NEW MODELS IN GENERAL RELATIVITY AND EINSTEIN-GAUSS-BONNET GRAVITY

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New models in general relativity and Einstein-Gauss-Bonnet gravity

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Abstract

We generate the Einstein-Gauss-Bonnet field equations in five dimensions for a spherically symmetric static spacetime. The matter distributions considered are both neutral and charged. The introduction of a coordinate transformation brings the condition of isotropic pressure to a single master ordinary differential equation that is an Abel equation of the second kind. We demonstrate that the master equation can be reduced to a first order nonlinear canonical differential equation. Firstly, we consider uncharged gravitating matter. Several new classes of exact solutions are found in explicit and implicit forms. One of the potentials is determined completely. The second potential satisfies a constraint equation. Secondly, we study charged gravitating matter with Maxwell's equations. We find new classes of exact charged solutions in explicit and implicit forms using two approaches. In the first approach, we can find new exact models without integration. In the second approach the Abelian pressure isotropy equation has to be integrated, which we demonstrate is possible in a number of cases. The inclusion of the electromagnetic field provides an extra degree of freedom that leads to viable exact solutions. An interesting feature characterising the new models is that a general relativity limit does not exist. Our new solutions exist only in Einstein-Gauss-Bonnet gravity. In addition, we have considered the dynamics of a shear-free fluid in Einstein gravity in higher dimensions with nonvanishing heat flux in a spherically symmetric manifold. This endeavour generates new exact models, being a generalisation of models developed in earlier treatments.

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Chapter 1

Introduction

General relativity is currently the most successful and widely accepted theory describing gravity, viz., the curving of spacetime rather than an invisible force that attracts objects to one another. Simply put, in the words of John Wheeler, general relativity explains the unique marriage between matter and curvature for which matter tells spacetime how to bend and spacetime tells matter how to move. Therefore, one cannot merely associate the notion of gravity to that of a force but rather to a more transcendent, mathematical structure that is spacetime: a four-dimensional separable time-oriented, Hausdorff, paracompact, C^∞ pseudo-Riemannian manifold with a Lorentzian signature (Hawking and Ellis 1973). Moreover, this theory provides us with the foundation for analysing the dynamics of stellar objects, structures and regions of space containing stars, galaxies, black holes and dark matter because general relativity describes the interactions of bodies in the Universe as a result of their gravitational fields. Observational evidence for general relativity is supported by many factors including the perihelion precession of Mercury, the gravitational redshift and the bending of light. As such it is widely used to model compact objects that include neutron stars (like pulsars or magnetars), white dwarfs and black holes on the astrophysical frontier, and it is supported by the existence of gravitational waves in the cosmological context.

Furthermore, general relativity couples the geometric properties of gravity (curvature of spacetime) to the energy and momentum of a physical system in the form of the Einstein field equations. Spherically symmetric manifolds and the Einstein field equations form the basis for studying relativistic stellar models in astrophysics and cosmology in the presence of a gravitational field. In general, it is difficult solve the field equations as they comprise a system of ten nonlinear coupled, partial differential equations. Particular solutions can be generated using a number of different techniques. These include methods such as the geometric Lie analysis, harmonic analysis, imposing an equation of state, the use of conformal symmetries and ad hoc approaches restricting the matter variables or gravitational potentials. We also require that exact solutions be mathematically feasible and physically acceptable, and they must be used in conjunction with other fundamental theories such as thermodynamics and electromagnetism in order to determine the associated physical features so as to provide a deeper insight into the behaviour of the gravitational field.

Consequently, several exact solutions for static spherically symmetric manifolds have been found. The first and most well known solutions are the Schwarzschild exterior and interior solutions (Schwarzschild 1916a, 1916b). The Schwarzschild exterior solution represents a vacuum solution that describes the gravitational field outside a spherically symmetric uncharged body, that is static and asymptotically flat. The Schwarzschild interior solution describes the interior of the body where the mass density is constant. The Reissner-Nordström solution describes the gravitational field outside a spherical, non-rotating, charged object and reduces to the Schwarzschild exterior solution when the electromagnetic field vanishes (Reissner 1916, Weyl 1917, Nordström 1918, Jeffrey 1921). It has been 105 years since Albert Einstein formulated this beautiful geometric theory which generalised the theory of special relativity and refined Newton's law of universal gravitation, yet it continues to be a central paradigm of physics today.

A shortcoming of general relativity, despite modelling relativistic matter success-fully, is that it cannot fully explain certain recent observed phenomena, in particular the late time expansion of the universe which has been found in astronomical observations and reported in the Wilkinson Microwave Anisotropy Probe (WMAP) data. Hence the notion of modified theories of gravity is required. One such approach to modify conventional gravity is to introduce higher order curvature terms. Some examples of such theories are $f(R)$ gravity, scalar-tensor theories, Lovelock gravity, and a special case, namely Einstein-Gauss-Bonnet (hereafter EGB) gravity. EGB gravity is the most widely studied of the higher dimensional curvature theories, and belongs to a class of second order polynomials discovered by Lovelock (1971) where the corresponding action principle is a modification of the Einstein-Hilbert action. EGB gravity arises in the low energy limit in string theories. The Gauss-Bonnet action is composed of quadratic forms of the Riemann tensor, Ricci tensor and Ricci scalar. As a result, the field equations appear as second order quasilinear differential equations with these higher order quantities having no impact in four dimensions, and in the absence of such terms Einstein gravity is regained. The advantage of EGB gravity over other higher order curvature theories is that it avoids the problem of ghost terms. For the above reasons, EGB gravity is a promising theory of modified gravity.

A significant amount of research in this framework of gravity have been conducted in the context of compact stars, gravitational collapse and static spherically symmetric models. Some exterior solutions for EGB gravity have been found, these include models by Boulware and Deser (1985), and Anabalon *et al* (2009). Boulware and Deser (1985) obtained higher dimensional vacuum solutions analogous to the Schwarzchild exterior spacetime from general relativity. Similarly, Anabalon *et al* (2009) constructed a vacuum solution for the Kerr-Schild model in five dimensions. Nevertheless, finding exact interior solutions for such a regime is a different research problem in its entirety, because of curvature corrections and as such has not been extensively studied. Thus far, a substantially small number of interior static solutions in EGB gravity have been reported. For example, Maharaj *et al* (2015) obtained an interior exact solution by specifying a form for one gravitational potential in order to find the second potential, and with the use of the Frobenius method built a class of models that admit an equation of state. A similar approach was adopted by Chilambwe *et al* (2015).

The concept of higher dimensions was first brought forward by Kaluza and Klein, independently, who wanted to unify gravity with the electromagnetic field by the introduction of an extra dimension (Kaluza 1921, Klein 1926). This notion of higher dimensions provides us with a platform to understand the nature of the early universe and plays a crucial role in describing the gravitational dynamics of stellar objects not only in conventional general relativity but also in modified gravity theories such as Lovelock gravity and string theories. As a consequence, a number of studies in the higher dimensional regime have been reported in the literature. For example, Tangherlini (1963) investigated the Schwarzschild spacetime in higher dimensions. Myers and Perry (1986) examined black hole solutions to the Einstein's equations and generalised the Reissner and Nordström, and Kerr spacetimes to higher dimensions. Furthermore, Iyer and Vishweshwara (1989), and Chatterjee *et al* (1990) generalised the Vaidya metric and presented a model for a radiating star in the uncharged and charged case respectively. In the framework of Lovelock gravity, Brassel *et al* (2018) investigated the continual gravitational collapse of a spherically symmetric radiation shell in five dimensional EGB gravity and found that the final fate of such a collapse is an extended and weak curvature conical singularity at the centre, which was initially naked for a time before being covered by an apparent horizon. Brassel *et al* (2019) analysed higher dimensional radiating black holes in EGB gravity and established that collapse terminates with a strong curvature singularity which could be naked; the gravitational dynamics are affected by the presence of higher dimensions. Many authors have studied the implications of higher dimensions on the Einstein field equations. These include some works developed by Chatterjee (1990), Sil and Chatterjee (1994) and Ghosh and Deshkar (2007).

In light of the above, it is interesting to investigate spherically symmetric radiating spacetimes with vanishing shear in higher dimensional general relativity. These spacetimes are important for applications in astrophysics and cosmology, and have been widely studied to model relativistic stars which dissipate null radiation in the form of a radial heat flow. Heat flux is a necessary component in modelling a complete description of radiating relativistic stars, and it is crucial for astrophysical applications involving gravitational collapse, singularities in manifolds and black hole physics, as pointed out by Kransinski (1997). Several exact solutions for shear-free relativistic fluid models with nonvanishing heat flux were obtained. These include some earlier works by Bergmann (1981), Maiti (1982) and Modak (1984) in a cosmological setting, and by Govender and Thirukkanesh (2009) and Maharaj *et al* (2011), in the field of astrophysics. On similar grounds Msomi *et al* (2011) found a five-parameter family of transformations that mapped existing solutions into new ones using the Lie group theoretic approach. Banerjee *et al* (1989) obtained conformally flat radiating solutions. Investigations involving the process of gravitational collapse for such heat flow models are contained in the seminal works of de Oliveira *et al* (1985), Glass (1990) and Deng and Mannheim (1990) in dimensions of four, and by Bhui *et al* (1995) and Banerjee and Chatterjee (2005) in the context of higher dimensional cosmological models. In addition to this, Nyonyi *et al* (2013) considered the contributions of an electromagnetic field and provided analytical solutions using the Lie analysis. Nyonyi *et al* (2014) extended this endeavour to higher dimensional manifolds.

The number of spacetime dimensions has been shown to influence certain features of astronomical objects. In particular the mass-radius ratio depends on the number of dimensions. Examples of this nature include some pioneering works by Ponce de Leon and Cruz (2000), and by Paul (2001) who determined specific bounds for the mass-radius relationship in general relativity. Hence it is important to make use of the correct value of the gravitational coupling constant (κ_N), which is defined in terms of arbitrary dimensions (N), and directly influences the Einstein field equations. Additionally, the surface area ($\mathcal{A}_{N-2}$), which is also a function of the spacetime dimensions, becomes important when considering the effects of an electromagnetic field. The general definition for the coupling constant and surface area was investigated by Mansouri and Nayeri (1998).

In this thesis we aim to seek new classes of exact solutions to the EGB field equations in a five dimensional regime for spherically symmetric gravitational fluids (neutral and charged) which are applicable to stellar objects in relativistic astrophysics. We also investigate models with vanishing shear and nonzero heat flux in higher dimensional general relativity. New families of exact solutions are found by transforming the condition of pressure isotropy equation to canonical form.

Chapter 2

Fundamental concepts in general relativity

2.1 Introduction

Einstein's theory of general relativity is a remarkable theory of gravity and serves as the backbone for relativistic astrophysics and cosmology. Therefore, in this chapter we will provide the background theory and fundamental concepts of differential geometry that is essential in generating a spherically symmetric model for stellar objects or gravitating systems. In section 2.2, we introduce the concept of a differentiable manifold and its associated elements as well as the important components of differential geometry such as the Riemann tensor, Ricci tensor, Ricci scalar and the Einstein tensor. In section 2.3, we provide an outline of the matter distribution and a special case of the energy momentum tensor. In section 2.4, the expressions of the Einstein field equations are given for both the uncharged and charged cases respectively. Finally in section 2.5, we introduce the constituents for a special case of Lovelock gravity, that is EGB gravity.

2.2 Differential geometry

Spacetime in general relativity is modelled by a N dimensional ($N-1$ dimensional space and one dimensional time), differentiable, pseudo-Riemannian manifold $\boldsymbol{M}$ on which a metric tensor field $\boldsymbol{g}$ is imposed. This tensor field is described to be symmetric and nondegenerate and entails the dynamics associated with the gravitational field. Individual points in a N dimensional manifold are labelled by a system of N real coordinates as

$$(x^a) = \left(x^0, x^1, x^2, ..., x^{N-1}\right), \tag{2.2.1}$$

where $x^0 = ct$ (c is the speed of light in a vacuum taken to be unity) represents the timelike coordinate and $\left(x^1, x^2, ..., x^{N-1}\right)$ are the spacelike coordinates. Furthermore, the infinitesimal distance between two neighbouring points on a N dimensional manifold is denoted by the line element

$$ds^2 = g_{ab}dx^a dx^b. \tag{2.2.2}$$

In the above g_{ab} is the metric tensor.

The metric connection coefficient $\mathbf{\Gamma}$ is defined in terms of the metric tensor $\boldsymbol{g}$ and its derivatives with coefficients

$$\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}\left(g_{cd,b} + g_{db,c} - g_{bc,d}\right), \tag{2.2.3}$$

where commas denote partial differentiation. These coefficients are also known as Christoffel symbols of the second kind which are symmetric in their lower indices.

The spacetime curvature is contained in the Riemann tensor $R^d{}_{abc}$ which is defined by

$$R^d{}_{abc} = \Gamma^d{}_{ac,b} - \Gamma^d{}_{ab,c} + \Gamma^d{}_{eb}\Gamma^e{}_{ac} - \Gamma^d{}_{ec}\Gamma^e{}_{ab}. \tag{2.2.4}$$

Contraction of (2.2.4) leads to the Ricci tensor given by

$$\begin{aligned} R_{ab} &= R^c{}_{acb} \\ &= \Gamma^c{}_{ab,c} - \Gamma^c{}_{ac,b} + \Gamma^c{}_{dc}\Gamma^d{}_{ab} - \Gamma^c{}_{db}\Gamma^d{}_{ac}. \end{aligned} \tag{2.2.5}$$

We note that the Ricci tensor is symmetric and upon contraction, we acquire the Ricci scalar which is written as

$$\begin{aligned} R &= R^a{}_a \\ &= g^{ab} R_{ab}. \end{aligned} \tag{2.2.6}$$

The Einstein tensor $\boldsymbol{G}$ is then constructed with the definitions of the Ricci tensor (2.2.5), Ricci scalar (2.2.6) and the metric tensor $\boldsymbol{g}$ to give

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}, \tag{2.2.7}$$

which is symmetric by definition with zero divergence

$$G^{ab}{}_{;b} = 0. \tag{2.2.8}$$

2.3 Matter

The matter content of spacetime can be described by a relativistic fluid, and it is expressed by the total energy momentum tensor $\mathcal{T}^{(Total)}$ that contains contributions from charged matter and neutral barotropic fluids.

2.3.1 Uncharged matter

The energy momentum tensor for uncharged matter is defined by the symmetric tensor

$$T_{ab} = (\rho + p) u_a u_b + p g_{ab} + q_a u_b + q_b u_a + \pi_{ab}, \tag{2.3.1}$$

where ρ is the energy density, p is the isotropic pressure, q_a is the heat flux vector and π_{ab} is the anisotropic pressure (stress) tensor. All these quantities are measured relative to a comoving fluid velocity $\boldsymbol{u}$ which is unit and timelike ($u^a u_a = -1$). In addition, we note that $q^a u_a = 0$ and $\pi^{ab} u_a = \pi^a{}_a = 0$. In perfect fluids, the heat flux and anisotropic stress are absent ($q_a = 0$ and $\pi_{ab} = 0$). Hence the energy momentum tensor becomes

$$T_{ab} = (\rho + p) u_a u_b + p g_{ab}, \tag{2.3.2}$$

for a perfect fluid.

We can introduce the radial pressure $p_{\|}$ and tangential pressure $p_{\perp}$ by defining

$$p = \frac{1}{N-1}\left(p_{\|} + (N-2)\,p_{\perp}\right), \tag{2.3.3}$$

in N dimensions. In four dimensions $p = \frac{1}{3}\left(p_{\|} + 2p_{\perp}\right)$ for the isotropic pressure. Important results about the energy momentum tensor and energy conditions in N dimensions are contained in the works of Maharaj and Brassel (2021) and Brassel *et al* (2021).

2.3.2 Charged matter

When the matter distribution contains electric charge, we must consider the contribution of the electromagnetic field to the total energy momentum tensor $\mathcal{T}^{(Total)}$. The electromagnetic field tensor $\boldsymbol{F}$, also known as the Faraday tensor, is defined in terms of the electromagnetic potential $\boldsymbol{A}$ by

$$F_{ab} = A_{b;a} - A_{a;b}. \tag{2.3.4}$$

We note that this tensor F_{ab} is skew-symmetric.

The electromagnetic matter tensor $\boldsymbol{E}$ is composed of the Faraday tensor and is written as

$$E_{ab} = \frac{1}{\mathcal{A}_{N-2}}\left(F_{ac}F_{b}{}^{c} - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right), \tag{2.3.5}$$

where $\mathcal{A}_{N-2}$ is the surface area of an $N-2$ sphere denoted by

$$\mathcal{A}_{N-2} = \frac{2\pi^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)}. \tag{2.3.6}$$

In the above $\Gamma(...)$ is the gamma function. In four dimensions, the surface area becomes

$$\mathcal{A}_{2} = 4\pi. \tag{2.3.7}$$

The electromagnetic field is governed by Maxwell's equations. These fundamental equations are expressed covariantly as

$$F_{ab;c} + F_{bc;a} + F_{ca;b} \;=\; 0, \tag{2.3.8a}$$

$$F^{ab}{}_{;b} \;=\; \mathcal{A}_{N-2} J^{a}. \tag{2.3.8b}$$

In the above J^a is the current density defined by

$$J^a \;=\; \sigma u^a, \tag{2.3.9}$$

for a non-conducting fluid, and σ is the proper charge density.

Thus the total energy momentum tensor $\mathcal{T}^{(Total)}$ is given by

$$\mathcal{T}^{(Total)}_{ab} \;=\; T_{ab} + E_{ab}, \tag{2.3.10}$$

and it is divergence free

$$\mathcal{T}^{ab}{}_{;b} \;=\; 0,$$

$$\left(T^{ab} + E^{ab}\right)_{;b} \;=\; 0. \tag{2.3.11}$$

2.4 Field equations

The Einstein field equations in the absence of charge are given by

$$G_{ab} = \kappa_N T_{ab}, \tag{2.4.1}$$

where κ_N is the coupling constant defined by

$$\kappa_N = \frac{2\,(N-2)\,\pi^{\frac{N-1}{2}} G}{c^4\,(N-3)\left(\frac{N-1}{2} - 1\right)!}. \tag{2.4.2}$$

Equation (2.4.1) governs the nature and dynamical interaction between the matter distribution and curvature. In four dimensions the coupling constant evaluates to

$$\kappa_4 (= \kappa) = 8\pi. \tag{2.4.3}$$

Note that we use geometrized units for which the gravitational constant and the speed of light are unity ($G = c = 1$). This is the procedure followed in subsequent chapters.

The Einstein-Maxwell system of equations can be generated with the use of the Einstein tensor (2.2.7), the total energy momentum tensor (2.3.10), the electromagnetic tensor (2.3.5) along with Maxwell's equations (2.3.8) in the form

$$G_{ab} = \kappa_N \left(T_{ab} + E_{ab}\right), \tag{2.4.4a}$$

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0, \tag{2.4.4b}$$

$$F^{ab}{}_{;b} = \mathcal{A}_{N-2} J^a. \tag{2.4.4c}$$

2.5 Einstein-Gauss-Bonnet (EGB) gravity

An action will generate field equations in any gravity theory. The Einstein-Hilbert action generates the well known Einstein field equations in any dimensions; a modification of this very action results in the Gauss-Bonnet action that is required to produce the EGB field equations. The Gauss-Bonnet action in N dimensions has the form

$$S = \int \sqrt{-g}\left[\frac{1}{2}(R - 2\Lambda + \alpha L_{GB})\right] d^N x + S_{matter}, \tag{2.5.1}$$

where the parameter α represents the Gauss-Bonnet coupling constant, g is the determinant of the metric tensor $\boldsymbol{g}$, R is the Ricci scalar, Λ is the cosmological constant and L_{GB} is the Lovelock term. An interesting feature of this action is that the field equations appear as second order differential equations which are quasilinear in the highest derivative. In addition, the action is valid in arbitrary spacetime dimensions but the Gauss-Bonnet term makes no contribution in dimensions of four or less.

Varying the action (2.5.1), the EGB field equations for uncharged matter are derived in the form

$$G_{ab} - \frac{\alpha}{2} H_{ab} = \kappa_N T_{ab}. \tag{2.5.2}$$

In the above T_{ab} is the energy momentum tensor for neutral matter and H_{ab} is the Gauss-Bonnet tensor which reads as

$$H_{ab} = g_{ab} L_{GB} - 4RR_{ab} + 8R_{ac}R^c{}_b + 8R^{cd}R_{acbd} - 4R_a{}^{cde}R_{bcde}, \tag{2.5.3}$$

and the Lovelock term is written as

$$L_{GB} = R^2 + R_{abcd}R^{abcd} - 4R_{cd}R^{cd}. \tag{2.5.4}$$

Note that (2.5.4) is quadratic in the Riemann tensor, Ricci tensor and the Ricci scalar. The field equations (2.5.2) are highly nonlinear. We note that negative values for α leads to unphysical behaviour; thus it is essential to work with α being greater than zero. It is also important to note that a variety of other actions exist, e.g. in $f(R)$ gravity the gravitational action is a well defined function of the Ricci scalar as described by Capozziello and De Laurentis (2011) and Goswami *et al* (2015).

Furthermore equation (2.5.2) can be extended to charged matter for which the system of charged EGB field equations are defined by

$$G_{ab} - \frac{\alpha}{2}H_{ab} = \kappa_N \left(T_{ab} + E_{ab}\right), \tag{2.5.5a}$$

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0, \tag{2.5.5b}$$

$$F^{ab}{}_{;b} = \mathcal{A}_{N-2}J^a. \tag{2.5.5c}$$

When $\alpha = 0$ we regain the N dimensional field equations of general relativity given by (2.4.4).

Chapter 3

Static spacetimes

3.1 Introduction

Spherically symmetric static spacetimes have been widely studied in the modelling of relativistic compact objects. Some well known early examples of this nature include the models developed by Schwarzchild (1916) for neutral fluids, and by Reissner and Nordström (1917, 1918) for charged matter. Several families of exact solutions have been subsequently found which may be used to model dense stars with strong gravitational fields. Therefore, in this chapter we provide the interior spacetime metrics for describing spherically symmetric static spacetimes. In section 3.2, we present the Einstein field equations in four dimensions and the components of the Christoffel symbols, the Ricci tensor and Ricci scalar. We also extend this to include the electromagnetic field to generate the Einstein-Maxwell field equations. Furthermore, we consider the N dimensional metric for general relativity and generate the respective field equations in section 3.3. We provide the EGB and charged EGB field equations, sections 3.4 and 3.5 respectively, in dimensions of five and six. This chapter is thus a basis for the research undertaken in the proceeding chapters.

