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A Generalized of S_λ-I-Convergence of Complex Uncertain Double Sequences

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Abstract
In this paper, we introduce the λ_{I_2}-statistically convergence sequence concepts which are namely λ_{I_2}-statistically convergence almost surely ($S_\lambda(I_2)$ a.s.), λ_{I_2}-statistically convergence in measure, λ_{I_2}-statistically convergence in mean, λ_{I_2}-statistically convergence in distribution and λ_{I_2}-statistically convergence uniformly almost surely ($S_\lambda(I_2)$ u.a.s.). Additionally, decomposition theorems and relationships among them are presented, further, when reciprocal of one theorem is not satisfied, an counterexample is shown to support the result.

Keywords: λ_{I_2}-convergence; uncertainty theory; complex uncertain variable; ideal spaces.

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Una generalización de S_λ-I-convergencia de sucesiones dobles complejas inciertas.

Resumen
En este artículo, se introduce las nociones de sucesiones estadísticamente λ_{I_2}-convergences las cuales son llamadas estadísticamente λ_{I_2}-convergence casi seguro ($S_\lambda(I_2)$ c.s.), estadísticamente λ_{I_2}-convergente en medida, estadísticamente λ_{I_2}-convergente en media, estadísticamente λ_{I_2}-convergente en distribución y estadísticamente λ_{I_2}-convergente uniformemente casi seguro ($S_\lambda(I_2)$ u.c.s.). Adicionalmente, presentamos algunos teoremas y relaciones existentes entre las nociones mencionadas anteriormente, asu vez, cuando el reciproco de un teorema no se satisface, se presenta un contra ejemplo para soportar el resultado.

Palabras clave: λ_{I_2}-convergencia; teoría de la incertidumbre; variable incierta compleja; espacios de ideales.

1 Introduction

Freedman and Sember introduced the concept of a lower asymptotic density and defined the concept of convergence in density, in [1]. Taking this definition, we can give the definition of statistical convergence which has been formally introduced by Fast [2] and Steinhaus [3]. Schoenberg reintroduced this concept independently [4]. A number sequence (x_n) is statistically convergent to L provided that for every $\varepsilon > 0$, $d\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} = 0$ or equivalently there exists a subset $K \subseteq \mathbb{N}$ with $d(K) = 1$ and $m_0(\varepsilon)$ such that $n \geq m_0(\varepsilon)$ and $n \in K$ imply that $|x_n - L| < \varepsilon$. In this case, we write $st\text{-}\lim x_n = L$. From the definition, we can easily show that any convergent sequence is statistically convergent, but not conversely.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$. Mursaleen [5] defined λ-statistical convergence by using the λ sequence. He denoted this new method by S_λ. A number $x = (x_n)$ is said to be λ-statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{n \in I_n : |x_n - L| \geq \varepsilon\}| = 0,$$

where $I_n = [n - \lambda_n + 1, n]$. It is denoted by $st\text{-}\lim x_n = L$. Now, let Λ be the set of all non-decreasing sequences $\lambda = (\lambda_n)$ of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n$ and $\lambda_1 = 1$.

The concept of I-convergence of real sequences is a generalization of statistical convergence which is based on the structure of the ideal I of subsets of the set of natural numbers. P. Kostyrko et al. [6] introduced the concept of I-convergence of sequences in a metric space and studied some properties of this convergence. Later, it was further studied by Salat, Tripathy and Ziman ([7], [8]) and many others. Recently, Das, Savas and Ghosal [9] introduced new notions, namely I-statistical convergence and I-lacunary statistical convergence by using ideal. Then, by using the notions of double sequences introduced by Mursaleen and Edely [10] and I-convergence, Das et al. [11] presented the idea of I-convergence for double sequences via ideals.

In order to deal with belief degree, an uncertainty theory was founded by Liu [12] in 2007, and refined by Liu [13] in 2010 which based on an uncertain measure which satisfies normality, duality, subadditivity, and product axioms. Thereafter, a concept of uncertain variable was proposed to represent the uncertain quantity and a concept of uncertainty distribution to describe uncertain variables. Up to now, uncertainty theory has successfully been applied to uncertain programming (Liu [14], Liu and Chen [15]), uncertain risk analysis and uncertain reliability analysis (Liu [16]), uncertain logic (Liu [17]), uncertain differential equation (Liu [18], Yao and Chen [19]), uncertain graphs (Gao and Gao [20], Zhang and Peng [?]), uncertain calculus (Liu [21]) and uncertain finance (Chen [22], Liu [21], Liu [23]), Omer Kisi ([24, 25, 26]).

In real life, uncertainty not only appears in real quantities but also in complex quantities. In order to model complex uncertain quantities, Peng [27] presented the concepts of complex uncertain variable and complex uncertainty distribution, and also the expected value was proposed to measure a complex uncertain variable in 2012. Since sequence convergence plays an important role in the fundamental theory of mathematics, there are also many convergence concepts in uncertainty theory. In 2007, Liu [12] first introduced convergence in measure, convergence in mean, convergence almost surely (a.s.) and convergence in distribution and their relationships were also discussed. You [28] introduced another type of convergence named convergence uniformly almost surely and showed the relationships among those convergence concepts. Zhang [?] proved some theorems on the convergence of a sequence of uncertain complex variables. After that, Guo and Xu [29] gave the concept of convergence in mean square for a sequence of uncertain complex variables and showed that an uncertain sequence converged in mean square if and only if it was a Cauchy sequence. Tripathy and Kumar [30] introduced statistical convergence of complex uncertain sequence. Aditionally, Das et al. [31] in 2020 introduced and studied the notion of statistical convergence of complex uncertain doubl sequence. On the other hand, Kisi and Guler [32] defined λ-statistically convergence of a sequence of uncertain complex variables. Then, inspired Kisi [33] introduced and studied the convergence concepts of λ_I-statistically convergence of a sequence of uncertain complex variables by using ideal. In this paper, motivate by mentioned above, we make a generalization of λ_I-convergence of complex uncertain variables on double sequences by using ideal spaces and discuss some relations among them.

