC_R(t) = Cov(R_1, R_2) \begin{cases} e^{-t/\tau_1}, & \text{if } t \geq 0 \\ e^{-|t|/\tau_2}, & \text{if } t < 0 \end{cases} \tag{4.13}$$

To derive $Cov(R_1, R_2)$, we employ similar methods for σ_R^2 (Eq. 3.11):

$$\begin{aligned} \mathbb{E}[R_1(t)R_2(t)] &= \mathbb{E}_\nu\Big[\mathbb{E}[R_1R_2|\nu]\Big] \\ &= \mu_{S_1}\mu_{S_2}\mathbb{E}_\nu[\nu_1\nu_2](\Delta t)^2 \\ &= \mu_{S_1}\mu_{S_2}\Big(c\bar{\gamma}a_1a_2\frac{\tau_1\tau_2}{\tau_1+\tau_2}(\Delta t)^2 + (a_1\tau_1\gamma_1\Delta t)(a_2\tau_2\gamma_2\Delta t)\Big), \end{aligned} \tag{4.14}$$

to get:

$$Cov(R_1, R_2) = \mu_{S_1}\mu_{S_2}c\bar{\gamma}a_1a_2\frac{\tau_1\tau_2}{\tau_1+\tau_2}(\Delta t)^2. \tag{4.15}$$

Using the same method as before to relate the autocovariance of R to the variance of cumulative losses in larger time windows T_w, we relate the cross-covariance function to the covariance of cumulative losses in T_w:

$$\begin{aligned} Cov(\mathcal{R}_1, \mathcal{R}_2) &= \int_{-T_w}^{T_w} CC_R(t)\Big(T_w - |t|\Big)\,dt \\ &= Cov(R_1, R_2)\frac{\tau_1\Big(T_w + \tau_1(e^{-T_w/\tau_1}-1)\Big) + \tau_2\Big(T_w + \tau_2(e^{-T_w/\tau_2}-1)\Big)}{\Delta t} \qquad (4.16) \\ &= c\bar{\gamma}\mu_{S_1}\mu_{S_2}a_1a_2\frac{\tau_1\tau_2}{\tau_1+\tau_2}\Big[\tau_1\Big(T_w + \tau_1(e^{-T_w/\tau_1}-1)\Big) + \tau_2\Big(T_w + \tau_2(e^{-T_w/\tau_2}-1)\Big)\Big]\Delta t \qquad (4.17) \end{aligned}$$

Recall that the (point-wise) covariance of the frequency distributions is:

$$Cov(\nu_1, \nu_2) = c\bar{\gamma}a_1a_2\frac{\tau_1\tau_2}{\tau_1+\tau_2}. \tag{4.18}$$

It follows naturally from the above calculations for $Cov(\mathcal{R}_1, \mathcal{R}_2)$ that $Cov(\nu_1, \nu_2)$ in larger time windows T_w is:

$$Cov = c\bar{\gamma}a_1a_2\frac{\tau_1\tau_2}{\tau_1+\tau_2}\left[\tau_1\Big(T_w+\tau_1(e^{-T_w/\tau_1}-1)\Big)+\tau_2\Big(T_w+\tau_2(e^{-T_w/\tau_2}-1)\Big)\right] \tag{4.19}$$

We state this equation for completeness, and also because it will be used in our application.

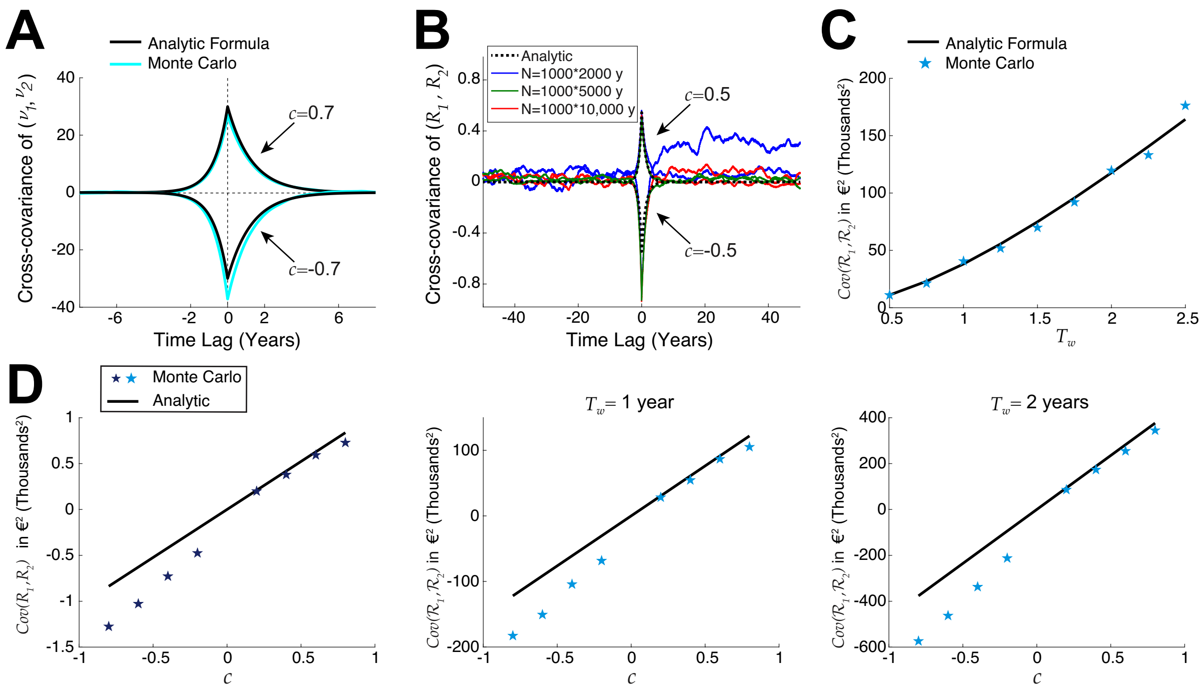

Fig. 10.: Analytic theory vs Monte Carlo simulations of cross-covariance of cumulative losses in different time windows.