3.2 General relativity

The line element for a spherically symmetric static spacetime in four dimensions with metric signature $(-+++)$ is given by

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.2.1}$$

where $\nu(r)$ and $\lambda(r)$ are arbitrary functions of r and represent gravitational potentials. We use spacetime coordinates $(x^a) = (t, r, \theta, \phi)$. With the use of equation (2.2.3), we can obtain the nonzero Christoffel symbols for the metric element (3.2.1) as

$$\begin{aligned}
\Gamma^0{}_{01} &= \nu' & \Gamma^1{}_{11} &= \lambda' \\
\Gamma^1{}_{33} &= -re^{-2\lambda}\sin^2\theta & \Gamma^2{}_{33} &= -\sin\theta\cos\theta \\
\Gamma^1{}_{00} &= \nu' e^{2(\nu-\lambda)} & \Gamma^2{}_{12} &= \Gamma^3{}_{13} = \tfrac{1}{r} \\
\Gamma^1{}_{22} &= -re^{-2\lambda} & \Gamma^3{}_{23} &= \cot\theta.
\end{aligned}$$

Note that primes denote differentiation with respect to the radial coordinate r.

Substituting these quantities in the definition (2.2.5) we obtain the nonvanishing components of the Ricci tensor as

$$\begin{aligned}
R^0{}_0 &= -\frac{1}{r}\left[e^{-2\lambda}\left(r\nu'' + r(\nu')^2 - r\nu'\lambda' + 2\nu'\right)\right], & (3.2.2a)\\
R^1{}_1 &= -\frac{1}{r}\left[e^{-2\lambda}\left(r\nu'' + r(\nu')^2 - r\nu'\lambda' - 2\lambda'\right)\right], & (3.2.2b)\\
R^2{}_2 &= \frac{1}{r^2}\left[-re^{-2\lambda}\nu' + r\lambda' e^{-2\lambda} + 1 - e^{-2\lambda}\right], & (3.2.2c)\\
R^3{}_3 &= R^2{}_2. & (3.2.2d)
\end{aligned}$$

Using system (3.2.2), we can generate the Ricci scalar using (2.2.6) as

$$R = 2\left[\frac{1}{r^2} - e^{-2\lambda}\left(\nu'' + \nu'^2 - \lambda'\nu' - \frac{2}{r}\lambda' + \frac{2}{r}\nu' + \frac{1}{r^2}\right)\right]. \tag{3.2.3}$$

Then using (3.2.2) and (3.2.3) in (2.2.7), the nonvanishing Einstein tensor components can be generated. These are given by

$$G^0{}_0 = \frac{1}{r^2}e^{-2\lambda} - \frac{2\lambda'}{r}e^{-2\lambda} - \frac{1}{r^2}, \tag{3.2.4a}$$

$$G^1{}_1 = \frac{1}{r^2}e^{-2\lambda} + \frac{2\nu'}{r}e^{-2\lambda} - \frac{1}{r^2}, \tag{3.2.4b}$$

$$G^2{}_2 = e^{-2\lambda}\left[\nu'' + \nu'^2 + \frac{\nu'}{r} - \frac{\lambda'}{r} - \nu'\lambda'\right], \tag{3.2.4c}$$

$$G^3{}_3 = G^2{}_2. \tag{3.2.4d}$$

For the static spherically symmetric spacetime (3.2.1), the fluid four-velocity is defined by $u^a = e^{-\nu}\delta^a{}_0$. The nonzero matter components of the energy momentum tensor for a neutral fluid (2.3.2) are then

$$T^0{}_0 = -\rho, \tag{3.2.5a}$$

$$T^1{}_1 = p_{\parallel}, \tag{3.2.5b}$$

$$T^2{}_2 = p_{\perp}, \tag{3.2.5c}$$

$$T^3{}_3 = T^2{}_2, \tag{3.2.5d}$$

where ρ is the energy density, $p_{\parallel}$ is the radial pressure and $p_{\perp}$ is the tangential pressure.

Equating the nonvanishing components of the Einstein tensor (3.2.4) to the nonvanishing components of the energy momentum tensor (3.2.5), the Einstein field equations are obtained as

$$\rho = \frac{1}{8\pi}\left[-\frac{1}{r^2}e^{-2\lambda} + \frac{2\lambda'}{r}e^{-2\lambda} + \frac{1}{r^2}\right], \tag{3.2.6a}$$

$$p_{\parallel} = \frac{1}{8\pi}\left[\frac{1}{r^2}e^{-2\lambda} + \frac{2\nu'}{r}e^{-2\lambda} - \frac{1}{r^2}\right], \tag{3.2.6b}$$

$$p_{\perp} = \frac{1}{8\pi}e^{-2\lambda}\left[\nu'' + \nu'^2 + \frac{\nu'}{r} - \frac{\lambda'}{r} - \nu'\lambda'\right]. \tag{3.2.6c}$$

In the above we have a set of three equations with five unknowns being ρ, $p_{\parallel}$, $p_{\perp}$, ν and λ. This system is underdetermined and solutions can be obtained by postulating one variable in order to determine the remaining two. Another approach is to specify an equation of state.

In addition we can extend the above equations to include charged matter. Components of the four-potential $\boldsymbol{A}$ are chosen as

$$A_a = (\Phi(r), 0, 0, 0) . \tag{3.2.7}$$

Use of (3.2.7) in equation (2.3.4) yields only one nonzero component of the Faraday tensor and it is given by

$$F_{01} = -\Phi'(r). \tag{3.2.8}$$

The contravariant component of the Faraday tensor then has the form

$$F^{01} = e^{-2(\nu+\lambda)}\Phi'(r) = e^{-(\nu+\lambda)}E(r), \tag{3.2.9}$$

where the quantity $E(r)$ is the electrostatic field intensity defined in terms of the gravitational potentials ν and λ, and the electric potential Φ by

$$E(r) = e^{-(\nu+\lambda)}\Phi'(r). \tag{3.2.10}$$

Substituting the above quantities in definition (2.3.5), we can obtain the nonzero components of the electromagnetic tensor as

$$E^a{}_b = \mathrm{diag}\left(-\frac{1}{8\pi}E^2, -\frac{1}{8\pi}E^2, \frac{1}{8\pi}E^2, \frac{1}{8\pi}E^2\right). \tag{3.2.11}$$

Thus the Einstein-Maxwell field equations, (2.4.4), may now be expressed as

$$8\pi\left(\rho + \frac{E^2}{8\pi}\right) = -\frac{1}{r^2}e^{-2\lambda} + \frac{2\lambda'}{r}e^{-2\lambda} + \frac{1}{r^2}, \tag{3.2.12a}$$

$$8\pi\left(p_{\|} - \frac{E^2}{8\pi}\right) = \frac{1}{r^2}e^{-2\lambda} + \frac{2\nu'}{r}e^{-2\lambda} - \frac{1}{r^2}, \tag{3.2.12b}$$

$$8\pi\left(p_{\perp} + \frac{E^2}{8\pi}\right) = e^{-2\lambda}\left[\nu'' + \nu'^2 + \frac{\nu'}{r} - \frac{\lambda'}{r} - \nu'\lambda'\right], \tag{3.2.12c}$$

$$e^{-\lambda}\left[r^2 E\right]' = 4\pi\sigma r^2. \tag{3.2.12d}$$

Equations (3.2.12) reduce to (3.2.6) for neutral matter.

3.3 N dimensional general relativity

The line element for a spherically symmetric static spacetime in N dimensions is given by

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2 d\Omega_{N-2}^2, \tag{3.3.1}$$

where the $(N-2)$-sphere is denoted by

$$\begin{aligned} d\Omega_{N-2}^2 &= d\theta_1^2 + \sin^2(\theta_1)d\theta_2^2 + \sin^2(\theta_1)\sin^2(\theta_2)d\theta_3^2 \\ &\quad + \cdots + \sin^2(\theta_1)\sin^2(\theta_2)\sin^2(\theta_3)\cdots\sin^2(\theta_{N-3})d\theta_{N-2}^2 \\ &= \sum_{i=1}^{N-2}\left[\prod_{j=1}^{i-1}\sin^2(\theta_j)\right](d\theta_i)^2. \end{aligned} \tag{3.3.2}$$

The nonvanishing Ricci components for the metric in (3.3.1) read as

$$R^0{}_0 = e^{-2\lambda}\left[\nu'\lambda' - (\nu')^2 - \nu'' - \frac{(N-2)\,\nu'}{r}\right], \tag{3.3.3a}$$

$$R^1{}_1 = e^{-2\lambda}\left[\nu'\lambda' - (\nu')^2 - \nu'' + \frac{(N-2)\,\lambda'}{r}\right], \tag{3.3.3b}$$

$$R^2{}_2 = e^{-2\lambda}\left[\frac{\lambda'}{r} - \frac{\nu'}{r} - \frac{(N-3)}{r^2}\right] + \frac{N-3}{r^2}, \tag{3.3.3c}$$

$$R^{N-1}{}_{N-1} = R^{N-2}{}_{N-2} = \cdots = R^2{}_2. \tag{3.3.3d}$$

Using system (3.3.3) the resulting Ricci scalar becomes

$$\begin{aligned} R &= 2\Bigg[\frac{(N-2)(N-3)}{2r^2} - e^{-2\lambda}\Bigg(\nu'' + \nu'^2 - \lambda'\nu' \\ &\quad - \frac{(N-2)\,\lambda'}{r} + \frac{(N-2)\,\nu'}{r} + \frac{(N-3)(N-2)}{2r^2}\Bigg)\Bigg]. \end{aligned} \tag{3.3.4}$$

Therefore the nonzero Einstein tensor components in N dimensions are generated as

$$G^0{}_0 = (N-2)\left[\frac{(N-3)\,e^{-2\lambda}}{2r^2} - \frac{e^{-2\lambda}\lambda'}{r} - \frac{(N-3)}{2r^2}\right], \tag{3.3.5a}$$

$$G^1{}_1 = (N-2)\left[\frac{(N-3)\,e^{-2\lambda}}{2r^2} + \frac{e^{-2\lambda}\nu'}{r} - \frac{(N-3)}{2r^2}\right], \tag{3.3.5b}$$

$$\begin{aligned} G^2{}_2 &= e^{-2\lambda}\Bigg[\nu'' + \nu'^2 + \frac{(N-3)\,\nu'}{r} - \frac{(N-3)\,\lambda'}{r} - \nu'\lambda' \\ &\quad + \frac{(N-4)(N-3)}{2r^2}\Bigg] - \frac{(N-4)(N-3)}{2r^2}, \end{aligned} \tag{3.3.5c}$$

$$G^{N-1}{}_{N-1} = G^{N-2}{}_{N-2} = \cdots = G^2{}_2. \tag{3.3.5d}$$

The nonzero matter components for a neutral fluid are expressed by

$$\mathrm{diag}\left(T^{a}{}_{b}\right) = \left(-\rho, p_{\|}, p_{\perp}, p_{\perp}, ..., p_{\perp}\right), \tag{3.3.6}$$

in the absence of anisotropic pressure and heat. As a result the higher dimensional Einstein field equations read as

$$\rho = \frac{(N-2)}{\kappa_N}\left[\frac{e^{-2\lambda}\lambda'}{r} + \frac{(N-3)}{2r^2} - \frac{(N-3)\,e^{-2\lambda}}{2r^2}\right], \tag{3.3.7a}$$

$$p_{\|} = \frac{(N-2)}{\kappa_N}\left[\frac{(N-3)\,e^{-2\lambda}}{2r^2} + \frac{e^{-2\lambda}\nu'}{r} - \frac{(N-3)}{2r^2}\right], \tag{3.3.7b}$$

$$\begin{aligned} p_{\perp} = {} & \frac{1}{\kappa_N}\left(e^{-2\lambda}\left[\nu'' + \nu'^2 + \frac{(N-3)\,\nu'}{r} - \frac{(N-3)\,\lambda'}{r} - \nu'\lambda'\right.\right. \\ & \left.\left. + \frac{(N-4)\,(N-3)}{2r^2}\right] - \frac{(N-4)\,(N-3)}{2r^2}\right). \end{aligned} \tag{3.3.7c}$$

This applies for uncharged matter. When $N = 4$, equations (3.3.7) reduce to (3.2.6).

We can extend the above to include a charged matter distribution. We find that the Einstein-Maxwell system of equations in N dimensions becomes

$$\begin{aligned} \kappa_N\left(\rho + \frac{E^2}{2\mathcal{A}_{N-2}}\right) = {} & \frac{(N-2)\,e^{-2\lambda}\lambda'}{r} + \frac{(N-2)\,(N-3)}{2r^2} \\ & - \frac{(N-2)\,(N-3)\,e^{-2\lambda}}{2r^2}, \end{aligned} \tag{3.3.8a}$$

$$\begin{aligned} \kappa_N\left(p - \frac{E^2}{2\mathcal{A}_{N-2}}\right) = {} & \frac{(N-2)\,(N-3)\,e^{-2\lambda}}{2r^2} + \frac{(N-2)\,e^{-2\lambda}\nu'}{r} \\ & - \frac{(N-2)\,(N-3)}{2r^2}, \end{aligned} \tag{3.3.8b}$$

$$\begin{aligned} \kappa_N\left(p + \frac{E^2}{2\mathcal{A}_{N-2}}\right) = {} & e^{-2\lambda}\left[\nu'' + \nu'^2 + \frac{(N-3)\,\nu'}{r} - \frac{(N-3)\,\lambda'}{r}\right. \\ & \left. -\nu'\lambda' + \frac{(N-4)\,(N-3)}{2r^2}\right] - \frac{(N-4)\,(N-3)}{2r^2}, \end{aligned} \tag{3.3.8c}$$

$$e^{-\lambda}\left[r^{(N-2)}E\right]' = \mathcal{A}_{N-2}\sigma r^{(N-2)}. \tag{3.3.8d}$$

With $N = 4$, $\kappa_4 = 8\pi$ and $\mathcal{A}_2 = 4\pi$ we find that equations (3.3.8) reduce to (3.2.12).

3.4 EGB gravity in 5D

Five and six spacetime dimensions are the critical dimensions in EGB gravity. A pioneering work in this direction was conducted by Wiltshire (1986). We first consider EGB gravity in five dimensions as this reflects the gravitational behaviour in odd spacetime dimensions. The line element for a spherically symmetric static spacetime in five dimensions with metric signature $(-++++)$ is given by

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2 + \sin^2\theta \sin^2\phi d\psi^2), \qquad (3.4.1)$$

where $\nu(r)$ and $\lambda(r)$ are arbitrary functions of r that represent gravitational potentials. We use spacetime coordinates $(x^a) = (t, r, \theta, \phi, \psi)$. The nonzero Christoffel symbols for the above metric are

$$\begin{array}{ll}
\Gamma^0{}_{01} = \nu' & \Gamma^1{}_{11} = \lambda' \\
\Gamma^1{}_{33} = -re^{-2\lambda}\sin^2\theta & \Gamma^2{}_{33} = -\sin\theta\cos\theta \\
\Gamma^1{}_{00} = \nu' e^{2(\nu-\lambda)} & \Gamma^2{}_{12} = \Gamma^3{}_{13} = \Gamma^4{}_{14} = \frac{1}{r} \\
\Gamma^1{}_{22} = -re^{-2\lambda} & \Gamma^3{}_{23} = \Gamma^4{}_{24} = \cot\theta \\
\Gamma^1{}_{44} = -re^{-2\lambda}\sin^2\theta\sin^2\phi & \Gamma^2{}_{44} = -\sin\theta\cos\theta\sin^2\phi \\
\Gamma^4{}_{34} = \cot\phi & \Gamma^3{}_{44} = -\sin\phi\cos\phi.
\end{array}$$

Using equation (2.2.5), the nonvanishing Ricci tensor components can be obtained as

$$R^0{}_0 = -\frac{1}{r}\left[e^{-2\lambda}(-r\nu'\lambda' + r(\nu')^2 + r\nu'' + 3\nu')\right], \qquad (3.4.2a)$$

$$R^1{}_1 = -\frac{1}{r}\left[e^{-2\lambda}(-r\nu'\lambda' + r(\nu')^2 + r\nu'' - 3\lambda')\right], \qquad (3.4.2b)$$

$$R^2{}_2 = -\frac{1}{r^2}\left[re^{-2\lambda}\nu' - re^{-2\lambda}\lambda' - 2 + 2e^{-2\lambda}\right], \qquad (3.4.2c)$$

$$R^3{}_3 = R^4{}_4 = R^2{}_2, \qquad (3.4.2d)$$

where once again primes represent differentiation with respect to r.

The Ricci scalar reads

$$\begin{aligned} R &= -\frac{1}{r^2}\left[2\left(-r^2e^{-2\lambda}\lambda'\nu' + r^2e^{-2\lambda}\nu'^2 + r^2e^{-2\lambda}\nu'' - 3re^{-2\lambda}\lambda'\right.\right. \\ &\left.\left. +3re^{-2\lambda}\nu' + 3e^{-2\lambda} - 3\right)\right]. \end{aligned} \tag{3.4.3}$$

Then using (3.4.2) and (3.4.3) in (2.2.7), the nonvanishing Einstein tensor components are given by

$$G^0{}_0 = \frac{3}{r^2}e^{-2\lambda} - \frac{3\lambda'}{r}e^{-2\lambda} - \frac{3}{r^2}, \tag{3.4.4a}$$

$$G^1{}_1 = \frac{3}{r^2}e^{-2\lambda} + \frac{3\nu'}{r}e^{-2\lambda} - \frac{3}{r^2}, \tag{3.4.4b}$$

$$\begin{aligned} G^2{}_2 &= e^{-2\lambda}\nu'' + e^{-2\lambda}\nu'^2 + \frac{2\nu'}{r}e^{-2\lambda} - \frac{2\lambda'}{r}e^{-2\lambda} \\ &\quad -e^{-2\lambda}\nu'\lambda' + \frac{e^{-2\lambda}}{r^2} - \frac{1}{r^2}, \end{aligned} \tag{3.4.4c}$$

$$G^3{}_3 = G^4{}_4 = G^2{}_2. \tag{3.4.4d}$$

Equation (2.5.3) yields the nonzero components of the Gauss-Bonnet tensor as

$$H^0{}_0 = \frac{-24e^{-2\nu}\lambda'\left(e^{-4\lambda+2\nu} - e^{-2\lambda+2\nu}\right)}{r^3}, \tag{3.4.5a}$$

$$H^1{}_1 = \frac{24e^{-2\lambda}\nu'\left(e^{-2\lambda} - 1\right)}{r^3}, \tag{3.4.5b}$$

$$\begin{aligned} H^2{}_2 &= \frac{1}{r^2}\left[8e^{-2\lambda}\left(-3e^{-2\lambda}\lambda'\nu' + e^{-2\lambda}\nu'^2\right.\right. \\ &\quad \left.\left. +e^{-2\lambda}\nu'' + \nu'\lambda' - \nu'^2 - \nu''\right)\right], \end{aligned} \tag{3.4.5c}$$

$$H^3_3 = H^4_4 = H^2{}_2, \tag{3.4.5d}$$

for the line element (3.4.1). Then (3.4.4) and (3.4.5) give the three independent components

$$\begin{aligned} G^0{}_0 - \frac{\alpha}{2}H^0{}_0 &= \frac{3}{r^2}e^{-2\lambda} - \frac{3\lambda'}{r}e^{-2\lambda} - \frac{3}{r^2} \\ &\quad -\frac{\alpha}{2}\left[\frac{-24e^{-2\nu}\lambda'\left(e^{-4\lambda+2\nu} - e^{-2\lambda+2\nu}\right)}{r^3}\right], \end{aligned} \tag{3.4.6a}$$

$$\begin{aligned} G^1{}_1 - \frac{\alpha}{2}H^1{}_1 &= \frac{3}{r^2}e^{-2\lambda} + \frac{3\nu'}{r}e^{-2\lambda} - \frac{3}{r^2} \\ &\quad -\frac{\alpha}{2}\left[\frac{24e^{-2\lambda}\nu'\left(e^{-2\lambda} - 1\right)}{r^3}\right], \end{aligned} \tag{3.4.6b}$$

$$\begin{aligned}
G^2{}_2 - \frac{\alpha}{2}H^2{}_2 &= e^{-2\lambda}\nu'' + e^{-2\lambda}\nu'^2 + \frac{2\nu'}{r}e^{-2\lambda} - \frac{2\lambda'}{r}e^{-2\lambda} \\
&\quad - e^{-2\lambda}\nu'\lambda' + \frac{e^{-2\lambda}}{r^2} - \frac{1}{r^2} \\
&\quad - \frac{\alpha}{2}\left[\frac{1}{r^2}\left[8e^{-2\lambda}(-3e^{-2\lambda}\lambda'\nu' + e^{-2\lambda}\nu'^2\right.\right. \\
&\quad \left.\left. + e^{-2\lambda}\nu'' + \nu'\lambda' - \nu'^2 - \nu'')\right]\right], \qquad (3.4.6c)
\end{aligned}$$

which gives the curvature part of the field equations.

The fluid five-velocity is described by $u^a = e^{-\nu}\delta^a{}_0$ for the line element (3.4.1). Therefore the nonzero components of the energy momentum tensor for uncharged matter can be obtained using definition (2.3.2). These read as

$$\mathrm{diag}\,(T^a{}_b) = \left(-\rho,\ p_\parallel,\ p_\perp,\ p_\perp,\ p_\perp\right), \qquad (3.4.7)$$

giving

$$\begin{aligned}
T^0{}_0 &= -\rho, & (3.4.8a) \\
T^1{}_1 &= p_\parallel, & (3.4.8b) \\
T^2{}_2 &= p_\perp, & (3.4.8c)
\end{aligned}$$

for the independent neutral matter components.

We now equate equations (3.4.6) and (3.4.8) to generate the EGB field equations

$$\begin{aligned}
\rho &= \frac{1}{3\pi^2}\left[\frac{3}{e^{4\lambda}r^3}\left(r^2\lambda'e^{2\lambda} + re^{4\lambda} - 4\alpha\lambda' + 4\alpha\lambda'e^{2\lambda} - re^{2\lambda}\right)\right], & (3.4.9a) \\
p_\parallel &= \frac{1}{3\pi^2}\left[\frac{3}{e^{4\lambda}r^3}\left(r^2\nu'e^{2\lambda} - re^{4\lambda} - 4\alpha\nu' + 4\alpha\nu'e^{2\lambda} + re^{2\lambda}\right)\right], & (3.4.9b) \\
p_\perp &= \frac{1}{3\pi^2}\left[\frac{1}{e^{4\lambda}r^2}\left(-e^{4\lambda} - 4\alpha\nu'' - 4\alpha\nu'^2 + 12\alpha\nu'\lambda'\right)\right. \\
&\quad + \frac{1}{e^{2\lambda}r^2}\left(1 - r^2\lambda'\nu' + r^2\nu'^2 + r^2\nu'' - 4\alpha\nu'\lambda'\right. \\
&\quad \left.\left. + 4\alpha\nu'^2 + 4\alpha\nu'' + 2r\nu' - 2r\lambda'\right)\right], & (3.4.9c)
\end{aligned}$$

where we determined the value of the coupling constant κ_N in five dimensions as $\kappa_5 = 3\pi^2$. Equations (3.4.9) are a system of three equations with five unknowns ρ,

$p_\parallel$, $p_\perp$, ν and λ. These equations are highly nonlinear, and the addition of the Gauss-Bonnet terms increases the complexity of the system. To find a solution to the system we can impose an equation of state on physical grounds.

