Phase-space representation for Galilean quantum particles of arbitrary spin

José M. Gracia-Bondía and Joseph C. Várilly

Escuela de Matemática, Universidad de Costa Rica, 11501 San José, Costa Rica

J. Phys. A **21** (1988), L879–L893

Abstract

The phase-space approach to quantization is extended to incorporate spinning particles with Galilean symmetry. The appropriate phase space is the coadjoint orbit $\mathbb{R}^6 \times \mathbb{S}^2$. From two basic principles, traciality and Galilean covariance, the Weyl symbol calculus is constructed. Then the Galilean-equivariant twisted products of functions on this phase space are identified.

In the conventional description, the states of a quantum-mechanical, nonrelativistic particle are identified to elements of the Hilbert space $\mathcal{H}^j = \mathbb{C}^{2j+1} \otimes L^2(\mathbb{R}^3, d\boldsymbol{\xi})$. Since the pioneering work by Weyl [1], Wigner [2] and foremost, Moyal [3], phase-space realizations of such a physical system, for $j = 0$, have attracted considerable attention. In this letter we extend the phase-space approach to cover spinning particles as well, within the framework of nonrelativistic mechanics.

Let $\mathbb{S}^2$ denote the manifold of states of a "classical spin", i.e., the sphere. Let U^j be a physical (i.e., projective) unitary representation of the Galilei group G on $\mathcal{H}^j$, write $g \cdot A := U^j(g) A U^j(g^{-1})$ for any operator A on $\mathcal{H}^j$, and let $g \cdot u$ denote the action of G on the phase-space $\mathbb{R}^6 \times \mathbb{S}^2$, with coordinates $u := (\boldsymbol{q}, \boldsymbol{p}; \boldsymbol{n})$. By a "Stratonovich–Weyl correspondence" we mean a rule assigning to every operator A a function W_A on the phase space $\mathbb{R}^6 \times \mathbb{S}^2$, satisfying the following postulates:

(a) The correspondence is linear and one-to-one.

(b) Self-adjoint operators are mapped into real functions.

(c) The identity operator is mapped into the constant function 1.

(d) *Traciality*. For a suitable multiple $d\mu^j$ of the ordinary measure on $\mathbb{R}^6 \times \mathbb{S}^2$, the equation $\int W_A(u) W_B(u)\, d\mu^j(u) = \operatorname{Tr} AB$ holds whenever both sides make sense.

(e) *Covariance*. $W_{g\cdot A}(u) = W_A(g^{-1} \cdot u)$ for $g \in G$, $u \in \mathbb{R}^6 \times \mathbb{S}^2$.

The problem of finding a Stratonovich–Weyl correspondence for a Galilean particle with arbitrary spin has an essentially unique solution. We collect first the necessary formulae for the Galilei group. Lévy-Leblond's notation [4] is employed throughout. A Galilean transformation, defined by

$$(b, \boldsymbol{a}, \boldsymbol{v}, R)(\boldsymbol{x}, t) := (R\boldsymbol{x} + \boldsymbol{v}t + \boldsymbol{a}, t + b),$$

where $b \in \mathbb{R}$, $\boldsymbol{a}, \boldsymbol{v} \in \mathbb{R}^3$, $R \in SO(3)$ and $(\boldsymbol{x}, t)$ are spacetime coordinates, has inverse $(b, \boldsymbol{a}, \boldsymbol{v}, R)^{-1} = (-b, R^{-1}(b\boldsymbol{v} - \boldsymbol{a}), -R^{-1}\boldsymbol{v}, R^{-1})$. It acts on phase space [5] by:

$$(b, \boldsymbol{a}, \boldsymbol{v}, R) \cdot (\boldsymbol{q}, \boldsymbol{p}; \boldsymbol{n}) := \big(R(\boldsymbol{q} - \tfrac{b}{m}\boldsymbol{p}) + \boldsymbol{a} - b\boldsymbol{v}, R\boldsymbol{p} + m\boldsymbol{v}, R\boldsymbol{n}\big). \tag{1}$$

Here m is the mass of the particle.

