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# Solidarity in games with a coalition structure

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ABSTRACT

A new axiomatic characterization of *the two-step Shapley value* Kamijo (2009) is presented based on a *solidarity* principle of the members of any union: when the game changes due to the addition or deletion of players outside the union, all members of the union will share the same gains/losses.



## 1. Introduction

In the framework of cooperative games, there are many natural settings in which players organize themselves into groups for the purpose of payoff bargaining. Those include syndicates, unions, cartels, parliamentary coalitions, cities, countries, etc. This fact is incorporated into the game by a *coalition structure*, which is an exogenous partition of players into a set of groups or unions. The evaluation of players' expectations in the game is given by a *coalitional value* which takes into account the fact that agents interact on two levels: firstly, among the unions, and secondly, within each union.

Games with a coalition structure were first considered by Aumann and Drèze (1974). These authors extend the value introduced in Shapley (1953) in such a way that the game splits into subgames played by the unions in isolation, and every player receives his Shapley value ($Sh$) in the subgame played within his union. A different approach was followed by Owen (1977). In his case, unions play a quotient game among themselves, and each union receives a payoff that is shared among its players in an internal game. The payoffs of unions in the quotient game and the payoffs of players within each union, are both given by the Shapley value. This gives rise to the Owen value.

Alternative coalitional values have been considered. In Owen (1982), the Banzhaf (1965) value ($Bz$) was used to solve both the game among unions and the game within each union. In Alonso-Meijide and Fiestras-Janeiro (2002), the symmetric coalitional Banzhaf value was introduced, the Banzhaf value being applied in the quotient game, and the Shapley value within unions. In Amer et al. (2002) an example was introduced as its counterpart (reversing the application of the Shapley and Banzhaf values). These four values cover the possible variations of the application of the Shapley and Banzhaf values at the two levels of interaction: $(Sh, Sh)$, $(Bz, Bz)$, $(Bz, Sh)$, and $(Sh, Bz)$. Axiomatic characterizations of these values can be found in Alonso-Meijide et al. (2007). These values fall into a wider family of $(\psi, \phi)$-coalitional values considered in Albizuri and Zarzuelo (2004), where $\psi$ is the value applied in the game among unions, and $\phi$ is the value applied within each union.

The standard motivation for incorporating a coalition structure into a game is that players are interested in joining a union in order to improve their bargaining position in the game. This is, for example, the point of view given in Hart and Kurz (1983, Section 1, page 1048):

> "With this view in mind, coalitions do not form in order to obtain their "worth" and then "leave" the game. But rather, they "stay" in the game and bargain as a unit with all the other players. This means that coalitions try to obtain as much as possible by not letting the others exploit their (individual) weaknesses when they are separated. As an everyday example of such a situation, "I will have to check this with my wife/husband" may (but not necessarily) lead to a better bargaining position, due to the fact that the other party has to convince *both* the player and the spouse".

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Hence, when a union is formed, all its members commit themselves to bargaining with the others as a unit. A critical question here is how to share the gains (or losses) obtained by the players in a union. For example, suppose that in a certain marital situation, the couple follows a rule $\Psi$ in order to decide what each one initially contributes to the union. And also suppose that if they separate in the future, $\Psi$ will be used too to determine how to share their common future wealth. We believe that if $\Psi$ was consistent with the marriage vows that joined them "for better or for worse", then $\Psi$ should share their wealth variation equally between them. We informally express this *solidarity* property as follows: if the data of the game change due to factors external to the union, then all members of the union change their value in an equal amount. Although many types of changes can be considered, this paper will only focus on addition to or deletion from the set of players outside the union.

It is easy to see that the Owen value does not satisfy this solidarity property. This is because the payoff to each member is determined by the Shapley value of an auxiliary game, played by all the members of the union (and only by them). In this auxiliary game, the worth of each subcoalition is given by its payoff (Shapley value) in a modified quotient game played by itself and the other unions.[1] If we delete players outside the union, this auxiliary game changes and the internal redistribution of the wealth obtained by the union also changes, even if the total payoff that the union obtains is unchanged.

In Kamijo (2009) a new coalitional value, named *the two-step Shapley value*, is considered and axiomatized. This value satisfies most of the properties that support the Shapley value in the setting of games without coalition structure. Therefore, it can be considered as an alternative value extension to the coalition structure setting. Our goal is the characterization of the two-step Shapley value by explicitly introducing this solidarity principle in the axiomatic system. This yields additional support for the two-step Shapley value as an interesting alternative to the Owen value whenever solidarity matters.

The rest of the paper is organized as follows. Section 2 is devoted to some preliminary definitions and notation and the two-step Shapley value is presented. Section 3 introduces the solidarity axiom and shows that the Owen value does not satisfy this axiom. We give the axiomatic characterization of the two-step Shapley value based on this axiom. In Section 4, (i) we show that the set of axioms is independent, (ii) and we provide an alternative characterization using the axiom of balanced contributions.