We regain the Einstein version for gravitating spherically symmetric perfect fluids in a five-dimensional static spacetime when $\alpha = 0$. The field equations in this limit become

$$\rho = \frac{1}{3\pi^2}\left[\frac{3}{e^{4\lambda}r^3}\left(\lambda' e^{2\lambda}r^2 + e^{4\lambda}r - re^{2\lambda}\right)\right], \tag{3.4.10a}$$

$$p_\parallel = \frac{1}{3\pi^2}\left[\frac{3}{e^{4\lambda}r^3}\left(\nu' e^{2\lambda}r^2 - e^{4\lambda}r + re^{2\lambda}\right)\right], \tag{3.4.10b}$$

$$p_\perp = \frac{1}{3\pi^2}\left[\frac{1}{e^{2\lambda}r^2}\left(1 - r^2\lambda'\nu' + \nu'^2 r^2 + \nu'' r^2 + 2r\nu' - 2r\lambda'\right) - \frac{1}{r^2}\right], \tag{3.4.10c}$$

which has been widely studied in the past.

We now extend the above results to the charged case. Components of the five-potential $\mathbf{A}$ are chosen as

$$A_a = (\Phi(r), 0, 0, 0, 0)\,. \tag{3.4.11}$$

Use of equation (2.3.4) yields only one nonzero component of the Faraday tensor in this five dimensional regime, and it is given by

$$F_{01} = -\Phi'(r). \tag{3.4.12}$$

The contravariant component becomes

$$F^{01} = e^{-2(\nu+\lambda)}\Phi'(r) = e^{-(\nu+\lambda)}E(r), \tag{3.4.13}$$

where the electrostatic field intensity is defined by

$$E(r) = e^{-(\nu+\lambda)}\Phi'(r). \tag{3.4.14}$$

The surface area in five dimensions evaluates to $\mathcal{A}_3 = 2\pi^2$, and using (3.4.12) and (3.4.13) in the definition (2.3.5), the nonzero components of the electromagnetic field tensor can be generated. These are given by

$$\mathrm{diag}(E^a{}_b) = \left(-\frac{1}{4\pi^2}E^2, -\frac{1}{4\pi^2}E^2, \frac{1}{4\pi^2}E^2, \frac{1}{4\pi^2}E^2, \frac{1}{4\pi^2}E^2\right). \tag{3.4.15}$$

System (2.5.5) yields the field equations as

$$\begin{aligned}
3\pi^2\left(\rho + \frac{1}{4\pi^2}E^2\right) &= \frac{3}{e^{4\lambda}r^3}\left(r^2\lambda' e^{2\lambda} + re^{4\lambda} - 4\alpha\lambda'\right. \\
&\quad \left. + 4\alpha\lambda' e^{2\lambda} - re^{2\lambda}\right), &(3.4.16a)\\
3\pi^2\left(p_\parallel - \frac{1}{4\pi^2}E^2\right) &= \frac{3}{e^{4\lambda}r^3}\left(r^2\nu' e^{2\lambda} - re^{4\lambda} - 4\alpha\nu'\right. \\
&\quad \left. + 4\alpha\nu' e^{2\lambda} + re^{2\lambda}\right), &(3.4.16b)\\
3\pi^2\left(p_\perp + \frac{1}{4\pi^2}E^2\right) &= \frac{1}{e^{4\lambda}r^2}\left(-e^{4\lambda} - 4\alpha\nu'' - 4\alpha\nu'^2 + 12\alpha\nu'\lambda'\right) \\
&\quad + \frac{1}{e^{2\lambda}r^2}\left(1 - r^2\lambda'\nu' + r^2\nu'^2 + r^2\nu'' + 4\alpha\nu'^2\right. \\
&\quad \left. - 4\alpha\nu'\lambda' + 4\alpha\nu'' + 2r\nu' - 2r\lambda'\right), &(3.4.16c)\\
e^{-\lambda}\left[r^3E\right]' &= 2\pi^2 r^3\sigma, &(3.4.16d)
\end{aligned}$$

which is the charged version of (3.4.9).

3.5 EGB gravity in 6D

Six spacetime dimensions is the second critical dimension in EGB gravity. We now consider EGB gravity in six dimensions as it reflects gravitational behaviour in even dimensions. The line element for a spherically symmetric static spacetime in six dimensions with metric signature $(-+++++)$ is described by

$$\begin{aligned}
ds^2 &= -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2 + \sin^2\theta\sin^2\phi d\psi^2\right. \\
&\quad \left. + \sin^2\theta\sin^2\phi\sin^2\psi d\varphi^2\right), &(3.5.1)
\end{aligned}$$

where $\nu(r)$ and $\lambda(r)$ are arbitrary functions of r that represent gravitational potentials. We use spacetime coordinates $(x^a) = (t, r, \theta, \phi, \psi, \varphi)$. The nonzero Christoffel symbols for the above metric are

$\Gamma^0{}_{01} = \nu'$ $\quad\quad$ $\Gamma^1{}_{11} = \lambda'$

$\Gamma^1{}_{33} = -re^{-2\lambda}\sin^2\theta$ $\quad\quad$ $\Gamma^2{}_{33} = -\sin\theta\cos\theta$

$\Gamma^1{}_{00} = \nu' e^{2(\nu-\lambda)}$ $\quad\quad$ $\Gamma^2{}_{12} = \Gamma^3{}_{13} = \Gamma^4{}_{14} = \Gamma^5{}_{15} = \frac{1}{r}$

$\Gamma^1{}_{22} = -re^{-2\lambda}$ $\quad\quad$ $\Gamma^3{}_{23} = \Gamma^4{}_{24} = \Gamma^5{}_{25} = \cot\theta$

$\Gamma^1{}_{44} = -re^{-2\lambda}\sin^2\theta\sin^2\phi$ $\quad\quad$ $\Gamma^2{}_{44} = -\sin\theta\cos\theta\sin^2\phi$

$\Gamma^4{}_{34} = \Gamma^5{}_{35} = \cot\phi$ $\quad\quad$ $\Gamma^3{}_{44} = -\sin\phi\cos\phi$

$\Gamma^1{}_{55} = -re^{-2\lambda}\sin^2\theta\sin^2\phi\sin^2\psi$ $\quad\quad$ $\Gamma^2{}_{55} = -\sin\theta\cos\theta\sin^2\phi\sin^2\psi$

$\Gamma^3{}_{55} = -\sin\phi\cos\phi\sin^2\psi$ $\quad\quad$ $\Gamma^4{}_{55} = -\sin\psi\cos\psi$

$\Gamma^5{}_{45} = \cot\phi.$

Using (2.2.5) we obtain the nonvanishing Ricci tensor components as

$$R^0{}_0 = -\frac{1}{r}\left[e^{-2\lambda}\big(-r\nu'\lambda' + r(\nu')^2 + r\nu'' + 4\nu'\big)\right], \quad (3.5.2a)$$

$$R^1{}_1 = -\frac{1}{r}\left[e^{-2\lambda}\big(-r\nu'\lambda' + r(\nu')^2 + r\nu'' - 4\lambda'\big)\right], \quad (3.5.2b)$$

$$R^2{}_2 = -\frac{1}{r^2}\left[e^{-2\lambda}r\nu' - r\lambda' e^{-2\lambda} - 3 + 3e^{-2\lambda}\right], \quad (3.5.2c)$$

$$R^3{}_3 = R^4{}_4 = R^5{}_5 = R^2{}_2. \quad (3.5.2d)$$

The Ricci scalar reads as

$$R = -\frac{1}{r^2}\left[2\big(-r^2e^{-2\lambda}\lambda'\nu' + r^2e^{-2\lambda}\nu'^2 + r^2e^{-2\lambda}\nu'' - 4re^{-2\lambda}\lambda' + 4re^{-2\lambda}\nu' + 6e^{-2\lambda} - 6\big)\right]. \quad (3.5.3)$$

Then using (3.5.2) and (3.5.3) in (2.2.7), the nonvanishing Einstein tensor components are given by

$$G^0{}_0 = \frac{6}{r^2}e^{-2\lambda} - \frac{4\lambda'}{r}e^{-2\lambda} - \frac{6}{r^2}, \quad (3.5.4a)$$

$$G^1{}_1 = \frac{6}{r^2}e^{-2\lambda} + \frac{4\nu'}{r}e^{-2\lambda} - \frac{6}{r^2}, \quad (3.5.4b)$$

$$G^2{}_2 = \nu''e^{-2\lambda} + \nu'^2e^{-2\lambda} + \frac{3\nu'}{r}e^{-2\lambda} - \frac{3\lambda'}{r}e^{-2\lambda} - \nu'\lambda'e^{-2\lambda} + \frac{3e^{-2\lambda}}{r^2} - \frac{3}{r^2}, \quad (3.5.4c)$$

$$G^3{}_3 = G^4{}_4 = G^5{}_5 = G^2{}_2. \quad (3.5.4d)$$

With the use of equation (2.5.3), we can obtain the nonzero components of the Gauss-Bonnet tensor as

$$\begin{aligned}
H^0{}_0 &= \frac{24}{r^3}\left[\frac{1}{r}\left(e^{-4\lambda}-2e^{-2\lambda}+1\right)+4\lambda' e^{-2\lambda}\left(1-e^{-2\lambda}\right)\right], && (3.5.5a)\\
H^1{}_1 &= \frac{24}{r^3}\left[\frac{1}{r}\left(e^{-4\lambda}-2e^{-2\lambda}+1\right)+4\nu' e^{-2\lambda}\left(e^{-2\lambda}-1\right)\right], && (3.5.5b)\\
H^2{}_2 &= \frac{24e^{-2\lambda}}{r^2}\Bigg[\left(\nu''+(\nu')^2\right)\left(e^{-2\lambda}-1\right)+\nu'\lambda'\left(1-3e^{-2\lambda}\right)\\
&\quad +\frac{\left(1-e^{-2\lambda}\right)\left(\nu'-\lambda'\right)}{r}\Bigg], && (3.5.5c)\\
H^3{}_3 &= H^4{}_4 = H^5{}_5 = H^2{}_2. && (3.5.5d)
\end{aligned}$$

Then system (3.5.4) and (3.5.5) provide the three independent components as

$$\begin{aligned}
G^0{}_0-\frac{\alpha}{2}H^0{}_0 &= \frac{6}{r^2}e^{-2\lambda}-\frac{4\lambda'}{r}e^{-2\lambda}-\frac{6}{r^2}\\
&\quad -\frac{12\alpha}{r^3}\left[\frac{1}{r}\left(e^{-4\lambda}-2e^{-2\lambda}+1\right)+4\lambda' e^{-2\lambda}\left(1-e^{-2\lambda}\right)\right], && (3.5.6a)\\
G^1{}_1-\frac{\alpha}{2}H^1{}_1 &= \frac{6}{r^2}e^{-2\lambda}+\frac{4\nu'}{r}e^{-2\lambda}-\frac{6}{r^2}\\
&\quad -\frac{12\alpha}{r^3}\left[\frac{1}{r}\left(e^{-4\lambda}-2e^{-2\lambda}+1\right)+4\nu' e^{-2\lambda}\left(e^{-2\lambda}-1\right)\right], && (3.5.6b)\\
G^2{}_2-\frac{\alpha}{2}H^2{}_2 &= \nu'' e^{-2\lambda}+\nu'^2 e^{-2\lambda}+\frac{3\nu'}{r}e^{-2\lambda}-\frac{3\lambda'}{r}e^{-2\lambda}-\nu'\lambda' e^{-2\lambda}\\
&\quad +\frac{3e^{-2\lambda}}{r^2}-\frac{3}{r^2}-\frac{12\alpha e^{-2\lambda}}{r^2}\Bigg[\left(\nu''+(\nu')^2\right)\left(e^{-2\lambda}-1\right)\\
&\quad +\nu'\lambda'\left(1-3e^{-2\lambda}\right)+\frac{\left(1-e^{-2\lambda}\right)\left(\nu'-\lambda'\right)}{r}\Bigg]. && (3.5.6c)
\end{aligned}$$

The fluid six-velocity is described by $u^a = e^{-\nu}\delta^a{}_0$ for the line element (3.5.1). Therefore the nonzero components of the energy momentum tensor for uncharged matter can be obtained using definition (2.3.1). In the absence of heat and anisotropic stress, these read as

$$\mathrm{diag}\left(T^a{}_b\right) = \left(-\rho,\ p_{\|},\ p_{\perp},\ p_{\perp},\ p_{\perp},\ p_{\perp}\right), \qquad (3.5.7)$$

giving

$$T^0{}_0 = -\rho, \qquad (3.5.8a)$$

$$T^1{}_1 = p_{\parallel}, \quad (3.5.8b)$$

$$T^2{}_2 = p_{\perp}, \quad (3.5.8c)$$

for the independent neutral matter components.

The coupling constant in six dimensions is given by $\kappa_6 = \frac{32\pi^2}{9}$, and equating systems (3.5.6) and (3.5.8) provides the EGB field equations as

$$\begin{aligned}
\rho\left(\frac{32\pi^2}{9}\right) &= \frac{1}{e^{4\lambda}r^4}\left[\left(4r^3e^{2\lambda} - 48\alpha r\left(1-e^{2\lambda}\right)\right)\lambda'\right.\\
&\quad \left. -6r^2e^{2\lambda}\left(1-e^{2\lambda}\right) + 12\alpha\left(e^{2\lambda}-1\right)^2\right], && (3.5.9a)\\
p_{\parallel}\left(\frac{32\pi^2}{9}\right) &= \frac{1}{e^{4\lambda}r^4}\left[\left(1-e^{2\lambda}\right)\left(6r^2e^{2\lambda} - 48\alpha r\nu' + 12\alpha e^{2\lambda} - 12\alpha\right)\right.\\
&\quad \left. +4r^3e^{2\lambda}\nu'\right], && (3.5.9b)\\
p_{\perp}\left(\frac{32\pi^2}{9}\right) &= \frac{1}{e^{4\lambda}r^2}\left[\left(12\alpha\left(e^{2\lambda}-1\right) + r^2e^{2\lambda}\right)\left(\nu'' + \nu'^2 - \nu'\lambda'\right)\right.\\
&\quad \left. +24\alpha\nu'\lambda'\right] + \frac{1}{e^{4\lambda}r^3}\left[\left(3r^2e^{2\lambda} + 12\alpha\left(e^{2\lambda}-1\right)\right)\right.\\
&\quad \left. \times\left(\nu' - \lambda'\right) + 3re^{2\lambda}\left(1-e^{2\lambda}\right)\right], && (3.5.9c)
\end{aligned}$$

for neutral matter.

We now include the effects of the electromagnetic field. Components of the six-potential **A** are chosen as

$$A_a = \left(\Phi(r), 0, 0, 0, 0, 0\right). \quad (3.5.10)$$

Use of equation (2.3.4) yields only one nonzero component of the Faraday tensor in this six dimensional regime, and it is given by

$$F_{01} = -\Phi'(r). \quad (3.5.11)$$

The contravariant component reads as

$$F^{01} = e^{-2(\nu+\lambda)}\Phi'(r) = e^{-(\nu+\lambda)}E(r), \quad (3.5.12)$$

where the electrostatic field intensity is defined by

$$E(r) = e^{-(\nu+\lambda)}\Phi'(r). \quad (3.5.13)$$

The surface area in six dimensions evaluates to $\mathcal{A}_4 = \frac{8}{3}\pi^2$, and thus the nonzero components of the electromagnetic tensor are given by

$$\text{diag}(E^a{}_b) = \left(-\frac{3}{16\pi^2}E^2, -\frac{3}{16\pi^2}E^2, \frac{3}{16\pi^2}E^2, \frac{3}{16\pi^2}E^2, \frac{3}{16\pi^2}E^2\right). \tag{3.5.14}$$

Then system (2.5.5) yields the charged EGB field equations as

$$\begin{aligned}
\left(\rho + \frac{3}{16\pi^2}E^2\right)\left(\frac{32\pi^2}{9}\right) &= \frac{1}{e^{4\lambda}r^4}\Big[\left(4r^3e^{2\lambda} - 48\alpha r\left(1-e^{2\lambda}\right)\right)\lambda' \\
&\quad -6r^2e^{2\lambda}\left(1-e^{2\lambda}\right) + 12\alpha\left(e^{2\lambda}-1\right)^2\Big], &(3.5.15a)\\
\left(p_{\parallel} - \frac{3}{16\pi^2}E^2\right)\left(\frac{32\pi^2}{9}\right) &= \frac{1}{e^{4\lambda}r^4}\Big[\left(1-e^{2\lambda}\right)\left(6r^2e^{2\lambda} - 48\alpha r\nu'\right. \\
&\quad \left.+12\alpha e^{2\lambda} - 12\alpha\right) + 4r^3e^{2\lambda}\nu'\Big], &(3.5.15b)\\
\left(p_{\perp} + \frac{3}{16\pi^2}E^2\right)\left(\frac{32\pi^2}{9}\right) &= \frac{1}{e^{4\lambda}r^2}\Big[\left(12\alpha\left(e^{2\lambda}-1\right) + r^2e^{2\lambda}\right) \\
&\quad \times\left(\nu'' + \nu'^2 - \nu'\lambda'\right) + 24\alpha\nu'\lambda'\Big] \\
&\quad +\frac{1}{e^{4\lambda}r^3}\Big[\left(3r^2e^{2\lambda} + 12\alpha\left(e^{2\lambda}-1\right)\right) \\
&\quad \times\left(\nu' - \lambda'\right) + 3re^{2\lambda}\left(1-e^{2\lambda}\right)\Big], &(3.5.15c)\\
e^{-\lambda}\left[r^4E\right]' &= \frac{8}{3}\pi^2r^4\sigma. &(3.5.15d)
\end{aligned}$$

for the line element (3.5.1). When $E = 0$ we regain the uncharged EGB field equations (3.5.9).

Chapter 4

Spherical EGB models

4.1 Introduction

Several exact solutions in general relativity are known for static interiors of spherical stars. However, little is known about the interiors of compact objects in EGB gravity. Recent investigations (Maharaj *et al* 2015, Chilambwe *et al* 2015, Hansraj *et al* 2015) have reported new solutions to the EGB field equations in five dimensions for a static spherically symmetric interior of a perfect fluid. These models have a simple form and are expressable in terms of elementary functions. Some other exact solutions have been found in the presence of charge in five dimensional EGB gravity and for the case of neutral fluids in six dimensional EGB. Two recent examples are given by Hansraj (2017) and Hansraj and Mkhize (2020).

4.2 Isotropic pressure condition equation

The spacetime of spherically symmetric static models in five dimensions is represented by

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2 + \sin^2\theta \sin^2\phi d\psi^2). \qquad (4.2.1)$$

as noted before in chapter 3.

We now equate equations (3.4.9b) and (3.4.9c) with $p_{\|} = p_{\perp} = p$ to generate the condition of pressure isotropy. This gives the result

$$\begin{aligned}
&\frac{1}{e^{4\lambda}r^3}\left[12\alpha\nu' - 4\alpha r\nu'' - 4\alpha r\nu'^2 + 12\alpha r\nu'\lambda'\right] \\
&+\frac{1}{e^{2\lambda}r^2}\left[r^2\nu'' + 4\alpha\nu'' + 4\alpha\nu'^2 + r^2\nu'^2 - r^2\lambda'\nu' + 2e^{2\lambda}\right. \\
&\left.-2 - 4\alpha\nu'\lambda' - r\nu' - 2r\lambda'\right] = 0, \qquad (4.2.2)
\end{aligned}$$

in canonical coordinates.

In order to simplify system (3.4.9), we apply the transformation

$$e^{2\nu(r)} = y^2(x), \quad e^{-2\lambda(r)} = Z(x), \quad x = r^2, \qquad (4.2.3)$$

first introduced by Durgapal and Bannerji (1983) in general relativity. The EGB field equations with isotropic pressure can then be written as

$$\rho = \frac{1}{3\pi^2}\left[\frac{3}{x}(1-Z)\left(1-4\alpha\dot{Z}\right) - 3\dot{Z}\right], \qquad (4.2.4a)$$

$$p = \frac{1}{3\pi^2}\left[\frac{3}{x}(Z-1)\left(1-\frac{8\alpha\dot{y}Z}{y}\right) + 6Z\frac{\dot{y}}{y}\right], \qquad (4.2.4b)$$

$$\begin{aligned}
p = \frac{1}{3\pi^2}&\left[2\dot{Z} + \frac{Z-1}{x} - \frac{4\alpha\dot{y}Z(Z-1)}{xy}\right. \\
&-\frac{4\alpha}{y}\left(4\ddot{y}Z^2 + 6\dot{y}\dot{Z}Z - 4\ddot{y}Z - 2\dot{y}\dot{Z}\right) \\
&\left.+\frac{1}{y}\left(4x\ddot{y}Z + 2x\dot{Z}\dot{y} + 6Z\dot{y}\right)\right], \qquad (4.2.4c)
\end{aligned}$$

where dots denote differentiation with respect to x. Equating (4.2.4b) and (4.2.4c) yields

$$\begin{aligned}
&\ddot{y}\left[2xZ\left(4\alpha(1-Z)+x\right)\right] + \dot{y}\left[x^2\dot{Z} - 2\alpha\left(4Z - 4Z^2 - 2x\dot{Z} + 6x\dot{Z}Z\right)\right] \\
&+y\left[1 - Z + x\dot{Z}\right] = 0, \qquad (4.2.5)
\end{aligned}$$

which is equivalent to the consistency condition of isotropic pressure (4.2.2), now in terms of new variables y, Z and x. Equation (4.2.5) is the master equation for the system. It is a second order linear differential equation in y if Z is specified. We can rearrange equation (4.2.5) to obtain a first order nonlinear differential equation in Z, which is an equivalent form of the master equation. We get

$$\begin{aligned} & \left(-12\alpha x\dot{y}Z + 4\alpha x\dot{y} + x^2\dot{y} + xy\right)\dot{Z} - 8\alpha\left(x\ddot{y} - \dot{y}\right)Z^2 \\ & + \left(2x^2\ddot{y} + 8\alpha x\ddot{y} - 8\alpha\dot{y} - y\right)Z + y = 0. \end{aligned} \tag{4.2.6}$$

Here we observe that the above equation is an Abelian differential equation in Z. To study the dynamics of the model we need to find exact solutions of (4.2.5) or its transformed versions.