4.2 Summary of Multiple Loss Categories

We can see from figure 10 that our analytic theory captures covariance of cumulative losses in different time windows. In **A)** we see comparisons of cross-covariance of frequency rates $(\nu_1, \nu_2), CC_\nu(t)$. In cyan is the Monte Carlo, with very few realizations, and in solid black is our analytic theory (Eq (4.12)), with input correlations $c = \pm 0.7$. In **B)** we see comparisons of cross-covariance, the function of actual loss time series $CC_R(t)$, and Monte Carlo in colors, with fixed $N = 1000$ but total time 2000 years (blue), 5000 years (red), 10,000 years

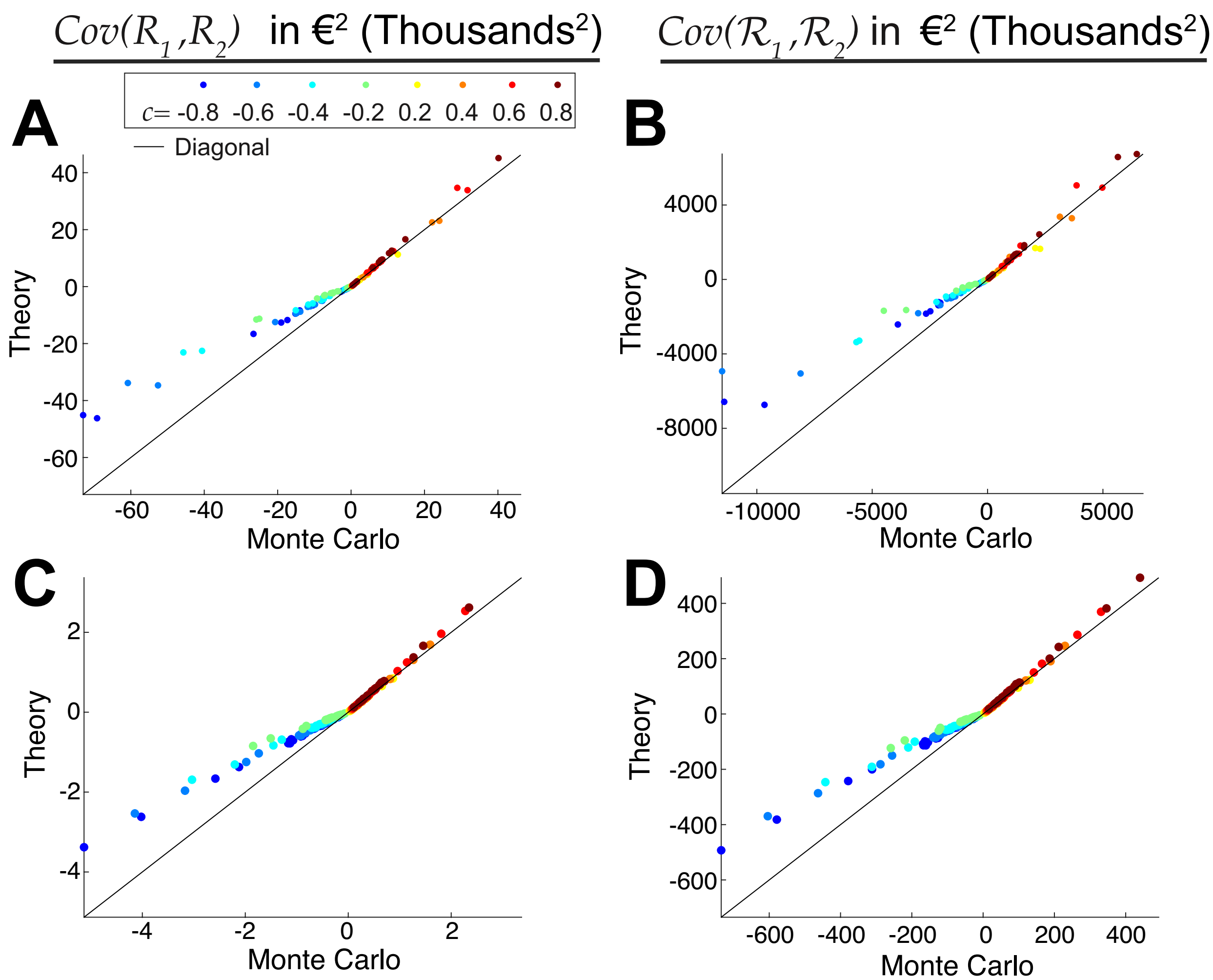

Fig. 11.: More demonstrations of the analytic theory for covariance of cumulative losses with $T_w = 1$ year throughout, but varying the input correlation c and randomly choosing the severity distribution parameters.

(green). The analytic theory is the dashed black, Eq (4.13), with input correlations $c = \pm 0.5$. Unsurprisingly, our calculations more accurately capture the Monte Carlo simulations as we increase the number of years. This is reminiscent of our findings with the autocovariance of R ($A_R(t)$), where we found more realizations leading to better accuracy between analytical formulas and Monte Carlo simulations. **C**) gives us a picture of the accuracy of our analytical equation for varying time windows T_w for a fixed $c = 0.25$. The fixed parameters are: $a_1 = 1.5, \tau_1 = 1.3\,\text{years}, \gamma_1 = 30\,\text{years}^{-1}, a_2 = 2, \tau_2 = 0.75\,\text{years}, \gamma_2 = 40\,\text{years}^{-1}$; $S_1 \sim$GPD($k = .15, \sigma = 50$), $S_2 \sim$Weibull($a = 5, b = 0.4$). Lastly, **D**) displays comparisons for $Cov(R_1, R_2)$ and $Cov(\mathcal{R}_1, \mathcal{R}_2)$ with $T_w = 1$ year and $T_w = 2$ years, with the same format as before. The input correlation c varies across a wide-range. A positive correlation is apparent between our analytical and Monte Carlo simulations.

