Mathematical Social Sciences 71 (2014) 6–11

Contents lists available at ScienceDirect

Mathematical Social Sciences

journal homepage: www.elsevier.com/locate/econbase

Axiomatic characterizations of the weighted solidarity values

Emilio Calvo [a], Esther Gutiérrez-López [b,*]

[a] Department of Economic Analysis and ERI-CES, Universitat de Valencia, Spain
[b] Departamento de Economía Aplicada IV, Universidad del País Vasco U.P.V./E.H.U., Spain

HIGHLIGHTS

- We define and characterize the family of all weighted solidarity values.
- We present two axiomatizations, one with additivity, and the other, without it.
- We study the behavior of these values in the class of monotonic games.

ARTICLE INFO

Article history:
Received 7 October 2013
Received in revised form
21 February 2014
Accepted 21 March 2014
Available online 13 April 2014

ABSTRACT

We define and characterize the class of all *weighted solidarity values*. Our first characterization employs the classical axioms determining the solidarity value (except *symmetry*), that is, *efficiency*, *additivity* and *the A-null player axiom*, and two new axioms called *proportionality* and *strong individual rationality*. In our second axiomatization, the *additivity* and *the A-null player* axioms are replaced by a new axiom called *average marginality*.

1. Introduction

When a cooperative solution is considered from an axiomatic point of view, asymmetric versions of the value appear when the property of *symmetry* is dropped from the set of axioms that characterizes the value. The grounds justifying each asymmetric value depend on the context at hand. It could be differences in the negotiation ability of players or because they are representatives of groups of different size, etc. Thus, it seems to be more realistic to introduce some "weights" associated to the players in order to measure these differences.

The first nonsymmetric generalization of a value in coalitional form games with transferable utility is due to Shapley (1953a). He defines the family of weighted Shapley values associated to positive *weights* for the players. Kalai and Samet (1987) extend the notion of "weights" to "weight systems", allowing a weight of zero for some players. They also characterize the family of all weighted Shapley values axiomatically using *efficiency*, *additivity*, *null player axiom*, and two new axioms called *positivity* and *partnership consistency*. Hart and Mas-Colell (1989) provide a different axiomatization with *monotonicity* and *consistency*, among other axioms, but without additivity. Nowak and Radzik (1995), assuming that the weights of the players are given exogenously, provide two axiomatic characterizations of the corresponding weighted Shapley value: the first one, using the classical axioms determining the Shapley value, but replacing symmetry by a new axiom called ω*—mutual dependence*; the second one, adding a property called *marginality*, introduced by Young (1985), but removing additivity and the null player axiom. They also provide a characterization of the family of all weighted Shapley values.

The basic principle behind the weighted Shapley values is to pay players according to their productivity. A direct consequence is that null players always receive zero payoff, and this is the content of the null player axiom. Nevertheless, it is very easy to find real-life examples where a greater degree of solidarity among players seems to be natural.

There are several values that do not satisfy the null player axiom. We focus here on the *solidarity value*, introduced by Sprumont (1990). The Shapley value is based on the individual marginal contributions of a player to the coalitions she belongs to. In the solidarity value the individual marginal contribution is replaced by the average of the marginal contributions of all players which are in the coalition. This means that the individual contribution of each player is also shared among her partners in the game, being this a feasible way to express a certain degree of solidarity between the players in the cooperative game. Nowak and Radzik (1994) characterize this value axiomatically by means of the same axioms as

* Corresponding author.
E-mail addresses: emilio.calvo@uv.es (E. Calvo), mariaester.gutierrez@ehu.es (E. Gutiérrez-López).

http://dx.doi.org/10.1016/j.mathsocsci.2014.03.005
0165-4896/

the Shapley value but replacing the null player axiom by the A-null player axiom (A-null stands average null). In this case, a player receives zero if the average of the marginal contributions is zero for all the coalitions he belongs to.

This paper defines and characterizes the family of all *weighted solidarity values* associated to positive weights for the players. Our first characterization (Theorem 6) employs the classical axioms determining the solidarity value (except *symmetry*), that is, *efficiency*, *additivity* and *the A-null player axiom*, and two new axioms called *proportionality* and *strong individual rationality*. This result is analogous to that of Kalai and Samet (1987), since the *proportionality* axiom can be seen as a variant of their axiom of *partnership consistency* and *strong individual rationality* is a weaker version of *positivity*. In our second axiomatization (Theorem 7), the *additivity* and *the A-null player* axioms are dropped and replaced by a new axiom called *average marginality*. This last property is similar to that of Young (1985) but with the average of the marginal contributions instead of the individual marginal contribution. In these two results, axioms imply the existence of a weight system such that the value is precisely the corresponding weighted solidarity value. Thus, weights are obtained endogenously. Finally, we study the behavior of the weighted solidarity values in the class of monotonic games. Contrary to the weighted Shapley values, there is a positive relationship between players' weights and their bargaining power (Theorem 8).

The paper is organized as follows. Section 2 is devoted to some preliminaries. Section 3 defines the family of all weighted solidarity values. Finally, we provide the axiomatic characterizations in Section 4.