In the remainder of this chapter we consider some known exact solutions of the EGB field equations with neutral matter.

4.3 Exact models

We present known solutions to the master equations (4.2.5) and (4.2.6). These have been studied by Chilambwe *et al* (2015), Hansraj *et al* (2015) and Maharaj *et al* (2015). We follow their notation.

4.3.1 $y^2 = A^2$

We set y to be constant

$$y^2 = A^2, \tag{4.3.1}$$

so then equation (4.2.6) becomes

$$x\dot{Z} - Z + 1 = 0. \tag{4.3.2}$$

Equation (4.3.2) is identified as a first order linear ordinary differential equation in Z which has the solution

$$Z = C_1 x + 1. \tag{4.3.3}$$

This solution is characterized by the equation of state

$$\rho = 4\pi^2\alpha p^2 - 2p, \tag{4.3.4}$$

relating the energy density ρ and isotropic pressure p.

4.3.2 $y = a + kx$

We take a linear form of y given by

$$y = a + kx. \tag{4.3.5}$$

Equation (4.2.6) becomes

$$\left[2kx^2 + ax - 4\alpha kx\,(3Z-1)\right]\dot{Z} + 8\alpha kZ^2 - [8\alpha k + a + kx]\,Z$$
$$+a + kx = 0. \tag{4.3.6}$$

Here we observe that equation (4.3.6) is a nonlinear Abelian differential equation of the second kind in Z. Despite its complexity an exact solution can be found and, it is given by

$$Z = \frac{1}{3\beta k}[(1+2Q)k\beta \pm (a+2kx)(1-Q)], \tag{4.3.7}$$

where

$$\beta = 4\alpha, \tag{4.3.8}$$

$$Q = \frac{(80C_1^2x^2 - F)^{\frac{1}{2}}\left[(80C_1^2x^2 - F)^{\frac{1}{2}} - 4\sqrt{5}C_1x\right]^{\frac{1}{3}}}{F^{\frac{1}{3}}\left[\left((80C_1^2x^2 - F)^{\frac{1}{2}} - 4\sqrt{5}C_1x\right)^{\frac{2}{3}} - F^{\frac{1}{3}}\right]}, \tag{4.3.9}$$

and

$$F = a^3 - 6a^2k\beta + 12ak^2\beta^2 - 8k^3\beta^3 + 6k(a^2 + 4k^2\beta^2 - 4ak\beta)x$$
$$+12k^2(a - 2k\beta)x^2 + 8k^3x^3. \tag{4.3.10}$$

Thus we see that the metric function Z has a complicated form but can be expressed in terms of elementary functions.

4.3.3 $x\ddot{y} - \dot{y} = 0$

Setting the coefficient of Z^2 to zero in equation (4.2.6) gives

$$x\ddot{y} - \dot{y} = 0. \tag{4.3.11}$$

This equation is a second order linear differential equation in y which can be integrated to obtain

$$y = \frac{1}{2}C_1x^2 + C_2, \tag{4.3.12}$$

where C_1 and C_2 are constants of integration. We now insert (4.3.12) into (4.2.6) to get

$$\begin{aligned} &\left(\frac{3}{2}C_1x^3 + 4\alpha C_1x^2 + C_2x - 12\alpha C_1x^2Z\right)\dot{Z} \\ &+\left(\frac{3}{2}C_1x^2 - C_2\right)Z + \frac{1}{2}C_1x^2 + C_2 = 0. \end{aligned} \tag{4.3.13}$$

Here we note that this equation is a simpler form of (4.2.6). However the use of (4.3.12) does not remove its nonlinearity. Equation (4.3.13) is a first order nonlinear ordinary differential equation which can be written in the form

$$\begin{aligned} &\left(3\epsilon x^3 + 8\beta\epsilon x^2 + 2x - 24\beta\epsilon x^2Z\right)\dot{Z} \\ &+\left(3\epsilon x^2 - 2\right)Z + \epsilon x^2 + 2 = 0, \end{aligned} \tag{4.3.14}$$

where

$$\beta = \alpha, \tag{4.3.15}$$

$$\epsilon = \frac{C_1}{C_2}. \tag{4.3.16}$$

A solution to equation (4.3.14) can be found by testing the exactness of the differential equation. Thus, we let

$$\tilde{M}(x,Z) = \left(3\epsilon x^2 - 2\right)Z + \epsilon x^2 + 2, \tag{4.3.17}$$

$$\tilde{H}(x,Z) = 3\epsilon x^3 + 8\beta\epsilon x^2 + 2x - 24\beta\epsilon x^2Z. \tag{4.3.18}$$

It can be seen that $\frac{\partial \tilde{M}(x,Z)}{\partial Z} \neq \frac{\partial \tilde{H}(x,Z)}{\partial x}$, therefore (4.3.13) is an inexact differential equation. However, an integrating factor, $\tilde{K}(x)$ can be found such that

$$\begin{aligned} &\tilde{K}\left(3\epsilon x^3 + 8\beta\epsilon x^2 + 2x - 24\beta\epsilon x^2 Z\right)\dot{Z} \\ &+\tilde{K}\left[\left(3\epsilon x^2 - 2\right)Z + \epsilon x^2 + 2\right] = 0, \end{aligned} \tag{4.3.19}$$

becomes an exact differential equation. This yields the condition

$$\begin{aligned} \frac{\partial(\tilde{K}(x)\tilde{M}(x,Z))}{\partial Z} &= \frac{\partial(\tilde{K}(x)\tilde{H}(x,Z))}{\partial x}, \\ \tilde{K}(x)\left(3C_1x^2 - 2C_2\right) &= \frac{d\tilde{K}}{dx}\left(3\epsilon x^3 + 8\beta\epsilon x^2 + 2x - 24\beta\epsilon x^2 Z\right) \\ &\quad +\tilde{K}(x)\left(9\epsilon x^2 + 16\beta\epsilon x + 2 - 48\beta\epsilon x Z\right). \end{aligned} \tag{4.3.20}$$

As a result, the integrating factor takes on the form

$$\tilde{K}(x) = \frac{1}{x^2}. \tag{4.3.21}$$

Using the above equation in (4.3.19) yields

$$\epsilon + 3\epsilon Z - \frac{2\,(Z-1)}{x^2} + \left(\frac{2}{x} + 3\epsilon x + 8\beta\epsilon\,(-3Z+1)\right)\dot{Z} = 0. \tag{4.3.22}$$

Thus we have transformed the inexact differential equation (4.3.14) to an exact ordinary differential equation. The solution to equation (4.3.22) is then expressed as

$$\epsilon x\,(3Z+1) + \frac{2\,(Z-1)}{x} + 8\beta\epsilon Z\left(1 - \frac{3Z}{2}\right) = C_3. \tag{4.3.23}$$

Therefore, the solution to equation (4.3.13) is given by

$$Z = \frac{3\epsilon x^2 + 8\beta\epsilon x + 2 \pm \Upsilon}{24\beta\epsilon x}, \tag{4.3.24}$$

where

$$\begin{aligned} \Upsilon &= \left[4(1 - 16\beta\epsilon x) + 4\epsilon(16\beta^2\epsilon + 3 - 12\beta C_3)x^2 \right. \\ &\quad \left. +3\epsilon^2(3x^4 + 32\beta x^3)\right]^{\frac{1}{2}}, \end{aligned} \tag{4.3.25}$$

for which C_3 is a constant. This class of solution is expressible in terms of elementary functions.

4.3.4 $Z = a$

If we take the form

$$Z = a, \tag{4.3.26}$$

where a is constant, then equation (4.2.5) reduces to

$$\left[8\alpha x - 8\alpha a x + 2x^2\right]\ddot{y} + \left[8\alpha\left(a-1\right)\right]\dot{y} + \left(\frac{1-a}{a}\right)y \;=\; 0. \tag{4.3.27}$$

This equation can be transformed to

$$z(z-1)\frac{d^2y}{dz^2} - \frac{dy}{dz} + By \;=\; 0, \tag{4.3.28}$$

where we have set

$$z = \frac{x+A}{A},\; A = 4\alpha(1-a),\; B = \frac{1-a}{2a}. \tag{4.3.29}$$

The gravitational behaviour for this category of models is governed by the well known hypergeometric differential equation. The solutions of (4.3.28) are given in terms of hypergeometric functions in general. Maharaj *et al* (2015) showed that

$$\begin{aligned} y \;=\;& C_1\left[\sum_{n=1}^{\infty}\frac{2a_0(-1)^n}{A^n}\frac{1}{n!(n+2)!}\prod_{j=1}^{n}(j(j+1)+B)x^{n+2}\right] \\ &+C_2\left[\mu\left(\sum_{n=1}^{\infty}\frac{2a_0(-1)^n}{A^n}\frac{1}{n!(n+2)!}\prod_{j=1}^{n}(j(j+1)+B)x^{n+2}\right)\ln x \right.\\ &\left.+\sum_{n=0}^{\infty}b_nx^n\right], \qquad n\geq 1, \end{aligned} \tag{4.3.30}$$

which is a series solution. In the above the symbol $\prod$ denotes multiplication and a_0, A, μ and B are all constants. For $B = -m$, an integer, we obtain the particular case

$$\begin{aligned} y \;=\;& C_1\left[\sum_{n=1}^{m}\frac{2a_0}{A^n}\frac{1}{n!(n+2)!}\prod_{j=1}^{n}(m-j(j+1))x^{n+2}\right] \\ &+C_2\left[\mu\left(\sum_{n=1}^{m}\frac{2a_0}{A^n}\frac{1}{n!(n+2)!}\prod_{j=1}^{n}(m-j(j+1))x^{n+2}\right)\ln x \right.\\ &\left.+\sum_{n=0}^{\infty}b_nx^n\right]. \end{aligned} \tag{4.3.31}$$

Here the hypergeometric series in (4.3.31) terminates when $C_2 = 0$ and we obtain a solution in terms of simple polynomial functions.

Chapter 5

New models in EGB gravity: neutral matter

5.1 Introduction

Particular solutions in EGB gravity have been reported in the literature. For example, Bhawal (1990) studied the higher dimensional analogue of the Boulware and Deser spacetime metric in geodesic motion. These results were then compared to the geometry of the higher dimensional Schwarszchild spacetime. Efforts have been made by Ghosh *et al* (2014) to study, in dimensions of five, the gravitational contraction of a spherical cloud that is made up of inhomogeneous dust. Cai (2002) discussed the thermodynamic properties and phase structures of black hole solutions in Einstein gravity with a Gauss-Bonnet term. Moreover, Dadhich *et al* (2010) established that the gravitational field inside a uniform density fluid sphere is independent of the spacetime dimensions not only in Einstein gravity but also in EGB gravity, enabling the universality of the Schwarszchild solution in EGB theory. These endeavours show that EGB gravity has been studied and applied to many aspects of relativistic astrophysics. However little is known about the interiors of static spherically symmetric stellar objects in this regime of gravity. Therefore, in this chapter we aim to seek out new static interior solutions for spherically symmetric compact bodies. In section 5.2, we make use of a substitution to analyse the pressure isotropy condition. This is a new approach and is not contained in earlier works. In sections 5.3, 5.4, 5.5 and 5.6 we present new classes of models for the gravitational potentials.

5.2 Abel equations

Particular solutions to (4.2.6) have been derived in the past using arbitrary approaches. We show that this equation may be studied systematically. In spite of its complexity it may be reduced to a canonical differential equation. We first note that the master equation (4.2.6) is classified as an Abel differential equation of the second kind. It can be written in the form

$$\begin{aligned}\left(Z-\frac{1}{3}-\frac{x}{12\alpha}-\frac{y}{12\alpha\dot{y}}\right)\dot{Z}=\frac{2}{3}\left(\frac{1}{x}-\frac{\ddot{y}}{\dot{y}}\right)Z^2\\ +\left(\frac{2\ddot{y}}{3\dot{y}}+\frac{\ddot{y}x}{6\alpha\dot{y}}-\frac{2}{3x}-\frac{y}{12\alpha\dot{y}x}\right)Z+\frac{y}{12\alpha x\dot{y}}.\end{aligned} \tag{5.2.1}$$

This equation is a first order nonlinear ordinary differential equation in Z. It can be simplified by applying a transformation as suggested in Zaitsev and Polyanin (1994). We introduce the new variable

$$w = \left(Z-\frac{1}{3}-\frac{x}{12\alpha}-\frac{y}{12\alpha\dot{y}}\right)\left(\frac{\dot{y}}{x}\right)^{\frac{2}{3}}, \tag{5.2.2}$$

where $\dot{y}\neq 0$ and $\alpha\neq 0$. Substituting (5.2.2) into (5.2.1) we have

$$\begin{aligned}w\dot{w} &= w\left(\frac{\dot{y}}{x}\right)^{\frac{2}{3}}\left[-\frac{1}{18\alpha}+\frac{y}{36\alpha x\dot{y}}+\frac{x\ddot{y}}{18\alpha\dot{y}}+\frac{2\ddot{y}}{9\dot{y}}-\frac{y\ddot{y}}{36\alpha\dot{y}^2}-\frac{2}{9x}\right]\\ &\quad+\left(\frac{\dot{y}}{x}\right)^{\frac{4}{3}}\left[\frac{y}{432\alpha^2\dot{y}}-\frac{y^2\ddot{y}}{216\alpha^2\dot{y}^3}+\frac{\ddot{y}}{\dot{y}}\left(\frac{4}{27}+\frac{2x}{27\alpha}+\frac{x^2}{108\alpha^2}\right)-\frac{1}{54\alpha}\right.\\ &\quad\left.+\frac{y}{27\alpha x\dot{y}}-\frac{4}{27x}+\frac{x}{216\alpha^2}+\frac{xy\ddot{y}}{216\alpha^2\dot{y}^2}-\frac{y^2}{432\alpha^2x\dot{y}^2}+\frac{y\ddot{y}}{54\alpha\dot{y}^2}\right].\end{aligned} \tag{5.2.3}$$

This can be written in the form

$$w\dot{w} = wF_1+F_0, \tag{5.2.4}$$

where $w = w(x)$ and we have introduced the new functions F_1 and F_0 which depend on the potential y. They have the form

$$F_1 = \left(\frac{\dot{y}}{x}\right)^{\frac{2}{3}}\left[-\frac{1}{18\alpha}+\frac{y}{36\alpha x\dot{y}}+\frac{x\ddot{y}}{18\alpha\dot{y}}+\frac{2\ddot{y}}{9\dot{y}}-\frac{y\ddot{y}}{36\alpha\dot{y}^2}-\frac{2}{9x}\right], \tag{5.2.5}$$

$$\begin{aligned} F_0 = & \left(\frac{\dot{y}}{x}\right)^{\frac{4}{3}}\left[\frac{y}{432\alpha^2\dot{y}}-\frac{y^2\ddot{y}}{216\alpha^2\dot{y}^3}+\frac{\ddot{y}}{\dot{y}}\left(\frac{4}{27}+\frac{2x}{27\alpha}+\frac{x^2}{108\alpha^2}\right)-\frac{1}{54\alpha}\right. \\ & \left.+\frac{y}{27\alpha x\dot{y}}-\frac{4}{27x}+\frac{x}{216\alpha^2}+\frac{xy\ddot{y}}{216\alpha^2\dot{y}^2}-\frac{y^2}{432\alpha^2 x\dot{y}^2}+\frac{y\ddot{y}}{54\alpha\dot{y}^2}\right]. \end{aligned} \tag{5.2.6}$$

It is necessary to integrate (5.2.4) and find $w = w(x)$.

We have shown that equation (4.2.6) is reducible to the more elegant and standard form given in equation (5.2.4). Further, we can make the transformation

$$q = \int F_1 dx, \tag{5.2.7}$$

and define the parameterized function

$$R(q) = \frac{F_0}{F_1}. \tag{5.2.8}$$

Then (5.2.4) can be written as

$$w_q w = w + R(q), \tag{5.2.9}$$

where w is a function of q. We note that expression (5.2.9) is the original master equation (4.2.6) reduced to canonical form. It is difficult to solve the canonical form, equation (5.2.9), for $w = w(q)$ and regain the function $w = w(x)$ in general. Therefore, we are only concerned with (5.2.4) which is a first order nonlinear differential equation in w. Since F_1 and F_0 both depend on an arbitrary function of y in a complicated manner, it won't be possible to solve (5.2.4) in general. However, we note that there are special cases for which we are able to obtain w explicitly.

We now show that it is possible to integrate (5.2.4) in particular cases by restricting the functions F_1 and F_0.

5.3 Case I: $F_0 = 0$

We set

$$F_0 = 0, \tag{5.3.1}$$

so that

$$\left(\frac{\dot{y}}{x}\right)^{\frac{4}{3}}\left[\frac{y}{432\alpha^2\dot{y}} - \frac{y^2\ddot{y}}{216\alpha^2\dot{y}^3} + \frac{\ddot{y}}{\dot{y}}\left(\frac{4}{27} + \frac{2x}{27\alpha} + \frac{x^2}{108\alpha^2}\right) - \frac{1}{54\alpha}\right.$$
$$\left. + \frac{y}{27\alpha x\dot{y}} - \frac{4}{27x} + \frac{x}{216\alpha^2} + \frac{xy\ddot{y}}{216\alpha^2\dot{y}^2} - \frac{y^2}{432\alpha^2 x\dot{y}^2} + \frac{y\ddot{y}}{54\alpha\dot{y}^2}\right] = 0. \tag{5.3.2}$$

This is written as

$$2x^2\dot{y}^3 + 16\alpha y\dot{y}^2 - 2xy^2\ddot{y} + 64\alpha^2 x\dot{y}^2\ddot{y} + 4x^3\dot{y}^2\ddot{y} + 32\alpha x^2\dot{y}^2\ddot{y}$$
$$-64\alpha^2\dot{y}^3 + xy\dot{y}^2 + 2x^2y\dot{y}\ddot{y} - 8\alpha x\dot{y}^3 - y^2\dot{y} + 8\alpha xy\dot{y}\ddot{y} = 0. \tag{5.3.3}$$

Equation (5.2.4) is now written as

$$w\dot{w} \;=\; wF_1, \tag{5.3.4}$$

which can be identified as a separable differential equation. We integrate equation (5.3.4) to obtain

$$\begin{aligned} w &= \int F_1 dx \\ &= \int \left(\frac{\dot{y}}{x}\right)^{\frac{2}{3}}\left[-\frac{1}{18\alpha} + \frac{y}{36\alpha x\dot{y}} + \frac{x\ddot{y}}{18\alpha\dot{y}} + \frac{2\ddot{y}}{9\dot{y}} - \frac{y\ddot{y}}{36\alpha\dot{y}^2} - \frac{2}{9x}\right] dx \\ &\quad + C, \end{aligned} \tag{5.3.5}$$

where C is a constant of integration.

Substituting for w from (5.3.5) in equation (5.2.2) and isolating Z we acquire

$$\begin{aligned} Z &= \left(\int \left(\frac{\dot{y}}{x}\right)^{\frac{2}{3}}\left[-\frac{1}{18\alpha} + \frac{y}{36\alpha x\dot{y}} + \frac{x\ddot{y}}{18\alpha\dot{y}} + \frac{2\ddot{y}}{9\dot{y}} - \frac{y\ddot{y}}{36\alpha\dot{y}^2} - \frac{2}{9x}\right] dx + C\right) \\ &\quad \times \left(\frac{x}{\dot{y}}\right)^{\frac{2}{3}} + \frac{1}{3} + \frac{x}{12\alpha} + \frac{y}{12\alpha\dot{y}}. \end{aligned} \tag{5.3.6}$$

Therefore the metric function Z (and therefore the gravitational potential $\lambda(r)$) can be found explicitly for this class of models. Note that this case is subject to the condition that y satisfies equation (5.3.3).

5.4 Case II: $F_1 = 0$

We now let

$$F_1 = 0, \tag{5.4.1}$$

which yields the following constraint

$$\left(\frac{\dot{y}}{x}\right)^{\frac{2}{3}}\left[-\frac{1}{18\alpha}+\frac{y}{36\alpha x\dot{y}}+\frac{x\ddot{y}}{18\alpha\dot{y}}+\frac{2\ddot{y}}{9\dot{y}}-\frac{y\ddot{y}}{36\alpha\dot{y}^2}-\frac{2}{9x}\right] \;=\; 0. \tag{5.4.2}$$

This can be simplified to the form

$$-8\alpha\dot{y}^2 + x\dot{y}\ddot{y}\left(8\alpha+2x\right) - 2x\dot{y}^2 - xy\ddot{y} + y\dot{y} \;=\; 0. \tag{5.4.3}$$

This is a highly nonlinear ordinary differential equation that can be simplified to the form

$$\left(-2x\dot{y} - 8\alpha\dot{y} + y\right)\left(x\ddot{y} - \dot{y}\right) \;=\; 0. \tag{5.4.4}$$

In the above equation, we observe that it is a product of a first order and second order linear ordinary differential equation. As a result, we can obtain two solutions for the variable $y(x)$.

It remains to find Z if the condition (5.4.1) holds. Equation (5.2.4) becomes

$$w\dot{w} \;=\; F_0, \tag{5.4.5}$$

which is again a separable equation. Integrating we obtain the solution

$$w \;=\; \left(2\int F_0 \; dx + C\right)^{\frac{1}{2}}, \tag{5.4.6}$$

$$\begin{aligned} w \;=\; & \left[\left(2\int\left(\frac{\dot{y}}{x}\right)^{\frac{4}{3}}\left[\frac{y}{432\alpha^2\dot{y}} - \frac{y^2\ddot{y}}{216\alpha^2\dot{y}^3} + \frac{\ddot{y}}{\dot{y}}\left(\frac{4}{27}+\frac{2x}{27\alpha}+\frac{x^2}{108\alpha^2}\right)\right.\right.\right. \\ & +\frac{y}{27\alpha x\dot{y}} - \frac{4}{27x} + \frac{x}{216\alpha^2} + \frac{xy\ddot{y}}{216\alpha^2\dot{y}^2} - \frac{y^2}{432\alpha^2 x\dot{y}^2} \\ & \left.\left.\left. +\frac{y\ddot{y}}{54\alpha\dot{y}^2} - \frac{1}{54\alpha}\right] \; dx\right) + C\right]^{\frac{1}{2}}. \end{aligned} \tag{5.4.7}$$

The use of equation (5.2.2) and (5.4.7) then yields

$$\begin{aligned} Z &= \left[\left(2\int\left(\frac{\dot{y}}{x}\right)^{\frac{4}{3}}\left[\frac{y}{432\alpha^2\dot{y}}-\frac{y^2\ddot{y}}{216\alpha^2\dot{y}^3}+\frac{\ddot{y}}{\dot{y}}\left(\frac{4}{27}+\frac{2x}{27\alpha}+\frac{x^2}{108\alpha^2}\right)\right.\right.\right. \\ &\quad +\frac{y}{27\alpha x\dot{y}}-\frac{4}{27x}+\frac{x}{216\alpha^2}+\frac{xy\ddot{y}}{216\alpha^2\dot{y}^2}-\frac{y^2}{432\alpha^2 x\dot{y}^2} \\ &\quad \left.\left.\left.+\frac{y\ddot{y}}{54\alpha\dot{y}^2}-\frac{1}{54\alpha}\right]\,dx\right)+C\right]^{\frac{1}{2}}\left(\frac{x}{\dot{y}}\right)^{\frac{2}{3}}+\frac{1}{3}+\frac{x}{12\alpha}+\frac{y}{12\alpha\dot{y}}. \end{aligned} \tag{5.4.8}$$

Here, Z and consequently the metric potential $\lambda(r)$, are defined explicitly in terms of variables x and y. An analytic form for y must satisfy the constraint provided in equation (5.4.4): we show that this equation can be integrated in general.