2 Preliminaries

In this section, some fundamental concepts in uncertainty theory are introduced, which were used throughout the study.

Definition 2.1. (see [6]) A family of sets $I \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

1. $\emptyset \in I$,
2. for each $A, B \in I$ we have $A \cup B \in I$,
3. for each $A \in I$ and each $B \subseteq A$ we have $B \in I$.

Throughout this paper, $I_2 \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ is an ideal on $\mathbb{N} \times \mathbb{N}$.

Definition 2.2. (see [12]) Let $\mathcal{F}$ be a σ-algebra on a non-empty set Γ. A set function M is called an uncertain measure if it satisfies the following axioms:

1. $M\{\Gamma\} = 1$;

2. $M\{\Lambda\} + M\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{F}$;

3. For every countable sequence of $\{\lambda_n\} \in \mathcal{F}$, we have

$$M(\bigcup_{n=1}^{\infty} \lambda_n) \leq \sum_{n=1}^{\infty} M\{\lambda_n\}.$$

The triplet $(\Gamma, \mathcal{F}, M)$ is called an uncertainty space, and each element Λ in $\mathcal{F}$ is called an event.

Definition 2.3. (see [12]) An uncertain variable ξ is a measurable function from an uncertainty space $(\Gamma, \mathcal{F}, M)$ to the set of real numbers, i.e. for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma : \xi(\gamma) \in B\}$$

is an event.

Remark 2.1. A uncertainty distribution Φ of uncertain variable ξ is defined by $\Phi(x) = M\{\xi \leq x\}$, $\forall x \in \mathbb{R}$.

Definition 2.4. The double uncertain sequence $\{\zeta_{n,m}\}$ is said to be convergent almost surely (a.s.) to ζ if there exists an event Λ with $M(\Lambda) = 1$ such that

$$\lim_{n,m\to\infty} |\zeta_{n,m}(\gamma) - \zeta(\gamma)| = 0,$$

for every $\gamma \in \Lambda$. In this case we write $\zeta_{n,m} \to \zeta$, a.s.

Definition 2.5. The double uncertain sequence $\{\zeta_{n,m}\}$ is said to be convergent in measure to ζ if

$$\lim_{n,m\to\infty} M\{|\zeta_{n,m} - \zeta| \geq \varepsilon\} = 0,$$

for every $\varepsilon > 0$.

Definition 2.6. The double uncertain sequence $\{\zeta_{n,m}\}$ is said to be convergent in mean to ζ if

$$\lim_{n,m\to\infty} E[|\zeta_{n,m} - \zeta|] = 0.$$

Definition 2.7. Let $\Phi, \Phi_{n_1,m_1}, \Phi_{n_1,m_2}, ...$ be the double uncertainty distributions of uncertain variables $\zeta, \zeta_{n_1,m_1}, \zeta_{n_1,m_2}, ...$, respectively. We say that the double sequence $\{\zeta_{n,m}\}$ converges in distribution to ζ if

$$\lim_{n,m\to\infty} \|\Phi_{n,m}(x)\| = \Phi(x)$$

for all x which $\Phi(x)$ is continuous.

Definition 2.8. The double uncertain sequence $\{\zeta_{n,m}\}$ is said to be convergent uniformly almost surely (u.a.s.) to ζ if there exists a sequence of events $\{E_{j,k}\}$, $M\{E_{j,k}\} \to 0$ such that $\{\zeta_{n,m}\}$ converges uniformly to ζ in Γ-$E_{j,k}$, for any fixed j, k.

3 Main Results

In this section, we give new concepts and study their certain properties. Additionally, decomposition theorems and relationships among the concepts are discussed. Throughout this paper, $\lambda = (\lambda_n)$ and $\mu = (\mu_m)$ are two non-decreasing sequence of positive numbers tending to ∞ as n, m approach to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$ and $\mu_{m+1} \leq \mu_m + 1, \mu_1 = 1$. We shall denote $\lambda_n\mu_m$ by $\lambda_{n,m}$. Besides, $I_n = [n - \lambda_n + 1, n]$ and $J_m = [m - \mu_m + 1, m]$; and we denote $j \in I_n$ and $k \in J_m$ by $(j,k) \in I_{n,m}$,

Definition 3.1. Let $\{\zeta_{nm}\}$ be complex double uncertain variables The double sequence $\{\zeta_{n,m}\}$ is said to be λ_{I_2}-statistically convergent almost surely ($S_\lambda(I_2)$. a.s.) to ζ if for every $\varepsilon, \delta > 0$ there exists an event Λ with $M(\Lambda) = 1$ such that

$$\{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{n,m}} |\{(j,k) \in I_{nm} : \|\zeta_{j,k}(\gamma) - \zeta(\gamma)\| \geq \varepsilon\}| \geq \delta\} \in I_2,$$

for every $\gamma \in \Gamma$. In this case, we write $\zeta_{n,m} \to \zeta$ ($S_\lambda(I_2)$. a.s.).

