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On deconvolution methods

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Abstract

Several methods for solving efficiently the one-dimensional deconvolution problem are proposed. The problem is to solve the Volterra equation $\mathbf{k}u := \int_0^t k(t-s)u(s)ds = g(t)$, $0 \leq t \leq T$. The data, $g(t)$, are noisy. Of special practical interest is the case when the data are noisy and known at a discrete set of times. A general approach to the deconvolution problem is proposed: represent $\mathbf{k} = A(I + S)$, where a method for a stable inversion of A is known, S is a compact operator, and $I + S$ is injective. This method is illustrated by examples: smooth kernels $k(t)$, and weakly singular kernels, corresponding to Abel-type of integral equations, are considered. A recursive estimation scheme for solving deconvolution problem with noisy discrete data is justified mathematically, its convergence is proved, and error estimates are obtained for the proposed deconvolution method.

1 Introduction

In many applications one models the relation between input signal $u(t)$ and output signal $g(t)$ by the equation

$$\mathbf{k}u := \int_0^t k(t-s)u(s)ds := k * u = g(t), \quad 0 \leq t \leq T, \quad (1.1)$$

where $k(t)$, given for all $t \geq 0$, characterizes the linear system, $k*u$ is the convolution, $u(t) = k(t) = 0$ for $t < 0$ and the cases $T < \infty$ and $T = \infty$ are both of interest. In practice $g(t)$ is measured with some error, so $g_\delta(t)$ is known, $\|g_\delta(t) - g(t)\| \leq \delta$. The norm we use is $L^2(0, \infty; e^{-2\sigma t})$ norm, or $L^2(0, T)$, or $L^\infty(0, T)$, and the case $T < \infty$ can be reduced to the case $T = \infty$, as we show below.

If the operator \mathbf{k} in (1.1) is considered as an operator on $L^\infty(0, T)$, and $\int_0^T |k(t)|dt < \infty$, then \mathbf{k} is not boundedly invertible, so problem (1.1) is ill-posed. One can see this from the formula $\int_0^t k(t-s)e^{i\sigma s} ds \rightarrow 0$ as $n \rightarrow \infty$.

If $T < \infty$, one sets $u(s) = 0$ for $t > T$ and defines $g(t)$ for $t > T$ as the left-hand side of equation (1.1). If this is done, then (1.1) can be considered as an equation on $(0, \infty)$ and its solution equals to $u(t)$, the solution of (1.1) on $[0, T]$, when $t \in [0, T]$. We assume that $k(t) \not\equiv 0$ and $\int_0^\infty |k(t)|dt < \infty$. Then (1.1) has at most one solution in $L^1[0, T]$ if $k(t) \neq 0$ almost everywhere in $[0, T]$ ([10], p.327). The assumption $\int_0^\infty |k(t)|dt < \infty$ can be replaced without loss of generality by a weaker assumption $\int_0^\infty \exp(-\sigma t)|k(t)|dt < \infty$, with an arbitrary large fixed $\sigma > 0$. This weaker assumption can be reduced to the original one by changing variables.

A deconvolution method is a method to construct a stable approximation $u_\delta(t)$ of the solution $u(t)$ to equation (1.1), given g_δ :

$$\|u_\delta(t) - u(t)\| := \eta(\delta) \rightarrow 0 \quad \text{as } \delta \rightarrow 0. \quad (1.2)$$

An operator $R(\delta)$ which constructs such u_δ from g_δ , $u_\delta = R(\delta)g_\delta$, is called a regularizer (or a regularizing family, since $\delta \rightarrow 0$) if (1.2) holds.

There is a large literature on ill-posed problems. General methods for constructing regularizers have been developed. They include variational regularization, iterative regularization, method of quasisolutions, etc [9]. In Section 2 the specific form of equation (1.1) is used for constructing regularizers for equation (1.1). The emphasis is on the causality property of the regularizer. The idea is similar to the one in [6] and [3]. In Section 3 a simple general method to construct regularizers for equation (1.1) is proposed. This method is practically efficient. It is illustrated by two examples in which the results from [2]-[5], and [7] are used. In Section 4 we investigate a recursive algorithm proposed in [1] for solving equation (1.1) with noisy discrete data. Again, the emphasis is on the causality property of the estimate: we use only the data collected up to the time t in order to estimate the signal u at this time. If one would use a variational regularization for constructing a stable estimation of $u(t)$, one would have to use all the data collected on the full time interval $[0, T]$, and not only on the "current" time interval $[0, t]$. Our analysis is much shorter than in [1] and yields more detailed results. Moreover, In Section 6 we discuss briefly a generalization of these results to the case of operator-valued kernels, which includes, in particular, matrix-valued kernels, that is, systems of Volterra equations. In Section 5 proofs are given.

