The Stratonovich-Weyl correspondence: A general approach to Wigner functions

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Abstract

A formalism is proposed for developing phase-space representations of elementary quantum systems under general invariance groups. Several examples are discussed, including the usual Weyl calculus, the Moyal formulation for spin, Poincaré disk quantizations, and the phase-space calculus for Galilean spinning particles.

Introduction

Ever since the 1949 paper of Moyal [1], which showed how the Weyl correspondence [2] enables one to develop Quantum Mechanics as a theory of functions on phase space, composed according to the "twisted" or Moyal product, with states being represented by their Wigner functions [3], it has been thought useful to extend this formalism beyond the arena of nonrelativistic spinless particles. The case of spinning particles seemed for some time to be particularly troublesome. In fact, an early suggestion of Stratonovich [4] for the spin case contains the seed of a Moyal theory for spin, as has recently been shown [5].

In this paper, I develop the main idea of [5] as a general recipe, which I call the "Stratonovich–Weyl correspondence", linking elementary classical systems to the elementary quantum systems with the same invariance group. The basic property of a Moyal formulation, namely, that quantum expectation values should be computed "classically" by integrating over the phase space, turns out to be enough (together with group covariance) to identify the twisted products (and hence, the symbol calculus) for many invariance groups.

Examples are given to show how the Stratonovich–Weyl correspondence works for the "ordinary" Weyl calculus, for pure spins, for Poincaré-disk quantizations, and for Galilean spinning particles.

1 Moyal quantum mechanics

The Moyal approach to Quantum Mechanics, as a theory which sticks as closely as possible to classical mechanical formulations, may be considered to have five essential aspects.

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1. Both *observables* and *states* are (generalized) functions on a classical phase space M. For example, in the simplest case of a flat phase space $M = \mathbb{R}^{2N}$, with coordinates $u = (\boldsymbol{q}, \boldsymbol{p})$, where $\boldsymbol{q}$, $\boldsymbol{p}$ give the position and momentum coordinates in $\mathbb{R}^N$, we consider observables of the form $H = |\boldsymbol{p}|^2/2m + V(\boldsymbol{q})$, and states such as $\rho = e^{-(\boldsymbol{q}\cdot\boldsymbol{q}+\boldsymbol{p}\cdot\boldsymbol{p})/2}$.

2. *Expected values* should be computed "classically", that is, by integrating over the phase space:

$$\langle f\rangle_\rho = \int_M f(u)\,\rho(u)\,du \Bigg/ \int_M \rho(u)\,du.$$

3. Quantum mechanics enters via the rule for the *composition of observables*, which is given by a *nonlocal* "twisted product":

$$(f \times h)(u) = \int_M \int_M L(u, v, w)\, f(v)\, h(w)\, dv\, dw.$$

   Of course, if one takes $L(u, v, w) = \delta(u - v)\,\delta(u - w)$, one recovers the ordinary pointwise product on M; but since the uncertainty principle forbids localization at points of phase space, we must exclude this option and search for another trikernel L, which will not be local.

4. The twisted product is *equivariant under canonical symmetries* of M. That is, one is given a Lie group G, acting on the phase space M by symplectic transformations $u \mapsto g \cdot u$, in such a way that

$$f^g \times h^g = (f \times h)^g, \quad \text{with} \quad f^g(u) := f(g^{-1}\cdot u).$$

   In terms of the trikernel L, this condition becomes

$$L(g\cdot u, g\cdot v, g\cdot w) = L(u, v, w) \quad \text{for all} \quad g \in G,$$

   which, together with Condition 2, restricts the possibilities for L. For instance, in the case $M = \mathbb{R}^{2N}$, we can take $G = \mathrm{ISp}(2N,\mathbb{R})$, the group of inhomogeneous linear symplectic transformations of $\mathbb{R}^{2N}$, in which case one obtains:

$$\begin{aligned} &L(\boldsymbol{q}_1, \boldsymbol{p}_1; \boldsymbol{q}_2, \boldsymbol{p}_2; \boldsymbol{q}_3, \boldsymbol{p}_3) \\ &\quad = (\text{constant}) \exp\left\{\frac{2i}{\hbar}\left[\boldsymbol{q}_1\cdot\boldsymbol{p}_2 - \boldsymbol{q}_2\cdot\boldsymbol{p}_1 + \boldsymbol{q}_2\cdot\boldsymbol{p}_3 - \boldsymbol{q}_3\cdot\boldsymbol{p}_2 + \boldsymbol{q}_3\cdot\boldsymbol{p}_1 - \boldsymbol{q}_1\cdot\boldsymbol{p}_3\right]\right\}, \end{aligned}$$

   where the only remaining freedom is the choice of the Planck constant $\hbar$.

5. Some *correspondence rule* $f \mapsto \mathrm{Op}(f)$, of functions on M to Hilbert-space operators, links the theory to ordinary Quantum Mechanics, and establishes an equivalence of the two formalisms. For the flat case, this is of course the Weyl correspondence rule:

$$\mathrm{Op}(f) = \int_{\mathbb{R}^{2N}} \hat{f}(\boldsymbol{x}, \boldsymbol{y})\, W(\boldsymbol{x}, \boldsymbol{y})\, \frac{d^N\boldsymbol{x}\, d^N\boldsymbol{y}}{(2\pi\hbar)^N} \tag{1}$$

   where $W(\boldsymbol{x}, \boldsymbol{y}) = \exp\{i(\boldsymbol{x}\cdot\boldsymbol{Q} + \boldsymbol{y}\cdot\boldsymbol{P})/\hbar\}$ are the Weyl operators.