Definition 3.2. The double sequence $\{\zeta_{n,m}\}$ is said to be λ_{I_2}-statistically convergent in measure to ζ if

$$\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{n,m}}|\{(j,k)\in I_{nm}:M(\|\zeta_{j,k}-\zeta\|\geq\epsilon)\geq\delta\}|\geq\vartheta\}\in I_2,$$

for every $\varepsilon,\delta,\vartheta>0$.

Definition 3.3. The double sequence $\{\zeta_{n,m}\}$ is said to be statistically convergent in mean to ζ if

$$\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{n,m}}|\{(j,k)\in I_{nm}:E(\|\zeta_{j,k}-\zeta\|)\geq\varepsilon\}|\geq\delta\}\in I_2,$$

for every $\epsilon,\delta>0$.

Definition 3.4. Let $\Phi,\Phi_{n_1,m_1},\Phi_{n_1,m_2},...$ be the double uncertainty distributions of uncertain variables $\zeta,\zeta_{n_1,m_1},\zeta_{n_1m_2},...$, respectively. We say that the complex uncertain double sequence $\{\zeta_{n,m}\}$ λ_{I_2}-statistically converges in distribution to ζ if for every $\varepsilon,\delta>0$,

$$\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{n,m}}|\{(j,k)\in I_{n,m}:\|\Phi_{j,k}(c)=\Phi(c)\|\geq\varepsilon\}|\geq\delta\}\in I_2,$$

for all c at which $\Phi(c)$ is continuous.

Definition 3.5. The double sequence $\{\zeta_{n,m}\}$ is said to be λ_{I_2}-statistically convergent uniformly almost surely ($S_\lambda(I_2)$, u.a.s.) to ζ if for every $\epsilon,\sigma>0,\exists\delta>0,\forall x\in\mathbb{R}$ and a sequence of events $\{E^{'}_{j,k}\}$ such that

$$\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{n,m}}|(j,k)\in I_{n,m}:\|M(E^{'}_{j,k})-0|\geq\varepsilon\}|\geq\sigma\}\in I_2,$$

implies that

$$\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{n,m}}|(j,k)\in I_{n,m}:\|\zeta_{j,k}(x)-\zeta(x)|\geq\delta\}|\geq\sigma\}\in I_2.$$

Definition 3.6. The double sequence $\{\zeta_{,nm}\}$ is said to be λ_{I_2}-statistically convergent to ζ if for every $\epsilon>0$,

$$\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{n,m}}|(j,k)\in I_{n,m}:\|\zeta_{j,k}(\gamma)-\zeta(\gamma)|\geq\varepsilon\}|\geq\delta\}\in I_2,$$

for every $\gamma\in\Gamma$.

Now, we give the relationships among the convergence concepts of double sequence of uncertain complex variables defined above.

Proposition 3.1. If the double sequence $\{\zeta_{n,m}\}$ of complex uncertain variables λ_{I_2}-statistically converges in mean to ζ. Then, $\{zeta_{n,m}\}$ λ_{I_2}-statistically converges in measure to ζ.

Proof. It follows from Markov inequality that for any given $\epsilon, \delta, \vartheta > 0$, we have

$$\{(n,m) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_{n,m}}|\{(j,k) \in I_{n,m} : M(\|\zeta_{j,k}-\zeta\| \geq \varepsilon) \geq \delta\}| \geq \vartheta\} \subseteq$$
$$\{(n,m) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_{n,m}}|\{(j,k) \in I_{n,m} : (\frac{E(\|\zeta_{j,k}-\zeta\|)}{\varepsilon}) \geq \delta\}| \geq \vartheta\} \in I_2.$$

Therefore, $\{\zeta_{n,m}\}$ λ_{I_2}-statistically converges in measure to ζ. □

Converse of above theorem is not true. i.e. λ_{I_2}-statistical convergence in measure does not imply λ_{I_2}-statistical convergence in mean as can be seen in the following example.

Example 3.1. Let's suppose the uncertainty space $(\Gamma, \mathbb{F}, M)$ to be $\gamma_1, \gamma_2, \gamma_3, ...$ with

$$M(\Lambda) = \begin{cases} \sup\limits_{\gamma_{n,m}\in\Lambda} \frac{1}{(n+1)(m+1)} & if \quad \sup\limits_{\gamma_{n,m}\in\Lambda} \frac{1}{(n+1)(m+1)} < 0.5\} \\ 1 - \sup\limits_{\gamma_{n,m}\in\Lambda^c} \frac{1}{(n+1)(m+1)} & if \quad \sup\limits_{\gamma_{n,m}\in\Lambda^c} \frac{1}{(n+1)(m+1)} < 0.5\} \\ 0.5 & otherwise. \end{cases}$$

and the double complex uncertain variables be defined by

$$\zeta_{n,m}(\gamma) = \begin{cases} [(n+1)(m+1)]i & if \quad \gamma = \gamma_{n,m} \\ 0 & otherwise. \end{cases}$$

for $n,m=1,2,3,...$ and $\zeta\equiv 0$. For some small number $\epsilon,\delta,\vartheta>0$ and $n,m\geq 2$, we have

$$\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{nm,}}|\{(j,k)\in I_{n,m}:M(\|\zeta_{j,k}-\zeta\|\geq\varepsilon)\geq\delta\}|\geq\vartheta\}$$

$$=\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{n,m}}|\{(j,k)\in I_{n,m}:M(\gamma:\|\zeta_{j,k}-\zeta\|\geq\varepsilon)\geq\delta\}|\geq\vartheta\}$$

$$=\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{n,m}}|\{(j,k)\in I_{n,m}:M\{\gamma_{n,m}\}\geq\delta\}|\geq\vartheta\}\in I_2$$

