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On deconvolution problems: numerical aspects.

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Abstract. An optimal algorithm is described for solving the deconvolution problem of the form $\mathbf{k}u := \int_0^t k(t-s)u(s)ds = f(t)$ given the noisy data f_δ, $||f - f_\delta|| \leq \delta$. The idea of the method consists of the representation $\mathbf{k} = A(I+S)$, where S is a compact operator, $I + S$ is injective, I is the identity operator, A is not boundedly invertible, and an optimal regularizer is constructed for A. The optimal regularizer is constructed using the results of the paper MR 40#5130.

1. Introduction

Deconvolution problem consists of solving equation of the form

$$\mathbf{k}u := \int_0^t k(t-s)u(s)\,ds := k \star u = f(t), \quad 0 \leq t \leq T, \tag{1.1}$$

where $k(t)$, $t \geq 0$, is a kernel of linear integral equation (1.1), $k \star u$ is the convolution. It is important in many engineering applications, in physics, and other areas. There is a vast literature on deconvolution methods, see, for example, [6].

If the operator $\mathbf{k}$ in (1.1) is considered as an operator on $X := L^\infty(0,T)$, and $\int_0^T |k(t)|\,dt < \infty$, then $\mathbf{k}$ is not boundedly invertible, so problem (1.1) is ill-posed. Assume that the data f are noisy: f_δ is given, such that $||f - f_\delta|| \leq \delta$. In this case it is natural to seek an approximate solution of equation (1.1) in the class $Q_\delta := \{u \in X : \ ||\mathbf{k}u - f_\delta|| \leq \delta\}$. However, for ill-posed equation (1.1) an arbitrary element $u_\delta \in Q_\delta$ cannot be taken as an approximate solution to (1.1), since u_δ is not continuous with respect to δ in general. In order to select possible solutions one needs to use *a priori* information (usually available) about the solution, which may be of a quantitative or qualitative nature.

The usage of qualitative *a priori* information makes it possible to narrow the class of solutions, for example, to a compact set, so that the problem becomes stable under small changes in the data. This leads to a concept of a **quasisolution** [8]. Various algorithms for approximate determination of quasisolutions were studied in [8].

A priori information of a qualitative nature (for example, smoothness of the solution) generates different approaches. The one which is used often is **variational regularization** [20], [10], which allows one to construct stable approximate solutions to ill-posed problems by means of a stabilizing functional. The variational method has been extensively developed in [4], [3], and certain *a priori* and *a posteriori* choices of a regularization parameter $\varepsilon = \varepsilon(\delta)$ have been designed and implemented [9],[2].

One can also find approximate solutions to (1.1) **by iterations** (see [21], [1]), taking $x_n = R(f_\delta, x_{n-1}, ..., x_{n-k})$, where $k \leq n$. For these solutions to be stable under small changes of the data, the iteration number $n = n(\delta)$ yielding x_n must depend on the δ suitably.

Other important techniques in theory of ill-posed problems give regularizing operators by using Fourier, Laplace, Mellin, and other integral transforms, statistical regularization, and the dynamical systems method (DSM) [12], [13]).

In [14] some general new approaches are proposed for solving an ill-posed deconvolution problem. One of these approaches is based on the following idea. Assume that the operator $\mathbf{k}$ in (1.1) can be decomposed into a sum $\mathbf{k} := A + B$, where the operator $A^{-1}B := S$ is compact in the Banach space X, in which $\mathbf{k}$ acts, and $I + S$ is boundedly invertible. By the Fredholm alternative, it is equivalent to assuming that $\mathcal{N}(I + S) = \{0\}$, where $\mathcal{N}(A)$ is the null space of A. In this case $I + S$ is an isomorphism of X onto X, $\mathcal{R}(A) = \mathcal{R}(\mathbf{k})$, where $\mathcal{R}(A)$ is the range of the operator A, and

$$\mathbf{k}u = A(I + S)u = f_\delta. \tag{1.2}$$

If a regularizer for A is known, then (1.2) can be solved stably by the scheme

$$u_\delta = (I + S)^{-1} R(\delta) f_\delta, \tag{1.3}$$

and

$$||u - u_\delta|| \to 0 \quad \text{as} \quad \delta \to 0. \tag{1.4}$$

Since $I + S$ is an isomorphism, the error $||v - v_\delta||$ of the approximation of the solution of the equation $Av = f_\delta$ by the formula $v_\delta = R(\delta) f_\delta$ is of the same order as $||u_\delta - u||$. In this paper (see sections 2 and 3) we show that the proposed method is practically efficient and works better than the variational regularization.