5.4.1 Case A: $-2x\dot{y} - 8\alpha\dot{y} + y = 0$

From (5.4.4) we get

$$(2x+8\alpha)\,\dot{y}-y \;=\; 0. \tag{5.4.9}$$

Equation (5.4.9) is a first order linear ordinary differential equation for which we can obtain the solution as

$$y \;=\; \tilde{Q}\sqrt{x+4\alpha}, \tag{5.4.10}$$

where $\tilde{Q}$ is an integration constant.

The solution of (5.4.8) for Z in Case A is given by

$$Z \;=\; C^{\frac{1}{2}}\left(\frac{2x}{\tilde{Q}\,(x+4\alpha)^{-\frac{1}{2}}}\right)^{\frac{2}{3}}+1+\frac{x}{4\alpha}. \tag{5.4.11}$$

Therefore we have found a solution for Z in terms of elementary functions.

5.4.2 Case B : $x\ddot{y} - \dot{y} = 0$

In this case, from (5.4.4) we obtain

$$x\ddot{y}-\dot{y} \;=\; 0, \tag{5.4.12}$$

which is a second order linear ordinary differential equation. The solution to (5.4.12) can be easily expressed by

$$y = \frac{C_1 x^2}{2} + C_2, \tag{5.4.13}$$

where C_1 and C_2 are constants of integration.

The solution of (5.4.8), for this case, is expressed by

$$\begin{aligned} Z &= \left[\left(\frac{(C_1)^{\frac{4}{3}}}{\alpha}\right)\left[\frac{x^2}{64\alpha} + \frac{(C_2)^2}{144\alpha(C_1)^2x^2} + \frac{x}{6} - \frac{C_2}{C_1 9x}\right] + C\right]^{\frac{1}{2}} \left(\frac{1}{C_1}\right)^{\frac{2}{3}} \\ &\quad + \frac{1}{3} + \frac{x}{8\alpha} + \frac{C_2}{12\alpha C_1 x}. \end{aligned} \tag{5.4.14}$$

Thus we have obtained a solution for the potential Z in terms of elementary functions.

In this class of models we are able to obtain two possible analytic forms for the function y in terms of elementary functions. The constraint equation in (5.4.3) is satisfied. The solution of y obtained from Case A is not contained in earlier models and the potential Z is represented by elementary functions. *It is a new solution.* In Case B we are able to obtain a result, with the use of the substitution (5.2.2), for the potential y similar to that given in section 4.3.3 by equation (4.3.12). However the functional form of Z in (5.4.14) differs from the form presented by (4.3.24). In (4.3.24) we observe that the potential Z contains algebraic functions of x and powers of the variable x. In (5.4.14) the potential Z contains only the powers of the variable x. Hence the solution in Case B is *also a new solution.* We believe that the new solutions arise because of the transformation (5.2.2) introduced in our analysis to transform the condition of pressure isotropy to canonical form. It is also important to note that these two classes of new solutions exist only in EGB gravity. They do not have a 5-dimensional Einstein limit as $\alpha \neq 0$.

5.5 Case III: $F_1 = KF_0$

We choose F_1 to be proportional to F_0 where K is some constant. This gives the condition

$$F_1 \;=\; KF_0. \tag{5.5.1}$$

We find that (5.5.1) can be written explicitly as

$$\begin{aligned}
&96\alpha^2 x\dot{y}^2\ddot{y} - 24\alpha x\dot{y}^3 - 12\alpha xy\dot{y}\ddot{y} - 96\alpha^2\dot{y}^3 + 24\alpha x^2\dot{y}^2\ddot{y} + 12\alpha y\dot{y}^2 \\
&= K\left(\frac{\dot{y}}{x}\right)^{\frac{2}{3}}\left[\left(xy\dot{y}^2 - 2xy^2\ddot{y} + 4x\dot{y}^2\ddot{y}\left(16\alpha^2 + 8\alpha x + x^2\right) - 8\alpha x\dot{y}^3\right.\right. \\
&\left.\left. +16\alpha y\dot{y}^2 - 64\alpha^2\dot{y}^3 + 2x^2\dot{y}^3 + 2x^2y\dot{y}\ddot{y} - y^2\dot{y} + 8\alpha xy\dot{y}\ddot{y}\right)\right].
\end{aligned} \tag{5.5.2}$$

Then substituting (5.5.1) into equation (5.2.4) yields

$$w\dot{w} \;=\; F_0\left(K\,w + 1\right). \tag{5.5.3}$$

Integrating equation (5.5.3) we obtain

$$\frac{w}{K} - \frac{\ln\left(1 + K\,w\right)}{K^2} \;=\; \int F_0\,dx + C. \tag{5.5.4}$$

In terms of the variable Z we obtain

$$\begin{aligned}
&\frac{\left(Z - \frac{1}{3} - \frac{x}{12\alpha} - \frac{y}{12\alpha\dot{y}}\right)\left(\frac{\dot{y}}{x}\right)^{\frac{2}{3}}}{K} - \frac{\ln\left[1 + K\left(\left(Z - \frac{1}{3} - \frac{x}{12\alpha} - \frac{y}{12\alpha\dot{y}}\right)\left(\frac{\dot{y}}{x}\right)^{\frac{2}{3}}\right)\right]}{K^2} \\
&= \int\left(\frac{\dot{y}}{x}\right)^{\frac{4}{3}}\left[\frac{y}{432\alpha^2\dot{y}} - \frac{y^2\ddot{y}}{216\alpha^2\dot{y}^3} + \frac{\ddot{y}}{\dot{y}}\left(\frac{4}{27} + \frac{2x}{27\alpha} + \frac{x^2}{108\alpha^2}\right)\right. \\
&\left. + \frac{y}{27\alpha x\dot{y}} - \frac{4}{27x} + \frac{x}{216\alpha^2} + \frac{xy\ddot{y}}{216\alpha^2\dot{y}^2} - \frac{y^2}{432\alpha^2 x\dot{y}^2} + \frac{y\ddot{y}}{54\alpha\dot{y}^2} - \frac{1}{54\alpha}\right] dx \\
&+ C.
\end{aligned} \tag{5.5.5}$$

This solution must satisfy the constraint equation (5.5.2). In this case the quantity Z, and consequently the gravitational potential $\lambda(r)$, are given implicitly.

5.6 Case IV: $F_1 = Q_1$ and $F_0 = Q_2$

We now set F_1 and F_0 to be arbitrary constants

$$F_1 = Q_1, \tag{5.6.1}$$

$$F_0 = Q_2, \tag{5.6.2}$$

such that

$$Q_1 = \left(\frac{\dot{y}}{x}\right)^{\frac{2}{3}} \left[-\frac{1}{18\alpha} + \frac{y}{36\alpha x\dot{y}} + \frac{x\ddot{y}}{18\alpha\dot{y}} + \frac{2\ddot{y}}{9\dot{y}} - \frac{y\ddot{y}}{36\alpha\dot{y}^2} - \frac{2}{9x}\right], \tag{5.6.3}$$

$$\begin{aligned} Q_2 = {} & \left(\frac{\dot{y}}{x}\right)^{\frac{4}{3}} \left[\frac{y}{432\alpha^2\dot{y}} - \frac{y^2\ddot{y}}{216\alpha^2\dot{y}^3} + \frac{\ddot{y}}{\dot{y}}\left(\frac{4}{27} + \frac{2x}{27\alpha} + \frac{x^2}{108\alpha^2}\right) - \frac{1}{54\alpha} \right. \\ & \left. + \frac{y}{27\alpha x\dot{y}} - \frac{4}{27x} + \frac{x}{216\alpha^2} + \frac{xy\ddot{y}}{216\alpha^2\dot{y}^2} - \frac{y^2}{432\alpha^2 x\dot{y}^2} + \frac{y\ddot{y}}{54\alpha\dot{y}^2}\right]. \end{aligned} \tag{5.6.4}$$

Equation (5.2.4) is now written as

$$w\dot{w} = wQ_1 + Q_2, \tag{5.6.5}$$

which can be easily integrated to obtain the solution

$$\frac{w}{Q_1} - \frac{Q_2 \ln\left(wQ_1 + Q_2\right)}{Q_1^2} = x + C. \tag{5.6.6}$$

The use of equation (5.2.2) then produces

$$\frac{\left(Z - \frac{1}{3} - \frac{x}{12\alpha} - \frac{y}{12\alpha\dot{y}}\right)\left(\frac{\dot{y}}{x}\right)^{\frac{2}{3}}}{Q_1} - \frac{Q_2 \ln\left(\left[\left(Z - \frac{1}{3} - \frac{x}{12\alpha} - \frac{y}{12\alpha\dot{y}}\right)\left(\frac{\dot{y}}{x}\right)^{\frac{2}{3}}\right] Q_1 + Q_2\right)}{Q_1^2}$$
$$= x + C. \tag{5.6.7}$$

The quantity Z and the gravitational potential $\lambda(r)$ are found implicitly once again for the above solution for which analytic forms of y must satisfy the additional conditions in equations (5.6.3) and (5.6.4) simultaneously. It is unlikely that this case will lead to viable exact solutions.

Chapter 6

New models in EGB gravity: charged matter

6.1 Introduction

Charged fluid distributions are important in relativistic astrophysics as the presence of charge counteracts the process of gravitational collapse by the Coulombic repulsive force along with the pressure gradient. As a result, stellar models with the inclusion of an electromagnetic field have been analysed extensively in the literature. These models can be studied by finding static spherically symmetric interior solutions to the the Einstein-Maxwell field equations. Several investigations in this direction have been conducted. These include the works of Tikekar (1990), Maharaj and Leach (1996), and Komathiraj and Maharaj (2007) in general relativity, who found exact charged gravitating solutions that model neutron stars. Other charged treatments include the results of Maharaj and Thirukkanesh (2009), Varela *et al* (2010) and Hansraj *et al* (2013). In light of this, it is interesting to examine charged compact objects in modified gravity theories. Not many static interior solutions for charged spheres in EGB gravity are known. One such endeavour was conducted by Hansraj (2017) in five dimensions. Furthermore, Wiltshire (1988) derived the exterior metric for EGB gravity in the presence of charge. This is analogous to the Boulware and Deser exterior spacetime for neutral fluids. Therefore in this chapter we analyse the condition of pressure isotropy with an electromagnetic field. We then seek out new solutions to the transformed Abelian differential equation in the presence of charge.

6.2 Abel equations

We now equate equations (3.4.16b) and (3.4.16c) with $p_{\parallel} = p_{\perp} = p$ to produce the pressure isotropy condition with the inclusion of charge E as

$$\begin{aligned}&\frac{1}{e^{4\lambda}r^3}\left[12\alpha\nu' - 4\alpha r\nu'' - 4\alpha r\nu'^2 + 12\alpha r\nu'\lambda'\right]\\&+\frac{1}{e^{2\lambda}r^2}\left[r^2\nu'' + 4\alpha\nu'' + 4\alpha\nu'^2 + r^2\nu'2 - r^2\lambda'\nu' + 2e^{2\lambda}\right.\\&\left.-2 - 4\alpha\nu'\lambda' - r\nu' - 2r\lambda'\right] - \frac{3}{2}E^2 = 0,\end{aligned} \tag{6.2.1}$$

in terms of canonical coordinates.

In order to analyse system (3.4.16) and seek out exact solutions, we apply the Durgapal and Bannerji (1983) transformation

$$e^{2\nu(r)} = y^2(x), \quad e^{-2\lambda(r)} = Z(x), \quad x = r^2. \tag{6.2.2}$$

The charged EGB field equations (3.4.16) from earlier with isotropic pressure ($p_{\parallel} = p_{\perp} = p$) can then be written as

$$3\pi^2\left(\rho + \frac{1}{4\pi^2}E^2\right) = \frac{3}{x}(1-Z)\left(1-4\alpha\dot{Z}\right) - 3\dot{Z}, \tag{6.2.3a}$$

$$3\pi^2\left(p - \frac{1}{4\pi^2}E^2\right) = \frac{3}{x}(Z-1)\left(1-\frac{8\alpha\dot{y}Z}{y}\right) + 6Z\frac{\dot{y}}{y}, \tag{6.2.3b}$$

$$\begin{aligned}3\pi^2\left(p + \frac{1}{4\pi^2}E^2\right) = \;& 2\dot{Z} + \frac{Z-1}{x} - \frac{8\alpha\dot{y}Z(Z-1)}{xy}\\&-\frac{4\alpha}{y}\left(4\ddot{y}Z^2 + 6\dot{y}\dot{Z}Z - 4\ddot{y}Z - 2\dot{y}\dot{Z}\right)\\&+\frac{1}{y}\left(4x\ddot{y}Z + 2x\dot{Z}\dot{y} + 6Z\dot{y}\right),\end{aligned} \tag{6.2.3c}$$

$$\frac{Z}{4x}\left[2x\dot{E} + 3E\right]^2 = \pi^4\sigma^2, \tag{6.2.3d}$$

where dots denote differentiation with respect to x.

Equating (6.2.3b) and (6.2.3c) yields

$$\ddot{y}\left[4xZ\left(4\alpha\left(1-Z\right)+x\right)\right]+\dot{y}\left[2x^2\dot{Z}-8\alpha\left(2Z-2Z^2-x\dot{Z}+3xZ\dot{Z}\right)\right]$$
$$+y\left[2-2Z+2x\dot{Z}-\frac{3}{2}E^2x\right]=0, \tag{6.2.4}$$

which is the condition of pressure isotropy. Equation (6.2.4) represents the master equation for this system now with the inclusion of an electric field. It is still a second order linear differential equation with the variables y, Z and E. *If y and Z are specified then the electric field E is generated without integration.* It is interesting to observe that we can then obtain a general solution for the matter variables, electric field and proper charge density in terms of y and Z with the use of equation (6.2.4). These are written as follows

$$\rho = \frac{(1-Z)\left(1-4\alpha\dot{Z}\right)}{x\pi^2}-\frac{\dot{Z}}{\pi^2}-\frac{1}{6\pi^2}\left[\frac{\ddot{y}}{y}\left[4Z\left(4\alpha(1-Z)+x\right)\right]+\frac{\dot{y}}{xy}\left[2x^2\dot{Z}\right.\right.$$
$$\left.\left.-8\alpha\left(2Z-2Z^2-x\dot{Z}+3xZ\dot{Z}\right)\right]+\frac{2}{x}-\frac{2Z}{x}+2\dot{Z}\right], \tag{6.2.5}$$

$$p = \frac{1}{x\pi^2}\left(Z-1\right)\left(1-\frac{8\alpha\dot{y}Z}{y}\right)+3Z\frac{\dot{y}}{\pi^2 y}+\frac{1}{6\pi^2}\left[\frac{\ddot{y}}{y}\left[4Z\left(4\alpha(1-Z)+x\right)\right]\right.$$
$$\left.+\frac{\dot{y}}{xy}\left[2x^2\dot{Z}-8\alpha\left(2Z-2Z^2-x\dot{Z}+3xZ\dot{Z}\right)\right]+\frac{2}{x}-\frac{2Z}{x}+2\dot{Z}\right], \tag{6.2.6}$$

$$E^2 = \frac{2}{3}\left[\frac{\ddot{y}}{y}\left[4Z\left(4\alpha(1-Z)+x\right)\right]+\frac{\dot{y}}{xy}\left[2x^2\dot{Z}-8\alpha\left(2Z-2Z^2-x\dot{Z}+3xZ\dot{Z}\right)\right]\right.$$
$$\left.+\frac{2}{x}-\frac{2Z}{x}+2\dot{Z}\right], \tag{6.2.7}$$

$$\sigma^2 = \frac{Z}{4x\pi^4}\left[2x\left[\frac{2}{3}\left(\frac{\ddot{y}}{y}\left[4Z\left(4\alpha\left(1-Z\right)+x\right)\right]+\frac{\dot{y}}{xy}\left[2x^2\dot{Z}-8\alpha\left(2Z-2Z^2\right.\right.\right.\right.\right.$$
$$\left.\left.\left.-x\dot{Z}+3xZ\dot{Z}\right)\right]+\frac{2}{x}-\frac{2Z}{x}+2\dot{Z}\right)\right]^{-\frac{1}{2}}\times\left[\frac{1}{3}\left(2\ddot{Z}-\frac{2}{x^2}-\frac{2\dot{Z}}{x}+\frac{2Z}{x^2}\right.\right.$$
$$+\frac{\dddot{y}}{y}\left[4Z\left(4\alpha(1-Z)+x\right)\right]-\frac{\ddot{y}\dot{y}}{y^2}\left[4Z\left(4\alpha(1-Z)+x\right)\right]+\frac{\ddot{y}}{y}\left[16\alpha\dot{Z}\right.$$
$$\left.-32\alpha Z\dot{Z}+4\dot{Z}x+4Z\right]+\frac{\ddot{y}}{xy}\left[2x^2\dot{Z}-8\alpha\left(2Z-2Z^2-x\dot{Z}+3xZ\dot{Z}\right)\right]$$
$$-\frac{\dot{y}}{x^2y}\left[2x^2\dot{Z}-8\alpha\left(2Z-2Z^2-x\dot{Z}+3xZ\dot{Z}\right)\right]-\frac{\dot{y}\dot{y}}{xy^2}\left[2x^2\dot{Z}-8\alpha\left(2Z\right.\right.$$
$$\left.\left.-2Z^2-x\dot{Z}+3xZ\dot{Z}\right)\right]+\frac{\dot{y}}{xy}\left[4x\dot{Z}+2x^2\ddot{Z}-8\alpha\dot{Z}+8\alpha Z\dot{Z}+8\alpha x\ddot{Z}\right.$$

$$
\begin{aligned}
&-24\alpha x\dot{Z}^2 - 24\alpha xZ\ddot{Z}\Big]\Big)\Big] + 3\left[\frac{2}{3}\left(\frac{\ddot{y}}{y}\left[4Z\left(4\alpha\left(1-Z\right)+x\right)\right]\right.\right. \\
&+\frac{\dot{y}}{xy}\left[2x^2\dot{Z} - 8\alpha\left(2Z - 2Z^2 - x\dot{Z} + 3xZ\dot{Z}\right)\right] \\
&\left.\left.+\frac{2}{x} - \frac{2Z}{x} + 2\dot{Z}\right)\right]^{\frac{1}{2}}\Bigg]^2, \qquad (6.2.8)
\end{aligned}
$$

for the system (6.2.3).

A choice of the potentials y and Z may lead to a model with unphysical behaviour. Consequently in many investigations a choice for the electric field is made on physical grounds. For recent examples of this approach see the treatments of Mathias *et al* (2021), Lighuda *et al* (2021) and Mafa Takisa *et al* (2019). *If the electric field E is specified then the condition of pressure isotropy, an Abelian differential equation, has to be integrated.* We rearrange equation (6.2.4) to obtain a first order nonlinear differential equation in Z to get

$$
\begin{aligned}
&\left(-24\alpha x\dot{y}Z + 8\alpha x\dot{y} + 2x^2\dot{y} + 2xy\right)\dot{Z} - 16\alpha\left(x\ddot{y} - \dot{y}\right)Z^2 \\
&+\left(4x^2\ddot{y} + 16\alpha x\ddot{y} - 16\alpha\dot{y} - 2y\right)Z + 2y - \frac{3}{2}xyE^2 = 0. \qquad (6.2.9)
\end{aligned}
$$

This equation is classified as an Abel differential equation of the second kind in Z and the addition of the electric field does not remove its complex nonlinear nature. We rewrite (6.2.9) as

$$
\begin{aligned}
&\left(Z - \frac{1}{3} - \frac{x}{12\alpha} - \frac{y}{12\alpha\dot{y}}\right)\dot{Z} = \frac{2}{3}\left(\frac{1}{x} - \frac{\ddot{y}}{\dot{y}}\right)Z^2 \\
&+\left(\frac{2\ddot{y}}{3\dot{y}} + \frac{x\ddot{y}}{6\alpha\dot{y}} - \frac{2}{3x} - \frac{y}{12\alpha x\dot{y}}\right)Z + \frac{y}{12\alpha x\dot{y}} - \frac{yE^2}{16\alpha\dot{y}}. \qquad (6.2.10)
\end{aligned}
$$

We once again make use of the transformation suggested by Zaitsev and Polyanin (1994) given by (5.2.2) in Chapter 5

$$
w = \left(Z - \frac{1}{3} - \frac{x}{12\alpha} - \frac{y}{12\alpha\dot{y}}\right)\left(\frac{\dot{y}}{x}\right)^{\frac{2}{3}}, \qquad (6.2.11)
$$

where $\dot{y} \neq 0$ and $\alpha \neq 0$. Using (6.2.11) in (6.2.10) we obtain

$$
w\dot{w} = w\left(\frac{\dot{y}}{x}\right)^{\frac{2}{3}}\left[-\frac{1}{18\alpha} + \frac{y}{36\alpha x\dot{y}} + \frac{x\ddot{y}}{18\alpha\dot{y}} + \frac{2\ddot{y}}{9\dot{y}} - \frac{y\ddot{y}}{36\alpha\dot{y}^2} - \frac{2}{9x}\right]
$$

$$+\left(\frac{\dot{y}}{x}\right)^{\frac{4}{3}}\left[\frac{y}{432\alpha^2\dot{y}}-\frac{y^2\ddot{y}}{216\alpha^2\dot{y}^3}+\frac{\ddot{y}}{\dot{y}}\left(\frac{4}{27}+\frac{2x}{27\alpha}+\frac{x^2}{108\alpha^2}\right)\right.$$
$$+\frac{y}{27\alpha x\dot{y}}-\frac{4}{27x}+\frac{x}{216\alpha^2}+\frac{xy\ddot{y}}{216\alpha^2\dot{y}^2}-\frac{y^2}{432\alpha^2x\dot{y}^2}-\frac{1}{54\alpha}$$
$$\left.+\frac{y\ddot{y}}{54\alpha\dot{y}^2}-\frac{yE^2}{16\alpha\dot{y}}\right]. \tag{6.2.12}$$

This can be written in the form

$$w\dot{w} = wF_1+F_0, \tag{6.2.13}$$

where $w = w(x)$ and functions F_1 and F_0 now depend on the potential y and the electric field intensity E. They have the forms

$$F_1 = \left(\frac{\dot{y}}{x}\right)^{\frac{2}{3}}\left[-\frac{1}{18\alpha}+\frac{y}{36\alpha x\dot{y}}+\frac{x\ddot{y}}{18\alpha\dot{y}}+\frac{2\ddot{y}}{9\dot{y}}-\frac{y\ddot{y}}{36\alpha\dot{y}^2}-\frac{2}{9x}\right], \tag{6.2.14}$$

$$F_0 = \left(\frac{\dot{y}}{x}\right)^{\frac{4}{3}}\left[\frac{y}{432\alpha^2\dot{y}}-\frac{y^2\ddot{y}}{216\alpha^2\dot{y}^3}+\frac{\ddot{y}}{\dot{y}}\left(\frac{4}{27}+\frac{2x}{27\alpha}+\frac{x^2}{108\alpha^2}\right)\right.$$
$$+\frac{y}{27\alpha x\dot{y}}-\frac{4}{27x}+\frac{x}{216\alpha^2}+\frac{xy\ddot{y}}{216\alpha^2\dot{y}^2}-\frac{y^2}{432\alpha^2x\dot{y}^2}-\frac{1}{54\alpha}$$
$$\left.+\frac{y\ddot{y}}{54\alpha\dot{y}^2}-\frac{yE^2}{16\alpha\dot{y}}\right]. \tag{6.2.15}$$

In order to find a solution for $w = w(x)$, we must integrate (6.2.13) and make appropriate choices for E. Since F_1 and F_0 both depend on an arbitrary function of y in a complex manner and F_0 contains contributions from the electromagnetic field, it will not be possible to find a solution to (6.2.13) in general. As a result we restrict the functions F_1 and F_0 in specific cases where a solution for $w(x)$ is possible. These cases are provided below.