The reason for outlining the Moyal formalism in so abstract a setting is to provide a means of going beyond the "flat case" to incorporate spinning and relativistic particles as well. Since, as Moyal noted [1], the "opposite" of the Weyl correspondence is simply the assignment of a transition operator $|\Psi\rangle\langle\Phi|$ to its *Wigner function* $f_{\Phi\Psi}$, where

$$
\begin{aligned}
f_{\Phi\Psi}(\boldsymbol{q},\boldsymbol{p}) &= \int_{\mathbb{R}^N} e^{i\boldsymbol{p}\cdot\boldsymbol{x}/\hbar}\overline{\Phi(\boldsymbol{q}-\tfrac{1}{2}\boldsymbol{x})}\,\Psi(\boldsymbol{q}+\tfrac{1}{2}\boldsymbol{x})\,d^N\boldsymbol{x} \\
&= \int_{\mathbb{R}^N} e^{i\boldsymbol{q}\cdot\boldsymbol{y}/\hbar}\overline{\hat{\Phi}(\boldsymbol{p}+\tfrac{1}{2}\boldsymbol{y})}\,\hat{\Psi}(\boldsymbol{p}-\tfrac{1}{2}\boldsymbol{y})\,d^N\boldsymbol{y},
\end{aligned}
\tag{2}
$$

we may obtain "Wigner functions" for more general systems by inverting the generalized Weyl correspondence rule. This can be done for spinning particles in an essentially unique way [5,6], as I shall indicate shortly.

A key observation, made independently by Grossmann [7] and Royer [8], is that the Weyl correspondence can be recast more simply by, so to speak, applying the Fourier inversion formula to (1) to obtain:

$$
\operatorname{Op}(f) = \int_{\mathbb{R}^{2N}} f(\boldsymbol{q},\boldsymbol{p})\,\Pi(\boldsymbol{q},\boldsymbol{p})\,\frac{d^N\boldsymbol{q}\,d^N\boldsymbol{p}}{(2\pi\hbar)^N} \tag{3}
$$

where the "parity operators" $\Pi(\boldsymbol{q},\boldsymbol{p})$ are defined by:

$$
\big[\Pi(\boldsymbol{q},\boldsymbol{p})\Psi\big](\boldsymbol{\xi}) := 2^N \exp\Big\{\frac{2i}{\hbar}\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{\xi})\Big\}\Psi(2\boldsymbol{p}-\boldsymbol{\xi}) \tag{4}
$$

in the momentum-space representation. Henceforth we will consider only the form (3) of the Weyl correspondence rule.

The rule (3) is also the starting point for a "discrete approximation" to the Weyl calculus, considered by Cohendet *et al.* [9] (see also Wootters [10]). Here one takes $M = \mathbb{Z}_N \times \mathbb{Z}_N$, where $\mathbb{Z}_N$ is the cyclic group of order N; if N is odd, one may define

$$
\operatorname{Op}(f) := \frac{1}{N}\sum_{q,p\in\mathbb{Z}_N} f(q,p)\,\Delta_{qp}
$$

where the parity operators Δ_{qp} act on $\mathbb{C}^N = L^2(\mathbb{Z}_N)$ by

$$
\big[\Delta_{qp}\Psi\big](k) := \exp\Big\{\frac{4\pi i}{N}\,q(p-k)\Big\}\Psi(2p-k).
$$

The underlying symmetry group is the finite Heisenberg group A_N [11, 12]; however, as M is not a symplectic manifold, this case does not lie completely within the class of Moyal quantizations which we consider here.

2 Elementary classical systems

The underlying *phase space* M is taken, as is usual in classical mechanics, to be a *symplectic manifold* with a nondegenerate closed 2-form (symplectic form) ω_M. We also assume that a *symmetry group* G is given, which acts on M by symplectomorphisms (i.e., diffeomorphisms leaving ω_M invariant). An *elementary classical system* is a pair (G, M) where the action of G on M is transitive, that is, (M, ω_M) is a G-homogeneous symplectic manifold.

We shall henceforth assume that G is a connected Lie group. The classification of G-homogeneous symplectic manifolds is known: see [13, 14], for example. (M, ω_M) is locally isomorphic to an orbit of some affine action of G on the dual space $\mathfrak{g}^*$ of its Lie algebra $\mathfrak{g}$, and the linear part of this action is always the coadjoint action of G. The inhomogeneous part of the action is classified by the cohomology space $H^2(\mathfrak{g}, \mathbb{R})$; this part can be removed by lifting to a suitable *central extension* of G, which can be taken simply connected, so that (M, ω_M) will appear as just a coadjoint orbit of the extended group.

A different role of central extensions of G occurs in the study of *projective unitary irreducible representations* of G. These are also classified by $H^2(\mathfrak{g}, \mathbb{R})$. Indeed, if $\widetilde{G}$ is the simply connected covering group of G, the required extension $\overline{G}$ is given by the exact sequence [15]:

$$0 \longrightarrow H^2(\mathfrak{g}, \mathbb{R}) \longrightarrow \overline{G} \longrightarrow \widetilde{G} \longrightarrow 0. \tag{5}$$

$\overline{G}$ is called the "splitting group" of G, since any projective unitary irreducible representation of G can be lifted to a *linear* unitary irreducible representation of $\overline{G}$.

As an example of a splitting group, we may take G to be the abelian group $\mathbb{R}^{2N}$; then $\mathfrak{g} = \mathbb{R}^{2N}$ and $H^2(\mathfrak{g}, \mathbb{R}) = \mathbb{R}$. If $\{X_1, \dots, X_N, Y_1, \dots, Y_N\}$ is a basis for $\mathfrak{g}$, then a basis for the extended Lie algebra $\overline{\mathfrak{g}}$ is $\{X_1, \dots, X_N, Y_1, \dots, Y_N, Z\}$, where Z is central and $[X_i, Y_j] = \delta_{ij} Z$. The splitting group $\overline{G}$ is thus the $(2N+1)$-dimensional Heisenberg group $\mathbb{H}_{2N+1}$.