Theoretically the proposed method is optimal on the class of the data defined as a triple $\{\delta, f_\delta, M_2\}$, where $f \in C^2(0,T)$, $||f''|| \leq M_2$, and f is otherwise arbitrary, $f_\delta \in L^\infty(0,T)$ and $||f - f_\delta|| \leq \delta$ and f_δ is otherwise arbitrary.

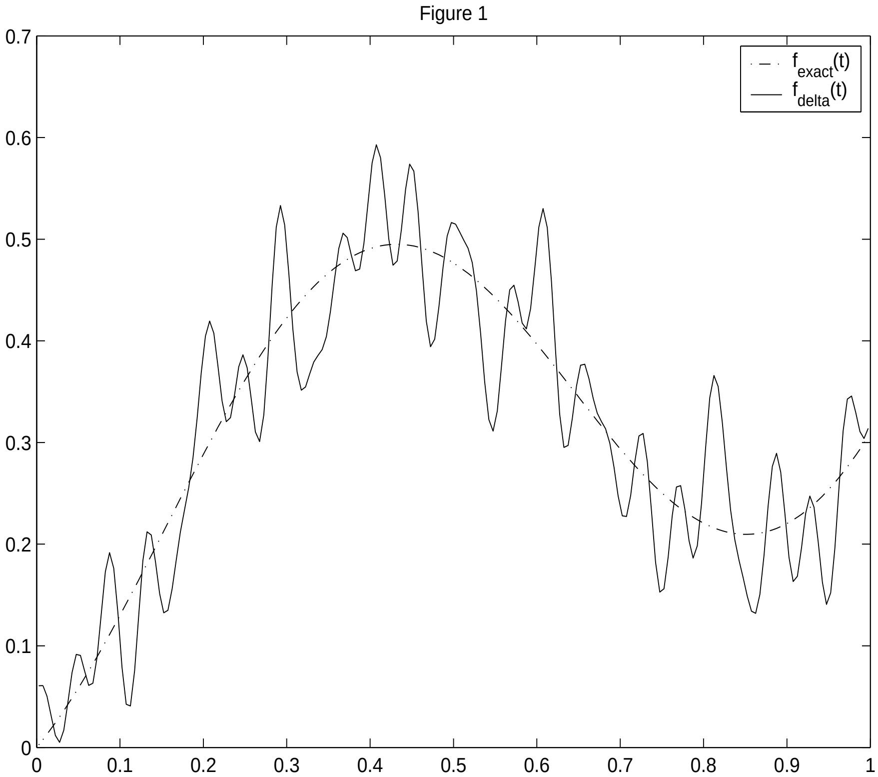
The operator $R(\delta)$, defined in (2.3) and originally proposed in [10] for stable numerical differentiation, yields an optimal estimate of f' in $L^\infty(0,T)-$norm in the following sense:

$$\inf_T \sup_{\{f_\delta :\, ||f - f_\delta|| \leq \delta,\ ||f|| \leq M_2\}} ||Tf_\delta - f'|| \geq (2M_2\delta)^{1/2},$$

where the *infimum* is taken over all, linear and non-linear, operators $T : X \to X$, $X = L^\infty(0,T)$, the *supremum* is taken over all f and f_δ subject to the conditions $f \in C^2(0,T)$, $||f''|| \leq M_2$, $||f - f_\delta|| \leq \delta$, and

$$||R(\delta) f_\delta - f'|| \leq (2M_2\delta)^{1/2},$$

(see e.g., [16],[17], [14]).



