5-2-2022

Signed Graphs with Maximum Nullity at Most Two

Fredrick S. Dahlgren
Georgia State University

Signed Graphs with Maximum Nullity at Most Two

by

F. Scott Dahlgren

Under the Direction of Hein van der Holst, PhD and Marina Arav, PhD

ABSTRACT

A signed graph is an ordered pair (G, Σ), where $G = (V, E)$ is a graph and $\Sigma \subseteq E$. The edges in Σ are called odd, and the edges in $E \setminus \Sigma$ are called even. The family of matrices $\mathcal{S}(G, \Sigma)$ is defined such that if $[a_{i,j}] = A \in \mathcal{S}(G, \Sigma)$, then $a_{i,j} < 0$ if there is at least one edge between i and j and if all edges between i and j are even; $a_{i,j} > 0$ if there is at least one edge between i and j and if all edges between i and j are odd; $a_{i,j} \in \mathbb{R}$ if there is at least one even edge and at least one odd edge between i and j; and $a_{i,j} = 0$ if there are no edges between i and j. The maximum nullity of a signed graph $M(G, \Sigma)$ is the largest corank(A) for $A \in \mathcal{S}(G, \Sigma)$. The matrix $A \in \mathcal{S}(G, \Sigma)$ has the Strong Arnold Property with respect to (G, Σ) if $X = 0$ is the only matrix such that $AX = 0$, and $x_{i,j} = 0$ if i is adjacent to j or

$i = j$. The stable maximum nullity of a signed graph $\xi(G, \Sigma)$ is the largest corank(A) for $A \in \mathcal{S}(G, \Sigma)$ where A has the Strong Arnold Property. Here, we present a combinatorial characterization of signed graphs with maximum nullity at most two, extending a result of Johnson, Loewy, and Smith. We also find the forbidden minors for signed graphs with stable maximum nullity at most two, extending a result of Hogben and van der Holst. We generalize the notion of zero forcing to signed graphs. We find the zero forcing number of signed graphs with maximum nullity at most two, extending a result of Row.

INDEX WORDS: Signed graphs, maximum nullity, zero forcing, inverse eigenvalue problem for a graph, linear algebra, combinatorial matrix theory.

Signed Graphs with Maximum Nullity at Most Two

by

F. Scott Dahlgren

A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

in the College of Arts and Sciences

Georgia State University

2022

Signed Graphs with Maximum Nullity at Most Two

by

F. Scott Dahlgren

Committee Chairs: Hein van der Holst

Marina Arav

Committee: Guantao Chen

Frank J. Hall

Zhongshan Li

Electronic Version Approved:

Office of Graduate Studies

College of Arts and Sciences

Georgia State University

May 2022

DEDICATION

To my family, for their love, patience, and support.

ACKNOWLEDGMENTS

I would like to thank the Faculty of the Department of Mathematics and Statistics for their continued support and for all they have taught me. I am forever in their debt. In particular, I would like to thank Dr. Hein van der Holst and Dr. Marina Arav, who have shown me great kindness and patience over the years. I am grateful for their support and care. I would also like to thank Dr. Zhongshan Li, Dr. Frank J. Hall, and Dr. Guantao Chen who were kind enough to give their time as committee members. I would again like to thank Dr. Li, as well as Dr. Michael Stewart, Dr. Florian Enescu, and Dr. Yi Jiang, for their service as Graduate Advisor. I would also like to thank Ms. Sandra Ahuama-Jonas and Ms. Katina Akins for their work ensuring each semester went smoothly for me. I would like to thank Dr. Alexandra Smirnova and again thank Dr. Guantao Chen for their leadership of the Department as Chair Professor. Finally, I would like to thank the many students of Georgia State University who I have had the pleasure to know and study with.

TABLE OF CONTENTS

ACKNOWLEDGMENTS . . . . **v**

LIST OF FIGURES . . . . **viii**

LIST OF ABBREVIATIONS . . . . **ix**

1 Background **1**

1.1 Matrices 5

1.2 Graphs 9

1.3 Signed Graphs 13

1.4 ΔY-Transformations 18

1.5 Zero Forcing 22

1.6 Graph Structures and Maximum Nullity 25

1.6.1 1-Separations 25

1.6.2 Trees and Forests 26

1.6.3 Linear 2-Trees 27

1.6.4 Graph of Two Parallel Paths 27

1.6.5 Thin Outs 29

1.6.6 T_3-Family of Graphs 30

1.7 2-Connected Signed Graphs with $M(G, \Sigma) \leq 2$ 32

1.8 Sign-Nonsingular Matrices 35

2 Signed Graphs with $\xi(G, \Sigma) \leq 2$ **38**

2.1 The Signed Four-Wheel 40

2.2 The $K_3^=$-Family of Signed Graphs 43

2.3 Partial Wide 2-Paths 47

2.3.1 Two Wide Separations 49

2.3.2 One Wide Separation 60

2.3.3 No Wide Separations 67

2.4 Proof of the Main Result 69

3 Signed Graphs with $M(G, \Sigma) \leq 2$ **71**

3.1 Global Structure of Signed Graphs (G, Σ) with $M(G, \Sigma) \leq 2$ 73

3.2 Pendant Paths on an Odd 4-Wheel 76

3.3 Pendant Paths Attached to 2-Connected Partial Wide 2-Paths . . . 77

3.3.1 Two Wide Separations 79

3.3.2 One Wide Separation 88

3.3.3 No Wide Separations 98

3.4 Proof of the Main Result 105

4 $Z(G, \Sigma)$ for Signed Graphs with $M(G, \Sigma) \leq 2$ **108**

4.1 Zero Forcing on Signed Graphs 109

4.2 Signed Graphs with $M(G, \Sigma) = Z(G, \Sigma) = 2$ 117

4.3 Signed Graphs with $M(G, \Sigma) = 2 \leq Z(G, \Sigma)$ 120

REFERENCES **126**

LIST OF FIGURES

Figure 1.1 A ΔY-transformation on a simple graph. 18

Figure 1.2 The T_3-family from Hogben and van der Holst [14]. 31

Figure 1.3 The odd 4-wheel. Solid edges are even, and dotted edges are odd. . . 32

Figure 1.4 The signed digraph of the maximal SNS-matrix of order 3 and the corresponding signed graph, where odd edges are dashed and even edges are solid. 37

Figure 2.1 The $K_3^=$-family of signed graphs. Solid edges are even; dotted edges are odd; and dashed lines may be odd or even. 43

Figure 2.2 The $K_3^=$-family and their minors. The first row are the members of the $K_3^=$ family. Below the horizontal rule, each column are proper minors of the member of the $K_3^=$-family in that column. Arrows to the right represent a subdivision of an edge followed by a ΔY-transformation. 46

Figure 2.3 The $K_{2,4}e_i^j$ family of signed graphs, for $i = e, o$ and $j = 0, 1, 2, 3, 4, 5$. Solid edges are even, and dotted edges are odd. Dashed edges are even for $K_{2,4}e_e^j$ and are odd for $K_{2,4}e_o^j$. 50

Figure 3.1 Examples of signed graphs with maximum nullity at most two. Solid edges are even; dotted edges are odd; and dashed lines may be even or odd. . 78

Figure 4.1 A signed graph (G, Σ) with $Z(G, \Sigma) < Z(G)$. Even edges are solid, and odd edges are dashed. 113

LIST OF ABBREVIATIONS

SAP	Strong Arnold property
SNS	Sign nonsingular
w.r.t.	With respect to
Δ	The symmetric difference
^{T}	Matrix transpose
$^{-1}$	Matrix inverse
$\times$	Cartesian product
$\circ$	Hadamard product
$\oplus$	Direct sum
$\preceq$	Minor relationship
$\prec$	Proper minor relationship
$\leftrightarrow$	Adjacency relationship in graphs
$\rightarrow$	Adjacency relationship in directed graphs
$\delta(V)$	Edges incident on V
$\overline{S}$	The complement of S

$A/A_{2,2}$	The Schur complement of $A_{2,2}$ in A
$A[\alpha, \beta]$	The submatrix of A with rows indexed by α and columns by β
$A[\alpha]$	The submatrix of A with rows indexed by α and columns by α
$\mathrm{corank}(A)$	The dimension of the kernel of a matrix A
$d_G(v)$	The degree of a vertex v in the graph G
$d_G(a, b)$	The distance between the vertex a and b in the graph G
$\dim(U)$	The dimension of a subpace U
$G = (V, E)$	A graph with vertex set V and edge set E
(G, Σ)	A signed graph on G with signature Σ
$G[\alpha]$	The induced subgraph of G on the vertices α
K_n	The complete graph on n vertices
$K_{n,m}$	The complete bipartitie graph on n and m vertices
$K_{i,j,k}$	The complete tripartitie graph on i, j, and k vertices
$\ker(A)$	The kernel of a matrix A
$l(P)$	The length of the path P, $l(P) = \|E(P)\|$
$\mathcal{M}(n \times m)$	The family of matrices with n rows and m columns

rank(A)	The dimension of the range of a matrix A
$\mathcal{S}(G)$	A family of symmetric matrices defined by the simple graph G
$\mathcal{S}(G, \Sigma)$	A family of symmetric matrices defined by the signed graph (G, Σ)

CHAPTER 1

Background

We start with a graph. We place real numbers as weights on the vertices and place weights on the edges. Then, we know the eigenvalues for our problem. The inverse eigenvalue problem for a graph asks 'which graphs have the real numbers $\lambda_1, \ldots, \lambda_n$ as eigenvalues?' An instructive question to start is 'which graphs can achieve any set of real numbers as their eigenvalues?'

Observation 1.1. The empty graph on n vertices is the only simple graph which may achieve any n-tuple of real numbers as eigenvalues.

Proof. First, we consider a matrix A with eigenvalues all equal to 1. Then, A is similar to the identity matrix I. That is, for some nonsingular matrix V, we have

$$A = VIV^{-1} = VV^{-1} = I.$$

Therefore, the empty graph is the only simple graph which has eigenvalues all equal to 1.

Next, we consider any n-tuple of real numbers $\lambda_1, \ldots, \lambda_n$, and we place them along the diagonal of Λ, where Λ is a diagonal matrix. The eigenvalues of Λ are exactly our n-tuple of real numbers. As Λ is diagonal and as our graph is simple, the graph of Λ is the empty graph on n vertices. Therefore, the empty graph on n vertices may achieve any n-tuple of real numbers as eigenvalues. Because the empty graph is the only simple graph with all eigenvalues equal to 1, the empty graph is the unique simple graph which may achieve any n-tuple of real numbers as eigenvalues. $\square$

Observation 1.1 shows us that adding a single edge e to the empty graph limits which eigenvalues are possible. In general, the inverse eigenvalue problem for a graph is very challenging. Related problems include the multiplicity of eigenvalues, inertia sets, and minimum rank. In this dissertation, we are primarily interested in the category of signed graphs with the possibility of parallel edges, instead of the category of simple graphs. Unlike for a simple graph, the family of matrices for a signed graph is closed under addition, which feels more natural in certain settings. While many results carry over nicely from simple graphs to signed graphs, others do not. As an example, we may add an odd edge f and a parallel even edge e to the empty graph; and, the resulting signed graph shows that Observation 1.1 does not hold for signed graphs with multiple edges.

Here, we build upon previous work of Arav, Hall, Li, and van der Holst who characterized 2-connected signed graphs with maximum nullity at most two [4]. In 2007, Hogben and van der Holst found the forbidden minors for graphs with stable maximum nullity at most two [14]. The existence of forbidden minors is guaranteed by the Graph Minor Theorem of Robertson and Seymour, which proves Wagner's Conjecture that every infinite family of graphs has a finite number of forbidden minors [18]. Geelen, Gerards, and Whittle extended the Graph Minor Theorem to include signed graphs; so, we may find a finite number of forbidden minors for signed graphs with stable maximum nullity at most two [11]. Here, we extend the results of Hogben and van der Holst [14] to signed graphs by finding the forbidden minors of signed graphs with stable maximum nullity at most two. In 2009, Johnson, Loewy, and Smith provided a combinatorial characterization of graphs with maximum nullity at most two [16]. Here, we extend their result to signed graphs with maximum nullity at most two. In 2012, Row found the zero forcing number of graphs with maximum nullity at most two

[20]. Here, we extend this result by finding the zero forcing number of signed graphs with maximum nullity at most two. We also generalize the notion of zero forcing on signed graphs by finding new color change rules for signed graphs, which may be of interest outside the inverse eigenvalue problems.

1.1 Matrices

We consider the family of real matrices $\mathcal{M}(n \times m)$ with n rows and m columns. We write $A = [a_{i,j}] \in \mathcal{M}(n \times m)$ when we wish to detail the *entries* of A: the entry $a_{i,j}$ lies in the i-th row and j-th column. If $n = m$, then we write $\mathcal{M}(n \times n)$ for the family of *square matrices.* We may find a *submatrix* of $A \in \mathcal{M}(n \times m)$ which includes $\alpha \subseteq \{1, 2, \ldots, n\}$ rows and $\beta \subseteq \{1, 2, \ldots, m\}$ columns, denoted $A[\alpha, \beta]$. If $A \in \mathcal{M}(n \times n)$ and $\alpha = \beta$, then we write $A[\alpha]$ instead of $A[\alpha, \alpha]$ when convenient. Recall that we may consider $A \in \mathcal{M}(n \times n)$ as a linear transformation from the vector space $\mathbb{R}^n$ into $\mathbb{R}^n$. The *range* of A are all those vectors $y \in \mathbb{R}^n$ for which there exists a vector $x \in \mathbb{R}^n$ such that $y = Ax$. In general, we write $\dim(U)$ for the dimension of a linear subspace U of $\mathbb{R}^n$, and $\operatorname{rank}(A)$ is the dimension of the range of the matrix $A \in \mathcal{M}(n \times n)$. The $\dim(U)$ is also the number of vectors in a *basis* of U. The *kernel* of A are all those vectors $x \in \mathbb{R}^n$ such that $Ax = 0$, denoted $\ker(A)$. The $\ker(A)$ is also a subspace of $\mathbb{R}^n$.

Definition 1.2. Suppose $A \in \mathcal{M}(n \times m)$. We name the dimension of the $\ker(A)$ the *corank of* A, denoted $\operatorname{corank}(A)$.

Lemma 1.3. Suppose $A \in \mathcal{M}(n \times n)$, and $\operatorname{corank}(A) = k$ with $0 < k \leq n$. Let $\alpha \subset \{1, \ldots, n\}$ be an index set such $|\alpha| = k - 1$. Define $U \subseteq \mathbb{R}^n$ where $u \in U$ if and only if $u[\alpha] = 0$. Then, we may find a vector $x \in \ker(A) \cap U$ such that $x \neq 0$.

Proof. As U and $\ker(A)$ are subspaces of $\mathbb{R}^n$,

$$\dim\big(U \cap \ker(A)\big) = \dim(U) + \dim\big(\ker(A)\big) - \dim\big(U \cup \ker(A)\big).$$

As, $\dim(U \cup \ker(A)) \leq n$,

$$\dim(U \cap \ker(A)) \geq \dim(U) + \dim(\ker(A)) - n.$$

As $\dim(U) = n - (k-1)$ and $\dim(\ker(A)) = k$,

$$\dim(U \cap \ker(A)) \geq (n - k + 1) + k - n = 1.$$

As $\dim(U \cap \ker(A)) > 0$, we may find $x \in U \cap \ker(A)$ such that $x \neq 0$. $\square$

A matrix $D = [d_{i,j}] \in \mathcal{M}(n \times n)$ is a *diagonal matrix* if $i \neq j$ implies $d_{i,j} = 0$, and the *diagonal* of D has real entries $d_{i,i}$. The *identity matrix* I is a diagonal matrix with a diagonal of all ones. When convenient, we write I_n to clarify that $I_n \in \mathcal{M}(n \times n)$. A matrix $U \in \mathcal{M}(n \times n)$ is *real orthogonal* if $U^T U = U U^T = I$. For $A \in \mathcal{M}(n \times n)$, if there exists $B \in \mathcal{M}(n \times n)$ such that $AB = I$, then B is the *inverse* of A, and we write $B = A^{-1}$. If A^{-1} exists, we say that A is *full rank*.

The *complement* of a subset $\alpha \subseteq \{1, 2, \ldots, n\}$ is the subset $\overline{\alpha} = \{i \in \{1, 2, \ldots, n\} \mid i \notin \alpha\}$. We *partition* a matrix A with index sets α, β such that

$$A = \begin{bmatrix} A[\alpha, \beta] & A[\alpha, \overline{\beta}] \\ A[\overline{\alpha}, \beta] & A[\overline{\alpha}, \overline{\beta}] \end{bmatrix} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}.$$

The result is a 2×2 *block matrix*. In a similar process, we may partition A into an $n' \times m'$ block matrix so long as $n' \leq n$ and $m' \leq m$. A *block diagonal* matrix is a block matrix where $A_{i,j} = 0$ whenever $i \neq j$. The *direct sum* of two matrices A and B is the block diagonal matrix D where A and B appear along the diagonal, and we write $D = A \oplus B$.

A matrix $A = [a_{i,j}] \in \mathcal{M}(n \times n)$ is a *symmetric matrix* if $A = A^T$; that is, if $a_{i,j} =$

$a_{j,i}$ $\forall i, j \in \{1, \ldots, n\}$. An *eigenvalue* λ of A and an *eigenvector* $x \neq 0$ of A satisfy $Ax = \lambda x$. We limit our discussion of eigenvalues and eigenvectors to symmetric matrices. A symmetric matrix has only real eigenvalues. If A is a symmetric matrix, then there exists a real orthogonal matrix U such that UAU^T is a diagonal matrix. Further, the diagonal entries of UAU^T are the n eigenvalues of A. The *multiplicity* of an eigenvalue is the number of times the eigenvalue appears on the diagonal of UAU^T. For symmetric matrices, the multiplicity of zero as an eigenvalue of A is exactly the corank(A). If the symmetric matrix A has an eigenvalue λ with multiplicity k, then corank$(\lambda I - A) = k$.

Definition 1.4. Suppose we partition $A \in \mathcal{M}(n \times n)$ such that

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}.$$

If $A_{2,2}$ is full rank, then the *Schur complement* of $A_{2,2}$ in A is

$$A/A_{2,2} = A_{1,1} - A_{1,2}A_{2,2}^{-1}A_{2,1}.$$

When the matrix $A_{2,2}$ is a single nonzero entry $a_{n,n}$, then we denote $A/A_{2,2} = A/a_{n,n}$ when convenient.

Observation 1.5. If A is a symmetric matrix, then corank$(A) =$ corank$(A/A_{2,2})$

Proof. We may write A as a product of an upper triangular matrix, a block diagonal matrix, and a lower triangular matrix.

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} = \begin{bmatrix} I & A_{1,2}A_{2,2}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A/A_{2,2} & 0 \\ 0 & A_{2,2} \end{bmatrix} \begin{bmatrix} I & 0 \\ A_{2,2}^{-1}A_{2,1} & I \end{bmatrix}.$$

As the triangular matrices have non-zero diagonal, each is full rank. As A and $A/A_{2,2}$ are real symmetric, the corank(A) = corank$(A/A_{2,2} \oplus A_{2,2})$ = corank$(A/A_{2,2})$ + corank$(A_{2,2})$. As $A_{2,2}$ is full rank, corank$(A_{2,2}) = 0$. Therefore, corank(A) = corank$(A/A_{2,2})$. $\square$

The *Hadamard* product of $A = B \circ C$ is the entrywise product $a_{i,j} = b_{i,j}c_{i,j}$. The following definition is from Barioli, Fallat, and Hogben [6].

Definition 1.6. Suppose $A, X \in \mathcal{M}(n \times n)$. We say X *fully annihilates* A if

- $AX = 0$,
- $A \circ X = 0$, and
- $I \circ X = 0$.

1.2 Graphs

A *graph* is an ordered pair $G = (V, E)$ where V is the vertex set and E is the edge set. When convenient, the vertex set is assumed to be $V = \{1, 2, \ldots, n\}$. If a graph has more than one edge between a pair of vertices, then these are *parallel edges*; the graph is a *multigraph*; and the graph has *multiple edges*. If an edge $e = ii$ for some vertex i, then e is a *loop*. A graph with no multiple edges and no loops is a *simple graph*.

An edge e is *incident* on the two *endvertices* u, v if $e = uv$. Two vertices u, v are *adjacent* if there is an edge between them, denoted $u \leftrightarrow v$. The *degree* of a vertex v is the number of edges incident on v, denoted $d_G(v)$ or simply $d(v)$. A *pendant vertex* is a vertex with $d(v) = 1$. Similarly, a *pendant edge* is an edge incident on a pendant vertex. A *path* is an alternating sequence of vertices and edges, $v_1 e_1 v_2 e_2 \ldots e_{k-1} v_k$, where each vertex is unique; and P has *endvertices* v_1 and v_k. The *length* of a path P is the number of edges in P, denoted $l(P) = |E(P)| = k - 1$. The *distance* between two vertices $a, b \in V$ is $d_G(a, b) = \min\{l(P) : P \text{ has endvertices } a \text{ and } b\}$. Similarly, if $U, W \subseteq V$, $d_G(U, W) = \min\{d_G(a, b) : a \in U, b \in W\}$.

Definition 1.7. A k-separation in a graph G is an ordered pair (G_1, G_2) such that

- $|V(G_1) \cap V(G_2)| = k$,
- $V(G_1) \cup V(G_2) = V(G)$, and
- $E(G_1) \cup E(G_2) = E(G)$.

Definition 1.8. Let $G = (V, E)$ be a simple graph with a cut vertex v. Suppose $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are subgraphs of G. If

- $E_1 \neq \emptyset$,
- $E_2 \neq \emptyset$,
- $V = V_1 \cup V_2$,
- $E = E_1 \cup E_2$, and
- $V_1 \cap V_2 = \{v\}$;

then, G is the *1-sum* of G_1 and G_2 at v.

Notice that a 1-sum of G at v defines a particular 1-separation of G.

Definition 1.9. The *adjacency matrix* A of the simple graph $G = (V, E)$ has entries

- $a_{i,j} = 0$ if $ij \notin E$, and
- $a_{i,j} = 1$ if $ij \in E$.

If $V' \subseteq V$ and $E' \subseteq E$ where E' has endvertices in V', then $H = (V', E')$ is a *subgraph* of G. Further, H is an *induced subgraph* of G if E' includes all edges from E with endvertices in V'. That is, we may construct a subgraph by deleting edges and vertices from a graph; and, whenever we delete an endvertex v, we also delete all edges of the form uv for $u \in V$. We may define an induced subgraph more precisely in terms of the adjacency matrix. If $\alpha \subseteq \{1, 2, \ldots, n\}$ and A is the adjacency matrix of G, then $A[\alpha, \alpha]$ is the adjacency matrix of the induced subgraph H of G on the vertex set α, denoted $H = G[\alpha]$.

Definition 1.10. The *Laplacian matrix* A of the simple graph $G = (V, E)$ has entries

- $a_{i,j} = 0$ if $i \neq j$ and $ij \notin E$,
- $a_{i,j} = -1$ if $i \neq j$ and $ij \in E$, and
- $a_{i,j} = d(i)$ if $i = j$.

Definition 1.11. The *generalized Laplacian matrix* A of the simple graph $G = (V, E)$ has entries

- $a_{i,j} = 0$ if $i \neq j$ and $ij \notin E$, and
- $a_{i,j} < 0$ if $i \neq j$ and $ij \in E$.

Definition 1.12. Let $G = (V, E)$ be a simple graph. Define $\mathcal{S}(G)$ to be the family of matrices such that $A \in \mathcal{S}(G)$ has entries

- $a_{i,j} = 0$ if $i \neq j$ and $ij \notin E$, and
- $a_{i,j} \neq 0$ if $i \neq j$ and $ij \in E$.

So, the adjacency matrix, the Laplacian matrix, and the generalized Laplacian matrices all belong to $\mathcal{S}(G)$.

Definition 1.13. The *maximum nullity* of a simple graph G is

$$M(G) = \max_{A \in \mathcal{S}(G)} \{\operatorname{corank}(A)\}.$$

The following theorem rephrases the results of Fiedler about tridiagonal matrices in terms of simple graphs [10].

Theorem 1.14. G is a path if and only if $M(G) = 1$.

The *contraction* of an edge $uv \in E(G)$ results in a graph H where $V(H) = (V(G) \setminus \{u, v\}) \cup w$ and $E(H) = E(G) \setminus uv$, where the edges in $E(G)$ which were adjacent to $\{u, v\}$ in G are now adjacent to $w \in V(H)$. We say H is a *minor* of G if we may obtain H from G by a sequence of contracting edges, deleting edges, or deleting isolated vertices, denoted $H \preceq G$. If $H \preceq G$ and H is not isomorphic to G, then H is a *proper minor* of G, denoted $H \prec G$.

Definition 1.15. We say $A \in \mathcal{S}(G)$ has the *Strong Arnold Property* (SAP) with respect to the simple graph G if X fully annihilates A implies $X = 0$.

The following lemma is from van der Holst [22].

Lemma 1.16. Suppose $G = (V, E)$ is a simple graph, $|V| = n$, $A \in \mathcal{S}(G)$, and $\{y_1, y_2, \ldots, y_k\}$ is a basis for $\ker(A)$. Form the matrix U such that

$$U = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} = \begin{bmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_k^T \end{bmatrix}.$$

A has the SAP if and only if the matrices $\{u_i u_i^T \mid i \in V\}$ and $\{u_i u_j^T + u_j u_i^T \mid ij \in E\}$ form a basis for $\mathcal{M}(k \times k)$.

The following definition is from Barioli, Fallat, and Hogben [7].

Definition 1.17. The *stable maximum nullity* of a simple graph G is

$$\xi(G) = \max\{\text{corank}(A) \mid A \in \mathcal{S}(G), A \text{ has the SAP}\}.$$

1.3 Signed Graphs

A *signed graph* is an ordered pair (G, Σ), where G is a graph and $\Sigma \subseteq E$. We call Σ the *signature* of (G, Σ). Edges in Σ are *odd edges*, and edges in $E \setminus \Sigma$ are *even edges*. While we do not allow loops, we do allow multiple edges.

Definition 1.18. Let (G, Σ) be a signed graph. Define $\mathcal{S}(G, \Sigma)$ to be the family of matrices such that $A \in \mathcal{S}(G, \Sigma)$ has entries

- $a_{i,j} < 0$ if $i \leftrightarrow j$ and all edges between i and j are even,
- $a_{i,j} > 0$ if $i \leftrightarrow j$ and all edges between i and j are odd,
- $a_{i,j} \in \mathbb{R}$ if there as at least one even edge and at least one odd edge between i and j, and
- $a_{i,j} = 0$ if $i \neq j$ and there are no edges between i and j.

For signed graphs, $A, B \in \mathcal{S}(G, \Sigma)$ implies that $A + B \in \mathcal{S}(G, \Sigma)$; that is, $\mathcal{S}(G, \Sigma)$ is a cone. The corresponding statement is not true for simple graphs, because $A, -A \in \mathcal{S}(G)$.

Definition 1.19. The *maximum nullity* of a signed graph is

$$M(G, \Sigma) = \max\{\text{corank}(A) \mid A \in \mathcal{S}(G, \Sigma)\}.$$

Observation 1.20. Let (G, Σ) be a signed graph where G is a simple graph. If $A \in \mathcal{S}(G, \Sigma)$, then $A \in \mathcal{S}(G)$. So, $M(G, \Sigma) \leq M(G)$.

Definition 1.21. The matrix $A \in \mathcal{S}(G, \Sigma)$ has the *Strong Arnold Property* (SAP) with respect to (w.r.t) (G, Σ) if $X = 0$ is the only matrix such that

- $AX = 0$, and
- $x_{i,j} = 0$ if $i \not\leftrightarrow j$ or $i = j$.

Definition 1.22. The *stable maximum nullity* of a signed graph (G, Σ) is

$$\xi(G, \Sigma) = \max\{\text{corank}(A) \mid A \in \mathcal{S}(G, \Sigma), A \text{ has the SAP w.r.t. } (G, \Sigma)\}.$$

If (G, Σ) has no parallel edges of opposite sign, then $\xi(G, \Sigma) \leq \xi(G)$ because $\mathcal{S}(G, \Sigma) \subseteq \mathcal{S}(G)$. The following lemma is from Arav, Hall, Li, and van der Holst (Corollary 20 in [3]).

Lemma 1.23. If $(H, \Omega) \preceq (G, \Sigma)$, then $\xi(H, \Omega) \leq \xi(G, \Sigma)$.

If we *switch around* a vertex v, then the resulting signed graph is $(G, \Sigma\Delta\delta(v))$, where $\delta(v)$ are the edges incident on v and Δ is the symmetric difference. Similarly, if $U \subseteq V$, then we may also switch around U to obtain $(G, \Sigma\Delta\delta(U))$, where $\delta(U)$ are the edges between U and $V \setminus U$. Two signed graphs (G, Σ_1) and (G, Σ_2) are *switching equivalent* if there exists $U \subseteq V$ such that $\Sigma_2 = \Sigma_1\Delta\delta(U)$.

Lemma 1.24. Switching around vertices does not change the maximum nullity nor the stable maximum nullity of a signed graph.

Proof. Let $A \in \mathcal{S}(G, \Sigma)$. Let $U \subseteq V(G)$. Take D to be a diagonal matrix with $d_{i,i} = -1$ if $i \in U$; otherwise, $d_{i,i} = 1$. Define $B = DAD$.

For all $i, j \in V$, the entry $b_{i,j} = d_{i,i}a_{i,j}d_{j,j} = \pm a_{i,j}$. Further, $b_{i,j} = -a_{i,j}$ if and only if $ij \in \delta(U)$. Therefore, $B \in \mathcal{S}(G, \Sigma\Delta\delta(U))$. As D is real orthogonal, $\text{corank}(B) = \text{corank}(A)$. Suppose $\text{corank}(A) = M(G, \Sigma)$. Then, $M(G, \Sigma) = \text{corank}(B) \leq M(G, \Sigma\Delta\delta(U))$. Using the same argument, we may switch around U in $(G, \Sigma\Delta\delta(U))$ to obtain

$$M(G,\Sigma) \leq M(G,\Sigma\Delta\delta(U)) \leq M(G,\Sigma\Delta\delta(U)\Delta\delta(U)) = M(G,\Sigma\Delta\emptyset) = M(G,\Sigma).$$

Therefore, $M(G,\Sigma) = M(G,\Sigma\Delta\delta(U))$.