2 A deconvolution method

Let $K(\lambda) := \int_0^\infty e^{-\lambda t} k(t) dt$. By capital letters the Laplace transform is denoted. If u solves (1.1) then $U(\lambda) = K^{-1}(\lambda)G(\lambda)$. One has:

$$u_\delta(t) := \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t} K^{-1}(\lambda) G_\delta(\lambda) \frac{1}{\left(\frac{\lambda}{N} + 1\right)^m} d\lambda, \quad \lambda = \sigma + i\mu \in C_\sigma, \quad (2.1)$$

where C_σ is the contour $\Re\lambda = \sigma$, $\lambda = \sigma + i\mu$, $m > 0$ is a sufficiently large positive integer and $N > 0$ is a large parameter. We do not show the dependence of $u_\delta(t)$ on m and N to simplify the notations. We want to prove that one can choose $N = N(\delta) \rightarrow \infty$ as $\delta \rightarrow 0$, such that (1.2) holds. Note that $G_\delta(\lambda) = G(\lambda) + W(\lambda)$, where $|W(\lambda)| = \left| \int_0^\infty e^{-\lambda t} w(t) dt \right| \leq \frac{\delta}{\sigma}$, the noise $w(t)$ satisfies the inequality $|w(t)| \leq \delta$, $\sigma = \Re\lambda > 0$, and $G(\lambda) := \int_0^\infty e^{-\lambda t} g(t) dt$. When one uses $L^\infty(0, T)$ norm one assumes that

$$K^{-1}(\lambda)G(\lambda) \in L^1(C_\sigma), \quad (2.2)$$

This is an a priori assumption on $u(t)$.

We assume throughout this paper that:

$$|K(\lambda)| \geq c|\lambda|^{-a}, \quad \lambda \in C_\sigma, \quad a \in \mathbb{R}. \quad (2.3)$$

Here and below $c > 0$ denote various constants independent of δ and N . The constant a may be negative, but in many applications $a \geq 0$.

Theorem 2.1 *If $m > a + 0.5$ and (2.3) holds, then there exists $N(\delta) \rightarrow \infty$ as $\delta \rightarrow 0$, such that (1.2) holds with $L^2(0, \infty; e^{-2\sigma t})$ norm.*

If (2.2) and (2.3) hold, and $m > a + 1$, then (1.2) holds with $L^\infty(0, T)$ norm.

If $m > a + 1$, (2.3) holds, and

$$|K^{-1}(\lambda)G(\lambda)| \leq \frac{c}{1 + |\lambda|^{1+d}}, \quad d = \text{const} > 0, \lambda \in C_\sigma, \quad (2.4)$$

then $N(\delta) = O\left(\delta^{-\frac{1}{q}}\right)$, $q = a + d + 1$, $\eta(\delta) = O\left(\delta^{\frac{d}{q}}\right)$ if $0 < d < 1$, and the norm in (1.2) is $L^\infty(0, T)$ norm.

If $d > 1$, then $N(\delta) = O\left(\delta^{-\frac{1}{2+a}}\right)$ and $\eta(\delta) = O\left(\delta^{\frac{1}{a+2}}\right)$.

If $d = 1$, then $N(\delta) = O\left(\exp\left[\frac{|\log \delta|}{a+2}(1 + o(1))\right]\right)$, and $\eta(\delta) = O\left(\frac{|\log \delta|}{\exp\left[\frac{|\log \delta|}{a+2}(1 + o(1))\right]}\right)$.

We sketch proofs in the last Section.

3 A general approach to deconvolution

Suppose the operator \mathbf{k} in (1.1) can be decomposed into a sum $\mathbf{k} := A + B$, where $A^{-1}B := S$ is compact in the Banach space X , in which \mathbf{k} acts, and $I+S$ is boundedly invertible, or which is the same by the Fredholm alternative, $N(I+S) = \{0\}$, where $N(A)$ is the null space of A . In this case $I+S$ is an isomorphism of X onto X , $\mathcal{R}(A) = \mathcal{R}(\mathbf{k})$, and

$$\mathbf{k}u = A(I+S)u = g. \quad (3.1)$$

If a regularizer for A is known, then (3.1) can be solved stably by the scheme

$$u_\delta = (I+S)^{-1}R(\delta)g_\delta, \quad (3.2)$$

and (1.2) holds.

Since $I+S$ is an isomorphism, the error $\|v - v_\delta\|$ of the approximation of the stable solution of the equation $Av = g$ by the formula $v_\delta = R(\delta)g_\delta$ is of the same order as $\|u_\delta - u\|$.