A related example is the Galilean group $G = \mathbb{R}^4 \ltimes (\mathbb{R}^3 \ltimes SO(3))$, where $\ltimes$ denotes a semidirect product, parametrized by $(b, \boldsymbol{a}, \boldsymbol{v}, R)$ with $b \in \mathbb{R}$, $\boldsymbol{a}, \boldsymbol{v} \in \mathbb{R}^3$, $R \in SO(3)$, acting on $\mathbb{R}^4$ by $(b, \boldsymbol{a}, \boldsymbol{v}, R) \cdot (t, \boldsymbol{x}) = (t + b, R\boldsymbol{x} - \boldsymbol{v}t + \boldsymbol{a})$. Again $H^2(\mathfrak{g}, \mathbb{R}) = \mathbb{R}$, so the Lie algebra $\overline{\mathfrak{g}}$ of $\overline{G}$ is 11-dimensional and has a basis $\{H, P_1, P_2, P_3, J_1, J_2, J_3, K_1, K_2, K_3, M\}$ satisfying:

$$\begin{aligned} &[J_i, J_j] = \varepsilon_{ij}{}^k J_k, \qquad [J_i, P_j] = \varepsilon_{ij}{}^k P_k, \qquad [J_i, K_j] = \varepsilon_{ij}{}^k K_k, \\ &[K_i, H] = P_i, \qquad\quad [K_i, P_j] = \delta_{ij} M, \end{aligned} \tag{6}$$

where M is the central basis element. Notice that the subalgebra which is generated by the elements $\{K_1, K_2, K_3, P_1, P_2, P_3, M\}$ is the Heisenberg algebra $\mathfrak{h}_7$, so that the Heisenberg group is a subgroup of the splitting group $\overline{G}$ of the Galilei group.

A further example arises from the restricted Poincaré group $G = \mathbb{R}^4 \ltimes SO_0(3,1)$, with $\widetilde{G} = \mathbb{R}^4 \ltimes SL(2, \mathbb{C})$. Here $H^2(\mathfrak{g}, \mathbb{R}) = 0$, so that $\overline{G} = \widetilde{G}$ is again a 10-dimensional group.

The diagram (5) is also useful in a purely classical theory, as was noticed by Martínez-Alonso [16]: it enables us to lift affine symplectic actions of G to *linear* symplectic actions of $\overline{G}$. Thus, the G-homogeneous symplectic manifolds appear as coadjoint orbits of $\overline{G}$, which may then be called *elementary classical systems* [16].

A final ingredient of the theory should be a bridge between coadjoint orbits and linear unitary irreducible representations of $\overline{G}$. Indeed, for nilpotent groups, the Kirillov correspondence [17] provides this link.

For other classes of groups, such as semidirect products of abelian and semisimple groups, the situation is not so clear; suffice it to say that the Kirillov recipe for assigning orbits to representations is in general neither onto (the orbits must satisfy certain "integrality" conditions) nor one-to-one. The approach to be outlined here attempts to finesse this problem by assuming that in each particular case, an orbit and a representation have already been "matched" by some Kirillov-type recipe.

We turn now to explicit definitions. Let G be a connected Lie group (we will not consider discrete symmetries here), and let $\mathfrak{g}$ be its Lie algebra. Let $\mathfrak{g}^*$ denote the dual vector space of $\mathfrak{g}$. We take note of the *adjoint* and *coadjoint actions* of G, which are the representations:

$$\begin{aligned} \mathrm{Ad}\colon G \to \mathrm{End}(\mathfrak{g}), &\quad \text{determined by} \quad \exp[t(\mathrm{Ad}\, g)X] = g(\exp tX)g^{-1}, \\ \mathrm{Coad}\colon G \to \mathrm{End}(\mathfrak{g}^*), &\quad \text{defined by} \quad \langle(\mathrm{Coad}\, g)u, X\rangle = \langle u, (\mathrm{Ad}\, g^{-1})X\rangle, \end{aligned} \tag{7}$$

for $g \in G$, $X \in \mathfrak{g}$, $u \in \mathfrak{g}^*$. We will abbreviate $g \cdot u \equiv (\mathrm{Coad}\, g)u$.

In terms of the linear coordinate functions on $\mathfrak{g}^*$:

$$\xi_X(u) := \langle u, X\rangle, \quad \text{for} \quad X \in \mathfrak{g}, \tag{8}$$

the coadjoint action is determined by

$$\xi_X(g \cdot u) = \xi_{(\mathrm{Ad}\, g^{-1})X}(u). \tag{9}$$

Thus, in practice, one computes Ad by (7) and uses (9) to determine Coad in terms of the coordinate functions on $\mathfrak{g}^*$.

Now $\mathfrak{g}^*$ carries a natural *Poisson bracket* structure, given by

$$\{f, h\}_P(u) := \langle u, [df(u), dh(u)]\rangle,$$

since a covector $df(u)$ at $u \in \mathfrak{g}^*$ may be identified with an element of $\mathfrak{g}$. In particular,

$$\{\xi_X, \xi_Y\}_P(u) = \langle u, [X, Y]\rangle = \xi_{[X,Y]}(u). \tag{10}$$

If we write $x_i := \xi_{X_i}$ as X_i runs through a basis for $\mathfrak{g}$, then the Poisson brackets may be explicitly computed by

$$\begin{aligned} \{f, h\}_P &= \frac{\partial f}{\partial x_i}\frac{\partial h}{\partial x_j}\{\xi_{X_i}, \xi_{X_j}\}_P \\ &= \frac{\partial f}{\partial x_i}\frac{\partial h}{\partial x_j}\xi_{[X_i,X_j]} = c_{ij}^k \frac{\partial f}{\partial x_i}\frac{\partial h}{\partial x_j} x_k \end{aligned} \tag{11}$$

where c_{ij}^k are the structure constants for $\mathfrak{g}$.

The Poisson bracket structure is G-equivariant: if $f^g(u) := f(g^{-1} \cdot u)$, then

$$\big(\{f, h\}_P\big)^g = \{f^g, h^g\}_P\,,$$

as can be seen directly from (9) and (10). Thus this structure is foliated by the coadjoint orbits, and its restriction to any orbit M defines there a symplectic form ω_M (in particular, the dimension of M is always even); see [14, 18] for details.