Additionally, we provide more demonstrations of the analytic theory for covariance of cumulative losses in figure 11. Here we set $T_w = 1\,\text{year}$ throughout, but we vary the input correlation c (see legend for coloring) and randomly choose the severity distribution parameters, all uniform distributions (independent) with the same ranges as in Figs 6,8. The horizontal axes is the Monte Carlo, vertical is the analytic theory; the diagonal line is solid black, so perfect accuracy are points that are on the black line. **A**–**B** $S_1 \sim$Lognormal, $S_2 \sim$GPD, with Δt windows (**A**) and $T_w = 1\,\text{year}$ (**B**). **C**–**D** $S_1 \sim$Weibull, $S_2 \sim$Burr, with Δt windows (**C**) and $T_w = 1\,\text{year}$ (**D**). The fixed parameters are: $a_1 = 1.5, \tau_1 = 1.3\,\text{years}, \gamma_1 = 30\,\text{years}^{-1}, a_2 = 2, \tau_2 = 0.75\,\text{years}, \gamma_2 = 40\,\text{years}^{-1}$. In either figure, we notice that when $c < 0$, our analytical and Monte Carlo don't match. This is likely due to how we simulate negative correlations in $\nu_{1,2}$, where a fraction (c) of the instantaneous jumps are in opposite directions – we did not fully explore the other ways to simulate negative correlations because positive correlations are the main focus operational risk.

CHAPTER 5

APPLYING THE MODEL TO INDUSTRY-WIDE AVERAGES

5.1 Extracting Data from a Real-world Source

From a statistical point of view, the data scarcity of operational risk losses is still one of the major problems to be solved to adequately estimate model parameters and test model assumptions[8]. Furthermore, obtaining actual operational risk loss data with granular details for a particular institution is extremely difficult due to proprietary reasons. Loss information could reveal vulnerabilities in operations that institutions do not want to be publicly available, especially to business competitors. [19] notes that there are three well-known providers of external loss data for banks, namely the Fitch Group, the SAS Institute and the Operational Riskdata eXchange Association (ORX). Fitch and SAS construct their databases from publicly available information (media reports, regulatory filings, legal judgments, etc,) about operational losses over US$1 million, and the data can be purchased from these vendors. The most easily attainable and detailed operational risk loss data we found was provided by **ORX**, an organization that facilitates sharing of actual operational risk losses among its member institutions in a secure and anonymous platform [20]. ORX provides information to the public about the total count of loss events and the amount of losses in prior years, as well as the number of member institutions contributing data in a given year [21, 22]. Moreover, the cumulative frequency and severity losses data are available by risk categories. While the ORX data provides useful and intuitive data, it was important that we extract only the data relevant to the models we wish to provide. With our focus on financial, operational risk, we only utilized data related to retail, private and commercial banking and the categories of loss provided, which are presented in Table 2. These categories of risk in the banking business lines of the ORX report coincide with our earlier definition of operational risk.

Table 2.: Risk Categories and Abbreviation

The convention used for Operational Risk loss categories are segmented into the 7 categories below. Although the names of each category can vary slightly, the actual descriptions are equivalent; we adopt the naming convention used in ORX [21, 22].

Risk Categories

Abbreviation	**Definition**
IF	Internal Fraud
EF	External Fraud
EPWS	Employment Practices, Workplace Safety
CPBP	Clients, Products, Business Practices
DPS	Disasters and Public Safety
TIF	Technology and Infrastructure Failure
EDPM	Execution, Delivery and Process Management

Table 3.: Extracted data of frequency of events with loss category proportions [21].

Number of losses from 2014-2019 by Type								
	Internal Fraud	External Fraud	Employment Practices & Workplace Safety (EPWS)	Clients, Products & Business Practices (CPBP)	Disasters & Public Safety (DPS)	Technology & Infrastructure Failure (TIF)	Execution, Delivery & Process Management (EDPM)	Totals
Retail Banking	4561	100361	35026	39991	2509	2116	51352	235916
Private Banking	140	1671	1794	3268	23	110	3947	10953
Commercial Banking	327	8949	2033	7891	206	631	13462	33499
Totals	5028	110981	38853	51150	2738	2857	68761	280368
% of Total Bank Loss	2%	40%	14%	18%	1%	1%	25%	
% Bank Losses to Gross	73%							

Table 4.: Extracted data of severity of events with loss category proportions [21].

Total loss in Millions from 2014-2019 by Type								
	Internal Fraud	External Fraud	Employment Practices & Workplace Safety (EPWS)	Clients Products & Business Practices (CPBP)	Disasters & Public Safety (DPS)	Technology & Infrastructure Failure (TIF)	Execution, Delivery & Process Management (EDPM)	Totals
Retail Banking	950	7045.2	3394.7	26504.3	389.7	844.4	13655.6	52783.9
Private Banking	161.1	304.4	233.8	2456.2	2.1	27.6	886.6	4071.8
Commercial Banking	158.4	4514	240	12962	30.3	678.4	5425.1	24008.2
Totals	1269.5	11863.6	3868.5	41922.5	422.1	1550.4	19967.3	80863.9
% of Total Bank Loss	2%	15%	5%	52%	1%	2%	25%	
% Bank Losses to Gross	56%							

There was a total gross loss of 144.6 €–billion, and a total frequency of events 383,652, for the years 2014-2019. We pulled this data from page 6 of the ORX Annual Banking Lost Report [21], by summing both the yearly data for frequency and gross loss. This is of importance, as we would use these numbers to ensure that our breakdown of the data per category matched the proportion of frequency of loss events, 73 percent, and, in regards to severity, the proportion of total gross loss, 56 percent, which were both reported on pages 10 and 12 respectively, in the banking business lines of the report [21]. The final piece of extracted data would be the frequency of loss and the severity of loss in the banking lines of the report on pages 11 and 13 [21]. With this in hand we determine the proportions of each loss category. The following tables, Table 3 & Table 4 respectively, outline the extracted data from pages 11 and 13. To ensure that the development of our tables were accurate, we calculated these proportions and displayed them in row 6, column 2 of Table 3 & Table 4. Since we want to focus on bank loss, as opposed to the inclusion of all loss types and categories, we needed to specifically obtain the proportion of each type of bank loss relative to total bank loss. Row 5 of Table 3 & Table 4 capture this information by simply dividing our row 4 totals by the value found in row 4 column 8 (sum total of severity and frequency respectively). This we be an imperative step for assumptions we would be forced to use moving forward. We give further insight of the ORX data with the representation of Figures 12 & 13.