Suppose A has the SAP with respect to (G,Σ). Suppose for a contradiction that B does not have the SAP. Then, we may find $X \neq 0$ such that such that $BX = 0$ and $x_{i,j} = 0$ if $i \leftrightarrow j$ or $i = j$. Because $DADX = BX = 0$, we have $A(DXD) = 0$. That is, A does not have the SAP. Hence, A has the SAP if and only if B has the SAP. Therefore, $\xi(G,\Sigma) = \xi(G,\Sigma\Delta\delta(U))$. □

A subgraph of a signed graph is *odd* if it has an odd number of odd edges. In particular, an *odd cycle* in a signed graph is a cycle with an odd number of odd edges. Similarly, an *even cycle* is a cycle with an even number of odd edges.

Zaslavsky proved the following theorem about signed graphs (Proposition 3.2 in [23]).

Theorem 1.25. Two signed graphs (G,Σ_1) and (G,Σ_2) are switching equivalent if and only if they have the same set of odd cycles.

The following theorem is from Arav, Hall, Li, and van der Holst (Theorem 49 in [3]).

Theorem 1.26. A signed graph (G,Σ) has $\xi(G,\Sigma) \leq 1$ if and only if (G,Σ) is switching equivalent to $(H,\emptyset)$, where H is a graph whose underlying simple graph is a disjoint union of paths.

Later, we use the following corollary about trees, and we provide a proof here for completeness.

Corollary 1.27. If T is a tree, then $M(T, \Sigma) = M(T)$ for any signature Σ.

Proof. Let (T, Σ) be a signed graph where T is a tree. Suppose $A \in \mathcal{S}(T)$ with $\text{corank}(A) = M(T)$. Because $M(T) \geq M(T, \Sigma)$, we only need to show $M(T) \leq M(T, \Sigma)$.

Define the signature Ω_0 such that $A \in \mathcal{S}(T, \Omega_0)$. Denote $k = |\Omega_0 \Delta \Sigma|$. If $k = 0$, then our proof is complete. So, we may assume there exists an edge $e_0 \in \Omega_0 \Delta \Sigma$. If we define $\Omega_1 = \Omega_0 \Delta e_0$, then we may find a real orthogonal diagonal matrix D_0 such that $D_0 A D_0 \in \mathcal{S}(T, \Omega_1)$. As $|\Omega_1 \Delta \Sigma| = k - 1$, we may repeat this process for the sequence of edges $\{e_0, \ldots e_{k-1}\}$, and $\Omega_k = \Sigma$. Finally, we observe that

$$M(T) = \text{corank}(A) = \text{corank}(D_k \ldots D_0 A D_0 \ldots D_k) \leq M(T, \Sigma).$$

$\square$

From Theorem 1.14, we know that $M(G) = 1$ implies that G is a path. Together with Corollary 1.27 above, we also know that $M(G, \Sigma) = 1$ implies G is a path. This result is the prototype for our work: a complete characterization of the signed graphs with $M(G, \Sigma) = 1$.

A signed graph (H, Ω) is a *minor* of (G, Σ) if (H, Ω) may be obtained from (G, Σ) by a sequence of sign switchings, deleting edges, deleting isolated vertices, or contracting an edge. We write $(H, \Omega) \preceq (G, \Sigma)$ when (H, Ω) is a minor of (G, Σ). If $(H, \Omega) \preceq (G, \Sigma)$ but $G \neq H$, then (H, Ω) is a *proper minor* of (G, Σ), denoted $(H, \Omega) \prec (G, \Sigma)$.

Arav, Hall, Li, and van der Holst showed that $\xi(G, \Sigma) = 1$ implies that G is a disjoint union of paths [3]. Additionally, they proved the following theorem which we use later.

Lemma 1.28. Suppose (G, Σ) is a signed graph, $|V| = n$, $A \in \mathcal{S}(G, \Sigma)$, and $\{y_1, y_2, \ldots, y_k\}$ is a basis for $\ker(A)$. Form the matrix U such that

$$U = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} = \begin{bmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_k^T \end{bmatrix}.$$

A has the SAP with respect to (G, Σ) if and only if the matrices $\{u_i u_i^T \mid i \in V\}$ and $\{u_i u_j^T + u_j u_i^T \mid ij \in E\}$ form a basis for $\mathcal{M}(k \times k)$.

Lemma 1.29. Let (G, Σ) be a signed graph. Then, $M(G, \Sigma) = M(G, E \setminus \Sigma)$ and $\xi(G, \Sigma) = \xi(G, E \setminus \Sigma)$.

Proof. Let $A \in \mathcal{S}(G, \Sigma)$ such that $\operatorname{corank}(A) = M(G, \Sigma)$. Because $-A \in \mathcal{S}(G, E \setminus \Sigma)$ and because $\operatorname{corank}(A) = \operatorname{corank}(-A)$, $M(G, \Sigma) \leq M(G, E \setminus \Sigma)$. If $A \in \mathcal{S}(G, \Sigma)$ has the SAP with respect to (G, Σ), then $-A \in \mathcal{S}(G, E \setminus \Sigma)$ has the SAP with respect to $(G, E \setminus \Sigma)$. Otherwise, we found a matrix $X \neq 0$ such that $x_{i,j} = 0$ if $i \leftrightarrow j$ or $i = j$ and $-AX = 0$; yet, $AX \neq 0$. So, $\xi(G, E) \leq \xi(G, E \setminus \Sigma)$. Because $\Sigma = E \setminus (E \setminus \Sigma)$, a second application of our argument implies $M(G, \Sigma) \geq M(G, E \setminus \Sigma)$ and $\xi(G, \Sigma) \geq \xi(G, E \setminus \Sigma)$. Therefore, $M(G, \Sigma) = M(G, E \setminus \Sigma)$ and $\xi(G, \Sigma) = \xi(G, E \setminus \Sigma)$ $\square$

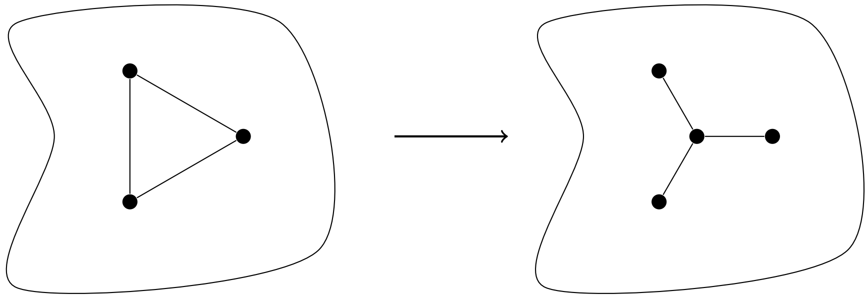


Figure 1.1 A ΔY-transformation on a simple graph.

1.4 ΔY-Transformations

Suppose G is a simple graph, and G has a triangle (K_3) which we label T. Then, we may perform a ΔY*-transformation* on G to obtain a new graph H by deleting the edges of T, adding a new vertex v, and adding edges between v and the vertices of T (Figure 1.4).

The following lemma from Hogben and van der Holst shows that ΔY-transformations do not decrease the stable maximum nullity of a graph (Lemma 2.1 in [14]).

Lemma 1.30. Let G be a simple graph, and let H be obtained from G by applying a ΔY-transformation. Then, $\xi(G) \leq \xi(H)$.

Their proof of this lemma relies on building a matrix in $\mathcal{S}(H)$ with the SAP, beginning with a matrix in $\mathcal{S}(G)$ which has the SAP. We adapt these proofs to signed graphs with triangles for the maximum nullity and stable maximum nullity. First, we provide the corresponding definition for signed graphs.

Definition 1.31. Suppose (G, Σ) is a signed graph, and (G, Σ) has a triangle which we label T. Then, we may perform a ΔY*-transformation* on (G, Σ) to obtain a new signed graph

(H, Ω) by deleting the edges of T, adding a new vertex v, and adding odd edges between v the vertices of T.

Lemma 1.32. Let (G, Σ) be a signed graph, and let (H, Ω) be obtained from G by applying a ΔY-transformation. Then, $M(G, \Sigma) \leq M(H, \Omega)$ and $\xi(G, \Sigma) \leq \xi(H, \Omega)$.

Proof. Let (G, Σ) be a signed graph on n vertices with a triangle T. For clarity, we assume that $V(T) = \{1, 2, 3\}$ and denote $\overline{V(T)} = V(G) \setminus V(T) = \{4, 5, \ldots, n\}$. First, we want to show that we may assume that T has no odd edges. If T is an odd cycle, then we may instead consider $(G, E(G) \setminus \Sigma)$ where T is an even cycle. By Lemma 1.29, $M(G, \Sigma) = M(G, E(G) \setminus \Sigma)$ and $\xi(G, \Sigma) = \xi(G, E(G) \setminus \Sigma)$. So, we assume that T is an even cycle. If T has two odd edges, then they must be incident on a vertex $t \in T$. Then, we may switch around t, and $M(G, \Sigma) = M(G, \Sigma \Delta \delta(t))$ and $\xi(G, \Sigma) = \xi(G, \Sigma \Delta \delta(t))$ by Lemma 1.24. As T has no odd edges in $(G, \Sigma \Delta \delta(t))$, we may assume T has no odd edges in (G, Σ) for the rest of the proof.

Let (H, Ω) to be the result of applying a ΔY-transformation on T in (G, Σ). Let $A \in \mathcal{S}(G, \Sigma)$. Then, we may partition A as

$$A = \begin{bmatrix} K + R & A[V(T), \overline{V(T)}] \\ A[\overline{V(T)}, V(T)] & A[\overline{V(T)}] \end{bmatrix},$$

where K and R describe the adjacency of T, as follows:

- $r_{i,j} < 0$ if ij is an edge in T,
- $k_{i,j} = 0$ if there are no edges between $\{i, j\}$ and $\overline{V(T)}$,
- $k_{i,j} > 0$ if there are only even edges between $\{i, j\}$ and $\overline{V(T)}$,

- $k_{i,j} < 0$ if there are only odd edges between $\{i, j\}$ and $\overline{V(T)}$, and
- $k_{i,j} \in \mathbb{R}$ if there is both an even edge and an odd edge between $\{i, j\}$ and $\overline{V(T)}$.

We want to construct a matrix in $\mathcal{S}(H, \Omega)$ with the same corank as A. Because the edges of T are even, $r_{1,2}r_{2,3}r_{1,3} < 0$. So, we want to find positive real numbers b, c, d such that $r_{1,2} = -bc$, $r_{1,3} = -bd$, and $r_{2,3} = -cd$. The solution follows from $b = \sqrt{-r_{1,2}r_{1,3}/r_{2,3}}$, $c = \sqrt{-r_{1,2}r_{2,3}/r_{1,3}}$, and $d = \sqrt{-r_{2,3}r_{1,3}/r_{1,2}}$. With these real numbers a, b, c we augment A with an $(n+1)$-th row and column, to construct a matrix B:

$$B = \begin{bmatrix} r_{1,1} + b^2 & k_{1,2} & k_{1,3} & A[1, \overline{V(T)} & b \\ k_{2,1} & r_{2,2} + c^2 & k_{2,3} & A[2, \overline{V(T)} & c \\ k_{3,1} & k_{3,2} & r_{3,3} + d^2 & A[3, \overline{V(T)} & d \\ A[\overline{V(T)}, 1] & A[\overline{V(T)}, 2] & A[\overline{V(T)}, 3] & A[\overline{V(T)}] & 0 \\ b & c & d & 0 & 1 \end{bmatrix}.$$

Then, the Schur complement of $b_{n+1,n+1}$ in B is:

$$B/b_{n+1,n+1} = \begin{bmatrix} r_{1,1} + b^2 & k_{1,2} & k_{1,3} & A[1, \overline{V(T)} \\ k_{2,1} & r_{2,2} + c^2 & k_{2,3} & A[2, \overline{V(T)} \\ k_{3,1} & k_{3,2} & r_{3,3} + d^2 & A[3, \overline{V(T)} \\ A[\overline{V(T)}, 1] & A[\overline{V(T)}, 2] & A[\overline{V(T)}, 3] & A[\overline{V(T)}] \end{bmatrix} - \begin{bmatrix} b^2 & bc & bd & 0 \\ cb & c^2 & cd & 0 \\ db & dc & d^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A.$$

From Lemma 1.5, $\operatorname{corank}(A) = \operatorname{corank}(B)$ because $B/b_{n+1,n+1} = A$. Further, $B \in \mathcal{S}(H, \Omega)$, so $M(G, \Sigma) \leq M(H, \Omega)$.

Next, we want to investigate ξ and the SAP. Let Y be a symmetric matrix such that $y_{i,j} = 0$ if $i \leftrightarrow j$ or $i = j$ and such that $BY = 0$. Then we may partition Y:

$$Y = \begin{bmatrix} 0 & y_{1,2} & y_{2,3} & Y[1, \overline{V(T)}] & 0 \\ y_{2,1} & 0 & y_{2,3} & Y[2, \overline{V(T)}] & 0 \\ y_{3,1} & y_{3,2} & 0 & Y[3, \overline{V(T)}] & 0 \\ Y[\overline{V(T)}, 1] & Y[\overline{V(T)}, 2] & Y[\overline{V(T)}, 3] & Y[\overline{V(T)}] & Y[\overline{V(T)}, n+1] \\ 0 & 0 & 0 & Y[n+1, \overline{V(T)}] & 0 \end{bmatrix}.$$

As $BY = 0$, the entries $y_{1,2} = y_{2,3} = y_{3,1} = 0$. Let $X \in \mathcal{M}(n \times n)$ such that

$$X = \begin{bmatrix} 0 & 0 & 0 & Y[1, \overline{V(T)}] \\ 0 & 0 & 0 & Y[2, \overline{V(T)}] \\ 0 & 0 & 0 & Y[3, \overline{V(T)}] \\ Y[\overline{V(T)}, 1] & Y[\overline{V(T)}, 2] & Y[\overline{V(T)}, 3] & Y[\overline{V(T)}] \end{bmatrix}.$$

Because $BY = 0$, $AX = 0$. Suppose A has the SAP. Because $x_{i,j} = 0$ if $i \leftrightarrow j$ or $i = j$ and because A has the SAP, we have $X = 0$. Because $X = 0$ and $BY = 0$, $Y[\overline{V(T)}, n+1] = 0$. Hence, $Y = 0$, and B has the SAP. Therefore, $\xi(G, \Sigma) \leq \xi(H, \Omega)$. $\square$

1.5 Zero Forcing

We start this section with a game on a simple graph $G = (V, E)$, taken from [1]. Before the game begins, all the vertices of G are colored white. First, we color some nonempty subset of vertices $B \subseteq V$ blue. Then, we apply the *color change rule*: if b is a blue vertex and if w is the only white vertex in the neighborhood of b, then we color w blue. We have a new set of blue vertices after applying the color rule once, $B^{(1)} = B \cup w$. We keep applying the color change rule until we may no longer color any white vertices blue. The game ends after $s < n$ steps. If $B^{(s)} = V$, then B is a *zero forcing set* of G. Denote the family of all zero forcing sets of G with $\mathcal{B}$. We say B is a *minimum zero forcing set* if $|B| = \min_{X \in \mathcal{B}} |X|$, and the *zero forcing number* of G is $Z(G) = |B|$.

Next, we tie zero forcing to the algebraic properties of $A \in \mathcal{S}(G)$. The following lemma is from the Special Graphs Workshop [1].

Lemma 1.33. For a simple graph G, $M(G) \leq Z(G)$.

Proof. Let $G = (V, E)$ be a simple graph. Suppose for a contradiction that $M(G) > Z(G)$. Then, we may find a zero forcing set B and a matrix $A \in \mathcal{S}(G)$ with $n \geq \operatorname{corank}(A) = M(G) > Z(G) = |B|$. As $\operatorname{corank}(A) > |B|$, from Lemma 1.3, we may find $x \in \ker A$ such that $x_b = 0$ for all $b \in B$, but $x \neq 0$.

Here, we begin an iterative argument using the size of B. As B is a zero forcing set and $|B| < n$, there is some blue vertex $v \in B$ which colors some white vertex $w \notin B$ when applying the color change rule: $w \in B^{(1)}$. As $x \in \ker A$, we have $\sum_{j=1}^{n} a_{v,j} x_j = 0$. Because $x_b = 0$ for all $b \in B$, we have $a_{v,w} x_w = 0$. As $vw \in E$, we have $a_{v,w} \neq 0$. So, $x_w = 0$. That is,

$x_b = 0$ for all $b \in B^{(1)}$. As B is a zero forcing set, $B^{(s)} = V$ after applying the color change rule s times. So, we may repeat the same argument until $x_b = 0$ for all $b \in B^{(s)} = V$. That is, $x = 0$. Because we assumed $x \neq 0$, we have our contradiction. Therefore, $M(G) \leq Z(G)$. $\square$

The proof of Lemma 1.33 clarifies the choice of the color change rule for simple graphs. We make use of a simple corollary, which follows directly from Lemma 1.33 and Observation 1.20.

Corollary 1.34. If (G, Σ) is a signed graph and G is a simple graph, then

$$\xi(G, \Sigma) \leq M(G, \Sigma) \leq M(G) \leq Z(G).$$

Definition 1.35. The *path covering number* of a graph G, denoted $P(G)$, is the minimum number of vertex-disjoint paths that cover $V(G)$, such that each path in the covering is an induced subgraph of G.

The following theorem is from Hogben (Theorem 2.13 in [13]) and follows from the fact that applying the color change rule to a zero forcing set constructs a path cover.

Theorem 1.36. For any graph G, $P(G) \leq Z(G)$.

Goldberg and Berman studied a variant of zero forcing for sign pattern matrices [12]. We will define sign pattern matrices in section 1.8, and we will use a different definition than Goldberg and Berman. For this section alone, a sign pattern matrix P is a matrix whose entries are from the set $\{-, 0, +, ?\}$. Their rules for zero forcing require the diagonal entries to have known signs (that is, from the set $\{-, 0, +\}$), which they call a *sign pattern* *matrix with fixed periphery*. So, their results are not perfectly applicable to our work, as $A \in \mathcal{S}(G, \Sigma)$ has no restriction on the diagonal entries. They derive new zero forcing rules for sign pattern matrices with fixed periphery and a zero forcing number $Z_{\pm}(P)$. We are interested in the zero forcing number

$$Z(G, \Sigma) = \min\{|\alpha| : \alpha \subseteq V(G), \forall A \in \mathcal{S}(G, \Sigma)\ \forall x \in \ker A, x[\alpha] = 0 \implies x = 0\}.$$

Similar to here, they define the maximum nullity $M(P) = \max\{\operatorname{corank}(A) \mid \operatorname{sign}(A) = P\}$. Despite these differences, we can apply one result from their work (Theorem 3.2 and Rule 2 in [12]):

Lemma 1.37. If P is a sign pattern matrix, then $M(P) \leq Z_{\pm}(P)$.

Translating this to our work, we have the following lemma.

Lemma 1.38. If (G, Σ) is a signed graph, then $M(G, \Sigma) \leq Z(G, \Sigma) \leq Z(G)$.

Proof. Let (G, Σ) be a signed graph. Let P be a sign pattern matrix with fixed periphery such that (G, Σ) is the signed graph of P and $M(P) = M(G, \Sigma)$. If $\operatorname{sign}(A) = P$, then $A \in \mathcal{S}(G, \Sigma)$; hence $Z_{\pm}(P) \leq Z(G, \Sigma)$. Because $\mathcal{S}(G, \Sigma) \subseteq \mathcal{S}(G)$, we have $Z(G, \Sigma) \leq Z(G)$. From Lemma 1.37, we have $M(P) \leq Z_{\pm}(P)$. Hence, we may write

$$M(G, \Sigma) \leq M(P) \leq Z_{\pm}(P) \leq Z(G, \Sigma) \leq Z(G).$$

□

1.6 Graph Structures and Maximum Nullity

This section details combinatorial structures of graphs and signed graphs related to the maximum nullity or stable maximum nullity.

1.6.1 1-Separations

Originally in terms of the partial inertia sets of a signed graph allowing loops, the following is from Arav, van der Holst, and Sinkovic (Formula (3) in [5]).

Lemma 1.39. Let $[(G_1,\Sigma_1),(G_2,\Sigma_2)]$ be a 1-separation of a signed graph (G,Σ) with $v = V(G_1)\cap V(G_2)$. Then,

$$M(G,\Sigma) = \max\left\{M(G_1,\Sigma_1)+M(G_2,\Sigma_2)-1, M((G_1,\Sigma_1)-v)+M((G_2,\Sigma_2)-v)-1\right\}.$$

Proof. Let $|V(G)| = n$. Taking Formula (3) in [5], we have

$$\begin{aligned}\min\left\{\mathcal{I}(G,\Sigma)\right\} = \min\Big\{&\Big(\mathcal{I}\big((G_1,\Sigma_1)-v\big)+\mathcal{I}\big((G_2,\Sigma_2)-v\big)+\{(1,1)\}\Big)\\ &\cup\Big(\mathcal{I}(G_1,\Sigma_1)+\mathcal{I}(G_1,\Sigma_1)\Big)\Big\},\end{aligned}$$

where $\mathcal{I}(G,\Sigma)$ are all possible ordered pairs (p,q) such that there exists $A\in\mathcal{S}(G,\Sigma)$, A has p positive eigenvalues, and A has q negative eigenvalues. As $|V(G)-M(G,\Sigma)|$ is the sum of the number of positive eigenvalues and the number of negative eigenvalues, we have

$$\begin{aligned}n-M(G,\Sigma) = \min\{&|V(G_1-v)|+|V(G_2-v)|-M((G_1,\Sigma_1)-v)-M((G_2,\Sigma_2)-v)+2,\\ &|V(G_1)|+|V(G_2)|-M(G_1,\Sigma_1)-M(G_2,\Sigma_2)\}.\end{aligned}$$

Because $[(G_1,\Sigma_1),(G_2,\Sigma_2)]$ is a 1-separation, we count each vertex of $G-v$ exactly one time in $|V(G_1-v)|+|V(G_2-v)|$. Similarly, we count v twice and all other vertices of G once in $|V(G_1)|+|V(G_2)|$. So, we have

$$\begin{aligned} n - M(G,\Sigma) = \min\{&n-1-M((G_1,\Sigma_1)-v)-M((G_2,\Sigma_2)-v)+2, \\ &n+1-M(G_1,\Sigma_1)-M(G_2,\Sigma_2)\}. \end{aligned}$$

Now, we simplify to obtain our result:

$$\begin{aligned} -M(G,\Sigma) &= \min\{-M((G_1,\Sigma_1)-v)-M((G_2,\Sigma_2)-v)+1, -M(G_1,\Sigma_1)-M(G_2,\Sigma_2)+1\}. \\ M(G,\Sigma) &= \max\{M((G_1,\Sigma_1)-v)+M((G_2,\Sigma_2)-v)-1, M(G_1,\Sigma_1)+M(G_2,\Sigma_2)-1\}. \end{aligned}$$

□

1.6.2 Trees and Forests

Barioli, Fallat, and Hogben proved the following theorem about forests (Theorem 3.7 in [7]).

Theorem 1.40. If F is a forest, then $\xi(F) \leq 2$.

The following Theorem is the main result of Johnson and Duarte and relates the path covering number to the maximum nullity for trees [15].

Theorem 1.41. If T is a tree, then $M(T) = P(T)$.

1.6.3 Linear 2-Trees

We may iteratively construct a tree on $n > 2$ vertices, as follows. We start with a K_2, two vertices joined by a single edge. Then, we grow the tree by identifying one of the vertices in the graph with one of the vertices of a second copy of K_2. So, we have a P_3. We continue identifying vertices of our graph with a vertex of a new copy of K_2 until we have our tree.

Instead of using copies of K_2 to build a 1-tree, we may use copies of K_{k+1} to build a *k-tree* using the same process. At each step, we identify a K_k in the graph with a K_k in the new copy of K_{k+1}. In particular, we are interested in 2-trees, built by identifying a K_2 in the graph with a K_2 in a new copy of a triangle. A *2-path* is a 2-tree whose dual is a path. A *partial 2-path* is a subgraph of a 2-path. A *linear 2-tree* is a 2-connected partial 2-path, introduced by Hogben and van der Holst in [14] and under another name by Johnson, Loewy, and Smith in [16]. Sinkovic proved the following theorem (Theorem 3.13 in [21]).

Theorem 1.42. If G is a partial 2-path, then $M(G) = P(G)$.

1.6.4 Graph of Two Parallel Paths

Johnson, Loewy, and Smith provided two equivalent definitions of a graph of two parallel paths [16]. First, we provide their definition using matrices from $\mathcal{S}(G)$.

Definition 1.43. A graph G is *a graph of two parallel paths* if there exists $A \in \mathcal{S}(G)$ such that

$$A = \begin{bmatrix} T_1 & B \\ B^T & T_2 \end{bmatrix},$$

where T_1 and T_2 are irreducible and tridiagonal. Further, B satisfies the following:

- If $b_{i,j} \neq 0$, then $b_{k,l} = 0$ for $k > i$ and $l < j$, and for $k < i$ and $l > j$.

- If $B \neq 0$ and $b_{k_1,k_1+1} \neq 0$, then B has a nonzero entry other than b_{k_1,k_1+1}.

Their other definition of a graph of two parallel paths uses the existence of a particular embedding in the plane [16]. Specifically, a graph G is a graph of two parallel paths if we may draw G such that

- the two paths P_1 and P_2 cover the vertices of G,

- P_1 and P_2 are independent, induced paths,

- any edges between P_1 and P_2 do not cross, and

- the vertices of G are all drawn on the infinite face.

Together with a family of exceptional graphs, Johnson, Loewy, and Smith showed the graphs of two parallel paths are the only graphs with $M(G) = 2$ [16]. They used the following lemma in the proof of their main result (Lemma 3.7 in [16]).

Lemma 1.44. Suppose G is a graph of two parallel paths. Then, $M(G) = 2$.

The following theorem from Row fully characterizes graphs with a zero forcing number of two (Theorem 2.3 in [20]).

Theorem 1.45. For a simple graph G, $Z(G) = 2$ if and only if G is a graph of two parallel paths.

1.6.5 Thin Outs

The *blocks* of a 1-connected simple graph G are the maximal connected subgraphs with no cut vertex. At one extreme, a 2-connected graph has only a single block. At the other extreme, every K_2 of a tree is a block. A *thin out* of G is a block B of G with a pendant vertex added to each vertex of B which is also a cut vertex of G. At one extreme, a 2-connected graph is isomorphic to its own thin out. At the other extreme, every thin out of a tree with at least 3 vertices is either a P_3 or a P_4. The following lemma is from Arav, Hall, Li, and van der Holst (Corollary 42 in [3]).

Lemma 1.46. If (G, Σ) is the disjoint union of (G_1, Σ_1) and (G_2, Σ_2), then

$$\xi(G, \Sigma) = \max\left\{\xi(G_1, \Sigma_1), \xi(G_2, \Sigma_2)\right\}.$$

The following lemma follows from a result of Arav, Hall, Li, and van der Holst, originally phrased in terms of inertia sets (Theorem 43 in [3]).

Lemma 1.47. Let (G, Σ) be a connected signed graph and suppose (G, Σ) is the 1-sum of (G_1, Σ_1) and (G_2, Σ_2) at v, with both $E(G_1)$ and $E(G_2)$ nonempty. For $i = 1, 2$, let (H_i, Ω_i) be the signed graph obtained from G_i and K_2 by identifying the vertex v with a vertex of K_2. Then,

$$\xi(G, \Sigma) = \max\left\{\xi(H_1, \Omega_1), \xi(H_2, \Omega_2)\right\}.$$

Lemma 1.48. Let (G, Σ) be a signed graph with $\xi(G, \Sigma) \geq 3$. Then, there exists a 2-connected block B of G such that a thin out (H, Ω) of B in (G, Σ) satisfies $\xi(H, \Omega) \geq 3$.

Proof. Let (G, Σ) be a signed graph with $\xi(G, \Sigma) \geq 3$. By Lemma 1.46, we may assume (G, Σ) is connected. If (G, Σ) is 2-connected, then (G, Σ) is exactly one block. So, we may assume that G has at least 1 cut vertex v. If $v \in V(G_1)$ for some block of G, then we found a 1-sum (G_1, Σ_1) and (G_2, Σ_2) at v. If we identify a vertex of K_2 with v in (G_i, Σ_i), then we obtain the signed graph (H_i, Ω_i) for $i = 1, 2$. By Lemma 1.47, we may assume that $\xi(H_1, \Omega_1) \geq 3$. We repeat this process for a new cut vertex of H_1 until we have our thinout (H, Ω). $\square$

1.6.6 T_3*-Family of Graphs*

Hogben and van der Holst introduced the T_3-family to fully characterize the graphs with $\xi(G) \leq 2$ [14]. They defined the T_3 graph as the result of deleting the edges of a triangle from $K_{2,2,2}$. The graphs $T_3(\Delta Y)^i$ are the result of applying a series of i ΔY-transformations to T_3. The T_3-*family* includes K_4, $K_{2,3}$, and $T_3(\Delta Y)^i$ (Figure 1.2).

The following theorem and corollary are some of the main results of Hogben and van der Holst [14].

Theorem 1.49. Let $G = (V, E)$ be a simple graph. Then, $\xi(G) \leq 2$ if and only if G has no minor isomorphic to a graph in the T_3-family.

Corollary 1.50. Let $G = (V, E)$ be a 2-connected simple graph on n vertices. The following are equivalent:

- $\xi(G) = 2$.

- $M(G) = 2$.

