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Induced polynomial structures on generalized geometry

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Abstract: In this paper, we study different geometric structures that can be defined as section endomorphisms of the generalized tangent bundle $\mathbb{T}M := TM \oplus T^*M \to M$. This vector bundle admits some structures that arise canonically and other that can be induced from geometric structures defined on the manifold. We comment some well-known examples and present new structures, focusing on the polynomial structures that can be induced in the generalized tangent bundle.

Key words: Generalized tangent bundle, almost symplectic structure, almost complex structure, almost paracomplex structure, almost product structure, almost tangent structure

1. Introduction

The notion of generalized tangent bundle was first introduced by N. Hitchin [6] in 2003 and further developed by M. Gualtieri in his PhD published in 2011 in [5]. This vector bundle, $\mathbb{T}M := TM \oplus T^*M \to M$, is defined as the Whitney sum of the tangent and the cotangent bundle of a manifold M. Generalized complex structures were the first point of study in $\mathbb{T}M$. Almost complex and almost symplectic structures defined on the base manifold M induce generalized almost complex structures, thus allowing to see them as similar mathematical objects. The study of these structures has been useful in the development of other science fields, such as the geometric context of string theory [6]. They can also be used to obtain different geometric structures in other vector bundles (e.g., an induced Norden structure on the cotangent bundle T^*M [14]).

While the first analysis of the generalized tangent bundle was focused on generalized complex structures, it did not take long to study other geometric structures over it. A. Wade proposed in [17] the study of generalized paracomplex structures over $\mathbb{T}M$ in a similar way to the study of generalized complex structures. Both generalized complex and paracomplex structures have been widely investigated since then (for example, complex structures in [7] and paracomplex structures in [4]), using different points of view and studying specific examples. Other structures, such as generalized tangent structures, were subsequently studied (e.g., [1]).

Although originally generalized complex structures were forced to be compatible with a metric that emerges naturally within the vector bundle, later studies such as [11] by A. Nannicini suggested that this condition may be omitted. This suggestion agrees with other specialized texts in vector bundles, such as [15]

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by W. A. Poor. The advantage of studying these geometric structures in this way is that the majority of the concepts can be defined for any vector bundle $E \to M$. Then, if we take the tangent bundle $E = TM \to M$, we recover the well-known geometric structures on a manifold M. In our case, we are interested in studying the case $E = \mathbb{T}M \to M$. Following this line, we will use terminology as close as possible to that of structures on manifolds instead of that used in the majority of studies in generalized geometry. This perspective allows us to induce more generalized structures. For example, we can construct a generalized complex structure and a generalized paracomplex structure using a metric over the manifold.

When analyzing the generalized tangent bundle, different structures arise naturally, without the need of adding more structures to the base manifold. These structures are a pseudo-Riemannian metric, a symplectic structure, and a paracomplex structure. This should not be surprising: other vector bundles also have canonical structures. For example, every vector bundle $E \to M$ has a vector field $L \in \mathfrak{X}(E)$, called the Liouville or dilation vector field, such that locally it is defined as $L_{(x,e)} = e^j \frac{\partial}{\partial e^j}$ for any natural chart (x^i, e^j) of E. Another example is the canonical 1-form $\theta \in \Lambda^1(T^*M)$ that can always be defined on the cotangent bundle $T^*M \to M$. Locally, it is $\theta_{(x,w)} = w_i dx^i$ for any natural chart (x^i, w_j) in T^*M. This 1-form generates a canonical symplectic 2-form $\omega = d\theta$ on the cotangent bundle. However, the Liouville field as well as the canonical 1-form are structures defined on the tangent bundle of E and the cotangent bundle of T^*M, respectively, while the generalized canonical structures indicated are defined on $\mathbb{T}M$ as a vector bundle over M.

In this document, we want to study different geometric structures defined on $\mathbb{T}M$. Some of the structures that are presented here are familiar examples, and others are original. The paper is structured as follows:

In Section 2, inspired in classical texts as [9, 15], we present the definitions of different geometric structures over any vector bundle $E \to M$. We define Riemannian and pseudo-Riemannian metrics, symplectic structures and polynomial structures, in particular almost complex, almost product, almost paracomplex, and almost tangent structures. After that, we check the possible interactions between a polynomial structure and a metric in the form of (α, ε)-metric structures, previously studied over the tangent bundle in studies such as [2, 3].

Section 3 takes the geometric structures defined in Section 2 for any vector bundle $E \to M$ and specializes them for the generalized tangent bundle $E = \mathbb{T}M$, also called big tangent bundle in some references (e.g., [16]). Firstly, we study the geometric structures that arise naturally in the bundle, without introducing more additional structures on the base manifold M. These canonical structures are a canonical pseudo-Riemannian metric $\mathcal{G}_0$, a symplectic structure Ω_0 (both introduced in [5, 6]) and a paracomplex structure $\mathcal{F}_0$ (presented in [17]). We also show how these three generalized structures interact between them.

Finally, in Section 4, we obtain different induced generalized structures by a structure on the base manifold, according this one is a metric, an almost symplectic structure, a polynomial structure, or a polynomial structure with a compatible metric. Thus, we enlarge the original vision of generalized structures which allowed to see almost complex and almost symplectic structures on the base manifold, both of them inducing generalized almost complex structures. Besides, we show the richness of structures of the generalized tangent bundle.

It is worth noting that in this document we do not analyze the integrability of these structures. Therefore, each polynomial structure is preceded by the adverb “almost” (e.g., generalized almost complex structures).

2. Geometric structures on vector bundles

In this section, we work with any smooth manifold M and a vector bundle $E \to M$ over it, denoting the fiber in each point $p \in M$ as $E_p \subset E$. Several geometric structures will be defined over the $\mathfrak{F}(M)$-module of sections of E, $\Gamma(E)$, where $\mathfrak{F}(M)$ denotes the ring of differentiable functions over M. Most of these structures can be found in reference books specialized in vector bundles (for example, in [15]).

Definition 2.1 ([15, Def. 3.1]) *A metric defined on the vector bundle E is a differentiable morphism $g\colon \Gamma(E)\times\Gamma(E)\to\mathfrak{F}(M)$ that is bilinear, symmetric, and nondegenerate.*

The morphism g can also be seen as a section $g\in\Gamma(E^*\otimes E^*)$, such that for each $p\in M$ the morphism $g_p\colon E_p\times E_p\to\mathbb{R}$ must fulfill the indicated properties over the fiber E_p as a vector space. If the metric is positive definite, it is named a Riemannian metric, whilst if it is not positive definite, it is called pseudo-Riemannian metric. In the latter case, for each fiber E_p, it is possible to find an orthonormal basis $\{e_1,\dots,e_{r+s}\}$ of E_p in such a way that $g_p(e_i,e_i)>0$ for $i=1,\dots,r$ and $g_p(e_j,e_j)<0$ for $j=r+1,\dots,r+s$. Then, the pair (r,s) is called the signature of g. In the specific case when $r=s$, the metric signature is said to be neutral.

When we work with the tangent bundle $E=TM$, the usual concepts of Riemannian manifold and pseudo-Riemannian manifold arise. If we work over the generalized tangent bundle $E=\mathbb{T}M$, we will talk about generalized Riemannian and pseudo-Riemannian metrics.

Definition 2.2 ([15, Def. 8.4]) *A symplectic structure on the vector bundle E is a differentiable morphism $\omega\colon\Gamma(E)\times\Gamma(E)\to\mathfrak{F}(M)$ that is bilinear, skew-symmetric, and nondegenerate.*

As for the metrics, a symplectic structure can also be considered an element $\omega\in\Gamma(E^*\otimes E^*)$, such that for each $p\in M$, the morphism $\omega_p\colon E_p\times E_p\to\mathbb{R}$ must meet all the requirements indicated in the previous definition as a structure over the vector space E_p. In this case, the rank of the bundle E must be even.

If we take the particular case when $E=TM$, we retrieve the concept of almost symplectic manifold. An almost symplectic manifold is called a symplectic manifold if the almost symplectic structure ω is closed (that is, $d\omega=0$). In the case $E=\mathbb{T}M$, these morphisms will be called generalized symplectic structures.