While all of this data is useful, it is cumulative from 2014 - 2019. To better fit and test our model with data we need to be able to view the data over different time windows, i.e., years, months, weeks, etc. For our purposes we want to partition the data down to be analyzed per year. Secondly, to show that the model is applicable to specific events that a financial institution would find interest in, we want to narrow the data down to specific types of loss, as opposed to total operational loss, which can result from a vast number of loss categories.

In order to accomplish this task we utilize the percentage breakdown for banking event types, and make an assumption that, outside of major outlier events, the percentage breakdown would remain in approximately the same percentage for each of the five years. We then ensured that these numbers total back up to the cumulative numbers and percentages that were reported by ORX [21]. Specifically, we took the total number of events and the severity of those events, and recorded them in the second column of Table 5 & Table 6. In column 3 of these same tables we include the number of banks that participated in the ORX study. This gives relevance to the data as it applies to a true picture of banking loss around the world. This piece of data also allows us to calculate averages per bank, which we present in table 7. Next we take the proportion of each type of bank loss reported from row 5 of table 3 & table 4 and divide them by the respective values in column 2. While this method may not reflect the actual values for each individual year, it does provide a very realistic approach to apply data that is otherwise inaccessible. This data is formulated and represented in Table 5 & Table 6 below.

Table 5.: Frequency of Losses by Year

	Total Number of Events	**Number of firms**	**Assuming 73% of loss due to Bank Loss**	**Assuming 2% due to Internal Loss**	**Assuming 15% due to External fraud**	**Assuming 5% due to EPWS**	**Assuming 52% due to CPBP**	**Assuming 1% due to DPS**	**Assuming 2% due to TIF**	**Assuming 25% due to EDPM**
2014	65766	80	48009.18	960.1836	19203.67	6721.2852	8641.652	480.0918	480.0918	12002.3
2015	69519	85	50748.87	1014.977	20299.55	7104.8418	9134.797	507.4887	507.4887	12687.22
2016	65797	92	48031.81	960.6362	19212.72	6724.4534	8645.726	480.3181	480.3181	12007.95
2017	63491	96	46348.43	926.9686	18539.37	6488.7802	8342.717	463.4843	463.4843	11587.11
2018	59642	97	43538.66	870.7732	17415.46	6095.4124	7836.959	435.3866	435.3866	10884.67
2019	59437	100	43389.01	867.7802	17355.6	6074.4614	7810.022	433.8901	433.8901	10847.25
[redacted]										
mean			46,678	934	18,671	6,535	8,402	467	467	11,669
standard deviation			2862.1094	57.24219	1144.844	400.69531	515.1797	28.621094	28.621094	715.5273
variance			53.498686	7.565857	33.83554	20.017375	22.69757	5.3498686	5.3498686	26.74934

Lastly, in order to analyze means, variance and covariance of the data as it relates to different institutions, we succinctly describe this data as total industry-wide frequency and severity by risk categories. We then divide by the number of institutions contributing data

Table 6.: Severity of Events by Year

	Total Gross loss in Millions	Number of firms	Assuming 73% of loss due to Bank Loss	Assuming 2% due to Internal Loss	Assuming 15% due to External fraud	Assuming 5% due to EPWS	Assuming 52% due to CPBP	Assuming 1% due to DPS	Assuming 2% due to TIF	Assuming 25% due to EDPM
2014	37600	80	21,056	421.12	3158.4	1052.8	10949.12	210.56	421.12	5264
2015	25100	85	14,056	281.12	2108.4	702.8	7309.12	140.56	281.12	3514
2016	28500	92	15,960	319.2	2394	798	8299.2	159.6	319.2	3990
2017	20000	96	11,200	224	1680	560	5824	112	224	2800
2018	17600	97	9,856	197.12	1478.4	492.8	5125.12	98.56	197.12	2464
2019	15800	100	8,848	176.96	1327.2	442.4	4600.96	88.48	176.96	2212
[redacted]										
mean			13,496	270	2,024	675	7,018	135	270	3,374
standard deviation			4553.32	91.0664	682.998	227.666	2367.726	45.5332	91.0664	1138.33
variance			67.478293	9.542872	26.13423	15.08861	48.65929	6.747829	9.5428717	33.739146

to get approximate loss data per institution. Even though institutions vary in size and have different realized operational loss statistics, this data is likely the best and most detailed data that is publicly available.

The columns of table 7 show: The average frequency (# events in a year, per institution) and the variances (absent from Fig. 14A), segmented by the 7 risk categories. The last column is the severity (€-Millions) **per event**, which is the only information we have access to for fitting our models to μ_S. The overall average frequency and severity without regard to category are: 514.13 events/year and 321.68 €-Millions/event (resp.). Same ORX data the same source as in Figure 14.

5.2 Fitting the Data to the Model

In practice the Poisson distribution is widely used for frequency distribution and its parameter is estimated independently from the internal and/or external loss data, whereas, the parameters of alternative severity distributions are estimated from the internal and/or external loss data [19]. So fitting our model to ORX data mainly requires dealing with the frequency distribution, i.e., fitting $c_{j,k}(j \neq k), a_j, \tau_j, \gamma_j$, giving a total of 42 parameters to determine with the 7 loss categories (7 different a, τ, γ and 21 input correlations $c_{j,k}(j \neq k)$).

Table 7.: **Statistics by Risk Categories.** The average frequency (# events in a year, per institution) and the variances (absent from Fig. 14A), segmented by the 7 risk categories. The last column is the severity (€-Millions) **per event**.