This argument shows that our "deconvolution" method for stable solution of (1.1) is optimal on the above data set: the operator $R(\delta)$ gives an optimal (on the above data set) approximation of f'. Inversion of an isomorphism $I + S$, where S is a compact operator, can be done very accurately by a projection method, for example, so that the total error of the solution is of the same order as the error obtained by applying $R(\delta)$.

2. **The case** $k(t) \in C^1(0, T)$

Let $k(t) \in C^1(0, T)$ and $k(0) \neq 0$. Then without loss of generality one can take $k(0) = 1$. As in [14], write (1.1) as

$$\mathbf{k}u = \int_0^t u(s)\,ds + \int_0^t [k(t-s) - 1]u(s) := Au + Bu = f. \tag{2.1}$$

Assume that $f(x)$ is given by its δ-approximation, i.e. one knows $f_\delta(x)$ such that $||f - f_\delta||_X \leq \delta$. In the experiments of this section $\delta = 0.1$. Let $A^{-1}B := S$. Then

$$\mathbf{k}u = A(I + S)u = f. \tag{2.2}$$

Stable inversion of A is equivalent to stable numerical differentiation of noisy data, and therefore as a regularizer $R(\delta)f_\delta$ for A one can use (see [11],[17],[18], [19], and also [15],[16])

$$R(\delta)f_\delta := \frac{f_\delta(t+h(\delta)) - f_\delta(t-h(\delta))}{2h(\delta)}, \tag{2.3}$$

with $h(\delta) = \left(\frac{2\delta}{M_2}\right)^{1/2}$, $\quad ||f''||_{L^\infty_{(0,T)}} \le M_2$. Hence

$$(I+S)u_\delta = R(\delta)f_\delta, \tag{2.4}$$

where S is a Volterra operator: $Su_\delta = \int_0^t k'(t-s)u_\delta(s)\,ds$. To test numerical efficiency of the above deconvolution algorithm, we take

$$k(y) = \exp(ay), \quad f(t) = \frac{(b+a)(\exp(at) - \cos(bt)) + (b-a)\sin(bt)}{a^2+b^2}. \tag{2.5}$$

Then equation (1.1) has the exact solution:

$$u_{orig}(t) = \sin(bt) + \cos(bt). \tag{2.6}$$

The graphs of f and its δ-approximation, f_δ, for $T=1,\ a=1,\ b=2\pi,$ are presented in Figure 1. The perturbation was generated as a sum of five sinusoids with various periods and amplitudes in such a way that $||f - f_\delta||_X \le 0.1$. For $\delta = 0.1$ and for the above choice of f, T, a, and b, one has $h(\delta) = \left(\frac{2\delta}{M_2}\right)^{1/2} = 0.1253$. Since in practice often only an estimate for M_2 may be available, our first experiment was done with the approximate value of $h(\delta)$, namely $h = 0.105$. The goal of the first experiment was to compare the results obtained by the deconvolution method suggested in [14] and by the variational regularization with a choice of the parameter by the Morozov discrepancy principle. The integral in (1.1) was calculated by the corrected trapezoid formula (see [7]) with the number of node points $n = 200$ on the interval $[0,1]$. The graphs of $u_{\text{disc}}(t)$ and $u_{\text{deconv}}(t)$ as well as the graph of the original solution, $u_{\text{orig}}(t)$, for $h(\delta) = 0.105$ and $n = 200$ are given in Figure 2. One can see from the picture that method [14] provides higher quality of reconstruction.