Example 3.1. Let $k(t) \in C^1(0, T)$ and assume $k(0) \neq 0$. Then, without loss of generality, one may assume $k(0) = 1$. Write (1.1) as

$$\mathbf{k}u = \int_0^t u(s)ds + \int_0^t [k(t-s) - 1]u(s)ds := Au + Bu = g.$$

Here stable inversion of A is equivalent to stable numerical differentiation of noisy data. This problem has been solved in [2] (see also [3] - [5], [7]), and the results of these works yield the following theorem:

Theorem 3.1 *Assume that $\|u\|_{W^{2,\infty}} \leq m_2 < \infty$. Then the operator*

$$R(\delta)g_\delta := \frac{g_\delta(t+h(\delta)) - g(t)}{h(\delta)}$$

is a regularizer for the operator A if $h(\delta) = 2\left(\frac{\delta}{m_2}\right)^{1/2}$. One has: $\|R(\delta)g_\delta - u\|_{L^\infty_{[0, T-h]}} \leq 2\sqrt{m_2\delta}$.

In [5] weaker a priori assumption on u is used: $\|u\|_a \leq m_a$, $0 < a \leq 1$, where the Hölder-space norm is defined as $\|u\|_a := \sup_{x \neq y, x, y \in [0, T]} \frac{|u(x) - u(y)|}{|x - y|^a} + \sup_{0 \leq t \leq T} |u(t)|$.

In this example our method yields the equation

$$(I+S)u_\delta = R(\delta)g_\delta,$$

where S is a Volterra operator: $Su_\delta := \int_0^t k'(t-s)u_\delta(s)ds$. Therefore u_δ can be easily found by iterations.

Example 3.2. Let $k(t) = \frac{t^{-\gamma}}{\Gamma(1-\gamma)} + m(t)$, $0 < \gamma < 1, m(t) \in C^1$,

$$Au = \frac{t^{-\gamma}}{\Gamma(1-\gamma)} * u, \quad Bu = m * u.$$

One has $A^{-1}g = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{g'(s)ds}{(t-s)^{1-\gamma}}$. Define

$$R_1(\delta)g_\delta := \frac{1}{\Gamma(\gamma)} \int_0^t \frac{(R(\delta)g_\delta)(s)}{(t-s)^{1-\gamma}} ds,$$

where $R(\delta)$ is defined in Theorem 3.1.

Theorem 3.2 *The operator-function $R_1(\delta)$ is a regularizer for equation $Au = g$, the operator $S := A^{-1}B$ is compact in $L^2(0, T)$ and $N(I + S) = \{0\}$.*

Compactness of S is clear from its definition. The operator $R_1(\delta)$ is a regularizer for A because $R(\delta)$ is a regularizer for the operator of differentiation. Finally $N(I+S) = \{0\}$ because $A^{-1}B$ is a Volterra operator. Therefore u_δ can be easily computed by our general method and (1.2) holds.

4 Recursive estimation given discrete noisy data

Assume that

$$g_\delta(nh) := \xi_n = g(nh) + w_n$$

are noisy measurements of the data $g(t)$ at the time moments nh , $h > 0$ is small number, $|w_n| \leq \delta$ is noise. One wishes to estimate stably $u(t)$, the solution to (1.1), given the data ξ_1, \dots, ξ_n . The following estimation method was essentially proposed in [1]. Set $v_0 = \frac{\xi_1 - \xi_0}{h}$. Define v_j by recursive formulas:

$$\alpha v_n + \sum_{j=0}^{n-1} \int_{jh}^{(j+1)h} k(nh - s) ds v_j = \xi_n, \quad v_0 = \frac{\xi_1 - \xi_0}{h}. \quad (4.1)$$

In this Section we assume that $k(t) \in L^1(\mathbb{R}_+) \cap L^2_{loc}(\mathbb{R}_+)$ and $u(t) \in L^2_{loc}(\mathbb{R}_+)$. Then $g(t) \in C_{loc}(\mathbb{R}_+)$, as the following lemma claims.

Lemma 4.1 *If $u, k \in L^2_{loc}(\mathbb{R}_+)$ then $k * u \in C_{loc}(\mathbb{R}_+)$.*

Proof. One has

$$\begin{aligned} |(k * u)(t+h) - (k * u)(t)| &\leq \int_0^{t+h} |k(t+h-s) - k(t-s)| |u| ds \\ &+ \int_t^{t+h} |k(t-s)| |u| ds := I_1 + I_2. \end{aligned}$$

Now,

$$I_1^2 \leq \int_0^{t+h} |k(t+h-s) - k(t-s)|^2 ds \int_0^{t+h} |u|^2 ds \rightarrow 0 \text{ as } h \rightarrow 0,$$

and

$$I_2^2 \leq \int_t^{t+h} |k(t-s)|^2 ds \int_t^{t+h} |u(s)|^2 ds \rightarrow 0 \text{ as } h \rightarrow 0.$$