Risk Category	**Statistics (over 6 years)**		**Grand Average**
	Frequency Mean	**Frequency Var**	**Severity**
	(# events per year)	(# events per year)^2	(€-Millions per event)
IF	9.22	1.67	250.91
EF	203.51	815.19	106.23
EPWS	71.25	99.91	98.95
CPBP	93.8	173.16	814.51
DPS	5.02	0.496	153.3
TIF	5.24	0.54	539.31
EDPM	1.26	312.93	288.58

The superiority of frequency measures relative to measures based on total losses (such as the average annual total losses or the standard deviation of quarterly total losses) is likely the outcome of frequency measures being more stable proxies for risk exposure, as they do not fluctuate significantly when new tail losses are incurred [17]. We ideally would want the mean frequency from the model $\mu_{\nu_j} = a_j \tau_j \gamma_j T_w$ to equal the yearly frequency average from ORX data (Table 7, 1st column), similarly for the variances (Table 7, 2nd column) and covariances (Fig 14A). Thus our objective is to determine the value of our parameters in the the following system:

$$\begin{aligned} a_j \tau_j \gamma_j T_w &= \mu_{\nu_j,ORX} \ ; \text{for } j = 1, \ldots, 7 \\ a_j^2 \gamma_j \tau_j^2 \Big(T_w + \tau_j (e^{-T_w/\tau_j} - 1) \Big) &= \sigma^2_{\nu_j,ORX} \ ; \text{for } j = 1, \ldots, 7 \end{aligned}$$

$$c\bar{\gamma}a_j a_k \frac{\tau_j\tau_k}{\tau_j+\tau_k}\Big[\tau_j\Big(T_w+\tau_j(e^{-T_w/\tau_j}-1)\Big)+\tau_k\Big(T_w+\tau_k(e^{-T_w/\tau_k}-1)\Big)\Big] = Cov(\nu_{j,ORX},\nu_{k,ORX}) \text{ ; for } j\neq k$$

The process, though tedious in nature, was not overly complicated. We simply developed an objective function to minimize an equation that sets our calculated statistic of frequency with the known data from Table7. Specifically, we want to minimize

$$\sum_j \Big|\mu_{\nu_j}-\mu_{\nu_j,ORX}\Big| + \Big|\sigma^2_{\nu_j}-\sigma^2_{\nu_j,ORX}\Big| + \sum_{j\neq k}\Big|Cov(\nu_j,\nu_k)-Cov(\nu_{j,ORX},\nu_{k,ORX})\Big|,$$

and since the ORX data provides yearly averages, we set $T_w = 1\,\text{year}$. Utilizing Matlab we are able to ascertain our needed parameters. Taking a look at how our parameters fit from Figure 15, we see in **A**) the 3 yearly frequency statistics we fit: mean μ_ν, variance σ^2_ν, and covariance $Cov(\nu_j,\nu_k)$. In **B**), which is the same as **A** but with vertical axes on log-scale to enhance the differences; one of the model fits has slightly negative covariance (dashed lines) but it is very small (-0.018). The model fits both under- and over- estimates the actual mean and variances of yearly frequencies in the ORX data. For the covariances of frequencies, the model fits tend to under-estimate the data. The model fits are overall good but not perfect (Fig 15). This may not be surprising given that we have an under-constrained system of 35 equations and 42 unknowns, but note that the parameters themselves have constraints: $a_j,\tau_j,\gamma_j > 0$ and $c_{j,k}\in(-1,1)$, which may explain why we do not have perfect fits.

5.3 Effects of time window on covariance of loss distributions

After finding parameters that capture the frequency distribution well, we can very quickly assess how the covariance $Cov(\mathcal{R}_j,\mathcal{R}_k)$ of the actual loss distributions varies with a large range of time windows; this would usually require time-consuming Monte Carlo simulations but our analytic calculations circumvent that. From Eq (4.16):

$$Cov(\mathcal{R}_j, \mathcal{R}_k) = c_{j,k}\bar{\gamma}\mu_{S_j}\mu_{S_k}a_j a_k \frac{\tau_j \tau_k}{\tau_j + \tau_k}\Big[\tau_j\Big(T_w + \tau_j(e^{-T_w/\tau_j} - 1)\Big) + \tau_k\Big(T_w + \tau_k(e^{-T_w/\tau_k} - 1)\Big)\Big]\Delta t.$$

We only need to use the mean severity per event ($\mu_{S_{j/k}}$) from Table 7 (far right column).

The results are summarized in Figure 16.In **A**) we see the resulting $Cov(\mathcal{R}_j, \mathcal{R}_k)$ for all 21 pairs as a function of time window T_w (log scale on vertical axes). The statistics can vary a few orders of magnitude or more for these times. In **B**) we show the order of smallest and largest covariance pairs, setting $T_w = 1\,\text{year}$ for exposition purposes. The ordering does not change as T_w varies. Here we find that the largest covariance is (CPBP, EDPM), and the smallest is with (IF,DPS). We note that the univariate statistics for both the frequency and average severity per event (Table 7) are not at all indicative of which pairs would have the highest covariance, and neither are the frequencies in Fig 14A (i.e., the largest/smallest covariance in frequencies do not correspond to the largest/smallest covariance of losses in Fig 16). This indicates that obtaining an aggregate loss distribution combining frequency and severity distributions, and the resulting covariances, are not entities one could glean a priori.

The results in Figure 16 indicate that for all 21 covariances, care must be taken; systems should use a consistent time window, otherwise the covariances can vary by at least a few orders of magnitude when treating daily statistics as yearly statistics.