The corresponding volume form

$$\lambda = (\text{constant})\, \omega_M \wedge \omega_M \wedge \cdots \wedge \omega_M \quad (\tfrac{1}{2}\dim M \text{ times})$$

is a G-invariant measure (the "Liouville measure") on M. We are free to choose the normalization constant as we think fit; it will be convenient to hold this choice in reserve for the moment.

3 Stratonovich–Weyl kernels

We shall now take as given: (i) a connected Lie group G (the splitting group of the original invariance group); (ii) a linear unitary irreducible representation U of G on some separable Hilbert space $\mathcal{H}$; and (iii) a coadjoint orbit M of G.

A *Stratonovich–Weyl kernel* for the triple (G, U, M) is a function Ω from M to a space of operators on $\mathcal{H}$, such that, for all $u \in M$:

$$\Omega(u) \text{ is selfadjoint;} \tag{12a}$$

$$\operatorname{Tr}[\Omega(u)] = 1; \tag{12b}$$

$$U(g)\,\Omega(u)\,U(g)^{-1} = \Omega(g \cdot u), \quad \text{for all} \quad g \in G; \tag{12c}$$

$$\int_M \operatorname{Tr}[\Omega(u)\,\Omega(v)]\,\Omega(v)\,d\lambda(v) = \Omega(u). \tag{12d}$$

In general, as we shall see, $\Omega(u)$ need not be trace-class, so that (12b) and (12d) should be understood in a weak sense, e.g., by considering distributions over M. However, we may evade this technical problem by using an alternative formulation. Write:

$$W_A(u) := \operatorname{Tr}[A\,\Omega(u)]. \tag{13}$$

The operator-function correspondence $A \mapsto W_A$ is what I shall call the *Stratonovich–Weyl correspondence*. We may supplement (12) with the requirement that $A \mapsto W_A$ be one-to-one (which follows automatically in many cases). Then (13) has an *inversion formula*:

$$A = \int_M W_A(u)\,\Omega(u)\,d\lambda(u). \tag{14}$$

To see this, let B be the right-hand side of (14), and notice that (13) and (12d) give

$$\begin{aligned} W_B(u) = \operatorname{Tr}[\Omega(u)\,B] &= \int_M W_A(v)\operatorname{Tr}[\Omega(u)\,\Omega(v)]\,d\lambda(v) \\ &= \operatorname{Tr}\left[A \int_M \Omega(v)\ \operatorname{Tr}[\Omega(u)\,\Omega(v)]\,d\lambda(v)\right] \\ &= \operatorname{Tr}[A\,\Omega(u)] = W_A(u), \end{aligned}$$

so that $B = A$. It turns out that the Weyl correspondence (3) is a particular case of (14); and it was Stratonovich [4] who emphasized the importance of going back and forth between functions and operators with the same operator kernel, as in (13) and (14).

From (12), (13) and (14), it is clear that the "symbols" W_A of self-adjoint operators A must be *real* functions on M; that W_1 is the constant function 1 (so that $\operatorname{Tr}[\Omega(u)] = 1$ can be replaced by the equivalent distributional identity $\int_M \Omega(u)\,d\lambda(u) = 1$); that (12c) yields the *covariance condition*:

$$W_{U(g)AU(g)^{-1}}(g \cdot u) \equiv W_A(u);$$

and that (12d) yields the *tracial property*:

$$\int_M W_A(u)\,W_B(u)\,d\lambda(u) = \operatorname{Tr}[AB]. \tag{15}$$

The equation (15) is the centerpiece of a Moyal formulation: it means that the quantum expectation value $\operatorname{Tr}[AB]$ is computed by integrating the product of the corresponding symbols W_A, W_B over phase space. If B is a density matrix, W_B is its "Wigner function"; and in general we may expect it to take some negative values, which emphasizes its non-classical nature. (For a discussion of the nonnegativity of Wigner functions in the "flat case", see [19–21].)

The traces of products of the operators $\Omega(u)$ define interesting functions [4]. Postulate (12d) says that

$$K(u,v) := \operatorname{Tr}[\Omega(u)\,\Omega(v)]$$

is a reproducing kernel for a space of symbols W_A. It turns out that the trikernel

$$L(u,v,w) := \operatorname{Tr}[\Omega(u)\,\Omega(v)\,\Omega(w)] \tag{16}$$

is all we need to define a *twisted product* over M:

$$(f \times h)(u) := \int_M \int_M L(u,v,w)\, f(v)\, h(w)\, d\lambda(v)\, d\lambda(w). \tag{17}$$

The cyclicity of the kernel (16), together with (15), gives the required associativity of the twisted product. The covariance postulate (12c) gives *equivariance* of the twisted product:

$$(f \times h)^g = f^g \times h^g.$$

The remaining postulates of (12) give supplementary information about the twisted product [22]. Thus, (12a) yields $\overline{L(u,v,w)} = L(u,w,v)$, and (12b) gives

$$\int_M L(u,v,w)\, d\lambda(u) = K(v,w),$$

which in turn yields the *tracial identity* for the twisted product:

$$\int_M (f \times h)(u)\, d\lambda(u) = \int_M f(u)\, h(u)\, d\lambda(u).$$

This tracial identity is the essential property of the twisted product – indeed, it is a version of the Moyal property (15) – and has many interesting mathematical consequences, such as the possibility of extending the twisted product to distributions on M [23, 24], or developing harmonic analysis in the phase-space framework [5]. We shall not explore this issue further here.