Lemma 4.1 is proved. \square

This lemma makes it reasonable to assume that $g(t) \in C_{loc}(\mathbb{R}_+)$. In [1] the case $g \in L^2_{loc}(\mathbb{R}_+)$ is discussed, when g is not defined pointwise. It is proposed in [1] to use a mollification of g around the points nh instead of using $g(nh)$. However, this mollification requires a knowledge of g in a neighborhoods of all points nh , and this is an information different from the one assumed at the beginning, namely ξ_1, \dots, ξ_n . By this reason and because of Lemma 4.1, we assume that $g(t) \in C_{loc}(\mathbb{R}_+)$, the space of functions continuous on any compact subinterval of \mathbb{R}_+ .

Our assumptions in this section are:

A) $k(t) \in L^1(\mathbb{R}_+) \cap L^2_{loc}(\mathbb{R}_+)$ and $u(t) \in L^2_{loc}(\mathbb{R}_+)$,

B) (2.3) holds,

C) the union of the spectra of $\{K(\lambda)\}_{\forall \lambda \in C_\sigma}$ does not contain the set $\{z : z \in \mathbb{C}, \pi - \varphi < \arg z < \pi + \varphi, |z| < r\}$, where $\varphi > 0$ and $r > 0$ are arbitrary small fixed numbers.

If the assumptions **A)**, **B)**, and **C)** hold, then our result is Theorem 4.1 below. If, in addition, k and g are Hoelder-continuous, then our result is Theorem 4.2 below, which gives the rate of convergence in (1.2).

We prove that if $\alpha = \alpha(\delta)$ and $h = h(\delta)$ are chosen suitably, then the function $v_\delta(t)$, defined by the formula:

$$v_\delta(t) = v_j \quad \text{for } jh \leq t \leq (j+1)h,$$

approximates stably $u(t)$, so that $\|v_\delta(t) - u(t)\| \rightarrow 0$ as $\delta \rightarrow 0$, where $\|\cdot\|$ is $L^\infty(0, T)$ -norm. The rate of convergence is estimated in Theorem 4.2 under additional a priori assumptions.

Let

$$\alpha u_\alpha(t) + \int_0^t k(t-s)u_\alpha(s)ds = g(t), \quad 0 \leq t \leq T. \quad (4.2)$$

The function $v_\delta(t)$, defined above, solves the equation

$$\alpha v_\delta(t) + \int_0^t k(t-s)v_\delta(s)ds = f_\delta(t), \quad (4.3)$$

where $f_\delta(jh) = \xi_j$ for $t = jh$, and for other values of t the function $f_\delta(t)$ is defined as the left-hand side of (4.3):

$$f_\delta(t) := g_\delta(nh) - g(t) + \int_0^t [k(t-s) - k(nh-s)] v_\delta(s) ds + g(t) := \varphi_\delta(t) + g(t),$$

where $(n-1)h \leq t \leq nh$, $n = 1, 2, 3, \dots$

One has

$$|\varphi_\delta(t)| \leq \delta + \gamma_g(h) + c(T) \|v_\delta\| \gamma_k(h),$$

where

$$\gamma_g(h) := |g(nh) - g(t)| \rightarrow 0, \quad \gamma_k(h) := \int_0^t |k(t-s) - k(nh-s)| ds \rightarrow 0 \quad \text{as } h \rightarrow 0,$$

and $(n-1)h \leq t \leq nh$, $n = 1, 2, 3, \dots$

Denote $v_\delta - u_\alpha := w$. From (4.2) and (4.3) one gets: $w = (\alpha + \mathbf{k})^{-1} \varphi_\delta$, where $\alpha + \mathbf{k} := \alpha I + \mathbf{k}$, and I is the identity operator. One has $\|(\alpha + \mathbf{k})^{-1}\| \leq c\alpha^{-1}$, and $\|\varphi_\delta\| \leq \delta + \gamma_g(h) + c(T) \gamma_k(h) \|v_\delta\|$. Therefore:

$$\|v_\delta - u_\alpha\| \leq c(T) \frac{\delta + \gamma_g(h) + c(T) \gamma_k(h) \|v_\delta\|}{\alpha}.$$

Let us choose $h = h(\delta)$ such that $\frac{\gamma_k(h) + \gamma_g(h)}{\alpha} \rightarrow 0$ as $\alpha \rightarrow 0$, and $\alpha = \alpha(\delta)$ so that $\frac{\delta}{\alpha(\delta)} \rightarrow 0$ as $\delta \rightarrow 0$.