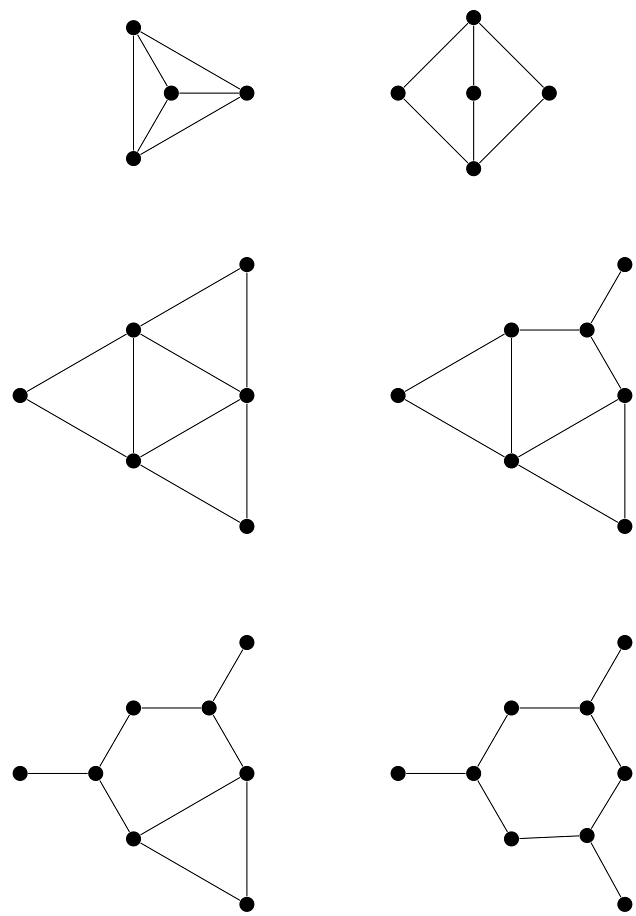


Figure 1.2 The T_3-family from Hogben and van der Holst [14].

- G has no K_4-minor, no $K_{2,3}$-minor, nor T_3-minor.

- G is a linear 2-tree.

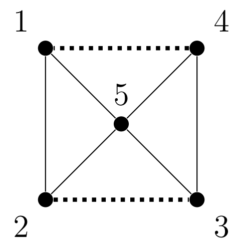


Figure 1.3 The odd 4-wheel. Solid edges are even, and dotted edges are odd.

1.7 2-Connected Signed Graphs with $M(G, \Sigma) \leq 2$

The main result of Arav, Hall, Li, and van der Holst fully characterizes 2-connected signed graphs with $M(G) \leq 2$ [4]. First, we need to define several signed graphs. The *odd 4-wheel* or W_4^o is drawn in Figure 1.3. We denote the signed graph $(K_n, \emptyset)$ by K_n^e; the signed graph $(K_n, E(K_n))$ by K_n^o; $(K_{2,3}, \emptyset)$ by $K_{2,3}^e$; and $(K_4, \{e\})$ for a single $e \in E(K_4)$ by K_4^i. By $K_n^=$, we denote the signed graph on n vertices where there is exactly one even edge and exactly one odd edge between each pair of vertices. The following lemma is from Arav, Hall, Li, and van der Holst (Lemmas 1–4 in [4]), and it follows more general results in their previous work [3].

Lemma 1.51.

1. $M(K_n^=) = \xi(K_n^=) = n$,

2. $M(K_n^e) = \xi(K_n^e) = n - 1$,

3. $M(K_4^i) = \xi(K_4^i) = 2$,

4. $M(K_{2,3}^e) = \xi(K_{2,3}^e) = 3$, and

5. $M(K_{2,3}^i) = \xi(K_{2,3}^i) = 2$.

Next, we need to define a class of signed graphs which we construct similarly to a linear 2-tree. A pair of edges $\{e, f\} \in E(K_4)$ are *split* if both e and f belong to an even and odd triangle in K_4^i. We construct a *sided wide 2-path* $[(G, \Sigma), \mathcal{F}]$ recursively:

1. Let (G, Σ) be an even cycle, an odd cycle, or a K_4^i.

   (a) If (G, Σ) is a cycle, let $\mathcal{F}$ be two distinct edges in G.

   (b) If (G, Σ) is a K_4^i, let $\mathcal{F}$ be a split pair of edges in K_4^i.

   Then, $[(G, \Sigma), \mathcal{F}]$ is a sided wide 2-path.

2. Let $[(G, \Sigma), \mathcal{F}]$ be a sided wide 2-path. Let e and f be distinct edges in an even or odd cycle C. If (H, Ω) is obtained from (G, Σ) by identifying the edge f of C with an edge h in $\mathcal{F}$, then $[(H, \Omega), (F - h) \cup e]$ is a sided wide 2-path.

3. Let $[(G, \Sigma), \mathcal{F}]$ be a sided wide 2-path. Let $\{e, f\}$ be a split pair in K_4^i. If (H, Ω) is obtained from (G, Σ) by identifying the edge f of K_4^i with an edge h in $\mathcal{F}$, then $[(H, \Omega), (F - h) \cup e]$ is a sided wide 2-path.

For a sided wide 2-path $[(G, \Sigma), \mathcal{F}]$, the edges in $\mathcal{F}$ are the *sides* of $[(G, \Sigma), \mathcal{F}]$. For a signed graph (G, Σ), if there exists a set $\mathcal{F}$ of two distinct edges such that $[(G, \Sigma), \mathcal{F}]$ is a sided wide 2-path, then (G, Σ) is a *wide 2-path.* A *partial wide 2-path* is a spanning subgraph of a wide 2-path. We note that if G is a partial 2-path, then (G, Σ) is a partial wide 2-path.

Now, we may state the main result of Arav, Hall, Li, and van der Holst [4].

Theorem 1.52. Let (G, Σ) be a 2-connected signed graph. Then, the following are equivalent:

- $M(G, \Sigma) \leq 2$.
- $\xi(G, \Sigma) \leq 2$.
- (G, Σ) has no minor isomorphic to $K_3^{=}$, K_4^e, K_4^o, or $K_{2,3}^e$.
- (G, Σ) is a partial wide 2-path or is isomorphic to W_4^o.

1.8 Sign-Nonsingular Matrices

We take this section largely from Brualdi and Shader's text [8]. A *sign pattern matrix* is a matrix with entries from the set $\{0, +, -\}$. The *qualitative class* of an $n \times m$ sign pattern matrix S is

$$\mathcal{Q}(S) = \Big\{ A \in \mathcal{M}(n \times m) : \text{sign}(a_{i,j}) = s_{i,j} \ \forall i \in \{1, \dots, n\},\ \forall j \in \{1, \dots, m\} \Big\}.$$

If for every matrix $A \in \mathcal{Q}(S)$, A has independent rows and independent columns, then S is a *sign-nonsingular matrix*, abbreviated SNS-matrix. That is, the linear system $Ax = b$ has a unique solution if $A \in \mathcal{Q}(S)$ and S is a SNS matrix. A SNS-matrix is a *maximal SNS-matrix* if changing any 0 to either + or − results in a sign pattern matrix which is not a SNS-matrix.

A *directed graph* or *digraph* is an ordered pair $D = (V, E)$, where V are the vertices and E are the directed edges. Here, we do not allow loops. The directed edges are ordered pairs from the Cartesian product $V \times V$. If $ij \in E$, then we say i is incident on j and write $i \rightarrow j$. A *signed digraph* is a digraph where we label each directed edge with one sign from the set $\{+, -\}$. If S is a sign pattern matrix, then $D(S)$ is the signed digraph of S, a digraph where the directed edge ij is labeled with the sign $s_{i,j}$ whenever $s_{i,j} \neq 0$. A *directed cycle* in a digraph is a sequence of directed edges such that

$$i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_k \rightarrow i_1,$$

and the vertices $i_1, i_2, \dots, i_k$ are all distinct. The sign of a directed cycle $i_1 \rightarrow i_2 \rightarrow \dots i_k \rightarrow i_1$ is the product

$$\text{sign}(i_1 i_2)\text{sign}(i_2 i_3) \cdots \text{sign}(i_{k-1} i_k)\text{sign}(i_k i_1).$$

The following theorem defines the relationship between signed digraphs and SNS matrices (Theorem 3.2.1 in [8]).

Theorem 1.53. Let S be a square sign pattern matrix with negative diagonal entries, that is $s_{i,i} = -$ for all i. Then, S is a SNS-matrix if and only if every directed cycle of the signed digraph $D(S)$ is negative.

We illustrate an application of this theorem to a maximal SNS-matrix (6.5 in [8]),

$$\begin{bmatrix} - & + & 0 \\ - & - & + \\ - & - & - \end{bmatrix}, \tag{1.1}$$

with Figure 1.4. The sign pattern in (1.1) also corresponds to a signed graph as shown in Figure 1.4, as follows. The rows of (1.1) index the vertices u_i, and the columns index the vertices v_j. We place an odd edge between u_i and v_j if the corresponding entry of (1.1) is positive, and an even edge if the corresponding entry is negative. We may also construct the sign pattern in (1.1) from our bipartite signed graph, using a method originally from Little, as follows [17]. First, we switch around vertices so that we have a perfect matching with odd edges. Then, we label our edges with $+$ if the edge is odd and $-$ if the edge is even. Next, we direct each edge in our signed graph from the vertices u_i to the vertices v_j. Finally, we contract the edges in our perfect matching. We will use this construction to obtain zero forcing rules on signed graphs.

Little found a characterization of SNS-matrices [17], but first we need a definition. We may *subdivide* a graph G by replacing a $P_2 = ue_0w$ with a $P_3 = ue_1ve_2w$ to obtain a *subdivision of* G, the resulting graph H. If we subdivide each edge an even number of times,

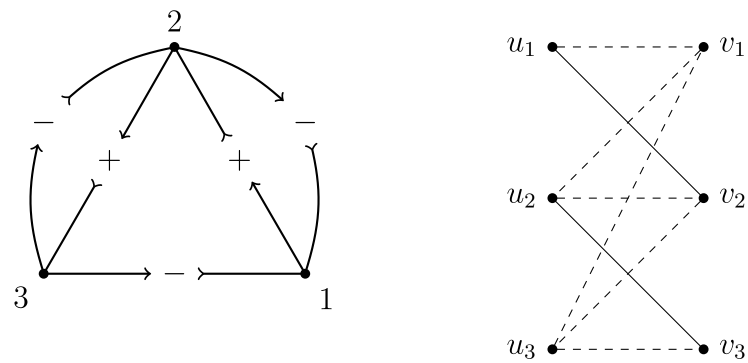


Figure 1.4 The signed digraph of the maximal SNS-matrix of order 3 and the corresponding signed graph, where odd edges are dashed and even edges are solid.

then H is an *even subdivision* of G. Although originally a result of Little, we also rely on results of Robertson, Seymour, and Thomas to state the following theorem [17, 19].

Theorem 1.54. Let S be a sign pattern matrix. Let (G, Σ) be the bipartite signed graph corresponding to S. Suppose G has a perfect matching. Then, $D(S)$ has no positive directed cycle if and only if G has no even subdivision of $K_{3,3}$.

CHAPTER 2

Signed Graphs with Stable Maximum Nullity at Most Two

This chapter contains the results of Arav, Dahlgren, and van der Holst [2]. The following theorem is the main result of this chapter. The last section of this chapter contains the proof.

Theorem 2.1. A signed graph (G,Σ) has $\xi(G,\Sigma) \leq 2$ if and only if (G,Σ) has no minor isomorphic to K_4^e, K_4^o, or a signed graph in the $K_3^=$-family.

The above theorem extends the result of Hogben and van der Holst, where they found the forbidden minors for graphs with stable maximum nullity at most two [14].

2.1 The Signed Four-Wheel

Lemma 2.2. Let (G, Σ) be a signed graph with a pendant vertex v, such that $\xi(G, \Sigma) = k$. Suppose $A \in \mathcal{S}(G, \Sigma)$ has the SAP, and $\operatorname{corank}(A) = k$. If $a_{v,v} \neq 0$, then $\xi(G - v, \Sigma \setminus \delta(v)) = k$.

Proof. Let (G, Σ) be a signed graph with a pendant vertex v. Denote with $(H, \Omega) = (G - v, \Sigma \setminus \delta(v))$. Suppose there exists a matrix $A \in \mathcal{S}(G, \Sigma)$ such that $\operatorname{corank}(A) = \xi(G, \Sigma) = k$ and $a_{v,v} \neq 0$. Suppose A has the SAP with respect to (G, Σ). Write A as

$$A = \begin{bmatrix} a_{v,v} & A[v, \overline{v}] \\ A[\overline{v}, v] & A[\overline{v}] \end{bmatrix}.$$

The Schur complement of $a_{v,v}$ in A is

$$B = A/a_{v,v} = A[\overline{v}] - a_{v,v}^{-1} A[\overline{v}, v] A[v, \overline{v}].$$

From Observation 1.5, $\operatorname{corank}(B) = k$. As $A[\overline{v}, v]A[v, \overline{v}]$ is zero except for one diagonal entry and as $A[\overline{v}] \in \mathcal{S}(H, \Omega)$, we also have $B \in \mathcal{S}(H, \Omega)$.

Suppose for a contradiction that B does not have the SAP with respect to (H, Ω). Then, there exists a non-zero real symmetric matrix X such that $BX = 0$ and $x_{i,j} = 0$ if $i \leftrightarrow j$ or $i = j$. Take Y such that

$$Y = \begin{bmatrix} 0 & -a_{v,v}^{-1} A[v, \overline{v}] X \\ -a_{v,v}^{-1} X A[\overline{v}, v] & X \end{bmatrix}.$$

As X has a zero diagonal and $A[v, \overline{v}]$ has one non-zero element, $A[v, \overline{v}] X A[\overline{v}, v] = 0$. Further, $A[\overline{v}] X A[\overline{v}, v] = BXA[\overline{v}, v] + a_{v,v}^{-1} A[\overline{v}, v] A[v, \overline{v}] X A[\overline{v}, v] = 0$. So,

$$AY = \begin{bmatrix} a_{v,v} & A[v, \overline{v}] \\ A[\overline{v}, v] & A[\overline{v}] \end{bmatrix} \begin{bmatrix} 0 & -a_{v,v}^{-1} A[v, \overline{v}] X \\ -a_{v,v}^{-1} X A[\overline{v}, v] & X \end{bmatrix}$$

$$= \begin{bmatrix} -a_{v,v}^{-1}A[v,\overline{v}]XA[\overline{v},v] & -A[v,\overline{v}]X + A[v,\overline{v}]X \\ -a_{v,v}^{-1}A[\overline{v}]XA[\overline{v},v] & BX \end{bmatrix} = 0.$$

Because X has a zero diagonal, by construction $y_{i,j} = 0$ if $i = j$. As X has a zero diagonal, $A[v,\overline{v}]X$ has a zero corresponding to the pendant edge of v: $y_{v,j} = 0$ if $v \leftrightarrow j$. Because $x_{i,j} = 0$ if $i \leftrightarrow j$ for the vertices of H, we have $y_{i,j} = 0$ if $i \leftrightarrow j$. That is, A does not have the SAP with respect to (G, Σ). Thus, we have our contradiction, and B has the SAP with respect to (H, Ω). Hence, $\xi(H, \Omega) \geq k$. As (H, Ω) is a minor of (G, Σ), $\xi(H, \Omega) \leq k$. Finally, $\xi(G - v, \Sigma \setminus \delta(v)) = k$.

□

Lemma 2.3. Let (G, Σ) be a W_4^o with single pendant edges attached to some of the vertices of W_4^o. Then, $\xi(G, \Sigma) = 2$.

Proof. We may take $A \in \mathcal{S}(G, \Sigma)$ which has the SAP and corank$(A) = \xi(G, \Sigma)$. By Lemma 2.2, the entry $a_{v,v} = 0$ whenever v is a pendant vertex. If (G, Σ) has no pendant vertices, then $(G, \Sigma) = W_4^o$. From Theorem 1.52, corank$(A) = \xi(W_4^o) \leq 2$. If (G, Σ) has exactly 1 pendant vertex v adjacent to u, then we may take the the matrix B to be the Schur complement of the submatrix with columns and rows $\{u, v\}$ in $A \in \mathcal{S}(G, \Sigma)$. The graph of B is either a 4-cycle or a 4-cycle with a chord edge, and corank(A) = corank$(B) \leq 2$. If (G, Σ) has more than 1 pendant vertex, then we may sequentially apply the Schur complement as in the previous case to obtain the matrix B. The graph of B is a subgraph of either a 4-cycle or a 4-cycle with a chord edge, and corank(A) = corank$(B) \leq 2$. For all our cases, corank$(A) \leq 2$ and A has the SAP, so $\xi(G, \Sigma) \leq 2$. Because (G, Σ) has an odd cycle, Theorem 1.26 implies

$\xi(G, \Sigma) \geq 2$. Therefore, $\xi(G, \Sigma) = 2$. $\square$

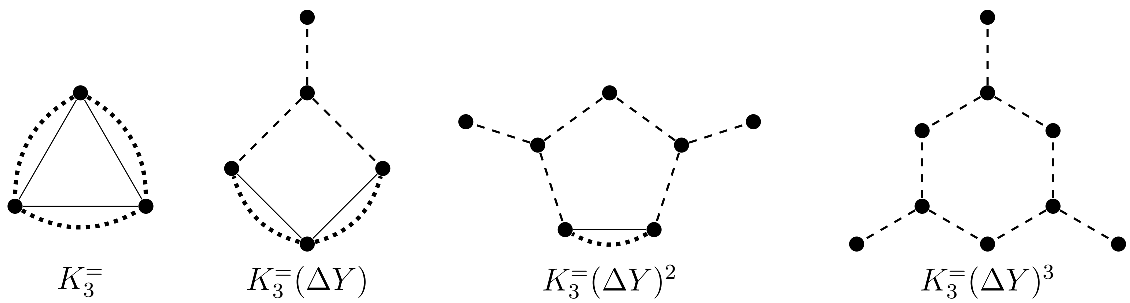


Figure 2.1 The $K_3^=$-family of signed graphs. Solid edges are even; dotted edges are odd; and dashed lines may be odd or even.

2.2 The $K_3^=$-Family of Signed Graphs

Starting with $K_3^=$, we construct the $K_3^=$-family of signed graphs by repeating the following: selecting a pair of parallel edges, subdividing one of these parallel edges, assigning the resulting two edges any sign, and applying a ΔY-transformation on the the resulting triangle. Figure 2.1 depicts the members of the $K_3^=$-family. When convenient, we keep track of the members of the $K_3^=$-family by the number of ΔY-transformations: $K_3^=$, $K_3^=(\Delta Y)$, $K_3^=(\Delta Y)^2$, and $K_3^=(\Delta Y)^3$.

Lemma 2.4. Every member (G, Σ) of the $K_3^=$-family has $\xi(G, \Sigma) = 3$.

Proof. From Theorem 1.52, we know $\xi(K_3^=) \geq 3$ because $K_3^=$ is 2-connected. By definition, every (G, Σ) in the $K_3^=$-family may be formed from $K_3^=$ by a sequence of the following on pairs of multiple edges: subdivide an edge and perform a ΔY-transformation on the subsequent triangle. As a signed graph is a proper minor of its own subdivision, Lemmas 1.32 and 1.23 imply $\xi(K_3^=(\Delta Y)^3) \geq \xi(G, \Sigma) \geq \xi(K_3^=) \geq 3$.

Consider the simple graph H associated with $K_3^=(\Delta Y)^3$. We construct a zero forcing set

B to show $Z(H) \leq 3$. First, we select two pendant vertices u and v. Next, we select the vertex w along the shortest path from u to v with $d(w) = 2$. So, $B = \{u, v, w\}$ is a zero forcing set for H. From Corollary 1.34, we have $\xi(K_3^{=}(\Delta Y)^3) \leq Z(H) \leq |B| = 3$. Because $3 \leq \xi(G, \Sigma) \leq \xi(K_3^{=}(\Delta Y)^3)$, we conclude $\xi(G, \Sigma) = 3$. □

The following corollary is immediate from Lemma 2.4

Corollary 2.5. If (G, Σ) has a minor isomorphic to a member of the $K_3^{=}$-family, then $M(G, \Sigma) \geq \xi(G, \Sigma) \geq 3$.

Lemma 2.6. If (G, Σ) is a member of the $K_3^{=}$-family and $(H, \Omega) \prec (G, \Sigma)$, then $\xi(H, \Omega) < 3$.

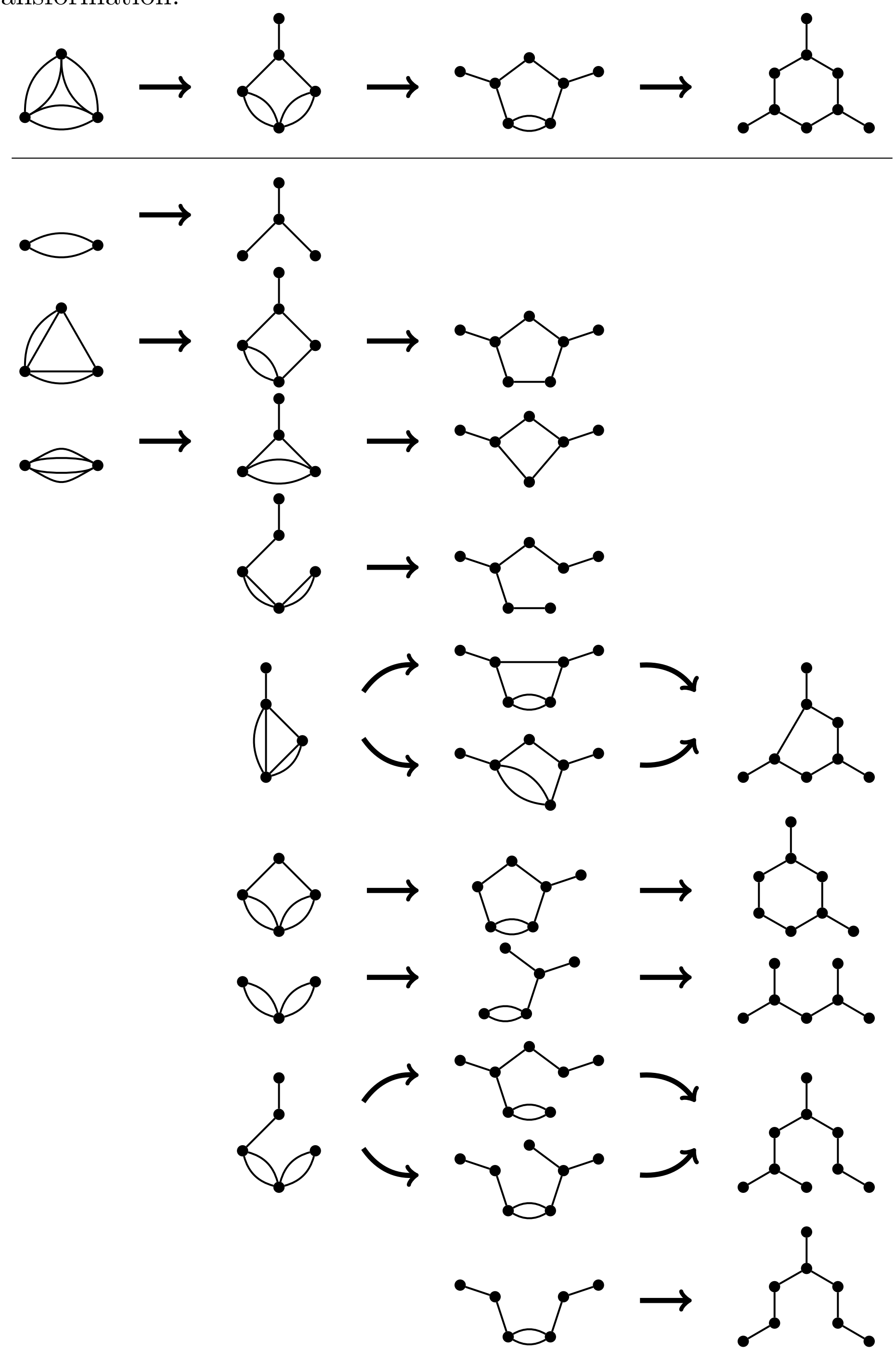
Proof. Suppose (H, Ω) is a proper minor of a member of the $K_3^{=}$-family. We only need to consider the components of (H, Ω) with at least 3 vertices to show $\xi(H, \Omega) < 3$. We proceed by a case study on whether (H, Ω) has multiple edges.

Suppose (H, Ω) has no multiple edges. From Theorem 1.40, every forest F has $\xi(F) \leq 2$. So we may assume (H, Ω) has exactly one cycle C; otherwise, $\xi(H, \Omega) \leq 2$. If there exists two pendant vertices a and b such that $d_H(a, b) = 3$, then $\{a, b\}$ form a zero forcing set for (H, Ω). Otherwise (H, Ω) has at least one pendant vertex a, and $\{a, b\}$ form a zero forcing set where $d_H(a, b) = 2$. So the zero forcing number of (H, Ω) is at most 2 when (H, Ω) has a cycle, and Corollary 1.34 implies $\xi(H, \Omega) \leq 2$. Therefore, $\xi(H, \Omega) < 3$ when (H, Ω) has no multiple edges.

Suppose (H, Ω) has multiple edges. For each pair of multiple edges, replace the odd edge with a path of length two consisting of odd edges and apply a ΔY-transformation on the resulting triangle to form the signed graph (H', Ω'). As (H', Ω') has no multiple edges and is again a proper minor of (G, Σ), the above argument holds and $\xi(H', \Omega') < 3$. Lemmas 1.32 and 1.23 imply $\xi(H, \Omega) \leq \xi(H', \Omega') < 3$ when (H, Ω) has multiple edges. $\square$

Figure 2.2 illustrates the case study of the proof of Lemma 2.6.

Figure 2.2 The $K_3^{=}$-family and their minors. The first row are the members of the $K_3^{=}$ family. Below the horizontal rule, each column are proper minors of the member of the $K_3^{=}$-family in that column. Arrows to the right represent a subdivision of an edge followed by a ΔY-transformation.



2.3 Partial Wide 2-Paths

Definition 2.7. Let (G, Σ) be a signed graph. A pair $[G_1, G_2]$ of subgraphs of G is a *wide separation* of (G, Σ) if there exists an odd 4-cycle C_4 such that

- $G_1 \cup C_4 \cup G_2 = G$,
- $E(G_1) \cap E(C_4) = \emptyset$,
- $E(G_2) \cap E(C_4) = \emptyset$,
- $V(G_1) \cap V(G_2) = \emptyset$,
- $V(G_1) \cap C_4 = \{r_1, r_2\}$, and
- $V(G_2) \cap C_4 = \{s_1, s_2\}$;

where r_1 and r_2 are not adjacent in C_4; and, s_1 and s_2 are not adjacent in C_4. We call r_1, r_2 the *vertices of attachment* of G_1 and s_1, s_2 the vertices of attachment of G_2.

Lemma 2.8. Let (G, Σ) be a signed graph with pendant vertices s_1, s_2 adjacent to u_1, u_2 where $u_1 \neq u_2$. If

- there exists $A \in \mathcal{S}(G, \Sigma)$ such that
  - $\operatorname{corank}(A) = \xi(G, \Sigma)$,
  - A has the SAP, and
  - $a_{s_1,s_1} = a_{s_2,s_2} = 0$; and

- $G - \{u_1, u_2, s_1, s_2\}$ has three components, each a path,

then $\xi(G, \Sigma) \leq 2$.

Proof. We relabel the vertices u_1, u_2, s_1, s_2 as $n-3, n-2, n-1, n$. We take the Schur complement $B = A/A[\{n-3, n-2, n-1, n\}]$. From Observation 1.5, $\operatorname{corank}(A) = \operatorname{corank}(B)$. The graph $H = G(B)$ has three components, each a path: P_1, P_2, and P_3. We again relabel the vertices such that $i \in P_j$, $i' \in P_{j'}$, and $j < j'$ implies $i < i'$.

From Theorem 1.14, $\operatorname{corank}(B) \leq M(H) = 3$. Suppose for a contradiction that $\operatorname{corank}(B) = 3$. Then, we may find non-zero vectors $x_1 \in \ker(B[P_1])$, $x_2 \in \ker(B[P_2])$, and $x_3 \in \ker(B[P_3])$. Because $B = B[P_1] \oplus B[P_2] \oplus B[P_3]$, the vectors $\{[x_1\ 0\ 0]^T, [0\ x_2\ 0]^T, [0\ 0\ x_3]^T\}$ form a basis for $\ker B$. Then, we may write U

$$U = \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \\ y_{n-3,1} & y_{n-3,2} & y_{n-3,3} \\ y_{n-2,1} & y_{n-2,2} & y_{n-2,3} \\ y_{n-1,1} & y_{n-1,2} & y_{n-1,3} \\ y_{n,1} & y_{n,2} & y_{n,3} \end{bmatrix}.$$

where the columns of U are a basis for $\ker(A)$ for some choice of $y_{i,j}$ for $i = n-3, n-2, n-1, n$ and $j = 1, 2, 3$. As $a_{n-1,n-1} = 0$, the pendant edge $(n-1)(n-3)$ forces $y_{n-3,j} = 0$. Similarly, $a_{n,n} = 0$, and the pendant edge $(n-2)(n)$ forces $y_{n-2,j} = 0$. So, we have

$$U = \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_{n-1,1} & y_{n-1,2} & y_{n-1,3} \\ y_{n,1} & y_{n,2} & y_{n,3} \end{bmatrix}.$$

We denote the i-th row of U with u_i. Because each edge is incident on $\{n-3, n-2\}$ or within the paths P_1, P_2, P_3, the span of the matrices $\{u_i u_i^T \mid i \in V\}$ and $\{u_i u_j^T + u_j u_i^T \mid ij \in E\}$ has dim ≤ 5. Because $\operatorname{corank}(A) = 3$ and A has the SAP, Lemma 1.28 implies this dimension must be 6. We have our contradiction; and $\operatorname{corank}(B) \leq 2$. Therefore, we have

$$\xi(G, \Sigma) = \operatorname{corank}(A) = \operatorname{corank}(B) \leq 2.$$

□

2.3.1 Two Wide Separations

The $K_{2,4}e_i^j$ family of signed graphs is presented in Figure 2.3.

Lemma 2.9. $\xi(K_{2,4}e_i^0) = 2$ for $i = o, e$.

Proof. Because $K_{2,4}e_i^0$ has an odd cycle for $i = o, e$, Theorem 1.26 implies $\xi(K_{2,4}e_i^0) \geq 2$ for $i = o, e$. Because $K_{2,4}e_i^0$ is a partial wide 2-path for $i = o, e$, Theorem 1.52 implies $\xi(K_{2,4}e_i^0) \leq 2$ for $i = o, e$. □

Lemma 2.10. $\xi(K_{2,4}e_i^1) \leq 2$ for $i = o, e$.