The nondegeneracy of these two structures leads to isomorphisms between $\Gamma(E)$ and $\Gamma(E^*)$. These morphisms are commonly known as musical isomorphisms. If we take a symplectic structure ω (the construction for a metric g is analogous), these isomorphisms are defined as follows.

Definition 2.3 ([15, Def. 3.8]) *The flat isomorphism $\flat_\omega\colon\Gamma(E)\to\Gamma(E^*)$ with respect to ω transforms each element $X\in\Gamma(E)$ to the section $\flat_\omega X\in\Gamma(E^*)$, defined as $(\flat_\omega X)(Y):=\omega(X,Y)$ for every $Y\in\Gamma(E)$. The sharp isomorphism with respect to ω, $\sharp_\omega\colon\Gamma(E^*)\to\Gamma(E)$, is defined as the inverse of the flat isomorphism.*

It can be easily proven that the sharp isomorphism satisfies the relation $\omega(\sharp_\omega\xi,X)=\xi(X)$ for all $\xi\in\Gamma(E^*)$ and $X\in\Gamma(E)$. These morphisms of sections can also be seen as bundle isomorphisms between E and its dual bundle E^* over the identity map $id\colon M\to M$. For example, the flat isomorphism can be thought as the bundle morphism $\flat_\omega\colon E\to E^*$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\flat_\omega} & E^* \\ \downarrow & & \downarrow \\ M & \xrightarrow{id} & M \end{array}$$

commutes and $(\flat_\omega X_p)(Y_p):=\omega_p(X_p,Y_p)$ for every $p\in M$ and $X_p,Y_p\in E_p$. In other words, if $X_p\in E_p$ then $\flat_\omega X_p\in E_p^*$ for every $p\in M$.

The main focus of this document is to work with polynomial structures on generalized geometry. Therefore, we introduce now this type of geometric structures for any vector bundle E.

Definition 2.4 *A polynomial structure over the vector bundle E is defined as a section endomorphism $J\colon \Gamma(E) \to \Gamma(E)$ given a minimal polynomial P, such that $P(J) = 0$. When $P = x^2 + 1$ (that is, $J^2 = -Id$) the endomorphism J is called an almost complex structure over E [15, Def. 1.58]; if $P = x^2 - 1$ (in other words, $J^2 = +Id$) we say that J is an almost product structure over the bundle; and when $P = x^2$ (that is to say, $J^2 = 0$) and the rank of J is half of the rank of E, J is an almost tangent structure.*

The morphism J can also be seen as a bundle endomorphism $J\colon E \to E$ over the identity with the same minimal polynomial, that is, the diagram

$$\begin{tikzcd} E \arrow[r, "J"] \arrow[d] & E \arrow[d] \\ M \arrow[r, "id"] & M \end{tikzcd}$$

commutes (in other words, $JX_p \in E_p$ for each $X_p \in E_p$).

Depending on the given polynomial P, the structure that arises over the bundle E present different properties. It is relevant to study the eigenbundles associated to each eigenvalue of an almost complex and almost product structure over E. We analyze these three structures separately:

- For an almost product structure F defined on E we can find the eigenbundles $L_F^+, L_F^- \subset E$ associated, respectively, to the eigenvalues $+1, -1$ of the endomorphism. These two subbundles are the following:

$$\begin{aligned} L_F^+ &:= \{Y_p \in E\colon FY_p = Y_p\} = \{X_p + FX_p \in E\colon X_p \in E\}, \\ L_F^- &:= \{Y_p \in E\colon FY_p = -Y_p\} = \{X_p - FX_p \in E\colon X_p \in E\}. \end{aligned} \tag{2.1}$$

  The projections over each eigenbundle, P_F^+, P_F^- (seen as section endomorphisms), are given by

$$P_F^+ X = \frac{1}{2}(X + FX), \qquad P_F^- X = \frac{1}{2}(X - FX), \tag{2.2}$$

  for every $X \in \Gamma(E)$. In principle, the dimensions of L_F^+ and L_F^- do not have to be equal. When this happens (that is, when $\dim L_F^+ = \dim L_F^-$), it is said that the endomorphism F is an almost paracomplex structure on the vector bundle E.

- When working with an almost complex structure J over the bundle E, it is required to introduce the complexified bundle

$$E_{\mathbb{C}} := E \otimes \mathbb{C} = \{X_p + iY_p\colon X_p, Y_p \in E, p \in M\}.$$

  The endomorphism J is extended to $E_{\mathbb{C}}$ defining $J(iX_p) := iJX_p$. Then, as the minimal polynomial associated to J is $P = x^2 + 1$, its eigenvalues are $+i, -i$. If we denote their respective eigenbundles as $L_J^{1,0}, L_J^{0,1} \subset E_{\mathbb{C}}$, they are determined by

$$\begin{aligned} L_J^{1,0} &:= \{Y_p \in E_{\mathbb{C}}\colon JY_p = iY_p\} = \{X_p - iJX_p \in E_{\mathbb{C}}\colon X_p \in E\}, \\ L_J^{0,1} &:= \{Y_p \in E_{\mathbb{C}}\colon JY_p = -iY_p\} = \{X_p + iJX_p \in E_{\mathbb{C}}\colon X_p \in E\}. \end{aligned} \tag{2.3}$$

The projections over each of these eigenbundles, $P_J^{1,0}, P_J^{0,1}$, are defined for every $X \in \Gamma(E)$ as

$$P_J^{1,0}X = \frac{1}{2}(X - iJX), \qquad P_J^{0,1}X = \frac{1}{2}(X + iJX), \tag{2.4}$$

and extended to $\Gamma(E_{\mathbb{C}})$ with $P_J^{1,0}(iX) = iP_J^{1,0}(X)$ and $P_J^{0,1}(iX) = iP_J^{0,1}(X)$. It is immediate to check that the dimensions of the subbundles $L_J^{1,0}, L_J^{0,1}$ are equal, so the rank of a vector bundle with an almost complex structure defined on it must be even.

- Finally, for an almost tangent structure S on the bundle E, it is straightforward that the rank of E must be even and
$$\mathrm{Im}\, S = \ker S.$$

In the particular case when we work with the tangent bundle $E = TM$, the usual concepts of almost complex, almost product, almost paracomplex, and almost tangent manifolds arise. When we work with the generalized tangent bundle $E = \mathbb{T}M$, these structures will be called generalized almost complex, generalized almost product, generalized almost paracomplex and generalized almost tangent.

Before continuing, since the generalized tangent bundle is the Whitney sum of two vector bundles, it is worthwhile showing that the complexification of a vector bundle behaves correctly with the Whitney sum of vector bundles.

Proposition 2.5 *If $E, F \to M$ are two vector bundles over a manifold M, then $(E \oplus F)_{\mathbb{C}} \cong E_{\mathbb{C}} \oplus F_{\mathbb{C}}$.*

Proof The isomorphism between the two bundles is clear: each element $(e_{1p}+f_{1p})+i(e_{2p}+f_{2p})$ of $(E\oplus F)_{\mathbb{C}}$, such that $e_{1p}, e_{2p} \in E$ and $f_{1p}, f_{2p} \in F$ for a certain $p \in M$, is assigned to the element $(e_{1p}+ie_{2p})+(f_{1p}+if_{2p})$, belonging to $E_{\mathbb{C}} \oplus F_{\mathbb{C}}$. □

A polynomial structure J on a vector bundle E induces a geometric structure J^* over the dual vector bundle E^*. This endomorphism is defined as follows.

Definition 2.6 *Let $J\colon \Gamma(E) \to \Gamma(E)$ be a polynomial structure. Then, the dual structure of J is defined as the endomorphism $J^*\colon \Gamma(E^*) \to \Gamma(E^*)$ such that for each $\xi \in \Gamma(E^*)$, the section $J^*\xi$ is defined as $(J^*\xi)(X) := \xi(JX)$ for every $X \in \Gamma(E)$.*

It is straightforward to check that the minimal polynomial of a dual structure J^* is the same that the one associated to the endomorphism J.

Until now, we have described each geometric structure (metrics, symplectic structures, and polynomial structures) separately. We take now two different geometric structures such that they interact in different ways. In particular, we study the possible interactions between a metric g and a polynomial structure J with minimal polynomial $P = x^2 \pm 1$ (that is, J is almost complex or almost product). If this endomorphism interacts with the metric g as an isometry or antiisometry, we reach the following definition.