2. Preliminaries

A *cooperative game* with transferable utility (TU-game) is a pair (N, v) where $N \subsetneq \mathbb{N}$[1] is a nonempty and finite set and $v : 2^N \to \mathbb{R}$ is a *characteristic function*, satisfying $v(\emptyset) = 0$. An element i of N is called a *player* and every nonempty subset S of N a *coalition*. The real number $v(S)$ is called the *worth* of coalition S, and it is interpreted as the total payoff that the coalition S, if it forms, can obtain for its members. Let $\mathcal{G}^N$ denote the set of all cooperative TU-games with player set N and let $\mathcal{G}$ denote the set of all games, that is, $\mathcal{G} = \cup_{\emptyset \neq N \subsetneq \mathbb{N}} \mathcal{G}^N$.

For all $S \subseteq N$, we denote the restriction of (N, v) to S as (S, v). For simplicity, we write $S \cup i$ instead of $S \cup \{i\}$, $N \backslash i$ instead of $N \backslash \{i\}$, and $v(i)$ instead of $v(\{i\})$. For each vector $x \in \mathbb{R}^N$, let $x(S) := \sum_{i \in S} x_i$ for each $S \subseteq N$.

A *value* is a function γ which assigns to every TU-game (N, v) and every player $i \in N$, a real number $\gamma_i(N, v)$, which represents an assessment made by i of his gains from participating in the game. A *payoff configuration* is an element of $\prod_{S \subseteq N} \mathbb{R}^S$.

Let (N, v) be a game. For all $S \subseteq N$ and all $i \in S$, define

$$\Delta^i(v, S) := v(S) - v(S \setminus i).$$

We call $\Delta^i(v, S)$ the *marginal contribution of player i to coalition S in the TU-game* (N, v). The *Shapley value* (Shapley, 1953b) of the game (N, v) is the payoff vector $Sh(N, v) \in \mathbb{R}^N$ defined by

$$Sh_i(N, v) = \sum_{S \subseteq N : i \in S} \frac{(n-s)!\,(s-1)!}{n!} \Delta^i(v, S), \quad \text{for all } i \in N,$$

where $s = |S|$ and $n = |N|$.

Two players $i, j \in N$ are *symmetric* in (N, v) if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \backslash \{i, j\}$. Player $i \in N$ is a *null player* in (N, v) if $v(S \cup i) = v(S)$ for all $S \subseteq N \backslash i$. For any two games (N, v) and (N, v'), the game $(N, v + v')$ is defined by $(v + v')(S) = v(S) + v'(S)$ for all $S \subseteq N$.

Consider the following properties of a value γ in $\mathcal{G}^N$:

Efficiency: for all (N, v), $\sum_{i \in N} \gamma_i(N, v) = v(N)$.

Additivity: for all (N, v) and (N, v'), $\gamma(N, v + v') = \gamma(N, v) + \gamma(N, v')$.

Symmetry: for all (N, v) and all $\{i, j\} \subseteq N$, if i and j are symmetric players in (N, v), then $\gamma_i(N, v) = \gamma_j(N, v)$.

Null player axiom: for all (N, v) and all $i \in N$, if i is a null player in (N, v), then $\gamma_i(N, v) = 0$.

The following theorem is due to Shapley (1953b).

Theorem 1 (*Shapley, 1953b*)**.** *A value γ on $\mathcal{G}^N$ satisfies* efficiency, additivity, symmetry *and* null player axiom *if, and only if, γ is the Shapley value.*

For all $\emptyset \neq T \subseteq N$, *the unanimity game of the coalition* T, (N, u_T), is defined by

$$u_T(S) = \begin{cases} 1 & \text{if } S \supseteq T, \\ 0 & \text{otherwise.} \end{cases}$$

It is well known that the family of games $\{(N, u_T)\}_{\emptyset \neq T \subseteq N}$ is a basis for $\mathcal{G}^N$. This allows an alternative definition of the Shapley value as the linear mapping $Sh : \mathcal{G}^N \longrightarrow \mathbb{R}^N$, which is defined for all unanimity game (N, u_T) as follows

$$Sh_i(N, u_T) = \begin{cases} \dfrac{1}{|T|} & \text{if } i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

The *solidarity value*, *Sl*, was introduced by Sprumont (1990), Section 5, in a recursive way. Let (N, v) be a game. For all $S \subseteq N$, define

$$\Delta^{av}(v, S) := \frac{1}{s} \sum_{i \in S} \Delta^i(v, S).$$

Thus, $\Delta^{av}(v, S)$ is the *average of the marginal contributions* of players within coalition S in the game (N, v). Then, the solidarity value is defined by

$$Sl_i(S, v) = \frac{1}{s} \Delta^{av}(v, S) + \sum_{j \in S \setminus i} \frac{1}{s} Sl_i(S \setminus j, v), \quad \text{for all } i \in S \subseteq N, \tag{1}$$

starting with

$$Sl_i(\{i\}, v) = v(i), \quad \text{for all } i \in N.$$

Later on, Nowak and Radzik (1994) yield a different definition of *Sl*, similar to that of the Shapley value, but with the average of the marginal contributions instead of the individual marginal contribution:

$$Sl_i(N, v) = \sum_{S \subseteq N : i \in S} \frac{(n-s)!\,(s-1)!}{n!} \Delta^{av}(v, S), \quad \text{for all } i \in N. \tag{2}$$

Calvo (2008) shows that both definitions, (1) and (2), are equivalent.

The solidarity value satisfies some solidarity principle, since null players can obtain positive payoffs (see Example 1.1 in Nowak and Radzik (1994)). They introduce a variation of the null player axiom in order to characterize *Sl* on $\mathcal{G}^N$. Player $i \in N$ is an *A-null player* in (N, v) if $\Delta^{av}(v, S) = 0$ for all coalition $S \subseteq N$ containing i. There is clearly no relation between the null player and the A-null player concepts. The solidarity value satisfies the following axiom.