Therefore, the double sequence $\{\zeta_{n,m}\}$ λ_{I_2}-statistically converges in measure to ζ. Nevertheless, for each $n,m\geq 2$, we have the double uncertainty distribution of uncertain variable $\|\zeta_{n,m}-\zeta\|=\|\zeta_{n,m}\}$ is

$$\Phi_{n,m}(x)=\begin{cases}0 & if \quad x<0;\\ 1-\frac{1}{(n+1)(m+1)} & if \quad 0\leq 1<(n+1)(m+1);\\ 1 & if \quad x\geq(n+1)(m+1).\end{cases}$$

Thus, for each $n,m\geq 2$, and for every $\epsilon,\delta>0$, we have

$$\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{n,m}}|\{(j,k)\in I_{n,m}:E(\|\zeta_{j,k}-\zeta\|-1)\geq\varepsilon\}|\geq\delta\}$$

$$=\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{n,m}}|(j,k)\in I_{n,m}:$$
$$([\int_0^{(n+1)(m+1)}1-(1-\frac{1}{(n+1)(m+1)})dx]-1)\geq\varepsilon\}|\geq\delta\},$$

which is correct. Hence, the double sequence $\{\zeta_{n,m}\}$ does not λ_{I_2}-statistically converge in mean to ζ.

Proposition 3.2. Let $\{\zeta_{n,m}\}$ be a double sequence of complex uncertain variables with real part $\{\xi_{n,m}\}$ and imaginary part $\{\gamma_{n,m}\}$, respectively, for $n, m = 1, 2, 3, ...$ If the double sequences $\{\xi_{n,m}\}$ and $\{\gamma_{n,m}\}$ λ_{I_2}-statistically converge in measure to ξ and γ, respectively. Then, the double sequence $\{\zeta_{n,m}\}$ λ_{I_2}-statistically converges in measure to $\zeta = \xi + i\gamma$.

Proof. By definition of λ_{I_2}-statistical convergence in a measure of sequence of double uncertain complex variables that for any small numbers $\varepsilon, \delta, \vartheta > 0$,

$$\{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{n,m}} |\{(j,k) \in I_{n,m} : M(\|\xi_{n,m} - \xi\| \geq \frac{\varepsilon}{\sqrt{2}}) \geq \delta\}| \geq \vartheta\} \in I_2,$$

$$\{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{n,m}} |\{(j,k) \in I_{n,m} : M(\|\gamma_{j,k} - \gamma\| \geq \frac{\varepsilon}{\sqrt{2}}) \geq \delta\}| \geq \vartheta\} \in I_2.$$

We can check $\|\zeta_{n,m} - \zeta\| = \sqrt{|\xi_{j,k} - \xi|^2 + |\gamma_{n,m} - \gamma|^2}$. Therefore, we have

$$\{\|\zeta_{n,m} - \zeta\| \geq \varepsilon\} \subset \{\|\xi_{n,m} - \xi\| \geq \frac{\varepsilon}{\sqrt{2}} \cup \|\gamma_{n,m} - \gamma\| \geq \frac{\varepsilon}{\sqrt{2}}\}.$$

Using the subadditivity axiom of an uncertain measure, we obtain

$$\{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{n,m}} |(j,k) \in I_{n,m} : M(\|\zeta_{j,k} - \zeta\| \geq \varepsilon) \geq \delta\}| \geq \vartheta\}$$

$$\subseteq \{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{n,m}} |(j,k) \in I_{n,m} : M(\|\xi_{j,k} - \xi\| \geq \frac{\varepsilon}{\sqrt{2}}) \geq \delta\}| \geq \vartheta\} \cup$$
$$\{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{n,m}} |(j,k) \in I_{n,m} : M(\|\gamma_{j,k} - \gamma\| \geq \frac{\varepsilon}{\sqrt{2}}) \geq \delta\}| \geq \vartheta\} \in I_2.$$

Therefore, we have

$$\{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{n,m}} |(j,k) \in I_{n,m} : M(\|\zeta_{j,k} - \zeta\| \geq \varepsilon) \geq \delta\}| \geq \vartheta\} \in I_2.$$

Hence, we have proved that $\{\zeta_{n,m}\}$ λ_{I_2}-statistically converges in measure to ζ. □

Proposition 3.3. Let $\{\zeta_{n,m}\}$ be a double sequence of complex uncertain variables with real part $\{\xi_{n,m}\}$ and imaginary part $\{\gamma_{n,m}\}$, respectively, for $n, m = 1, 2, 3, ...$ If the double sequences $\{\xi_{n,m}\}$ and $\{\gamma_{n,m}\}$ λ_{I_2}-statistically converge in measure to ξ and γ, respectively. Then, the double sequence $\{\zeta_{n,m}\}$ λ_{I_2}-statistically converges in distribution to $\zeta = \xi + i\gamma$.