It is well known that the projective representations of G are restrictions of true representations of central extensions $\widetilde{G}_m$ of the universal covering group $\widetilde{G}$ [4] (which we will denote also by U^j). They act on $\mathcal{H}^j$ (which can be thought of as momentum space for the j-spin particle) by:

$$[U^j(b, \boldsymbol{a}, \boldsymbol{v}, \widetilde{R})\Phi]_s(\boldsymbol{\xi}) := \exp\biggl[\frac{i}{\hbar}\Bigl(\frac{b|\boldsymbol{\xi}|^2}{2m} - \boldsymbol{\xi} \cdot \boldsymbol{a} + \tfrac{1}{2}m\boldsymbol{a} \cdot \boldsymbol{v}\Bigr)\biggr] \sum_{t=-j}^{j} \mathcal{D}^j_{st}(\widetilde{R})\, \Phi_t\big(R^{-1}(\boldsymbol{\xi} - m\boldsymbol{v})\big), \tag{2}$$

where $\widetilde{R} \in \mathrm{SU}(2)$, the covering group of SO(3), R is the rotation matrix corresponding to $\widetilde{R}$, and the $\mathcal{D}^j_{st}(\widetilde{R})$ are the usual matrix elements $\langle js \mid \pi_j(\widetilde{R}) \mid jt\rangle$ [6], where π_j denotes the irreducible representation of SU(2) on $\mathbb{C}^{2j+1}$, and $s = -j, \dots, j-1, j$.

The system of factors is

$$\omega_m(g, g') = \frac{m}{2\hbar}\,(-b'\boldsymbol{v} \cdot R\boldsymbol{v}' + \boldsymbol{v} \cdot R\boldsymbol{a}' - \boldsymbol{a} \cdot R\boldsymbol{v}')$$

if $g = (b, \boldsymbol{a}, \boldsymbol{v}, \widetilde{R})$, $g' = (b', \boldsymbol{a}', \boldsymbol{v}', \widetilde{R}')$ belong to $\widetilde{G}$. It restricts nicely to the exponent of the canonical commutation relations in Weyl form [1], on considering the subgroup of $\widetilde{G}$ of elements such that $b = 0$, $\widetilde{R} = 1$. We note that (1) comes naturally from Kirillov–Souriau theory [7, 8], as $\mathbb{R}^6 \times \mathbb{S}^2$ is an orbit of the coadjoint action of $\widetilde{G}$ corresponding to a Casimir element $m > 0$.

By condition (a), we may write

$$W_A(u) = \mathrm{Tr}(A\,\Gamma^j(u))$$

for some operator-valued function Γ^j on $\mathbb{R}^6 \times \mathbb{S}^2$. Now, by the tracial condition (d):

$$\begin{aligned}\mathrm{Tr}\,AB &= \int W_A(u) W_B(u)\, d\mu^j(u) = \int \mathrm{Tr}(A \cdot \Gamma^j(u)) W_B(u)\, d\mu^j(u) \\ &= \mathrm{Tr}\Bigl(A \int W_B(u)\, \Gamma^j(u)\, d\mu^j(u)\Bigr),\end{aligned}$$

which implies

$$B = \int W_B(u)\, \Gamma^j(u)\, d\mu^j(u) \quad \text{for any } B.$$

Thus the tracial condition, which is obviously imposed to assure the equality of standard quantum-mechanical and phase-space averages, has the important consequence that the correspondence $A \rightleftarrows W_A$ can be implemented with the *same* operator kernel Γ^j.

We now show that $\Gamma^j(u)$ is a tensor product of operators $\Delta^j(\boldsymbol{n})$ acting on $\mathbb{C}^{2j+1}$ and $\Pi(\boldsymbol{q}, \boldsymbol{p})$ acting on $L^2(\mathbb{R}^3, d\boldsymbol{\xi})$: $\Gamma^j(\boldsymbol{q}, \boldsymbol{p}; \boldsymbol{n}) = \Delta^j(\boldsymbol{n}) \otimes \Pi(\boldsymbol{q}, \boldsymbol{p})$. Introduce the following functions over the sphere:

$$Z^j_{rs}(\boldsymbol{n}) := \frac{\sqrt{4\pi}}{2j+1} \sum_{l=0}^{2j} \sqrt{2l+1}\,\big\langle j\,l\,r\,(s-r) \mid j\,s\big\rangle\, Y_{l,s-r}(\boldsymbol{n}), \tag{3}$$

