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ELECTROMAGNETIC WAVE SCATTERING BY A SMALL IMPEDANCE PARTICLE: THEORY AND MODELING

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Abstract. Electromagnetic (EM) waves, scattered by a small impedance particle of arbitrary shape, embedded in a homogeneous medium, are calculated by a new analytic formula. The range of applicability and the accuracy of this formula are illustrated by numerical results. The formula was derived in (*) A.G.Ramm, Optics Communications, 284,(2011), 3872-3877. The accuracy of the new formula is estimated by a comparison with the Mie-type solution for an impedance sphere.

The novelty of our paper is in the demonstration of the range of applicability of the new formula and its practical value, by the numerical results and their comparison with the exact solution for EM wave scattering by impedance spheres. The exact solution is obtained in the form of Mie-type series, and is new. Estimate of the error of this series, in which five terms are kept, shows that the relative error of this solution is less than 10^{-3} for the parameters' range considered. The numerical results obtained are of interest to a wide audience, and the novelty of the formula from (*) is in its appicability to wave scattering by small particles of arbitrary shapes, when Mie-type solution is not applicable.

Key words: Electromagnetic wave scattering; small impedance particles; particles of arbitrary shapes.

1 Introduction

Electromagnetic (EM) wave scattering by small bodies has a long history, starting from Rayleigh (1871), see [1], [2] [9]. In [2] analytic formulas for the $S-$matrix for acoustic and EM wave scattering by small bodies of arbitrary shapes are derived. In particular, analytic formulas for the polarizability tensors for homogeneous bodies of arbitrary shapes are obtained. Analytic solution

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of the EM wave scattering problem for perfectly conducting and dielectric homogeneous spheres was given by Mie in 1908 (see [3]). It was used and developed by Bruning and Lo [4], [5], Mackowski [6], Xu [7], and others.

The aim of this paper is to test numerically asymptotic formulas (1)-(3) from the paper [8] to compare the accuracy of these formulas with Mie-type solution for an impedance sphere, and to find numerically the range of applicability of formulas derived in [8].

The results of this paper are new. They show that the formulas, derived in [8], can be used in many applications.

In this paper theoretical and numerical results for EM wave scattering by a small impedance particle D of an arbitrary shape, embedded in a homogeneous medium, are obtained. The medium is described by a constant permittivity $\varepsilon_0 > 0$, permeability $\mu_0 > 0$ and, possibly, constant conductivity $\sigma_0 \geq 0$. The theory and analytic formulas (21)-(24) (see below) for the solution to this problem are derived in [8].

The scattering problem is formulated and studied in Sections 2 and 3. In Section 2 analytic formulas from [8] for the scattered EM field are given and explained. The distance d from the small scatterer D, of the characteristic size a, (where $a = 0.5 diamD$, and $diamD$ stands for the diameter of the domain D), till the far zone is of the order $10a$, and this is very small distance if a is very small. In particular, $d \ll \lambda$, if a is sufficiently small. The practical conclusion is: if the scatterer is small, i.e., $ka \ll 1$, then the far zone for this scatterer starts very close to the scatterer, at the distances of the order $O(10a)$ from the scatterer. So, formula (24) (see below) is valid everywhere except a neighborhood of the scatterer of the order $O(10a)$.

In section 3 explicit analytic formulas for EM field, scattered by an impedance sphere, are derived. This derivation is similar to the derivations in Mie's theory, which treated perfectly conducting or homogeneous dielectric spheres.

Section 4 contains numerical results that demonstrate accuracy of the asymptotic formulas from [8] in a wide range of values ka and ζ, where ζ is the boundary impedance of the small scatterer. The explicit analytic solution, obtained in Section 3, is used for comparison with the asymptotic solution from [8]. These results show the accuracy of formula (24) and give its applicability limits.

In Section 5 the conclusions are formulated.

2 EM wave scattering by one small impedance particle of arbitrary shape

Let $a = 0.5 diamD$ be the radius of small particle D, $k > 0$ be a wavenumber, $k = \frac{2\pi}{\lambda}$, $ka \ll 1$, λ be the wavelength of the incident EM wave. The particle D is embedded in a homogeneous medium with constant parameters ε_0, μ_0. Let $k^2 = \omega^2 \varepsilon_0 \mu_0$, where ω is the frequency. Our arguments remain valid if one assumes that the medium has a constant conductivity $\sigma_0 > 0$. In this case ε_0 is replaced by $\varepsilon_0 + i\frac{\sigma_0}{\omega}$. Let S denote the boundary of D, $[E, H] = E \times H$ denote the cross product of two vectors, and $(E, H) = E \cdot H$ the dot product of two vectors.

EM wave scattering problem consists of finding vectors E and H satisfying the Maxwell equations:

$$\nabla \times E = i\omega\mu_0 H, \quad \nabla \times H = -i\omega\varepsilon_0 E \qquad \text{in} \qquad D' = \mathbb{R}^3 \backslash D, \tag{1}$$

the impedance boundary condition:

$$[N, [E, N]] = \zeta [H, N] \qquad \text{on} \qquad S, \tag{2}$$

and the radiation condition:

$$E = E_0 + v_E, \ \ H = H_0 + v_H, \tag{3}$$

where ζ is the boundary impedance of the particle, N is the unit normal to S pointing out of D, E_0, H_0 are the incident fields satisfying equations (1) in all of $\mathbb{R}^3$, v_E and v_H are the scattered fields, satisfying the Sommerfeld radiation condition, $r(\frac{\partial v}{\partial r} - ikv) = o(1)$ as $r := |x| \to \infty$. We assume that the incident wave is a plane wave, i.e., $E_0 = \Upsilon e^{ik\alpha\cdot x}$, Υ is a constant vector, $\alpha \in S^2$ is a unit vector, S^2 is the unit sphere in $\mathbb{R}^3$, $\alpha \cdot \Upsilon = 0$.

In general, the impedance ζ can be a constant, $\mathrm{Re}\zeta \geq 0$, or a 2×2 matrix function acting on the tangential to S vector fields, such that

$$\mathrm{Re}(\zeta E_t, E_t) \geq 0 \qquad \forall E_t \in T. \tag{4}$$

We assume in this paper that ζ is a constant. This simplifies numerical calculations.