Table 1.

t	$u_{\text{exact}}(t)$	$u_{\text{disc}}(t)$	$u_{\text{deconv}}(t)$
0.05	1.26007351067010	0.88613219081253	1.61104047434242
0.15	1.39680224666742	0.77345683250358	1.16771020714854
0.25	1.00000000000000	0.78531546607804	0.97567292365993
0.35	0.22123174208247	0.32264819143761	0.46901890046136
0.45	-0.64203952192021	-0.01522580641369	-0.94010917284100
0.55	-1.26007351067010	-0.72058597578420	-1.39931254313538
0.65	-1.39680224666742	-0.65363525334725	-1.13246945274454
0.75	-1.00000000000000	-0.84181827797783	-1.26012127085008
0.85	-0.22123174208247	-0.48659287989254	-0.24854842261471
0.95	0.64203952192021	-0.25478764331776	0.99713489435843

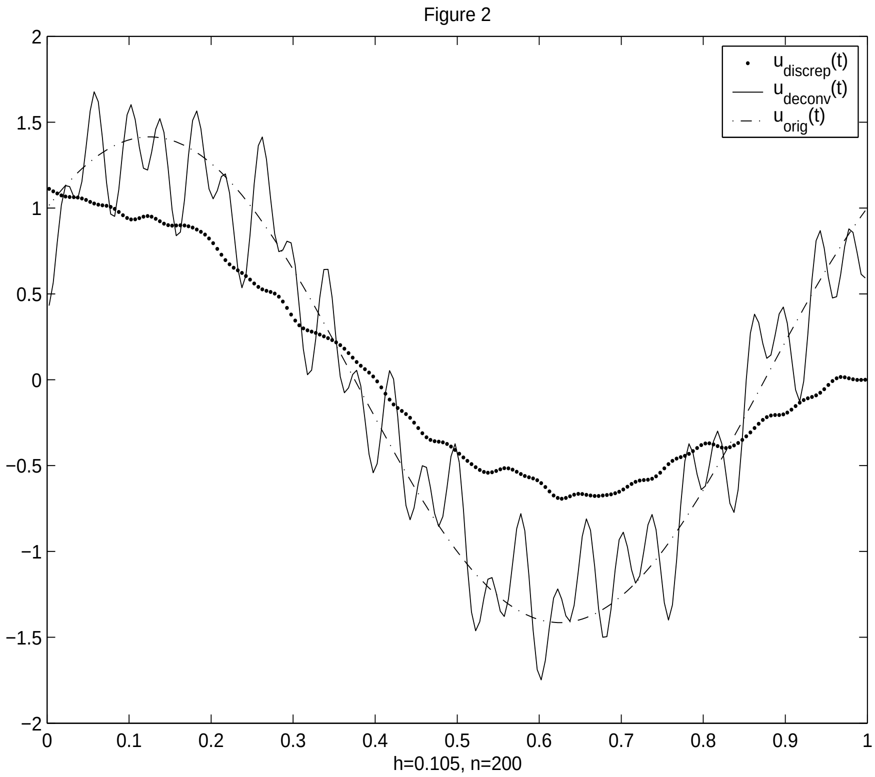


Table 1 allows one to analyze the computed values of $u_{\text{disc}}(t)$ and $u_{\text{deconv}}(t)$ for $h(\delta) = 0.1$ and $n = 10$. The regularization parameter for the variational regularization calculated by the Morozov discrepancy principle, $\varepsilon_{\text{disc}}$, is equal to 0.0275 for our particular f_δ. The functions $u_{\text{disc}}(t)$ and $u_{\text{deconv}}(t)$ approximate the exact solution $u_{\text{orig}}(t)$ with the relative errors $\delta_{\text{disc}} = \mathbf{0.5216}$ and $\delta_{\text{deconv}} = \mathbf{0.2470}$, respectively, for $n = 10$.

Table 2.

n	δ_{disc}	δ_{deconv}
10	0.52160739359373	0.24703402714545
50	0.47882066139400	0.29886579484582
100	0.49421933901812	0.29887532874922

The deconvolution procedure [14] is applicable when the constant M_a, $a > 1$ is known. Here M_a is the bound on the $f^{(a)}$, $a > 0$ is a real number, and $f^{(a)}$ is