where Y_{lm} denotes the usual spherical harmonics and $\langle j\, l\, r(s-r) \mid j\, s\rangle$ is a Clebsch–Gordan coefficient. Using the well-known formula [6] for transforming spherical harmonics:

$$Y_{lm}(R\boldsymbol{n}) = \sum_{n=-l}^{l} \mathcal{D}^{l*}_{mn}(\widetilde{R})\, Y_{ln}(\boldsymbol{n}),$$

one derives [9], after some calculation:

$$Z^j_{rs}(R\boldsymbol{n}) = \sum_{p,q=-j}^{j} \mathcal{D}^j_{rp}(\widetilde{R})\, \mathcal{D}^{j*}_{sq}(\widetilde{R})\, Z^j_{pq}(\boldsymbol{n}).$$

Define

$$\Delta^j(\boldsymbol{n}) := \sum_{r,s=-j}^{j} Z^j_{rs}(\boldsymbol{n})\, |jr\rangle\langle js|. \tag{4}$$

As $Z^j_{rs} = \overline{Z}^j_{sr}$, the Δ^j are selfadjoint. One computes easily that

$$\operatorname{Tr}\Delta^j(\boldsymbol{n}) = 1, \tag{5a}$$

$$\operatorname{Tr}\big(\Delta^j(\boldsymbol{m})\Delta^j(\boldsymbol{n})\big) = \frac{4\pi}{2j+1}\sum_{l=0}^{2j}\sum_{s=-l}^{l} Y_{ls}(\boldsymbol{m})Y^*_{ls}(\boldsymbol{n}) =: \frac{4\pi}{2j+1} K^j(\boldsymbol{m},\boldsymbol{n}). \tag{5b}$$

Here K^j is the *reproducing kernel* of the space of spherical harmonics of degree $\leqslant 2j$. Now introduce

$$\Pi(\boldsymbol{q},\boldsymbol{p})\Phi(\boldsymbol{\xi}) := 2^3 \exp\Big(\frac{2i}{\hbar}\,\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{\xi})\Big)\Phi(2\boldsymbol{p}-\boldsymbol{\xi}),$$

and compute [10, 11]:

$$\operatorname{Tr}\Pi(\boldsymbol{q},\boldsymbol{p}) = 1, \tag{6a}$$

$$\operatorname{Tr}\big(\Pi(\boldsymbol{q},\boldsymbol{p})\Pi(\boldsymbol{q}',\boldsymbol{p}')\big) = (2\pi\hbar)^3\,\delta(\boldsymbol{q}-\boldsymbol{q}')\,\delta(\boldsymbol{p}-\boldsymbol{p}'). \tag{6b}$$

We note that the geometrical meaning of the $\Pi(\boldsymbol{q},\boldsymbol{p})$ as reflection operators was only uncovered some years ago by Grossmann [12] and Royer [13]. It is easily seen that the $\Pi(\boldsymbol{q},\boldsymbol{p})$ are self-adjoint.

From (5) and (6) it follows that the $\Gamma^j(u) = \Delta^j(\boldsymbol{n})\otimes\Pi(\boldsymbol{q},\boldsymbol{p})$ are selfadjoint, and

$$\operatorname{Tr}\Gamma^j(u) = 1, \tag{7a}$$

$$\operatorname{Tr}\big(\Gamma^j(u)\Gamma^j(u')\big) = \frac{4\pi}{2j+1}(2\pi\hbar)^3\,\delta^j(u-u'), \tag{7b}$$

with an obvious meaning for $\delta^j(u-u')$.

Now, our initial set of postulates is readily seen to translate into the following conditions for the family of operators $\Gamma^j(u)$:

(i) The $\Gamma^j(u)$ are selfadjoint;

(ii) $\displaystyle\int_{\mathbb{R}^6\times\mathbb{S}^2} \Gamma^j(u)\, d\mu^j(u) = 1;$

(iii) $\displaystyle\int_{\mathbb{R}^6\times\mathbb{S}^2} \operatorname{Tr}\bigl(\Gamma^j(u)\,\Gamma^j(u')\bigr)\,\Gamma^j(u')\,d\mu^j(u') = \Gamma^j(u)$;

(iv) $\Gamma^j(g\cdot u) = U^j(g)\,\Gamma^j(u)\,U^j(g)^{-1}$, whenever $g = (b,\boldsymbol{a},\boldsymbol{v},\widetilde{R}) \in \widetilde{G}$ and $g\cdot u$ is given by (1), with R being the rotation determined by $\widetilde{R} \in \mathrm{SU}(2)$.