In formula (4), T is the set of all tangential to S continuous vector fields such that $\mathrm{Div} E_t = 0$, where Div is the surface divergence, and E_t is the tangential component of E.

We define the tangential component E_t by the formula:

$$E_t = E - N(E, N) = [N, [E, N]]. \tag{5}$$

This definition corresponds to the geometrical meaning of the tangential component of E.

In the literature (see, for example, [17], p. 11) one may find $[N, E]$ as the definition of the tangential component of E. Such a definition is somewhat misleading: it gives a vector, rotated from E_t by an angle $\pi/2$ in the plane, tangential to the surface.

Problem (1)-(4) is equivalent to problem (6), (7), (3), (4), where

$$\nabla \times \nabla \times E = k^2 E \quad \text{in} \quad D', \qquad H = \frac{\nabla \times E}{i\omega\mu_0}, \tag{6}$$

$$[N, [E, N]] = \frac{\zeta}{i\omega\mu_0} [\nabla \times E, N] \qquad \text{on} \qquad S. \tag{7}$$

We have reduced our problem to finding one vector $E(x)$. If $E(x)$ is found, then H is found by the formula $H = \frac{\nabla \times E}{i\omega\mu_0}$, and the pair E and H solves the Maxwell equations and satisfies the impedance boundary condition.

Let us look for E of the form

$$E = E_0 + \nabla \times \int_S g(x,t)\sigma(t)dt, \qquad g(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}, \tag{8}$$

where g is the standard Green's function, $t \in S$ and dt is an element of the area of S, $\sigma(t)$ is a tangential vector field on S. This E solves equation (6) in D for any continuous $\sigma(t)$ because E_0 solves (6). To check this statement one uses the identity $\nabla \cdot \nabla \times E = 0$, valid for any smooth vector field E, and the formula

$$-\nabla^2 g(x,y) = k^2 g(x,y) + \delta(x-y). \tag{9}$$

The integral $\int_S g(x,t)\sigma(t)dt$ satisfies the radiation condition. Thus, formula (8) solves problem (6), (7), (3), (4), provided that $\sigma(t)$ is chosen so that boundary condition (7) is satisfied.

Let $O \in \mathbb{R}^3$ be a point inside D, the origin. To derive an integral equation for $\sigma = \sigma(t)$, substitute $E(x)$ from (8) into impedance boundary condition (7), follow the argument in [8], and get the following equation:

$$\sigma(t) = A\sigma + f, \qquad A\sigma = -2[N_s, B\sigma]. \tag{10}$$

Here A is a linear Fredholm-type integral operator. Formulas for the operators A, B and function f were derived in [8]. Equation (10) can be rewritten as

$$\sigma(s) = 2[f_e(s), N_s] - 2[N_s, B\sigma] := A\sigma + f. \tag{11}$$

The operator A is linear and compact in the space $C(S)$, so that equation (11) is of Fredholm type. Therefore, equation (11) is solvable for any $f \in T$ if its homogeneous version has only the trivial solution $\sigma = 0$. In this case the solution σ to equation (11) is of the order of the right-hand side f, that is, $O(a^{-\kappa})$ as $a \to 0$.

We assume that

$$\zeta = \frac{h}{a^\kappa}, \tag{12}$$

where $\mathrm{Re}h \geq 0$, and $\kappa \in [0,1)$ is a constant.

Let us rewrite (8) as

$$E(x) = E_0(x) + [\nabla_x g(x,O), Q] + \nabla \times \int_S (g(x,t) - g(x,O))\sigma(t)dt, \tag{13}$$

where

$$Q := \int_S \sigma(t)dt. \tag{14}$$

Consequently, the scattering problem is solved if vector Q is found. This simplifies the solution drastically, compared with the standard approach in which one solves numerically boundary integral equations (BIE) for the unknown vector-function σ.

Since $\sigma = O(a^{-\kappa})$, one has $Q = O(a^{2-\kappa})$. In [8], it was explained that the third term on the right of (13) is negligible compared with the second one. Therefore, equation (13) can be written in form

$$E(x) = E_0(x) + [\nabla_x g(x, O), Q]. \tag{15}$$

with an error that tends to zero as $a \to 0$ under our assumptions.

Note that the relation $|x| \gg ka^2$, that holds in far zone, is satisfied for $d = O(a)$ if $ka \ll 1$. Thus, formula (15) is applicable in a wide region.

Let us estimate Q asymptotically, as $a \to 0$. Integrating equation (11) over S, we get

$$Q = 2\int_S [f_e(s), N_s]ds - 2\int_S [N_s, B\sigma]ds. \tag{16}$$

It was shown in [8] that the second term in the right-hand side of the above equation is equal to $-Q$ plus terms negligible compared with $|Q|$ as $a \to 0$. Thus,

$$Q = \int_S [f_e(s), N_s]ds, \qquad a \to 0. \tag{17}$$

Let us estimate the integral in the right-hand side of (17). The expression $[N_s, f_e]$ one can rewrite as (see formula (33) in [8]):

$$[N_s, f_e] = [N_s, E_0] - \frac{\zeta}{i\omega\mu_0}[N_s, [\nabla \times E_0, N_s]]. \tag{18}$$

If E_0 tends to a finite limit as $a \to 0$, then formula (18) implies that

$$[N_s, f_e] = O(\zeta) = O(\frac{1}{a^\kappa}), \qquad a \to 0. \tag{19}$$

By Lemma 2 from [8], the operator $(I - A)^{-1}$ is bounded, so $\sigma = O(\frac{1}{a^\kappa})$, and

$$Q = O(a^{2-\kappa}), \qquad a \to 0, \tag{20}$$

because the integration over S adds factor $O(a^2)$. It follows from our arguments that Q does not vanish.