A-Null player axiom: for all (N, v) and all $i \in N$, if i is an A-null player, then $\gamma_i(N, v) = 0$.

[1] $\mathbb{N}$ is the set of the nature numbers.

The following theorem is due to Nowak and Radzik (1994).

Theorem 2 (*Nowak and Radzik, 1994*). *A value γ on $\mathcal{G}^N$ satisfies* efficiency, additivity, symmetry *and* the A-null player axiom *if, and only if, γ is the solidarity value.*

Nowak and Radzik (1994) define a new basis for $\mathcal{G}^N$, denoted by $\{(N, b_T)\}_{\emptyset \neq T \subseteq N}$. For all $\emptyset \neq T \subseteq N$, (N, b_T) is defined by

$$b_T(S) = \begin{cases} \binom{|S|}{|T|}^{-1} & \text{if } S \supseteq T \\ 0 & \text{otherwise.} \end{cases} \qquad (3)$$

They prove that all players in $N \diagdown T$ are A-null players in the game (N, b_T), so they receive a zero payoff, and all players in T are symmetric so they receive the same payoff. Thus, it holds that

$$Sl_i(N, b_T) = \begin{cases} \frac{1}{|T|} b_T(N) & \text{if } i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 1. It is shown in the recent paper of Radzik (2013) that the solidarity value Sl has a very close relationship with the *equal split value* defined as the value of the form $\phi_i^{Eq}(N, v) = v(N)/|N|$ for all $i \in N$. Namely, it turns out that for large $|N|$ the approximation $Sl(N, v) \approx \phi^{Eq}(N, v)$ can be justified for some wide subsets of games in $\mathcal{G}^N$. In that paper, the general problem of asymptotic equivalence between both values is also studied.

3. Weighted solidarity values and its basic properties

In this section, we define the *weighted solidarity values* in two different ways and we prove that both definitions are equivalent.

A *system of positive weights* is a function $\omega : \mathbb{N} \to \mathbb{R}$ with $\omega(i) > 0$ for all $i \in \mathbb{N}$. We denote $\omega_i = \omega(i)$. For each $i \in N$, ω_i is the *weight* of player i. A *weighted value* γ^ω is a function that assigns to every game (N, v) and every weight $\omega \in \mathbb{R}^N_{++}$[2] a vector $\gamma^\omega(N, v)$ in $\mathbb{R}^N$. We say that a weighted value γ^ω *extends* a value γ if $\gamma^\omega(N, v) = \gamma(N, v)$ for all (N, v) and all weight vector ω with $\omega_i = \omega_j$ for all $i, j \in N$. The most important weighted generalization of the Shapley value is the *weighted Shapley value* Sh^ω (Shapley, 1953a; Kalai and Samet, 1987). Let ω be a system of positive weights; then the *weighted Shapley value* Sh^ω is the linear mapping defined for each unanimity game (N, u_T), $\emptyset \neq T \subseteq N \subsetneq \mathbb{N}$, as follows

$$Sh_i^\omega(N, u_T) = \begin{cases} \frac{\omega_i}{\omega(T)} & \text{if } i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

The weighted Shapley value Sh^ω satisfies efficiency, additivity and the null player axiom, but not symmetry.

The above definition of the weighted Shapley value is based on the unanimity games (N, u_T) which play an essential role in the axiomatization of the classical Shapley value. A similar role for the solidarity value plays the games (N, b_T) of the form (3). Below in Definition 1, we will use this analogy to propose the definition of the *weighted solidarity value*.

Definition 1. Let ω be a system of positive weights. The *weighted solidarity value* Sl^ω is the linear mapping defined for each game (N, b_T), $\emptyset \neq T \subseteq N \subsetneq \mathbb{N}$, as follows

$$Sl_i^\omega(N, b_T) = \begin{cases} \frac{\omega_i}{\omega(T)} b_T(N) & \text{if } i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

[2] $\mathbb{R}^N_{++} = \{x \in \mathbb{R}^N : x_i > 0 \text{ for all } i \in N\}$.

Therefore, the weighted solidarity value Sl^ω satisfies efficiency and additivity, but not symmetry.

Next we show that the weighted solidarity value can also be defined recursively.

Definition 2. Let ω be a system of positive weights. For each game (N, v), we define recursively the following payoff configuration:

$$a_i^\omega(S, v) = \frac{\omega_i}{\omega(S)} \Delta^{av}(v, S) + \frac{1}{s} \sum_{j \in S \setminus i} a_i^\omega(S \setminus j, v),$$

$$\text{for all } S \subseteq N \text{ and all } i \in S, \qquad (4)$$

starting with

$$a_i^\omega(\{i\}, v) = v(i), \quad \text{for all } i \in N.$$

Theorem 3. *For all games (N, v) and all $\omega \in \mathbb{R}^N_{++}$, we have that $Sl^\omega(N, v) = a^\omega(N, v)$.*

Proof. Since both Sl^ω and a^ω are linear mappings, we only have to prove that $Sl^\omega(N, b_T) = a^\omega(N, b_T)$ for all $\emptyset \neq T \subseteq N$.