Taking

$$d\mu^j(u) := (2\pi\hbar)^{-3}\frac{2j+1}{4\pi}\,d\boldsymbol{q}\,d\boldsymbol{p}\,d\boldsymbol{n},$$

the conditions (ii) and (iii) follow from (7). Now (iv) is verified by a direct calculation, using (2):

$$\begin{aligned}
&2^{-3}[U^j(b,\boldsymbol{a},\boldsymbol{v},\widetilde{R})\,\Gamma^j(\boldsymbol{q},\boldsymbol{p};\boldsymbol{n})\,U^j((b,\boldsymbol{a},\boldsymbol{v},\widetilde{R})^{-1})\Phi]_s(\boldsymbol{\xi})\\
&\quad= 2^{-3}\exp\biggl[\frac{i}{\hbar}\Bigl(\frac{b|\boldsymbol{\xi}|^2}{2m} - \boldsymbol{\xi}\cdot\boldsymbol{a} + \tfrac{1}{2}m\boldsymbol{a}\cdot\boldsymbol{v}\Bigr)\biggr]\\
&\qquad\times\sum_{t=-j}^{j}\mathcal{D}^j_{st}(\widetilde{R})[\Gamma^j(\boldsymbol{q},\boldsymbol{p};\boldsymbol{n})\,U^j((b,\boldsymbol{a},\boldsymbol{v},\widetilde{R})^{-1})\Phi]_t(R^{-1}(\boldsymbol{\xi}-m\boldsymbol{v}))\\
&\quad= \exp\biggl[\frac{i}{\hbar}\Bigl(\frac{b|\boldsymbol{\xi}|^2}{2m} - \boldsymbol{\xi}\cdot\boldsymbol{a} + \tfrac{1}{2}m\boldsymbol{a}\cdot\boldsymbol{v} + 2\boldsymbol{q}\cdot[\boldsymbol{p} - R^{-1}(\boldsymbol{\xi}-m\boldsymbol{v})]\Bigr)\biggr]\\
&\qquad\times\sum_{t,u=-j}^{j}\mathcal{D}^j_{st}(\widetilde{R})\,Z^j_{tu}(\boldsymbol{n})[U^j((b,\boldsymbol{a},\boldsymbol{v},\widetilde{R})^{-1})\Phi]_u(2\boldsymbol{p} - R^{-1}(\boldsymbol{\xi}-m\boldsymbol{v}))\\
&\quad= \exp\biggl[\frac{i}{\hbar}\Bigl(\frac{b|\boldsymbol{\xi}|^2}{2m} - \boldsymbol{\xi}\cdot\boldsymbol{a} + \tfrac{1}{2}m\boldsymbol{a}\cdot\boldsymbol{v} + 2(R\boldsymbol{q})\cdot(R\boldsymbol{p} + m\boldsymbol{v} - \boldsymbol{\xi})\\
&\qquad\qquad - \frac{b}{2m}|2R\boldsymbol{p} + m\boldsymbol{v} - \boldsymbol{\xi}|^2 - (b\boldsymbol{v}-\boldsymbol{a})\cdot(2R\boldsymbol{p} + m\boldsymbol{v} - \boldsymbol{\xi}) - \tfrac{1}{2}m(b\boldsymbol{v}-\boldsymbol{a})\cdot\boldsymbol{v}\Bigr)\biggr]\\
&\qquad\times\sum_{t,u,v=-j}^{j}\mathcal{D}^j_{st}(\widetilde{R})\,Z^j_{tu}(\boldsymbol{n})\,\mathcal{D}^{j*}_{vu}(\widetilde{R})\,\Phi_v(2R\boldsymbol{p} + 2m\boldsymbol{v} - \boldsymbol{\xi})\\
&\quad= \exp\biggl\{\frac{i}{\hbar}\Bigl[2\bigl(R\boldsymbol{q} - \tfrac{b}{m}R\boldsymbol{p} + \boldsymbol{a} - b\boldsymbol{v}\bigr)\cdot(R\boldsymbol{p} + m\boldsymbol{v} - \boldsymbol{\xi})\Bigr]\biggr\}\sum_{v=-j}^{j}Z^j_{sv}(R\boldsymbol{n})\,\Phi_v(2R\boldsymbol{p} + 2m\boldsymbol{v} - \boldsymbol{\xi})\\
&\quad= 2^{-3}\bigl[\Gamma^j\bigl(R(\boldsymbol{q} - \tfrac{b}{m}\boldsymbol{p}) + \boldsymbol{a} - b\boldsymbol{v}, R\boldsymbol{p} + m\boldsymbol{v}, R\boldsymbol{n}\bigr)\Phi\bigr]_s(\boldsymbol{\xi}).
\end{aligned}$$