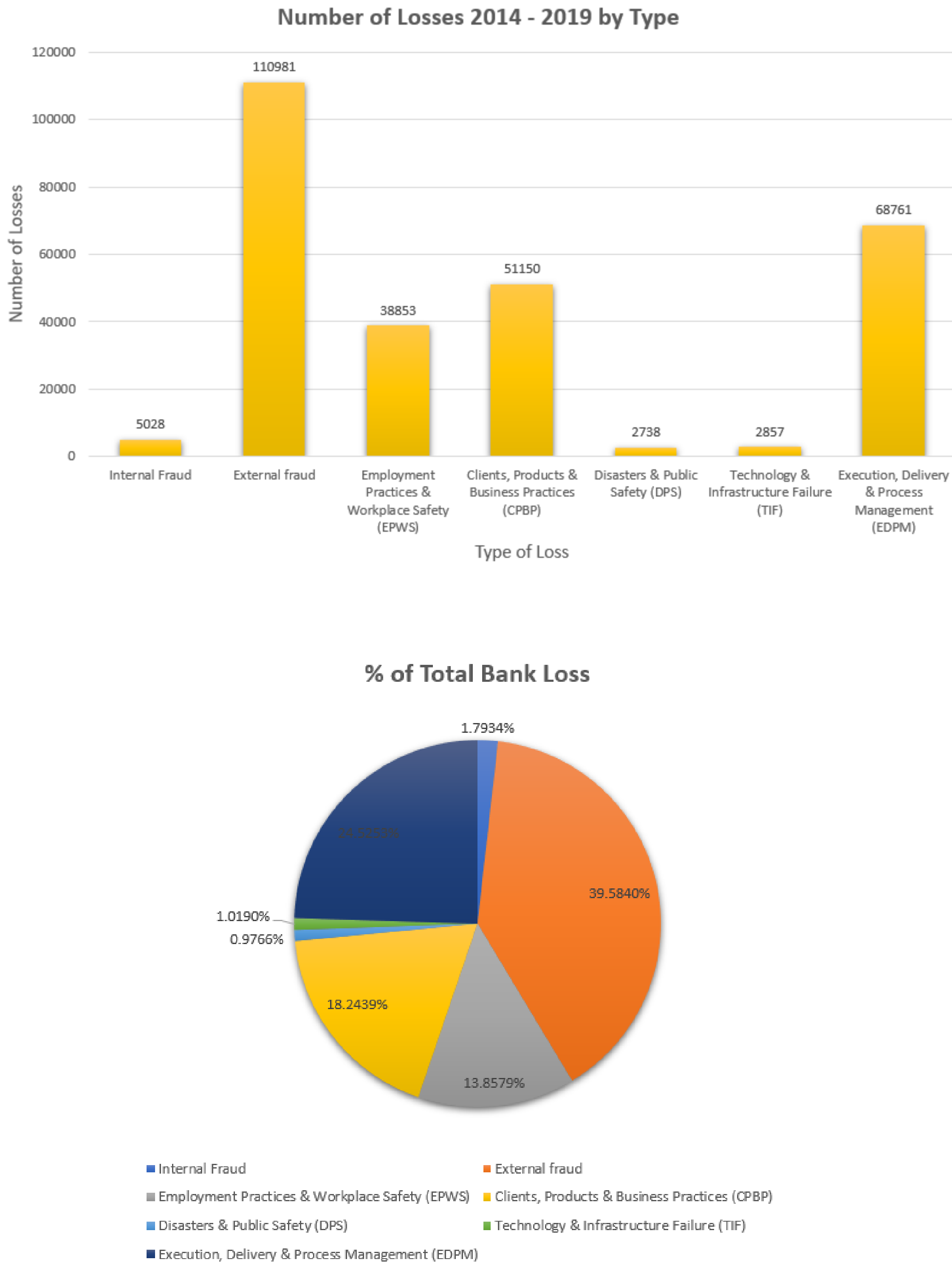

Fig. 12.: Financial Losses and percentage of financial losses by event type as reported by ORX 2019 [21].