Proof. Let $c = a+ib$ be a given continuity point of the complex uncertainty distribution Φ. Otherwise, for any $\alpha > a, \beta >> b$, we have

$$\{\xi_{n,m} \leq a, \gamma_{n,m} \leq b\} = \{\xi_{n,m} \leq a, \gamma_{n,m} \leq b, \xi \leq \alpha, \gamma \leq \beta\} \cup \{\xi_{n,m} \leq a, \gamma_{n,m} \leq b, \xi > \alpha, \gamma > \beta\} \cup \{\xi_{n,m} \leq a, \gamma_{n,m} \leq b, \xi \leq \alpha, \gamma > \beta\} \cup \{\xi_{n,m} \leq a, \gamma_{n,m} \leq b, \xi > \alpha, \gamma \leq \beta\}$$

$$\subset \{\xi \leq a, \gamma \leq b\} \cup \{|\xi_{n,m} - \xi| \geq \alpha - a\} \cup \{|\gamma_{n,m} - \gamma| \geq \beta - b\}.$$

It follows from subadditivity axiom that

$$\Phi_{n,m}(c) = \Phi_{n,m}(a + ib) \leq \Phi(\alpha + i\beta) + M\{|\xi_{n,m} - \xi| \geq \alpha - a\} + M\{|\gamma_{n,m} - \gamma| \geq \beta - b\}.$$

Since $\{\xi_{n,m}\}$ and $\{\gamma_{n,m}\}$ λ_{I_2}-statistically converge in measure to ξ and γ, respectively. Then, for any small numbers $\varepsilon, \delta > 0$ we have

$$(n, m) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_{n,m}}|\{(j, k) \in I_{n,m} : M(\|\xi_{j,k}-\xi\| \geq \alpha-a) \geq \varepsilon\}| \geq \delta\} \in I_2,$$

$$(n, m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{n,m}}|\{(j, k) \in I_{n,m} : M(\|\gamma_{j,k} - \gamma\| \geq \beta - b) \geq \varepsilon\}| \geq \delta\} \in I_2.$$

Therefore, we get $I_2\text{-}\limsup_{n,m\to\infty} \Phi_{n,m}(c) \leq \Phi(\alpha + i\beta)$ for any $\alpha > a, \beta > b$. Letting $\alpha + i\beta \to a + ib$, we have

$$I_2 - \limsup_{n,m\to\infty} \Phi_{n,m}(c) \leq \Phi(c). \tag{1}$$

On the other hand, for any $x < a, y < b$ we have

$$\{\xi \leq x, \gamma \leq y\} = \{\xi_{n,m} \leq a, \gamma_{n,m} \leq b, \xi \leq x, \gamma \leq y\} \cup \{\xi_{n,m} \leq a, \gamma_{n,m} \leq b, \xi > x, \gamma > y\} \cup \{\xi_{n,m} \leq a, \gamma_{n,m} \leq b, \xi \leq x, \gamma > y\} \cup \{\xi_{n,m} \leq a, \gamma_{n,m} \leq b, \xi > a, \gamma \leq b\}$$

$$\subset \{\xi \leq a, \gamma \leq b\} \cup \{|\xi_{n,m} - \xi| \geq a - x\} \cup \{|\gamma_{n,m} - \gamma| \geq b - y\}.$$

This implies,

$$\Phi(x+iy) \leq \Phi_{n,m}(a+ib)+M(\{\|\xi_{n,m}-\xi\| \geq a-x\}+M(\{\|\gamma_{n,m}-\gamma\| \geq b-y.$$

Since

$$(n,m) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_{n,m}}|\{(j,k) \in I_{n,m} : M(\|\xi_{j,k}-\xi\| \geq a-x) \geq \varepsilon\}| \geq \delta\} \in I_2,$$

$$(n,m) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_{n,m}}|\{(j,k) \in I_{n,m} : M(\|\gamma_{j,k}-\gamma\| \geq b-y) \geq \varepsilon\}| \geq \delta\} \in I_2,$$

we obtain

$$\Phi(x+iy) \leq I_2\text{-}\liminf_{n,m\to\infty} \Phi_{n,m}(a+ib)$$

for any $x < a, y < b$. Taking $x + iy \to a + ib$, we obtain

$$\Phi(c) \leq I_2 - \liminf_{n,m\to\infty} \Phi_{n,m}(c) \tag{2}$$

By (1) and (2) we have $\Phi_{n,m}(c) \to \Phi(c)$ as $n \to \infty$. Therefore, the double sequence $\{\zeta_{n,m}\}$ is λ_{I_2}-statistically convergent in distribution to $\zeta = \xi + i\gamma$. □

Converse of above proposition is not true. i.e. λ_{I_2}-statistical convergence in distribution does not imply λ_{I_2}-statistical convergence in measure as can be seen in the following example.

Example 3.2. Consider the uncertainty space $(\Gamma, \mathcal{F}, M)$ be $\{\gamma_1, \gamma_2\}$ with $M(\gamma_j) = \frac{1}{2}$, $j = 1, 2$. We define

$$\zeta(\gamma) = \begin{cases} i & if \quad \gamma = \gamma_1; \\ \\ -i & if \quad \gamma = \gamma_2. \end{cases}$$

Besides, we also define $\zeta_{n,m} = -\zeta$ for $n, m = 1, 2, 3, ...$ Then, $\zeta_{n,m}$ and ζ have the same distribution

$$\Phi_{n,m}(c) = \Phi_{n,m}(a + ib) = \begin{cases} 0 & if \quad a < 0, \ -\infty < b < +\infty; \\ \\ 0 & if \qquad a \geq 0, b < -1; \\ \\ \frac{1}{2} & if \qquad a \geq 0, \ -1 < b < 1; \\ \\ 1 & if \qquad a \geq 0, \ b \geq 1. \end{cases}$$

Then, $\{\zeta_{n,m}\}$ is λ_{I_2}-statistically in distribution to ζ. Nevertheless, for a given $\varepsilon > 0$, we have

$$\{(n, m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{n,m}} |\{(j, k) \in I_{n,m} : M(\|\xi_{j,k} - \xi\| \geq \varepsilon) \geq 1\}|\delta\}$$

$$= \{(n, m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{n,m}} |\{(j, k) \in I_{n,m} : M(\gamma : \|\xi_{j,k}(\gamma) - \xi(\gamma)\| \geq \varepsilon) \geq 1\}|\delta\} \in I_2.$$

Therefore, the double sequence $\{\zeta_{n,m}\}$ does not λ_{I_2}-statistically converge in measure to ζ. By Proposition 3.4, the real part and imaginary part of $\{\zeta_{n,m}\}$ also do not λ_{I_2}-statistically converge in measure. Additionally, since $\zeta_{n,m} = -\zeta$ for $n, m = 1, 2, 3, ...$, the double sequence $\{\zeta_{n,m}\}$ does not λ_{I_2}-statistically converge a.s. to ζ.