Definition 2.7 *The (α, ε)-metric structures over E, with α, ε taking values in $\{+1, -1\}$, are defined as the structure (E, J, g) composed by a polynomial structure $J\colon \Gamma(E) \to \Gamma(E)$ and a metric g over the bundle such that*

$$J^2 = \alpha Id, \qquad g(JX, JY) = \varepsilon g(X, Y), \tag{2.5}$$

for every $X, Y \in \Gamma(E)$. Depending on the values of α and ε, an (α, ε)-metric structure receives a different name:

- *If $\alpha = +1$, $\varepsilon = +1$, (E, J, g) is called almost product Riemannian structure. In this case, g is required to be Riemannian. If J is almost paracomplex, then (E, J, g) is named almost para-Norden structure.*
- *If $\alpha = +1$, $\varepsilon = -1$, (E, J, g) is called almost para-Hermitian structure. Because of the compatibility condition between J and g, the metric must be pseudo-Riemannian with neutral signature.*
- *If $\alpha = -1$, $\varepsilon = +1$, (E, J, g) is called almost Hermitian structure. The metric g is requested to be Riemannian.*
- *If $\alpha = -1$, $\varepsilon = -1$, (E, J, g) is called almost Norden structure. As in the case of the almost para-Hermitian structures, the metric g must be pseudo-Riemannian with neutral signature.*

Using the metric g and the endomorphism J it is possible to define a new geometric structure called fundamental tensor associated to the (α, ε)-structure. This structure $\Phi\colon \Gamma(E) \times \Gamma(E) \to \mathfrak{F}(M)$ is defined as

$$\Phi(X, Y) := g(JX, Y), \tag{2.6}$$

for all $X, Y \in \Gamma(E)$. It is immediate to check that if $\alpha\varepsilon = +1$ the fundamental tensor is a metric over the vector bundle (called twin metric), whereas if $\alpha\varepsilon = -1$ the morphism Φ is a symplectic structure over E (called fundamental symplectic structure).

As Φ is nondegenerate, its musical isomorphisms can be studied. These isomorphisms are related to the musical isomorphisms of the metric g, as it is stated in the following proposition.

Proposition 2.8 *Let (E, J, g) be an (α, ε)-metric structure over the fiber bundle E with fundamental tensor Φ. Then, the following equalities hold:*

$$\begin{aligned} \flat_\Phi &= \flat_g J = \alpha\varepsilon J^* \flat_g, \\ \varepsilon \sharp_\Phi &= \sharp_g J^* = \alpha\varepsilon J \sharp_g. \end{aligned} \tag{2.7}$$

Proof We start checking the first equations. If we take $X, Y \in \Gamma(E)$, using the definition of the flat isomorphism then

$$(\flat_\Phi X)(Y) = \Phi(X, Y) = g(JX, Y) = (\flat_g JX)(Y).$$

This happens for every $Y \in \Gamma(E)$; hence, $\flat_\Phi X = \flat_g JX$ for all $X \in \Gamma(E)$. In respect of the second equality,

$$(\flat_g JX)(Y) = g(JX, Y) = \alpha\varepsilon g(X, JY) = \alpha\varepsilon(\flat_g X)(JY) = \alpha\varepsilon(J^* \flat_g X)(Y),$$

for all $X, Y \in \Gamma(E)$. Therefore, the first equations hold. For the other equalities, taking any $\eta \in \Gamma(E^*)$ and $Y \in \Gamma(E)$, then

$$\begin{aligned} g(\varepsilon\sharp_\Phi \eta, Y) &= g(J\sharp_\Phi \eta, JY) = \Phi(\sharp_\Phi \eta, JY) = \eta(JY) = (J^*\eta)(Y) = g(\sharp_g J^* \eta, Y), \\ g(\sharp_g J^* \eta, Y) &= (J^*\eta)(Y) = \eta(JY) = g(\sharp_g \eta, JY) = g(\alpha\varepsilon J \sharp_g \eta, Y). \end{aligned}$$

The metric g is nondegenerate, so Eq. (2.7) holds. □

This proposition is useful, for example, in order to study the eigenbundles associated to a dual structure. In particular, we study briefly the case of an almost product structure F.

Proposition 2.9 *Let F be an almost product structure on a vector bundle E, and F^* its dual structure on E^*. Then, the dimensions of the eigenbundles $L^+_F, L^+_{F^*}$ are the same (and, consequently, the dimensions of $L^-_F, L^-_{F^*}$ coincide). In particular, if F is almost paracomplex on E then F^* is almost paracomplex on E^*.*

Proof If we build an auxiliar metric g over E such that $g(FX, FY) = g(X, Y)$ (just take any metric G over E and define $g(X, Y) = G(X, Y) + G(FX, FY)$), it can be checked that $\flat_g(L^+_F) = L^+_{F^*}$. In the first place, if we take an element $X_p + FX_p \in L^+_F$ and name $\xi_p = \flat_g X_p$, because of the Eq. (2.7), we have $\flat_g(X_p + FX_p) = \xi_p + F^*\xi_p$; therefore, $\flat_g(L^+_F) \subset L^+_{F^*}$. Conversely, if $\xi_p + F^*\xi_p \in L^+_{F^*}$ and we name $X_p = \sharp_g\xi_p$ then $\flat_g(X_p + FX_p) = \xi_p + F^*\xi_p$ and $L^+_{F^*} \subset \flat_g(L^+_F)$. □

These structures have been widely studied in the tangent bundle (see, for example, [2, 3]). A manifold endowed with an (α, ε)-metric structure is called an (α, ε)-metric manifold.

3. Canonical structures on the generalized tangent bundle

We work now with the generalized or big tangent bundle $\mathbb{T}M := TM \oplus T^*M$. In this section, we describe some geometric structures that can be defined over this $2n$-rank vector bundle (with $n = \dim M$), following the definitions given in Section 2. This line is followed in recent studies as, for example, A. Nannicini's studies in the field [10–14].

As in Section 2, the geometric structures that are presented here are defined as morphisms of sections. The sections of the bundle $\mathbb{T}M$ can be described as combinations of vector fields and 1-forms over the manifold. In other words, $\Gamma(\mathbb{T}M) = \Gamma(TM \oplus T^*M) = \mathfrak{X}(M) \oplus \Lambda^1(M)$, where $\mathfrak{X}(M)$ denotes the $\mathfrak{F}(M)$-module of vector fields and $\Lambda^1(M)$ the $\mathfrak{F}(M)$-module of 1-forms on M.

Firstly, it is necessary to establish the matrix notation that will be used from now on. If we have a section endomorphism $\mathcal{K}\colon \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$, then we can write

$$\mathcal{K} = \begin{pmatrix} H & \alpha \\ \beta & K \end{pmatrix}, \tag{3.1}$$

where $H\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$, $\alpha\colon \Lambda^1(M) \to \mathfrak{X}(M)$, $\beta\colon \mathfrak{X}(M) \to \Lambda^1(M)$, $K\colon \Lambda^1(M) \to \Lambda^1(M)$. This means that if we take $X + \xi \in \Gamma(\mathbb{T}M)$, then

$$\mathcal{K}(X + \xi) = \begin{pmatrix} H & \alpha \\ \beta & K \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} = \begin{pmatrix} HX + \alpha\xi \\ \beta X + K\xi \end{pmatrix} = (HX + \alpha\xi) + (\beta X + K\xi).$$

This matrix notation is useful in order to look for restrictions. For example, if we want $\mathcal{K}$ to be a generalized almost product or almost complex structure, computing $\mathcal{K}^2$, we obtain

$$\mathcal{K}^2 = \begin{pmatrix} H & \alpha \\ \beta & K \end{pmatrix} \begin{pmatrix} H & \alpha \\ \beta & K \end{pmatrix} = \begin{pmatrix} H^2 + \alpha\beta & H\alpha + \alpha K \\ \beta H + K\beta & \beta\alpha + K^2 \end{pmatrix} = \begin{pmatrix} \pm Id & 0 \\ 0 & \pm Id \end{pmatrix}, \tag{3.2}$$

where Id and 0 denote the identity and null morphisms, respectively (with respect to the bundles referred in Eq. (3.1)).