The Q can be expressed in terms of E_0. If S is a sphere of radius a then (see [8])

$$Q = -\frac{8\pi i a^{2-\kappa}}{3\omega\mu_0} h\nabla \times E_0(O). \tag{21}$$

The factor $\frac{8\pi}{3}$ appears if D is a sphere. Otherwise a tensorial factor τ_{jp} appears:

$$Q_j := (Q, e_j) = -\frac{i\zeta|S|}{\omega\mu_0}\tau_{jp}(\nabla \times E_0(O))_p, \tag{22}$$

where over repeated indices p summation from 1 to 3 is understood, and

$$\tau := \tau_{jp} = \delta_{jp} - b_{jp}, \qquad b_{jp} := \frac{1}{|S|}\int_S N_j N_p dS, \tag{23}$$

where δ_{jp} is the Kronecker delta, and b_{jp} depends on the shape of S. If S is a sphere, then $b_{jp} = \frac{1}{3}\delta_{jp}$. In this case one gets formula (21), where ζ is calculated by (12).

From equations (22) and (23) one obtains

$$E(x) = E_0(x) - \frac{i\zeta|S|}{\omega\mu_0}[\nabla_x g(x, O), \tau\nabla \times E_0(O)]. \tag{24}$$

Formula (24) gives representation for the field $E(x)$ in the region $r \gg a$, i.e., in the far zone. As we have already argued, if a is sufficiently small, then the far zone starts very close to the small scatterer, say, at a distance $10a$.

3 EM wave scattering by impedance sphere

In this section, we present the explicit analytic formulas for EM field scattered on the impedance sphere, based on the Mie theory [3], [7]. The governing equations for the EM wave scattering problem similarly to Section 2, are (1)-(3). Define a ball B:

$$B := \{y \in \mathbb{R}^3 : |y - x_0| < a\}, \tag{25}$$

where x_0 is the origin O, the notation used in Section 2.

The boundary condition (2) is:

$$E_t = \zeta[H_t, N] \qquad \text{on} \qquad S := \partial B, \tag{26}$$

where

$$E_t = [N, [E, N]] = E - N(E, N), \quad H_t = [N, [H, N]] = H - N(H, N), \tag{27}$$

N is the unit normal to S, pointing out of B, and ζ is the impedance. Here E_t and H_t are the tangential component of the vector fields E and H, respectively.

We look for a solution of the form:

$$E = E_0 + E_s, \ H = H_0 + H_s, \tag{28}$$

where E_0, H_0 are the initial incident fields satisfying (1), and E_s and H_s are the scattered fields satisfying the radiation condition. From (27), we obtain the relations

$$[N, E_t] = [N, E] \tag{29}$$

The boundary condition (26) can be rewritten as

$$[N, E] = \zeta[N, [H, N]], \tag{30}$$

Substituting (28) into (30), one gets

$$[N, E_s] - \zeta[N, [H_s, N] = \zeta[N, [H_0, N]] - [N, E_0], \tag{31}$$

We use boundary condition (31) for finding the expansion coefficients a_n and b_n in series (43) and (44) below.

The assumption $ka \ll 1$ holds, and $E_0 = e^{ikx_3}e_1$, where k is the wave number with $k^2 = \omega^2\mu_0\varepsilon_0$, and e_j, $j = 1, 2, 3$, are unit vectors along the coordinate axes x_j.

Let us assume that the particle D is a sphere centered at the origin, of radius a, and define

$$\psi_{emn} = cos(m\varphi)P_{nm}(cos(\theta))z_n(kr), \tag{32}$$

and

$$\psi_{omn} = sin(m\varphi)P_{nm}(cos(\theta))z_n(kr), \tag{33}$$

where P_{nm} are the associated Legendre functions, $z_n(kr)$ is the spherical Bessel function, which can be $j_n(\rho)$ or $h^{(1)}(\rho)$, $\rho := kr$, $r := |x|$, and

$$j_n(\rho) = \sqrt{\frac{\pi}{2\rho}}J_{n+1/2}(\rho),\ y_n(\rho) = \sqrt{\frac{\pi}{2\rho}}Y_{n+1/2}(\rho),\ h_n^{(1)} = j_n + iy_n, \tag{34}$$

where J_ν and Y_ν are the Bessel functions of the first and second kind, respectively. Let us denote by hat over a vector the corresponding unit vector $\widehat{x} := \frac{x}{|x|}$. Define the following vector spherical harmonics:

$$M_{emn} = \nabla \times (r\widehat{x}\psi_{emn}),\ M_{omn} = \nabla \times (r\widehat{x}\psi_{omn}), \tag{35}$$

$$N_{emn} = \frac{\nabla \times M_{emn}}{k},\ N_{omn} = \frac{\nabla \times M_{omn}}{k}. \tag{36}$$

The plane wave E_0 can be represented as a spherical harmonics expansion (cf [10]):

$$E_0 = \sum_{n=1}^{\infty} E_n(M_{o1n}^{(1)} - iN_{e1n}^{(1)}), \tag{37}$$

where $E_n = i^n\frac{2n+1}{n(n+1)}$, and

$$M_{e1n} = -\pi_n(\cos(\theta))\sin(\varphi)z_n(\rho)\widehat{e}_\theta - \tau_n(\cos(\theta))\cos(\varphi)z_n(\rho)\widehat{e}_\varphi, \tag{38}$$

$$M_{o1n} = \pi_n(\cos(\theta))\cos)z_n(\rho)\widehat{e}_\theta - \tau_n(\cos(\theta))\sin(\varphi)z_n(\rho)\widehat{e}_\varphi, \tag{39}$$

$$N_{e1n} = \tfrac{z_n(\rho)}{\rho}n(n+1)\pi_n(\cos(\theta))\sin(\theta)\cos(\varphi)\widehat{e}_r + \tfrac{[\rho z_n(\rho)]'}{\rho}\tau_n(\cos(\theta))\cos(\varphi)\widehat{e}_\theta - \tfrac{[\rho z_n(\rho)]'}{\rho}\pi_n(\cos(\theta))\sin(\varphi)\widehat{e}_\varphi, \tag{40}$$

$$N_{o1n} = \tfrac{z_n(\rho)}{\rho}n(n+1)\pi_n(\cos(\theta))\sin(\theta)\sin(\varphi)\widehat{e}_r + \tfrac{[\rho z_n(\rho)]'}{\rho}\tau_n(\cos(\theta))\sin(\varphi)\widehat{e}_\theta + \tfrac{[\rho z_n(\rho)]'}{\rho}\pi_n(\cos(\theta))\cos(\varphi)\widehat{e}_\varphi, \tag{41}$$

the prime indicates differentiation with respect to the argument in parentheses, $\pi_n = \frac{P_n^1}{\sin(\theta)}$ and $\tau_n = \frac{dP_n^1}{d\theta}$. In formulas (38)-(41), $\widehat{e}_r$, $\widehat{e}_\theta$, and $\widehat{e}_\varphi$ are unit vectors in the spherical coordinate system.