Let $\emptyset \neq T \subseteq N$. For all $i \in N \diagdown T$ we have that $a_i^\omega(\{i\}, b_T) = b_T(i) = 0$ and $\Delta^{av}(b_T, S) = 0$ for all $S \subseteq N$ containing i, as $i \in N \diagdown T$ is an A-null player in (N, b_T). Therefore, applying Definitions 1 and 2, we deduce that $a_i^\omega(N, b_T) = 0 = Sl_i^\omega(N, b_T)$ for all $i \in N \diagdown T$.

Let $i \in T$. It holds that $\Delta^{av}(b_T, S) = 0$ for all $S \not\supseteq T$, and then $a_i^\omega(S, b_T) = 0$ for all $S \not\supseteq T$ containing i. Moreover, $\Delta^{av}(b_T, T) = 1$ so then

$$a_i^\omega(T, b_T) = \frac{\omega_i}{\omega(T)} = Sl_i^\omega(T, b_T).$$

Suppose now that $T \subsetneq N$. Then, $\Delta^{av}(b_T, S) = 0$ for all $S \supsetneq T$, as S contains A-null players, that is, players that belong to $N \diagdown T$. Thus, for any $j \in N \diagdown T$,

$$\begin{aligned} a_i^\omega(T \cup j, b_T) &= \frac{\omega_i}{\omega(T \cup j)} \Delta^{av}(b_T, T \cup j) + \frac{1}{t+1} a_i^\omega(T, b_T) \\ &= \frac{1}{t+1} \frac{\omega_i}{\omega(T)} = Sl_i^\omega(T \cup j, b_T). \end{aligned}$$

Suppose by induction that $a_i^\omega(S, b_T) = Sl_i^\omega(S, b_T)$ for all $S \subsetneq N$ containing i. Then,

$$a_i^\omega(N, b_T) = \frac{1}{n} \sum_{k \in N \setminus i} a_i^\omega(N \setminus k, b_T) = \frac{1}{n} \sum_{k \in N \setminus T} a_i^\omega(N \setminus k, b_T). \qquad (5)$$

By induction hypothesis, $a_i^\omega(N \setminus k, b_T) = \frac{\omega_i}{\omega(T)} b_T(N \setminus k)$ for all $k \in N \setminus T$. Thus, following (5):

$$\begin{aligned} a_i^\omega(N, b_T) &= \frac{1}{n} \sum_{k \in N \setminus T} \frac{\omega_i}{\omega(T)} \binom{n-1}{t}^{-1} \\ &= \frac{1}{n} \frac{\omega_i}{\omega(T)} (n-t) \binom{n-1}{t}^{-1} \\ &= \frac{\omega_i}{\omega(T)} \binom{n}{t}^{-1} = Sl_i^\omega(N, b_T). \quad \blacksquare \end{aligned}$$

Hence, the weighted solidarity value Sl^ω also satisfies the A-null player axiom, which trivially follows from (4). It turns out that this value satisfies the three basic properties (excluding symmetry) of the solidarity value.

4. Axiomatic characterizations

Following the analogy between the Shapley value and the solidarity value, we here propose two axiomatizations of the family of all weighted solidarity values.

Kalai and Samet (1987) first characterize the family of all weighted Shapley values using *efficiency*, *additivity*, *the null player axiom*, and two axioms called *positivity* and *partnership consistency*. Hart and Mas-Colell (1989) provide a different axiomatization with *monotonicity* and *consistency*, among other axioms, but without additivity. Furthermore, Nowak and Radzik (1995) also provide another axiomatization without *additivity* and *the null player axiom*, by adding *positivity, mutual dependence* and *marginality*.

Kalai and Samet (1987) introduce the following concept and axioms. Let (N, v) be a game. A coalition $S \subseteq N$ is a *partnership* in (N, v) if for each $T \subsetneq S$ and each $R \subseteq N \setminus S$, $v(R \cup T) = v(R)$.

Partnership Consistency: for all game (N, v), if $S \subseteq N$ is a partnership in (N, v), then $\gamma_i(N, v) = \gamma_i(N, \gamma(v)(S)u_S)$, for all $i \in S$, where $\gamma(v)(S)$ denotes $\sum_{i\in S} \gamma_i(N, v)$.

Partnership consistency expresses the following idea: suppose we want to reallocate $\gamma(v)(S)$ among the members of a partnership S. Since each proper subcoalition of S is powerless, it is natural to reallocate $\gamma(v)(S)$ by applying γ to the unanimity game $\gamma(v)(S)u_S$. This axiom says that each player in S receives after the reallocation exactly what he received in the original game (N, v).

Positivity: if (N, v) is monotonic (i.e., $v(T) \le v(S)$ for all $T, S \subseteq N$ such that $T \subseteq S$) and has no null players, then $\gamma_i(N, v) > 0$ for all $i \in N$.

Theorem 4 (*Kalai and Samet, 1987*)**.** *A value γ on $\mathcal{G}^N$ satisfies* efficiency, additivity, the null player axiom, *partnership consistency and* positivity *if, and only if, there exists a weight system $\omega \in \mathbb{R}^N_{++}$ such that γ is the weighted Shapley value Sh^{ω}.*

Nowak and Radzik (1995) formulate some new axioms in order to provide another axiomatization without *additivity*. They introduce the following concept. Let (N, v) be a game and let $i, j \in N$ $(i \neq j)$. If $v(S \cup i) = v(S) = v(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$, then the players i and j are *mutually dependent* in (N, v).