We search now canonical structures, that is, structures that arise naturally within the generalized tangent bundle $\mathbb{T}M$ without adding any additional structure over the tangent or the cotangent bundle. The main structures that are defined on this bundle are the canonical metric and the canonical symplectic structure.

Definition 3.1 ([5, Sec. 1]) *The canonical pairing or natural generalized metric over the generalized tangent bundle* $\mathbb{T}M$ *is defined as* $\mathcal{G}_0\colon \Gamma(\mathbb{T}M)\times\Gamma(\mathbb{T}M)\to\mathfrak{F}(M)$ *such that for each* $X,Y\in\mathfrak{X}(M)$ *and* $\xi,\eta\in\Lambda^1(M)$*,*

$$\mathcal{G}_0(X+\xi,Y+\eta):=\frac{1}{2}(\xi(Y)+\eta(X)). \tag{3.3}$$

Definition 3.2 ([10, Sec. 2]) *The canonical or natural generalized symplectic structure over the bundle* $\mathbb{T}M$ *is defined as* $\Omega_0\colon \Gamma(\mathbb{T}M)\times\Gamma(\mathbb{T}M)\to\mathfrak{F}(M)$ *such that for each* $X,Y\in\mathfrak{X}(M)$ *and* $\xi,\eta\in\Lambda^1(M)$*,*

$$\Omega_0(X+\xi,Y+\eta):=\frac{1}{2}(\xi(Y)-\eta(X)). \tag{3.4}$$

The canonical generalized metric is a pseudo-Riemannian with neutral signature. Associated to each of these two structures, we can study their musical isomorphisms. These isomorphisms can be seen as bundle morphisms between the generalized tangent bundle $\mathbb{T}M$ and its dual bundle $\mathbb{T}^*M:=(\mathbb{T}M)^*$ over the identity, that is to say, $\flat_{\mathcal{G}_0},\flat_{\Omega_0}\colon\mathbb{T}M\to\mathbb{T}^*M$ and $\sharp_{\mathcal{G}_0},\sharp_{\Omega_0}\colon\mathbb{T}^*M\to\mathbb{T}M$. Then, the following proposition is true.

Proposition 3.3 *The generalized tangent bundle* $\mathbb{T}M$ *is canonically isomorphic to its dual bundle,* $\mathbb{T}^*M$*.*

This is an enormous difference with respect to the tangent bundle TM: it is isomorphic to the cotangent bundle T^*M, but not canonically (it is necessary to define previously a metric g on the tangent bundle). On the contrary, the existence of a canonical metric over the generalized tangent bundle avoids this disadvantage.

There is another canonical structure that can be studied on the generalized tangent bundle. This morphism is an almost paracomplex structure. We present it as an endomorphism of $\Gamma(\mathbb{T}M)$.

Definition 3.4 ([17, Ex. 1]) *The canonical or natural generalized almost paracomplex structure is defined as the polynomial structure* $\mathcal{F}_0\colon\Gamma(\mathbb{T}M)\to\Gamma(\mathbb{T}M)$ *such that* $\mathcal{F}_0(X+\xi):=-X+\xi$ *for all* $X\in\mathfrak{X}(M)$ *and* $\xi\in\Lambda^1(M)$*. In matrix notation, it is*

$$\mathcal{F}_0=\begin{pmatrix}-Id & 0\\ 0 & Id\end{pmatrix}. \tag{3.5}$$

According to the Eqs. (2.1, 2.2), the eigenbundles $\mathcal{L}^+_{\mathcal{F}_0},\mathcal{L}^-_{\mathcal{F}_0}\subset\mathbb{T}M$ associated to the eigenvalues $+1,-1$ of $\mathcal{F}_0$ are, respectively,

$$\mathcal{L}^+_{\mathcal{F}_0}=T^*M,\qquad \mathcal{L}^-_{\mathcal{F}_0}=TM,$$

and the projections over these subbundles are the expected ones:

$$\mathcal{P}^+_{\mathcal{F}_0}(X+\xi)=\xi,\qquad \mathcal{P}^-_{\mathcal{F}_0}(X+\xi)=X.$$

The following result shows the relation that exists between the three canonical structures on $\mathbb{T}M$ recalled above.

Proposition 3.5 *The three canonical generalized structures are related between them with the expression*

$$\Omega_0(X+\xi,Y+\eta)=\mathcal{G}_0(\mathcal{F}_0(X+\xi),Y+\eta), \tag{3.6}$$

for every $X+\xi,Y+\eta\in\Gamma(\mathbb{T}M)$*.*

4. Induced structures on the generalized tangent bundle

If the base manifold M presents certain geometric structures defined for vector fields of the manifold, various generalized geometric structures can be induced on $\mathbb{T}M$. We describe in this section different induced structures that can be defined on $\mathbb{T}M$. We focus on generalized almost complex, generalized almost product and generalized almost tangent structures that arise from structures in TM. We introduce the examples based on the geometric structure on the base manifold.

4.1. Generalized polynomial structures induced from a metric

When the base manifold is endowed with a Riemannian or pseudo-Riemannian metric (M, g), the musical isomorphisms associated to g induce different generalized polynomial structures on the generalized tangent bundle. These morphisms are two generalized almost tangent structures, a generalized almost complex structure and a generalized almost paracomplex structure. We present these endomorphisms in the following propositions.

Proposition 4.1 *Let (M, g) be a Riemannian or pseudo-Riemannian manifold. Then, the metric g induces two generalized almost tangent structures, defined as $\mathcal{S}_{g,\flat}(X+\xi) = \flat_g X$ and $\mathcal{S}_{g,\sharp}(X+\xi) = \sharp_g \xi$. In matrix notation, they are*

$$\mathcal{S}_{g,\flat} = \begin{pmatrix} 0 & 0 \\ \flat_g & 0 \end{pmatrix}, \qquad \mathcal{S}_{g,\sharp} = \begin{pmatrix} 0 & \sharp_g \\ 0 & 0 \end{pmatrix}. \tag{4.1}$$

The image and the kernel of each structure are given by

$$\mathrm{Im}\ \mathcal{S}_{g,\flat} = \ker \mathcal{S}_{g,\flat} = T^*M, \qquad \mathrm{Im}\ \mathcal{S}_{g,\sharp} = \ker \mathcal{S}_{g,\sharp} = TM.$$

Proposition 4.2 ([8, Ex. 3.1], [11, Sec. 4]) *Let (M, g) be a Riemannian or pseudo-Riemannian manifold. Then, we can induce a generalized almost complex structure and a generalized almost paracomplex structure. These structures are, respectively,*

$$\mathcal{J}_g = \begin{pmatrix} 0 & -\sharp_g \\ \flat_g & 0 \end{pmatrix}, \tag{4.2}$$

$$\mathcal{F}_g = \begin{pmatrix} 0 & \sharp_g \\ \flat_g & 0 \end{pmatrix}. \tag{4.3}$$

The $+i, -i$-eigenbundles of the generalized almost complex structure $\mathcal{J}_g$ are

$$\mathcal{L}^{1,0}_{\mathcal{J}_g} = \{X_p - i\flat_g X_p \in \mathbb{T}M_{\mathbb{C}} \colon X_p \in TM\}, \qquad \mathcal{L}^{0,1}_{\mathcal{J}_g} = \{X_p + i\flat_g X_p \in \mathbb{T}M_{\mathbb{C}} \colon X_p \in TM\},$$

and the projections to each subbundle are

$$\mathcal{P}^{1,0}_{\mathcal{J}_g}(X+\xi) = \frac{1}{2}\left((X - i\flat_g X) + (\xi + i\sharp_g \xi)\right), \qquad \mathcal{P}^{0,1}_{\mathcal{J}_g}(X+\xi) = \frac{1}{2}\left((X + i\flat_g X) + (\xi - i\sharp_g \xi)\right).$$

On the other hand, the $+1, -1$-eigenbundles of the generalized almost paracomplex structure $\mathcal{F}_g$ are

$$\mathcal{L}^{+}_{\mathcal{F}_g} = \{X_p + \flat_g X_p \in \mathbb{T}M \colon X_p \in TM\}, \qquad \mathcal{L}^{-}_{\mathcal{F}_g} = \{X_p - \flat_g X_p \in \mathbb{T}M \colon X_p \in TM\},$$

and the projections to each subbundle are

$$\mathcal{P}^{+}_{\mathcal{F}_g}(X+\xi) = \frac{1}{2}\left((X + \flat_g X) + (\xi + \sharp_g \xi)\right), \qquad \mathcal{P}^{-}_{\mathcal{F}_g}(X+\xi) = \frac{1}{2}\left((X - \flat_g X) + (\xi - \sharp_g \xi)\right).$$

4.2. Generalized polynomial structures induced from an almost symplectic structure

Similarly to Subsection 4.1, when there is an almost symplectic structure ω defined on the manifold M, using the musical isomorphisms $\flat_\omega, \sharp_\omega$ we can generate two generalized almost tangent structures, a generalized almost complex structure and a generalized almost paracomplex structure. These structures are shown in the next two propositions.