In (37) we have attached the superscript $^{(1)}$ to vector spherical harmonics for which the radial dependence of the generating functions is specified by $z_n = j_n$. Applying the operator $\nabla\times$ to (37), one gets

$$H_0 = -\frac{k}{\omega\mu_0}\sum_{n=1}^{\infty} E_n(N_{o1n}^{(1)} + M_{e1n}^{(1)}). \tag{42}$$

From the boundary condition, formula (37), orthogonality property of the functions M_{enm}, M_{omn}, N_{emn} and N_{omn}, one obtains that the scattered fields E_s and H_s can be expressed by the formulas:

$$E_s = \sum_{n=1}^{\infty} E_n(ia_n N_{e1n}^{(2)} - b_n M_{o1n}^{(2)}), \tag{43}$$

$$H_s = \frac{k}{\omega\mu_0}\sum_{n=1}^{\infty} E_n(a_n M_{e1n}^{(2)} + ib_n N_{o1n}^{(2)}), \tag{44}$$

where a_n and b_n are the unknown coefficients which will be obtained later, and we attach the superscript $^{(2)}$ to vector spherical harmonics for which the radial dependence of the generating functions is specified by $z_n = h_n^{(1)}$. We refer to the general terms in the series (43) and (44) as s_n (see Tables 3 and 4 below).

Let us derive the formulas for the unknown coefficients a_n and b_n used in (43) and (44). In the spherical coordinates the boundary condition can be written as

$$[\widehat{e}_r, Es] - \zeta a[\widehat{e}_r, [H_s, \widehat{e}_r]] = a\zeta[\widehat{e}_r, [H_0, \widehat{e}_r] - [\widehat{e}_r, E_0], \tag{45}$$

where we have used the identity $N_s = a\widehat{e}_r$.

Let

$$\psi_n(\rho) = \rho j_n(\rho) \qquad \text{and} \qquad \xi_n(\rho) = i\rho y_n(\rho), \tag{46}$$

where $\rho = kr$. Then calculating the cross products $[\widehat{e}_r, E_s]$, $[\widehat{e}_r, [H_s, \widehat{e}_r]]$, $[\widehat{e}_r, E_i]$, $[\widehat{e}_r, [H_i, \widehat{e}_r]]$ and substituting these values into (45), one can get the following linear algebraic system (LAS) with respect to coefficients a_n and b_n:

$$a_n G_1(a, \rho, \theta, \varphi) + b_n G_2(a, \rho, \theta, \varphi) = F_1(a, \rho, \theta, \varphi), \tag{47}$$

$$a_n G_3(a, \rho, \theta, \varphi) + b_n G_4(a, \rho, \theta, \varphi) = F_2(a, \rho, \theta, \varphi), \tag{48}$$

where the functions $G_1, G_2, F_1, G_3, G_4, F_2$ are expressed via the known functions of the parameters a, ρ, θ, φ, and ζ (see [7]).

Using the above expressions, one finds the coefficients a_n and b_n:

$$a_n = -E_n \frac{i\psi_n'(\rho) + \frac{\rho\zeta}{\omega\mu_0}\psi_n(\rho)}{\frac{\rho\zeta}{\omega\mu_0}(\psi_n(\rho) + \xi_n(\rho)) + i(\psi_n'(\rho) + \xi_n'(\rho))}, \tag{49}$$

$$b_n = E_n \frac{\psi_n(\rho) + \frac{i\rho\zeta}{\omega\mu_0}\psi_n'(\rho)}{\frac{i\rho\zeta}{\omega\mu_0}(\psi_n'(\rho) + \xi_n'(\rho)) + (\psi_n(\rho) + \xi_n(\rho))}, \tag{50}$$

where $E_n = i^n \frac{2n+1}{n(n+1)}$.

Let us derive a formula for the EM field in far zone. One has (see [11])

$$h_1^{(1)}(\rho) \approx \frac{-e^{i\rho}}{\rho}, \quad \rho \gg 1, \qquad \text{and} \qquad \frac{dh_1^{(1)}(\rho)}{d\rho} \approx \frac{-ie^{i\rho}}{\rho}, \qquad \rho \gg 1. \tag{51}$$

Let $\rho = kr$ and $\rho \gg ka$. Using (51) and the identities $P_1^1(\cos\theta) = -\sin\theta$, $\pi_1(\cos\theta) = -1$ and $\tau_1(\cos\theta) = -\cos\theta$, one gets

$$\begin{aligned} E(\rho) &= e^{i\rho}\cos(\theta)e_1 + 2ia_1\tfrac{e^{i\rho}}{\rho^2}\sin(\theta)\cos(\varphi)\widehat{e}_r \\ &+(-b_1\tfrac{e^{i\rho}}{\rho}\cos(\varphi) + ia_1\tfrac{e^{\rho}(1+i\rho)}{\rho^2}\cos(\theta)\cos(\varphi))\widehat{e}_\theta \\ &+(b_1\tfrac{e^{i\rho}}{\rho}\cos(\theta)\sin(\varphi) - ia_1\tfrac{e^{i\rho}(1+i\rho)}{\rho^2}\sin(\varphi))\widehat{e}_\varphi, \end{aligned} \tag{52}$$

where e_1 is the unit vector along x_1-axis, $e_1 = \sin(\theta)\cos(\varphi)\widehat{e}_r + \cos(\theta)\cos(\varphi)\widehat{e}_\theta - \sin(\varphi)\widehat{e}_\varphi$. Formula (52) will be used in the next Section for numerical comparison with asymptotic formula (24).