Figure 3

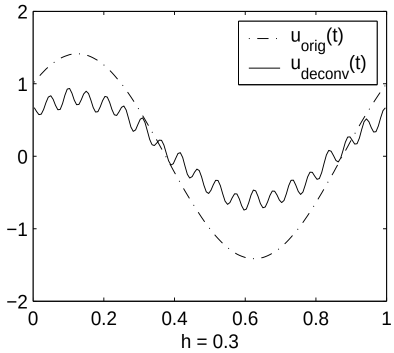


Figure 4

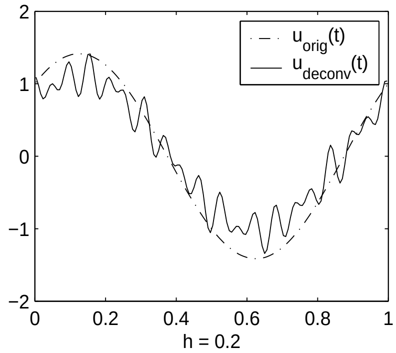


Figure 5

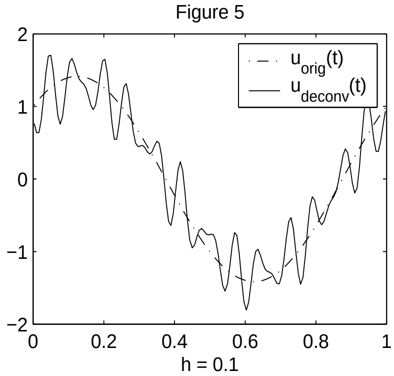
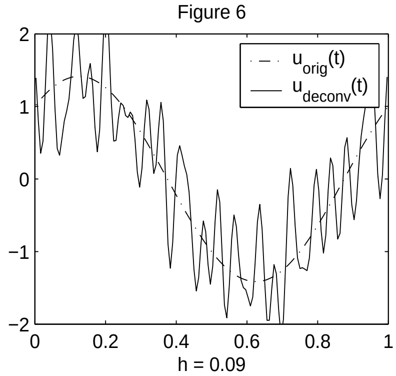


Figure 6



the (fractional order) derivative of f (see [16] for details). Figures 3-6 show the dependence of the quality of calculations provided by the deconvolution technique for different values of $h(\delta)$ with the same f_δ that is given in Figure 1. The level of reconstruction is acceptable for all values of $h(\delta) \in (0.09, 0.3)$, but the best quality is attained for the near-optimal values: $h(\delta) = 0.1$ and $h(\delta) = 0.2$. Outside the interval $(0.09, 0.3)$ the reconstruction by the variational regularization works better because $h(\delta)$ is away from its optimal value.

Table 2 contains relative errors, δ_{disc} and δ_{deconv}, for values of $n = 10,\ 50,\ 100$. In both cases the relative errors are not decaying further as n increases, because the major component in these errors come from the noise level, and not from the error of the computational methods.

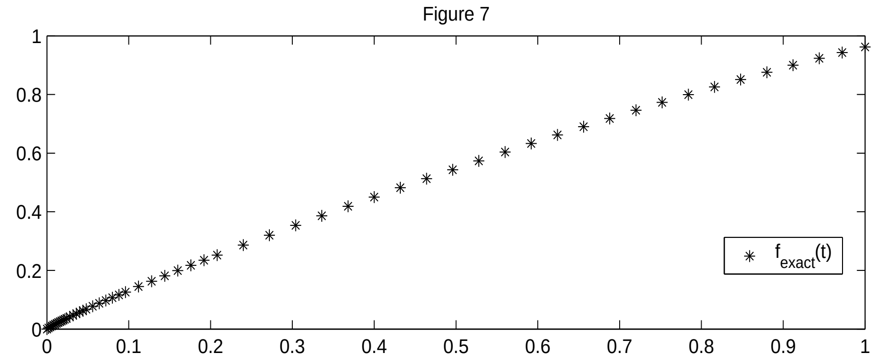
3. **Kernel of the type** $k(t) = \frac{t^{\gamma-1}}{\Gamma(\gamma)} + m(t), \quad 0 < \gamma < 1, \quad m(t) \in C^1$