Proposition 4.3 *Let (M,ω) be an almost symplectic manifold. Then, ω induces two generalized almost tangent structures, namely $\mathcal{S}_{\omega,\flat}(X+\xi)=\flat_\omega X$ and $\mathcal{S}_{\omega,\sharp}(X+\xi)=\sharp_\omega\xi$. In matrix notation, they are*

$$\mathcal{S}_{\omega,\flat}=\begin{pmatrix} 0 & 0 \\ \flat_\omega & 0 \end{pmatrix}, \qquad \mathcal{S}_{\omega,\sharp}=\begin{pmatrix} 0 & \sharp_\omega \\ 0 & 0 \end{pmatrix}. \tag{4.4}$$

The image and the kernel of each structure are given by

$$\operatorname{Im}\,\mathcal{S}_{\omega,\flat}=\ker\mathcal{S}_{\omega,\flat}=T^*M, \qquad \operatorname{Im}\,\mathcal{S}_{\omega,\sharp}=\ker\mathcal{S}_{\omega,\sharp}=TM.$$

Proposition 4.4 ([5, Sec. 3], [17, Ex. 2]) *Let (M,ω) be an almost symplectic manifold. Then, we can induce a generalized almost complex structure and a generalized almost paracomplex structure. These structures are, respectively,*

$$\mathcal{J}_\omega=\begin{pmatrix} 0 & -\sharp_\omega \\ \flat_\omega & 0 \end{pmatrix}, \tag{4.5}$$

$$\mathcal{F}_\omega=\begin{pmatrix} 0 & \sharp_\omega \\ \flat_\omega & 0 \end{pmatrix}. \tag{4.6}$$

The $+i,-i$-eigenbundles of the generalized almost complex structure $\mathcal{J}_\omega$ are

$$\mathcal{L}^{1,0}_{\mathcal{J}_\omega}=\{X_p-i\flat_\omega X_p\in\mathbb{T}M_{\mathbb{C}}\colon X_p\in TM\}, \qquad \mathcal{L}^{0,1}_{\mathcal{J}_\omega}=\{X_p+i\flat_\omega X_p\in\mathbb{T}M_{\mathbb{C}}\colon X_p\in TM\},$$

and the projections to each subbundle are

$$\mathcal{P}^{1,0}_{\mathcal{J}_\omega}(X+\xi)=\frac{1}{2}\left((X-i\flat_\omega X)+(\xi+i\sharp_\omega\xi)\right), \qquad \mathcal{P}^{0,1}_{\mathcal{J}_\omega}(X+\xi)=\frac{1}{2}\left((X+i\flat_\omega X)+(\xi-i\sharp_\omega\xi)\right).$$

On the other hand, the $+1,-1$-eigenbundles of the generalized almost paracomplex structure $\mathcal{F}_\omega$ are

$$\mathcal{L}^{+}_{\mathcal{F}_\omega}=\{X_p+\flat_\omega X_p\in\mathbb{T}M\colon X_p\in TM\}, \qquad \mathcal{L}^{-}_{\mathcal{F}_\omega}=\{X_p-\flat_\omega X_p\in\mathbb{T}M\colon X_p\in TM\},$$

and the projections to each subbundle are

$$\mathcal{P}^{+}_{\mathcal{F}_\omega}(X+\xi)=\frac{1}{2}\left((X+\flat_\omega X)+(\xi+\sharp_\omega\xi)\right), \qquad \mathcal{P}^{-}_{\mathcal{F}_\omega}(X+\xi)=\frac{1}{2}\left((X-\flat_\omega X)+(\xi-\sharp_\omega\xi)\right).$$

4.3. Generalized polynomial structures induced from a polynomial structure

When we have a polynomial structure defined on the manifold, it is possible to induce a generalized polynomial structure with the same minimal polynomial. This generalization can be made using diagonal matrices. The following propositions gather the generalized structures that are induced using an almost complex, almost product, or almost tangent structure on the base manifold, respectively.

Proposition 4.5 *Let* (M,J) *be an almost complex manifold. Then, the following generalized almost complex structures are induced, with* $\lambda\in\{+1,-1\}$ *:*

$$\mathcal{J}_{\lambda,J}=\begin{pmatrix} J & 0\\ 0 & \lambda J^*\end{pmatrix}. \tag{4.7}$$

When $\lambda=+1$ *, the eigenbundles* $\mathcal{L}^{1,0}_{\mathcal{J}_{+1,J}},\mathcal{L}^{0,1}_{\mathcal{J}_{+1,J}}\subset\mathbb{T}M_{\mathbb{C}}$ *associated to the eigenvalues* $+i,-i$ *of* $\mathcal{J}_{+1,J}$ *are*

$$\mathcal{L}^{1,0}_{\mathcal{J}_{+1,J}}=L^{1,0}_J\oplus L^{1,0}_{J^*},\qquad \mathcal{L}^{0,1}_{\mathcal{J}_{+1,J}}=L^{0,1}_J\oplus L^{0,1}_{J^*},$$

where $L^{1,0}_J,L^{0,1}_J\subset TM_{\mathbb{C}}$ *are the* $+i,-i$ *-eigenbundles of* J *; and* $L^{1,0}_{J^*},L^{0,1}_{J^*}\subset T^*M_{\mathbb{C}}$ *are the* $+i,-i$ *-eigenbundles of* J^* *. The projections over each subbundle are, respectively,*

$$\mathcal{P}^{1,0}_{\mathcal{J}_{+1,J}}(X+\xi)=P^{1,0}_JX+P^{1,0}_{J^*}\xi,\qquad \mathcal{P}^{0,1}_{\mathcal{J}_{+1,J}}(X+\xi)=P^{0,1}_JX+P^{0,1}_{J^*}\xi,$$

where $P^{1,0}_J,P^{0,1}_J$ *are the respective projections into* $L^{1,0}_J,L^{0,1}_J$ *, and* $P^{1,0}_{J^*},P^{0,1}_{J^*}$ *are the corresponding ones into* $L^{1,0}_{J^*},L^{0,1}_{J^*}$ *. Likewise, for* $\lambda=-1$ *these subbundles are*

$$\mathcal{L}^{1,0}_{\mathcal{J}_{-1,J}}=L^{1,0}_J\oplus L^{0,1}_{J^*},\qquad \mathcal{L}^{0,1}_{\mathcal{J}_{-1,J}}=L^{0,1}_J\oplus L^{1,0}_{J^*},$$

and the projections over each subbundle are

$$\mathcal{P}^{1,0}_{\mathcal{J}_{-1,J}}(X+\xi)=P^{1,0}_JX+P^{0,1}_{J^*}\xi,\qquad \mathcal{P}^{0,1}_{\mathcal{J}_{-1,J}}(X+\xi)=P^{0,1}_JX+P^{1,0}_{J^*}\xi.$$

It can be easily checked that $\mathcal{J}_{-1,J}$ is isometric with respect to the canonical metric $\mathcal{G}_0$, while $\mathcal{J}_{+1,J}$ is antiisometric. The example associated to $\lambda=-1$ was one of the first examples presented by M. Gualtieri in [5, Sec. 3]. Although $\mathcal{J}_{+1,J}$ is not isometric with respect to $\mathcal{G}_0$, following our terminology both morphisms $\mathcal{J}_{\lambda,J}$ for $\lambda\in\{+1,-1\}$ are generalized almost complex structures.