Proof. Let $A \in \mathcal{S}(K_{2,4}e_i^1)$ such that A has the SAP and $\operatorname{corank}(A) = \xi(K_{2,4}e_i^1)$. If $a_{12,12} \neq 0$, then we may take the Schur complement $A/a_{12,12}$. The signed graph of $A/a_{12,12}$ is exactly

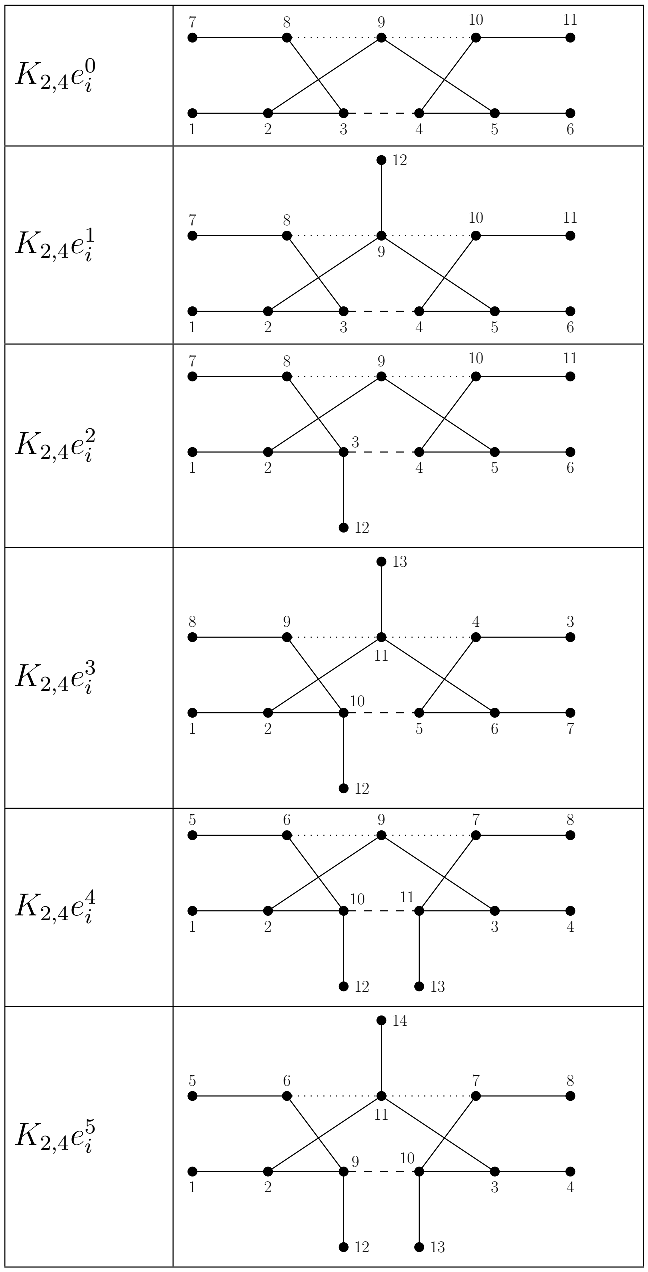


Figure 2.3 The $K_{2,4}e_i^j$ family of signed graphs, for $i = e, o$ and $j = 0, 1, 2, 3, 4, 5$. Solid edges are even, and dotted edges are odd. Dashed edges are even for $K_{2,4}e_e^j$ and are odd for $K_{2,4}e_o^j$.

$K_{2,4}e_i^0$, and Lemmas 2.2 and 2.9 imply $\xi(K_{2,4}e_i^1) = \xi(K_{2,4}e_i^0) = 2$. If $a_{12,12} = 0$, then we consider the Schur complement $B = A/A[\{9, 12\}]$. The graph of B is a tree T with $P(T) = 2$, and Theorem 1.41 implies $\text{corank}(B) \leq 2$. Therefore, we have

$$\xi(K_{2,4}e_i^1) = \text{corank}(A) = \text{corank}(B) \leq 2.$$

□

Lemma 2.11. $\xi(K_{2,4}e_i^2) \leq 2$ for $i = o, e$.

Proof. Let $A \in \mathcal{S}(K_{2,4}e_i^2)$ such that A has the SAP and $\text{corank}(A) = \xi(K_{2,4}e_i^2)$. If $a_{12,12} \neq 0$, then we may take the Schur complement $A/a_{12,12}$. The signed graph of $A/a_{12,12}$ is exactly $K_{2,4}e_i^0$, and Lemmas 2.2 and 2.9 imply $\xi(K_{2,4}e_i^2) = \xi(K_{2,4}e_i^0) = 2$. If $a_{12,12} = 0$, then we consider the Schur complement $B = A/A[\{9, 12\}]$. The graph of B is a graph on two parallel paths, denoted H. So, we may apply Lemma 1.44, and $\text{corank}(B) \leq M(H) = 2$. Therefore, we have

$$\xi(K_{2,4}e_i^2) = \text{corank}(A) = \text{corank}(B) \leq 2.$$

□

Lemma 2.12. $\xi(K_{2,4}e_i^3) \leq 2$ for $i = o, e$.

Proof. Let $A \in \mathcal{S}(K_{2,4}e_i^3)$ such that A has the SAP and $\text{corank}(A) = \xi(K_{2,4}e_i^3)$. If $a_{12,12} \neq 0$, then Lemma 2.10 implies $\text{corank}(A) \leq 2$. If $a_{13,13} \neq 0$, then Lemma 2.11 implies $\text{corank}(A) \leq 2$. So, we may assume both $a_{12,12} = a_{13,13} = 0$. We take the Schur complement $B = A/A[\{10, 11, 12, 13\}]$. The graph $H = G(B)$ has three components, each a path. Therefore, Lemma 2.8 implies $\xi(K_{2,4}e_i^3) \leq 2$. □

Lemma 2.13. $\xi(K_{2,4}e_i^4) \leq 2$ for $i = o, e$.

Proof. Let $A \in \mathcal{S}(K_{2,4}e_i^3)$ such that A has the SAP and $\operatorname{corank}(A) = \xi(K_{2,4}e_i^3)$. If $a_{12,12} \neq 0$, then Lemma 2.12 implies $\operatorname{corank}(A) \leq 2$. If $a_{13,13} \neq 0$, then Lemma 2.12 implies $\operatorname{corank}(A) \leq 2$. So, we may assume both $a_{12,12} = a_{13,13} = 0$. We take the Schur complement $B = A/A[\{10, 11, 12, 13\}]$. Then, we may obtain the tree $T = G(B)$ by subdividing each edge of $K_{1,4}$ exactly once. As $P(T) = 3$, Theorem 1.41 implies $\operatorname{corank}(B) \leq P(T) = 3$.

If $B[\{7, 8\}]$ is full rank, then we may take the Schur complement $C = B/B[\{7, 8\}]$. The resulting tree $T' = G(C)$ has $P(T') = 2$. From Theorem 1.41, $\operatorname{corank}(C) \leq M(T') = 2$. Therefore,

$$\operatorname{corank}(A) = \operatorname{corank}(B) = \operatorname{corank}(C) \leq M(T') = 2.$$

So, we may assume $\operatorname{corank}(B[\{7, 8\}]) \geq 1$ for the remainder of the proof. As $\operatorname{corank}(B[\{7, 8\}]) \geq 1$ and as $b_{7,8} \neq 0$, all entries of $B[\{7, 8\}]$ are nonzero. Then, the $\det B[\{7, 8, 9\}] = b_{7,9}^2 b_{8,8} \neq 0$. By symmetry, the same argument applied to the other branches of T implies $\operatorname{corank}(B[\{1, 2\}]) \geq 1$, $\operatorname{corank}(B[\{3, 4\}]) \geq 1$, and $\operatorname{corank}(B[\{5, 6\}]) \geq 1$.

Take the Schur complement $C' = B/B[\{7, 8, 9\}]$ and the graph $F = G(C')$. The forest F is three disjoint paths each on two vertices. Suppose for a contradiction that $\operatorname{corank}(B) = 3$. Then, the $\operatorname{corank}(C') = \operatorname{corank}(B) = 3$. We may find non-zero vectors $x_1 \in \ker(C'[\{1, 2\}])$, $x_2 \in \ker(C'[\{3, 4\}])$, and $x_3 \in \ker(C'[\{5, 6\}])$. Because $C' = C'[\{1, 2\}] \oplus C'[\{3, 4\}] \oplus C'[\{5, 6\}]$, the vectors $\{[x_1\ 0\ 0]^T, [0\ x_2\ 0]^T, [0\ 0\ x_3]^T\}$ form a basis for $\ker C'$. Then, the columns of W,

$$W = \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \\ y_{7,1} & y_{7,2} & y_{7,3} \\ y_{8,1} & y_{8,2} & y_{8,3} \\ y_{9,1} & y_{9,2} & y_{9,3} \end{bmatrix},$$

form a basis for $\ker B$ for some $y_{i,j}$, $i = 7, 8, 9$ and $j = 1, 2, 3$, where we may take $[y_{7,1}\ y_{7,2}\ y_{7,3}] \propto [y_{8,1}\ y_{8,2}\ y_{8,3}]$. Because $b_{2,9} \neq 0$, the coordinates $y_{9,j} = 0$ for $j = 1, 2, 3$. Then, the columns of U,

$$U = \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \\ y_{7,1} & y_{7,2} & y_{7,3} \\ y_{8,1} & y_{8,2} & y_{8,3} \\ 0 & 0 & 0 \\ y_{10,1} & y_{10,2} & y_{10,3} \\ y_{11,1} & y_{11,2} & y_{11,3} \\ y_{12,1} & y_{12,2} & y_{12,3} \\ y_{13,1} & y_{13,2} & y_{13,3} \end{bmatrix},$$

form a basis for $\ker(A)$, where we may take $[y_{12,1}\ y_{13,1}] \propto [y_{12,3}\ y_{13,3}]$. As $a_{12,12} = 0$, the pendant edge $v_{10}v_{12}$ forces $y_{10,j} = 0$. Similarly, $a_{13,13} = 0$, and the pendant edge $v_{11}v_{13}$ forces $y_{11,j} = 0$. As $\dim(U) = 3$, we may take $y_{12,2} = 0$, and we may write:

$$U = \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \\ y_{7,1} & y_{7,2} & y_{7,3} \\ y_{8,1} & y_{8,2} & y_{8,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_{12,1} & 0 & y_{12,3} \\ y_{13,1} & y_{13,2} & y_{13,3} \end{bmatrix}.$$

We denote the i-th row of U with u_i. The span of the matrices $\{u_i u_i^T \mid i \in V\}$ and $\{u_i u_j^T +$

$u_j u_i^T \mid ij \in E\}$ has dim ≤ 5. Because corank$(A) = 3$ and A has the SAP, Lemma 1.28 implies this dimension must be 6. We have our contradiction; and corank$(B) \leq 2$. Therefore, we have

$$\xi(K_{2,4}e_i^4) = \text{corank}(A) = \text{corank}(B) \leq 2.$$

□

Lemma 2.14. $\xi(K_{2,4}e_i^5) \leq 2$ for $i = o, e$.

Proof. Let $A \in \mathcal{S}(K_{2,4}e_i^5)$ such that A has the SAP and corank$(A) = \xi(K_{2,4}e_i^5)$. If $a_{14,14} \neq 0$, then Lemmas 2.2 and 2.13 imply corank$(A/a_{14,14}) \leq 2$. If $a_{13,13} \neq 0$, then Lemmas 2.2 and 2.12 imply corank$(A/a_{13,13}) \leq 2$. If $a_{12,12} \neq 0$, then Lemmas 2.2 and 2.12 imply corank$(A/a_{12,12}) \leq 2$. So, we may assume $a_{12,12} = a_{13,13} = a_{14,14} = 0$. Take the Schur complement $B = A/A[\{9, 10, 11, 12, 13\}]$. Then, the graph of B is four copies of K_2, and Corollary 1.14 implies corank$(B) \leq 4$.

Suppose for a contradiction that corank$(B) = 4$. Then, we may find nonzero vectors $x_1 \in \ker(B[\{1, 2\}])$, $x_2 \in \ker(B[\{3, 4\}])$, $x_3 \in \ker(B[\{5, 6\}])$, and $x_4 \in \ker(B[\{7, 8\}])$. Then, the columns of U,

$$U = \begin{bmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & x_4 \\ y_{9,1} & y_{9,2} & y_{9,3} & y_{9,4} \\ y_{10,1} & y_{10,2} & y_{10,3} & y_{10,4} \\ y_{11,1} & y_{11,2} & y_{11,3} & y_{11,4} \\ y_{12,1} & y_{12,2} & y_{12,3} & y_{12,4} \\ y_{13,1} & y_{13,2} & y_{13,3} & y_{13,4} \\ y_{14,1} & y_{14,2} & y_{14,3} & y_{14,4} \end{bmatrix},$$

form a basis for the $\ker(A)$ for $y_{i,j}$ where $i = 9, \ldots, 14$ and $j = 1, 2, 3, 4$. Because $a_{12,12} = 0$, the pendant edge v_9v_{12} forces $y_{9,j} = 0$ for $j = 1, 2, 3, 4$. Similarly, $y_{10,j} = y_{11,j} = 0$ for $j = 1, 2, 3, 4$. So, we may write

$$U = \begin{bmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & x_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ y_{12,1} & y_{12,2} & y_{12,3} & y_{12,4} \\ y_{13,1} & y_{13,2} & y_{13,3} & y_{13,4} \\ y_{14,1} & y_{14,2} & y_{14,3} & y_{14,4} \end{bmatrix}.$$

We denote the i-th row of U with u_i. The span of the matrices $\{u_iu_i^T \mid i \in V\}$ and $\{u_iu_j^T + u_ju_i^T \mid ij \in E\}$ has dim ≤ 7. Because $\text{corank}(B) = \text{corank}(A) = 4$ and A has the SAP, Lemma 1.28 implies this dimension must be 10. We have our contradiction; and $\text{corank}(B) \leq 3$.

We assume for a contradiction that $\text{corank}(B) = 3$. So, one of the four paths in the graph of B corresponds to a full rank 2×2 matrix, and we may assume that $\text{corank}(B[\{7, 8\}]) = 0$. So, we may find nonzero vectors $q_1 \in \ker(B[\{1, 2\}])$, $q_2 \in \ker(B[\{3, 4\}])$, and $q_3 \in$

$\ker(B[\{5,6\}])$. With a similar argument, the columns of W,

$$W = \begin{bmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ z_{12,1} & 0 & z_{12,3} \\ z_{13,1} & z_{13,2} & 0 \\ z_{14,1} & 0 & z_{14,3} \end{bmatrix},$$

form a basis for $\ker(A)$ for some $z_{12,1}, z_{12,3}, z_{13,1}, z_{13,2}, z_{14,1}, z_{14,3}$. We denote the i-th row of W with w_i. The span of the matrices $\{w_i w_i^T \mid i \in V\}$ and $\{w_i w_j^T + w_j w_i^T \mid ij \in E\}$ has $\dim \leq 5$. Because $\text{corank}(B) = \text{corank}(A) = 3$ and A has the SAP, Lemma 1.28 implies this dimension must be 6. We have our contradiction; and $\text{corank}(B) \leq 2$. Therefore, we have

$$\xi(K_{2,4}e_i^5) = \text{corank}(A) = \text{corank}(B) \leq 2.$$

$\square$

Lemma 2.15. Let (G, Σ) be a signed graph such that the removal of pendant vertices yields a 2-connected partial wide 2-path (H, Ω). Let $[H_1, H_2]$ and $[H_3, H_4]$ be distinct wide separations of (H, Ω) such that $H_1 \subseteq H_3$ and $H_4 \subseteq H_2$. Let r_1, r_2 be the vertices of attachment of H_2 and let s_1, s_2 be the vertices of attachment of H_3. Let P_1 and P_2 be disjoint paths between $\{r_1, r_2\}$ and $\{s_1, s_2\}$, where P_i has endvertices r_i and s_i for $i = 1, 2$. If a pendant edge is incident with a vertex on P_1 or P_2, then either

- (G, Σ) has a minor isomorphic to a member of the $K_3^=$-family or

- $\xi(G,\Sigma) \leq 2$ and both H_1 and H_2 are disconnected.

Proof. If (G,Σ) has a pendant vertex adjacent to an internal vertex of P_1 or P_2, then we found a minor of (G,Σ) isomorphic to $K_3^{=}(\Delta Y)$. So, we may assume every pendant vertex is adjacent to an endvertex of P_1 or P_2.

Suppose that a pendant vertex is adjacent to an endvertex of P_1. If P_1 has at least two edges, then we found a minor of (G,Σ) isomorphic to $K_3^{=}(\Delta Y)^3$. So, we may assume that P_1 has 1 edge or $r_1 = s_1$. If P_2 has at least two edges, then we found a minor of (G,Σ) isomorphic to $K_3^{=}(\Delta Y)^3$. So, we may assume that P_2 has 1 edge or $r_2 = s_2$. If both P_1 and P_2 each have exactly 1 edge, then we found a minor of (G,Σ) isomorphic to $K_3^{=}(\Delta Y)^2$. So, we may assume that $r_1 = s_1$ or $r_2 = s_2$.

Suppose that P_1 has exactly 1 edge and $r_2 = s_2$. If H_1 or H_4 is connected and if a pendant vertex is adjacent to r_1 or s_1, then (G,Σ) has a minor isomorphic to $K_3^{=}(\Delta Y)$. If H_1 or H_4 is connected and if a pendant vertex is adjacent to $r_2 = s_2$, then (G,Σ) has a minor isomorphic to $K_3^{=}(\Delta Y)^2$. So, we may assume that both H_1 and H_4 are disconnected. Then, (G,Σ) is isomorphic to a minor of $K_{2,4}e_i^5$, and Lemma 2.14 implies $\xi(G,\Sigma) \leq \xi(K_{2,4}e_i^5) \leq 2$. We may apply the same argument by symmetry for the case that P_2 has exactly 1 edge and $r_1 = s_1$.

Next, we assume that $r_1 = s_1$ and $r_2 = s_2$. If H_1 and H_4 are connected, then we found a minor of (G,Σ) isomorphic to $K_3^{=}(\Delta Y)$. If both H_1 and H_4 are disconnected, then we found a minor of (G,Σ) isomorphic to $K_{2,4}e_i^5$. Lemma 2.14 implies $\xi(G,\Sigma) \leq \xi(K_{2,4}e_i^5) \leq 2$. So, we may assume that H_1 is disconnected and H_4 is connected.

By symmetry, we may assume that (G, Σ) has a pendant edge t_1r_1 incident on P_1. If H_4 contains a cycle, then we found a minor of (G, Σ) isomorphic to $K_3^{=}(\Delta Y)^2$. So, we may assume that H_4 has no cycles. As H_4 is connected, we may find a path Q between q_1 and q_2, the vertices of attachment of H_4 in $[H_3, H_4]$. If (G, Σ) has a pendant edge incident on an internal vertex of Q, then we found a minor of (G, Σ) isomorphic to $K_3^{=}(\Delta Y)^3$. If Q has at least two edges and (G, Σ) has pendant edges incident on both q_1 and q_2, then we found a minor of (G, Σ) isomorphic to $K_3^{=}(\Delta Y)^2$. So, we may assume that Q has exactly one edge or that (G, Σ) has at most one pendant vertex adjacent to Q.

Let $A \in \mathcal{S}(G, \Sigma)$ such that $\text{corank}(A) = \xi(G, \Sigma)$ and A has the SAP. If the entry $a_{t_1,t_1} \neq 0$, then Lemma 2.2 implies $\xi(G, \Sigma) = \xi(G - t_1, \Sigma \setminus t_1r_1)$. As $(G - t_1, \Sigma \setminus t_1r_1)$ is a minor of a 2-connected partial wide 2-path, Theorem 1.52 and Lemma 1.23 imply $\xi(G - t_1, \Sigma \setminus t_1r_1) \leq 2$. Hence, $\xi(G, \Sigma) \leq 2$. So, we may assume that $a_{t_1,t_1} = 0$. We take the Schur complement $B = A/A[\{t_1, r_1\}]$. Next, we construct a zero forcing set for $G - \{t_1, r_1\}$. Let v_1, v_2 be the vertices of attachment of H_1 for the wide separation $[H_1, H_2]$. If (G, Σ) has a pendant vertex adjacent to v_1, then we take z_1 to be this pendant vertex; otherwise, we take $z_1 = v_1$. If (G, Σ) has a pendant vertex adjacent to q_1 or q_2, then we take z_2 to be this pendant vertex; otherwise, we take $z_2 = q_2$. Then, $Z = \{z_1, z_2\}$ is a zero set for $G - \{t_1, r_1\}$, and by Lemma 1.33 $\text{corank}(B) \leq M(G - \{t_1, r_1\}) \leq Z(G - \{t_1, r_1\}) \leq |Z| = 2$. From Observation 1.5, we have $\text{corank}(A) = \text{corank}(B) \leq 2$. Therefore, $\xi(G, \Sigma) = \text{corank}(A) \leq 2$.

Suppose (G, Σ) has a pendant edge t_2r_2 incident on P_2. By Lemma 2.2, we may assume that entries $a_{t_1,t_1} = a_{t_2,t_2} = 0$, corresponding to the pendant vertices t_1 and t_2. Then,

$G - \{t_1, r_1, t_2, r_2\}$ has three components, each a path. From Lemma 2.8, $\xi(G, \Sigma) \leq 2$. $\square$

Lemma 2.16. Let (G, Σ) be a signed graph such that removing pendant vertices yields a 2-connected partial 2-path (H, Ω) with at least two wide separations. Then, either (G, Σ) has a minor isomorphic to a member of the $K_3^{=}$-family or $\xi(G, \Sigma) \leq 2$.

Proof. Let $[H_1, H_2]$ and $[H_3, H_4]$ be two distinct wide separations of (H, Ω) such that there are no wide separations $[F_1, F_2]$ and $[F_3, F_4]$ with $F_1 \subset H_1$ and $F_4 \subset H_4$. Let u_1, u_2 be the vertices of attachment of H_1; let r_1, r_2 be the vertices of attachment of H_2; let s_1, s_2 be the vertices of attachment of H_3. Because (H, Ω) is 2-connected, we may find disjoint paths P_1 and P_2 between $\{r_1, r_2\}$ and $\{s_1, s_2\}$ in (G, Σ). If any pendant vertex is adjacent to P_1 or P_2, then Lemma 2.15 implies (G, Σ) has a minor isomorphic to a member of the $K_3^{=}$-family or $\xi(G, \Sigma) \leq 2$. So, we may assume no pendant vertex is adjacent to P_1 or P_2.

Suppose H_1 contains a cycle C. Without loss of generality, we may assume that C is at the end of the partial wide 2-path H, and we have a 2-separation (C, F) of H. So, we may find exactly two vertices $\{v_1, v_2\} = V(C) \cap V(F)$, and we may assume that $v_1 \leftrightarrow v_2$. Let Q_1 be a path from v_1 and u_1; let Q_2 be a path from v_2 and u_2. As (H, Ω) is 2-connected, we may take Q_1 disjoint from Q_2. If (G, Σ) has a pendant vertex adjacent to $(Q_1 - v_1) \cup (Q_2 - v_2)$, then we found a minor of (G, Σ) isomorphic to $K_3^{=}(\Delta Y)$. We take the path P from C by removing the edge $v_1 v_2$, and P has endvertices v_1 and v_2.

Suppose H_1 contains no cycle. Then, we take the path $P \subseteq H_1$ with endvertices u_1 and u_2.

If (G, Σ) has two pendant vertices adjacent to vertices $p_i, p_{i+2} \in V(P)$ and $d_P(p_i p_{i+2}) \geq 2$, then we found a minor of (G, Σ) isomorphic to $K_3^{=}(\Delta Y)^2$. Hence, (G, Σ) has at most two pendant vertices adjacent to $p_j, p_{j+1} \in V(P)$ where $d_P(p_j, p_{j+1}) = 1$. If there are two such pendant vertices $w_1 \leftrightarrow p_j$ and $w_2 \leftrightarrow p_{j+1}$, then we add the edge w_1w_2 to obtain $(G^{(1)}, \Sigma^{(1)}) = (G, \Sigma \cup w_1w_2)$. If there is only one such pendant vertex $w_1 \leftrightarrow p_j$, then we add the edge w_1p_{j+1} to obtain $(G^{(1)}, \Sigma^{(1)}) = (G, \Sigma \cup w_1p_{j+1})$. If H_1 is disconnected, then H_1 consists of two isolated vertices $\{v_1, v_2\}$, because (H, Ω) is 2-connected. If (G, Σ) has no pendant edge at v_1, then take $x_1 = v_1$; otherwise, we take this pendant vertex to be x_1. Define x_2 similarly. Take $(G^{(2)}, \Sigma^{(2)}) = (G^{(1)}, \Sigma^{(1)} \cup x_1x_2)$.

We apply our argument on H_1 to H_4 in $(G^{(2)}, \Sigma^{(2)})$. The resulting signed graph $(G^{(3)}, \Sigma^{(3)})$ is a 2-connected partial wide 2-path. By Theorem 1.52, we have $\xi(G^{(3)}, \Sigma^{(3)}) \leq 2$. As $(G, \Sigma) \preceq (G^{(1)}, \Sigma^{(1)}) \preceq (G^{(2)}, \Sigma^{(2)}) \preceq (G^{(3)}, \Sigma^{(3)})$, Lemma 1.23 implies $\xi(G, \Sigma) \leq (G^{(3)}, \Sigma^{(3)})$. Therefore, $\xi(G, \Sigma) \leq 2$. $\square$

2.3.2 One Wide Separation

Lemma 2.17. Let (G, Σ) be a signed graph such that the removal of all pendant vertices yields a partial wide 2-path (H, Ω) with exactly one wide separation $[H_1, H_2]$. Let u_1, u_2 be the vertices of attachment of H_1; let w_1, w_2 be the vertices of attachment of H_2. Suppose that

- H_1 and H_2 are paths;
- there are pendant vertices adjacent to u_1 and u_2;
- there are no pendant vertices adjacent to the internal vertices of H_2;

- if there is a pendant vertex adjacent to an internal vertex of H_1, then $l(H_1) = 2$ and either
  - at most one pendant vertex is incident with $\{w_1, w_2\}$ or
  - H_2 has one edge w_1w_2.

Then, $\xi(G,\Sigma) \leq 2$.

Proof. Let $A \in \mathcal{S}(G,\Sigma)$ such that $\operatorname{corank}(A) = \xi(G,\Sigma)$ and A has the SAP. Let s_1, s_2 be the pendant vertices such that $s_1 \leftrightarrow u_1$ and $s_2 \leftrightarrow u_2$.

Suppose $a_{s_1,s_1} = a_{s_2,s_2} = 0$. Then, we may take the Schur complement $B = A/A[s_1, s_2, u_1, u_2]$, and by Observation 1.5 we have $\operatorname{corank}(A) = \operatorname{corank}(B)$. The forest $F = G(B)$ consists of two disjoint paths, and Theorem 1.41 implies $\operatorname{corank}(B) \leq P(F) = 2$. Hence, $\xi(G,\Sigma) = \operatorname{corank}(A) = \operatorname{corank}(B) \leq 2$. So, we may assume that $a_{s_1,s_1} \neq 0$. We take $(G^{(1)}, \Sigma^{(1)}) = (G - s_1, \Sigma \setminus s_1u_1)$. By Lemma 2.2, we know that $\xi(G,\Sigma) = \xi(G^{(1)}, \Sigma^{(1)})$. If a pendant vertex of (G,Σ) is adjacent to an internal vertex of H_1, then we may add edges to $(G^{(1)}, \Sigma^{(1)})$ to obtain a 2-connected partial wide 2-path $(G^{(2)}, \Sigma^{(2)})$. By Theorem 1.52, $\xi(G^{(2)}, \Sigma^{(2)}) \leq 2$. Because $(G^{(1)}, \Sigma^{(1)}) \preceq (G^{(2)}, \Sigma^{(2)})$, Lemma 1.23 implies $\xi(G^{(1)}, \Sigma^{(1)}) \leq \xi(G^{(2)}, \Sigma^{(2)})$. Hence, $\xi(G,\Sigma) = \xi(G^{(1)}, \Sigma^{(1)}) \leq \xi(G^{(2)}, \Sigma^{(2)}) \leq 2$. We may therefore assume no pendant vertex of (G,Σ) is adjacent to an internal vertex of H_1. Similarly, if at most one pendant vertex of (G,Σ) is adjacent to $\{w_1, w_2\}$, then we may add edges to extend $(G^{(1)}, \Sigma^{(1)})$ to a 2-connected partial wide 2-path $(G^{(3)}, \Sigma^{(3)})$. With a similar argument, $\xi(G,\Sigma) = \xi(G^{(1)}, \Sigma^{(1)}) \leq \xi(G^{(3)}, \Sigma^{(3)}) \leq 2$. We may therefore assume that (G,Σ) has pendant vertices t_1 and t_2 adjacent to w_1 and w_2, where $t_1 \leftrightarrow w_1$ and $t_2 \leftrightarrow w_2$. If $a_{t_1,t_1} = a_{t_2,t_2} = 0$, then we may take the Schur complement $C = A/A[\{t_1, t_2, w_2, w_2\}]$. By Observation 1.5, we have $\mathrm{corank}(A) = \mathrm{corank}(C)$. The forest $F' = G(C)$ consists of two disjoint paths. So, we may apply Theorem 1.41 to obtain $\mathrm{corank}(C) \leq P(F') = 2$. Hence, we have $\xi(G, \Sigma) = \mathrm{corank}(A) = \mathrm{corank}(C) \leq 2$. We may therefore assume that $a_{t_1,t_1} \neq 0$. We may delete vertex t_1 from (G, Σ) and apply Lemma 2.2 to obtain $\xi(G, \Sigma) = \xi(G - t_1, \Sigma \setminus \delta(t_1))$. We may extend $(G - t_1, \Sigma \setminus \delta(t_1))$ by adding edges to obtain a 2-connected partial 2-path; and by Theorem 1.52 and Lemma 1.23 we have $\xi(G - t_1, \Sigma \setminus \delta(t_1)) \leq 2$. Therefore, $\xi(G, \Sigma) = \xi\big(G - t_1, \Sigma \setminus \delta(t_1)\big) \leq 2$.