6.3 Case I: $F_0 = 0$

We set

$$F_0 = 0, \tag{6.3.1}$$

so that

$$\left(\frac{\dot{y}}{x}\right)^{\frac{4}{3}}\left[\frac{y}{432\alpha^2\dot{y}}-\frac{y^2\ddot{y}}{216\alpha^2\dot{y}^3}+\frac{\ddot{y}}{\dot{y}}\left(\frac{4}{27}+\frac{2x}{27\alpha}+\frac{x^2}{108\alpha^2}\right)\right.$$

$$+\frac{y}{27\alpha x\dot{y}} - \frac{4}{27x} + \frac{x}{216\alpha^2} + \frac{xy\ddot{y}}{216\alpha^2\dot{y}^2} - \frac{y^2}{432\alpha^2 x\dot{y}^2} - \frac{1}{54\alpha}$$
$$+\left.\frac{y\ddot{y}}{54\alpha\dot{y}^2} - \frac{yE^2}{16\alpha\dot{y}}\right] = 0, \quad (6.3.2)$$

which can be rewritten as

$$2x^2\dot{y}^3 + 16\alpha y\dot{y}^2 - 2xy^2\ddot{y} + 64\alpha^2 x\dot{y}^2\ddot{y} + 4x^3\dot{y}^2\ddot{y} + 32\alpha x^2\dot{y}^2\ddot{y} - y^2\dot{y}$$
$$-64\alpha^2\dot{y}^3 + xy\dot{y}^2 + 2x^2y\dot{y}\ddot{y} - 8\alpha x\dot{y}^3 + 8\alpha xy\dot{y}\ddot{y} - 27\alpha xy\dot{y}^2E^2 = 0. \quad (6.3.3)$$

From equation (6.3.3) we can obtain a general form for the electric field intensity E as

$$E = \left[-\frac{64\alpha\dot{y}}{27xy} + \frac{16}{27x} + \frac{1}{27\alpha} - \frac{2y\ddot{y}}{27\alpha\dot{y}^2} + \frac{64\alpha\ddot{y}}{27y} - \frac{8\dot{y}}{27y}\right.$$
$$\left.+\frac{4x^2\ddot{y}}{27\alpha y} + \frac{32x\ddot{y}}{27y} + \frac{2x\dot{y}}{27\alpha y} + \frac{2x\ddot{y}}{27\alpha\dot{y}} - \frac{y}{27\alpha x\dot{y}} + \frac{8\ddot{y}}{27\dot{y}}\right]^{\frac{1}{2}}. \quad (6.3.4)$$

Equation (6.2.13) is now written as

$$w\dot{w} = wF_1, \quad (6.3.5)$$

which is a separable differential equation. We integrate equation (6.3.5) to obtain

$$w = \int F_1 dx$$
$$= \int \left(\frac{\dot{y}}{x}\right)^{\frac{2}{3}} \left[-\frac{1}{18\alpha} + \frac{y}{36\alpha x\dot{y}} + \frac{x\ddot{y}}{18\alpha\dot{y}} + \frac{2\ddot{y}}{9\dot{y}} - \frac{y\ddot{y}}{36\alpha\dot{y}^2} - \frac{2}{9x}\right] dx$$
$$+ C, \quad (6.3.6)$$

where C is a constant of integration. This yields the potential

$$Z = \left(\int \left(\frac{\dot{y}}{x}\right)^{\frac{2}{3}} \left[-\frac{1}{18\alpha} + \frac{y}{36\alpha x\dot{y}} + \frac{x\ddot{y}}{18\alpha\dot{y}} + \frac{2\ddot{y}}{9\dot{y}} - \frac{y\ddot{y}}{36\alpha\dot{y}^2} - \frac{2}{9x}\right] dx + C\right)$$
$$\times \left(\frac{x}{\dot{y}}\right)^{\frac{2}{3}} + \frac{1}{3} + \frac{x}{12\alpha} + \frac{y}{12\alpha\dot{y}}, \quad (6.3.7)$$

in terms of the function y.

Hence we have solved the charged condition of pressure isotropy when $F_0 = 0$. *Any choice of the potential y leads to an exact solution of the system of charged EGB field equations* (6.2.3). Note that this property arises because of the presence of the electromagnetic field. For neutral matter considered in Chapter 5, section 5.3, there is an additional constraint given by (5.3.3) that has to be satisfied.

To demonstrate an explicit exact solution we choose y in the form

$$y = ax^n + l, \tag{6.3.8}$$

where a, l and n are real numbers. Then the potential Z and the charge E are given by

$$\begin{aligned} Z &= \frac{x}{6\alpha(2n-1)}\left(n + \frac{1}{2n} - \frac{5}{2}\right) + \frac{lx^{1-n}}{6\alpha a(n+1)}\left(\frac{1}{n} - \frac{1}{2}\right) + \frac{Cx^{\frac{4-2n}{3}}}{(an)^{\frac{2}{3}}} \\ &+\frac{2}{3} + \frac{x}{12\alpha} + \frac{(ax^n+l)\,x^{(1-n)}}{12\alpha an}, \qquad (6.3.9) \\ E &= \left[\frac{8}{27x} - \frac{1}{27\alpha} + \frac{64\alpha nax^{n-2}}{27(ax^n+l)}(n-2) + \frac{2n}{27\alpha} + \frac{8n}{27x}\right. \\ &\left.+\frac{2nax^n(2n-1)}{27\alpha(ax^n+l)} + \frac{nax^{n-1}(32n-40)}{27(ax^n+l)} + \frac{(ax^n+l)(1-2n)}{27\alpha anx^n}\right]^{\frac{1}{2}}, \qquad (6.3.10) \end{aligned}$$

which are expressed in terms of elementary functions.

6.4 Case II: $F_1 = 0$

This case is similar to the uncharged EGB solutions considered in section 5.4. We now let

$$F_1 = 0, \tag{6.4.1}$$

which yields the following constraint

$$\left(\frac{\dot{y}}{x}\right)^{\frac{2}{3}}\left[-\frac{1}{18\alpha} + \frac{y}{36\alpha x\dot{y}} + \frac{x\ddot{y}}{18\alpha\dot{y}} + \frac{2\ddot{y}}{9\dot{y}} - \frac{y\ddot{y}}{36\alpha\dot{y}^2} - \frac{2}{9x}\right] = 0. \tag{6.4.2}$$

The above equation can be written as

$$\dot{y}\ddot{y}\left(8\alpha x + 2x^2\right) - 8\alpha\dot{y}^2 - 2x\dot{y}^2 - xy\ddot{y} + y\dot{y} = 0. \tag{6.4.3}$$

This is a highly nonlinear ordinary differential equation that can be simplified to the form

$$(-2x\dot{y} - 8\alpha\dot{y} + y)(x\ddot{y} - \dot{y}) = 0. \tag{6.4.4}$$

In the above equation, we observe that it is a product of a first order and second order linear ordinary differential equation. As a result, we can thus obtain two functional forms for the variable $y(x)$.

With $F_1 = 0$ we have to solve

$$w\dot{w} = F_0, \tag{6.4.5}$$

which is a separable equation. Integrating we obtain

$$\begin{aligned} w &= \left(2\int F_0 \; dx + C\right)^{\frac{1}{2}} \\ &= \left(\left[2\int\left(\frac{\dot{y}}{x}\right)^{\frac{4}{3}}\left[\frac{y}{432\alpha^2\dot{y}} - \frac{y^2\ddot{y}}{216\alpha^2\dot{y}^3} + \frac{\ddot{y}}{\dot{y}}\left(\frac{4}{27} + \frac{2x}{27\alpha} + \frac{x^2}{108\alpha^2}\right)\right.\right.\right. \\ &\quad + \frac{y}{27\alpha x\dot{y}} - \frac{4}{27x} + \frac{x}{216\alpha^2} + \frac{xy\ddot{y}}{216\alpha^2\dot{y}^2} - \frac{y^2}{432\alpha^2 x\dot{y}^2} - \frac{1}{54\alpha} \\ &\quad \left.\left.\left. + \frac{y\ddot{y}}{54\alpha\dot{y}^2} - \frac{yE^2}{16\alpha\dot{y}}\right] dx\right] + C\right)^{\frac{1}{2}}, \end{aligned} \tag{6.4.6}$$

which contains the electric field intensity E. When $E = 0$ we regain equation (5.4.7) considered in section 5.4. In terms of Z we have

$$\begin{aligned} Z &= \left(\left(2\int\left(\frac{\dot{y}}{x}\right)^{\frac{4}{3}}\left[\frac{y}{432\alpha^2\dot{y}} - \frac{y^2\ddot{y}}{216\alpha^2\dot{y}^3} + \frac{\ddot{y}}{\dot{y}}\left(\frac{4}{27} + \frac{2x}{27\alpha} + \frac{x^2}{108\alpha^2}\right)\right.\right.\right. \\ &\quad + \frac{y}{27\alpha x\dot{y}} - \frac{4}{27x} + \frac{x}{216\alpha^2} + \frac{xy\ddot{y}}{216\alpha^2\dot{y}^2} - \frac{y^2}{432\alpha^2 x\dot{y}^2} - \frac{1}{54\alpha} \\ &\quad \left.\left.\left. + \frac{y\ddot{y}}{54\alpha\dot{y}^2} - \frac{yE^2}{16\alpha\dot{y}}\right] dx\right) + C\right)^{\frac{1}{2}}\left(\frac{x}{\dot{y}}\right)^{\frac{2}{3}} + \frac{1}{3} + \frac{x}{12\alpha} + \frac{y}{12\alpha\dot{y}}. \end{aligned} \tag{6.4.7}$$

Here, Z and consequently the metric potential $\lambda(r)$, are defined explicitly in terms of variables x and y. The solution to Z can only be found in terms of elementary functions if the electric field E is specified.

6.4.1 Case A: $-2x\dot{y} - 8\alpha\dot{y} + y = 0$

From (6.4.4) we get

$$(2x + 8\alpha)\,\dot{y} - y = 0. \tag{6.4.8}$$

Equation (6.4.8) is a first order linear ordinary differential equation for which we can obtain the solution as

$$y = \tilde{Q}\sqrt{x + 4\alpha}, \tag{6.4.9}$$

where $\tilde{Q}$ is an integration constant.

For this form of y we can find the potential Z as

$$\begin{aligned} Z = & \left(2\left(\frac{\tilde{Q}}{2}\right)^{\frac{4}{3}}\left(-\int \frac{(x+4\alpha)^{\frac{1}{3}}E^2}{8\alpha x^{\frac{4}{3}}}\,dx\right) + C\right)^{\frac{1}{2}}\left(\frac{2x}{\tilde{Q}\,(x+4\alpha)^{-\frac{1}{2}}}\right)^{\frac{2}{3}} \\ & +1 + \frac{x}{4\alpha}. \end{aligned} \tag{6.4.10}$$

6.4.2 Case B : $x\ddot{y} - \dot{y} = 0$

In this case from (6.4.4) we obtain

$$x\ddot{y} - \dot{y} = 0, \tag{6.4.11}$$

which is a second order linear ordinary differential equation. Its solution can be easily expressed by

$$y = \frac{C_1 x^2}{2} + C_2, \tag{6.4.12}$$

where C_1 and C_2 are constants of integration.

For the potential y given above we obtain the potential Z as

$$\begin{aligned} Z = & \left[\left(\frac{(C_1)^{\frac{4}{3}}}{\alpha}\right)\left[\frac{x^2}{64\alpha} + \frac{C_2^2}{144\alpha C_1^2 x^2} + \frac{x}{6} - \frac{C_2}{C_1 9x}\right]\right. \\ & \left. -\left(\int (C_1)^{\frac{4}{3}}\frac{\left(C_1\frac{x^2}{2} + C_2\right)E^2}{8\alpha C_1 x}\,dx\right) + C\right]^{\frac{1}{2}}\left(\frac{1}{C_1}\right)^{\frac{2}{3}} + \frac{1}{3} + \frac{x}{8\alpha} \\ & + \frac{C_2}{12\alpha C_1 x}. \end{aligned} \tag{6.4.13}$$

Observe that when $E = 0$ then (6.4.7), (6.4.10) and (6.4.13) for the potential Z reduces to the uncharged cases considered in section 5.4. The expressions (6.4.7), (6.4.10) and (6.4.13) are a charged generalisation. *Any choice of the electric field E generates a new exact solution for this class of models.* Clearly the choice made for the electric field should satisfy the physical criteria for a compact spherically symmetric star (Mafa Takisa *et al* 2019). We note that these two new classes of charged solutions exist only in EGB gravity. Since $\alpha \neq 0$ we cannot regain the Einstein limit.

6.5 Case III: $F_1 = KF_0$

We choose F_1 to be proportional to F_0 where K is some constant. This gives the condition

$$F_1 = KF_0. \tag{6.5.1}$$

We find that (6.5.1) can be written explicitly as

$$\begin{aligned}
&96\alpha^2 x\dot{y}^2\ddot{y} - 24\alpha x\dot{y}^3 - 12\alpha xy\dot{y}\ddot{y} - 96\alpha^2\dot{y}^3 + 24\alpha x^2\dot{y}^2\ddot{y} + 12\alpha y\dot{y}^2 \\
&= K\left(\frac{\dot{y}}{x}\right)^{\frac{2}{3}}\Big[xy\dot{y}^2 - 2xy^2\ddot{y} + 4x\dot{y}^2\ddot{y}\left(16\alpha^2 + 8\alpha x + x^2\right) - 8\alpha x\dot{y}^3 - y^2\dot{y} \\
&+16\alpha y\dot{y}^2 - 64\alpha^2\dot{y}^3 + 2x^2\dot{y}^3 + 2x^2y\dot{y}\ddot{y} + 8\alpha xy\dot{y}\ddot{y} - 27\alpha xyE^2\dot{y}^2\Big].
\end{aligned} \tag{6.5.2}$$

From (6.5.2) we obtain

$$\begin{aligned}
E = &\Bigg[\left(\frac{1}{K}\right)\left(\frac{x}{\dot{y}}\right)^{\frac{2}{3}}\left[\frac{8\dot{y}}{9y} - \frac{4}{9x} - \frac{8x\ddot{y}}{9y} - \frac{32\alpha\ddot{y}}{9y} + \frac{4\ddot{y}}{9\dot{y}} + \frac{32\alpha\dot{y}}{9xy}\right] \\
&+\frac{1}{27\alpha} - \frac{2y\ddot{y}}{27\alpha\dot{y}^2} + \frac{4\ddot{y}}{27\alpha y}\left(16\alpha^2 + 8\alpha x + x^2\right) - \frac{8\dot{y}}{27y} \\
&+\frac{16}{27x} - \frac{64\alpha\dot{y}}{27xy} + \frac{2x\dot{y}}{27\alpha y} - \frac{y}{27\alpha x\dot{y}} + \frac{2x\ddot{y}}{27\alpha\dot{y}} + \frac{8\ddot{y}}{27\dot{y}}\Bigg]^{\frac{1}{2}},
\end{aligned} \tag{6.5.3}$$

which gives the electric field intensity.

Then substituting (6.5.1) into equation (6.2.13) yields

$$w\dot{w} = F_0\left(K\,w + 1\right). \tag{6.5.4}$$

This equation can be written as

$$\frac{1}{K}\left(1 - \frac{1}{Kw + 1}\right)dw = F_0\;dx, \tag{6.5.5}$$

which is a separable equation. Integrating equation (6.5.5) we obtain

$$\frac{w}{K} - \frac{\ln(1+K\,w)}{K^2} = \int F_0\,dx + C, \tag{6.5.6}$$

and in terms of potential Z, we acquire

$$\frac{\left(Z-\frac{1}{3}-\frac{x}{12\alpha}-\frac{y}{12\alpha\dot{y}}\right)\left(\frac{\dot{y}}{x}\right)^{\frac{2}{3}}}{K} - \frac{\ln\left[1+K\left(\left(Z-\frac{1}{3}-\frac{x}{12\alpha}-\frac{y}{12\alpha\dot{y}}\right)\left(\frac{\dot{y}}{x}\right)^{\frac{2}{3}}\right)\right]}{K^2}$$
$$= \int\left(\frac{\dot{y}}{x}\right)^{\frac{4}{3}}\left[\frac{y}{432\alpha^2\dot{y}} - \frac{y^2\ddot{y}}{216\alpha^2\dot{y}^3} + \frac{\ddot{y}}{\dot{y}}\left(\frac{4}{27}+\frac{2x}{27\alpha}+\frac{x^2}{108\alpha^2}\right)\right.$$
$$+\frac{y}{27\alpha x\dot{y}} - \frac{4}{27x} + \frac{x}{216\alpha^2} + \frac{xy\ddot{y}}{216\alpha^2\dot{y}^2} - \frac{y^2}{432\alpha^2 x\dot{y}^2} - \frac{1}{54\alpha}$$
$$\left.+\frac{y\ddot{y}}{54\alpha\dot{y}^2} - \frac{yE^2}{16\alpha\dot{y}}\right]dx + C. \tag{6.5.7}$$

Thus we have solved the charged condition of pressure isotropy when $F_1 = KF_0$. The integration in (6.5.7) can be completed once a functional form for y is known. *Any choice of the potential y results in an exact solution of the system of charged EGB field equations* (6.2.3). This arises from the fact that we have now included the presence of an electric field while in the uncharged case, given in section 5.5, an additional constraint equation, (5.5.2) must be satisfied to produce exact solutions in closed form.

To illustrate an explicit exact solution we choose an analytic form of y as done in section 6.3. This is given by

$$y = ax^n + l, \tag{6.5.8}$$

where a, l and n are real numbers. As a result the potential Z in (6.5.7) is given implicitly as

$$\frac{\left(Z-\frac{1}{3}-\frac{x}{12\alpha}-\frac{ax^n+l}{12\alpha nax^{n-1}}\right)\left(an\,x^{n-2}\right)^{\frac{2}{3}}}{K}$$
$$-\frac{\ln\left[1+K\left(\left(Z-\frac{1}{3}-\frac{x}{12\alpha}-\frac{ax^n+l}{12\alpha nax^{n-1}}\right)\left(anx^{n-2}\right)^{\frac{2}{3}}\right)\right]}{K^2}$$
$$= \frac{(an)^{\frac{2}{3}}}{K}\left[\frac{x^{\frac{2n-1}{3}}}{6\alpha(1-2n)}\left(\frac{5}{2}-\frac{1}{2n}-n\right) + \frac{lx^{\frac{-(n+1)}{3}}}{6\alpha a(n+1)}\left(\frac{1}{n}-\frac{1}{2}\right)\right.$$
$$\left.+\frac{x^{\frac{2n-4}{3}}}{3}\right] + C. \tag{6.5.9}$$

Inserting (6.5.8) in equation (6.5.3) we get

$$\begin{aligned} E &= \left[\left(\frac{1}{K}\right)\left(\frac{x^{2-n}}{an}\right)^{\frac{2}{3}}\left[\frac{8anx^{n-1}(2-n)}{9(ax^n+l)} - \frac{8}{9x} + \frac{4n}{9x}\right.\right. \\ &\quad \left. + \frac{32\alpha n(2-n)ax^{n-2}}{9(ax^n+l)}\right] - \frac{1}{27\alpha} + \frac{8}{27x} - \frac{(ax^n+l)(2n-1)}{27\alpha anx^n} \\ &\quad + \frac{8n(4n-5)ax^{n-1}}{27(ax^n+l)} + \frac{64\alpha anx^{n-2}(n-2)}{27(ax^n+l)} \\ &\quad \left. + \frac{2anx^n(2n-1)}{27\alpha(ax^n+l)} + \frac{2n}{27}\left(\frac{1}{\alpha} + \frac{4}{x}\right)\right]^{\frac{1}{2}}, \end{aligned} \tag{6.5.10}$$

for the electric field intensity.