Proposition 4.6 ([8, Sec. 3.2]) *Let* (M,F) *be an almost product manifold. Then, the following generalized almost product structures are induced, with* $\lambda\in\{+1,-1\}$ *:*

$$\mathcal{F}_{\lambda,F}=\begin{pmatrix} F & 0\\ 0 & \lambda F^*\end{pmatrix}. \tag{4.8}$$

When $\lambda=+1$ *, the eigenbundles associated to the eigenvalues* $+1,-1$ *of* $\mathcal{F}_{\lambda,F}$ *are, respectively,*

$$\mathcal{L}^{+}_{\mathcal{F}_{+1,F}}=L^{+}_F\oplus L^{+}_{F^*},\qquad \mathcal{L}^{-}_{\mathcal{F}_{+1,F}}=L^{-}_F\oplus L^{-}_{F^*},$$

and the projections over each of these subbundles are

$$\mathcal{P}^{+}_{\mathcal{F}_{+1,F}}(X+\xi)=P^{+}_FX+P^{+}_{F^*}\xi,\qquad \mathcal{P}^{-}_{\mathcal{F}_{+1,F}}(X+\xi)=P^{-}_FX+P^{-}_{F^*}\xi.$$

Likewise, for $\lambda=-1$ *the subbundles associated to each eigenvalue are*

$$\mathcal{L}^{+}_{\mathcal{F}_{-1,F}}=L^{+}_F\oplus L^{-}_{F^*},\qquad \mathcal{L}^{-}_{\mathcal{F}_{-1,F}}=L^{-}_F\oplus L^{+}_{F^*},$$

with respective projections

$$\mathcal{P}^{+}_{\mathcal{F}_{-1,F}}(X+\xi)=P^{+}_FX+P^{-}_{F^*}\xi,\qquad \mathcal{P}^{-}_{\mathcal{F}_{-1,F}}(X+\xi)=P^{-}_FX+P^{+}_{F^*}\xi.$$

Analyzing the dimensions of each subbundle, as $\dim L_F^+ = \dim L_{F^*}^+$ (see Proposition 2.9) then $\mathcal{F}_{+1,F}$ is a generalized almost paracomplex structure if and only if F is almost paracomplex, while $\mathcal{F}_{-1,F}$ is always a generalized almost paracomplex structure, regardless of whether F is almost paracomplex or just almost product.

Proposition 4.7 *Let (M,S) be an almost tangent manifold. Then, the following generalized almost tangent structures are induced, with $\lambda \in \{+1,-1\}$:*

$$\mathcal{S}_{\lambda,S} = \begin{pmatrix} S & 0 \\ 0 & \lambda S^* \end{pmatrix}. \tag{4.9}$$

Independently of the value of λ, we have that

$$\operatorname{Im} \mathcal{S}_{\lambda,S} = \ker \mathcal{S}_{\lambda,S} = \ker S \oplus \ker S^*.$$

The structure corresponding to $\lambda = -1$ (studied in [1, Ex. 3.3]) is skew-symmetric with respect to the canonical metric $\mathcal{G}_0$, while for $\lambda = +1$ the structure $\mathcal{S}_{+1,S}$ is symmetric. In other words,

$$\mathcal{G}_0(\mathcal{S}_{\lambda,S}(X+\xi), Y+\eta) = \lambda \mathcal{G}_0(X+\xi, \mathcal{S}_{\lambda,S}(Y+\eta)).$$

4.4. Generalized polynomial structures induced from an (α,ε)-manifold

The examples shown so far are induced from the most basic geometries in the base manifold. We want now to induce generalized polynomial structures from manifolds endowed with more complicated structures. In particular, we work with a manifold with a polynomial structure and a compatible metric, that is, an (α,ε)-manifold.

If we denote the metric as g and its fundamental tensor as Φ (Φ is the twin metric or the fundamental almost symplectic structure), we can always induce the generalized almost tangent structures $\mathcal{S}_{g,\flat}, \mathcal{S}_{g,\sharp}, \mathcal{S}_{\Phi,\flat}, \mathcal{S}_{\Phi,\sharp}$ from Eqs. (4.1, 4.4), the generalized almost complex structures $\mathcal{J}_g, \mathcal{J}_\Phi$ from Eqs. (4.2, 4.5) and the generalized almost paracomplex structures $\mathcal{F}_g, \mathcal{F}_\Phi$ from Eqs. (4.3, 4.6). Furthermore, when $\alpha = -1$ the generalized almost complex structures $\mathcal{J}_{\lambda,J}$ defined in Eq. (4.7) can be generated from the manifold (M,J,g), while for $\lambda = +1$ the generalized almost product structures $\mathcal{F}_{\lambda,F}$ from Eq. (4.8) are induced from the manifold (M,F,g).

However, there are more generalized polynomial structures that can be defined using both the polynomial structure and the metric at the same time. In [11, Sec. 4], A. Nannicini induces a generalized almost complex structure from an almost Norden manifold (M,J,g), as follows:

$$\mathcal{J} = \begin{pmatrix} J & 0 \\ \flat_g & -J^* \end{pmatrix}.$$

In a similar way, we propose the following "triangular" examples of generalized almost complex structures that can be induced from any (α,ε)-metric manifold such that $\alpha = -1$.

Proposition 4.8 *Let (M,J,g) be an (α,ε)-metric manifold with $\alpha = -1$ (that is, J is an almost complex structure). Then, the following endomorphisms are generalized almost complex structures:*

$$\mathcal{J}_{J,g,\flat} = \begin{pmatrix} J & 0 \\ \flat_g & \varepsilon J^* \end{pmatrix}, \qquad \mathcal{J}_{J,g,\sharp} = \begin{pmatrix} J & \sharp_g \\ 0 & \varepsilon J^* \end{pmatrix}. \tag{4.10}$$

The eigenbundles of these structures associated to the eigenvalues $+i, -i$ *are, respectively,*

$$\mathcal{L}^{1,0}_{\mathcal{J}_{J,g,\flat}} = \{X_p - i(JX_p + \flat_g X_p) \in \mathbb{T}M_{\mathbb{C}} \colon X_p \in TM\}, \qquad \mathcal{L}^{0,1}_{\mathcal{J}_{J,g,\flat}} = \{X_p + i(JX_p + \flat_g X_p) \in \mathbb{T}M_{\mathbb{C}} \colon X_p \in TM\},$$
$$\mathcal{L}^{1,0}_{\mathcal{J}_{J,g,\sharp}} = \{\xi_p - i(\sharp_g \xi_p + \varepsilon J^* \xi_p) \in \mathbb{T}M_{\mathbb{C}} \colon \xi_p \in T^*M\}, \qquad \mathcal{L}^{0,1}_{\mathcal{J}_{J,g,\sharp}} = \{\xi_p + i(\sharp_g \xi_p + \varepsilon J^* \xi_p) \in \mathbb{T}M_{\mathbb{C}} \colon \xi_p \in T^*M\}.$$

Therefore, the projections to each subbundle of $\mathbb{T}M_{\mathbb{C}}$ *(using the notation from Eq.* (2.4)*) are*

$$\mathcal{P}^{1,0}_{\mathcal{J}_{J,g,\flat}}(X+\xi) = P^{1,0}_J X + P^{1,0}_{\varepsilon J^*}\xi - \frac{i}{2}\flat_g X, \qquad \mathcal{P}^{0,1}_{\mathcal{J}_{J,g,\flat}}(X+\xi) = P^{0,1}_J X + P^{0,1}_{\varepsilon J^*}\xi + \frac{i}{2}\flat_g X,$$
$$\mathcal{P}^{1,0}_{\mathcal{J}_{J,g,\sharp}}(X+\xi) = P^{1,0}_J X + P^{1,0}_{\varepsilon J^*}\xi - \frac{i}{2}\sharp_g \xi, \qquad \mathcal{P}^{0,1}_{\mathcal{J}_{J,g,\sharp}}(X+\xi) = P^{0,1}_J X + P^{0,1}_{\varepsilon J^*}\xi + \frac{i}{2}\sharp_g \xi.$$