Mutual Dependence: for all (N, v) and (N, v'), and all $\{i, j\} \subseteq N$ with $i \neq j$, if i and j are mutually dependent players in both (N, v) and (N, v'), then $\gamma_i(N, v)\, \gamma_j(N, v') = \gamma_i(N, v')\, \gamma_j(N, v)$.

Note that if players i and j are mutually dependent, then i becomes a null player when j is excluded from the game and the same concerns j if i is out of the game. The *mutual dependence* axiom says that if i and j are mutually dependent in both games (N, v) and (N, v'), then their payoffs are in the same proportion. One can easily see that two players i and j are mutually dependent if and only if the set $Q = \{i, j\}$ is a two-person partnership. Thus, *mutual dependence* is a variant of *partnership consistency*.

Moreover, they prove that, for each $\omega \in \mathbb{R}^N_{++}$, the weighted Shapley value Sh^{ω} satisfies the following axiom.

ω-Mutual Dependence: for all (N, v) and all $\{i, j\} \subseteq N$ with $i \neq j$, if i and j are mutually dependent players in (N, v), then

$$\frac{\gamma_i(N, v)}{\omega_i} = \frac{\gamma_j(N, v)}{\omega_j}.$$

Marginality[3]: for all (N, v) and (N, v'), if for some player $i \in N$, we have $v(S \cup i) - v(S) = v'(S \cup i) - v'(S)$, for all $S \subseteq N \setminus i$, then $\gamma_i(N, v) = \gamma_i(N, v')$.

This axiom is close to the *strong monotonicity* postulated in Young (1985) in order to characterize the Shapley value without *additivity* and the *null player axiom*.

Theorem 5 (*Nowak and Radzik, 1995*)**.** *A value γ on $\mathcal{G}^N$ satisfies* efficiency, positivity, mutual dependence *and* marginality *if, and only if, there exists a weight system $\omega \in \mathbb{R}^N_{++}$ such that γ is the weighted Shapley value Sh^{ω}.*

[3] In Nowak and Radzik (1995), this axiom is called *Marginal Contributions.*

We now define some new axioms in order to characterize the family of all weighted solidarity values.

Definition 3. Let (N, v) be a game. A coalition $S \subseteq N$ with $|S| \ge 2$ is a *team* in (N, v) if $\Delta^{av}(v, T) = 0$ for all $T \not\supseteq S$.

Definition 4. Two players $i, j \in N$ $(i \neq j)$ are *team mates* in (N, v) if $\Delta^{av}(v, S \cup i) = 0 = \Delta^{av}(v, S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$.

Proportionality: for all (N, v) and (N, v') and all $S \subseteq N$ with $|S| \ge 2$, if S is a team in both (N, v) and (N, v'), then $\gamma_i(N, v)\, \gamma_j(N, v') = \gamma_i(N, v')\, \gamma_j(N, v)$ for all $i, j \in S$.

This axiom says that if $S \subseteq N$ is a team in both games (N, v) and (N, v'), then the payoffs of players in S are in the same proportion in both games.

If players i and j are team mates, then i becomes an A-null player when j is excluded from the game and the same concerns j if i is out of the game. Note that if $S \subseteq N$ is a *team* in (N, v), then every two members of S are team mates. Nevertheless, if two players i and j are team mates, it is not necessary true that $S = \{i, j\}$ is a team. Indeed, consider the game (N, b_T) with $\emptyset \neq T \subsetneq N$ and two players $i, j \in N \setminus T$, and then i and j are team mates, as they are A-null players in (N, b_T), but $S = \{i, j\}$ is not a team, as $\Delta^{av}(b_T, T) = 1 \neq 0$. Thus, *proportionality* is weaker than the following axiom.

Mateship: for all (N, v) and (N, v') and all $\{i, j\} \subseteq N$ with $i \neq j$, if i and j are team mates in both (N, v) and (N, v'), then $\gamma_i(N, v)\, \gamma_j(N, v') = \gamma_i(N, v')\, \gamma_j(N, v)$.

The concept of team mates is parallel to the concept of mutually dependent players, but with the average of the marginal contributions instead of the individual marginal contribution. Thus, *mateship* is parallel to *mutual dependence*. Nevertheless, the concept of a team is in certain sense similar to a partnership, since if $S \subseteq N$ is a team, any coalition that does not contain S is completely powerless.

Next lemma shows how the players in a team are.

Lemma 1. *Let (N, v) be a game and let $S \subseteq N$ with $|S| \ge 2$ be a team in (N, v). Then, either all players in S are A-null players or no player in S is the A-null player.*

Proof. Suppose that there is a player $j \in S$ such that j is not an A-null player. Then, there exists a coalition $T \subseteq N$ with $j \in T$ and $\Delta^{av}(v, T) \neq 0$. Since S is a team in (N, v), it holds necessarily that $S \subseteq T$. Thus, all players $i \in S \subseteq T$ are not A-null players either. ∎

The next axiom is weaker than *positivity*.

Strong individual rationality: for all game (N, v), if $v(S) = 0$ for all $S \neq N$ and $v(N) > 0$, then $\gamma_i(N, v) > 0$ for all $i \in N$.

We are now ready to offer our first axiomatic characterization of the family of weighted solidarity values. This is analogous to that of Kalai and Samet (1987), since the concept of a team can be seen as a variant of the notion of partnership and the *proportionality* axiom, as a variant of *partnership consistency*.