6.6 Case IV: $F_1 = Q_1$ and $F_0 = Q_2$

We now set F_1 and F_0 to be arbitrary constants

$$F_1 = Q_1, \tag{6.6.1}$$

$$F_0 = Q_2, \tag{6.6.2}$$

such that

$$Q_1 = \left(\frac{\dot{y}}{x}\right)^{\frac{2}{3}}\left[-\frac{1}{18\alpha} + \frac{y}{36\alpha x\dot{y}} + \frac{x\ddot{y}}{18\alpha\dot{y}} + \frac{2\ddot{y}}{9\dot{y}} - \frac{y\ddot{y}}{36\alpha\dot{y}^2} - \frac{2}{9x}\right], \tag{6.6.3}$$

$$\begin{aligned} Q_2 &= \left(\frac{\dot{y}}{x}\right)^{\frac{4}{3}}\left[\frac{y}{432\alpha^2\dot{y}} - \frac{y^2\ddot{y}}{216\alpha^2\dot{y}^3} + \frac{\ddot{y}}{\dot{y}}\left(\frac{4}{27} + \frac{2x}{27\alpha} + \frac{x^2}{108\alpha^2}\right)\right. \\ &\quad + \frac{y}{27\alpha x\dot{y}} - \frac{4}{27x} + \frac{x}{216\alpha^2} + \frac{xy\ddot{y}}{216\alpha^2\dot{y}^2} - \frac{y^2}{432\alpha^2 x\dot{y}^2} - \frac{1}{54\alpha} \\ &\quad \left. + \frac{y\ddot{y}}{54\alpha\dot{y}^2} - \frac{yE^2}{16\alpha\dot{y}}\right]. \end{aligned} \tag{6.6.4}$$

Equation (6.2.13) is now written as

$$w\dot{w} = wQ_1 + Q_2, \tag{6.6.5}$$

and upon integration, we obtain the solution

$$\frac{w}{Q_1} - \frac{Q_2\ln(wQ_1+Q_2)}{Q_1^2} = x + C. \tag{6.6.6}$$

The use of equation (6.2.11) then produces

$$\frac{\left(Z-\frac{1}{3}-\frac{x}{12\alpha}-\frac{y}{12\alpha\ddot{y}}\right)\left(\frac{\dot{y}}{x}\right)^{\frac{2}{3}}}{Q_1}-\frac{Q_2\ln\left(\left[\left(Z-\frac{1}{3}-\frac{x}{12\alpha}-\frac{y}{12\alpha\ddot{y}}\right)\left(\frac{\dot{y}}{x}\right)^{\frac{2}{3}}\right]Q_1+Q_2\right)}{Q_1^2}$$
$$= x + C. \tag{6.6.7}$$

The quantity Z and the corresponding gravitational potential λ can be found implicitly for this particular case, provided that the potential y satisfies the constraints in equations (6.6.3) and (6.6.4) simultaneously.

6.7 Other exact models

Several other families of exact solutions are possible to the condition of pressure isotropy for charged matter. In the remainder of this chapter, we present certain solutions to the master equation given by equation (6.2.4) by transforming it to an equivalent form. To demonstrate this we make the choice

$$Z = a, \tag{6.7.1}$$

where a is constant. Then equation (6.2.4) reduces to

$$\left[4ax\left(4\alpha\left(1-a\right)+x\right)\right]\ddot{y}+\left[16\alpha a\left(a-1\right)\right]\dot{y}+\left[2-2a-\frac{3}{2}E^2x\right]y \quad = \quad 0. \tag{6.7.2}$$

This equation can be transformed to

$$z(z-1)\frac{d^2y}{dz^2}-\frac{dy}{dz}+\left[\frac{4-4a-3E^2A\left(z-1\right)}{8a}\right]y \quad = \quad 0, \tag{6.7.3}$$

where we have set

$$z=\frac{x+A}{A},\ A=4\alpha\left(1-a\right). \tag{6.7.4}$$

Equation (6.7.3) can be solved if E^2 is specified in particular cases. Below we present three families of exact solutions which have a simple analytic form. Clearly other choices of E^2 can be made, and this will be dictated by the physical requirements for a well behaved gravitating sphere in EGB gravity.

6.7.1 Case I

The isotropic pressure condition (6.7.3) for this case has the form

$$z\,(z-1)\,\frac{d^2y}{dz^2}-\frac{dy}{dz} \;=\; 0. \tag{6.7.5}$$

This equation is classified as a second order linear ordinary differential equation. It can be solved by reducing the order of the equation, i.e. if we substitute

$$\tilde{T}(z)=\frac{dy}{dz}. \tag{6.7.6}$$

where $\tilde{T}(z)$ is a function of z. Then (6.7.5) now takes on the form

$$\frac{d\tilde{T}(z)}{dz} \;=\; \frac{\tilde{T}(z)}{z\,(z-1)}. \tag{6.7.7}$$

Equation (6.7.7) is identified as a separable ordinary differential equation and can be integrated with the use of partial fractions. This equation can then be written as

$$\frac{d\tilde{T}(z)}{\tilde{T}(z)} \;=\; \int\left(-\frac{1}{z}+\frac{1}{z-1}\right)dz. \tag{6.7.8}$$

The electric field intensity is given by

$$E^2 \;=\; \left[\frac{4\,(1-a)}{12\alpha(1-a)\,(z-1)}\right]. \tag{6.7.9}$$

The potentials are expressed as

$$y \;=\; D_2+D_1\,(z-\ln z)\,, \tag{6.7.10}$$

$$Z \;=\; a. \tag{6.7.11}$$

6.7.2 Case II

The isotropic pressure condition (6.7.3) becomes

$$z\,(z-1)\,\frac{d^2y}{dz^2}-\frac{dy}{dz}+\tilde{B}y \;=\; 0. \tag{6.7.12}$$

Electric field intensity reads as

$$E^2 \;=\; \left[\frac{4\left(1-a-2a\tilde{B}\right)}{12\alpha\,(1-a)\,(z-1)}\right]. \tag{6.7.13}$$

The potentials in this case can be written as

$$\begin{aligned}
y &= C_1\left[\sum_{n=1}^{\infty}\frac{2a_0(-1)^n}{A^n}\frac{1}{n!(n+2)!}\prod_{j=1}^{n}\left(j\left(j+1\right)+\tilde{B}\right)\left(A\left(z-1\right)\right)^{n+2}\right] \\
&\quad +C_2\left[\mu\left(\sum_{n=1}^{\infty}\frac{2a_0(-1)^n}{A^n}\frac{1}{n!(n+2)!}\prod_{j=1}^{n}\left(j\left(j+1\right)+\tilde{B}\right)\left(A\left(z-1\right)\right)^{n+2}\right)\right. \\
&\quad \left.\times\ln\big(A\left(z-1\right)\big)+\sum_{n=0}^{\infty}b_n\big(A\left(z-1\right)\big)^n\right], \qquad n\geq 1, \qquad (6.7.14) \\
Z &= a. \qquad (6.7.15)
\end{aligned}$$

Here the potential y is a series solution. In the above a_0, A, μ and $\tilde{B}$ are all constants. We can make the observation that by setting the electric field to zero, that is $E = 0$, then the constant $\tilde{B} = B$ as expected for neutral matter in section 4.3.4. If $\tilde{B} = -m$, an integer then with $C_2 = 0$ the series (6.7.14) terminates and we obtain polynomial functions.

6.7.3 Case III

The isotropy condition equation takes on the form

$$-M+\frac{Mz+g}{4a}\left[2-2a-\frac{3}{2}E^2A\left(z-1\right)\right] = 0, \qquad (6.7.16)$$

where M and g represent constants. The electric field intensity is found to be

$$E^2 = \left[\left(\frac{2}{3}\left(2-2a\right)-\frac{8Ma}{3\left(Mz+g\right)}\right)\frac{1}{A\left(z-1\right)}\right]. \qquad (6.7.17)$$

The metric potentials y and Z are presented by

$$y = Mz+g, \qquad (6.7.18)$$

$$Z = a. \qquad (6.7.19)$$

Chapter 7

Shear-free fluids in higher dimensions

7.1 Introduction

Shear-free radiating spacetimes are important in the modelling process for the interior of relativistic stars. The heat flows outward from the hot centre to the surface of the star. Models containing heat flow in the field of astrophysics are applied to problems of gravitational collapse, thermodynamic processes at the stellar surface, and the formation of singularities. Investigations of shear-free stellar models in the presence of heat flux were conducted by Deng (1989), Wagh *et al* (2001), Herrera *et al* (2006) and Abebe *et al* (2015), to name a few. In an earlier treatment Kolassis *et al* (1988) obtained the first exact solution for a radiating star with the effects of dissipation, zero shear and nonzero heat that travels along geodesics. Deng (1989) developed a general method to determine solutions to the Einstein field equations with heat flux and obtained new classes of solutions in the process. On similar grounds, Wagh *et al* (2001) reported solutions by imposing a barotropic equation of state. Herrera *et al* (2006) generated analytical solutions to Einstein's equations describing spherically radiating collapsing spheres in the diffusion approximation with zero shear. Abebe *et* *al* (2015) performed an analysis on the junction condition, relating the radial pressure with heat flow of a shear-free radiating model with anisotropic pressure, using the Lie group theoretic approach, and presented several new exact solutions. Higher dimensions also play a role in astrophysical processes as shown by Brassel *et al* (2021) and Maharaj and Brassel (2021). It is therefore important to study shear-free relativistic fluids in a higher dimensional setting in general relativity. In section 7.2 we present the Einstein field equations in N dimensions. We consider the matter distribution to be a shear-free heat conducting fluid. In section 7.3 we show that the works of Brassel *et al* (2015) can be extended to arbitrary dimensions. In section 7.4 a generalisation of the study developed by Govender *et al* (2018) is presented.

7.2 Radiating spacetime

The line element for a spherically symmetric spacetime in the absence of shear in N dimensions reads as

$$ds^2 = -A^2 dt^2 + B^2 \left(dr^2 + r^2 d\Omega^2_{N-2}\right), \tag{7.2.1}$$

where $A = A(r,t)$ and $B = B(r,t)$ represent the metric functions. The $(N-2)$-sphere, as before, is denoted by

$$\begin{aligned} d\Omega^2_{N-2} &= d\theta_1^2 + \sin^2(\theta_1) d\theta_2^2 + \sin^2(\theta_1)\sin^2(\theta_2) d\theta_3^2 \\ &\quad + \cdots + \sin^2(\theta_1)\sin^2(\theta_2)\sin^2(\theta_3)\cdots\sin^2(\theta_{N-3}) d\theta^2_{N-2} \\ &= \sum_{i=1}^{N-2}\left[\prod_{j=1}^{i-1}\sin^2(\theta_j)\right](d\theta_i)^2. \end{aligned} \tag{7.2.2}$$

The nonzero Ricci components for the spacetime (7.2.1) are given by

$$\begin{aligned} R^0{}_0 &= \frac{1}{AB}\left[\frac{(N-1)\ddot{B}}{A} - \frac{(N-1)\dot{A}\dot{B}}{A^2} - \frac{A''}{B} - \frac{(N-3)A'B'}{B^2} \right. \\ &\quad \left. - \frac{(N-2)A'}{rB}\right], \end{aligned} \tag{7.2.3a}$$

$$R^0{}_1 = \frac{(N-2)}{AB}\left[\frac{\dot{B}'}{A} - \frac{\dot{B}A'}{A^2} - \frac{\dot{B}B'}{AB}\right], \tag{7.2.3b}$$

$$
\begin{aligned}
R^1{}_1 &= \frac{\ddot{B}}{A^2 B} - \frac{\dot{B}\dot{A}}{A^3 B} + \frac{(N-2)\,\dot{B}^2}{A^2 B^2} - \frac{A''}{AB^2} - \frac{(N-2)\,B''}{B^3} \\
&\quad + \frac{A'B'}{AB^3} + \frac{(N-2)\,B'^2}{B^4} - \frac{(N-2)\,B'}{rB^3}, && (7.2.3c) \\
R^2{}_2 &= \frac{1}{AB}\left[\frac{\ddot{B}}{A} - \frac{\dot{A}\dot{B}}{A^2} + \frac{(N-2)\,\dot{B}^2}{AB}\right] \\
&\quad + \frac{1}{B^2}\left[(5-2N)\,\frac{B'}{rB} - \frac{A'}{rA}\right. \\
&\quad \left. - \frac{(N-4)\,B'^2}{B^2} - \frac{B''}{B} - \frac{A'B'}{AB}\right], && (7.2.3d) \\
R^{N-1}{}_{N-1} &= R^{N-2}{}_{N-2} = \cdots = R^2{}_2, && (7.2.3e)
\end{aligned}
$$

where dots represent differentiation with respect to the time coordinate t and the primes represent differentiation with respect to radial coordinate r. The Ricci scalar is written as

$$
\begin{aligned}
R &= 2\left[\frac{(N-1)\,\ddot{B}}{A^2 B} + \frac{(N-2)\,(N-1)\,\dot{B}^2}{2A^2 B^2} - \frac{(N-1)\,\dot{A}\dot{B}}{A^3 B}\right. \\
&\quad - \frac{(N-2)\,B''}{B^3} - \frac{A''}{AB^2} - \frac{(N-5)\,(N-2)\,B'^2}{2B^4} \\
&\quad \left. - \frac{(N-3)\,A'B'}{AB^3} - \frac{(N-2)^2\,B'}{rB^3} - \frac{(N-2)\,A'B'}{rAB^2}\right]. && (7.2.4)
\end{aligned}
$$

As a result, the nonvanishing Einstein tensor components are generated as

$$
\begin{aligned}
G^0{}_0 &= (N-2)\left[\frac{B''}{B^3} + \frac{(N-2)\,B'}{rB^3} + \frac{(N-5)\,B'^2}{2B^4}\right. \\
&\quad \left. - \frac{(N-1)\,\dot{B}^2}{2A^2 B^2}\right], && (7.2.5a) \\
G^0{}_1 &= \frac{(N-2)}{AB}\left[\frac{\dot{B}'}{A} - \frac{\dot{B}A'}{A^2} - \frac{\dot{B}B'}{AB}\right], && (7.2.5b) \\
G^1{}_1 &= \frac{(N-2)}{A^2}\left[\frac{\dot{A}\dot{B}}{AB} - \frac{\ddot{B}}{B} - \frac{(N-3)}{2}\frac{\dot{B}^2}{B^2}\right] \\
&\quad + \frac{(N-2)}{B^2}\left[\frac{(N-3)}{2}\frac{B'^2}{B^2} + \frac{A'B'}{AB} + \frac{A'}{rA} + (N-3)\,\frac{B'}{rB}\right], && (7.2.5c) \\
G^2{}_2 &= \frac{(N-3)\,B''}{B^3} + \frac{A''}{AB^2} + \frac{(N-3)\,A'}{rAB^2} + \frac{(N-3)^2\,B'}{rB^3} \\
&\quad + \frac{(N-3)\,(N-6)\,B'^2}{2B^4} + \frac{(N-4)\,A'B'}{AB^3} \\
&\quad - \frac{(N-2)}{A^2}\left[\frac{\ddot{B}}{B} + \frac{(N-3)}{2}\frac{\dot{B}^2}{B^2} - \frac{\dot{A}\dot{B}}{AB}\right], && (7.2.5d)
\end{aligned}
$$

$$G^{N-1}{}_{N-1} = G^{N-2}{}_{N-2} = \cdots = G^{2}{}_{2}. \tag{7.2.5e}$$

The matter distribution is described by a shear-free imperfect fluid and the fluid N-velocity is given by $u^a = \frac{1}{A}\delta^a{}_0$. The heat flux $\boldsymbol{q}$ satisfies the condition $u^a q_a = 0$ so that $q^a = \frac{1}{B} q \delta^a{}_1$. The nonzero matter components read

$$T^0{}_0 = -\rho, \tag{7.2.6a}$$

$$T^0{}_1 = \frac{qB}{A}, \tag{7.2.6b}$$

$$T^1{}_1 = p_{\parallel}, \tag{7.2.6c}$$

$$T^2{}_2 = p_{\perp}, \tag{7.2.6d}$$

$$T^{N-1}{}_{N-1} = T^{N-2}{}_{N-2} = \cdots = T^2{}_2. \tag{7.2.6e}$$

We now equate the matter components and curvature components to obtain the field equations in N dimensions in the form

$$\kappa_N \rho = (N-2)\left[\frac{(N-1)\dot{B}^2}{2A^2B^2} - \frac{B''}{B^3} - \frac{(N-2)B'}{rB^3} - \frac{(N-5)B'^2}{2B^4}\right], \tag{7.2.7a}$$

$$\begin{aligned}\kappa_N p_{\parallel} &= \frac{(N-2)}{A^2}\left[\frac{\dot{A}\dot{B}}{AB} - \frac{\ddot{B}}{B} - \frac{(N-3)}{2}\frac{\dot{B}^2}{B^2}\right], \\ &\quad + \frac{(N-2)}{B^2}\left[\frac{(N-3)}{2}\frac{B'^2}{B^2} + \frac{A'B'}{AB} + \frac{A'}{rA} + (N-3)\frac{B'}{rB}\right],\end{aligned} \tag{7.2.7b}$$

$$\begin{aligned}\kappa_N p_{\perp} &= \frac{(N-3)B''}{B^3} + \frac{A''}{AB^2} + \frac{(N-3)A'}{rAB^2} + \frac{(N-3)(N-6)B'^2}{2B^4} \\ &\quad + \frac{(N-3)^2 B'}{rB^3} + \frac{(N-4)A'B'}{AB^3} \\ &\quad - \frac{(N-2)}{A^2}\left[\frac{\ddot{B}}{B} + \frac{(N-3)}{2}\frac{\dot{B}^2}{B^2} - \frac{\dot{A}\dot{B}}{AB}\right],\end{aligned} \tag{7.2.7c}$$

$$\kappa_N q = \frac{N-2}{AB}\left[\frac{\dot{B}'}{B} - \frac{\dot{B}A'}{AB} - \frac{\dot{B}B'}{B^2}\right]. \tag{7.2.7d}$$

These equations are a system of highly nonlinear coupled partial differential equations that describe the dynamics and evolution of shear-free gravitating fluids with heat flow. When $N = 4$ we obtain the equations considered by Brassel *et al* (2015) and Govender *et al* (2018). We can also observe that the dimension N affects the curvature and dynamics quite profoundly. The appearance of the $(N-5)$ and $(N-6)$ terms implies that certain terms in the field equations vanish in dimensions five and six.

The assumption of isotropic pressure ($p_{\parallel} = p_{\perp} = p$) gives the rise to the condition

$$\frac{A''}{A} + \frac{B''}{B} = \left(\frac{2B'}{B} + \frac{1}{r}\right)\left(\frac{A'}{A} + (N-3)\frac{B'}{B}\right), \tag{7.2.8}$$

where we have equated (7.2.7b) and (7.2.7c). This equation governs the gravitational behaviour of shear-free fluids with nonvanishing heat in a radiating spacetime of N dimensions. It necessary to solve this equation and find exact solutions in order to study the dynamics of the model.

In general it is difficult to solve equation (7.2.8) due to its nonlinear nature. Therefore we should reduce it to a simpler form, or an alternative form, that could lead to exact solutions. Interestingly it contains derivatives with respect to the radial coordinate r only, and there is no explicit dependence on the time coordinate t so equation (7.2.8) can be treated as an ordinary differential equation. It is thus suitable to introduce the new variable

$$x = r^2. \tag{7.2.9}$$

The isotropic pressure condition (7.2.8) can now be expressed as

$$\left(\frac{1}{B}\right) A_{xx} + 2A_x \left(\frac{1}{B}\right)_x - (N-3)\, A \left(\frac{1}{B}\right)_{xx} = 0, \tag{7.2.10}$$

where subscripts represent differentiation with respect to the variable x. This equation is our master equation and is a very useful form of the isotropic pressure condition because we can make appropriate choices for one potential in order to determine the other. We observe that equation (7.2.10) is a linear ordinary differential equation in A if we specify a form for the variable $\frac{1}{B}$ and vice versa. When $N = 4$ we regain the four dimensional pressure isotropy equation as found by Brassel *et al* (2015) and Govender *et al* (2018).

7.3 BMG solutions in N dimensions

In this section we present several exact solutions to the isotropic pressure condition equation (7.2.10). These solutions extend the Brassel *et al* (2015) models to a higher dimensional regime.

7.3.1 Solution I: $B = \alpha x^{\beta n}$

We let the potential B take the form

$$B(x,t) = \alpha x^{\beta n}, \tag{7.3.1}$$

where $\alpha = \alpha(t)$, $\beta = \beta(t)$ and $n \in \mathbb{R}$, for which equation (7.2.10) becomes

$$x^2 A_{xx} - (2\beta n) x A_x - (N-3)\beta n(\beta n + 1)A \;=\; 0. \tag{7.3.2}$$

We identify this equation as a second order Euler-Cauchy differential equation in the variable A.

It can be solved using the standard procedure of setting $A = x^m$. We then obtain the characteristic equation

$$m^2 - (2\beta n + 1)\, m - (N-3)\left(\beta^2 n^2 + \beta n\right) \;=\; 0, \tag{7.3.3}$$

that has roots

$$m \;=\; \frac{1}{2}\left[(2\beta n + 1) \pm \sqrt{1 + (4N-8)\left(\beta^2 n^2 + \beta n\right)}\right]. \tag{7.3.4}$$

Hence the solution to (7.3.2) is given by

$$\begin{aligned} A(x,t) \;&=\; \tau(t) x^{\frac{1}{2}\left[(2\beta n+1)+\sqrt{1+(4N-8)(\beta^2 n^2+\beta n)}\right]} \\ &\quad + \chi(t) x^{\frac{1}{2}\left[(2\beta n+1)-\sqrt{1+(4N-8)(\beta^2 n^2+\beta n)}\right]}, \end{aligned} \tag{7.3.5}$$

where $\tau(t)$ and $\chi(t)$ are functions of integration. In terms of the original variable r, we get

$$A(r,t) \;=\; \tau(t) r^{(2\beta n+1)+\sqrt{1+(4N-8)(\beta^2 n^2+\beta n)}}$$

$$+\chi(t)r^{(2\beta n+1)-\sqrt{1+(4N-8)(\beta^2 n^2+\beta n)}}, \quad (7.3.6)$$

for the potential A.

7.3.2 Solution II: $B^{-1} = \alpha k^{\beta x+\gamma}$

We now choose an exponential form for potential B such that

$$\frac{1}{B} = \alpha k^{\beta x+\gamma}, \quad (7.3.7)$$

where $\alpha = \alpha(t)$, $\beta = \beta(t)$, $\gamma = \gamma(t)$ and $k \,\epsilon\, \mathbb{R}$. Then (7.2.10) reduces to

$$A_{xx} + 2\beta\,(\ln k)\, A_x - (N-3)\,\beta^2\,(\ln k)^2\, A = 0, \quad (7.3.8)$$

which is a second order linear ordinary differential equation with constant coefficients.