□

Lemma 2.18. Let (G, Σ) be a signed graph such that removal of all pendant vertices yields a partial wide 2-path (H, Ω) with exactly one wide separation $[H_1, H_2]$. If both H_1 and H_2 are connected, then either (G, Σ) has a minor isomorphic to a member of the $K_3^=$-family or $\xi(G, \Sigma) \leq 2$.

Proof. Let u_1, u_2 be the vertices of attachment of H_1, and let w_1, w_2 be the vertices of attachment of H_2.

Suppose that H_1 contains a cycle C. We may assume that C is the cycle at the end of the partial wide 2-path (H, Ω), and we found a 2-separation (C, F) of H. Let $\{v_1, v_2\} = V(C) \cap V(F)$. Because (H, Ω) is 2-connected, we may find disjoint paths Q_1 and Q_2 between $\{v_1, v_2\}$ and $\{u_1, u_2\}$. If a pendant vertex of (G, Σ) is adjacent to $V(H_1) \setminus V(C)$, then we found a minor of (G, Σ) isomorphic to $K_3^=(\Delta Y)$ (and another minor of (G, Σ) isomorphic to $K_3^=(\Delta Y)^2$).

So, we may assume that any pendant vertices adjacent to H_1 are adjacent to $V(C)$. Let $P_1 \subset C$ be the path obtained from C by removing the edge v_1v_2. If (G, Σ) has two pendant vertices adjacent to vertices $a_1, a_2 \in V(P_1)$, $d_{P_1}(a_1, a_2) \geq 2$, and $\{a_1, a_2\} \setminus \{u_1, u_2\} \neq \emptyset$; then we found a minor of (G, Σ) isomorphic to $K_3^=(\Delta Y)^2$. Hence, either

- there are at most two pendant vertices adjacent to vertices $a_1, a_2 \in V(P_1)$ and $d_{P_1}(a_1, a_2) = 1$;
- there are three pendant vertices adjacent to the vertices of P_1, $l(P_1) = 2$, and the endvertices of P_1 are u_1 and u_2; or
- there are two pendant vertices of (G, Σ) adjacent to the endvertices of P_1, and the endvertices of P_1 are u_1 and u_2.

Suppose that H_1 has no cycle. Then, H_1 is a path from u_1 to u_2. We take $P_1 = H_1$. If (G, Σ) has two pendant vertices adjacent to vertices $b_1, b_2 \in V(P_1)$, $d_{P_1}(b_1, b_2) \geq 2$, and $\{b_1, b_2\} \setminus \{u_1, u_2\} \neq \emptyset$; then we found a minor of (G, Σ) isomorphic to $K_3^=(\Delta Y)^2$. Hence, either there are at most two pendant vertices adjacent to vertices $b_1, b_2 \in V(P_1)$ and $d_{P_1}(b_1, b_2) = 1$; there are three pendant vertices adjacent to the vertices of P_1, $l(P_1) = 2$, and the endvertices of P_1 are u_1 and u_2; or there are two pendant vertices of (G, Σ) adjacent to the endvertices of P_1, and the endvertices of P_1 are u_1 and u_2.

We apply our case study on H_1 to H_2, and we have defined $P_2 \subseteq H_2$. For $i = 1, 2$, if (G, Σ) has at most two pendant vertices adjacent to $c_1, c_2 \in V(P_i)$ and $d_{P_i}(c_1, c_2) = 1$, then (G, Σ) is a minor of a partial wide 2-path. Hence, Theorem 1.52 and Lemma 1.23 imply

$\xi(G,\Sigma) \leq 2$. So, we may assume that either there are pendant vertices adjacent with both endvertices of P_1, the endvertices of P_1 are u_1 and u_2, and $l(P_1) \geq 2$; or there are pendant vertices adjacent with both endvertices of P_2, the endvertices of P_2 are w_1 and w_2, and $l(P_2) \geq 2$. By symmetry, we may assume that there are pendant vertices adjacent with both endvertices of P_1, the endvertices of P_1 are u_1 and u_2, and $l(P_1) \geq 2$. If H_2 has a cycle, then we found a minor of (G,Σ) isomorphic to $K_3^{=}(\Delta Y)^2$. So, we may assume that H_2 has no cycle, and $P_2 = H_2$ is a path with endvertices w_1 and w_2.

If (G,Σ) has a pendant vertex adjacent to $V(P_2) \setminus \{w_1, w_2\}$, then we found a minor of (G,Σ) isomorphic to $K_3^{=}(\Delta Y)^3$. So, we may assume that any pendant vertex of (G,Σ) adjacent to P_2 is adjacent to w_1 or w_2. If $l(P_1) \geq 3$ and (G,Σ) has a pendant vertex adjacent to $P_1 - \{u_1, u_2\}$, then we found a minor of (G,Σ) isomorphic to $K_3^{=}(\Delta Y)^2$. If $l(P_1) = 2$, then (G,Σ) has a pendant vertex adjacent to $P_1 - \{u_1, u_2\}$, (G,Σ) has pendant vertices adjacent to w_1 and w_2, and $l(P_2) \geq 2$; and we found a minor of (G,Σ) isomorphic to $K_3^{=}(\Delta Y)^3$. So, we may assume that $l(P_2) = 1$. Hence, if $l(P_1) = 2$ and (G,Σ) has a pendant vertex adjacent $P_1 - \{u_1, u_2\}$, then either $l(P_2) = 1$ or (G,Σ) has at most one pendant vertex adjacent to $\{w_1, w_2\}$. By Lemma 2.17, we have $\xi(G,\Sigma) \leq 2$. □

Lemma 2.19. Let (G,Σ) be a signed graph such that the removal of pendant vertices yields a 2-connected partial wide 2-path (H,Ω) with exactly one wide separation $[H_1, H_2]$. Let u_1 and u_2 be the vertices of attachment of H_1. Let P be a path in $l(H_1) \geq 2$, with endvertices u_1 and u_2. If H_2 is disconnected, then either (G,Σ) has a minor isomorphic to a member of the $K_3^{=}$-family or $\xi(G,\Sigma) \leq 2$.

Proof. Suppose that H_1 contains a cycle C. As (H, Ω) is 2-connected, we may find disjoint paths Q_1 and Q_2 between $\{u_1, u_2\}$ and $V(C)$. Let $v_i = V(Q_i) \cap V(C)$ for $i = 1, 2$. If (G, Σ) has a pendant vertex adjacent to an internal vertex of Q_1 or Q_2, then we found a minor of (G, Σ) isomorphic to $K_3^{=}(\Delta Y)$. □

Lemma 2.20. Let (G, Σ) be a signed graph such that the removal of pendant vertices yields a 2-connected partial wide 2-path (H, Ω) with exactly one wide separation. Then, either (G, Σ) has a minor isomorphic to a member of the $K_3^{=}$-family or $\xi(G, \Sigma) \leq 2$.

Proof. Let $[H_1, H_2]$ be the wide separation of (G, Σ). If H_1 and H_2 are connected, then Lemma 2.19 implies either (G, Σ) has a minor in the $K_3^{=}$-family or $\xi(G, \Sigma) \leq 2$. So, we may assume that H_2 is disconnected. Let u_1 and u_2 be the vertices of attachment of H_1. If $l(H_1) \geq 2$, then Lemma 2.20 implies either (G, Σ) has a minor in the $K_3^{=}$-family or $\xi(G, \Sigma) \leq 2$. Hence, we may assume either $E(H_1) = \{u_1 u_2\}$ or H_1 is disconnected. That is, H is a graph of two parallel paths. Lemma 1.44 implies $\xi(H, \Omega) \leq M(H) \leq 2$. From Lemma 2.2, $\xi(G, \Sigma) = \xi(H, \Omega) \leq 2$. So for $i = 1, 2$, we may assume that any pendant vertex of (G, Σ) which is adjacent to $Q_i - v_i$ is adjacent to u_i.

Suppose (G, Σ) has a pendant vertex s_1 adjacent to u_1. Then, $v_1 \neq u_1$, and $l(Q_1) \geq 1$. If both $l(Q_1) \geq 1$ and $l(Q_2) \geq 1$, then we found a minor of (G, Σ) isomorphic to $K_3^{=}(\Delta Y)^2$. So, we may assume that $l(Q_2) = 0$, and $u_2 = v_2$. Then, $H_1 - \{u_1, u_2\}$ has no cycle. If $H_1 - \{u_1, u_2\}$ is not a path, then it contains a $K_{1,3}$, and we found a minor of (G, Σ) isomorphic to $K_3^{=}(\Delta Y)^3$. So, we may assume that $H - \{u_1, u_2\}$ is a path. Suppose that no pendant vertex is adjacent to u_2.

Let $A \in \mathcal{S}(G, \Sigma)$ with $\operatorname{corank}(A) = \xi(G, \Sigma)$ such that A has the SAP. We may assume that $a_{s_1,s_1} = 0$; otherwise, we may delete the pendant vertex s_1 by Lemma 2.2. Then, we may take the Schur complement $B = A/A[\{u_1, s_1\}]$. By Lemma 1.5, $\operatorname{corank}(A) = \operatorname{corank}(B)$. Because $G - \{u_1, s_1\}$ is a partial 2-path with a path cover number of two, by Theorem 1.42, $\operatorname{corank}(B) \leq 2$; and $\xi(G, \Sigma) = \operatorname{corank}(A) = \operatorname{corank}(B) \leq 2$. If (G, Σ) has a pendant vertex s_2 adjacent to u_2, then we may assume that a_{s_2,s_2} by Lemma 2.2. Because $G - \{u_1, u_2, s_1, s_2\}$ consists of three paths, we may apply Lemma 2.8, and $\xi(G, \Sigma) \leq 2$. Therefore, (G, Σ) has no pendant vertex adjacent to $V(Q_i) \setminus v_i$ for $i = 1, 2$. So, any pendant vertex of (G, Σ) adjacent to H_1 is adjacent to $V(C) \setminus \{u_1, u_2\}$.

Let P_1 be the path obtained from C by deleting the edge v_1v_2. If (G, Σ) has pendant vertices adjacent to v_1 and v_2 and if $l(P_1) \geq 2$, then we found a minor of (G, Σ) isomorphic to $K_3^=(\Delta Y)^2$. If (G, Σ) has two pendant vertices adjacent to $a_1, a_2 \in V(P_1)$, $\{a_1, a_2\} \neq \{v_1, v_2\}$, and $d_{P_1}(a_1, a_2) \geq 2$; then we found a minor of (G, Σ) isomorphic to $K_3^=(\Delta Y)^2$. Hence, (G, Σ) has at most two pendant vertices adjacent to $a_1, a_2 \in V(P_1)$ and $d_{P_1}(a_1, a_2) = 1$. We may add an edge between our pendant vertices to (G, Σ), and the resulting signed graph is a 2-connected partial wide 2-path. By Theorem 1.52, $\xi(G, \Sigma) \leq 2$. So, whenever H_1 contains a cycle, either $\xi(G, \Sigma) \leq 2$ or (G, Σ) has a minor isomorphic to a member of the $K_3^=$-family.

Suppose that H_1 contains no cycle. As (H, Ω) is 2-connected, H_1 must be a path with endvertices u_1 and u_2. Suppose there are pendant vertices adjacent to $b_1, b_2 \in V(H_1)$ and $d_{H_1}(b_1, b_2) \geq 2$. Suppose $b_1 = u_1$. If $H_1 - \{u_1, u_2\}$ is not a path, then it must contain $K_{1,3}$; and we found a minor of (G, Σ) isomorphic to $K_3^=(\Delta Y)^3$. So, we may assume that (G, Σ)

has no pendant vertex adjacent to u_1 or u_2; and we found a minor of (G, Σ) isomorphic to $K_3^{=}(\Delta Y)^2$. Hence, we may assume that $d_{H_1}(b_1, b_2) = 1$. We may add an edge between our pendant vertices to (G, Σ), and the resulting signed graph is a 2-connected partial wide 2-path. By Theorem 1.52, we have $\xi(G, \Sigma) \leq 2$. So, whenever H_1 contains no cycle, either $\xi(G, \Sigma) \leq 2$ or (G, Σ) has a minor isomorphic to a member of the $K_3^{=}$-family. □

2.3.3 No Wide Separations

The proof of the following lemma, originally in terms of simple graphs, is from Hogben and van der Holst (Theorem 5.1 in [14]).

Lemma 2.21. Let (G, Σ) be a signed graph such that the removal of pendant vertices yields a 2-connected partial 2-path (H, Ω). If (G, Σ) has no wide separation, then either (G, Σ) has a minor isomorphic to a member of the $K_3^{=}$-family or $\xi(G, \Sigma) \leq 2$.

Proof. Let $A \in \mathcal{S}(G, \Sigma)$ such that $\text{corank}(A) = \xi(G, \Sigma)$ and A has the SAP. Let $W \subseteq V(G)$ be the pendant vertices of (G, Σ), and let $S \subseteq V(G)$ be those vertices adjacent to the pendant vertices W. By Lemma 2.2, we may assume that $a_{s,s} = 0 \; \forall s \in S$.

Because (H, Ω) is a partial 2-path with no wide separations, H is outerplanar; therefore, G is outerplanar. So, we may embed G in the plane with every vertex on the infinite face. Let $\mathcal{B}$ be the collection of cycles bounding the finite faces. Then, the dual of G is a path P whose vertices correspond to the finite faces of our embedding with an edge $pq \in E(P)$ whenever p and q share a common edge in our embedding of G. Let p_1, p_2 be the endvertices of P.

If (G, Σ) has a pendant vertex $s \in \left(\bigcup_{q \neq p_1, p_2} B_q \right) \setminus (B_{p_1} \cup B_{p_2})$, then we found a minor of (G, Σ) isomorphic to $K_3^{=}(\Delta Y)$. Hence, we may assume $S \subseteq B_{p_1} \cup B_{p_2}$.

Suppose there is a vertex $s_1 \in S$ such that $s_1 \in \bigcap_{B \in \mathcal{B}} V(B)$. Let $w_1 \in W$ such that $w_1 \leftrightarrow s_1$. Take the Schur complement $A' = A/A[S \cup W]$. If the graph $F = G(A')$ has at least 3 components, then we found a minor of (G, Σ) isomorphic to $K_3^{=}(\Delta Y)^3$. So, F has at most 2 components. Because $s_1 \in \bigcap_{B \in \mathcal{B}} V(B)$, these two components are both paths. By Theorem 1.41, $\text{corank}(A') \leq M(F) = P(F) = 2$. By Lemma 1.5, $\text{corank}(A) = \text{corank}(A') \leq 2$. Hence, we may assume that $S \cap \left(\bigcap_{B \in \mathcal{B}} V(B) \right) = \emptyset$.

If $p_1 = p_2$, then $S = \emptyset$ and (G, Σ) has no pendant vertices. So, (G, Σ) is a 2-connected partial wide 2-path, and Theorem 1.52 implies $\xi(G, \Sigma) \leq 2$. Hence, we may assume that $p_1 \neq p_2$. If there are two vertices $s_2, s_3 \in S \cap V(B_{p_1})$ with $d_H(s_2, s_3) \geq 2$, then we found a minor of (G, Σ) isomorphic to $K_3^{=}(\Delta Y)^2$. Similarly, if there are two vertices $s_4, s_5 \in S \cap V(B_{p_2})$ with $d_H(s_4, s_5) \geq 2$, then we found a minor of (G, Σ) isomorphic to $K_3^{=}(\Delta Y)^2$. Hence, there is an edge $e_1 \in E(B_{p_1})$ and an edge $e_2 \in E(B_{p_2})$ such that $\{e_1, e_2\}$ is incident on every vertex of S. If we identify an edge of a copy of C_4 with e_1 and identify an edge of another copy of C_4 with e_2, then the resulting signed graph $(G^{(1)}, \Sigma^{(1)})$ is a 2-connected 2-path. By Theorem 1.52, $\xi(G^{(1)}, \Sigma^{(1)}) \leq 2$. As $(G, \Sigma) \preceq (G^{(1)}, \Sigma^{(1)})$, Lemma 1.23 implies $\xi(G, \Sigma) \leq \xi(G^{(1)}, \Sigma^{(1)}) \leq 2$.

$\square$

2.4 Proof of the Main Result

Proof. Suppose (G,Σ) has a minor isomorphic to a member of the $K_3^{=}$-family, K_4^e, K_4^o, or $K_{2,3}^e$. If (G,Σ) has a minor isomorphic to a member of the $K_3^{=}$-family, then Lemma 2.4 implies that $\xi(G,\Sigma) \geq 3$. From Lemma 1.51, if (G,Σ) has a minor isomorphic to K_4^e, K_4^o, or $K_{2,3}^e$, then $\xi(G,\Sigma) \geq 3$.

Suppose $\xi(G,\Sigma) \geq 3$. By Lemma 1.48, there exists a thin out (H,Ω) of a block of (G,Σ) with $\xi(H,\Omega) \geq 3$. Let (H',Ω') be obtained from (H,Ω) by removing pendant vertices. If (H',Ω') has a minor isomorphic to a member of the $K_3^{=}$-family, K_4^e, K_4^o, or $K_{2,3}^e$, then (G,Σ) has a minor isomorphic to a member of the $K_3^{=}$-family, K_4^e, K_4^o, or $K_{2,3}^e$. Therefore, we may assume that (H',Ω') has no minor isomorphic to $K_3^{=}$-family, K_4^e, K_4^o, or $K_{2,3}^e$. By Theorem 1.52, we know that (H',Ω') is either a W_4^o or a partial wide 2-path. From Lemma 2.3, we know that (H',Ω') is not W_4^o. Therefore, (H',Ω') is a partial wide 2-path. If (H',Ω') has at least two wide separations, then Lemma 2.16 implies (H,Ω) has a minor isomorphic to a member of the $K_3^{=}$-family. If (H',Ω') has exactly one wide separation, then Lemma 2.20 implies that (H,Ω) has a minor isomorphic to a member of the $K_3^{=}$-family. If (H',Ω') has no wide separations, then Lemma 2.21 implies (H,Ω) has a minor isomorphic to a member of the $K_3^{=}$-family. Because $(H,\Omega) \preceq (G,\Sigma)$, no matter how many wide separations are in (H',Ω'), (G,Σ) has a minor isomorphic to a member of the $K_3^{=}$-family.

$\square$

CHAPTER 3

Signed Graphs with Maximum Nullity at Most Two

The following theorem is the main result of this chapter. The last section of this chapter contains the proof.

Theorem 3.1. Let (G, Σ) be a signed graph. Then, $M(G, \Sigma) \leq 2$ if and only if one of the following holds:

1. (G, Σ) is a signed graph with two parallel paths;
2. (G, Σ) is a Seahorse;
3. (G, Σ) is a Starfish;
4. (G, Σ) is a Sea Anemone;
5. (G, Σ) is a Mollusk;
6. (G, Σ) is a Stingray;
7. (G, Σ) is obtained from W_4^o by attaching single pendant paths to some of the vertices of W_4^o;
8. (G, Σ) is obtained from attaching at most two pendant paths to each vertex of K_2; or
9. (G, Σ) is obtained from attaching at most two pendant paths to each vertex of $K_2^=$.

The above theorem extends the result of Johnson, Loewy, and Smith, a combinatorial characterization of the graphs with maximum nullity at most two [16]. We depict examples in Figure 3.1.

3.1 Global Structure of Signed Graphs (G,Σ) with $M(G,\Sigma) \le 2$

Lemma 3.2. Let (G,Σ) be the disjoint union of (G_1,Σ_1) and (G_2,Σ_2). Then,

$$M(G,\Sigma) = M(G_1,\Sigma_1) + M(G_2,\Sigma_2).$$

Proof. First, we observe that $\mathrm{corank}(A) = \mathrm{corank}(A_1) + \mathrm{corank}(A_2)$ whenever $A = A_1 \oplus A_2$. If $A_1 \in \mathcal{S}(G_1,\Sigma_1)$ and $A_2 \in \mathcal{S}(G_2,\Sigma_2)$, then $A_1 \oplus A_2 \in \mathcal{S}(G,\Sigma)$. Hence, $M(G_1,\Sigma_1) + M(G_2,\Sigma_2) \le M(G,\Sigma)$. If $A \in \mathcal{S}(G,\Sigma)$, then we may partition $A = A_1 \oplus A_2$ such that $A_1 \in \mathcal{S}(G_1,\Sigma_1)$ and $A_2 \in \mathcal{S}(G_2,\Sigma_2)$. Hence, $M(G,\Sigma) \le M(G_1,\Sigma_1) + M(G_2,\Sigma_2)$. $\square$

Lemma 3.3. Let (G,Σ) be a disconnected signed graph with $M(G,\Sigma) = 2$. Then, (G,Σ) is a disjoint union of two paths.

Proof. For each component (G_i,Σ_i) of (G,Σ), we know $M(G_i,\Sigma_i) \ge 1$. From Lemma 3.2, we know that $M(G_i,\Sigma_i) = 1$ for each component. So, (G,Σ) has exactly two components. From Theorem 1.14, (G_i,Σ_i) is a path for $i = 1,2$. That is, (G,Σ) is the disjoint union of two paths. $\square$

Definition 3.4. Let (G_1,Σ_1) and (G_2,Σ_2) be signed graphs. We may form the signed graph (G,Σ) by identifying a vertex $v_1 \in V(G_1)$ with a vertex $v_2 \in V(G_2)$ and name the vertex $v \in V(G)$. Then, (G,Σ) is the 1-sum of (G_1,Σ_1) and (G_2,Σ_2) at v. We say that (G,Σ) is obtained by *attaching* v_2 in (G_2,Σ) to v_1 in (G_1,Σ). If G_2 is a path, then we say that (G,Σ) is obtained by *attaching a path* to (G_1,Σ_1). If G_2 is a path and $d_{G_2}(v_2) = 1$, then we say that (G,Σ) is obtained by *attaching a pendant path* to (G_1,Σ_1); and (G,Σ) *has a pendant path attached to* v.

Lemma 3.5. Let (G, Σ) be a connected signed graph containing a cycle. If $M(G, \Sigma) \leq 2$, then either

1. (G, Σ) is obtained from a 2-connected signed graph (H, Ω) with $M(H, \Omega) \leq 2$ by attaching pendant paths at vertices of (H, Ω); or

2. (G, Σ) is obtained from a $K_2^=$ by attaching pendant paths to $K_2^=$.

Further, at each vertex of H at most two pendant paths can be attached.

Proof. If (G, Σ) has no cut vertex, then we may apply Theorem 1.52 to the case where $(G, \Sigma) = (H, \Omega)$. So, we may assume that (G, Σ) has a cut vertex, and we may find a 1-separation of (G, Σ). Suppose for a contradiction that $[(H_1, \Omega_1), (H_2, \Omega_2)]$ is a 1-separation of (G, Σ) such that H_1 contains a cycle and H_2 contains a cycle. From Lemma 1.39, we have $M(G, \Sigma) \geq M(H_1, \Omega_1) + M(H_2, \Omega_2) - 1 \geq 2 + 2 - 1 = 3$, but $M(G, \Sigma) \leq 2$. So, we may assume that H_1 has a cycle; and we may assume H_2 has no cycle. From Lemma 1.39, we have $M(H_1, \Omega_1) \leq M(G, \Sigma) - M(H_2, \Omega_2) + 1 \leq 2 - 1 + 1 = 2$.

Suppose for a contradiction that H_2 contains a vertex v such that $d_{H_2}(v) \geq 3$. Hence, we have $P(H_1) \geq 2$. As H_2 is a tree, Theorem 1.41 implies $M(H_1) = P(H_1) \geq 2$. From Lemma 1.27, we have $M(H_1, \Omega_1) = M(H_1) \geq 2$. From Lemma 1.39, we have

$$M(G, \Sigma) \geq M(H_1, \Omega_1) + M(H_2, \Omega_2) - 1 \geq 2 + 2 - 1 = 3.$$

Therefore, H_2 has no vertex v with $d_{H_2}(v) \geq 3$. That is, H_2 is a path. If (G, Σ) is the 1-sum of (H_1, Ω_1) and (H_2, Ω_2) at w, then w is either a pendant vertex of H_2 or an internal vertex of H_2. So, at most two pendant paths may be attached to H_1 at w. $\square$

Lemma 3.6. Let (G,Σ) be the 1-sum of (G_1,Σ_1) and (P,Σ_2) at v, where P is a path with endvertex v. Then,

$$M(G,\Sigma) = \max\{M(G_1,\Sigma_1), M(G_1 - v, \Sigma_1 \setminus \delta(v))\} \geq M(G_1,\Sigma_1).$$

Proof. From Lemma 1.39, we have

$$M(G,\Sigma) = \max\{M(G_1,\Sigma_1) + M(P,\Sigma_2) - 1, M(G_1 - v, \Sigma_1 \setminus \delta(v)) + M(P - v, \Sigma_2 \setminus \delta(v)) - 1\}.$$

Because $P - v$ is a path, $M(P,\Sigma_2) = M(P - v, \Sigma_2 \setminus \delta(v)) = 1$. So,

$$M(G,\Sigma) = \max\{M(G_1,\Sigma_1), M(G_1 - v, \Sigma_1 \setminus \delta(v))\} \geq M(G_1,\Sigma_1).$$

□

3.2 Pendant Paths on an Odd 4-Wheel

Lemma 3.7. If (G, Σ) is obtained from W_4^o by attaching single pendant paths to some of the vertices of W_4^o, then $M(G, \Sigma) = 2$. If (G, Σ) is obtained from W_4^o by attaching two pendant paths to a vertex of W_4^o, then $M(G, \Sigma) > 2$.

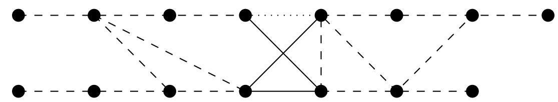
Proof. Suppose first that no vertex of W_4^o has more than one pendant path attached. Let $S \subseteq V(W_4^o)$ be the vertices with single pendant paths attached. If $S = \emptyset$, then Theorem 1.52 implies $M(W_4^o) = 2$. If $S \neq \emptyset$, then $W_4^o - S$ is a graph on two parallel paths, and Lemma 1.44 implies $M(W_4^o - S) \leq 2$. As (G, Σ) is formed from W_4^o adding single pendant paths, Lemma 3.6 implies $M(G, \Sigma) \leq 2$. Because G is not a path, Theorem 1.14 implies $M(G, \Sigma) = 2$.

Suppose next that we form (G, Σ) by attaching two pendant paths to $v \in V(W_4^o)$. Then, (G, Σ) is the 1-sum of W_4^o and P at v, where $P - v$ consists of two disjoint paths. So, $M(P - v) = 2$. From Lemma 1.39,

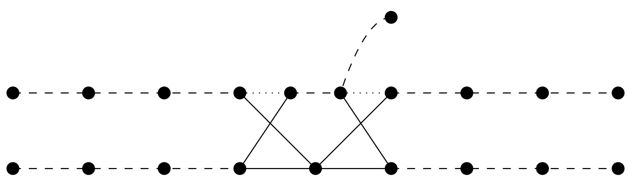
$$\begin{aligned} M(G, \Sigma) &= \max\{M(W_4^o) + M(P) - 1, M(W_4^o - v) + M(P - v) - 1\} \\ &= \max\{2 + 1 - 1, 2 + 2 - 1\} = 3. \end{aligned}$$

$\square$

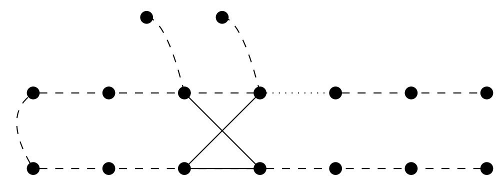
3.3 Pendant Paths Attached to 2-Connected Partial Wide 2-Paths



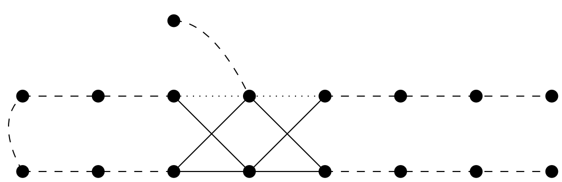
Signed graph with two parallel paths.



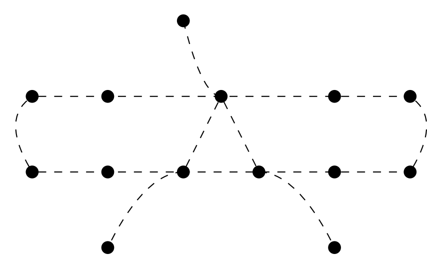
Stingray with $l(P_1) + l(P_2) = 1$.



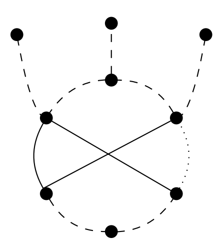
A Mollusk.



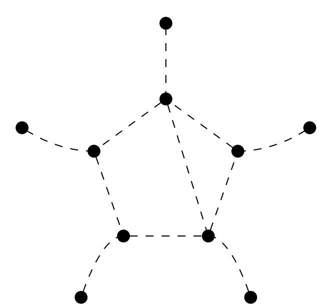
Stingray with $l(P_1) + l(P_2) = 0$.



A Seahorse.



A Sea Anemone with $l(P) = 2$.



A Starfish.

Figure 3.1 Examples of signed graphs with maximum nullity at most two. Solid edges are even; dotted edges are odd; and dashed lines may be even or odd.

3.3.1 Two Wide Separations

Definition 3.8. Let (G, Σ) be a signed graph obtained from adding pendant paths to (H, Ω), where (H, Ω) is a 2-connected partial wide 2-path. Suppose

- $[H_1, H_2]$ and $[H_3, H_4]$ are distinct wide separations of (H, Ω);
- $H_1 \subseteq H_3$ and $H_2 \subseteq H_4$;
- $\{r_1, r_2\}$ are the vertices of attachment of H_2;
- $\{s_1, s_2\}$ are the vertices of attachment of H_3; and
- P_1 and P_2 are vertex disjoint paths where
  - P_1 has endvertices r_1 and s_1, and
  - P_2 has endvertices r_2 and s_2.