4 Examples

Example 1 (The flat case)**.** Here $G = \mathbb{H}_{2N+1}$, the $(2N+1)$-dimensional Heisenberg group; a typical element is $g = (\boldsymbol{a}, \boldsymbol{b}, c)$ with $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^N$, $c \in \mathbb{R}$, and

$$(\boldsymbol{a}_1, \boldsymbol{b}_1, c_1)(\boldsymbol{a}_2, \boldsymbol{b}_2, c_2) = \big(\boldsymbol{a}_1 + \boldsymbol{a}_2, \boldsymbol{b}_1 + \boldsymbol{b}_2, c_1 + c_2 + \tfrac{1}{2}(\boldsymbol{a}_1 \cdot \boldsymbol{b}_2 - \boldsymbol{a}_2 \cdot \boldsymbol{b}_1)\big).$$

One takes a basis $\{X_1, \dots, X_N, Y_1, \dots, Y_N, Z\}$ for $\mathfrak{g}$, so that $[X_i, Y_j] = \delta_{ij} Z$. By writing $g = \exp(\boldsymbol{a} \cdot \boldsymbol{X}) \exp(\boldsymbol{b} \cdot \boldsymbol{Y}) \exp(cZ)$, one computes, using (9), that

$$(\boldsymbol{a}, \boldsymbol{b}, c) \cdot (\boldsymbol{x}, \boldsymbol{y}, z) = (\boldsymbol{x} + z\boldsymbol{b}, \boldsymbol{y} - z\boldsymbol{a}, z) \tag{18}$$

where $x_1, \dots, x_N, y_1, \dots, y_N, z$ are the coordinate functions (8) on $\mathfrak{g}^*$ for the chosen basis of $\mathfrak{g}$. Thus, as is well known, the coadjoint orbits of $\mathbb{H}_{2N+1}$ are either points $(\boldsymbol{x}, \boldsymbol{y}, 0)$ or hyperplanes M_z with z = nonzero constant.

Fix $z \neq 0$, and define *orbit coordinates* $(\boldsymbol{q}, \boldsymbol{p}) \in M_z$ by

$$\boldsymbol{p} := \boldsymbol{y}, \quad \boldsymbol{q} := \frac{\boldsymbol{x}}{z}\,.$$

Then (11) gives the Poisson brackets:

$$\{q_i, q_j\}_P = \{p_i, p_j\}_P = 0, \quad \{q_i, p_j\}_P = 1,$$

so that $(\boldsymbol{q}, \boldsymbol{p})$ are canonical coordinates, and $M_z \simeq \mathbb{R}^{2N}$ as a symplectic manifold. The coadjoint action (18) is expressed in the orbit coordinates as:

$$(\boldsymbol{a}, \boldsymbol{b}, c) \cdot (\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{q} + \boldsymbol{b}, \boldsymbol{p} - z\boldsymbol{a}). \tag{19}$$

For simplicity, we shall take the "Planck constant" $z = 1$. Then, if we consider the unitary irreducible representation U of $\mathbb{H}_{2N+1}$ acting on $\mathcal{H} = L^2(\mathbb{R}^N, d^N\boldsymbol{\xi})$ by

$$\big[U(\boldsymbol{a}, \boldsymbol{b}, c)\Psi\big](\boldsymbol{\xi}) := \exp\{-i(c + \boldsymbol{b} \cdot \boldsymbol{\xi} + \tfrac{1}{2}\boldsymbol{a} \cdot \boldsymbol{b})\}\Psi(\boldsymbol{\xi} + \boldsymbol{a}),$$

it may be verified directly that the Grossmann–Royer operators (4), with $\hbar = 1$, satisfy (12): that is, $\Omega(\boldsymbol{q}, \boldsymbol{p}) = \Pi(\boldsymbol{q}, \boldsymbol{p})$. Thus the Weyl correspondence is a special case of (14).

Example 2 (The finite Heisenberg group)**.** The finite Heisenberg group A_N [11, 12] is a central extension of the abelian group $\mathbb{Z}_N \times \mathbb{Z}_N$ by $\mathbb{Z}_N$. The coadjoint orbits have no place here, of course, but one may consider the group A_N acting on the finite torus $\mathbb{Z}_N \times \mathbb{Z}_N$ by

$$(a, b, c) \cdot (x, y) := (x + b, y - ca)$$

in a formal analogy with (19). Then, for $c = 1$ and for odd N, one may obtain a solution $\Omega(x, y) = \Delta_{xy} \in \mathbb{C}^{N\times N}$, where Δ_{xy} are the "Fano operators":

$$\Delta_{xy}(k) := \exp\Big\{\frac{4\pi i}{N} x(y - k)\Big\}\Psi(2y - k).$$

However, there are several other solutions, obtained by including an extra factor of the form $\exp\{2\pi i\eta(x, y)/N\}$ on the right hand side, so that the postulates (12) do not single out a unique "best" kernel in this discrete case [25].

Example 3 (The pure spin case)**.** The details for $G = \mathrm{SU}(2)$, the invariance group of spin, have been worked out by J. M. Gracia-Bondía and myself [5]. The Moyal formulation thereby obtained allows one to study spin dynamics as a flow on the 2-sphere acting on suitable Wigner functions. Here I shall summarize the method to obtain a Stratonovich–Weyl kernel.

We first note that $\mathrm{SU}(2)/\{\pm 1\} = \mathrm{SO}(3)$ and the covering homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$ can be identified with the coadjoint action, since $\mathfrak{g}^* \simeq \mathbb{R}^3$. If $\widetilde{R} \in \mathrm{SU}(2)$, and if $R \in \mathrm{SO}(3)$ denotes the corresponding rotation, then $\widetilde{R} \cdot \boldsymbol{x} = R\boldsymbol{x} \in \mathbb{R}^3$. Thus the coadjoint orbits are the point $\{0\}$ and spheres centred at 0. Fix the sphere $\mathbb{S}^2$ of radius 1, writing its elements as $\boldsymbol{n} = (\theta, \phi)$ in spherical coordinates.