The following example shows that λ_{I_2}-statistically convergence a.s. does not imply λ_{I_2}-statistically convergence in measure.

Example 3.3. Consider the uncertainly space $(\Gamma, \mathcal{F}, M)$ to be $\gamma_{1,1}, \gamma_{1,2}, ...$ with

$$M(\Lambda) = \begin{cases} \sup\limits_{\gamma_{n,m}\in\Lambda} \dfrac{nm}{2(n+m)+1} & if \quad \sup\limits_{\gamma_{n,m}\in\Lambda} \dfrac{nm}{2(n+m)+1} < 0.5; \\ 1 - \sup\limits_{\gamma_{n,m}\in\Lambda^c} \dfrac{nm}{2(n+m)+1} & if \quad \sup\limits_{\gamma_{n,m}\in\Lambda^c} \dfrac{nm}{2(n+m)+1} < 0.5; \\ 0.5 & otherwise. \end{cases}$$

and we define a double complex uncertain variables as

$$\zeta_{n,m}(\gamma) = \begin{cases} inm & if \quad \gamma = \gamma_{n,m} \\ 0 & otherwise. \end{cases}$$

for $n, m = 1, 2, 3...$ and $\zeta \equiv 0$- Then, the double sequence $\{\zeta_{n,m}\}$ λ_{I_2}-statistically converges a.s. to ζ. Nevertheless, for some small numbers $\varepsilon, \delta > 0$, we have

$$\{(n,m) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_{n,m}}|\{(j,k) \in I_{n,m} : M(\|\xi_{j,k} - \xi\| \geq \varepsilon) \geq \frac{1}{2}\}| \geq \delta\}$$

$$= \{(n,m) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_{n,m}}|\{(j,k) \in I_{n,m} : M(\gamma : \|\zeta_{j,k}(\gamma) - \zeta(\gamma)\| \geq \varepsilon) \geq \frac{1}{2}\}| \geq \delta\}$$

$$= \{(n,m) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_{n,m}}|\{(j,k) \in I_{n,m} : M\{\gamma_{n,m}\} \geq \frac{1}{2}\}| \geq \delta\}.$$

Therefore, the double sequence $\{\zeta_{n,m}\}$ does not λ_{I_2}-statistically converges in measure to ζ.

The following example shows that λ_{I_2}-statistically convergence in measure does not imply λ_{I_2}-statistically convergence a.s.

Example 3.4. Consider the uncertainty space $(\Gamma, \mathcal{F}, M)$ to be $[0,1]$ with the Borel algebra and the Lebesgue measure. For any positives integers

n and m, there is there are integers p and q such that $n = 2^p + k$ and $m = 2^q + j$ where k and j are integers between 0 and $2^p - 1$ and $2^q - 1$, respectively. Then, we define a double complex uncertain variable by

$$\zeta_{n,m}(\gamma) = \begin{cases} i & if \quad kj2^{-(p+q)} \leq \gamma \leq (k+1)(j+1)2^{-(p+q)}; \\ 0 & otherwise. \end{cases}$$

for $n, m = 1, 2, ...$ and $\zeta \equiv 0$. For some small numbers $\varepsilon, \delta, \vartheta > 0$ and $n, m \geq 2$, we have

$$\{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{n,m}} |\{(j,k) \in I_{n,m} : M(\|\xi_{j,k} - \xi\| \geq \varepsilon) \geq \delta\}| \geq \vartheta\}$$

$$= \{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{n,m}} |\{(j,k) \in I_{n,m} : M(\gamma : \|\zeta_{j,k}(\gamma) - \zeta(\gamma)\| \geq \varepsilon) \geq \delta\}| \geq \vartheta\}$$

$$= \{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{n,m}} |\{(j,k) \in I_{n,m} : M\{\gamma_{n,m}\} \geq \delta\}| \geq \vartheta\}.$$

Therefore, the double sequence $\{\zeta_{n,m}\}$ λ_{I_2}-statistically converges in measure to ζ. Additionally, for every $\varepsilon, \delta > 0$ we have $\{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{n,m}} |\{(j,k) \in I_{n,m} : E(\|\zeta_{j,k} - \zeta\|) \geq \varepsilon\}| \geq \delta\} \in I_2$. In consequence, the double sequence $\{\zeta_{n,m}\}$ λ_{I_2}-statistically converges in mean to ζ. Nevertheless, for any $\gamma \in [0,1]$, there there is an infinite numbers of intervals of the form $[kj2^{-(p+q)}, (k+1)(j+1)2^{-(p+q)}]$ containing γ. Hence, $\zeta_{n,m}(\gamma)$ does not λ_{I_2}-statistically converges to 0, i.e. the double sequence $\{\zeta_{n,m}\}$ does not λ_{I_2}-statistically converge a.s. to ζ.