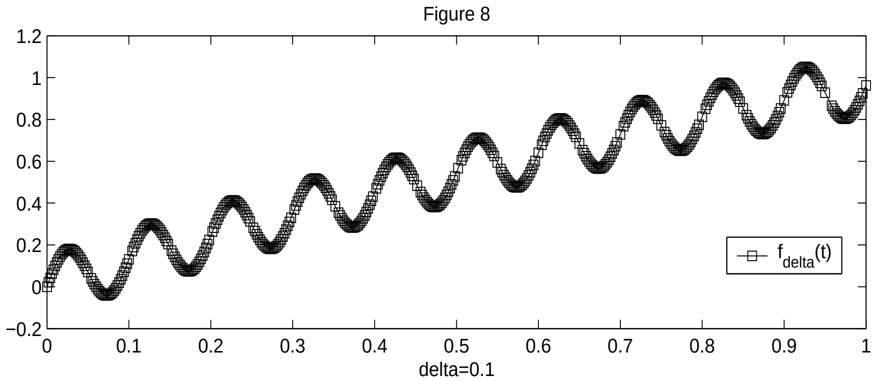
In this section we solve (1.1) with the kernel $k(t)$ of the form

$$k(t) = \frac{t^{\gamma-1}}{\Gamma(\gamma)} + m(t), \quad 0 < \gamma < 1, \quad m(t) \in C^1 \tag{3.1}$$

As in [14], write (1.1) as

$$\mathbf{k}u := Au + Bu = f, \tag{3.2}$$





where

$$Au := \frac{t^{\gamma-1}}{\Gamma(\gamma)} \star u, \quad Bu := m \star u. \tag{3.3}$$

One has ([5], pp.117-118) $A^{-1}f = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{f'(s)}{(t-s)^\gamma} ds$. Since the right-hand side f is given by its δ-approximation f_δ, $||f - f_\delta||_X \le \delta$, we replace A^{-1} by the regularizer $R_1(\delta)$ (see [14]):

$$R_1(\delta) f_\delta := \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{(R(\delta)f_\delta)(s)}{(t-s)^\gamma} ds. \tag{3.4}$$

The operator $R(\delta)$ in (3.4) is defined by formula (2.3) with $h = \mathbf{0.12}$. One gets

$$(I + S)u_\delta = R_1(\delta) f_\delta. \tag{3.5}$$

and

$$Su_\delta := A^{-1}Bu_\delta = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{m(0)u_\delta(s) + \int_0^s m'(s-p)u_\delta(p)dp}{(t-s)^\gamma} ds. \tag{3.6}$$

The goal of the experiment was to compare two numerical methods for solving (1.1)-(3.1): deconvolution method (3.5)-(3.6) and variational regularization with a choice of the parameter by the discrepancy principle.

The function

$$f(t) = \frac{t^\gamma}{\Gamma(1+\gamma)} \left(1 - \frac{2t^2}{(1+\gamma)(2+\gamma)}\right) + \frac{t^3}{3}\left(1 - \frac{t^2}{10}\right), \quad t \in [0,1],$$