The corresponding characteristic equation is then expressed as

$$m^2 + 2\,(\ln k)\,\beta m - (N-3)\,(\ln k)^2\beta^2 = 0. \quad (7.3.9)$$

It has roots

$$m = -\beta(\ln k) \pm \beta\ln(k)\sqrt{N-2}. \quad (7.3.10)$$

As a result, the general solution to equation (7.3.8) is given by

$$A(x,t) = \nu(t)k^{\beta(t)\left[-1-\sqrt{N-2}\right]x} + \kappa(t)k^{\beta(t)\left[-1+\sqrt{N-2}\right]x}. \quad (7.3.11)$$

Note that $\nu(t)$ and $\kappa(t)$ are functions of integration. In terms of the original variables (7.3.11) is written as

$$A(r,t) = \nu(t)k^{\beta(t)\left[-1-\sqrt{N-2}\right]r^2} + \kappa(t)k^{\beta(t)\left[-1+\sqrt{N-2}\right]r^2}. \quad (7.3.12)$$

7.3.3 Solution III: Rational function

In this case we make a rational functional choice for potential $1/B$ given by

$$\frac{1}{B} = \frac{\alpha x^2}{\beta x + 1}, \tag{7.3.13}$$

where $\alpha = \alpha(t)$, $\beta = \beta(t)$. Equation (7.2.10) now becomes

$$x^2 (\beta x + 1)^2 A_{xx} + 2x (\beta x + 1) (\beta x + 2) A_x - 2(N-3)A = 0. \tag{7.3.14}$$

Equation (7.3.14) is classified as a second order linear ordinary differential equation with variable coefficients. It is still a nontrivial task to produce solutions to this equation as it is not obvious to find a transformation that could reduce it to a standard form.

We proceed to solve this equation by utilising the following transformation

$$\Omega = i\sqrt{2}\left[\ln(x) - \ln(\beta x + 1)\right], \tag{7.3.15}$$

where $i = \sqrt{-1}$. Equation (7.3.14) is written as

$$\frac{e^{i\sqrt{2}\Omega}}{\left(e^{\frac{i\Omega}{\sqrt{2}}} - \beta\right)^4} A_{xx} + 2e^{\frac{i\Omega}{\sqrt{2}}} \frac{\left(2e^{\frac{i\Omega}{\sqrt{2}}} - \beta\right)}{\left(e^{\frac{i\Omega}{\sqrt{2}}} - \beta\right)^3} A_x - 2A(N-3) = 0. \tag{7.3.16}$$

With the use of the chain rule and after some calculations, equation (7.3.16) reduces to

$$-2A_{\Omega\Omega} + 3i\sqrt{2}A_{\Omega} - 2A(N-3) = 0. \tag{7.3.17}$$

We can observe that this equation is a second order linear ordinary differential equation with constants as coefficients. It can be solved easily. The corresponding characteristic equation is given by

$$2m^2 - 3i\sqrt{2}m + 2(N-3) = 0, \tag{7.3.18}$$

with roots

$$m = \frac{3i}{2\sqrt{2}} \pm \frac{i}{2}\sqrt{8N - 15}. \tag{7.3.19}$$

Equation (7.3.17) then has the solution

$$A(\Omega) \quad = \quad \gamma e^{\left(\frac{3i}{2\sqrt{2}}+\frac{i}{2}\sqrt{8N-15}\right)\Omega} + \varphi e^{\left(\frac{3i}{2\sqrt{2}}-\frac{i}{2}\sqrt{8N-15}\right)\Omega}. \qquad (7.3.20)$$

Therefore the general solution to equation (7.3.14) is expressed by

$$\begin{aligned} A(x,t) \quad = \quad & \gamma(t)\left(\frac{x}{\beta(t)x+1}\right)^{-\frac{1}{2}\left(3+\sqrt{8N-15}\right)} \\ & +\varphi(t)\left(\frac{x}{\beta(t)x+1}\right)^{-\frac{1}{2}\left(3-\sqrt{8N-15}\right)}, \end{aligned} \qquad (7.3.21)$$

in which we substituted for Ω from (7.3.15). Note that $\gamma(t)$ and $\varphi(t)$ are integration functions that depend on time. We also make the observation that a singularity exists when $x = 0$. In terms of original variables we obtain

$$\begin{aligned} A(r,t) \quad = \quad & \gamma(t)\left(\frac{r^2}{\beta(t)r^2+1}\right)^{-\frac{1}{2}\left(3+\sqrt{8N-15}\right)} \\ & +\varphi(t)\left(\frac{r^2}{\beta(t)r^2+1}\right)^{-\frac{1}{2}\left(3-\sqrt{8N-15}\right)}. \end{aligned} \qquad (7.3.22)$$

A remarkable and interesting feature of this case is that the use of a complex transformation given by (7.3.15) leads to the output of a real exact solution for the potential A. This is due to the fact that the complex quantity i has the property that its multiplicative inverse is its additive inverse and the fact that its square is a real number. The metric functions are expressed in terms of real functions only. If we set $N = 4$, we regain the potential A that was found by Brassel *et al* (2015).

7.3.4 Solution IV: $B^{-1} = \alpha(t)A^{\beta(t)}$

We now consider the dependence of the metric potentials by coupling A and B together (as done by Brassel (2014)) such that

$$\frac{1}{B} \quad = \quad \alpha(t)A^{\beta(t)}. \qquad (7.3.23)$$

Equation (7.2.10) becomes

$$(1-\beta(N-3))\,AA_{xx} + (2\beta-(N-3)\,\beta(\beta-1))\,A_x^2 \quad = \quad 0. \qquad (7.3.24)$$

This equation only contains the potential A and its derivatives with respect to x. It is a highly nonlinear second order ordinary differential equation. However it can be rewritten as

$$\frac{dA_x^2}{A_x^2} = \frac{2\left(\left(N-3\right)\beta\left(\beta-1\right)-2\beta\right)}{\left(1-\beta\left(N-3\right)\right)}\frac{dA}{A}, \tag{7.3.25}$$

The above equation is now a separable differential equation. Integrating we get

$$\frac{dA}{dx} = \eta A^{\frac{((N-3)\beta(\beta-1)-2\beta)}{(1-\beta(N-3))}}. \tag{7.3.26}$$

We observe that equation (7.3.24) now reduces to a first order linear ordinary differential equation with the solution

$$A(x,t) = \left[\frac{\left(N-3\right)\beta(t)^2-2\beta-1}{\left(\left(N-3\right)\beta(t)-1\right)}\left(\xi(t)x+\tau(t)\right)\right]^{\frac{((N-3)\beta(t)-1)}{(N-3)\beta(t)^2-2\beta-1}}, \tag{7.3.27}$$

where $\xi(t)$ and $\tau(t)$ are integration functions with dependence on time.

It is interesting to note that when $N=4$, we obtain four dimensional solutions as described by Brassel (2014).

7.4 GBM solutions in N dimensions

In this section we present a particular type of solution for the choice of the potential B. This extends the Govender *et al* (2018) models to N dimensions. We generate a solution for A in higher dimensions for three different cases and if we set $N=4$, the four dimensional case, we obtain the potentials A as described in Govender *et al* (2018).

7.4.1 Solution V: $1/B = \alpha(a+bx)^k$

We set

$$\frac{1}{B} = \alpha\left(a+bx\right)^k, \tag{7.4.1}$$

where $a, b \neq 0$, α is a function of time and $k \neq 0$ is a real parameter. As a result equation (7.2.10) now reduces to

$$(a+bx)^2 A_{xx} + 2bk(a+bx) A_x - b^2 k(k-1)(N-3) A = 0. \tag{7.4.2}$$

We introduce the new dependent variable

$$z = a + bx. \tag{7.4.3}$$

Therefore, equation (7.4.2) can be written as

$$z^2 \tilde{A}_{zz} + 2kz\tilde{A}_z - k(k-1)(N-3)\tilde{A} = 0, \tag{7.4.4}$$

where $\tilde{A} = \tilde{A}(z,t)$. We can identify this equation as a second order Euler-Cauchy ordinary differential equation.

It can be solved by making the substitution

$$\tilde{A} = z^m. \tag{7.4.5}$$

The associated characteristic equation then reads

$$m^2 + (2k-1)m - \left(k^2 - k\right)(N-3) = 0, \tag{7.4.6}$$

with roots

$$m = \frac{(1-2k)}{2} \pm \frac{\sqrt{8k - 8k^2 + 1 + 4kN(k-1)}}{2}. \tag{7.4.7}$$

We obtain three cases of solutions depending on $8k - 8k^2 + 1 + 4kN(k-1)$ which could be positive, negative or zero.

Case I: Repeated roots

We acquire repeated roots if

$$8k - 8k^2 + 1 + 4kN(k-1) = 0, \tag{7.4.8}$$

with values of k being

$$k = \frac{1}{2} \pm \frac{\sqrt{6 - 5N + N^2}}{2(N-2)}, \tag{7.4.9}$$

for $N \geq 4$. Consequently we get

$$m_1 = m_2 = \frac{1}{2} - k. \tag{7.4.10}$$

The solution that satisfies equation (7.4.4) is thus

$$\tilde{A} = [c + d \ln z] \, z^{\frac{(1-2k)}{2}}. \tag{7.4.11}$$

In terms of the variable x we obtain

$$A(x,t) = [c(t) + d(t) \ln(a + bx)] \, (a + bx)^{\frac{(1-2k)}{2}}, \tag{7.4.12}$$

where $c(t)$ and $d(t)$ are integration functions of time. It is interesting to note that this potential contains no dependence on dimensions N.

Case II: Real distinct roots

If k lies within the range

$$\frac{1}{2} + \frac{\sqrt{6 - 5N + N^2}}{2(N-2)} < k < \frac{1}{2} - \frac{\sqrt{6 - 5N + N^2}}{2(N-2)}, \tag{7.4.13}$$

we obtain distinct and real roots to equation (7.4.6). Thus, the solution to equation (7.4.4) is

$$\begin{aligned} \tilde{A}(z) &= c z^{\left[(1-2k)+\sqrt{8k-8k^2+1+4kN(k-1)}\right]/2} \\ &\quad + d z^{\left[(1-2k)-\sqrt{8k-8k^2+1+4kN(k-1)}\right]/2}. \end{aligned} \tag{7.4.14}$$

In terms of the original variables

$$\begin{aligned} A(x,t) &= c(t) \, (a + bx)^{\left[(1-2k)+\sqrt{8k-8k^2+1+4kN(k-1)}\right]/2} \\ &\quad + d(t) \, (a + bx)^{\left[(1-2k)-\sqrt{8k-8k^2+1+4kN(k-1)}\right]/2}. \end{aligned} \tag{7.4.15}$$

Note that the range of k in (7.4.13) corrects the mistake in Govender *et al* (2018) when $N = 4$.

Case III: Complex roots

If k lies in the range

$$\frac{1}{2}-\frac{\sqrt{6-5N+N^2}}{2\left(N-2\right)}<k<\frac{1}{2}+\frac{\sqrt{6-5N+N^2}}{2\left(N-2\right)}, \tag{7.4.16}$$

then we obtain complex roots for m. The general solution to (7.4.4) is then written as

$$\begin{aligned}\tilde{A} &= z^{\frac{(1-2k)}{2}}\left[c\cos\left(\left(\sqrt{8k^2-8k-1-4kN\left(k-1\right)}\right)\left(\frac{\ln z}{2}\right)\right)\right.\\ &\left.\quad +d\sin\left(\left(\sqrt{8k^2-8k-1-4kN\left(k-1\right)}\right)\left(\frac{\ln z}{2}\right)\right)\right],\end{aligned} \tag{7.4.17}$$

where c and d are integration functions that depend on time and in terms of original variables x we get

$$\begin{aligned}A(x,t) &= (a+bx)^{\frac{(1-2k)}{2}}\left[c(t)\cos\left(\left(\sqrt{8k^2-8k-1-4kN\left(k-1\right)}\right)\right.\right.\\ &\quad\times\left.\left(\frac{\ln\left(a+bx\right)}{2}\right)\right)+d(t)\sin\left(\left(\sqrt{8k^2-8k-1-4kN\left(k-1\right)}\right)\right.\\ &\quad\times\left.\left.\left(\frac{\ln\left(a+bx\right)}{2}\right)\right)\right].\end{aligned} \tag{7.4.18}$$

Note that this potential A and (7.4.16) corrects the mistakes in Govender *et al* (2018).

Chapter 8

Conclusion

In this project we have studied spherically symmetric models in general relativity and EGB gravity. We investigated a perfect fluid with isotropic pressure in a five dimensional static spacetime in EGB gravity. This analysis was done for both neutral and charged fluid distributions. The EGB field equations for such matter distributions were generated. We showed that the EGB field equations in standard coordinates are highly nonlinear due to the addition of higher order curvature corrections. The consistency condition arising from the isotropy of fluid pressure, for the potentials y and Z, was analysed. We showed that it is an Abelian differential equation of the second kind. A coordinate transformation was introduced to reduce this differential equation to a simpler form. Three classes of new solutions were determined for the potential Z. Each model has to satisfy an additional constraint equation for the potential y. Remarkably, when a charged matter distribution is considered, an additional constraint equation for y is not required, and any form of y can generate an exact solution. Furthermore we have studied a shear-free imperfect fluid distribution with nonzero heat flux in higher dimensional general relativity. The Einstein field equations in this regime were generated. As a result dimensions play a significant role on the curvature and dynamics of these equations. We have found several new exact classes of solutions to the transformed isotropic pressure condition equation. These results contain the four dimensional known exact solutions.

We now provide an overview of what has been investigated:

- In chapter 2, we briefly introduced the relevant aspects and definitions in differential geometry and general relativity required for subsequent chapters. A description for relativistic fluids in both neutral and charged cases were provided. We also provided the general definitions required for studying models in EGB gravity. The field equations in general relativity and EGB gravity were developed.

- In chapter 3 we generated the field equations for four dimensional and N dimensional Einstein gravity in static spherically symmetric spacetimes. Furthermore we evaluated the five and six dimensional interior spacetime metrics in EGB gravity. The associated field equations were also produced.

- In chapter 4, an analysis on the five dimensional EGB field equations for neutral matter was provided. The isotropic pressure condition for such matter is given by

$$\begin{aligned}&\left(-12\alpha x\dot{y}Z+4\alpha x\dot{y}+x^2\dot{y}+xy\right)\dot{Z}-8\alpha\left(x\ddot{y}-\dot{y}\right)Z^2\\&+\left(2x^2\ddot{y}+8\alpha x\ddot{y}-8\alpha\dot{y}-y\right)Z+y=0.\end{aligned} \tag{8.0.1}$$

This equation is classified as an Abelian ordinary differential equation of the second kind in Z. We showed that this equation may be transformed to a canonical differential equation represented by

$$w\dot{w} \;=\; wF_1+F_0, \tag{8.0.2}$$

where the functions F_1 and F_0 depend on y and have the form

$$F_1 = \left(\frac{\dot{y}}{x}\right)^{\frac{2}{3}}\left[-\frac{1}{18\alpha}+\frac{y}{36\alpha x\dot{y}}+\frac{x\ddot{y}}{18\alpha\dot{y}}+\frac{2\ddot{y}}{9\dot{y}}-\frac{y\ddot{y}}{36\alpha\dot{y}^2}-\frac{2}{9x}\right], \tag{8.0.3}$$

$$\begin{aligned}F_0 = {}& \left(\frac{\dot{y}}{x}\right)^{\frac{4}{3}}\left[\frac{y}{432\alpha^2\dot{y}}-\frac{y^2\ddot{y}}{216\alpha^2\dot{y}^3}+\frac{\ddot{y}}{\dot{y}}\left(\frac{4}{27}+\frac{2x}{27\alpha}+\frac{x^2}{108\alpha^2}\right)-\frac{1}{54\alpha}\right.\\&\left.+\frac{y}{27\alpha x\dot{y}}-\frac{4}{27x}+\frac{x}{216\alpha^2}+\frac{xy\ddot{y}}{216\alpha^2\dot{y}^2}-\frac{y^2}{432\alpha^2x\dot{y}^2}+\frac{y\ddot{y}}{54\alpha\dot{y}^2}\right].\end{aligned} \tag{8.0.4}$$

Furthermore a review of known solutions to (8.0.1) and its equivalent forms were also presented.

- In chapter 5, new solutions to equation (8.0.2) were generated by placing restrictions on the functions F_1 and F_0. The following constraints were imposed

(i) $F_0 = 0$,

(ii) $F_1 = 0$,

(iii) $F_1 = KF_0$,

(iv) $F_1 = Q_1$ and $F_0 = Q_2$.

In case (i) the solution to the gravitational potential Z was found explicitly in the form of quadrature. In case (iii) the potential Z was given implicitly. These solutions are subject to a constraint on the metric function y. Interestingly case (ii) yielded two forms of the potential Z in closed form. The constraint equation in this class of models generated analytic forms of the function y. These are given by

$$y = \tilde{Q}\sqrt{x + 4\alpha}, \tag{8.0.5}$$

$$y = \frac{C_1 x^2}{2} + C_2. \tag{8.0.6}$$

As a result the solution to Z is given in terms of elementary functions for each solution of y. We demonstrated that this method leads to new classes of solutions. Moreover the fourth case leads to no viable exact solutions. It is important to note that $\alpha \neq 0$ and the general relativity limit cannot be regained; our classes of exact solutions always contain the higher order curvature corrections due to the Gauss-Bonnet terms.

- Chapter 6 follows the approach of chapter 5 but now with the inclusion of charge. The effects of the electromagnetic field were considered. The isotropic pressure condition with charge reads as

$$\begin{aligned}&\left(-24\alpha x\dot{y}Z + 8\alpha x\dot{y} + 2x^2\dot{y} + 2xy\right)\dot{Z} - 16\alpha\left(x\ddot{y} - \dot{y}\right)Z^2\\&+\left(4x^2\ddot{y} + 16\alpha x\ddot{y} - 16\alpha\dot{y} - 2y\right)Z + 2y - \frac{3}{2}xyE^2 = 0.\end{aligned} \tag{8.0.7}$$

This equation is an Abelian differential equation of the second kind in Z. If a choice for the potentials Z and y are made, then the electric field can be found without integration. However choices for these potentials can lead to models with unphysical behaviour. We observed that if the electric field intensity is specified then the Abelian differential equation has to be integrated. We transformed the master equation (8.0.7) to the following canonical differential equation

$$w\dot{w} = wF_1 + F_0, \tag{8.0.8}$$

where functions F_1 and F_0 now depend on the potential y and the electric field intensity E. They have the forms

$$F_1 = \left(\frac{\dot{y}}{x}\right)^{\frac{2}{3}} \left[-\frac{1}{18\alpha} + \frac{y}{36\alpha x\dot{y}} + \frac{x\ddot{y}}{18\alpha\dot{y}} + \frac{2\ddot{y}}{9\dot{y}} - \frac{y\ddot{y}}{36\alpha\dot{y}^2} - \frac{2}{9x}\right], \tag{8.0.9}$$

$$\begin{aligned} F_0 = {} & \left(\frac{\dot{y}}{x}\right)^{\frac{4}{3}} \left[\frac{y}{432\alpha^2\dot{y}} - \frac{y^2\ddot{y}}{216\alpha^2\dot{y}^3} + \frac{\ddot{y}}{\dot{y}}\left(\frac{4}{27} + \frac{2x}{27\alpha} + \frac{x^2}{108\alpha^2}\right)\right. \\ & + \frac{y}{27\alpha x\dot{y}} - \frac{4}{27x} + \frac{x}{216\alpha^2} + \frac{xy\ddot{y}}{216\alpha^2\dot{y}^2} - \frac{y^2}{432\alpha^2 x\dot{y}^2} - \frac{1}{54\alpha} \\ & \left. + \frac{y\ddot{y}}{54\alpha\dot{y}^2} - \frac{yE^2}{16\alpha\dot{y}}\right]. \end{aligned} \tag{8.0.10}$$

To study the dynamics of this model we imposed restrictions on the functions F_1 and F_0. The four cases that were considered are given by

(i) $F_0 = 0$,

(ii) $F_1 = 0$,

(iii) $F_1 = KF_0$,

(iv) $F_1 = Q_1$ and $F_0 = Q_2$.

We showed that case (i) and case (iii) lead to a general form of E where any choice of y yields new exact solutions for the potential Z. Case (ii) yielded two functional forms for the potential y as demonstrated in the uncharged case and any choice of the electric field E satisfying physical criteria generates a new exact solution for the potential Z. Thus we were able to obtain explicit forms for the potential y and Z in cases (i), (ii) and (iii). In these three cases exact solutions to the EGB field equations follow. These three cases exist only in EGB gravity, and a general relativity limit does not exist for these classes of models. Case (iv) has two constraint equations that have to be satisfied and it is unlikely to provide new classes of solutions.

- Finally, we studied a spherically symmetric shear-free spacetime in chapter 7. The fluid distribution considered was an imperfect fluid with nonvanishing heat flux. The associated Einstein field equations were generated in N dimensions. The assumption of isotropic pressure yielded the condition

$$\left(\frac{1}{B}\right) A_{xx} + 2A_x \left(\frac{1}{B}\right)_x - (N-3)\, A \left(\frac{1}{B}\right)_{xx} \quad = \quad 0. \tag{8.0.11}$$

We obtained several new exact solutions to equation (8.0.11) by specifying the potential $1/B$, in order to determine an integrable equation in the potential A. The following choices were made:

•Solution I

$$\frac{1}{B} \quad = \quad \frac{1}{\alpha(t) x^{\beta(t) n}}, \tag{8.0.12}$$

•Solution II

$$\frac{1}{B} \quad = \quad \alpha(t) k^{\beta(t)x+\gamma(t)}, \tag{8.0.13}$$

•Solution III

$$\frac{1}{B} \quad = \quad \frac{\alpha(t) x^2}{\beta(t)x+1}, \tag{8.0.14}$$

•Solution IV

$$\frac{1}{B} \quad = \quad \alpha(t) A^{\beta(t)}, \tag{8.0.15}$$

•Solution V

$$\frac{1}{B} \quad = \quad \alpha(t)(a+bx)^k. \tag{8.0.16}$$

In each case the condition of pressure isotropy could be solved and the potential A was found explicitly. When $N = 4$ in all of the above cases, we regained the models described by Brassel *et al* (2015) and Govender *et al* (2018). In particular, we demonstrated that the use of a complex transformation given by

$$\Omega \quad = \quad i\sqrt{2}\left[\ln(x) - \ln(\beta x + 1)\right], \tag{8.0.17}$$

led to a real exact solution for the potential A in the class of models provided by the choice of $1/B$ in Solution III. It is interesting to observe that a transformation containing the complex quantity i results in a real form for the potential.

Investigations in this thesis and solutions generated form an integral part of relativistic astrophysics and cosmology that are widely used in the framework of general relativity and EGB gravity to model compact relativistic stars. As a consequence physically meaningful studies can be conducted on such models. We can enhance the investigations in this dissertation by performing a detailed physical analysis on the new exact solutions obtained. In particular a treatment similar to that of Mafa Takisa *et al* (2019) in general relativity can be extended to EGB gravity. The physical admissibility of these solutions will be considered as a future endeavour.

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