First, we prove that, for each $\omega \in \mathbb{R}^N_{++}$, the weighted solidarity value Sl^{ω} satisfies the following property.

ω-Proportionality: for all (N, v) and all $S \subseteq N$ with $|S| \ge 2$, if S is a team in (N, v), then

$$\frac{\gamma_i(N, v)}{\omega_i} = \frac{\gamma_j(N, v)}{\omega_j}, \quad \text{for all } i, j \in S.$$

Proposition 1. *For each $\omega \in \mathbb{R}^N_{++}$, the weighted solidarity value Sl^{ω} satisfies ω-proportionality.*

Proof. Let (N, v) be a game and suppose that $S \subseteq N$ with $|S| \ge 2$ is a team. First, we shall prove that $Sl_i^{\omega}(T, v) = 0$ for all $T \subseteq N$ such that $T \not\supseteq S$ and $i \in T$. Indeed, taking into account Definition 2,

$Sl_i^\omega(\{i\}, v) = v(i) = \Delta^{av}(v, \{i\}) = 0$ for all $i \in N$. By induction over $|T|$, we have:

$$Sl_i^\omega(T, v) = \frac{\omega_i}{\omega(T)}\Delta^{av}(v, T) + \frac{1}{t}\sum_{k \in T\setminus i} Sl_i^\omega(T \setminus k, v) = 0,$$

for all $T \not\supseteq S$ and $i \in T$. (6)

Second, we shall prove that

$$\frac{Sl_i^\omega(T, v)}{\omega_i} = \frac{Sl_j^\omega(T, v)}{\omega_j}, \quad \text{for all } i, j \in S \text{ and } T \supseteq S. \tag{7}$$

Indeed, using Definition 2 and (6),

$$Sl_i^\omega(S, v) = \frac{\omega_i}{\omega(S)}\Delta^{av}(v, S) + \frac{1}{s}\sum_{k \in S\setminus i} Sl_i^\omega(S \setminus k, v)$$
$$= \frac{\omega_i}{\omega(S)}\Delta^{av}(v, S), \quad \text{for all } i \in S.$$

Thus, expression (7) is true for $T = S$. Suppose that it is true for $|T| \le k$ and let $T \supseteq S$ with $|T| = k + 1$. Then, using again (6),

$$Sl_i^\omega(T, v) = \frac{\omega_i}{\omega(T)}\Delta^{av}(v, T) + \frac{1}{t}\sum_{k \in T\setminus i} Sl_i^\omega(T \setminus k, v)$$
$$= \frac{\omega_i}{\omega(T)}\Delta^{av}(v, T) + \frac{1}{t}\sum_{k \in S\setminus i} Sl_i^\omega(T \setminus k, v)$$
$$+ \frac{1}{t}\sum_{k \in T\setminus S} Sl_i^\omega(T \setminus k, v)$$
$$= \frac{\omega_i}{\omega(T)}\Delta^{av}(v, T) + \frac{1}{t}\sum_{k \in T\setminus S} Sl_i^\omega(T \setminus k, v),$$
for all $i \in S$.

Therefore, by induction hypothesis:

$$\frac{Sl_i^\omega(T, v)}{\omega_i} = \frac{1}{\omega(T)}\Delta^{av}(v, T) + \frac{1}{t}\sum_{k \in T\setminus S} \frac{Sl_i^\omega(T \setminus k, v)}{\omega_i}$$
$$= \frac{1}{\omega(T)}\Delta^{av}(v, T) + \frac{1}{t}\sum_{k \in T\setminus S} \frac{Sl_j^\omega(T \setminus k, v)}{\omega_j}$$
$$= \frac{Sl_j^\omega(T, v)}{\omega_j}, \quad \text{for all } i, j \in S.$$

Thus, expression (7) is proved and Sl^ω satisfies *ω-proportionality*. ■

We now prove the first characterization.

Theorem 6. *A value γ on $\mathcal{G}^N$ satisfies* efficiency, additivity, the A-null player axiom, proportionality *and* strong individual rationality *if, and only if, there exists a weight vector $\omega \in \mathbb{R}^N_{++}$ such that $\gamma = Sl^\omega$.*

Proof. *Existence.* We already know that the weighted solidarity values satisfy *efficiency*, *additivity* and *the A-null player axiom*. Moreover, for each $\omega \in \mathbb{R}^N_{++}$, Sl^ω satisfies *ω-proportionality*, so it also satisfies *proportionality*. Finally, taking into account Definition 2, for all game (N, v), if $v(S) = 0$ for all $S \neq N$ and $v(N) > 0$, then it holds that

$$Sl_i^\omega(N, v) = \frac{\omega_i}{\omega(N)} v(N) > 0 \quad \text{for all } i \in N,$$

as $\Delta^{av}(v, N) = v(N) > 0$ and $\Delta^{av}(v, S) = 0$ for all $S \neq N$. Thus, for each $\omega \in \mathbb{R}^N_{++}$, Sl^ω satisfies *strong individual rationality*.

Uniqueness. Let γ be a value satisfying the above axioms. Let $\omega = \gamma(N, b_N)$. By *strong individual rationality*, $\omega_i > 0$ for all $i \in N$. We shall prove that $\gamma = Sl^\omega$. Because of the *additivity* axiom, it is sufficient to show that $\gamma(N, \alpha b_T) = Sl^\omega(N, \alpha b_T)$ for all $\emptyset \neq T \subseteq N$ and all $\alpha \in \mathbb{R}$.