Proof Using Proposition 2.8, it can be easily seen that $\mathcal{J}_{J,g,\flat}, \mathcal{J}_{J,g,\sharp}$ are generalized almost complex structures. We check now the eigenbundles of $\mathcal{J}_{J,g,\flat}$. Taking any $X_p + \xi_p \in \mathbb{T}M$, we compute $(X_p+\xi_p) - i\mathcal{J}_{J,g,\flat}(X_p+\xi_p)$ and $(X_p+\xi_p) + i\mathcal{J}_{J,g,\flat}(X_p+\xi_p)$:

$$(X_p+\xi_p) - i\mathcal{J}_{J,g,\flat}(X_p+\xi_p) = (X_p+\xi_p) - i(JX_p + \flat_g X_p + \varepsilon J^*\xi_p) = (X_p - i(JX_p + \flat_g X_p)) + (\xi_p - i\varepsilon J^*\xi_p),$$
$$(X_p+\xi_p) + i\mathcal{J}_{J,g,\flat}(X_p+\xi_p) = (X_p+\xi_p) + i(JX_p + \flat_g X_p + \varepsilon J^*\xi_p) = (X_p + i(JX_p + \flat_g X_p)) + (\xi_p + i\varepsilon J^*\xi_p).$$

It can be checked that for any element $\xi_p - i\varepsilon J^*\xi_p$ (likewise, $\xi_p + i\varepsilon J^*\xi_p$) there is a vector $Y_p \in TM_{\mathbb{C}}$ such that $Y_p - i(JY_p + \flat_g Y_p) = \xi_p - i\varepsilon J^*\xi_p$ (likewise, $Y_p + i(JY_p + \flat_g Y_p) = \xi_p + i\varepsilon J^*\xi_p$). We propose $Y_p = i\sharp_g(\xi_p - i\varepsilon J^*\xi_p) = \varepsilon\sharp_g J^*\xi_p + i\sharp_g\xi_p$ (analogously, $Y_p = -i\sharp_g(\xi_p + i\varepsilon J^*\xi_p) = \varepsilon\sharp_g J^*\xi_p - i\sharp_g\xi_p$). We compute both cases using Proposition 2.8:

$$\begin{aligned}
Y_p - i(JY_p + \flat_g Y_p) &= \varepsilon\sharp_g J^*\xi_p + i\sharp_g\xi_p - i^2 J\sharp_g(\xi_p - i\varepsilon J^*\xi_p) - i^2\flat_g\sharp_g(\xi_p - i\varepsilon J^*\xi_p) \\
&= \varepsilon\sharp_g J^*\xi_p + i\sharp_g\xi_p + J\sharp_g\xi_p + iJ^2\sharp_g\xi_p + \xi_p - i\varepsilon J^*\xi_p \\
&= \varepsilon\sharp_g J^*\xi_p + i\sharp_g\xi_p - \varepsilon\sharp_g J^*\xi_p - i\sharp_g\xi_p + \xi_p - i\varepsilon J^*\xi_p \\
&= \xi_p - i\varepsilon J^*\xi_p, \\
Y_p + i(JY_p + \flat_g Y_p) &= \varepsilon\sharp_g J^*\xi_p - i\sharp_g\xi_p - i^2 J\sharp_g(\xi_p + i\varepsilon J^*\xi_p) - i^2\flat_g\sharp_g(\xi_p + i\varepsilon J^*\xi_p) \\
&= \varepsilon\sharp_g J^*\xi_p - i\sharp_g\xi_p + J\sharp_g\xi_p - iJ^2\sharp_g\xi_p + \xi_p + i\varepsilon J^*\xi_p \\
&= \varepsilon\sharp_g J^*\xi_p - i\sharp_g\xi_p - \varepsilon\sharp_g J^*\xi_p + i\sharp_g\xi_p + \xi_p + i\varepsilon J^*\xi_p \\
&= \xi_p + i\varepsilon J^*\xi_p.
\end{aligned}$$

Therefore, $\mathcal{L}^+_{\mathcal{J}_{J,g,\flat}}$ and $\mathcal{L}^-_{\mathcal{J}_{J,g,\flat}}$ are the indicated ones. The proof of the eigenbundles of $\mathcal{J}_{J,g,\sharp}$ is analogous. □

Similarly, other "triangular" generalized structure is proposed by C. Ida and A. Manea in [8]. This example is generated from an almost para-Hermitian manifold (M, F, g) with F almost paracomplex. In this situation, the following generalized almost paracomplex structure is induced:

$$\mathcal{F} = \begin{pmatrix} F & 0 \\ \flat_g & F^* \end{pmatrix}.$$

Analogously to Proposition 4.8, we present now a result that gathers that example and other similar ones. The proof of this proposition is parallel to the proof of Proposition 4.8 and therefore will be omitted.

Proposition 4.9 *Let* (M, F, g) *be* (α, ε)*-metric manifold with* $\alpha = +1$ *(that is,* F *is an almost product structure). Then, the following endomorphisms are generalized almost product structures:*

$$\mathcal{F}_{F,g,\flat} = \begin{pmatrix} F & 0 \\ \flat_g & -\varepsilon F^* \end{pmatrix}, \qquad \mathcal{F}_{F,g,\sharp} = \begin{pmatrix} F & \sharp_g \\ 0 & -\varepsilon F^* \end{pmatrix}. \tag{4.11}$$

These structures are generalized almost paracomplex if and only if F *is almost paracomplex. The eigenbundles of these structures associated to the eigenvalues* $+1, -1$ *are, respectively,*

$$\mathcal{L}^+_{\mathcal{F}_{F,g,\flat}} = \{X_p + FX_p + \flat_g X_p \in \mathbb{T}M \colon X_p \in TM\}, \qquad \mathcal{L}^-_{\mathcal{F}_{F,g,\flat}} = \{X_p - FX_p - \flat_g X_p \in \mathbb{T}M \colon X_p \in TM\},$$

$$\mathcal{L}^+_{\mathcal{F}_{F,g,\sharp}} = \{\xi_p - \varepsilon F^*\xi_p + \sharp_g \xi_p \in \mathbb{T}M \colon \xi_p \in T^*M\}, \qquad \mathcal{L}^-_{\mathcal{F}_{F,g,\sharp}} = \{\xi_p + \varepsilon F^*\xi_p - \sharp_g \xi_p \in \mathbb{T}M \colon \xi_p \in T^*M\}.$$

Therefore, the projections to each subbundle of $\mathbb{T}Mc$ *(using the notation from Eq. (2.2)) are*

$$\mathcal{P}^+_{\mathcal{F}_{F,g,\flat}}(X+\xi) = P^+_F X + P^-_{\varepsilon F^*}\xi + \frac{1}{2}\flat_g X, \qquad \mathcal{P}^-_{\mathcal{F}_{F,g,\flat}}(X+\xi) = P^-_F X + P^+_{\varepsilon F^*}\xi - \frac{1}{2}\flat_g X,$$

$$\mathcal{P}^+_{\mathcal{F}_{F,g,\sharp}}(X+\xi) = P^+_F X + P^-_{\varepsilon F^*}\xi + \frac{1}{2}\sharp_g \xi, \qquad \mathcal{P}^-_{\mathcal{F}_{F,g,\sharp}}(X+\xi) = P^-_F X + P^+_{\varepsilon F^*}\xi - \frac{1}{2}\sharp_g \xi.$$

Until now, we have induced generalized almost complex structures using almost complex structures on M, and generalized almost product structures using almost product structures on M. We want now to elaborate richer structures starting with an (α, ε)-metric manifold. Specifically, we want to induce a new generalized almost complex structure using an almost product structure on the manifold, and a new generalized almost product structure from an almost complex structure on M.

Firstly, if we take an (α, ε)-metric manifold (M, J, g) such that $\alpha = -1$, we want to find a generalized almost product structure that looks like

$$\begin{pmatrix} \lambda_1 J & \lambda_2 \sharp_g \\ \lambda_3 \flat_g & \lambda_4 J^* \end{pmatrix}, \tag{4.12}$$

with $\lambda_i \in \mathfrak{F}(M)$ for $i = 1, 2, 3, 4$. We use Eq. (3.2) in order to obtain constraints for the functions λ_i. If we want each λ_i to be constant and fix λ_1 equal to $\lambda_1 = 1$, then we have $\lambda_2\lambda_3 = 2$. If we decide to take $\lambda_2 = \lambda_3 = \sqrt{2}$, then a direct computation using Eq. (2.7) gives the result $\lambda_4 = \varepsilon$. Therefore, we expose the following proposition.