Let $u_\delta := u_{\alpha(\delta)}$. Then (1.2) holds by Lemma 5.1 below:

$$\|u_\delta - u\| = \|u_{\alpha(\delta)} - u\| = \|(\alpha + \mathbf{k})^{-1} \mathbf{k}u - u\| := \varepsilon(\alpha) \rightarrow 0 \quad \text{as } \alpha := \alpha(\delta) \rightarrow 0.$$

From (4.3) one gets

$$v_\delta = (\alpha + \mathbf{k})^{-1} g + (\alpha + \mathbf{k})^{-1} [g_\delta(nh) - g(t)] + (\alpha + \mathbf{k})^{-1} \mathbf{Q} v_\delta, \quad (4.4)$$

where

$$\mathbf{Q} v_\delta := \int_0^t [k(t-s) - k(nh-s)] v_\delta ds.$$

Since $\|(\alpha + \mathbf{k})^{-1}\| \leq \frac{c}{\alpha}$ and $\|g_\delta(nh) - g(t)\| \leq c(T)(\delta + \gamma_g(h))$, one has:

$$\|(\alpha + \mathbf{k})^{-1} [g_\delta(nh) - g(t)]\| \leq c(T) \frac{\delta + \gamma_g(h)}{\alpha} \rightarrow 0 \quad \delta \rightarrow 0,$$

provided that $\alpha = \alpha(\delta)$ and $h = h(\delta)$ are chosen so that $\frac{\delta}{\alpha(\delta)} \rightarrow 0$ and $\frac{\gamma_g(h(\delta))}{\alpha(\delta)} \rightarrow 0$ as $\delta \rightarrow 0$. Let $\alpha = \alpha(\delta)$. Then $\|(\alpha + \mathbf{k})^{-1} \mathbf{Q} v_\delta\| \leq \frac{c}{\alpha} \gamma_k(h) \|v_\delta\|$, and if $\frac{\gamma_k(h(\delta))}{\alpha(\delta)} \rightarrow 0$ as $\delta \rightarrow 0$, then $\|(\alpha + \mathbf{k})^{-1} \mathbf{Q}\| \leq \frac{c \gamma_k(h)}{\alpha} \rightarrow 0$ as $\delta \rightarrow 0$. Thus

$$v_\delta = [I - (\alpha + \mathbf{k})^{-1} \mathbf{Q}]^{-1} \left[(\alpha + \mathbf{k})^{-1} g + O\left(\frac{\delta + \gamma_g(h)}{\alpha}\right) \right],$$

where $O\left(\frac{\delta + \gamma_g(h)}{\alpha}\right)$ denotes an element whose norm is $O\left(\frac{\delta + \gamma_g(h)}{\alpha}\right)$. Therefore, with $u_\delta := (\alpha(\delta) + \mathbf{k})^{-1}g$, one gets:

$$\|v_\delta - u_\delta\| \leq c(T) \frac{\delta + \gamma_g(h) + \gamma_k(h)}{\alpha}.$$

Consequently:

$$\|v_\delta - u\| \leq \|v_\delta - u_\delta\| + \|u_\delta - u\| \leq c(T) \frac{\delta + \gamma_g(h) + \gamma_k(h)}{\alpha} + \varepsilon(\alpha) \rightarrow 0 \quad \text{as } \delta \rightarrow 0. \quad (4.5)$$

We have proved:

Theorem 4.1 *Assume **A**), **B**), and **C**). Then there exist $h = h(\delta) \rightarrow 0$ and $\alpha = \alpha(\delta) \rightarrow 0$ such that (4.5) holds.*

If $g(t)$ and $k(t)$ are Hoelder-continuous, then $\gamma_g(h) + \gamma_k(h) = O(h^b)$, $0 < b \leq 1$. Put $\delta = h^b$. Then (4.5) can be written as:

$$\|v_\delta - u\| \leq c\left[\frac{\delta}{\alpha} + \varepsilon(\alpha)\right]. \quad (4.6)$$

Let us estimate $\varepsilon(\alpha) = \|u_\delta - u\|$. One has:

$$\begin{aligned} \varepsilon(\alpha) &= \int_{-\infty}^{\infty} |\alpha(\alpha + K(\lambda))^{-1}| |K^{-1}(\lambda)G(\lambda)| d\mu \\ &\leq \int_{-\infty}^{\infty} \alpha |(\alpha + K(\lambda))^{-1}| \frac{d\mu}{[1 + (\sigma^2 + \mu^2)^{1/2}]^{1+d}} \\ &= \int_{-M}^M + \int_{-\infty}^M + \int_M^{\infty} := I_1 + I_2 + I_3. \end{aligned}$$

Using (2.3), one gets:

$$I_1 \leq c\alpha \int_{-M}^M \frac{\mu^a d\mu}{(1 + |\mu|)^{d+1}} \leq c\alpha M^{a-d}.$$