Let $\emptyset \neq T \subseteq N$ and $\alpha \in \mathbb{R}$. By the *A-null player axiom*, $\gamma_i(N, \alpha b_T) = 0 = Sl_i^\omega(N, \alpha b_T)$ for all $i \in N \setminus T$. Thus, it only remains to show that $\gamma_i(N, \alpha b_T)$ is uniquely determined for players $i \in T$. If $|T| = 1$, by *efficiency* $\gamma_i(N, \alpha b_T) = \alpha b_T(N) = Sl_i^\omega(N, \alpha b_T)$ for $\{i\} = T$. Suppose that $|T| \ge 2$, and then T is a team in $(N, \alpha b_T)$ and in (N, b_N) ; therefore by *proportionality*,

$$\gamma_i(N, \alpha b_T)\gamma_j(N, b_N) = \gamma_i(N, b_N)\gamma_j(N, \alpha b_T), \quad \text{for all } i, j \in T,$$

that is,

$$\frac{\gamma_i(N, \alpha b_T)}{\omega_i} = \frac{\gamma_j(N, \alpha b_T)}{\omega_j} = C, \quad \text{for all } i, j \in T.$$

Therefore, by *efficiency*,

$$\alpha b_T(N) = \sum_{i \in T}\gamma_i(N, \alpha b_T) = C\omega(T)$$

and then,

$$\gamma_i(N, \alpha b_T) = C\omega_i = \frac{\omega_i}{\omega(T)}\alpha b_T(N) = Sl_i^\omega(N, \alpha b_T),$$

for all $i \in T$. ■

Our second characterization is similar to that of Nowak and Radzik (1995). The *additivity* and *the A-null player* axioms are dropped and replaced by a new axiom called *average marginality*.

Average marginality: for all (N, v) and (N, v'), if for some player $i \in N$, we have $\Delta^{av}(v, S \cup i) = \Delta^{av}(v', S \cup i)$, for all $S \subseteq N \setminus i$, then $\gamma_i(N, v) = \gamma_i(N, v')$.

If for a player i, the average of the marginal contributions is equal in two different games for all the coalitions he belongs to, he must receive the same payoff in both games. This last property is similar to that of Young (1985) but with the average of the marginal contributions instead of the individual marginal contribution.

We now prove the second characterization.

Theorem 7. *A value γ on $\mathcal{G}^N$ satisfies* efficiency, proportionality, strong individual rationality *and* average marginality *and if, and only if, there exists a weight vector $\omega \in \mathbb{R}^N_{++}$ such that $\gamma = Sl^\omega$.*

Proof. *Existence.* It only remains to prove that, for each $\omega \in \mathbb{R}^N_{++}$, Sl^ω satisfies *average marginality*, but this is straightforward taking into account Definition 2.

Uniqueness. Let γ be a value satisfying the above axioms. Let $\omega = \gamma(N, b_N)$. By *strong individual rationality*, $\omega_i > 0$ for all $i \in N$. We shall prove that $\gamma = Sl^\omega$.

Let (N, v_0) be the game defined as $v_0(S) = 0$ for all $S \subseteq N$. First, we prove that $\gamma_i(N, v_0) = 0$ for all $i \in N$. If $|N| = 1$, by *efficiency*, $\gamma(N, v_0) = 0$. If $|N| \ge 2$, N is a team in (N, v_0) and in (N, b_N), then by *proportionality* we have that

$$\gamma_i(N, v_0)\gamma_j(N, b_N) = \gamma_i(N, b_N)\gamma_j(N, v_0), \quad \text{for all } i, j \in N,$$

that is,

$$\frac{\gamma_i(N, v_0)}{\omega_i} = C, \quad \text{for all } i \in N,$$

and by *efficiency*, it implies $\gamma_i(N, v_0) = 0$ for all $i \in N$.

Let (N, v) be a game. If $i \in N$ is an A-null player in (N, v), then $\Delta^{av}(v, S \cup i) = 0 = \Delta^{av}(v_0, S \cup i)$, for all $S \subseteq N \setminus i$; thus by *average marginality*, $\gamma_i(N, v) = \gamma_i(N, v_0) = 0$. Hence, it only remains to show that $\gamma_i(N, v)$ is uniquely determined when $i \in N$ is not an A-null player.

Now consider the game $(N, \alpha b_T)$ with $\alpha \neq 0$ and $\emptyset \neq T \subseteq N$. If $|T| = 1$, by *efficiency*, $\gamma_i(N, \alpha b_T) = \alpha b_T(N)$ for $\{i\} = T$. Suppose that $|T| \ge 2$, and then T is a team in $(N, \alpha b_T)$ and in (N, b_N); then by *proportionality* we have that

$$\frac{\gamma_i(N, \alpha b_T)}{\omega_i} = C, \quad \text{for all } i \in T,$$

and, by *efficiency*,

$$\gamma_i(N, \alpha b_T) = \frac{\omega_i}{\omega(T)} \alpha b_T(N), \quad \text{for all } i \in T.$$