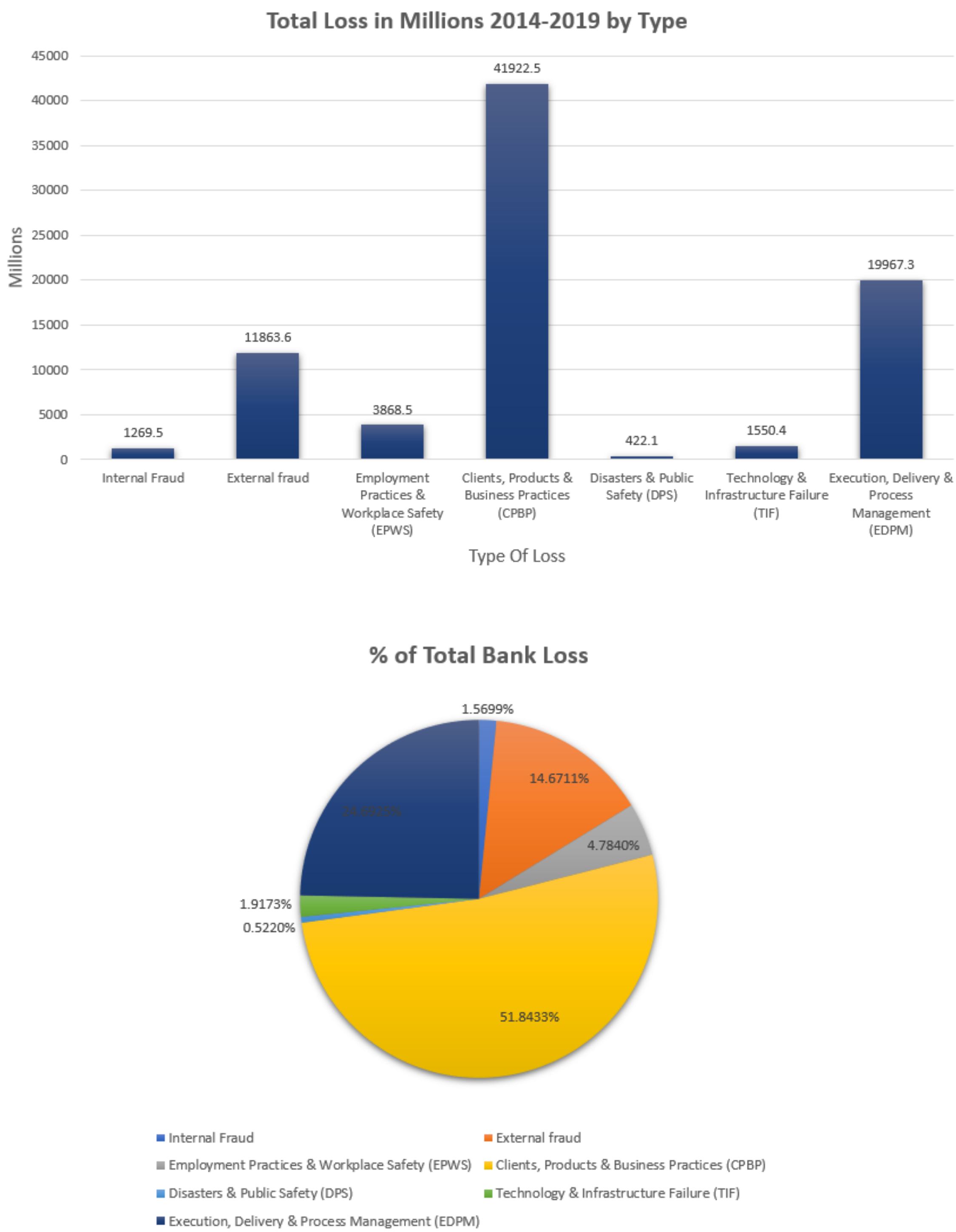

Fig. 13.: Amount of losses in millions, and percentage of amount of losses as reported by ORX 2019 [21].

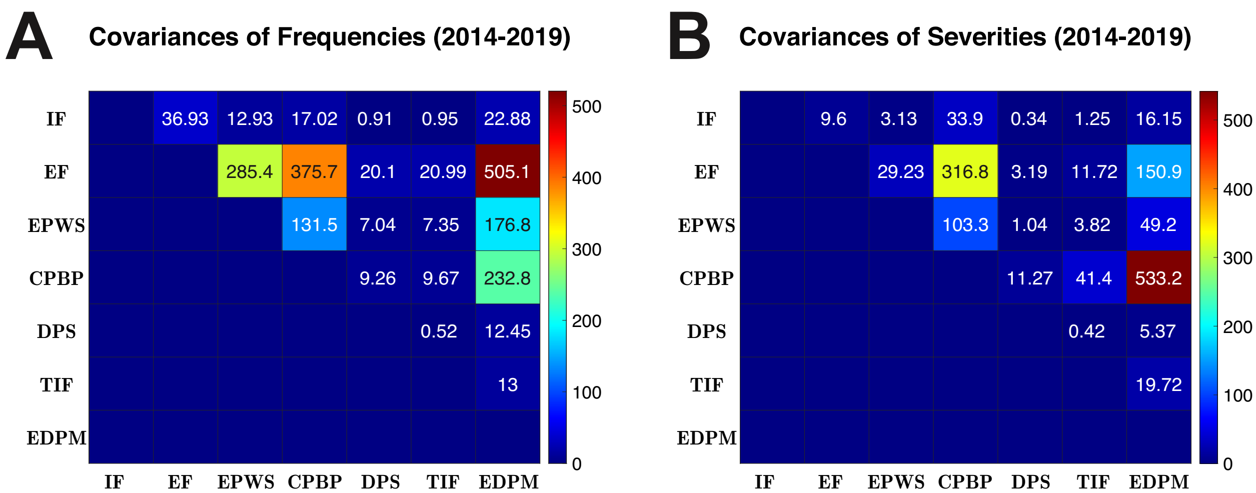

Fig. 14.: The year-to-year covariances of ORX data.

A) The covariances of the average frequencies per institution by risk category (see Table 2 for definition of abbreviations). **B**) Same as **A** but for average severities. Excluding diagonal (variances) and lower triangular portion because of symmetry.

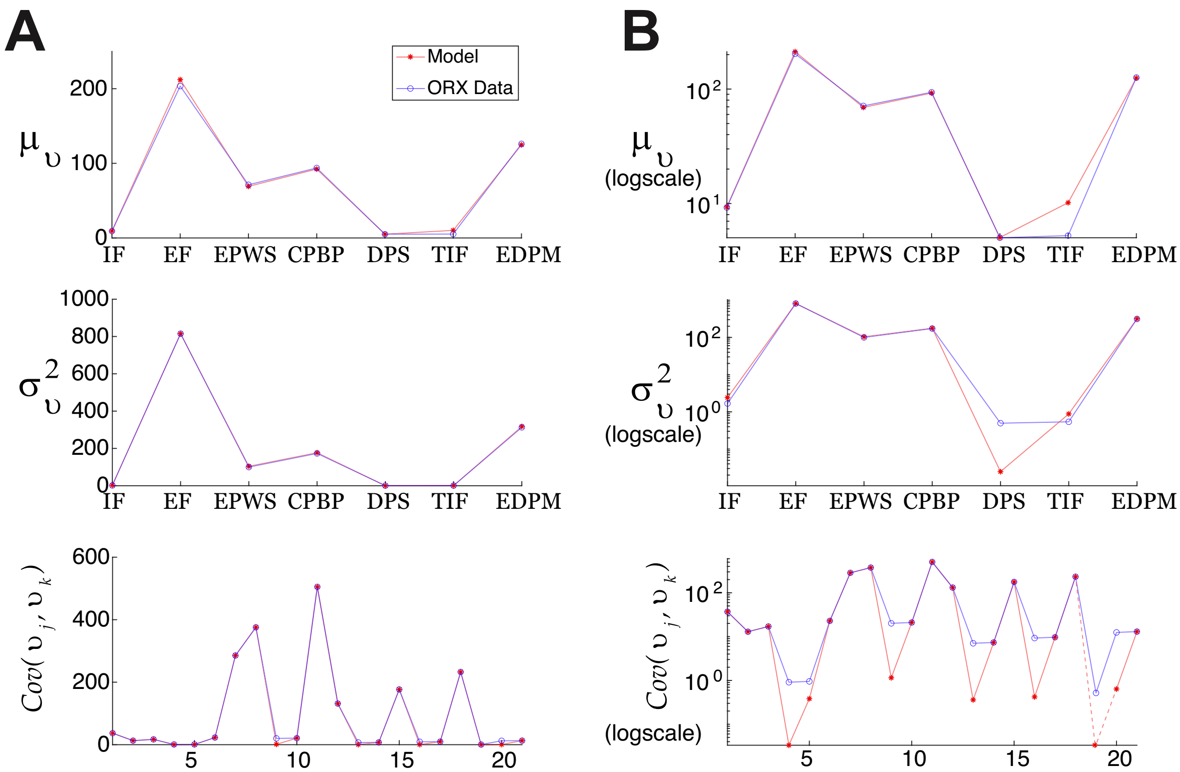

Fig. 15.: Fitting model to ORX data.