4 Numerical Simulation

In this section, we deal with the numerical calculations for comparison of the asymptotic formula (24) from Section 2 and analytic formula (52) from Section 3. The general approach to numerical solution of the wave scattering problem by many small particles was developed in [14],[15], [16], and illustrated by numerical results in [12]. We will use a series of algorithms developed in [12] and apply them for numerical modeling in this Section. Let us assume that the small particle D is a sphere of radius a. In this case, formula (24) for component $E(x)$ of EM field in far zone can be written as

$$E(x) = E_0(x) - \frac{8a^2\pi k\zeta}{3\omega\mu_0}[\nabla_x g(x, O), \nabla \times E_0(O)]. \tag{53}$$

Using the relations

$$\nabla_x g(x, O) = \nabla_x \frac{e^{ik|x|}}{4\pi|x|} = \frac{e^{ik|x|}}{4\pi|x|^2}(ik - \frac{1}{|x|})x \tag{54}$$

and

$$\nabla \times E_0(O) = \nabla \times \begin{pmatrix} e^{ikx_3} \\ 0 \\ 0 \end{pmatrix}_{|x=0} = \begin{pmatrix} 0 \\ ik \\ 0 \end{pmatrix} \tag{55}$$

one can rewrite (53) as follows

$$E(x) = e^{ikx_3}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{8a^2\pi k\zeta}{3\omega\mu_0}\frac{e^{ik|x|}}{4\pi|x|^2}(ik - \frac{1}{|x|})\begin{pmatrix} -x_3 \\ 0 \\ x_1 \end{pmatrix}, \tag{56}$$

where O is origin. Formula (56) gives the representation of components for electrical field E in the Carthesian coordinates. In order to compare (56) with (52), we should rewrite it in the spherical coordinates, using the know formulas for transformation of cartesian coordinates of vector $E(x)$ into spherical ones [13]

$$\begin{aligned} E_r &= E_1 \sin(\theta)\cos(\varphi) + E_2 \sin(\theta)\sin(\varphi) + E_3 \cos(\theta), \\ E_\theta &= E_1 \cos(\theta)\cos(\varphi) + E_2 \cos(\theta)\sin(\varphi) - E_3 \sin(\theta), \\ E_\varphi &= -E_1 \sin(\varphi) + E_2 \cos(\varphi), \end{aligned} \tag{57}$$

where $x = (x_1, x_2, x_3)$.

The following numerical expertiments are of practical importance:

a) checking the accuracy of series (43) and (44) for representation of components of the scattered field E_s and H_s;

b) investigation of the limits of applicability of the asymptotic formula (56) for various values of the radius a of the sphere, by comparing it with the Mie-type solution, given by formula (52);

c) determination of the optimal values of ζ which provide the smallest relative error of the new asymptotic solution (56).

4.1 Checking the accuracy of series (43) and (44)

Investigating convergence of the series (43) and (44), we calculate the coefficients a_n and b_n for a wide range of parameter ka. The dependence of $|a_1|$ and $|b_1|$ on the value of k is shown in Fig. 1 and Fig 2 at $a = 0.1$. The values of these coefficients grow at $k \to 1$, and at the same time, their maximal amplitude does not exceed 10 at $ka = 1$ in the considered range of ζ.

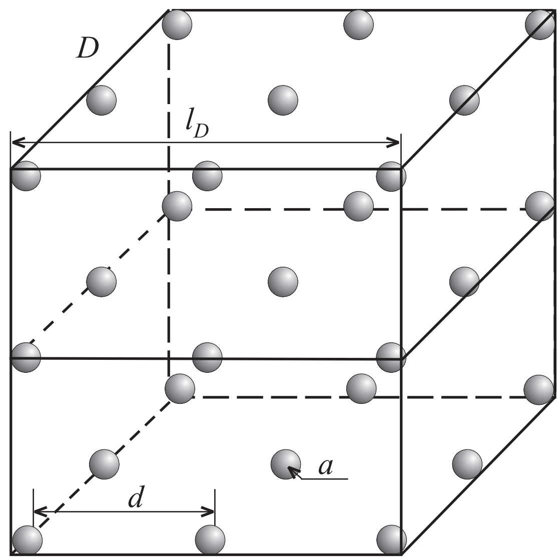

Figure 1: Modulus of a_1 versus k.

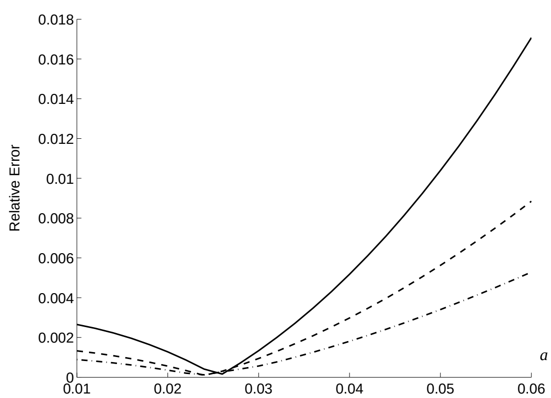

Figure 2: Modulus of b_1 versus k.

The numerical results show that the coefficients a_n and b_n in series (43), (44) decay quickly. The values of a_3 and b_3 are two orders smaller than these of a_1 and b_1. The moduli of the coefficients a_n and b_n for $n \geq 4$ have order less than 10^{-8}. The values of $|a_5|$ and $|b_5|$, presented in Table 1 and Table 2, show that no more than 5 coefficients a_n and b_n are needed in order to calculate the scattered field with high accuracy. The relative error does not exceed $10^{-3} - 10^{-4}$ in the range of considered parameters ka and ζ.

Table 1. Moduli of coefficient a_5 for various ζ, $a = 0.1$.

	$\zeta = 10$	$\zeta = 50$	$\zeta = 100$	$\zeta = 300$
$k = 0.5$	2.141×10^{-11}	2.141×10^{-11}	2.141×10^{-11}	2.139×10^{-11}
$k = 1.0$	4.115×10^{-8}	4.115×10^{-8}	4.113×10^{-8}	4.098×10^{-8}

Table 2. Moduli of coefficient b_5 for various ζ, $a = 0.1$.