The estimates of I_2 and I_3 are similar. Let us estimate, for example, I_3 :

$$I_3 \leq c \int_M^{\infty} \frac{d\mu}{(1 + \mu)^{d+1}} \leq cM^{-d}, \quad d > 0.$$

Thus

$$\varepsilon(\alpha) \leq c(\alpha M^{a-d} + M^{-d}).$$

If $d \geq a$, then $\varepsilon(\alpha) \leq c\alpha$, $\|v_\delta - u\| \leq c\left[\frac{\delta}{\alpha} + \varepsilon(\alpha)\right]$. Minimizing with respect to α , one gets $\|v_\delta - u\| \leq c\delta^{1/2}$ if $\alpha = \delta^{1/2}$. If $d < a$, then $\varepsilon(\alpha) \leq c(\alpha M^{a-d} + M^{-d})$. Minimizing with respect to M , one gets $\varepsilon(\alpha) \leq c\alpha^{d/a}$, $\|v_\delta - u\| \leq c(\delta\alpha^{-1} + \alpha^{d/a})$. Minimizing with respect to α , one gets $\|v_\delta - u\| \leq c\delta^{\frac{d}{d+a}}$.

Let us summarize the result:

Theorem 4.2 *Assume **A**), **B**), and **C**). If g and k are Hoelder-continuous, so that $\gamma_g(h) + \gamma_k(h) = O(h^b)$, $0 < b \leq 1$, then $\|v_\delta - u\| \leq c\delta^{\frac{d}{a+b}}$, provided that $h = \delta^{\frac{1}{b}}$, $\alpha = O(\delta^{\frac{a}{a+b}})$, and $d < a$. If $d \geq a$, then $\|v_\delta - u\| \leq c\delta^{\frac{1}{2}}$.*

In [8] a singular perturbation problem was solved for a class of one- and multi-dimensional integral equations. The problem we study in Sec. 4 contains a singular perturbation problem as a basic component: we are interested in the behavior of the operator $(\alpha + \mathbf{k})^{-1}$ as $\alpha \rightarrow +0$.

5 Proofs.

The norm below is $L^2(0, \infty; e^{-2\sigma t})$ norm, it is equivalent to $L^2(0, T)$ norm on $(0, T)$. By the spectrum of a scalar function $K(\lambda)$ we mean the set of its values, and if $K(\lambda)$ is an operator-valued function, then its spectrum is defined as usual.

Lemma 5.1 *Let (2.3) hold and assume that the union of the spectra of $\{K(\lambda)\}_{\forall \lambda \in C_\sigma}$ does not contain the set $\{z : z \in \mathbb{C}, \pi - \varphi < \arg z < \pi + \varphi, |z| < r\}$, where $\varphi > 0$ and $r > 0$ are arbitrary small fixed numbers. Then*

$$\varepsilon(\alpha) := \|(\alpha + \mathbf{k})^{-1}\mathbf{k}u - u\| \rightarrow 0 \quad \text{as } \alpha \rightarrow +0 \quad \text{for all } u \in L^2(0, \infty; e^{-2\sigma t}).$$

Proof. One has

$$\begin{aligned} \|(\alpha + \mathbf{k})^{-1}\mathbf{k}u - \mathbf{k}^{-1}\mathbf{k}u\|^2 &= \|(\alpha + \mathbf{k})^{-1}\alpha\mathbf{k}^{-1}\mathbf{k}u\|^2 \\ &= \|\alpha(\alpha + \mathbf{k})^{-1}u\|^2 = \left\| \frac{e^{\sigma t}}{2\pi} \int_{-\infty}^{\infty} e^{i\mu t} \alpha(\alpha + K(\lambda))^{-1} K^{-1}(\lambda) G(\lambda) d\mu \right\|^2 \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \alpha^2 |(\alpha + K(\lambda))^{-1}|^2 |K^{-1}(\lambda) G(\lambda)|^2 d\mu := \varepsilon(\alpha) \rightarrow 0 \quad \text{as } \alpha \rightarrow 0, \\ &\lambda = \sigma + i\mu \in C_\sigma. \end{aligned}$$

Here we have used: 1) Parseval's equality; 2) the assumption $u \in L^2(0, \infty; e^{-2\sigma t})$ which is equivalent to $K^{-1}(\lambda)G(\lambda) \in L^2(C_\sigma)$; 3) the estimate $\sup_{\lambda \in C_\sigma} |\alpha(\alpha + K(\lambda))^{-1}| \leq c$ which follows from the assumption about the range of $K(\lambda)$ on C_σ ; 4) estimate (2.3); and 5) the dominated convergence theorem. Lemma 5.1 is proved \square