Let $\mathcal{D}^j$ be the unitary irreducible representation of SU(2) on $\mathbb{C}^{2j+1}$ ($j = \frac{1}{2}, 1, \frac{3}{2}, \dots$). Then the Stratonovich–Weyl kernel $\boldsymbol{\Omega}^j$ for $(\mathrm{SU}(2), \mathcal{D}^j, \mathbb{S}^2)$ will be a $(2j+1) \times (2j+1)$-matrix-valued function, with matrix elements $Z^j_{rs}(\boldsymbol{n})$ ($r, s = -j, -j+1, \dots, j$). The covariance condition (12c) gives

$$Z^j_{rs}(R\boldsymbol{n}) = \sum_{p,q=-j}^{j} \mathcal{D}^j_{rp}(\widetilde{R})\, Z^j_{pq}(\boldsymbol{n}) \overline{\mathcal{D}}^j_{sq}(\widetilde{R}).$$

Writing

$$Y'_{lm}(\boldsymbol{n}) := \sum_{k=-j}^{j} \sqrt{\frac{2l+1}{4\pi}} \left\langle \begin{matrix} j & l \\ k & m \end{matrix} \middle| \begin{matrix} j \\ k+m \end{matrix} \right\rangle Z^j_{k,k+m}(\boldsymbol{n}),$$

where the $\left\langle \begin{matrix} j & l \\ k & m \end{matrix} \middle| \begin{matrix} j \\ k+m \end{matrix} \right\rangle$ are Clebsch–Gordan coefficients, we find that

$$Y'_{lm}(R\boldsymbol{n}) := \sum_{p=-l}^{l} \overline{\mathcal{D}}^l_{mp}(\widetilde{R})\, Y'_{lp}(\boldsymbol{n}),$$

so that $Y'_{lm}(\boldsymbol{n}) = \varepsilon^j_l Y_{lm}(\boldsymbol{n})$, where the latter are the usual spherical harmonics. Since $K^j(\boldsymbol{m}, \boldsymbol{n}) = \sum_{l=0}^{2j} \sum_{p=-l}^{l} Y_{lp}(\boldsymbol{m}) \overline{Y}_{lm}(\boldsymbol{n})$ is the appropriate reproducing kernel, the tracial postulate (12d) reduces to $|\varepsilon^j_l|^2 = l$. Now (12a) says that ε^j_l is real, so $\varepsilon^j_l = \pm 1$, and (12b) shows that $\varepsilon^j_0 = +1$. For convenience, we select $\varepsilon^j_l = +1$ for *all* l, and arrive at

$$Z^j_{rs}(\boldsymbol{n}) = \sum_{l=0}^{2j} \sqrt{\frac{4\pi(2l+1)}{2j+1}} \left\langle \begin{matrix} j & l \\ r & s-r \end{matrix} \middle| \begin{matrix} j \\ s \end{matrix} \right\rangle Y_{l,s-r}(\boldsymbol{n}), \tag{20}$$

which are the symbols $W_{|jr\rangle\langle js|}$ of the spin transitions $|jr\rangle\langle js|$. For $r = s$, we obtain the Wigner functions for the pure spin states:

$$Z^j_{rr}(\boldsymbol{n}) = \sum_{l=0}^{2j} \frac{2l+1}{2j+1} \left\langle \begin{matrix} j & l \\ r & 0 \end{matrix} \middle| \begin{matrix} j \\ r \end{matrix} \right\rangle P_l(\cos\theta). \tag{21}$$

The formulas (20) and (21) may be taken as the starting point for a theory of spin which makes no explicit mention of the operator formalism. For instance, the symbol of the J_z spin operator is

$$\sum_{m=-j}^{j} m Z^j_{mm}(\boldsymbol{n}) = \sqrt{j(j+1)}\, \cos\theta.$$

The trikernel for the twisted product is

$$L^j(\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{k}) = \sum_{r,s,t=-j}^{j} Z^j_{rs}(\boldsymbol{m})\, Z^j_{st}(\boldsymbol{n})\, Z^j_{tr}(\boldsymbol{k})$$

and the Liouville measure on $\mathbb{S}^2$ should be normalized as $d\lambda(\boldsymbol{n}) = \frac{2j+1}{4\pi} \sin\theta \, d\theta \, d\phi$, in order that (20) and (12d) be compatible. For instance, if $j = \frac{1}{2}$, we get

$$L^{1/2}(\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{k}) = \pi^2\big(1 + 3(\boldsymbol{m} \cdot \boldsymbol{n} + \boldsymbol{n} \cdot \boldsymbol{k} + \boldsymbol{k} \cdot \boldsymbol{n}) + 3\sqrt{3}\, i[\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{k}]\big).$$

In practice, the twisted product is computed from the observation that $Z^j_{rs} \times Z^j_{tu} = \delta_{st} Z^j_{ru}$, which makes the explicit evaluation of L^j superfluous. Several applications of the formalism to spin calculations are developed in [5]; in particular, the proof of the Majorana formula reduces to a few lines.

Example 4 (Poincaré disk quantization)**.** A favourite test case for those who propose quantization schemes is to quantize the Poincaré disk. For this case, the invariance group is usually taken to be $\mathrm{SL}(2, \mathbb{R})$. The Stratonovich–Weyl recipe also applies here. The material of this subsection is due to Héctor Figueroa [26].

Take $G = \mathrm{SL}(2, \mathbb{R})$. The commutation relations for $\mathfrak{g}$, with the basis:

$$Z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad W = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

are

$$[Z, X] = W, \quad [Z, W] = X, \quad [X, W] = -Z.$$

Any $g \in G$ can be written as $g = \exp(sZ)\exp(tX)\exp(\theta W)$, with $s, t \in \mathbb{R}$, $-\pi < \theta \leqslant \pi$. Letting $z = \xi_Z$, $x = \xi_X$, $w = \xi_W$ denote the coordinate functions on $\mathfrak{g}^* \simeq \mathbb{R}^3$, we compute from (9) that

$$\begin{aligned} \exp(sZ) \cdot (z, x, w) &= (z, x\cosh s - w\sinh s, w\cosh s - x\sinh s), \\ \exp(tX) \cdot (z, x, w) &= (z\cosh t + w\sinh t, x, w\cosh t + z\sinh t), \\ \exp(\theta W) \cdot (z, x, w) &= (z\cos\theta + x\sin\theta, x\cos\theta - z\sin\theta, w). \end{aligned}$$

From this it is clear that Coad is the homomorphism $\mathrm{SL}(2, \mathbb{R}) \to \mathrm{SO}(2, 1)$, and that the "Casimir function" $C = z^2 + x^2 - w^2$ is invariant. The coadjoint orbits of $\mathrm{SL}(2, \mathbb{R})$ are thus: halves of two-sheeted hyperboloids, for $C < 0$; one-sheeted hyperboloids, for $C > 0$; a point and two half-cones, for $C = 0$. The Kirillov correspondence [27] associates two principal-series representations to each orbit with $C > 0$, and a discrete-series representation to each half-hyperboloid with $C = -n^2$ ($n = 1, 2, 3, \dots$).