Proposition 4.10 *Let* (M, J, g) *be an* (α, ε)*-metric manifold with* $\alpha = -1$*. Then, the following generalized structure is almost paracomplex for the corresponding value of* ε*:*

$$\mathcal{F}_{J,g} = \begin{pmatrix} J & \sqrt{2}\,\sharp_g \\ \sqrt{2}\,\flat_g & \varepsilon J^* \end{pmatrix}. \tag{4.13}$$

The eigenbundles associated to the eigenvalues $+1, -1$ *of* $\mathcal{F}_{J,g}$ *are, respectively,*

$$\mathcal{L}^+_{\mathcal{F}_{J,g}} = \{X_p + JX_p + \sqrt{2}\,\flat_g X_p \in \mathbb{T}M \colon X_p \in TM\}, \qquad \mathcal{L}^-_{\mathcal{F}_{J,g}} = \{X_p - JX_p - \sqrt{2}\,\flat_g X_p \in \mathbb{T}M \colon X_p \in TM\}.$$

The projections over each subbundle are the following:

$$\mathcal{P}^+_{\mathcal{F}_{J,g}}(X+\xi) = \frac{1}{2}\left((X + JX + \sqrt{2}\,\flat_g X) + (\xi + \sqrt{2}\,\sharp_g \xi + \varepsilon J^*\xi)\right),$$

$$\mathcal{P}^-_{\mathcal{F}_{J,g}}(X+\xi) = \frac{1}{2}\left((X - JX - \sqrt{2}\,\flat_g X) + (\xi - \sqrt{2}\,\sharp_g \xi - \varepsilon J^*\xi)\right).$$

Proof We check the expressions proposed for the subbundles corresponding to each eigenvalue. To this end, we compute $(X_p+\xi_p)+\mathcal{F}_{J,g}(X_p+\xi_p)$ and $(X_p+\xi_p)-\mathcal{F}_{J,g}(X_p+\xi_p)$:

$$(X_p+\xi_p)+\mathcal{F}_{J,g}(X_p+\xi_p)=(X_p+JX_p+\sqrt{2}\,\flat_g X_p)+(\xi_p+\sqrt{2}\,\sharp_g\xi_p+\varepsilon J^*\xi_p),$$
$$(X_p+\xi_p)-\mathcal{F}_{J,g}(X_p+\xi_p)=(X_p-JX_p-\sqrt{2}\,\flat_g X_p)+(\xi_p-\sqrt{2}\,\sharp_g\xi_p-\varepsilon J^*\xi_p).$$

We want to verify that for any element $\xi_p+\sqrt{2}\,\sharp_g\xi_p+\varepsilon J^*\xi_p$ (likewise, $\xi_p-\sqrt{2}\,\sharp_g\xi_p-\varepsilon J^*\xi_p$) there is a $Y_p\in TM$ such that $Y_p+JY_p+\sqrt{2}\,\flat_g Y_p=\xi_p+\sqrt{2}\,\sharp_g\xi_p+\varepsilon J^*\xi_p$ (or $Y_p-JY_p-\sqrt{2}\,\flat_g Y_p=\xi_p-\sqrt{2}\,\sharp_g\xi_p-\varepsilon J^*\xi_p$). Our proposal is $Y_p=\frac{1}{\sqrt{2}}\sharp_g(\xi_p+\varepsilon J^*\xi_p)$ (analogously, $Y_p=-\frac{1}{\sqrt{2}}\sharp_g(\xi_p-\varepsilon J^*\xi_p)$). We compute both cases using Eq. (2.7):

$$\begin{aligned}
Y_p+JY_p+\sqrt{2}\,\flat_g Y_p &= \frac{1}{\sqrt{2}}\sharp_g\xi_p+\frac{\varepsilon}{\sqrt{2}}\sharp_g J^*\xi_p+\frac{1}{\sqrt{2}}J\sharp_g\xi_p+\frac{\varepsilon}{\sqrt{2}}J\sharp_g J^*\xi_p+\xi_p+\varepsilon J^*\xi_p\\
&= \frac{1}{\sqrt{2}}\sharp_g\xi_p-\frac{1}{\sqrt{2}}J\sharp_g\xi_p+\frac{1}{\sqrt{2}}J\sharp_g\xi_p-\frac{1}{\sqrt{2}}J^2\sharp_g\xi_p+\xi_p+\varepsilon J^*\xi_p\\
&= \frac{2}{\sqrt{2}}\sharp_g\xi_p+\xi_p+\varepsilon J^*\xi_p\\
&= \xi_p+\sqrt{2}\,\sharp_g\xi_p+\varepsilon J^*\xi_p,\\
Y_p-JY_p-\sqrt{2}\,\flat_g Y_p &= -\frac{1}{\sqrt{2}}\sharp_g\xi_p+\frac{\varepsilon}{\sqrt{2}}\sharp_g J^*\xi_p+\frac{1}{\sqrt{2}}J\sharp_g\xi_p-\frac{\varepsilon}{\sqrt{2}}J\sharp_g J^*\xi_p+\xi_p-\varepsilon J^*\xi_p\\
&= -\frac{1}{\sqrt{2}}\sharp_g\xi_p-\frac{1}{\sqrt{2}}J\sharp_g\xi_p+\frac{1}{\sqrt{2}}J\sharp_g\xi_p+\frac{1}{\sqrt{2}}J^2\sharp_g\xi_p+\xi_p-\varepsilon J^*\xi_p\\
&= -\frac{2}{\sqrt{2}}\sharp_g\xi_p+\xi_p-\varepsilon J^*\xi_p\\
&= \xi_p-\sqrt{2}\,\sharp_g\xi_p-\varepsilon J^*\xi_p.
\end{aligned}$$

Hence, $\mathcal{L}^{+}_{\mathcal{F}_{J,g}}$ and $\mathcal{L}^{-}_{\mathcal{F}_{J,g}}$ are the proposed ones. □

Following the same reasoning and similar calculations (the proof is analogous to the one indicated for Proposition 4.10; therefore, it will be omitted), it can be checked the following result when we start with an (α,ε)-metric manifold such that $\alpha=+1$.

Proposition 4.11 *Let (M,F,g) be an (α,ε)-metric manifold for $\alpha=+1$. Then, the following generalized structure is almost complex for the corresponding value of ε:*

$$\mathcal{J}_{F,g}=\begin{pmatrix} F & -\sqrt{2}\,\sharp_g\\ \sqrt{2}\,\flat_g & -\varepsilon F^*\end{pmatrix}. \tag{4.14}$$

The eigenbundles associated to the eigenvalues $+i,-i$ of $\mathcal{J}_{F,g}$ are, respectively,

$$\mathcal{L}^{1,0}_{\mathcal{J}_{F,g}}=\{X_p-i(FX_p+\sqrt{2}\,\flat_g X_p)\in\mathbb{T}M_{\mathbb{C}}\colon X_p\in TM\},\qquad \mathcal{L}^{0,1}_{\mathcal{J}_{F,g}}=\{X_p+i(FX_p+\sqrt{2}\,\flat_g X_p)\in\mathbb{T}M_{\mathbb{C}}\colon X_p\in TM\}.$$

The projections over each subbundle are the following:

$$\mathcal{P}^{1,0}_{\mathcal{J}_{F,g}}(X+\xi) = \frac{1}{2}\Big((X - i(FX + \sqrt{2}\ \flat_g X)) + (\xi + i(\sqrt{2}\ \sharp_g \xi + \varepsilon F^* \xi))\Big),$$

$$\mathcal{P}^{0,1}_{\mathcal{J}_{F,g}}(X+\xi) = \frac{1}{2}\Big((X + i(FX + \sqrt{2}\ \flat_g X)) + (\xi - i(\sqrt{2}\ \sharp_g \xi + \varepsilon F^* \xi))\Big).$$

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