The following example shows that λ_{I_2}-statistically convergence a.s. does not imply λ_{I_2}-statistically convergence in mean.

Example 3.5. Consider the uncertainty space $(\Gamma, \mathcal{F}, M)$ to be $\{\gamma_{1,1}, \gamma_{2,2}\}$ with $M(\Lambda) = \sum_{\gamma_{n,m} \in \Lambda} 3^{-(n+m)}$. The double complex uncertain variables are defined as follows

$$\zeta_{n,m}(\gamma) = \begin{cases} i3^{n+m} & if \quad \gamma = \gamma_{n,m}; \\ 0 & otherwise. \end{cases}$$

for $n, m = 1, 2, 3, ...$ and $\zeta \equiv 0$. Then, the double sequence $\{\zeta_{n,m}\}$ λ_{I_2}-statistically converges a.s. to ζ. Nevertheless, the double uncertain distributions of $\|\zeta_{n,m}\|$ are

$$\Phi_{n,m}(x) = \begin{cases} 0 & if \quad x < 0; \\ 1 - \dfrac{1}{3^{n+m}} & if \quad 0 \leq x < 3^{n+m}; \\ 1, & if \quad x \geq 3^{n+m}, \end{cases}$$

for $n, m = 1, 2, 3, ...$ Therefore, we have $\{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{n,m}}|\{(j,k) \in I_{n,m} : E(\|\zeta_{j,k} - \zeta\|) \geq 1\}| \geq \delta\} \in I_2$. Hence, the double sequence $\{\zeta_{n,m}\}$ does not λ_{I_2}-statistically converge in mean to ζ.

On the other hand, by this example, we can see that λ_{I_2}-statistically convergence in mean does not imply λ_{I_2}-statistically convergence a.s.

Now, we show some results that we obtained by using λ_{I_2}-statistically convergence uniformly a.s.

Proposition 3.4. Let $\zeta, \zeta_{1,1}, \zeta_{1,2}, ...$ be double complex uncertain variables. Then, $\{\zeta_{n,m}\}$ λ_{I_2}-statistically converges a.s. to ζ if and only if for any $\varepsilon, \delta, \vartheta > 0$, we have

$$\{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{n,m}}|\{(j,k) \in I_{n,m} : M(\bigcap_{j=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{n=j}^{\infty}\bigcup_{m=k}^{\infty} \|\zeta_{n,m} - \zeta\| \geq \varepsilon) \geq \delta\}| \geq \vartheta\} \in I_2.$$

Proof. By using the definition of λ_{I_2}-statistically convergence a.s., we have that there exists an event Λ with $M(\Lambda) = 1$ such that $\{(n,m) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{n,m}}|\{(j,k) \in I_{n,m} : \|\zeta_{j,k} - \zeta\| \geq \varepsilon\}| \geq \delta\} \in I_2$ for every $\varepsilon, \delta > 0$. Then, for any $\varepsilon, \vartheta > 0$, there exist j and k such that $\|\zeta_{n,m} - \zeta\| < \varepsilon$ where $n > j, m > k$ and for any $\gamma \in \Lambda$, that is equivalent to

$$\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{n,m}}|\{(j,k)\in I_{n,m}:M(\bigcap_{j=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{n=j}^{\infty}\bigcup_{m=k}^{\infty}\|\zeta_{n,m}-\zeta\|<\varepsilon)\geq 1\}|\geq\vartheta\}\in I_2.$$

It follows from the duality axiom of an uncertain measure that

$$\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{n,m}}|\{(j,k)\in I_{n,m}:M(\bigcap_{j=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{n=j}^{\infty}\bigcup_{m=k}^{\infty}\|\zeta_{n,m}-\zeta\|<\varepsilon)\geq 1\}|\geq\vartheta\}\in I_2.$$

□

Proposition 3.5. Let $\zeta,\zeta_{1,1},\zeta_{1,2},...$ be double complex uncertain variables. Then, $\{\zeta_{n,m}\}$ λ_{I_2}-statistically converges uniformly a.s. to ζ if and only if for any $\varepsilon,\delta,\vartheta>0$, we have

$$\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{n,m}}|\{(j,k)\in I_{n,m}:M(\bigcup_{n=j}^{\infty}\bigcup_{m=k}^{\infty}\|\zeta_{j,k}-\zeta\|\geq\varepsilon)\geq\delta\}|\geq\vartheta\}\in I_2.$$

Proof. If $\{\zeta_{n,m}\}$ λ_{I_2}-statistically converges uniformly a.s. to ζ, then for any $\vartheta>0$ there exists W such that $M\{W\}<\vartheta$ and $\{\zeta_{n,m}\}$ λ_{I_2}-statistically converges to ζ on $\Gamma-W$. Therefore, for any $\varepsilon>0$, there exist $j,k>0$ such that $\|\zeta_{j,k}-\zeta\|<\varepsilon$ where $n>j,m>k$ and for any $\gamma\in\Gamma-W$. This is

$$\bigcup_{n=j}^{\infty}\bigcup_{m=k}^{\infty}\{\|\zeta_{n,m}-\zeta\|\geq\varepsilon\}\subset W.$$

It follows from the subadditivity axiom that

$$\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{n,m}}|\{(j,k)\in I_{n,m}:\bigcup_{n=j}^{\infty}\bigcup_{m=k}^{\infty}\{\|\zeta_{j,k}-\zeta\|\geq\varepsilon\}|\geq\delta\}\subseteq\delta^{I_{2\theta}}M\{W\}\subseteq\vartheta.$$