Let M_n^+ be the orbit given by $C = -n^2$, $w \geqslant n$, for $n = 1, 2, 3, \dots$. One may choose coordinates on M_n^+ as $u = (t, \phi)$, where

$$u = (n\sinh t\cos\phi, n\sinh t\sin\phi, n\cosh t) = \exp(-\phi W)\exp(tX) \cdot (0, 0, n) \in \mathfrak{g}^*.$$

Alternatively, one could choose (w, ϕ) as coordinates, since they form a canonical pair: $\{w, \phi\}_P = 1$. The stereographic projection centred at $(0, 0, -1)$:

$$r = \frac{n\sinh t}{n\cosh t + 1}, \quad \phi = \phi$$

identifies $(t, \phi) \in M_n^+$ with the point with polar coordinates $(r, \phi) \in \mathbb{D}$, the Poincaré disk, so that the disk is indeed seen as a "phase space" for the action of $\mathrm{SL}(2, \mathbb{R})$, and there is a whole series of distinct quantizations of $\mathbb{D}$.

The discrete series representations of $\mathrm{SL}(2,\mathbb{R})$ are more conveniently expressed using the Cayley transform $C = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}$ to identify $\mathrm{SL}(2,\mathbb{R})$ with $\mathrm{SU}(1,1) = C\,\mathrm{SL}(2,\mathbb{R})\,C^{-1}$. Then

$$\exp(\theta W) \mapsto C\exp(\theta W)C^{-1} = \begin{pmatrix}e^{i\theta/2} & 0\\ 0 & e^{-i\theta/2}\end{pmatrix}.$$

The representation U_n acts on $\mathcal{H} = L^2_{\mathrm{hol}}\big(\mathbb{D}, (1-|\zeta|^2)^{n-1}\,d\zeta\,d\bar\zeta\big)$ according to

$$\left[U_n\begin{pmatrix}\alpha & \beta\\ \bar\beta & \bar\alpha\end{pmatrix}\Psi\right](\zeta) := (-\bar\beta\zeta + \alpha)^{-n-1}\Psi\left(\frac{\bar\alpha\zeta - \beta}{-\bar\beta\zeta + \alpha}\right).$$

Note that $[U_n(\exp(\theta W))\Psi](\zeta) = e^{-i(n+1)\theta/2}\Psi(e^{i\theta}\zeta)$, so that $\Psi_m : \zeta \mapsto \zeta^m$ is a joint eigenvector of this one-parameter subgroup ($m = 0,1,2,\dots$); traces are computed by $\operatorname{Tr} A = \sum_{m=0}^{\infty}\langle\Psi_m \mid A \mid \Psi_m\rangle$. With these tools, it is straightforward to verify that

$$\begin{aligned}&[\Omega_n(t,\phi)\Psi](\zeta)\\ &\quad := 2\left(\frac{\cosh\frac{t}{2} - ie^{i\phi}\zeta\sinh\frac{t}{2}}{\cosh\frac{t}{2} - i\zeta\sinh\frac{t}{2}}\right)^{n+1}(\cosh t - ie^{i\phi}\zeta\sinh t)^{-n-1}\Psi\left(\frac{\zeta\cosh t + ie^{-i\phi}\sinh t}{i\zeta\sinh t - e^{-i\phi}\cosh t}\right)\end{aligned}$$

satisfies the conditions (12) and so is a Stratonovich–Weyl kernel for $(\mathrm{SL}(2,\mathbb{R}), U_n, M_n^+)$.

In particular, notice that

$$[\Omega_n(0,0)\Psi](\zeta) = 2\Psi(-\zeta),$$

so that the parity operator of $\mathbb{D}$ yields the quantization recipe. This point of view has been exploited [28] to derive quantizations with noncompact phase spaces. Indeed, starting with Grossmann and Royer [7,8], the parity operator appears as a general talisman for quantization schemes; however, this Ansatz breaks down in compact cases, such as the spin case of the previous subsection, where a Stratonovich–Weyl kernel may nevertheless be constructed.

Example 5 (Galilean spinning particles)**.** For an elementary system corresponding to a nonrelativistic particle, the appropriate invariance group is the Galilean group; let G be its splitting group. An element of G may be written as

$$g = (\theta; b, \boldsymbol{a}, \boldsymbol{v}, \widetilde{R}) = (\exp(-\theta M); \exp(-bH)\exp(-\boldsymbol{a}\cdot\boldsymbol{P})\exp(-\boldsymbol{v}\cdot\boldsymbol{P})\widetilde{R})$$

with $\theta, b \in \mathbb{R}$, $\boldsymbol{a}, \boldsymbol{v} \in \mathbb{R}^3$, $\widetilde{R} \in \mathrm{SU}(2)$. The multiplication law is

$$\begin{aligned}&(\theta_1; b_1, \boldsymbol{a}_1, \boldsymbol{v}_1, \widetilde{R}_1)(\theta_2; b_2, \boldsymbol{a}_2, \boldsymbol{v}_2, \widetilde{R}_2)\\ &\qquad = \big(\theta_1 + \theta_2 + \tfrac12(\boldsymbol{a}_1\cdot R_1\boldsymbol{v}_2 - \boldsymbol{v}_1\cdot R_1\boldsymbol{a}_2 - b_2\boldsymbol{v}_1\cdot R_1\boldsymbol{v}_2); (b_1, \boldsymbol{a}_1, \boldsymbol{v}_1, \widetilde{R}_1)\,(b_2, \boldsymbol{a}_2, \boldsymbol{v}_2, \widetilde{R}_2)\big)\end{aligned}$$

with

$$(b_1, \boldsymbol{a}_1, \boldsymbol{v}_1, \widetilde{R}_1)\,(b_2, \boldsymbol{a}_2, \boldsymbol{v}_2, \widetilde{R}_2) = (b_1 + b_2, \boldsymbol{a}_1 + R_1\boldsymbol{a}_2 - b_2\boldsymbol{v}_1, \boldsymbol{v}_1 + R_1\boldsymbol{v}_2, \widetilde{R}_1\widetilde{R}_2).$$