	$\zeta = 10$	$\zeta = 50$	$\zeta = 100$	$\zeta = 300$
$k = 0.5$	1.806×10^{-11}	2.017×10^{-11}	2.099×10^{-11}	2.136×10^{-11}
$k = 1.0$	3.409×10^{-8}	3.642×10^{-8}	3.862×10^{-8}	4.074×10^{-8}

Convergence rate of the above series depends on the modulus of the coefficients a_n and b_n in the series. The "general term" s_n in the series (43) is $s_n = E_n(ia_n N^{(2)}_{e1n} - b_n M^{(2)}_{o1n})$. Because the function $h^1_n(\rho)$ increases rapidly as $\rho \to 0$, the convergence rate depends on the values of $a_n N^{(2)}_{e1n}$ and $b_n M^{(2)}_{o1n}$ in (43), and on the values of $a_n M^{(2)}_{e1n}$ and $b_n N^{(2)}_{o1n}$ in (44). The results, presented in Tables 3 and 4, show that the general terms s_n in series (43), (44) decay very fast for the considered range of ka. These results are presented for $a = 0.1$. It can be seen that terms with the index $n+1$ are by two or three orders less than the previous ones. Numerical results show that this rapid convergence holds, e.g., when $\rho \leq 2$: in this case the maximal relative error for E_r, calculated by formula (43) with 5 terms, is not more than 0.6×10^{-3}, and for E_θ and E_φ this error is not more than 0.2×10^{-3}, for any ζ between 5 and 300. The series (43) and (44) converge rapidly and yield high accuracy in calculations of the EM fields. Therefore, one can use the values E_s and H_s, calculated by series (43)-(44), as "exact" values of the scattered fields when comparing these values with the ones calculated by the asymptotic formula (56).

Table 3. Moduli of general terms s_n in series (43) for E_r component.

Table 3.

	s_1	s_2	s_3	s_4	s_5
$k = 0.1$	0.2516	0.0047	0.0001	4.819×10^{-7}	7.408×10^{-9}
$k = 0.05$	0.2514	0.0023	1.405×10^{-5}	6.016×10^{-8}	4.625×10^{-10}
$k = 0.01$	0.2501	0.0005	5.617×10^{-7}	4.811×10^{-10}	7.398×10^{-13}

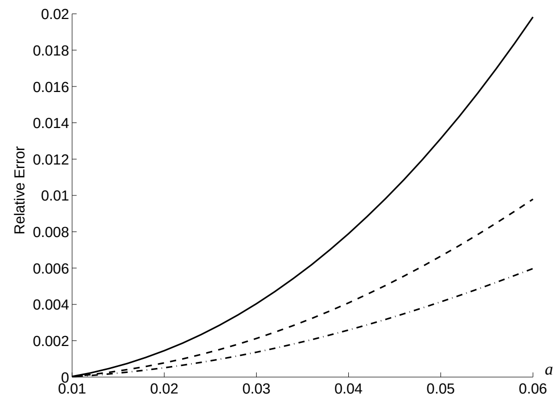

Figure 3: Relative error for E_r, E_θ and E_φ components at $ka \leq 1.0$.

Table 4. Moduli of general terms s_n in series (43) for E_φ component.

	s_1	s_2	s_3	s_4	s_5
$k = 0.1$	0.3445	0.0087	0.0001	1.043×10^{-6}	6.954×10^{-9}
$k = 0.05$	0.4933	0.0001	5.424×10^{-7}	1.163×10^{-9}	2.027×10^{-11}
$k = 0.01$	1.1076	0.0028	3.698×10^{-6}	3.287×10^{-9}	2.198×10^{-12}

The numerical results confirm the high accuracy of series (43) and (44) in a wide range of parameter ka. We carry out the calculations in order to establish the relative error of the solution calculated by using the above series for different value of impedance ζ (see Figure 3). Because the exact values S^* of the sums of series (43) and (44) are unknown, we use the following formula for calculating the relative error

$$\mathrm{RE} = |\frac{S_{n+1} - S_n}{S_{n+1}}|, \tag{58}$$

instead of generally used

$$\mathrm{RE} = |\frac{S^* - S_n}{S^*}|, \tag{59}$$

where S_n and S_{n+1} are the n-th and $n+1$-st partial sums of the series (43), (44).

The relative error for $0.01 \leq ka \leq 1.0$ does not exceed 0.8×10^{-3} for E_θ and E_φ components, and it does not exceed 0.16×10^{-3} for E_r component. This error is obtained when $n = 5$ terms of series (43) are kept for calculations. Further increase of n does not reduce the error practically. The optimal value of ζ, which provide minimal relative error, is ~ 100. The error grows when $\zeta > 100.7$.

The numerical results for values $1,4 \leq ka \leq 3.0$ are given in Table 5. The results are presented for $\zeta = 100.0$, the value of ζ which yields minimal error in the considered range of ka. Apparently, the relative error for E_r component increases more than for E_θ and E_φ components and attains its maximal value 1.7% at $ka = 3.0$. The error for E_θ and E_φ components does not exceed 1.15%. These errors stabilize at $n = 20$ terms in series (43).

Table 5. Relative error for series (43) at $ka > 1$.

ka	1.4	1.8	2.2	2.6	3.0
E_r	0.05 %	0.06 %	0.59 %	0.93 %	1.72 %
E_θ, E_φ	0.12 %	0.25 %	0.43 %	0.71 %	1.15 %

The relative error for H, i.e., the error for the series (44), is of the same order as for the series (43). Numerical results show that the relative errors of series (43) and (44) do not exceed 4.5% for $ka \leq 5$.

4.2 Error of the asymptotic formula (56)

The numerical results in this Section give the range of applicability of the asymptotic formula (56) for various values of parameters ka and ζ. The calculations were carried out not only for the $ka \ll 1$ but also for $ka = O(1)$. At $ka = 2.0$ the relative error exceeds 20% for all considered ζ. Analytic solution, given by formula (52), was considered as the exact solution, because its relative error was less than 10^{-3}.

In Fig. 4, the relative error of solution (56) for E_r component is shown for different values of a at $\zeta = 10$. One can see that at $a = 0.1$ the error decreases three times if a decreases five times (from 0.05 to 0.01). For small k at $a = 0.1$ the relative error is less than for $a = 0.05$, but for $k > 0.5$ this error is rapidly increasing. For example, at $k = 1$ for this a relative error exceeds 15%, It does not exceed 7% and 2% for $a = 0.05$ and $a = 0.01$ respectively. The curves presented in Fig. 5 show that the relative error diminishes if ζ increases, and at greater ζ it is less sensitive to change of a. For the considered range of a, the error does not change practically. The minimal relative error is attained at $a = 0.01$ and it is equal to 0.2% (see solid line in Fig. 5).