Lemma 5.2 *Under the assumptions of Lemma 5.1 one has:*

$$\|(\alpha + \mathbf{k})^{-1}g_\delta - u\| \leq \frac{c\delta}{\alpha} + \varepsilon(\alpha), \quad \varepsilon(\alpha) \rightarrow 0 \quad \text{as } \alpha \rightarrow 0, \quad c = \text{const} > 0.$$

Proof. One has

$$\|(\alpha + \mathbf{k})^{-1}g_\delta - u\| \leq \|(\alpha + \mathbf{k})^{-1}(g_\delta - g)\| + \|(\alpha + \mathbf{k})^{-1}g - u\| \leq \|(\alpha + \mathbf{k})^{-1}\|\delta + \varepsilon(\alpha).$$

By Lemma 5.1, $\varepsilon(\alpha) \rightarrow 0$ as $\alpha \rightarrow 0$, and $\|(\alpha + \mathbf{k})^{-1}\| \leq c\alpha^{-1}$, where $c = \text{const} > 0$ depends on φ , as follows from the proof of Lemma 5.1 and from the estimate

$$\sup_{\lambda \in \mathcal{C}_\sigma} |(\alpha + K(\lambda))^{-1}| \leq \frac{c}{\alpha}.$$

Lemma 5.2 is proved. \square

Proof of Theorem 2.1. If u_δ is defined in (2.1) and $\|\cdot\|$ is $L^2(0, \infty; e^{-2\sigma t})$ norm, then Parseval's equality yields

$$\begin{aligned} \|u_\delta - u\|^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| K^{-1}(\lambda)G_\delta(\lambda) \frac{1}{\left(\frac{\lambda}{N} + 1\right)^m} - K^{-1}(\lambda)G(\lambda) \right|^2 d\mu \\ &\leq \frac{1}{\pi} \int_{-\infty}^{\infty} |K^{-1}(\lambda)G(\lambda)|^2 \left| \frac{1}{\left(\frac{\lambda}{N} + 1\right)^m} - 1 \right|^2 d\mu + \frac{\delta^2}{\pi\sigma^2} \int_{-\infty}^{\infty} \frac{|K^{-1}(\lambda)|^2}{\left|\frac{\lambda}{N} + 1\right|^{2m}} d\mu \\ &:= I_1 + \delta^2 I_2, \end{aligned}$$

where we have used the formulas $G_\delta = G + W$, $|W| \leq \frac{\delta}{\sigma}$.

If $N \rightarrow \infty$ then $I_1 = I_1(N) \rightarrow 0$ by the dominated convergence theorem. Let us estimate I_2 assuming (2.3) and taking $m - a > 0.5$:

$$\begin{aligned} I_2 &\leq cN^{2m} \int_{-\infty}^{\infty} \frac{(\sigma^2 + \mu^2)^a d\mu}{[(\sigma + N)^2 + \mu^2]^m} \leq cN^{2m} \int_{-\infty}^{\infty} \frac{d\mu}{[(\sigma + N)^2 + \mu^2]^{m-a}} \\ &\leq c \frac{N^{2m}}{(N + \sigma)^{2m-2a-1}} \int_{-\infty}^{\infty} \frac{d\nu}{(1 + \nu^2)^{m-a}} \leq cN^{2a+1}. \end{aligned}$$

Thus, if $2a + 1 > 0$, then, using the estimate $(x + y)^{1/2} \leq x^{1/2} + y^{1/2}$, $x, y \geq 0$, one gets:

$$\|u_\delta - u\| \leq \delta c^{1/2} N^{a+\frac{1}{2}} + I_1^{\frac{1}{2}}(N) := \eta(\delta, N).$$

Minimizing $\eta(\delta, N)$ with respect to N for a fixed δ , denoting the minimizer by $N(\delta)$, $N(\delta) \rightarrow \infty$, as $\delta \rightarrow 0$, and the minimum by $\eta(\delta) := \eta(\delta, N(\delta))$, one gets $\eta(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. Thus, (1.2) is proved with $L^2(0, \infty; e^{-2\sigma t})$ norm. \square

If $L^\infty(0, T)$ norm is used for $\|u(t) - u_\delta(t)\|$, then one gets (1.2) if assumptions (2.2) and (2.3) are used. Namely,

$$\begin{aligned} \sup_{0 \leq t \leq T} |u_\delta(t) - u(t)| &\leq \frac{\exp(\sigma T)}{2\pi} \int_{-\infty}^{\infty} |K^{-1}G_\delta(1 + \frac{\lambda}{N})^{-m} - K^{-1}G| d\mu \\ &\leq c(T)(J_1 + J_2) := \eta, \end{aligned}$$

where

$$J_1 := J_1(N) = \int_{-\infty}^{\infty} |K^{-1}G| \left| \left(1 + \frac{\lambda}{N}\right)^{-m} - 1 \right| d\mu,$$

and

$$J_2 := J_2(N) = \frac{\delta}{\sigma} \int_{-\infty}^{\infty} |K^{-1}| \left| 1 + \frac{\lambda}{N} \right|^{-m} d\mu.$$