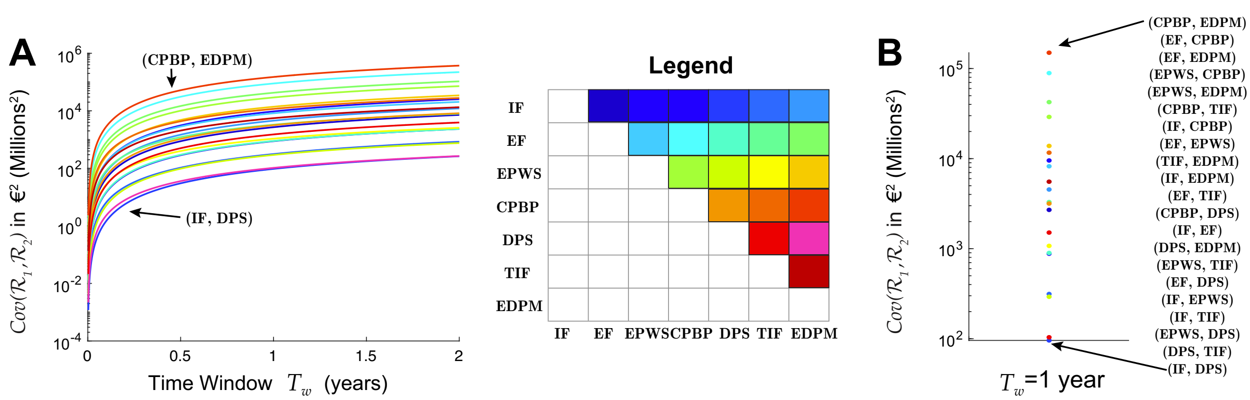

Fig. 16.: Covariance of loss distributions fit to ORX data.

CHAPTER 6

DISCUSSION AND CONCLUSIONS

6.1 Discussion

We present a dynamic stochastic differential equation model for the frequency distribution of Operational Risk loss events. This model is capable of capturing time-varying changes in the statistics of the frequency and is equipped with temporal correlations that do not exist in homogeneous Poisson Process models, that are commonly used. Our model was developed through a progression of calculating the 1st order statistics of a single time series, the cumulative statistics of a single time series, the statistics in arbitrary time windows of a single time series, as well as all of the latter with multiple time series. During this process, we introduce an application of methods relating the cumulative loss statistics to the statistics in smaller time windows, which is inspired by the successful usage in other disciplines (signal processing [10], computational neuroscience [11, 12, 13, 14, 15, 16]). We organically progressed from the common homogeneous calculations into the more complex inhomogeneous calculations, which is synonymous with real life applications of operational loss.

These modeling techniques were intentionally utilized to be specific to Operational Risk, thus yielding novel formulas and applications. As such, we were able to gather results, and then compare those results with simulations using Matlab. While the frequency distribution model contains complexities, it has only a few parameters, and is amenable to mathematical calculations that result in relatively accurate formulas. The detailed calculations and thorough comparisons with Monte Carlo simulations with a large range of parameters, have been provided in this thesis, and convey that our calculations do in fact coincide with the simulations, and become more accurate as we increase the number of realizations in the Monte Carlo simulations.

According to [8], different emphasis on each element may more closely reflect a specific loss history and risk profile but also complicate a comparison across banks. Thus, though the graphs are not provided here, care was taken to consider both the scale and weights of our parameters. Particularly, we take into consideration that one may want to place special significance on a particular variable as it reveals distinct characteristics of interest. In our case, we focus on the covariance as it bears relevance in its connection with copulas that are commonly used to aggregate loss distributions across different risk categories [23]. With minor alterations to our MATLAB code, which address both scaling and weighing of our univariate statistics, we find that our calculations well-fit the statistics ascertained from Monte Carlo simulations. Of most importance is the auto-covariance and cross-covariance, as this allows financial institutions to determine the relationships of aggregated losses of a time series.

Finally, we demonstrated how our loss distribution model could be fit to actual operational risk data, although the data was provided by an exchange (ORX) that contained industry-wide averages without the granular details of loss events that individual institutions have access to. Nevertheless, the ORX data provided information about the average severity of losses per event, segmented by the 7 common risk categories, as well as the year-to-year frequency of events from 2014–2019. Our model captured the mean and variance of yearly frequencies very well, which we then used to estimate the covariances of aggregate loss distributions. Our mathematical calculations enabled us to quickly assess how the covariances of the losses change with different time windows.

We do note that our findings would hold more weight with the ability to adequately benchmark this model with other state-of-the art models. We concur that benchmarks should be used extensively to justify model outputs, improve model stability, and maintain capital reasonableness [24]. However, all of the state-of-the art models rely on massive data to fit the both the frequency and severity distributions, which we do not have access to due to proprietary reasons. Additionally, while the absence of complicated computations make our model appetizing, it should be acknowledged that it's inability to be properly benchmarked with other models make it a sub-optimal choice as a model to be used for prediction. On the contrary, this is more of a backward looking model with ORX fits, and a proof of principle that incorrect correlations and covariances that feed into copulas could produce erroneous results [23]. Lastly, the model is not detailed enough to account for rare (tail-end) "black swan" events, as that is normally accounted for by fitting the tail of the severity distributions. Our work is predominantly a demonstration about how inaccurate statistics could come about with different time windows.

6.2 Conclusion

There lies financial power in being able to accurately predict the necessary capital based upon the relationships of loss and aggregated data over different time windows. We find that if institutions are not careful in their systems, and they are are not using the same time windows to estimate covariances (especially with risk categories with inherently different time-scales) that feed into copula calculations, that their results could be inaccurate. Further research is recommended to determine how this model can be applied to current industry modeling practices to provide more definitive predictability of necessary capital. Moreover, continued extensions and customization of this model should be developed. The implications of its ability to use time varying statistics could prove to be valuable to other dynamic economic and socioeconomic systems.

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