The numerical results show that formula (56) at $ka = 1.0$ yields the values of the EM fields with an error that does not exceed 15% for $\zeta = 10$ and 10% for $\zeta = 100$. This error decays slowly to 13 % and 9 % respectively if ζ increases up to 300, and it grows at $\zeta > 300$. The relative error for the components E_θ and E_φ is less than the one for E_r in all these cases.

4.3 Finding optimal values of ζ at fixed a

Numerical results presented in the previous section show that relative error of the asymptotic solution (56) depends on the value of impedance ζ. We found the optimal values of ζ that yield minimal error at some fixed values of ka. In Fig. 6 and Fig. 7 the relative error of the solution, calculated by formula (56), is shown for small and intermediate values of ζ, respectively. One can see that

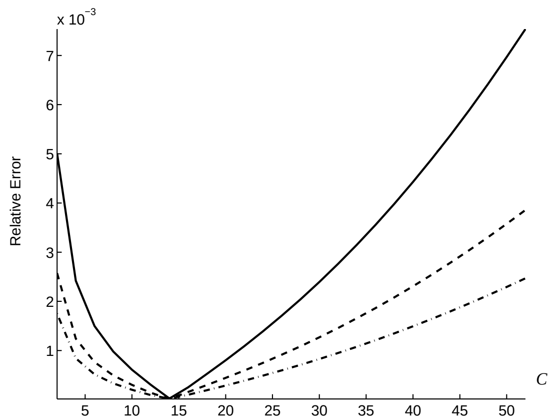

Figure 4: Relative error of solution to (56) (E_r component) for $\zeta = 10$.

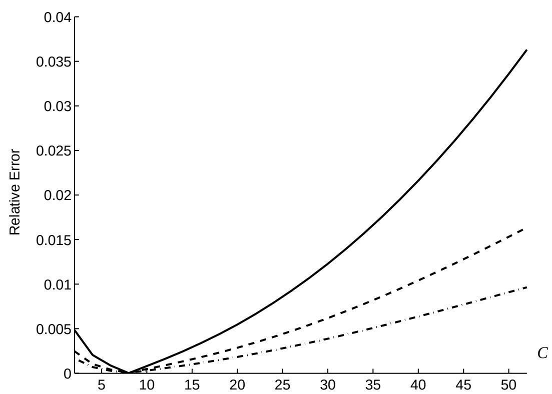

Figure 5: Relative error of solution to (56) (E_r component) for $\zeta = 100$.

the relative error decays when ζ increases. This error does not exceed 18 % for small ζ, and 13 % for the intermediate values of ζ. The error depends on a stronger at the smaller values of ζ. This dependence does not appear so strongly at the intermediate and big values of ζ. The error grows for $\zeta < 5$ and $\zeta > 300$. A local minimum of the relative error was found at $\zeta = 32.7$. This shows that the dependence of the relative error on the values of ζ is not monotone, and should be checked for each ka.

The amplitude and the radiation pattern of the scattered field depend on the values of k and ζ. Typical amplitudes of the components E_r, E_θ, and E_φ of the electric field are shown in the Figs. 8, 9, and 10 at $ka = 0.001$, $\zeta = 100.0$, $a = 1.0$, $r = 20$, where r is distance to far zone. These results are calculated by formula (56). The relative error of the numerical values calculated by formula (56) for this case is 3.1%, 3.6%, and 1.0% for components E_r, E_θ,

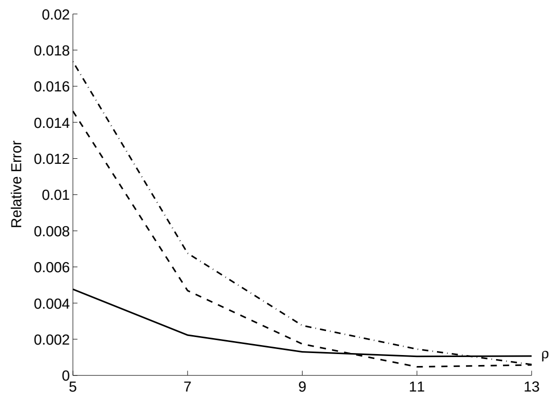

Figure 6: Relative error at small ζ for different a.

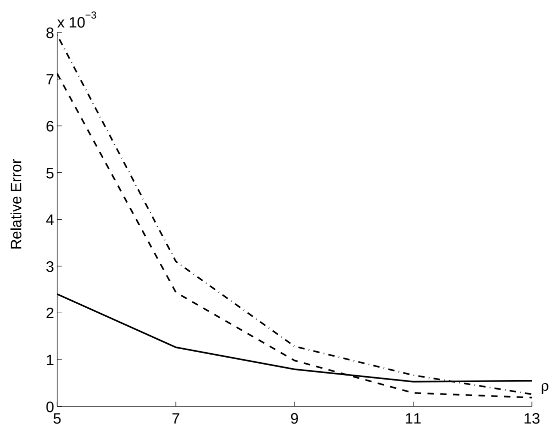

Figure 7: Relative error at middle ζ for different a.

and E_φ respectively. The error can be decreased to $0.1\% - 0.05\%$ by increasing r up to 50. Further increasing of r does not influence practically the decreasing of the error. The error of the same order is obtained also for the components H_r, H_θ, and H_φ.

5 Conclusions

Simple explicit analytic formula for calculation of EM waves, scattered by small impedance particle of arbitrary shape, is derived and tested numerically. The solution for a spherical particle can be obtained easily using our general formula, valid for small particles of arbitrary shapes.

To establish the limits of the applicability of the new formula, it is compared numerically with the results, obtained by Mie-type theory for spherical

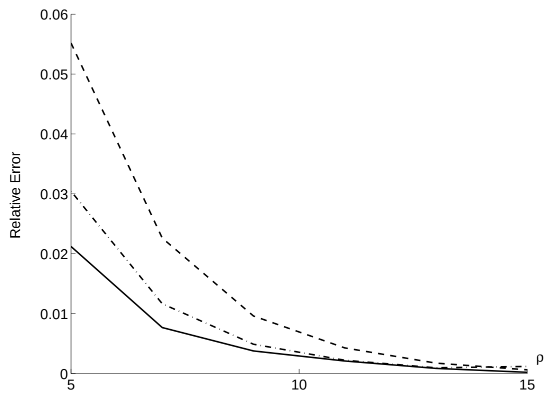

Figure 8: Typical amplitude of E_r component.