Then,

$$\{(n,m)\in\mathbb{N}\times\mathbb{N}: \frac{1}{\lambda_{n,m}}|\{(j,k)\in I_{n,m}: M(\bigcup_{n=j}^{\infty}\bigcup_{m=k}^{\infty}\|\zeta_{j,k}-\zeta\|\geq\varepsilon)\geq\delta\}|\geq \vartheta\}\in I_2.$$

On the contrary, if

$$\{(n,m)\in\mathbb{N}\times\mathbb{N}: \frac{1}{\lambda_{n,m}}|\{(j,k)\in I_{n,m}: M(\bigcup_{n=j}^{\infty}\bigcup_{m=k}^{\infty}\|\zeta_{j,k}-\zeta\|\geq\varepsilon)\geq\delta\}|\geq \vartheta\}\in I_2,$$

for any $\varepsilon,\delta,\vartheta>0$, then for given $\delta>0$ and $q,w\geq 1$, there exist q_j and w_k such that

$$\delta(M(\bigcup_{n=q_j}^{\infty}\bigcup_{m=wk}^{\infty}\{\|\zeta_{n,m}-\zeta\|\geq\frac{1}{q+w}\}))<\frac{\delta}{2^{q+w}}.$$

Let $W=\bigcup_{n=q_j}^{\infty}\bigcup_{m=wk}^{\infty}\{\|\zeta_{n,m}-\zeta\|\geq\frac{1}{q+w}\}$. Then,

$$\delta(M\{W\})\leq\sum_{q=1}^{\infty}\sum_{w=1}^{\infty}\delta(M(\bigcup_{n=q_j}^{\infty}\bigcup_{m=wk}^{\infty}\{\|\zeta_{n,m}-\zeta\|\geq\frac{1}{q+w}\}))\leq \sum_{q=1}^{\infty}\sum_{w=1}^{\infty}\frac{\delta}{2^{q+w}}.$$

Additionally, we obtain I_2-$\sup_{\gamma\in\Gamma-W}\|\zeta_{n,m}-\zeta\|<\frac{1}{q+w}$ for any $q,w=1,2,3,...$ and $n>q_j, m>w_k$.

□

Theorem 3.1. If a double sequence $\{\zeta_{n,m}\}$ of complex uncertain variables λ_{I_2}-statistically converges uniformly a.s. to ζ. Then, $\{\zeta_{n,m}\}$ λ_{I_2}-statistically converges a.s. to ζ.

Proof. From Proposition 3.5, we have that if $\{\zeta_{n,m}\}$ λ_{I_2}-statistically converges uniformly a.s. to ζ. Then,

$$\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{n,m}}|\{(j,k)\in I_{n,m}:M(\bigcup_{n=j}^{\infty}\bigcup_{m=k}^{\infty}\|\zeta_{j,k}-\zeta\|\geq\varepsilon)\geq\delta\}|\geq\vartheta\}\in I_2.$$

Since

$$\delta(M(\bigcap_{j=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{n=j}^{\infty}\bigcup_{m=k}^{\infty}\{\|\zeta_{n,m}-\zeta\|\geq\varepsilon\}))\leq\delta(M(\bigcup_{n=j}^{\infty}\bigcup_{m=k}^{\infty}\{\|\zeta_{n,m}-\zeta\|\geq\varepsilon\})),$$

taking the limit as $n,m\to\infty$ on both side of above inequality, we have

$$\delta(M(\bigcup_{n=j}^{\infty}\bigcup_{m=k}^{\infty}\{\|\zeta_{n,m}-\zeta\|\geq\varepsilon\}))=0.$$

By Proposition 3.4, $\{\zeta_{n,m}\}$ λ_{I_2}-statistically converges a.s. to ζ.

□

Theorem 3.2. If a double sequence $\{\zeta_{n,m}\}$ of complex uncertain variables λ_{I_2}-statistically converges uniformly a.s. to ζ. Then, $\{\zeta_{n,m}\}$ λ_{I_2}-statistically converges in measure to ζ.

Proof. If the double sequence $\{\zeta_{n,m}\}$ of complex uncertain variables λ_{I_2}-statistically converges uniformly a.s. to ζ. Then, from Proposition 3.5 and Theorem 3.1, we have

$$\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{n,m}}|\{(j,k)\in I_{n,m}:M(\bigcup_{n=j}^{\infty}\bigcup_{m=k}^{\infty}\|\zeta_{j,k}-\zeta\|\geq\varepsilon)\geq\delta\}|\geq\vartheta\}\in I_2,$$

$$\delta(M(\bigcap_{j=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{n=j}^{\infty}\bigcup_{m=k}^{\infty}\{\|\zeta_{n,m}-\zeta\|\geq\varepsilon\}))\leq\delta(M(\bigcup_{n=j}^{\infty}\bigcup_{m=k}^{\infty}\{\|\zeta_{n,m}-\zeta\|\geq\varepsilon\})).$$

Letting $n,m\to\infty$, we obtain that $\{\zeta_{n,m}\}$ λ_{I_2}-statistically converges in measure to ζ.

□

4 Conclusion

In this paper, we have introduced the notion of S_λ-I-Convergence by using uncertain theory. Besides, we showed some relations between λ_{I_2}-statistically convergence almost surely, λ_{I_2}-statistically convergence in measure, λ_{I_2}-statistically convergence in mean, λ_{I_2}-statistically convergence in distribution and λ_{I_2}-statistically convergence uniformly almost surely and when it was necessary, some counterexamples were shown to support our results. On the other hand, for reader we suggest to extend these notions in three dimension or higher, moreover these notions can be studied on neutrosophic normed spaces [34].

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