The conclusion is that there exists a phase-space representation for the description of a nonrelativistic spinning particle, as a theory of "Wigner functions" over $\mathbb{R}^6\times\mathbb{S}^2$. Full details of such a theory for spin are given in [9]; there it is seen that the $Z^j_{ss}(\boldsymbol{n})$ are the Wigner functions corresponding to the states $|js\rangle$.

Remark 1. The family Γ^j is essentially unique: unicity of the Π comes from the Stone–von Neumann theorem; in the definition of the Δ^j a few sign changes could be made, but it can be shown that only the definitions (3) and (4) make physical sense.

Remark 2. In the modern approach to phase-space quantum mechanics [14–19] the Stratonovich–Weyl correspondence is deemphasized in favor of the *twisted product* of two functions on phase space, corresponding to the usual product of two operators. In that way the theory is formulated autonomously as a calculus of functions on phase space. The twisted product, denoted by $\times$, is determined by the condition that $W_A \times W_B = W_{AB}$ for all operators A, B. Using the Stratonovich–Weyl correspondence, we find:

$$(f \times h)(u) = \int_{\mathbb{R}^6\times\mathbb{S}^2} \int_{\mathbb{R}^6\times\mathbb{S}^2} L^j(u,v,w)\, f(v)\, h(w)\, d\mu^j(v)\, d\mu^j(w)$$

where $L^j(u,v,w) := \operatorname{Tr}\bigl(\Gamma^j(u)\,\Gamma^j(v)\,\Gamma^j(w)\bigr)$. For instance,

$$\begin{aligned} L^{1/2}(u,v,w) = 16\bigl(1 + 3\boldsymbol{n}\cdot\boldsymbol{n}' + 3\boldsymbol{n}'\cdot\boldsymbol{n}'' + 3\boldsymbol{n}''\cdot\boldsymbol{n} + 3\sqrt{3}i[\boldsymbol{n},\boldsymbol{n}',\boldsymbol{n}'']\bigr) \\ \times \exp\Bigl(\frac{2i}{\hbar}\,(\boldsymbol{q}\cdot\boldsymbol{p}' - \boldsymbol{q}'\cdot\boldsymbol{p} + \boldsymbol{q}'\cdot\boldsymbol{p}'' - \boldsymbol{q}''\cdot\boldsymbol{p}' + \boldsymbol{q}''\cdot\boldsymbol{p} - \boldsymbol{q}\cdot\boldsymbol{p}'')\Bigr), \end{aligned}$$

if $u = (\boldsymbol{q},\boldsymbol{p};\boldsymbol{n})$, $v = (\boldsymbol{q}',\boldsymbol{p}';\boldsymbol{n}')$, $w = (\boldsymbol{q}'',\boldsymbol{p}'';\boldsymbol{n}'')$.