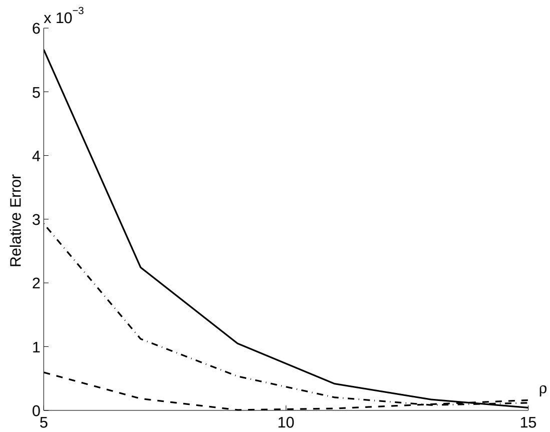

Figure 9: Typical amplitude of E_θ component.

impedance particle. The numerical results show that the Mie-type series for the solution converge rapidly: it is sufficient to use not more than 5 terms in order to obtain the solution with a relative error of the order 10^{-3}. We call the solution, obtained by Mie-type series, an analytic solution, and we use this solution as an "exact" solution when comparing with the numerical results obtained by the new asymptotic formula (56).

The comparison of asymptotic and analytic solutions show that relative error of asymptotic solution depends on the values of parameters ka and ζ. It is shown for the component E_r that the error less than 3.1 % is obtained for $ka = 0.001$ at $\zeta = 100.0$. The error grows slowly when ka grows, it is equal to 7.5 % at $ka = 1.0$, and it grows up to 13.6 %. at $ka = 2.0$. These results were obtained at $r/a = 20.0$.

Numerical results show that relative error of asymptotical solution can be

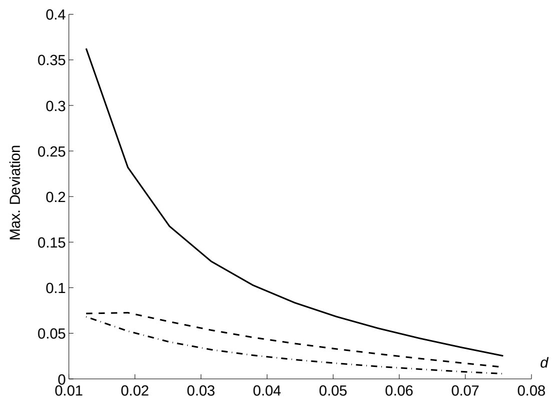

Figure 10: Typical amplitude of E_φ component.

reduced significantly by changing ζ at a fixed ka. For example, the relative error of E_r equal to 2.2% for $ka = 2.0$ is obtained at $\zeta = 300.0$. This relative error is larger for E_θ and smaller for E_φ.

The numerical results show that the range of optimal values of ζ is between $\zeta = 5.0$ and $\zeta = 300.0$. In this large range there exist several local minima of relative error as a function of ζ at a fixed ka. The values of these local minimums depend on ka considerably. In summary, one can concludes that the relative error of the new asymptotic formula (56) is in range 0.2% to 20.0% as ka varies in the range $0,001 - 2.0$ and ζ varies from 5.0 to 300.0. Thus, asymptotic formula (56) is applicable in a wide range of parameters.

References

[1] L. Landau, L. Lifshitz, *Electromagnetics of continuous media*, Pergamon Press, London, 1982.

[2] A. G. Ramm, *Wave scattering by small bodies of arbitrary shapes.* World Sci. Publisher, Singapore, 2005.

[3] G. Mie, Beitrage zur Optic Truber Medien speziell kolloidaler Metallosungen, *Ann. Phys.*, 25, 377-452, 1908.

[4] J. H. Bruning and Y. T. Lo, Multiple Scattering of EM waves by spheres, part I - Multiple expansion and ray-optical solutions, *IEEE Trans. Antennas Propag.* AP-19, 378-390, 1971.

[5] J. H. Bruning and Y. T. Lo, Multiple Scattering of EM waves by spheres, part II - Numerical and experimental results, *IEEE Trans. Antennas Propag.* AP-19, 391-400, 1971.

[6] D. W. Mackowski, Analysis of radiative scattering for multiple sphere configuration, *Proc. R. Soc. Ser. A*, London, 433, 599-614, 1991.

[7] Yu-Lin Xu, Electromagnetic scattering by an aggregate of spheres, *Applied Optics*, 34, No 21, 4573-4588, 1995.

[8] A. G. Ramm, Electromagnetic wave scattering by a small impedance particle of arbitrary shape, *Optics Communications*, 284, No. 16-17, (2011), 3872-3877.

[9] A. G. Ramm, *Scattering by obstacles*, D. Reidel, Dordrecht, 1986.

[10] C. F. Bohren and D.R. Huffman, *Absorption and scattering of light by small particles*, John Wiley and Sons, 1983.

[11] M. Abramowitz and R. A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, 1964.

[12] M. I. Andriychuk and A. G. Ramm, Scattering by many small particles and creating materials with a desired refraction coefcient, *Int. J. Computing Science and Mathematics*, Vol.3, Nos.1/2, pp.102-121, 2010.

[13] G. A. Korn and T. M. Korn, *Mathematical Handbook for Scientists and Engineers*, McGraw-Hill Book Company, 1968.

[14] A. G. Ramm, Wave scattering by many small particles embedded in a medium, *Phys. Lett. A*, 372/17, (2008), 3064-3070.

[15] A. G. Ramm, Wave scattering by many small bodies and creating materials with a desired refraction coefficient, *Afrika Matematika*, 22, N1, (2011), 33-55.

[16] A. G. Ramm, Scattering of scalar waves by many small particles, *AIP Advances*, 1, 022135, (2011).

[17] G. Markov, E. Vasiliev, *Mathematical methods of applied electrodynamics*, Sov. Radio, Moscow, 1970.