We call (G, Σ) a *Stingray* if the following hold:

1. No vertex of H is attached to two or more pendant paths.
2. There is exactly one pendant path attached to a vertex of P_1 or P_2.
3. $l(P_1) + l(P_2) \leq 1$.
4. If $l(P_1) + l(P_2) = 1$, then H_1 and H_4 are disconnected.
5. If $l(P_1) + l(P_2) = 0$, then
   - exactly one of H_1 or H_4 is disconnected, and the other one is a path Q;

- if $l(Q) \geq 2$, then there is at most one pendant path attached to an endvertex of Q, and there are no pendant paths attached to an internal vertex of Q.

Lemma 3.9. If (G, Σ) is a Stingray, then $M(G, \Sigma) = 2$.

Proof. Because (G, Σ) is not a path, Theorem 1.14 implies $M(G, \Sigma) \geq 2$. Let P be the pendant path attached to the vertex $v \in V(P_1) \cup V(P_2)$. Let $[G_1, P]$ be a 1-separation of G at v. By definition, $(G_1, E(G_1) \cap \Sigma)$ is a 2-connected partial wide 2-path, and Theorem 1.52 implies $M(G_1, E(G_1) \cap \Sigma) \leq 2$. Because $P(G_1 - P) = 2$, Theorem 1.42 implies $M(G_1 - P, E(G_1 - P) \cap \Sigma) \leq M(G_1 - P) = P(G_1 - P) = 2$. Hence, we may apply Lemma 3.6 to the 1-separation at v, and we have

$$M(G, \Sigma) = \max\{M(G_1, E(G_1) \cap \Sigma), M(G_1 - P, E(G_1 - P) \cap \Sigma)\} \leq 2.$$

$\square$

Lemma 3.10. Let (G, Σ) be a signed graph such that the removal of pendant paths yields a 2-connected partial wide 2-path (H, Ω) with at least two wide separations. If there is a vertex with at least two pendant paths attached in (G, Σ), then $M(G, \Sigma) \geq 3$.

Proof. Let v be the vertex of (G, Σ) with at least two pendant paths attached. Let (G_1, Σ_1) be a path with endvertices on these pendant paths. Then, we have a 1-separation $[(G_1, \Sigma_1), (G_2, \Sigma_2)]$ at v. For $i = 1, 2$, let $(G_i^{(1)}, \Sigma_i^{(1)})$ be the signed graph obtained from (G_i, Σ_i) by deleting the vertex v. Then, $G_1^{(1)}$ is a disjoint union of two paths. From Theorem 1.14 and Lemma 3.2, we have $M(G_1^{(1)}, \Sigma_1^{(1)}) = 2$. Because (G, Σ) has two wide separations, $G_2^{(1)}$ is not a path, so we have $M(G_2^{(1)}, \Sigma_2^{(1)}) \geq 2$. From Lemma 1.39, we have

$$M(G, \Sigma) \geq M(G_1^{(1)}, \Sigma_1^{(1)}) + M(G_2^{(1)}, \Sigma_2^{(1)}) - 1 \geq 2 + 2 - 1 = 3.$$

□

Lemma 3.11. Let (G, Σ) be a signed graph such that the removal of pendant paths yields a 2-connected partial wide 2-path (H, Ω). Let $[H_1, H_2]$ and $[H_3, H_4]$ be distinct wide separations in (H, Ω) such that $H_1 \subseteq H_3$ and $H_4 \subseteq H_2$. Let

- r_1, r_2 be the vertices of attachment of H_2,
- s_1, s_2 be the vertices of attachment of H_3, and
- P_1 and P_2 be vertex-disjoint paths between $\{r_1, r_2\}$ and $\{s_1, s_2\}$, where P_i has endvertices r_i and s_i for $i = 1, 2$.

Suppose a pendant path is incident with a vertex on P_1 or P_2. Then, $M(G, \Sigma) = 2$ if and only if (G, Σ) is a Stingray.

Proof. From Lemma 3.9, if (G, Σ) is a Stingray, then $M(G, \Sigma) = 2$.

Suppose $M(G, \Sigma) = 2$. We want to show that (G, Σ) is a Stingray. Because $\xi(G, \Sigma) \leq M(G, \Sigma) \leq 2$, Theorem 2.1 implies (G, Σ) has no minor isomorphic to a member of the $K_3^{=}$-family. From Lemma 3.10, no vertex of (G, Σ) has more than one pendant path attached.

Suppose a pendant path is attached to an internal vertex of P_1 or P_2. Then, we found a minor of (G, Σ) isomorphic to $K_3^{=}(\Delta Y)$. So, we may assume that any pendant path attached to a vertex of P_1 or P_2 is attached to an endvertex of P_1 or P_2.

Next, we want to prove that $l(P_1) + l(P_1) \leq 1$. By symmetry, we assume that (G, Σ) has a pendant path attached to an endvertex of P_1. If $l(P_1) \geq 2$, then we found a minor of (G, Σ) isomorphic to $K_3^=(\Delta Y)^3$. So, we may assume that $l(P_1) \leq 1$. If $l(P_2) \geq 2$, then we found a minor of (G, Σ) isomorphic to $K_3^=(\Delta Y)^3$. So, we may assume that $l(P_2) \leq 1$. If $l(P_1) = l(P_2) = 1$, then we found a minor of (G, Σ) isomorphic to $K_3^=(\Delta Y)^2$. So, we have $l(P_1) + l(P_2) \leq 1$.

Suppose that $l(P_1) + l(P_2) = 1$. By symmetry, we may assume that $l(P_1) = 1$ and $l(P_2) = 0$. Because $l(P_2) = 0$, we have $r_2 = s_2$. If H_1 is connected and there is a pendant path attached to $\{r_1, r_2 = s_2\}$, then we found a minor of (G, Σ) isomorphic to $K_3^=(\Delta Y)^2$. If H_1 is connected and there is a pendant path attached to s_1, then we found a minor of (G, Σ) isomorphic to $K_3^=(\Delta Y)$. Hence, H_1 is disconnected. By symmetry, we may apply the same argument to H_4. Hence, both H_1 and H_4 are disconnected. Suppose for a contradiction that more than one pendant path is attached to $V(P_1) \cup V(P_2)$. Let $(G^{(1)}, \Sigma^{(1)})$ be the graph obtained by removing these pendant paths and their vertices of attachment. Then, $P(G^{(1)}, \Sigma^{(1)}) \geq 3$. As $(G^{(1)}, \Sigma^{(1)})$ is a partial 2-path, Theorem 1.42 implies $M(G^{(1)}, \Sigma^{(1)}) = P(G^{(1)}\Sigma^{(1)}) \geq 3$. From Lemma 1.39, we have $M(G, \Sigma) \geq M(G^{(1)}, \Sigma^{(1)}) \geq 3$. Hence, there is at most one pendant path attached to $V(P_1) \cup V(P_2)$. That is, (G, Σ) is a Stingray.

Suppose that $l(P_1) + l(P_2) = 0$. Then, $r_1 = s_1$ and $r_2 = s_2$. If both H_1 and H_4 are connected, then we found a minor of (G, Σ) isomorphic to $K_3^=(\Delta Y)$. Suppose that both H_1 and H_4 are disconnected. Let $(G^{(1)}, \Sigma^{(1)})$ be the signed graph obtained by removing the pendant path P and its vertex of attachment from (G, Σ). Because $P(G^{(1)}, \Sigma^{(1)}) \geq 3$, Theorem 1.42 implies $M(G^{(1)}, \Sigma^{(1)}) = P(G^{(1)}, \Sigma^{(1)}) \geq 3$. From Lemma 1.39, we have $M(G, \Sigma) \geq M(G^{(1)}, \Sigma^{(1)}) \geq 3$. Because $M(G, \Sigma) = 2$, we may assume that H_1 is disconnected and H_4 is connected or that H_1 is connected and H_4 is disconnected. By symmetry, we may assume that H_1 is disconnected and H_4 is connected. Suppose (G, Σ) has more than one pendant path attached to $V(P_1) \cup V(P_2)$. Then, there must be one pendant path attached to r_1 and one pendant path attached to r_2. Let $(G^{(2)}, \Sigma^{(2)})$ be the signed graph obtained by removing these pendant paths and their vertices of attachment. Because $P(G^{(2)}, \Sigma^{(2)}) \geq 3$, Theorem 1.42 implies $M(G^{(2)}, \Sigma^{(2)}) = P(G^{(2)}, \Sigma^{(2)}) \geq 3$. From Lemma 1.39, we have $M(G, \Sigma) \geq M(G^{(2)}, \Sigma^{(2)}) \geq 3$. Hence, (G, Σ) has exactly one pendant path P attached to a vertex in $V(P_1) \cup V(P_2)$.

By symmetry, we may assume that (G, Σ) has a pendant path P attached to P_1. If H_4 has a cycle, then we found a minor of (G, Σ) isomorphic to $K_3^{=}(\Delta Y)^2$. Hence, we may assume that H_4 has no cycle. Let Q be the path in H_4 connecting the vertices of attachment of the wide separation $[H_3, H_4]$. If Q has a pendant path attached to an internal vertex of Q, then we found a minor of (G, Σ) isomorphic to $K_3^{=}(\Delta Y)^3$. Hence, we may assume any pendant path attached to a vertex of Q is attached to an endvertex of Q. If $l(Q) \geq 2$ and (G, Σ) has pendant paths attached to both endvertices of Q, then we found a minor of (G, Σ) isomorphic to $K_3^{=}(\Delta Y)^2$. Hence, we may assume that $l(Q) = 1$ or that (G, Σ) has at most one pendant path attached to an endvertex of Q. That is, (G, Σ) is a Stingray. $\square$

Lemma 3.12. Let (G, Σ) be a signed graph without multiple edges. Suppose some vertices are colored blue and other vertices are colored white. Suppose $(C_4, \{14\})$ is a subgraph of

(G, Σ), where $V(C_4) = \{1, 2, 3, 4\}$ and $E(C_4) = \{14, 42, 23, 31\}$. If the vertices $\{1, 2\}$ are colored blue and $\{3, 4\}$ are the only white vertices adjacent to $\{1, 2\}$ in (G, Σ), then we may color the vertices $\{3, 4\}$ blue in (G, Σ).

Proof. Let (H, Ω) be the induced subgraph on the vertices $\{1, 2, 3, 4\}$. Further, let (H, Ω) have an odd edge $\{14\}$ and even edges $\{42, 23, 31\}$. Let $[a_{i,j}] = A \in \mathcal{S}(H, \Omega)$. If $x \in \ker(A)$, then

$$Ax = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{1,2} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{1,3} & a_{2,3} & a_{3,3} & a_{3,4} \\ a_{1,4} & a_{2,4} & a_{3,4} & a_{4,4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0.$$

We color the vertices $\{1, 2\}$ blue. That is, $x_1 = x_2 = 0$. Then, we have the following equations

$$a_{1,3}x_3 + a_{1,4}x_4 = 0 \tag{3.1}$$

$$a_{2,3}x_3 + a_{2,4}x_4 = 0. \tag{3.2}$$

Because $a_{1,3} < 0$ and $a_{1,4} > 0$, x_3 and x_4 have the same sign. Because $a_{2,3} < 0$ and $a_{2,4} < 0$, x_3 and x_4 have opposite signs. Hence, $x_3 = x_4 = 0$; and, $x = 0$. That is, we may color $\{3, 4\}$ blue in (H, Ω). Let $B \in \mathcal{S}(G, \Sigma)$. Then, we may write

$$By = \begin{bmatrix} A & R \\ R^T & S \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} Rz \\ R^T x + Sz \end{bmatrix}.$$

If $\{3, 4\}$ are the only white neighbors of $\{1, 2\}$ in (G, Σ), then the nonzero entries of R correspond to blue vertices. If $y \in \ker(A)$, then we may assume the coordinates of y which correspond to these blue vertices are forced to be zero; that is, $r_{i,k} \neq 0$ implies $z_k = 0$.

Hence, $Rz = 0$. Therefore, we may color $\{3, 4\}$ blue in (G, Σ) whenever $\{3, 4\}$ are the only neighbors of $\{1, 2\}$ which are colored white in (G, Σ). □

Definition 3.13. A signed graph has *two parallel paths* if there exist two pairs of vertices $\{u_1, u_2\}$ and $\{v_1, v_2\}$ such that (G, Σ) is a spanning subgraph of a sided wide 2-path with sides u_1u_2 and v_1v_2, and there exists two disjoint paths with endvertices $\{u_1, v_1\}$ and $\{u_2, v_2\}$, respectively.

Lemma 3.14. Let (G, Σ) be a signed graph with two parallel paths. Then, $M(G, \Sigma) \leq 2$. If G is not a path, then $M(G, \Sigma) = 2$ and $Z(G, \Sigma) = 2$.

Proof. By definition, we may find a sided wide 2-path with sides u_1u_2 and v_1v_2 such that (G, Σ) is a spanning subgraph. If (G, Σ) has no wide separation, then $\{u_1, u_2\}$ is a zero forcing set for (G, Σ). Otherwise, we may continue coloring vertices blue until we color the vertices $\{r_1, r_2\}$ blue, where $\{r_1, r_2\}$ are the vertices of attachment for some wide separation. From Lemma 3.12, we may also color the other two vertices of attachment blue. Hence, $\{u_1, u_2\}$ is a zero forcing set for (G, Σ), and $Z(G, \Sigma) \leq 2$. From Lemma 1.38, $M(G, \Sigma) \leq Z(G, \Sigma) \leq 2$. If (G, Σ) is not a path, Theorem 1.14 implies $M(G, \Sigma) = 2$, and we also have $Z(G, \Sigma) = 2$. □

Lemma 3.15. Let (G, Σ) be a signed graph such that the removal of pendant paths yields a 2-connected partial wide 2-path (H, Ω), with at least two distinct wide separations. Then, $M(G, \Sigma) = 2$ if and only if

1. (G, Σ) is a signed graph with two parallel paths, or

2. (G, Σ) is a Stingray.

Proof. If (G, Σ) is a signed graph with two parallel paths, then Lemma 3.14 implies $M(G, \Sigma) = 2$. If (G, Σ) is a Stingray, then Lemma 3.9 implies $M(G, \Sigma) = 2$.

Suppose next that (G, Σ) is a signed graph such that the removal of pendant paths yields a 2-connected partial wide 2-path (H, Ω) with at least two distinct wide separations. Let $[H_1, H_2]$ and $[H_3, H_4]$ be distinct wide separations in (H, Ω) such that $H_1 \subseteq H_3$ and $H_4 \subseteq H_2$. We take $[H_1, H_2]$ in (H, Ω) such that there is no wide separation $[H_1^{(1)}, H_2^{(1)}]$ with $H_1^{(1)}$ a proper subgraph of H_1. Similarly, we take $[H_3, H_4]$ such that there is no wide separation $[H_3^{(1)}, H_4^{(1)}]$ with $H_4^{(1)}$ as a proper subgraph of H_4. Let $\{u_1, u_2\}$ be the vertices of attachment of H_1; let $\{r_1, r_2\}$ be the vertices of attachment of H_2; and let $\{s_1, s_2\}$ be the vertices of attachment of H_3. Let P_1 and P_2 be disjoint paths between $\{r_1, r_2\}$ and $\{s_1, s_2\}$ such that P_i has endvertices r_i and s_i for $i = 1, 2$. If a pendant path is attached to a vertex in $V(P_1) \cup V(P_2)$, then Lemma 3.11 implies that (G, Σ) is a Stingray. So, we may assume that no pendant path is attached to $V(P_1) \cup V(P_2)$.

Suppose H_1 contains a cycle C. We may assume that C is at the end of the partial wide 2-path H. That is, there is a 2-separation $[C, F]$ of H. Let $\{v_1, v_2\} = V(C) \cap V(F)$. Let Q_1 and Q_2 be two disjoint paths between $\{v_1, v_2\}$ and $\{u_1, u_2\}$, such that Q_i has endvertices v_i and u_i for $i = 1, 2$. If a pendant path is attached to a vertex of $Q_1 - v_1$ or to a vertex of $Q_2 - v_2$, then we found a minor of (G, Σ) isomorphic to $K_3^{=}(\Delta Y)$. So, any pendant path of (G, Σ) attached to Q_i is attached to v_i. Let P be the path obtained from C by removing the edge between v_1 and v_2.

Suppose H_1 contains no cycle. As H is 2-connected, H_1 is connected. Let P be the path in H_1 with endvertices u_1 and u_2.

Suppose there are two pendant paths attached to vertices $w_1, w_2 \in V(P)$. If $d_P(w_1, w_2) \geq 2$, then we found a minor of (G, Σ) isomorphic to $K_3^{=}(\Delta Y)^2$. Hence, there are at most two pendant paths attached to vertices $\{w_1, w_2\}$, and $d_P(w_1, w_2) = 1$. By symmetry, we may apply the same argument to the wide separation $[H_3, H_4]$. Hence, (G, Σ) is a signed graph with two parallel paths.

$\square$

3.3.2 One Wide Separation

Definition 3.16. Let (G, Σ) be a signed graph obtained from adding pendant paths to (H, Ω), where (H, Ω) is a 2-connected partial wide 2-path with exactly one wide separation $[H_1, H_2]$. Let $\{u_1, u_2\}$ be be the vertices of attachment of H_1, and let $\{w_1, w_2\}$ be the vertices of attachment of H_2. We call (G, Σ) a *Sea Anemone* if the following hold:

1. H_2 is a path.
2. The removal of any edge between u_1 and u_2 in H_1 yields a path P.
3. There is a single pendant path attached to each vertex in $\{u_1, u_2\}$.
4. If there is a pendant path attached to an internal vertex of P, then $l(P) = 2$.
5. There are no pendant paths attached to the internal vertices of H_2.
6. If $l(H_2) \geq 3$ and there are pendant paths attached to each vertex in $\{w_1, w_2\}$, then there is no pendant path attached to any internal vertex of P.
7. There is no vertex of (H, Ω) with two pendant paths attached in (G, Σ).

Lemma 3.17. If (G, Σ) is a Sea Anemone, then $M(G, \Sigma) = 2$.

Proof. Let P_1 and P_2 be the pendant paths attached to u_1 and u_2, respectively. Let $(G^{(1)}, \Sigma^{(1)}) = (G - P_1, \Sigma \setminus E(P_1))$, and let $(G^{(2)}, \Sigma^{(2)}) = (G - (P_1 - u_1), \Sigma \setminus E(P_1 - u_1))$. Then, $G^{(1)}$ is a partial 2-path with $P(G^{(1)}) = 2$. From Theorem 1.42, $M(G^{(1)}, \Sigma^{(1)}) \leq M(G^{(1)}) = 2$. From Lemma 1.39, we have $M(G, \Sigma) = \max\{M(G^{(1)}, \Sigma^{(1)}), M(G^{(2)}, \Sigma^{(2)})\}$. Hence, we may assume that $M(G, \Sigma) = M(G^{(2)}, \Sigma^{(2)})$.

Suppose $(G^{(2)}, \Sigma^{(2)})$ has a pendant path Q attached to a vertex $w \in \{w_1, w_2\}$. Let $(H, \Omega) = (G^{(2)} - Q, \Sigma^{(2)} \setminus E(Q))$, and let $(H^{(1)}, \Omega^{(1)}) = (G^{(2)} - (Q - w), \Sigma^{(2)} \setminus E(Q - w))$. Then, H is a partial 2-path with $P(H) = 2$. From Theorem 1.42, we have $M(H, \Omega) \leq M(H) = P(H) = 2$. From Lemma 1.39, we have $M(G^{(2)}, \Sigma^{(2)}) = \max\{M(H, \Omega), M(H^{(1)}, \Omega)\}$. Hence, we may assume that $M(G, \Sigma) = M(G^{(2)}, \Sigma^{(2)}) = M(H^{(1)}, \Omega^{(1)})$. If there is a pendant path attached to an internal vertex of P, then the pendant vertex of the pendant path attached to P and the pendant vertex of P_2 are a zero forcing set of $(H^{(1)}, \Omega^{(1)})$ by Lemma 3.12. Otherwise, the pendant vertex of P_2 and v, where $d_P(v, u_2) = 1$, are a zero forcing set of $(H^{(1)}, \Omega^{(1)})$ by Lemma 3.12. By Lemma 1.38, we have $M(H^{(1)}, \Omega^{(1)}) \leq Z(H^{(1)}, \Omega^{(1)}) \leq 2$. Hence, $M(G, \Sigma) = M(G^{(2)}, \Sigma^{(2)}) = M(H^{(1)}, \Omega^{(1)}) = 2$. So, we may assume that no pendant path Q is attached to a vertex $w \in \{w_1, w_2\}$. Whenever there is no pendant path Q, the same zero forcing sets for our argument on $(H^{(1)}, \Omega^{(1)})$ are also zero forcing sets for $(G^{(2)}, \Sigma^{(2)})$; and $M(G, \Sigma) = M(G^{(2)}, \Sigma^{(2)}) = 2$. Therefore, $M(G, \Sigma) = 2$. □

Lemma 3.18. Let (G, Σ) be a signed graph such that the removal of pendant paths yields a partial wide 2-path (H, Ω) with exactly one wide separation $[H_1, H_2]$. Suppose H_1 and H_2 are connected. Then, $M(G, \Sigma) = 2$ if and only if one of the following holds:

1. (G, Σ) is a signed graph with two parallel paths, or

2. (G, Σ) is a Sea Anemone.

Proof. If (G, Σ) is a Sea Anemone, then Lemma 3.17 implies $M(G, \Sigma) = 2$. If (G, Σ) is a signed graph with two parallel paths, then Lemma 3.14 implies $M(G, \Sigma) = 2$.

Suppose (G, Σ) is a signed graph with exactly one wide separation $[H_1, H_2]$ such that $M(G, \Sigma) = 2$. Let u_1, u_2 be the vertices of attachment of H_1, and let w_1, w_2 be the vertices of attachment of H_2. Suppose for a contradiction that two pendant paths R_1 and R_2 are attached to a vertex s. Let $(G^{(1)}, \Sigma^{(1)}) = (G - R_1, \Sigma \setminus E(R_1))$. Then, $G^{(1)}$ has at least two components, one of which is a not path; and Lemma 3.3 implies $M(G^{(1)}, \Sigma^{(1)}) \geq 3$. As (G, Σ) is the 1-sum of R_1 and $(G^{(1)}, \Sigma^{(1)})$ at s, Lemma 1.39 implies $M(G, \Sigma) \geq M(G^{(1)}, \Sigma^{(1)}) \geq 3$. Hence, no vertex of (G, Σ) has two or more pendant paths attached.

Suppose that H_1 has a cycle C. We may assume that C is at the end of the partial wide 2-path H. That is, there is a 2-separation (C, F) of H. Let $\{v_1, v_2\} = V(C) \cap V(F)$. Let Q_1 and Q_2 be disjoint paths between $\{v_1, v_2\}$ and $\{u_1, u_2\}$ such that Q_i has endvertices v_i and u_i for $i = 1, 2$. Suppose a pendant path in (G, Σ) is attached to a vertex in $V(H_1) \setminus V(C)$. That is, the pendant path is attached to a vertex of $V(Q_i) \setminus v_i$ for $i = 1$ or $i = 2$. Hence, we found a minor of (G, Σ) isomorphic to $K_3^{=}(\Delta Y)$. From Theorem 2.1, we have $M(G, \Sigma) \geq \xi(G, \Sigma) \geq 3$. Hence, we may assume that any pendant path attached to a vertex of H_1 is attached to a a vertex of C. Let $P_1 = C - \{v_1 v_2\}$. If there are two pendant paths attached to vertices $x_1, x_2 \in V(P)$ with $d_P(x_1, x_2) \geq 2$ and $\{x_1, x_2\} \neq \{u_1, u_2\}$, then we found a minor of (G, Σ) isomorphic to a $K_3^{=}(\Delta Y)^2$. From Theorem 2.1, we may assume that either

- (G, Σ) has at most two pendant paths attached to vertices of P_1;
- (G, Σ) has three pendant attached to vertices of P_1, $l(P_1) = 2$, and the endvertices of P_1 are u_1 and u_2; or
- (G, Σ) has two pendant paths attached to the endvertices of P_1.

Because both H_1 and H_2 are connected, we apply the same argument to H_2, and the above is also true for $P_2 \subseteq H_2$.

Suppose (G, Σ) has at most two pendant paths attached to the vertices $x_1, x_2 \in V(P_1)$ and at most two pendant paths attached to the vertices $y_1, y_2 \in V(P_2)$. If $d_{P_1}(x_1, x_2) = d_{P_2}(y_1, y_2) = 1$, then (G, Σ) has two parallel paths. Lemma 3.14 implies $M(G, \Sigma) = 2$. Hence, we may assume that either

- (G, Σ) has pendant paths attached to both endvertices of P_1, the endvertices of P_1 are u_1 and u_2, and $l(P_1) \geq 2$; or
- (G, Σ) has pendant paths attached to both endvertices of P_2, the endvertices of P_2 are w_1 and w_2, and $l(P_2) \geq 2$.

By symmetry, we may assume that (G, Σ) has pendant paths attached to both endvertices of P_1, the endvertices of P_1 are u_1 and u_2, and $l(P_2) \geq 2$. If H_2 contains a cycle, then we found a minor of (G, Σ) isomorphic to $K_3^{=}(\Delta Y)^2$. From Theorem 2.1, we may assume that H_2 has no cycle. Because H is 2-connected and because H_2 is connected, H_2 is a path with endvertices w_1 and w_2.

Suppose a pendant path of (G, Σ) is attached to an internal vertex of P_2. Then, we found a minor of (G, Σ) isomorphic to $K_3^{=}(\Delta Y)^3$. From Theorem 2.1, we may assume that no pendant path of (G, Σ) is attached to an internal vertex of P_2. If $l(P_1) > 2$ and there is a pendant path attached to an internal vertex of P_1, then we found a minor of (G, Σ) isomorphic to $K_3^{=}(\Delta Y)^2$. From Theorem 2.1, we may assume that if a pendant path is attached to an internal vertex of P_1, then $l(P_1) = 2$. Suppose that $l(P_1) = 2$, that (G, Σ)

has a pendant path attached to an internal vertex of P_1, and that $l(P_2) \geq 2$. If (G, Σ) has a pendant path attached to w_1 or w_2, then we found a minor of (G, Σ) isomorphic to $K_3^=(\Delta Y)^3$. From Theorem 2.1, we may assume that if (G, Σ) has a pendant path attached to an internal vertex of P_1 and if P_2 has an internal vertex, then (G, Σ) has no pendant paths attached to either w_1 or w_2. That is, (G, Σ) is a Sea Anemone. □

Definition 3.19. Let (G, Σ) be a signed graph such that removing all pendant paths yields a 2-connected partial wide 2-path (H, Ω) with exactly one wide separation $[H_1, H_2]$. Let u_1 and u_2 be the vertices of attachment of H_1. Suppose H_1 contains a path of length at least two with endvertices u_1 and u_2. Suppose H_2 is disconnected. We call (G, Σ) a Mollusk if each of the following hold.

1. There is a pendant path at u_1.
2. There is no pendant path at u_2.
3. $H_1 - \{u_1, u_2\}$ is a path.
4. Each pendant path attached to a vertex of $H_1 - \{u_1, u_2\}$ is attached to an endvertex of $H_1 - \{u_1, u_2\}$.
5. No vertex of H is attached to more than one pendant path.

Lemma 3.20. If (G, Σ) is a Mollusk, then $M(G, \Sigma) = 2$.

Proof. Let v_1 and v_2 be the vertices of attachment of H_2. Let P be the pendant path at u_1; let $(G^{(1)}, \Sigma^{(1)})$ be the signed graph obtained from (G, Σ) by removing the vertices

$V(P-u_1)$; and let $(G^{(2)},\Sigma^{(2)})$ be the signed graph obtained from (G,Σ) by removing the vertices $V(P)$. As G is the 1-sum of $G^{(2)}$ and P at u_1, Lemma 3.6 implies $M(G,\Sigma) = \max\{M(G^{(1)},\Sigma^{(1)}), M(G^{(2)},\Sigma^{(2)})\}$. Because $(G^{(1)},\Sigma^{(1)})$ has two parallel paths, Lemma 3.14 implies $M(G^{(1)},\Sigma) = 2$. Hence, we may assume that $M(G,\Sigma) = M(G^{(2)},\Sigma^{(2)})$.

The signed graph $(G^{(2)},\Sigma^{(2)})$ has two pendant paths attached to u_2. Let Q be one of these pendant paths. Let $(F^{(1)},\Psi^{(1)})$ be the signed graph obtained from $(G^{(2)},\Sigma^{(2)})$ by removing the vertices $V(Q-u_2)$, and let $(F^{(2)},\Psi^{(2)})$ be the signed graph obtained from $(G^{(2)},\Sigma^{(2)})$ by removing the vertices $V(Q)$. Because $(F^{(1)},\Sigma^{(1)})$ has two parallel paths, Lemma 3.14 implies $M(F^{(1)},\Psi^{(1)}) = 2$. Because $(F^{(2)},\Sigma^{(2)})$ has two parallel paths, Lemma 3.14 implies $M(F^{(2)},\Psi^{(2)}) = 2$. Because $G^{(2)}$ is the 1-sum of $F^{(2)}$ and Q, Lemma 3.6 implies $M(G^{(2)},\Sigma^{(2)}) = \max\{M(F^{(1)},\Psi^{(1)}), M(F^{(2)},\Psi^{(2)})\} = 2$. Therefore, $M(G,\Sigma) = M(G^{(2)},\Sigma^{(2)}) = 2$. □

Lemma 3.21. Let (G,Σ) be a signed graph such that the removal of pendant paths yields a 2-connected partial wide 2-path (H,Ω) with exactly one wide separation $[H_1,H_2]$. Suppose that H_2 is disconnected and H_1 is connected. Then, $M(G,\Sigma) = 2$ if and only if

1. (G,Σ) is a signed graph with two parallel paths; or

2. (G,Σ) is a Mollusk.

Proof. Suppose (G,Σ) is a signed graph with two parallel paths. Because (G,Σ) is not a path, Lemma 3.14 implies $M(G,\Sigma) = 2$. Suppose (G,Σ) is a Mollusk. Then, Lemma 3.20 implies $M(G,\Sigma) = 2$.