Figure 9

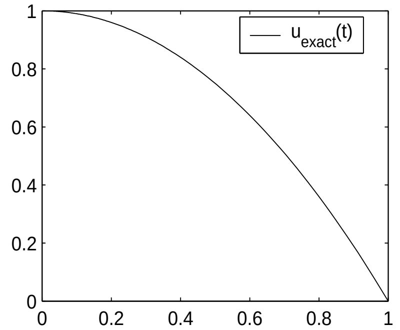


Figure 10

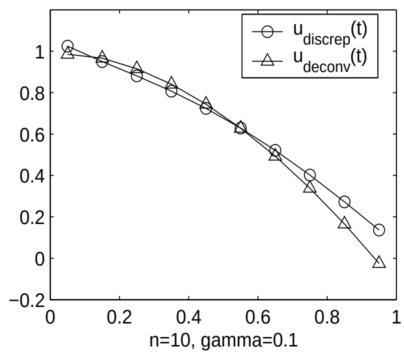


Figure 11

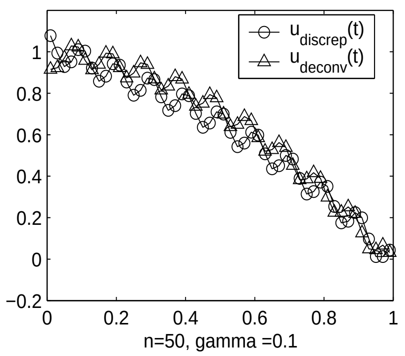
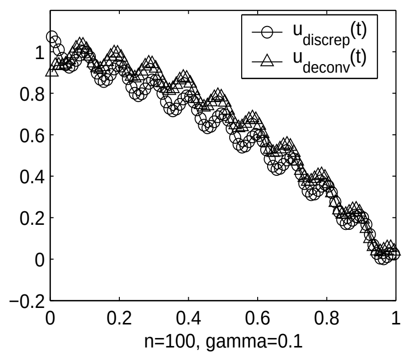


Figure 12



was chosen as the solution to direct problem (1.1)-(3.1) with $m(t) = t^2$ and the model function $u_{exact}(t) = 1 - t^2$. Then for the numerical tests the noisy function f_δ, $||f - f_\delta||_X \leq \delta$, $\delta = 0.1$, was used. The graphs of f and f_δ for $\gamma = 0.1$ are given in Figures 7 and 8.

Table 3.

t	$u_{\text{exact}}(t)$	$u_{\text{disc}}(t)$	$u_{\text{deconv}}(t)$
0.05	0.99000000000000	0.93361656127658	1.00281244943820
0.15	0.96000000000000	0.85983757148008	0.98540317766041
0.25	0.91000000000000	0.78772680932067	0.93442776030698
0.35	0.84000000000000	0.71362820985483	0.85896131974899
0.45	0.75000000000000	0.63504318796224	0.76170936024357
0.55	0.64000000000000	0.55028612377340	0.64396777696250
0.65	0.51000000000000	0.45824705747747	0.50654835394340
0.75	0.36000000000000	0.35823345537370	0.35004574315540
0.85	0.19000000000000	0.24979168725846	0.17493299888351
0.95	0	0.13214536398250	-0.01840021170081

Figure 13

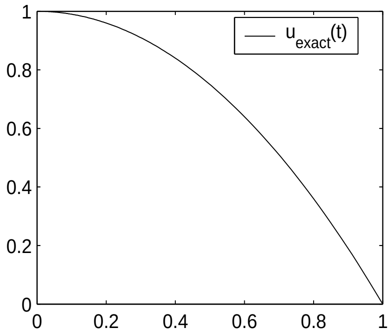


Figure 14

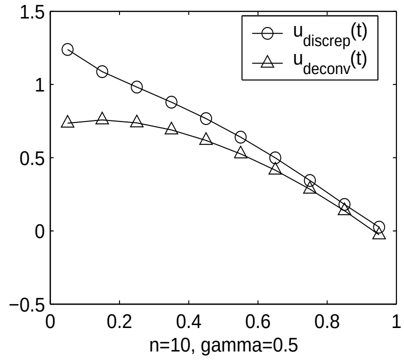


Figure 15

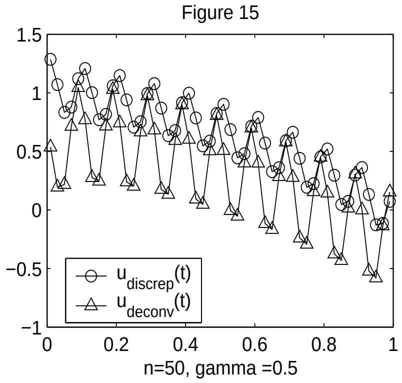
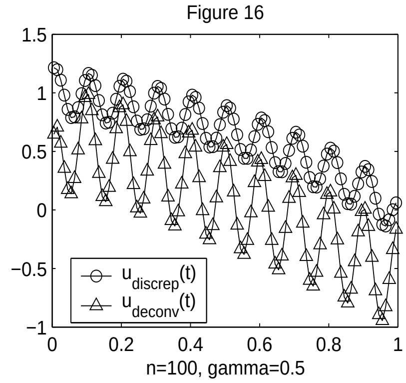