We now use the fact that the games $\{(N, b_T)\}_{\emptyset \neq T \subseteq N}$ form a basis for $\mathcal{G}^N$. Thus,

$$(N, v) = \sum_{\emptyset \neq T \subseteq N} (N, \alpha_T b_T),$$

where the constants α_T are uniquely determined by the game (N, v). Let $I(N, v) = \{T \subseteq N : \alpha_T \neq 0\}$. We proceed by induction over $|I(N, v)|$. We already know that $\gamma(N, v)$ is uniquely determined when $|I(N, v)| \leq 1$. Suppose that it is true for every game (N, v) with $|I(N, v)| \leq k$. Let (N, v) be a game with $|I(N, v)| = k + 1$. Then, we have $k + 1$ nonempty coalitions $T_1, \ldots, T_{k+1}$ such that

$$(N, v) = \sum_{j=1}^{k+1} \left(N, \alpha_{T_j} b_{T_j}\right).$$

Let $T = T_1 \cap \cdots \cap T_{k+1}$ and suppose that $i \notin T$. Define a new game (N, v') as

$$(N, v') = \sum_{j: i \in T_j} \left(N, \alpha_{T_j} b_{T_j}\right).$$

Then, $\left|I(N, v')\right| \leq k$ and $\Delta^{av}(v, S \cup i) = \Delta^{av}(v', S \cup i)$, for all $S \subseteq N \diagdown i$; thus by *average marginality*, $\gamma_i(N, v) = \gamma_i(N, v')$, but $\gamma_i(N, v')$ is uniquely determined by induction hypothesis. Suppose now that $i \in T$. If $|T| = 1$, by *efficiency*, $\gamma_i(N, v)$ is uniquely determined. If $|T| \geq 2$, T is a team in (N, v) as

$$\Delta^{av}(v, S) = \sum_{j=1}^{k+1} \alpha_{T_j} \Delta^{av}(b_{T_j}, S) = 0, \quad \text{for all } S \not\supseteq T,$$

and in (N, b_N); thus by *proportionality*,

$$\frac{\gamma_i(N, v)}{\omega_i} = C, \quad \text{for all } i \in T,$$

and by *efficiency*,

$$v(N) = C\omega(T) + \sum_{k \in N \diagdown T} \gamma_k(N, v).$$

Since $\gamma_k(N, v)$ is uniquely determined for all $k \in N \diagdown T$, we conclude that C and $\gamma_i(N, v)$ are also uniquely determined for all $i \in T$. ■

Remark 2. In these two characterizations, axioms imply the existence of a weight system such that the value is precisely the corresponding weighted solidarity value. Thus, weights are obtained endogenously.

Remark 3. Radzik (2012) shows that there is a problem with the interpretation of the weight system in the context of the weighted Shapley value (see Remark 4.7 and Examples 4.2 and 4.3 there). It is usual that weights are interpreted as a measure of the "importance" or "bargaining strength" that players have in the game. However, it turns out that there are monotonic games for which the behavior of the weighted Shapley value goes in opposite direction for some players' weights (Examples 4.2 and 4.3 there), that is, bigger weights correspond to lower payoffs. Contrary to the weighted Shapley value, in the Sl^{ω} value this positive relationship between weights and bargaining power is completely general for any monotonic game. This is the content of the next Theorem whose proof is a direct consequence of Definition 2 and is left to the reader.

Theorem 8. *Let (N, v) be a monotonic game and let $\omega, \omega' \in \mathbb{R}^N_{++}$ such that $\omega'_i \geq \omega_i$ and $\omega'_j = \omega_j$ for each $j \in N \setminus i$. Then $Sl_i^{\omega'}(N, v) \geq Sl_i^{\omega}(N, v)$.*

Acknowledgments

Emilio Calvo thanks the Ministry of Science and Technology and the European Feder Funds under project ECO2010-20584, and the Generalitat Valenciana under the Excellence Programs Prometeo 2009/068 and ISIC2012/021 for their financial support. Esther Gutiérrez-López wishes to thank financial support from the Spanish Ministry of Science and Technology and the European Regional Development Funds under project ECO2012-33618, and from UPV/EHU (UFI 11/51). The authors would like to thank an Associate Editor and two anonymous referees for their helpful comments and criticisms.

References

Calvo, E., 2008. Random marginal and random removal values. Internat. J. Game Theory 37, 533–564.
Hart, S., Mas-Colell, A., 1989. Potential, value and consistency. Econometrica 57, 589–614.
Kalai, E., Samet, D., 1987. On weighted Shapley values. Internat. J. Game Theory 16, 205–222.
Nowak, A.S., Radzik, T., 1994. A solidarity value for n-person transferable utility games. Internat. J. Game Theory 23, 43–48.
Nowak, A.S., Radzik, T., 1995. On axiomatizations of the weighted Shapley values. Games Econom. Behav. 8, 389–405.
Radzik, T., 2012. A new look at the role of players' weights in the weighted Shapley value. European J. Oper. Res. 223, 407–416.
Radzik, T., 2013. Is the solidarity value close to the equal split value?. Math. Social Sci. 65, 195–202.
Shapley, L.S., 1953a. Additive and Non-Additive Set Functions (Ph.D. thesis). Department of Mathematics, Princeton University.
Shapley, L.S., 1953b. A Value for n-Person Games. In: Kuhn, H.W., Tucker., A.W. (Eds.), Contributions to the theory of Games II. In: Annals of Mathematics Studies, vol. 28. Princeton University Press, Princeton, pp. 307–317.
Sprumont, Y., 1990. Population monotonic allocation schemes for cooperative games with transferable utility. Games Econom. Behav. 2, 378–394.
Young, P., 1985. Monotonic solutions for cooperative games. Internat. J. Game Theory 14, 65–72.