The tracial condition becomes

$$\int_{\mathbb{R}^6\times\mathbb{S}^2} (f \times h)(u)\, d\mu^j(u) = \int_{\mathbb{R}^6\times\mathbb{S}^2} f(u)\, h(u)\, d\mu^j(u),$$

and the covariance condition implies equivariance of the twisted product:

$$(f \times h)^g(u) = (f^g \times h^g)(u) \quad \text{for all} \quad g \in \widetilde{G}, \tag{8}$$

where $f^g(u) := f(g^{-1}\cdot u)$. In fact, (8) is true for the larger group $\mathrm{Sp}(6;\mathbb{R}) \rtimes \mathbb{R}^6$ of transformations of phase space (or its twofold covering $\mathrm{Mp}(6;\mathbb{R}) \rtimes \mathbb{R}^6$, to be precise). The canonical generators of this group are "distinguished" Hamiltonians, for which the quantum dynamics is rendered in the phase space essentially in classical terms.

Remark 3. Formulas (6) need some justification, as the operators Π are not of trace class. We intend to show elsewhere in the spirit of [20] that they hold in a distributional sense.

Remark 4. Generalization of the formalism developed here to any finite number of particles is straightforward.

In summary, the Moyal phase-space formalism now provides a self-contained approach to non-relativistic Quantum Mechanics, including both spatial and spin variables, which is fully covariant under the Galilei group.

We are grateful to José Cariñena for helpful discussions.

References

[1] H. Weyl, *Gruppentheorie und Quantenmechanik*, Hirzel, Leipzig, 1928.

[2] E. P. Wigner, "On the quantum correction for thermodynamic equilibrium", Phys. Rev. **40** (1932), 749–759.

[3] J. E. Moyal, "Quantum mechanics as a statistical theory", Proc. Cambridge Philos. Soc. **45** (1949), 99–124.

[4] J.-M. Lévy-Leblond, "Galilei group and Galilean invariance", in *Group Theory and its Applications*, E. M. Loebl, ed., Academic Press, New York, 1971; pp. 221–299.

[5] L. Martínez-Alonso, "Group-theoretical foundations of classical and quantum mechanics. II. Elementary systems", J. Math. Phys. **20** (1979), 219–230.

[6] L. C. Biedenharn and J. D. Louck, *Angular Momentum in Quantum Physics: Theory and Applications*, Addison-Wesley, Reading, MA, 1985.

[7] M. Vergne, "Representations of Lie groups and the orbit method", in *Emmy Noether in Bryn Mawr*, B. Srinivasan and J. Sally, eds., Springer, Berlin, 1983; pp. 59–101.

[8] J.-M. Souriau, *Structure des Systèmes Dynamiques*, Dunod, Paris, 1970.

[9] J. C. Várilly and J. M. Gracia-Bondía, "The Moyal representation for spin", Ann. Phys. **190** (1989), 107–148.

[10] R. L. Stratonovich, "On distributions in representation space", Sov. Phys. JETP **4** (1957), 891–898.

[11] S. R. de Groot and L. G. Suttorp, *Foundations of Electrodynamics*, North-Holland, Amsterdam, 1972.

[12] A. Grossmann, "Parity operators and quantization of δ-functions", Commun. Math. Phys. **48** (1976), 191–193.

[13] A. Royer, "Wigner function as the expectation value of a parity operator", Phys. Rev. A **15** (1977), 449–450.

[14] J. C. T. Pool, "Mathematical aspects of the Weyl correspondence", J. Math. Phys. **7** (1966), 66–76.

[15] F. Bayen, M. Flato, C. Frønsdal, A. Lichnerowicz and D. Sternheimer, "Deformation theory and quantization. I, II", Ann. Phys. **111** (1978), 61–110, 111–151.

[16] C. Frønsdal, "Some ideas about quantization", Rep. Math. Phys. **15** (1979), 111–145.

[17] J. M. Gracia-Bondía, "The Hydrogen atom in the phase-space formulation of quantum mechanics", Phys. Rev. A **30** (1984), 691–697.

[18] J. M. Gracia-Bondía and J. C. Várilly, "Algebras of distributions suitable for phase-space quantum mechanics. I", J. Math. Phys. **29** (1988), 869–879.

[19] J. C. Várilly and J. M. Gracia-Bondía, "Algebras of distributions suitable for phase-space quantum mechanics. II. Topologies on the Moyal algebra", J. Math. Phys. **29** (1988), 880–887.

[20] S. Albeverio, P. Blanchard and R. Høegh-Krohn, "Feynman path integrals and the trace formula for the Schrödinger operator", Commun. Math. Phys. **83** (1982), 49–76.