For the converse, suppose that $M(G,\Sigma)=2$. Let u_1 and u_2 be the vertices of attachment of H_1. We proceed with a case study on whether H_1 has a cycle or not.

Suppose that H_1 contains a cycle C. Let Q_1 and Q_2 be disjoint paths between $\{u_1,u_2\}$ and $V(C)$ such that Q_i has endvertices $\{u_i,v_i\}$ for $i=1,2$ and $v_1,v_2\in V(C)$. We proceed with a case study on whether (G,Σ) has a pendant path attached to a vertex in $V(Q_1)\setminus v_1$ or a vertex in $V(Q_2)\setminus v_2$.

Suppose there is a pendant path attached to a vertex in $V(Q_1)\setminus v_1$ or attached to a vertex in $V(Q_2)\setminus v_2$. If the pendant path is attached to an internal vertex of Q_1 or an internal vertex of Q_2, then we found a minor of (G,Σ) isomorphic to $K_3^=(\Delta Y)$. From Theorem 2.1, we have $M(G,\Sigma)\geq\xi(G,\Sigma)\geq 3$. So, the pendant path must be attached to u_1 or u_2. By symmetry, we may assume that a pendant path P_1 is attached to u_1 in (G,Σ). Then, $l(Q_1)\geq 1$. From Lemma 3.5, we know that P_1 is the only pendant path attached to u_1 in (G,Σ). If $l(Q_2)>0$, then we found a minor of (G,Σ) isomorphic to $K_3^=(\Delta Y)^2$. From Theorem 2.1, we may assume that $l(Q_2)=0$ and $u_2=v_2$. If $H_1-\{u_1,u_2\}$ has a cycle, then we found a minor of (G,Σ) isomorphic to $K_3^=(\Delta Y)^2$. From Theorem 2.1, we may assume than $H_1-\{u_1,u_2\}$ has no cycle. If there is a pendant path attached P_2 to u_2 in (G,Σ), then $G-\{P_1,P_2\}$ is a forest with $P(G-\{P_1,P_2\})=3$. From Theorem 1.41, $M(G-\{P_1,P_2\})=P(G-\{P_1,P_2\})=3$. From Lemma 1.39, we have

$$M(G,\Sigma)\geq M\big(G-\{P_1\},\Sigma\setminus E(P_1)\big)\geq M\Big(G-\{P_1,P_2\},\Sigma\setminus\big(E(P_1)\cup E(P_2)\big)\Big)=3.$$

Hence, we may assume that no pendant path is attached to u_2. Let G_1 be the graph obtained from H_1 by attaching the pendant paths of (G,Σ) to the vertices of H_1. If $G_1-\{u_1,u_2\}$ is not a path, then we found a minor of (G,Σ) isomorphic to $K_3^{=}(\Delta Y)^3$. Hence, $H_1 - \{u_1, u_2\}$ is a path. That is, (G,Σ) is a Mollusk.

Suppose next that there are no pendant paths attached to any vertex in $V(Q_1) \setminus v_1$ nor any vertex in $V(Q_2) \setminus v_2$. Hence, any pendant path in (G,Σ) attached to a vertex of H_1 is attached to $V(C) \setminus \{u_1, u_2\}$. Let P be the path formed by removing any edges between v_1 and v_2 in C. If there are pendant paths attached to $w_1, w_2 \in V(P)$ such that $d_P(w_1, w_2) \geq 2$ and $\{w_1, w_2\} \neq \{u_1, u_2\}$, then we found a minor of (G,Σ) isomorphic to $K_3^{=}(\Delta Y)^2$. From Theorem 2.1, we may assume that there are at most two pendant paths attached to $V(P)$ in (G,Σ) and $d_P(w_1, w_2) = 1$. That is, (G,Σ) is a signed graph with two parallel paths. Therefore, if H_1 has a cycle, then either (G,Σ) is a Mollusk or (G,Σ) is a signed graph with two parallel paths.

Suppose next that H_1 contains no cycle. Then, H_1 is a path P with endvertices u_1 and u_2. Suppose there are pendant paths attached to vertices $w_1, w_2 \in V(P)$ such that $d_P(w_1, w_2) \geq 2$. Then, $l(P) \geq 2$. Suppose that a pendant path P_1 is attached to u_1. If there is a pendant path P_2 attached to u_2, then $G - \{P_1, P_2\}$ is a forest with $P(G - \{P_1, P_2\}) = 3$. From Theorem 1.41, $M(G - \{P_1, P_2\}) = P(G - \{P_1, P_2\}) = 3$. From Lemma 1.39, we have

$$M(G,\Sigma) \geq M\big(G - \{P_1\}, \Sigma \setminus E(P_1)\big) \geq M\Big(G - \{P_1, P_2\}, \Sigma \setminus \big(E(P_1) \cup E(P_2)\big)\Big) = 3.$$

Hence, we may assume that no pendant path is attached to u_2. Let G_1 be the graph obtained from H_1 by attaching the pendant paths of (G,Σ) to the vertices of H_1. If $G_1 - \{u_1, u_2\}$ is not a path, then we found a minor of (G,Σ) isomorphic to $K_3^{=}(\Delta Y)^3$. From Theorem 2.1, we may assume that $G_1 - \{u_1, u_2\}$ is a path. Hence, $H_1 - \{u_1, u_2\}$ is a path. That is, (G,Σ)

is a Mollusk.

Suppose next that no pendant path is attached to the endvertices of P, which are u_1 and u_2. Then, we found a minor of (G, Σ) isomorphic to $K_3^=(\Delta Y)^2$. By Theorem 2.1, we may assume that $d_P(w_1, w_2) = 1$. Then, (G, Σ) has at most two pendant paths attached to two internal vertices of P and these two internal vertices of P are also neighbors in P. That is, (G, Σ) is a signed graph with two parallel paths. Therefore, if H_1 has no cycle then either (G, Σ) is a Mollusk or (G, Σ) is a signed graph with two parallel paths. □

Lemma 3.22. Let (G, Σ) be a signed graph such that the removal of pendant paths yields a 2-connected partial wide 2-path (H, Ω) with exactly one wide separation. Then, $M(G, \Sigma) \leq 2$ if and only if one of the following holds:

1. (G, Σ) is a signed graph with two parallel paths;

2. (G, Σ) is a Sea Anemone; or

3. (G, Σ) is a Mollusk.

Proof. If (G, Σ) is a signed graph with two parallel paths, then Lemma 3.14 implies $M(G, \Sigma) \leq 2$. If (G, Σ) is a Sea Anemone, then Lemma 3.17 implies $M(G, \Sigma) = 2$. If (G, Σ) is a Mollusk, then Lemma 3.20 implies $M(G, \Sigma) = 2$.

Suppose that $M(G, \Sigma) \leq 2$. Let $[H_1, H_2]$ be the wide separation of (H, Ω). Let u_1 and u_2 be the vertices of attachment of H_1. If H_1 is connected and H_2 is connected, then Lemma 3.18 implies (G, Σ) is a Sea Anemone or (G, Σ) is a signed graph with two parallel paths. If H_1 is connected and H_2 is disconnected, then Lemma 3.21 implies (G, Σ) is a Mollusk or

(G, Σ) is a signed graph with two parallel paths. If H_1 is disconnected and H_2 is connected, then Lemma 3.21 implies (G, Σ) is a Mollusk or (G, Σ) is a signed graph with two parallel paths. If H_1 is disconnected and H_2 is disconnected, then Lemma 3.5 implies (G, Σ) is a signed graph with two parallel paths. $\square$

3.3.3 No Wide Separations

Definition 3.23. Let (G, Σ) be a signed graph obtained from adding pendant paths to (H, Ω), where (H, Ω) is a 2-connected partial 2-path. Suppose

- C_1 and C_2 are distinct cycles in H such that there exists two 2-separations of H: (C_1, H_1) and (C_2, H_2);
- P_1 and P_2 are disjoint paths between C_1 and C_2 in H;
- $u_1 = P_1 \cap C_1$ and $u_2 = P_2 \cap C_1$; and
- $v_1 = P_1 \cap C_2$ and $v_2 = P_2 \cap C_2$.

We call (G, Σ) a *Seahorse* if the following hold:

1. $l(P_1) = 0$ and $l(P_2) = 1$.
2. There is a single pendant path attached to each vertex in $\{u_1 = v_1, u_2, v_2\}$.
3. If $l(C_1 - u_1u_2) \geq 3$, then no pendant path is attached to $C_1 \setminus \{u_1, u_2\}$.
4. If $l(C_2 - v_1v_2) \geq 3$, then no pendant path is attached to $C_2 \setminus \{v_1, v_2\}$.

Lemma 3.24. If (G, Σ) is a Seahorse, then $M(G, \Sigma) = 2$.

Proof. Let P be the pendant path attached to u_1. Let

$$(G^{(1)}, \Sigma^{(1)}) = \big(G - P, \Sigma \setminus E(P)\big),$$

and let

$$(G^{(2)}, \Sigma^{(2)}) = \big(G - (P - u_1), \Sigma \setminus E(P - u_1)\big).$$

Because $G^{(1)}$ is a tree with $P(G^{(1)}) = 2$, Theorem 1.41 implies $M(G^{(1)}, \Sigma^{(1)}) \leq M(G^{(1)}) = P(G^{(1)}) = 2$. Because $(G^{(2)}, \Sigma^{(2)})$ is a signed graph with two parallel paths, Lemma 3.14 implies $M(G^{(2)}, \Sigma^{(2)}) \leq 2$. From Lemma 1.39, we have

$$M(G, \Sigma) = \max\{M(G^{(1)}, \Sigma^{(1)}), M(G^{(2)}, \Sigma^{(2)})\} \leq 2.$$

As (G, Σ) is not a path, Theorem 1.14 implies $M(G, \Sigma) = 2$. □

Definition 3.25. If (G, Σ) is a signed graph obtained by either

- adding a single pendant path to each vertex of a signed C_5;
- adding a single pendant path to each vertex of a signed house graph; or
- adding a single pendant path to each vertex of a signed C_4 and identifying an edge of the signed C_4 with an edge of a signed cycle;

then (G, Σ) is a *Starfish*.

Lemma 3.26. If (G, Σ) is a Starfish, then $M(G, \Sigma) = 2$.

Proof. If (G, Σ) is formed from adding pendant paths to C_5, then let $v \in V(C_5)$ of (G, Σ). Otherwise, we let $v \in V(G)$ such that $d_G(v) = 4$. Let P be the pendant path attached to v. Let

$$(G^{(1)}, \Sigma^{(1)}) = \big(G - P, \Sigma \setminus E(P)\big),$$

and let

$$(G^{(2)}, \Sigma^{(2)}) = \big(G - (P - v), \Sigma \setminus E(P - v)\big).$$

Because $G^{(1)}$ is a tree with $P(G^{(1)}) = 2$, Theorem 1.41 implies $M(G^{(1)}, \Sigma^{(1)}) \leq M(G^{(1)}) = P(G^{(1)}) = 2$. Because $(G^{(2)}, \Sigma^{(2)})$ is a signed graph with two parallel paths, Lemma 3.14 implies $M(G^{(2)}, \Sigma^{(2)}) \leq 2$. From Lemma 1.39, we have

$$M(G, \Sigma) = \max\{M(G^{(1)}, \Sigma^{(1)}), M(G^{(2)}, \Sigma^{(2)})\} \leq 2.$$

As (G, Σ) is not a path, Theorem 1.14 implies $M(G, \Sigma) = 2$. □

Lemma 3.27. Let (G, Σ) be a signed graph such that the removal of pendant paths yields a 2-connected partial wide 2-path (H, Ω). Suppose (H, Ω) has no wide separation. Then, $M(G, \Sigma) \leq 2$ if and only if one of the following holds:

- (G, Σ) is a signed graph with two parallel paths;
- (G, Σ) is a Seahorse; or
- (G, Σ) is a Starfish.

Proof. If (G, Σ) is a signed graph with two parallel paths, then Lemma 3.14 implies $M(G, \Sigma) \leq 2$. If (G, Σ) is a Seahorse, then Lemma 3.24 implies $M(G, \Sigma) = 2$. If (G, Σ) is a Starfish, then Lemma 3.26 implies $M(G, \Sigma) = 2$.

Suppose $M(G, \Sigma) \leq 2$. From Lemma 3.5, we know that no vertex in $V(H)$ has more than one pendant path attached in (G, Σ). We proceed with a case study on the number of cycles in (H, Ω). Suppose (H, Ω) has at least two distinct cycles. Let C_1 and C_2 be distinct cycles in (H, Ω). Then, we have two 2-separations of H: (C_1, H_1) and (C_2, H_2). We may find two disjoint paths P_1 and P_2 between C_1 and C_2 such that P_1 has endvertices u_1 and v_1; P_2 has endvertices u_2 and v_2; $u_1, u_2 \in V(C_1)$; and $v_1, v_2 \in V(C_2)$.

If (G,Σ) has a pendant path attached to an internal vertex of P_1 or an internal vertex of P_2, then we found a minor of (G,Σ) isomorphic to $K_3^{=}(\Delta Y)$. By Theorem 2.1, $M(G,\Sigma) \geq \xi(G,\Sigma) \geq 3$. So, we may assume that any pendant path of (G,Σ) is attached to C_1 or C_2. If there are no pendant paths attached to $w_1, w_2 \in V(C_1)$ such that $d_{C_1-u_1u_2}(w_1,w_2) \geq 2$ and there are no pendant paths attached to $x_1, x_2 \in V(C_2)$ such that $d_{C_2-v_1v_2}(x_1,x_2) \geq 2$, then (G,Σ) is a signed graph with two parallel paths. Hence, we may assume that there is a pair of vertices w_1, w_2 with pendant paths attached in (G,Σ) such that $d_{C_1-u_1u_2}(w_1,w_2) \geq 2$ or there is a pair of vertices x_1, x_2 with pendant paths attached in (G,Σ) such that $d_{C_2-u_1u_2}(x_1,x_2) \geq 2$. By symmetry, we assume that the vertices w_1 and w_2 have pendant paths attached to $d_{C_1-u_1u_2}(w_1,w_2) \geq 2$. Let R_i be the pendant path attached to w_i for $i = 1,2$. Because of our construction of P_1, we may assume that $w_1 = u_1$ or that w_1 is between u_1 and w_2 along the path $C - u_1u_2$. If $w_1 \neq v_1$ and $w_2 \neq v_2$, then we found a minor of (G,Σ) isomorphic to $K_3^{=}(\Delta Y)^2$. From Theorem 2.1, we may assume either $w_1 = v_1$ or $w_2 = v_2$. By symmetry, we may assume $w_1 = v_1$. From our definition of P_1, we have that $u_1 = w_1 = v_1$. If there exists two distinct vertices $y_1, y_2 \in V(C_1) \setminus u_1$ attached to pendant paths in (G,Σ) such that $d_G(y_1,y_2) \geq 2$, then we found a minor of (G,Σ) isomorphic to $K_3^{=}(\Delta Y)^3$. From Theorem 2.1, we may assume that $d_G(y_1,y_2) = 1$ for any two distinct vertices $y_1, y_2 \in V(C_1) \setminus u_1$ attached to pendant paths in (G,Σ).

We continue with the case that (G,Σ) has at least two cycles, and we proceed with a case study on $l(P_2)$. Suppose $l(P_2) \geq 1$. Suppose a pendant path Q_2 is attached to a vertex in C_2 such that $d_{C_2-v_1v_2}(Q_2,u_1) \geq 2$. If R_2 is not attached to u_2, then we found a minor of (G,Σ)

isomorphic to $K_3^=(\Delta Y)^3$. From Theorem 2.1, the pendant path R_2 is attached to u_2. That is, $w_2 = u_2$. Because of the symmetry from $u_1 = v_1$, the same argument implies that Q_2 is attached to v_2. If $l(P_2) > 1$, then we found a minor of (G, Σ) isomorphic to $K_3^=(\Delta Y)^3$. From Theorem 2.1, we may assume $l(P_2) = 1$. If $l(C_1 - u_1u_2) \geq 3$ and a pendant path is attached to a vertex in $V(C_1) \setminus \{u_1, u_2\}$, then we found a minor of (G, Σ) isomorphic to $K_3^=(\Delta Y)^3$. From Theorem 2.1, we assume that if $l(C_1 - u_1u_2) \geq 3$, then no pedant path is attached to $C_1 - \{u_1, u_2\}$. By symmetry, we may assume that if $l(C_2 - v_1v_2) \geq 3$, then no pedant path is attached to $C_2 - \{v_1, v_2\}$. That is, (G, Σ) is a Seahorse. Hence, we may assume that $d_{C_2 - v_1v_2}(Q_2, u_1) = 1$. Because for any two distinct vertices $y_1, y_2 \in V(C_1) \setminus u_1$ attached to pendant paths in (G, Σ) we know $d_G(y_1, y_2) = 1$. That is, (G, Σ) is a signed graph with two parallel paths.

We continue with the next case when $l(P_2) = 0$. Then, $u_2 = v_2$. Recall that there is a pendant path R_i attached to w_i in (G, Σ) for $i = 1, 2$ such that $d_{C_1 - u_1u_2}(w_1, w_2) \geq 2$. If (G, Σ) has no pendant path attached to u_1 and no pendant path attached to u_2, then we found a minor of (G, Σ) isomorphic to $K_3^=(\Delta Y)^2$. If $w_1 = u_1$ and (G, Σ) has a pendant path attached to the vertex $v \in V(C_2) \setminus v_2$ such that $d_{C_2 - v_1v_2}(v, u_1) \geq 2$, then we found a minor of (G, Σ) isomorphic to $K_3^=(\Delta Y)^3$. If no pendant path is attached to u_2 and all pendant paths are attached to $\{u_1, v\} \in V(C_2) \setminus \{v_1, v_2\}$ such that $d(v, u_1) = 1$, then (G, Σ) is a signed graph with two parallel paths.

Hence, we may assume that (G, Σ) has a pendant path attached to u_1 and a pendant path attached to u_2. If (G, Σ) has a pendant path attached to a vertex $v \in V(C_1) \setminus \{u_1, u_2\}$ such that $d_{C_1-u_1u_2}(\{u_1,u_2\},v) \geq 2$, then we found a minor of (G,Σ) isomorphic to $K_3^{=}(\Delta Y)^3$. By Theorem 2.1, we may assume that if a pendant path in (G,Σ) is attached to a vertex $v \in V(C_1) \setminus \{u_1,u_2\}$, then $d_{C_1-u_1u_2}(u_1,v) = 1$ or $d_{C_1-u_1u_2}(u_2,v) = 1$. By symmetry, we may assume if a pendant path in (G,Σ) is attached to a vertex $v \in V(C_2) \setminus \{u_1,u_2\}$, then $d_{C_2-v_1v_2}(v_1,v) = 1$ or $d_{C_2-v_1v_2}(v_2,v) = 1$. If either pair of the following statements holds, then we found a minor of (G,Σ) isomorphic to $K_3^{=}(\Delta Y)^3$:

1. - (G,Σ) has a pendant path S_1 attached to a vertex in $V(C_1) \setminus \{u_1,u_2\}$ such that $d_{C_1-u_1u_2}(S_1,u_1) \geq 2$, and
   - (G,Σ) has a pendant path S_2 attached to a vertex in $V(C_2) \setminus \{u_1,u_2\}$ such that $d_{C_2-u_1u_2}(S_2,u_1) \geq 2$; or
2. - (G,Σ) has a pendant path S_1 attached to a vertex in $V(C_1) \setminus \{u_1,u_2\}$ such that $d_{C_1-u_1u_2}(S_1,u_2) \geq 2$, and
   - (G,Σ) has a pendant path S_2 attached to a vertex in $V(C_2) \setminus \{u_1,u_2\}$ such that $d_{C_2-u_1u_2}(S_2,u_2) \geq 2$.

From Theorem 2.1, if (G,Σ) has two pendant path attached to $V(C_1) \setminus \{u_1,u_2\}$, then $l(C_1 - u_1u_2) = 3$, and in addition, if there is a pendant path attached to a vertex of $V(C_2) \setminus \{u_1,u_2\}$, then $l(C_2 - u_1u_2) = 2$. If $l(P_1) = 0$, then (G,Σ) is a Starfish. If $l(P_1) > 0$, then (G,Σ) is a Seahorse. Similarly, if (G,Σ) has two pendant path attached to $V(C_2) \setminus \{u_1,u_2\}$, then $l(C_2 - u_1u_2) = 3$, and in addition, if there is a pendant path attached to a vertex of $V(C_1) \setminus \{u_1,u_2\}$, then $l(C_1 - u_1u_2) = 2$. If $l(P_1) = 0$, then (G,Σ) is a Starfish. If $l(P_1) > 0$, then (G,Σ) is a Seahorse.

We continue with the case when H has at most one cycle. Because H is 2-connected, $V(H)$ are on a cycle C and $|V(H)| \geq 3$. Let $P_1, \ldots, P_k$ be the pendant paths attached to the vertices of H, ordered around C. If $k \geq 6$, then we found a minor of (G, Σ) isomorphic to $K_3^=(\Delta Y)^3$. By Theorem 2.1, we may assume that $k \leq 5$. Suppose $k = 5$. If $d_C(P_i, P_{i+1}) = 2$ for some i, where $k + 1 = 1$, then we found a minor of (G, Σ) isomorphic to $K_3^=(\Delta Y)^3$. By Theorem 2.1, $d_C(P_i, P_{i+1}) = 1$ for all $i = 1, \ldots k$. That is, (G, Σ) is a Starfish. Suppose next that $k = 4$. If $d_C(P_i, P_{i+1}) \geq 2$ and $d_C(P_i, P_{i-1}) \geq 2$ for some i, then we found a minor of (G, Σ) isomorphic to $K_3^=(\Delta Y)^3$. By Theorem 2.1, we may assume that $d_C(P_i, P_{i+1}) = d_C(P_{i+2}, P_{i+3}) = 1$ for some i. That is, (G, Σ) is a signed graph with two parallel paths. Suppose next that $k = 3$. If $d_C(P_1, P_2) \geq 2$, $d_C(P_2, P_3) \geq 2$, and $d_C(P_3, P_1) \geq 3$, then we found a minor of (G, Σ) isomorphic to $K_3^=(\Delta Y)^3$. If $d_C(P_i, P_{i+1}) = 1$ for some i, then (G, Σ) is a signed graph with two parallel paths. Finally, suppose $k \leq 2$. Then, (G, Σ) is a signed graph with two parallel paths. $\square$

3.4 Proof of the Main Result

Proof. First, we prove the forward direction. If (G, Σ) is a signed graph with two parallel paths, then Lemma 3.15 implies $M(G, \Sigma) \leq 2$. If (G, Σ) is a Seahorse or a Starfish, then Lemma 3.27 implies $M(G, \Sigma) = 2$. If (G, Σ) is a Sea Anemone or a Mollusk, then Lemma 3.22 implies $M(G, \Sigma) = 2$. If (G, Σ) is a Stingray, then Lemma 3.15 implies $M(G, \Sigma) = 2$. If (G, Σ) is obtained from attaching single pendant paths to W_4^o, then Lemma 3.7 implies $M(G, \Sigma) = 2$. If (G, Σ) is obtained from attaching at most two pendant paths to each vertex of K_2, then G is a tree, and Lemma 1.41 implies $M(G, \Sigma) \leq M(G) = P(G) \leq 2$. If (G, Σ) is obtained from attaching at most two pendant paths to each vertex of $K_2^=$, then Lemma 3.5 implies $M(G, \Sigma) \leq 2$.

Next, we suppose that $M(G, \Sigma) \leq 2$. If $M(G, \Sigma) = 1$, then G is a path by Theorem 1.14. So, we may draw (G, Σ) as a signed graph with two parallel paths. So, we may assume that $M(G, \Sigma) = 2$. If G is disconnected, then Lemma 3.3 implies (G, Σ) consists of two disjoint paths; and, we may draw (G, Σ) as a signed graph with two parallel paths. So, we may assume that G is connected. If G has no cycle, then G is a tree. From Lemma 1.41, we may minimally cover the vertices of G with two paths; and, we may draw (G, Σ) as a signed graph with two parallel paths. So, we may assume that G has a cycle. By Lemma 3.5, we may assume that (G, Σ) is obtained from attaching pendant paths either (1) to K_2, (2) to $K_2^=$, or (3) to a 2-connected signed graph (H, Ω) with $M(H, \Omega) \leq 2$. Further, we may assume that no vertex has more than two pendant paths attached. The first two cases are listed. For the third case, Theorem 1.52 implies that (H, Ω) is either W_4^o or a partial wide 2-path. If $(H, \Omega) = W_4^o$, then Lemma 3.7 implies that no vertex has more than one pendant path attached. If (H, Ω) is a 2-connected partial wide 2-path, then we finish our proof with a case study on the number of wide separations in (H, Ω). If (H, Ω) has two wide separations, then Lemma 3.15 implies (G, Σ) is either a Stingray or a signed graph with two parallel paths. If (H, Ω) has exactly one wide separation, then Lemma 3.22 implies (G, Σ) is either a Sea Anemone, a Mollusk, or a signed graph with two parallel paths. If (H, Ω) has no wide separations, then Lemma 3.27 implies (G, Σ) is either a Seahorse, a Starfish, or a signed graph with two parallel paths. $\square$

CHAPTER 4

Zero Forcing Number for Signed Graphs with Maximum Nullity at Most Two

4.1 Zero Forcing on Signed Graphs

In this section, we generalize the notion of zero forcing on graphs by finding new color change rules for signed graphs. In the remaining sections of this chapter, we find the zero forcing number of signed graphs with maximum nullity at most two.

Consider a simple graph G. We let $A \in \mathcal{S}(G)$, and we let $x \in \ker(A)$. The color change rule for simple graphs comes from the fact that

$$a_{i,j}x_j = 0$$

implies $x_j = 0$ whenever the white vertex j is the only neighbor of the blue vertex i. For parallel edges, we have the possibility that $a_{i,j} = 0$ whenever there is an edge between i and j, which allows $x_j \neq 0$. For signed graphs, we may also consider a system of equations derived from the null space

$$\begin{aligned} a_{i,j}x_j + a_{i,k}x_k &= 0 \\ a_{l,j}x_j + a_{l,k}x_k &= 0. \end{aligned}$$

Here, we may also force $x_j = x_k = 0$, depending on the signature of our signed graph. In particular, we need to exclude the possibility that the determinant is zero.

Definition 4.1. Suppose (G, Σ) is a signed graph with some vertices colored blue and others colored white. If $S \subseteq V(G)$, then we partition the neighborhood $N(S)$ into the blue vertices $N_B(S)$ and the white vertices $N_W(S)$.

Lemma 4.2. Let (G, Σ) be a signed graph without parallel edges. Let S be the sign pattern matrix of $A \in \mathcal{S}(G, \Sigma)$. Suppose some vertices $B \subseteq V(G)$ are colored blue and the others are colored white. Then, we may color the vertices in $N_W(B)$ blue if and only if there exists a subset $B_0 \subseteq B$ such that $S[B_0, N_W(B)]$ is a SNS-matrix.

Proof. First, we observe that $S[B, N_W(B)]x = 0$ always has the solution $x = 0$. If $S[B_0, N_W(B)]$ is a SNS-matrix for some $B_0 \subseteq B$, then by definition, $x = 0$ is the only solution. That is, we may color the vertices in $N_W(B)$ blue.

Suppose next that we may color the vertices of $N_W(B)$ blue. That is, $x = 0$ is the unique solution to $S[B, N_W(B)]x = 0$. Then, $S[B, N_W(B)]$ has full column rank. Hence, we have $|B| \geq |N_W(B)|$. If $|B| = |N_W(B)|$, then $S[B, N_W(B)]$ also has full row rank, and $S[B, N_W(B)]$ is a SNS-matrix. If $|B| > |N_W(B)|$, then we found linear dependent rows in $S[B, N_W(B)]$. So, there is a subset of blue vertices $B_0 \subseteq B$ such that $S[B_0, N_V(B)]$ has full row rank. That is, $S[B_0, N_V(B)]$ is a SNS-matrix. $\square$

Observation 4.3. There is a natural bijection between the signed digraphs without a positive cycle and the coloring rules on signed graphs without parallel edges.

Proof. Suppose D is a signed digraph with no positive cycles. Let (H, Ω) be the signed graph obtained by decontracting the vertices of D. From Theorem 1.53, we have a SNS-matrix S. If we identify the rows of S as blue vertices and the columns of S as white vertices, then by Lemma 4.2, we may color all vertices blue. Suppose we have a signed graph (G, Σ) with some vertices B colored blue. If for some blue vertices $B_0 \subseteq B$, the induced subgraph on

$B_0 \cup N_W(B_0)$ is isomorphic to (H, Ω), then we may color the vertices $N_W(B_0)$ blue in (G, Σ). That is, D defines a unique coloring rule.

Suppose we may apply a coloring rule to the blue vertices B and color the white vertices $N_W(B)$ blue in a signed graph (G, Σ). Let $A \in \mathcal{S}(G, \Sigma)$, and let S be the sign-pattern matrix of A. From Lemma 4.2, we may find vertices $B_0 \subseteq B$ such $S[B_0, N_W(B)]$ is a SNS-matrix. Because $S[B_0, N_W(B)]$ is a SNS-matrix, $A[B_0, N_W(B)]$ is non-singular. So, the determinant of $A[B_0, N_W(B)]$ is non-zero. Hence, at least one term in the alternating sum of the determinant is nonzero for some permutation π of the vertices in $N_W(B)$:

$$a_{b_1,\pi(N_W(B))_1} a_{b_2,\pi(N_W(B))_2} \cdots a_{b_{|B|},\pi(N_W(B))_{|B|}}.$$

We may resign around vertices of B_0 such that these edges are even, and we have a SNS-matrix with negative entries along the diagonal. By Theorem 1.53, we have a signed digraph $D(S[B_0, N_W(B)])$ with no positive cycles. □

Lemma 4.4. Suppose (G, Σ) is a signed graph, and suppose the vertices $B = \{b_1, b_2, b_3\}$ are colored blue. If a subgraph on the vertex set $B \cup N_W(B)$ is isomorphic to $K_{3,3}$, such that one vertex partition is blue and the other is white, then no color change rule allows us to color the vertices in $N_W(B)$ blue.