If $K^{-1}G \in L^1(C_\sigma)$, then $\lim_{N \rightarrow \infty} J_1 = 0$ by the dominated convergence theorem. If $m > a + 1$ and (2.3) holds, then

$$J_2 \leq c\delta N^{a+1}.$$

Thus (1.2) holds with $L^\infty(0, T)$ -norm provided that $K^{-1}G \in L^1(C_\sigma)$, $m > a + 1$ and (2.3) holds.

If (2.4) holds, one can get a rate of decay. Namely

$$J_1 = \int_{-M}^M + \int_{|\mu| > M} = j_1 + j_2,$$

where

$$\begin{aligned} j_1 &\leq \int_{-M}^M (1 + |\lambda|^{d+1})^{-1} \left| \left(1 + \frac{\lambda}{N}\right)^{-m} - 1 \right| d\mu \\ &\leq c \int_0^M (1 + |\lambda|^{d+1})^{-1} \frac{|\lambda|}{N} d\mu \leq cM^{1-d}N^{-1}, \text{ if } 0 < d < 1, \end{aligned}$$

$$j_1 \leq cN^{-1} \text{ if } d > 1,$$

$$j_1 \leq c \frac{\log M}{N} \text{ if } d = 1,$$

and

$$j_2 \leq c \int_M^\infty (1 + |\lambda|^{d+1})^{-1} \left| \left(1 + \frac{\lambda}{N}\right)^{-m} - 1 \right| d\mu \leq c \int_M^\infty (1 + |\lambda|^{d+1})^{-1} \leq cM^{-d},$$

$c > 0$ stands for various constants, and $\lambda = \sigma + i\mu$. Thus

$$J_1 \leq c(M^{1-d}N^{-1} + M^{-d}) \text{ if } 0 < d < 1,$$

$$J_1 \leq c(N^{-1} + M^{-d}) \text{ if } d > 1,$$

$$J_1 \leq c\left(\frac{\log M}{N} + M^{-d}\right) \text{ if } d = 1.$$

If $0 < d < 1$ then choose $M = N$ and get $J_1 \leq cN^{-d}$. Therefore, if $m > a + 1$ and (2.3) and (2.4) hold, then

$$J_1 + J_2 \leq c(\delta N^{a+1} + N^{-d}) \text{ if } 0 < d < 1.$$

Minimizing with respect to N for a fixed $\delta > 0$, one gets the minimizer $N = N(\delta) = O(\delta^{-\frac{1}{1+a+d}})$ and the estimate $\eta \leq O(\delta^{\frac{d}{1+a+d}})$ if $0 < d < 1$. If $d > 1$ then choose $M^d = N$ and get $J_1 \leq cN^{-1}$, $J_1 + J_2 \leq c(\delta N^{a+1} + N^{-1})$, $N(\delta) = O(\delta^{-\frac{1}{2+a}})$, and the estimate $\eta \leq O(\delta^{\frac{1}{2+a}})$. If $d = 1$ then choose $M = N$ and get $J_1 \leq c\frac{\log N}{N}$, $J_1 + J_2 \leq c(\delta N^{a+1} + \frac{\log N}{N})$, $N(\delta) = O(\exp[\frac{|\log \delta|}{a+2}(1 + o(1))])$, and the estimate $\eta \leq O(\frac{|\log \delta|}{\exp[\frac{|\log \delta|}{2+a}(1+o(1))])}$. Theorem 2.1 is proved. \square

Remark 5.3. If $k(t) > 0$ and $k(t) \rightarrow 0$ monotonically, then $\Re K(\lambda) \geq 0$ for $\Re \lambda > 0$. If $\Re K(\lambda) \geq 0$ then $\left| \alpha(\alpha + K(\lambda))^{-1} \right| \leq 1$. Condition $\Re K(\lambda) \geq 0$ implies that the assumption of Lemma 5.1 holds with $\varphi = \frac{\pi}{2}$ and $r = \infty$.

6 Generalizations.

Most of our results and proofs remain valid for operator-valued functions $k(t)$, in particular for matrix-valued kernels, that is, for systems of Volterra equations. Let $k(t)$ be an operator in a Banach space, and $K(\lambda)$ be its Laplace transform. If one replaces the absolute values by the norms in (2.3), (2.4) and elsewhere in the proofs, then one gets Theorems 2.1, 3.1, 3.2, and 4.1 and lemmas 5.1 and 5.2 with operator-valued $k(t)$.

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