Once again, using (9) one can compute the coadjoint action explicitly; and it turns out that there appear three invariant "Casimir functions" of the coordinates $m, h, \boldsymbol{p}, \boldsymbol{j}, \boldsymbol{k}$ corresponding to the Lie algebra generators of (6). These are: m itself (the Galilean mass), $u = 2mh - \boldsymbol{p}\cdot\boldsymbol{p}$ and $|m\boldsymbol{j} + \boldsymbol{p}\times\boldsymbol{k}|^2$. The generic coadjoint orbits thus have dimension 8.

Select an orbit M by fixing $m > 0$, $u \in \mathbb{R}$ and $s := |\boldsymbol{j} + \frac{1}{m}\boldsymbol{p} \times \boldsymbol{k}| > 0$. On this orbit, coordinates may be chosen as follows: let

$$\boldsymbol{p} = \boldsymbol{p}, \quad \boldsymbol{q} := \boldsymbol{k}/m, \quad \boldsymbol{s} := (\boldsymbol{j} + \tfrac{1}{m}\boldsymbol{p} \times \boldsymbol{k})/s = (\boldsymbol{j} - \boldsymbol{q} \times \boldsymbol{p})/s.$$

Here $\boldsymbol{p}$ and $\boldsymbol{q}$ range freely over $\mathbb{R}^3$, while $\boldsymbol{s}$ is a unit vector in $\mathbb{S}^2$. Moreover, using (6) and (11), one may compute that:

$$\begin{aligned} \{q_i, q_j\}_P &= \{p_i, p_j\}_P = 0, \qquad \{q_i, p_j\}_P = \delta_{ij}\,, \\ \{q_i, s_j\}_P &= \{p_i, s_j\}_P = 0, \qquad \{s_i, s_j\}_P = \varepsilon_{ij}{}^k s_k\,. \end{aligned}$$

Thus $M \simeq \mathbb{R}^6 \times \mathbb{S}^2$ as a symplectic manifold.

The corresponding unitary representation of G may be taken to be

$$\begin{aligned} &\big[U(\theta; b, \boldsymbol{a}, \boldsymbol{v}, \widetilde{R})\Psi\big](\boldsymbol{\xi}) \\ &\quad := \exp\biggl\{i\Bigl(\theta + bu + \frac{b}{2m}\boldsymbol{\xi}\cdot\boldsymbol{\xi} - \boldsymbol{a}\cdot\boldsymbol{\xi} - \frac{m}{2}\boldsymbol{a}\cdot\boldsymbol{v}\Bigr)\biggr\}\mathcal{D}^j(\widetilde{R})\,\Psi\big(R^{-1}(\boldsymbol{\xi} + m\boldsymbol{v})\big), \end{aligned}$$

acting on $\mathcal{H} = L^2(\mathbb{R}^3, d^3\boldsymbol{\xi}) \otimes \mathbb{C}^{2j+1}$.

It only remains to produce the Stratonovich–Weyl kernel for (G, U, M). In a recent paper [6], J. M. Gracia-Bondía and I have established that (12) is satisfied by

$$[\Omega(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{s})\Psi](\boldsymbol{\xi}) := 2^3 \exp\bigl\{2i\boldsymbol{q}\cdot(\boldsymbol{p} - \boldsymbol{\xi})\bigr\}\Omega^j(\boldsymbol{s})\,\Psi(2\boldsymbol{p} - \boldsymbol{\xi}),$$

where $\Omega^j(\boldsymbol{s})$ is the spin kernel of Example 3. Thus, for Galilean spinning particles, the Stratonovich–Weyl kernel is a direct product of the Grossmann–Royer kernel on $\mathbb{R}^6$ and the j-spin kernel on $\mathbb{S}^2$.

5 Outlook

The above set of examples indicates that the concept of a Stratonovich–Weyl kernel is the appropriate one to build a Moyal quantum kinematics for free particles with any given connected invariance group. To justify this hope in a general setting requires some existence (and, where possible, uniqueness) theorems for Stratonovich–Weyl kernels. The above list, while very suggestive, does not provide this existence proof. It is therefore comforting to know that the same scheme also goes through for the Poincaré group: J. F. Cariñena, J. M. Gracia-Bondía and I have constructed a relativistic Stratonovich–Weyl kernel, yielding twisted products and Wigner functions for Klein–Gordon and Dirac particles [29]. In contrast to the usual procedure [30–32] of constructing "relativistic Wigner functions" by a Minkowskian analogy with the "flat" Wigner functions (2), our group-theoretic approach avoids the awkward phenomenon of "leaving the mass shell". However, several subtle points arise in the relativistic construction, so I shall not discuss it further here.

The precise role of the parity operators on noncompact phase spaces remains a puzzle. Indications exist that the postulates (12) suffice in many cases to prove a uniqueness theorem for Ω, via a procedure which yields a family of eigenvalues of Ω which in noncompact cases alternate between $+1$ and -1, from which a parity operator may be reconstructed. The compact case of Example 3, which is a subcase of the Galilean Example 5, shows however that in general the parity-operator Ansatz is misleading, and one must look deeper for a quantization recipe.

The uniqueness question is important since, insofar as uniqueness holds, there will be only one "correct" recipe for Wigner functions of a given type. (For $G = \mathrm{SU}(2)$, the undetermined signs $\varepsilon^j_1, \dots, \varepsilon^j_{2j}$ give a technical counterexample to uniqueness, so the general problem remains open.) These "correct" Wigner functions may then form the basis for moving beyond kinematical issues to develop Moyal Quantum Mechanics in more intricate contexts.

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