Figure 16

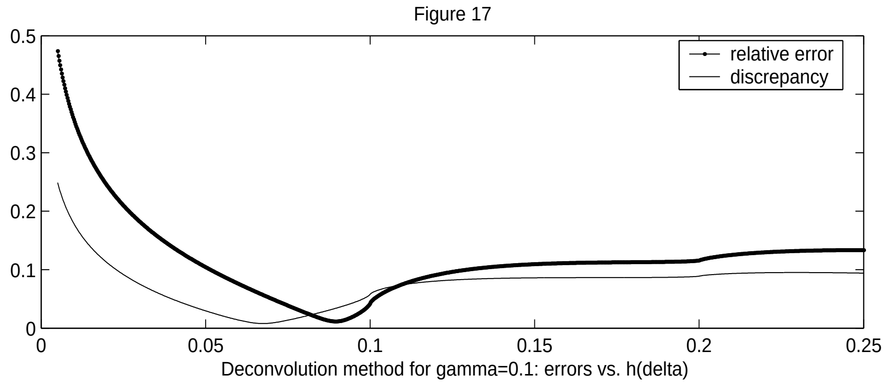


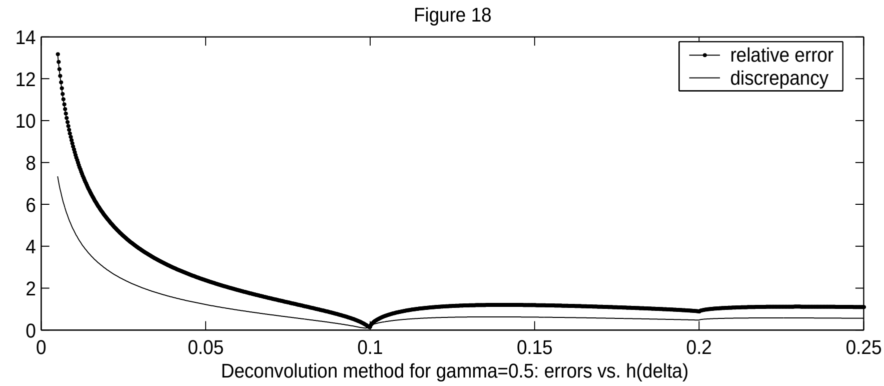
Figures 10-12 illustrate the numerical performance of method (3.5)-(3.6) and variational regularization for $\gamma = 0.1$. The solutions evaluated by formulas (3.5)-(3.6) and by variational regularization for $n = 10$ and $\gamma = 0.1$ are also presented in Table 3.

The results obtained for our particular test problem show that for small values of γ the deconvolution approach is superior to variational regularization both in terms of accuracy and stability. However as γ is getting bigger, the efficiency of the deconvolution method (as well as the efficiency of variational regularization) is getting worse. This is happening because when γ is close to 1, the ill-posedness of problem (1.1)-(3.1) grows due to the errors in calculations of the singular integral. One can compare Figures 9-12 and 13-16. Moreover, as γ changes from 0.1 to 0.9, method (3.5)-(3.6) becomes very sensitive to slight variations of $h(\delta)$. To illustrate this phenomena, we present the dependence of relative errors and discrepancies on $h(\delta)$ for $\gamma = 0.1$ and $\gamma = 0.5$ in Figures 17 and 18. For $\gamma = 0.1$ the relative error of the deconvolution method remains less than 10% when $h(\delta) \in (0.05, 0.12)$, while for $g = 0.5$ the relative error is only small for $h = 0.1$.

Finally, it is important to mention that CPU time for both methods, (3.5)-(3.6) and variational regularization, is approximately the same and it is very small: about 3-4 milliseconds for $n = 200$.

Conclusion. The paper presents numerical results of the implementation of the deconvolution method developed by AGR and presented together with other results





in [14]. The method is shown to be optimal in the sense explained in Section 1. The numerical results confirm the theoretical results on which the method is based. It is shown that the method is more accurate than the variational regularization method with the regularization parameter chosen by the discrepancy method.

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