Proof. We may always switch around a blue vertex such that we have negative entries along the diagonal of S, a 3×3 sign pattern matrix with no zeros. By Theorem 1.53, S is a SNS-matrix if and only if $D(S)$ has no positive cycle. So, ij and ji must be of opposite sign. Then, the signed digraph in Figure 1.4 is a subgraph of $D(S)$. That is, we have replaced a zero in the maximal SNS-matrix (1.1) with a + to obtain S. Therefore, S is not a SNS-matrix. By Lemma 4.2, there is no color change rule allowing us to color the vertices $N_W(B)$ blue. $\square$

We note that the previous Lemma also follows from Little's result in Theorem 1.54 because G contains an even subdivision of $K_{3,3}$.

Observation 4.5. The signed graph (G, Σ) in Figure 4.1 has $M(G, \Sigma) = Z(G, \Sigma) = 3$, while $Z(G) = 4$.

Proof. The vertices $\{7, 9, 11\}$ are a minimum zero forcing set for (G, Σ), which requires an application of (1.1), when the blue vertices $\{1, 3, 5\}$ color the white vertices $\{2, 4, 6\}$ blue. As (G, Σ) has a minor isomorphic to $K_3^=(\Delta)^3$, Theorem 2.1 implies $M(G, \Sigma) \geq \xi(G, \Sigma) \geq 3$. Therefore,

$$\xi(G, \Sigma) = M(G, \Sigma) = Z(G, \Sigma) = 3.$$

The vertices $\{1, 8, 10, 12\}$ are a minimum zero forcing set for G, by brute force [9]. $\square$

Lemma 4.6. Suppose (G, Σ) is a signed graph with $M(G, \Sigma) \leq 2$. The color change rule corresponding to the maximal SNS-matrix of order 3 is never applied to (G, Σ).

Proof. We consider the maximal SNS-matrix in (1.1) and the corresponding signed graph $(K_{3,3} - e, \Omega)$ in Figure 1.4. If we contract the even edges of $(K_{3,3} - e, \Omega)$, then we have a K_4^o. That is, $K_4^o \preceq (K_{3,3} - e, \Omega) \preceq (G, \Sigma)$. From Theorem 2.1, we have

$$M(G, \Sigma) \geq \xi(G, \Sigma) \geq \xi(K_4^o) \geq 3.$$

Yet, $M(G, \Sigma) \leq 2$; so, we never apply this rule to (G, Σ). $\square$

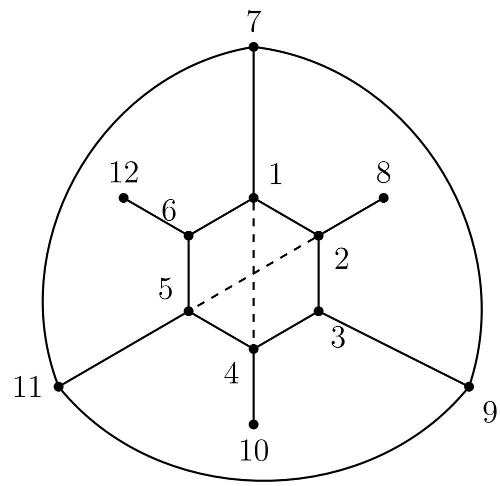


Figure 4.1 A signed graph (G, Σ) with $Z(G, \Sigma) < Z(G)$. Even edges are solid, and odd edges are dashed.

Lemma 4.7. Suppose we have a signed graph (G, Σ) with some vertices B colored blue. Suppose the white vertices $N_W(B)$ are on a cycle C_n. Suppose the vertices of C_n alternate blue and white: $w_1 \leftrightarrow b_1 \leftrightarrow w_2 \leftrightarrow \ldots \leftrightarrow b_n \leftrightarrow w_1$. Suppose no cord edge on C_n has a blue endvertex and a white endvertex. If $n = 4k$ and C_n is odd, then we may color $N_W(B)$ blue. If $n = 4k + 2$ and C_n is even, then we may color $N_W(B)$ blue.

Proof. Let D be a signed digraph which is a negative directed cycle. By Lemma 4.3, D defines a color change rule. If D has $2k$ vertices, then D has an odd number of directed edges labeled with $+$. Then, the corresponding signed graph (C_n, Σ) has an odd number of odd edges and $n = 4k$. If D has $2k + 1$ vertices, then D has an even number of directed edges labeled with $+$. Then, the corresponding signed graph (C_n, Σ) has an even number of odd edges and $n = 4k + 2$. $\square$

Lemma 4.8. Let (G, Σ) be a signed graph without parallel edges. Suppose $M(G, \Sigma) \leq 2$, and $Z(G,\Sigma) \leq 2$. We color some vertices of (G,Σ) blue and color others white. Then, we may apply the following color change rules to (G,Σ).

Rule 1 If w is the only white vertex adjacent to a blue vertex, then we may color w blue.

Rule 2 If $\{w_1, w_2\}$ are the only white neighbors of the blue vertices $\{b_1, b_2\}$ such that b_1w_1 is even, b_1w_2 is even, b_2w_1 is even, and b_2w_2 is odd, then we may color the vertices $\{w_1, w_2\}$ blue.

Rule 3 If $\{w_1, w_2\}$ are the only white neighbors of the blue vertices $\{b_1, b_2\}$ such that b_1w_1 is even, b_1w_2 is odd, b_2w_1 is odd, and b_2w_2 is odd, then we may color the vertices $\{w_1, w_2\}$ blue.

Proof. The first rule is the usual zero forcing rule for simple graphs. The other two rules follow from Lemma 3.12. Because $Z(G,\Sigma) \leq 2$, we do not consider cases where $|N_W(B_k)| > 2$, where B_k are the blue vertices after the k-th application of a coloring rule. Otherwise, we applied some coloring rule at step $j < k$ but did not color all the vertices in $N_W(B_j)$ blue. □

Lemma 4.9. Let (G,Σ) be a signed graph without parallel edges. Suppose $M(G,\Sigma) \leq 2$, and $Z(G,\Sigma) \leq 3$. We color some vertices of (G,Σ) blue and color others white. Suppose the vertices are on an even cycle

$$C_6 = b_1 \leftrightarrow w_1 \leftrightarrow b_2 \leftrightarrow w_2 \leftrightarrow b_3 \leftrightarrow w_3 \leftrightarrow b_1.$$

Suppose $N_W(\{b_1, b_2, b_3\}) = \{w_1, w_2, w_3\}$, Then, we may apply the following color change rule to (G,Σ).

Rule 4 If no cord edge between a blue and a white vertex of C_6 is on an even C_4, then we may color $\{w_1, w_2, w_3\}$ white.

Proof. As $Z(G,\Sigma) \leq 3$, we need not consider the case where the blue vertices B have $|N_W(B)| > 3$.

Because $M(G,\Sigma) \leq 2$, Lemma 4.6 implies we need not consider the maximal SNS-matrix on 3 vertices nor the corresponding signed digraph D, where D is edge maximal for having no positive cycle. Suppose we remove exactly one directed edge e from D. If e is $1j$, then we may apply rule 1 to b_1 and color w_j blue. If e is $i3$, then we may apply rule 1 to b_i and color w_1 blue. The only corresponding even cycle in C_6 is $b_2 \leftrightarrow w_1 \leftrightarrow b_3 \leftrightarrow w_2 \leftrightarrow b_2$. So, removing e from D corresponds to removing all even cycles from C_6. From Lemma 4.7, we may continue removing cord edges of C_6, and we may still color the vertices $\{w_1, w_2, w_3\}$ white.

To show that C_6 is minimal, we remove a single edge $b_i w_i$. Then, we apply rule 1 to b_i and color the white vertex w_{i-1} blue, where $w_0 = w_3$. □

Lemma 4.10. Let (G,Σ) be a signed graph with $M(G,\Sigma) \leq 2$. If (G,Σ) has no wide separation and no parallel edges, then $Z(G) = Z(G,\Sigma)$.

Proof. As (G,Σ) has no wide separation and no parallel edges, then only the first color change rule from Lemma 4.8 applies. That is, the collection of zero forcing sets for G is exactly the same collection as for (G,Σ), and $Z(G) = Z(G,\Sigma)$. □

Lemma 4.11. Let (G,Σ) be a signed graph with $M(G,\Sigma) \leq 2$. If $[G_1, G_2]$ is a wide separation, then $G_1 \cup C_4 \cup G_2$. Let the vertices of attachment of G_1 be $\{b_1, b_2\}$, and let

$\{w_1, w_2\}$ be the vertices of attachment of G_2. Let (H, Ω) be obtained by removing all edges from C_4 except u_1v_1 and u_2v_2. Then, $Z(G, \Sigma) = Z(H, \Omega)$.

Proof. Suppose $\{b_1, b_2\}$ are colored blue in both (G, Σ) and (H, Ω). Suppose $N_W(\{b_1, b_2\}) = \{w_1, w_2\}$ in both (G, Σ) and (H, Ω). In (G, Σ), we may apply rule 2 or 3 to $\{b_1, b_2\}$ and color $\{w_1, w_2\}$ blue. In (H, Ω), we first apply rule 1 to b_1 to color w_1 blue and again apply rule 1 to color w_2 blue. $\square$

4.2 Signed Graphs with $M(G, \Sigma) = Z(G, \Sigma) = 2$

Lemma 4.12. $Z(K_4^i) = 2$.

Proof. Let the vertices of K_4 be $\{1, 2, 3, 4\}$. Let the only odd edge of K_4^i have endvertices 1 and 4. We start by coloring the vertices $\{1, 2\}$ blue. Then, we may apply Lemma 3.12 and color the vertices $\{3, 4\}$ blue. Because $M(K_4^i) = 2$, we may apply Lemma 1.38, and we have $2 = M(K_4^i) \leq Z(K_4^i) \leq 2$. Hence, $Z(K_4^i) = 2$ □

Observation 4.13. $Z(K_4^i) < Z(K_4)$

Proof. From the previous Lemma 4.12, we know that $Z(K_4^i) = 2$. If we color two vertices of K_4 blue, then each blue vertex has two white neighbors. So, $Z(K_4) \geq 3$. If we color three vertices of K_4 blue, then each blue vertex has exactly one white neighbor. So, $Z(K_4) = 3$. □

Lemma 4.14. Let $[(G, \Sigma), \mathcal{F}]$ be a sided wide 2-path. Then, the endvertices of either edge in $\mathcal{F}$ are a zero forcing set for (G, Σ).

Proof. First, we start with (G_1, Σ_1) which is either an even cycle, an odd cycle, or a K_4^i. Then, $[(G_1, \Sigma_1), \mathcal{F}_1]$ is a sided wide 2-path with sides $\mathcal{F}_1$. If G_1 is a cycle, then the endvertices of $e \in \mathcal{F}_1$ are a zero forcing set. If $(G_1, \Sigma_1) = K_4^i$, then $\mathcal{F}_1$ is a split pair of edges. So, the endvertices of $e \in \mathcal{F}_1$ are a zero forcing set by Lemma 3.12. Next, we consider (G_2, Σ_2), which is either an even cycle, an odd cycle, or a K_4^i. If G_2 is a cycle, then we identify an edge $e \in E(G_2)$ with an edge in $\mathcal{F}_1$, and we may color the vertices of $V(G_2)$ blue. If $(G_2, \Sigma_2) = K_4^i$, then we identify an edge in $\mathcal{F}_1$ with an edge in a split pair of (G_2, Σ_2), and we may color the vertices of $V(G_2)$ blue. By definition, any sided wide 2-path may be constructed iteratively in this way. Therefore, the endvertices of either edge in the sides of a wide sided 2-path are a zero forcing set. □

Corollary 4.15. If (G, Σ) is a wide 2-path, then $Z(G, \Sigma) = 2$.

Proof. If (G, Σ) is a wide 2-path, then by definition there exists two distinct edges $\mathcal{F}$ such that $[(G, \Sigma), \mathcal{F}]$ is a sided wide 2-path. From Lemma 4.14, $Z(G, \Sigma) \leq 2$. From Theorems 1.14 and 1.52, $M(G, \Sigma) = 2$. From Lemma 1.38, $2 = M(G, \Sigma) \leq Z(G, \Sigma) \leq 2$. Therefore, $Z(G, \Sigma) = 2$. □

Lemma 4.16. If (G, Σ) is a 2-connected partial wide 2-path, then $Z(G, \Sigma) = 2$.

Proof. We begin with a wide 2-path (H, Ω) such that $V(G) = V(H)$, $E(G) \subseteq E(H)$, and $\Sigma \subseteq \Omega$. There exists a sided wide 2-path $[(H, \Omega), \mathcal{F}]$. For $e \in \mathcal{F}$, $H - e$ is not 2-connected. So, the side edges in $\mathcal{F}$ are edges of G. Let $\{u_1, u_2\}$ be the endvertices for an edge in $\mathcal{F}$. Color $\{u_1, u_2\}$ blue. For our first case, $\{u_1, u_2\}$ are vertices of attachment for a wide separation of (G, Σ). Then, we may color $\{v_1, v_2\}$ blue, where $\{v_1, v_2\}$ are the other two vertices of attachment in our wide separation blue, because of Lemma 3.12. For our second case, we assume that $\{u_1, u_2\}$ belong to a cycle C that has no vertices of attachment to a wide separation of (G, Σ). Then, we found a 2-separation of G, $[G_1, G_2]$ where G_1 is a cycle. We may color the vertices of G_1 blue, and we name the vertices $\{v_1, v_2\} = V(G_1) \cap V(G_2)$. If neither the first nor the second case is true, then we found a wide separation $[(G_1, \Sigma_1), (G_2, \Sigma_2)]$ where G_1 is a path or a cycle and $\{u_1, u_2\} \subseteq V(G_1)$. So, we may color the vertices of G_1 blue, and we apply rule 2 or 3 to the blue vertices of G_1 to color the vertices $\{v_1, v_2\}$ blue, where $\{v_1, v_2\}$ are the vertices of attachment of G_1. We may repeat our case study again starting with the blue vertices $\{v_1, v_2\}$ and ignoring all the other blue vertices. Eventually, we color all the vertices of (G, Σ) blue. That is, $\{u_1, u_2\}$ is a zero forcing set for (G, Σ), and Lemma 1.38 implies $2 = M(G, \Sigma) \leq Z(G, \Sigma) \leq 2$. Therefore, $Z(G, \Sigma) = 2$. $\square$

4.3 Signed Graphs with $M(G, \Sigma) = 2 \leq Z(G, \Sigma)$

Lemma 4.17. $Z(W_4^o) = 3$

Proof. As $\{1, 2, 3\}$ is a zero forcing set of W_4, Lemma 1.38 implies $Z(W_4^o) \leq Z(W_4) = 3$.

We may apply Rule 2 to $\{1, 3\}$ to color $\{4, 5\}$ blue only if 2 is also colored blue. By symmetry, other applications of Rule 2 are the same. As there are only two odd edges in W_4^o, we never apply Rule 3. Hence, the zero forcing sets for W_4 are the same as for W_4^o. That is, $Z(W_4^o) = 3$. □

Lemma 4.18. Let (G, Σ) be obtained by attaching pendant paths to vertices of W_4^o. If $M(G, \Sigma) = 2$, then $Z(G, \Sigma) = 3$

Proof. From Lemma 3.7, $M(G, \Sigma) = 2$ implies that no vertex of W_4^o has more than one pendant path attached in (G, Σ). If $Z(G, \Sigma) = 2$, then we found a zero forcing set with two vertices in W_4^o. However, Lemma 4.17 forbids a zero forcing set with only two vertices in W_4^o. We label the first five vertices of (G, Σ) as in Figure 1.3. For $i = 1, 2, 3$, we take v_i as i if no pendant path is attached to i in (G, Σ); otherwise, we take v_i as the pendant vertex of the pendant path attached to i. So, $\{v_1, v_2, v_3\}$ is a zero forcing set of G, and Lemma 1.38 implies $3 \leq Z(G, \Sigma) \leq Z(G) = 3$. □

Lemma 4.19. If (G, Σ) is a Stingray, then $Z(G, \Sigma) = 3$.

Proof. We use the notation from Definition 3.8. For $i = 1, 3$, we replace the C_4 in the wide separations $[H_i, H_{i+1}]$ of G with two edges. The resulting graph H is not a graph on two parallel paths. From Theorem 1.45, $Z(H) > 2$. From Lemma 4.11, $Z(G, \Sigma) = Z(H) \geq 3$.

Suppose $l(P_1)+l(P_2) = 1$. Then we take $v_1, v_2 \in V(H_1)$ such that v_i is the furthest vertex from the vertex of attachment a_i in H_1 for $i = 1, 2$. We take v_3 to be the pendant vertex of the pendant path attached to a vertex of P_1 or P_2. Then, $\{v_1, v_2, v_3\}$ is a zero forcing set of (G, Σ). So, $Z(G, \Sigma) \leq 3$. Hence, $Z(G, \Sigma) = 3$

Suppose $l(P_1)+l(P_2) = 0$. If there is a pendant path attached to the vertex of attachment q on the path Q, then we take v_1 to be the pendant vertex. If there is no pendant path attached to Q, then we take $v_1 = q$. We take v_2 to be the unique vertex such that $d_Q(v_1, v_2) = 1$. We take v_3 to be the pendant vertex of the pendant path attached to a vertex of P_1 or P_2. Then, $\{v_1, v_2, v_3\}$ is a zero forcing set of (G, Σ). So, $Z(G, \Sigma) \leq 3$. Hence, $Z(G, \Sigma) = 3$. □

Lemma 4.20. If (G, Σ) is a Sea Anemone, then $Z(G, \Sigma) = 3$.

Proof. We use the notation from Definition 3.16. First, we replace the C_4 in the wide separations of G with two edges. The resulting graph H is not a graph on two parallel paths. From Theorem 1.45, $Z(H) > 2$. From Lemma 4.11, $Z(G, \Sigma) = Z(H) \geq 3$.

Suppose first that (G, Σ) has a pendant path attached to an internal vertex of the path P in H_1. Name the pendant vertex of this pendant path p_1. By definition, there is a single pendant path attached to the vertex of attachment u_1 of the wide separation H_1. Similarly, name the pendant vertex of this pendant path p_2. If (G, Σ) has a pendant path attached to the vertex of attachment w_1 of H_2, then name this pendant vertex p_3. Then, $\{p_1, p_2, p_3\}$ is a zero forcing set of (G, Σ). If (G, Σ) has no pendant path attached to w_1, then $\{p_1, p_2, w_1\}$ is a zero forcing set of (G, Σ). So, $3 \leq Z(G, \Sigma) \leq 3$. Hence, $Z(G, \Sigma) = 3$.

Suppose next that (G, Σ) has no pendant path attached to an internal vertex of the path P in H_1. Then, we take the pendant vertex p_i of the pendant path attached to u_i for $i = 1, 2$. If (G, Σ) has a pendant path attached to the vertex of attachment w_1, then name this pendant vertex p_3. Then, $\{p_1, p_2, p_3\}$ is a zero forcing set of (G, Σ). If (G, Σ) has no pendant path attached to w_1, then $\{p_1, p_2, w_1\}$ is a zero forcing set of (G, Σ). So, $3 \leq Z(G, \Sigma) \leq 3$. Hence, $Z(G, \Sigma) = 3$.

□

Lemma 4.21. If (G, Σ) is a Mollusk, then $Z(G, \Sigma) \leq 3$. Further, $Z(G, \Sigma) = 2$ if and only if (G, Σ) is also a signed graph with two parallel paths.

Proof. We use the notation from Definition 3.19. First, we replace the C_4 in the wide separations of (G, Σ) with two edges. Name the resulting signed graph (H, Ω).

Suppose that there are two pendant paths attached to vertices of $H_1 - \{u_1, u_2\}$. Then, H is not a graph on two parallel paths, and (G, Σ) is not a signed graph with two parallel paths. From Theorem 1.45, $Z(H) > 2$. From Lemma 4.11, $Z(G, \Sigma) = Z(H) \geq 3$. We take v_1 and v_2 to be the pendant vertices of these two pendant paths. We take v_3 to be the pendant vertex of the pendant path attached at u_1. Hence, $\{v_1, v_2, v_3\}$ is a zero forcing set of (G, Σ). So, $3 \leq Z(G, \Sigma) \leq 3$. Hence, $Z(G, \Sigma) = 3$.

Suppose that there is exactly one pendant path attached to a vertex w of $H_1 - \{u_1, u_2\}$ such that $w \leftrightarrow u_1$. Then, H is a graph on two parallel paths, and (G, Σ) is a signed graph with two parallel paths. From Theorem 1.45, $Z(H) = 2$. From Lemma 4.11, $Z(G, \Sigma) = Z(H) = 2$. Hence, $Z(G, \Sigma) = 2$.

Suppose that there is exactly one pendant path attached to a vertex w of $H_1 - \{u_1, u_2\}$

such that $w \leftrightarrow u_2$. Then, H is not a graph on two parallel paths, and (G, Σ) is not a signed graph with two parallel paths. From Theorem 1.45, $Z(H) \geq 3$. From Lemma 4.11, $Z(G, \Sigma) = Z(H) \geq 3$. Take v_1 to be the pendant vertex of the pendant path attached at w. Take v_2 to be the pendant vertex attached at u_1. Take $v_3 = u_2$. Then, $\{v_1, v_2, v_3\}$ is a zero forcing set of (G, Σ). So, $3 \leq Z(G, \Sigma) \leq 3$. Hence, $Z(G, \Sigma) = 3$.

Suppose that there is no pendant path attached to any vertex of $H_1 - \{u_1, u_2\}$. Then, H is a graph on two parallel paths, and (G, Σ) is a signed graph with two parallel paths. From Theorem 1.45, $Z(H) = 2$. From Lemma 4.11, $Z(G, \Sigma) = Z(H) = 2$. Hence, $Z(G, \Sigma) = 2$.

Therefore, $Z(G, \Sigma) \leq 3$ if (G, Σ) is a Mollusk. Further, $Z(G, \Sigma) = 2$ if and only if (G, Σ) is also a signed graph with two parallel paths. □

Lemma 4.22. If (G, Σ) is a Seahorse, then $Z(G, \Sigma) = 3$.

Proof. Because G is not a graph on two parallel paths, Theorem 1.45 implies $Z(G) > 2$. As the pendant vertices of G are a zero forcing set, $Z(G) = 3$. As (G, Σ) has no parallel edges or wide separations, Lemma 4.10 implies $Z(G, \Sigma) = Z(G) = 3$. □

Lemma 4.23. If (G, Σ) is a Starfish, then $Z(G, \Sigma) = 3$.

Proof. We use the notation from Definition 3.25. For our first case, we suppose that (G, Σ) has 5 pendant paths. We take the pendant vertices $\{p_1, p_2, p_3\}$ of the pendant paths attached to $\{v_1, v_2, v_3\}$ such that $d_G(v_1, v_2) = d_G(v_2, v_3) = 1$ and $d(v_1) = d(v_3) = \max_{v \in V} d(v)$. Then, $\{p_1, p_2, p_3\}$ is a zero forcing set, and $Z(G) \leq 3$. As G has 5 pendant vertices, $P(G) \geq 3$.

For our second case, suppose that (G, Σ) has only 4 pendant paths. Then, we may find a 2-separation (C_4, C_k) of G where $\{v_1, v_2\} \in V(C_4) \cap V(C_k)$. We take $v_3 \in V(C_4)$ such that $v_3 \notin \{v_1, v_2\}$. Then, (G, Σ) has pendant paths attached to $\{v_1, v_2, v_3\}$, and we take the pendant vertices to be $\{p_1, p_2, p_3\}$. Then, $\{p_1, p_2, p_3\}$ is a zero forcing set for $Z(G)$. Suppose for a contradiction that $P(G) = 2$. Then, each path in the covering must end on a pendant vertex of G. If the edge v_1v_2 is along one of these two paths, then the path must have endvertices p_1 and p_2. Yet, any path covering with the edge v_1v_2 must have more than two paths. If the edge v_1v_2 is not in a path of our covering, then p_1 and p_2 are covered by different paths because the paths must be induced. Because the paths in our covering must have endvertices which are pendant vertices in G, the two paths cannot cover $V(C_k)\backslash\{v_1, v_2\}$. So, we have our contradiction, and $P(G) \geq 3$.

So, if (G, Σ) is a Starfish then $Z(G) \leq 3$ and $P(G) \geq 3$. From Theorem 1.36, $3 \geq Z(G) \geq P(G) \geq 3$. Hence, $Z(G) = 3$. Because (G, Σ) has no parallel edges and no wide separation, Corollary 4.10 implies $Z(G, \Sigma) = Z(G) = 3$.

□

We conclude this chapter with an extension of a result of Row to signed graphs [20].

Theorem 4.24. Let (G, Σ) be a signed graph where G is not a path. Then, (G, Σ) has $M(G, \Sigma) = Z(G, \Sigma) = 2$ if and only if (G, Σ) is a signed graph with two parallel paths.

Proof. First, suppose that (G, Σ) is a signed graph with two parallel paths and that G is not a path. From Lemma 3.14, we have that $M(G, \Sigma) = Z(G, \Sigma) = 2$.

Next, suppose that (G, Σ) is a signed graph such that $M(G, \Sigma) = Z(G, \Sigma) = 2$. Theorem 1.14 implies G is not a path. From Theorem 3.1, we know that (G, Σ) is a signed graph with two parallel paths, a Seahorse, a Starfish, a Sea Anemone, a Mollusk, a Stingray, or obtained from W_4^o by adding single pendant paths to some of the vertices of W_4^o. Because $Z(G,\Sigma) = 2$, Lemmas 4.22, 4.23, 4.20, and 4.19 implies (G,Σ) is not a Seahorse, a Starfish, a Sea Anemone, or a Stingray. Similarly, Lemma 4.18 implies (G,Σ) may not be obtained from W_4^o by adding single pendant paths to some of the vertices of W_4^o. If (G,Σ) is a Mollusk, then Lemma 4.21 implies that (G,Σ) is also a signed graph with two parallel paths; otherwise, $Z(G,\Sigma) = 3$. Therefore, $M(G,\Sigma) = Z(G,\Sigma) = 2$ implies (G,Σ) is a signed graph with two parallel paths. $\square$

REFERENCES

[1] AIM Minimum Rank — Special Graphs Work Group, *Zero forcing sets and the minimum rank of graphs*, Linear Algebra and its Applications **428** (2008), no. 7, 1628–1648.

[2] Marina Arav, F. Scott Dahlgren, and Hein van der Holst, *Signed graphs with stable maximum nullity at most two*, Linear Algebra and its Applications **620** (2021), 124–146.

[3] Marina Arav, Frank J. Hall, Zhongshan Li, and Hein van der Holst, *The inertia set of a signed graph*, Linear Algebra and its Applications **439** (2013), no. 5, 1506–1529.

[4] ———, *Two-connected signed graphs with maximum nullity at most two*, Linear Algebra and its Applications **611** (2021), 82–93.

[5] Marina Arav, Hein van der Holst, and John Sinkovic, *On the inertia set of a signed graph with loops*, Linear Algebra and its Applications **471** (2015), 169–183.

[6] Francesco Barioli, Shaun M. Fallat, and Leslie Hogben, *Computation of minimal rank and path cover number for certain graphs*, Linear Algebra and its Applications **392** (2004), 289–303.

[7] ———, *A variant on the graph parameters of Colin de Verdière: Implications to the minimum rank of graphs*, Electronic Journal of Linear Algebra **13** (2005), 387–404.

[8] Richard A. Brualdi and Bryan L. Shader, *Matrices of Sign-Solvable Linear Systems*, Cambridge University Press, New York, USA, 1995.

[9] Steve Butler, Laura DeLoss, Jason Grout, H. Tracy Hall, Josh LaGrange, Jephian C-H.

Lin, Tracy McKay, Jason Smith, and Geoff Tims, *Zero Forcing*, `https://github.com/jephianlin/mr_JG/blob/master/zero_forcing_64.pyx`, Sage Script, 2008.

[10] Miroslav Fiedler, *A characterization of tridiagonal matrices*, Linear Algebra and its Applications **2** (1969), no. 2, 191–197.

[11] Jim Geelen, Bert Gerards, and Geoff Whittle, *Solving Rota's Conjecture*, Notices of the American Mathematical Society **61** (2014), no. 7, 736–743.

[12] Felix Goldberg and Abraham Berman, *Zero forcing for sign patterns*, Linear Algebra and its Applications **447** (2014), 56–67.

[13] Leslie Hogben, *Minimum rank problems*, Linear Algebra and its Applications **432** (2010), no. 8, 1961–1974.

[14] Leslie Hogben and Hein van der Holst, *Forbidden minors for the class of graphs G with $\xi(G) \leq 2$*, Linear Algebra and its Applications **423** (2007), no. 1, 42–52.

[15] Charles R. Johnson and António Leal Duarte, *The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree*, Linear and Multilinear Algebra **46** (1999), no. 1–2, 139–144.

[16] Charles R. Johnson, Raphael Loewy, and Paul Anthony Smith, *The graphs for which the maximum multiplicity of an eigenvalue is two*, Linear and Multilinear Algebra **57** (2009), no. 7, 713–736.

[17] Charles H. C. Little, *A characterization of convertible (0, 1)-matrices*, Journal of Combinatorial Theory (B) **18** (1975), no. 3, 187–208.

[18] Neil Robertson and Paul D. Seymour, *Wagner's Conjecture*, Journal of Combinatorial

Theory (B) **92** (2004), no. 2, 325–357.

[19] Neil Robertson, Paul D. Seymour, and Robin Thomas, *Permanents, Pfaffin orientations, and even directed circuits*, Annals of Mathematics **150** (1999), no. 3, 929–975.

[20] Darren D. Row, *A technique for computing the zero forcing number of a graph with a cut-vertex*, Linear Algebra and its Applications **436** (2012), no. 12, 4423–4432.

[21] John Sinkovic, *Maximum nullity of outer planar graphs and the path cover number*, Linear Algebra and its Applications **432** (2010), no. 8, 2052–2060.

[22] Hein van der Holst, *On the Strong Arnold Hypothesis and the connectivity of graphs*, Electronic Journal of Linear Algebra **20** (2010), 574–585.

[23] Thomas Zaslavsky, *Signed graphs*, Discrete Applied Mathematics **4** (1982), no. 1, 47–74.