## 2. Notation and definitions

A *cooperative game* with transferable utility (TU-game) is a pair $(N, v)$ where $N$ is a nonempty and finite set and $v : 2^N \to \mathbb{R}$ is a *characteristic function*, defined on the power set of $N$, satisfying $v(\emptyset) = 0$. An element $i$ of $N$ is called a *player* and every nonempty subset $S$ of $N$ a *coalition*. The real number $v(S)$ is called the *worth* of coalition $S$, and it is interpreted as the total payoff that the coalition $S$, if it forms, can obtain for its members. Let $\mathcal{G}^N$ denote the set of all cooperative TU-games with player set $N$.

For each $S \subseteq N$, we denote the restriction of $(N, v)$ to $S$ as $(S, v)$. For simplicity, we write $S \cup i$ instead of $S \cup \{i\}$, $N \setminus i$ instead of $N \setminus \{i\}$, and $v(i)$ instead of $v(\{i\})$.

A *value* is a function $\gamma$ which assigns to every TU-game $(N, v)$ and every player $i \in N$, a real number $\gamma_i(N, v)$, which represents an assessment made by $i$ of his gains from participating in the game. One of the most important values is the *Shapley value* (Shapley, 1953). The Shapley value of the game $(N, v)$ is denoted $Sh(N, v)$.

Let $\Omega(N)$ be the set of all orders on $N$. The Shapley value of a game $(N, v)$ is given by the formula

$$Sh_i(N, v) = \frac{1}{|\Omega(N)|} \sum_{\omega \in \Omega(N)} \left[ v(P_i^\omega(N) \cup i) - v\left(P_i^\omega(N)\right) \right]$$

for each $i \in N$, where $P_i^\omega(N) = \{j \in N : \omega(j) < \omega(i)\}$ and $\omega(j)$ denotes the position of $j$ in the order $\omega$.

Two players $\{i, j\} \subseteq N$ are *symmetric* in $(N, v)$ if, for each $S \subseteq N \setminus \{i, j\}$: $v(S \cup i) = v(S \cup j)$. Player $i \in N$ is a *dummy player* in a game $(N, v)$ if, for each $S \subseteq N \setminus i$: $v(S \cup i) = v(S) + v(i)$. Player $i \in N$ is a *null player* in $(N, v)$ if, for each $S \subseteq N \setminus i$: $v(S \cup i) = v(S)$. For each two games $(N, v)$ and $(N, w) \in \mathcal{G}^N$, the game $(N, v + w)$ is defined as $(v + w)(S) = v(S) + w(S)$ for each $S \subseteq N$.

Consider the following properties of a value $\gamma$ in $\mathcal{G}^N$:

*Efficiency.* For each $(N, v) \in \mathcal{G}^N$: $\sum_{i \in N} \gamma_i(N, v) = v(N)$.

*Additivity.* For each $(N, v), (N', w) \in \mathcal{G}^N$ with $N = N'$: $\gamma(N, v + w) = \gamma(N, v) + \gamma(N, w)$.

*Symmetry.* For each $(N, v) \in \mathcal{G}^N$ and each $\{i, j\} \subseteq N$, if $i$ and $j$ are symmetric players in $(N, v)$, then $\gamma_i(N, v) = \gamma_j(N, v)$.

*Null player axiom.* For each $(N, v) \in \mathcal{G}^N$ and each $i \in N$, if $i$ is a null player in $(N, v)$, then $\gamma_i(N, v) = 0$.

The following theorem is due to Shapley (1953).

**Theorem 1** (*Shapley, 1953*). *A value $\gamma$ on $\mathcal{G}^N$ satisfies* efficiency, additivity, symmetry *and* null player axiom *if, and only if, $\gamma$ is the Shapley value.*

For each finite set $N$, a *coalition structure* over $N$ is a partition of $N$, i.e., $B = \{B_1, B_2, \ldots, B_m\} \subseteq 2^N$ is a coalition structure if it satisfies $\bigcup_{k \in M} B_k = N$, where $M = \{1, 2, \ldots, m\}$, and $B_k \cap B_l = \varnothing$ when $k \neq l$. We also assume $B_k \neq \emptyset$ for each $k \in M$. There are two trivial coalition structures: The first, which we denote by $B^N$, where only the grand coalition forms, that is, $B^N = \{N\}$; and the second is the discrete coalition structure, where each union is a singleton and is denoted by $B^n$, (i.e., $B^n = \{\{1\}, \{2\}, \ldots, \{n\}\}$). We denote the *game* $(N, v) \in \mathcal{G}^N$ *with coalition structure* $B$ as $(B, v)$. Let $\mathcal{CSG}^N$ denote the family of all TU-games with coalition structure with player set $N$, and let $\mathcal{CSG}$ denote the set of all TU-games with coalition structure.

For each game $(B, v) \in \mathcal{CSG}^N$, where $B = \{B_1, B_2, \ldots, B_m\}$, the *quotient game* is the *TU*-game $(M, v_B) \in \mathcal{G}^M$ where $v_B(T) = v(\bigcup_{i \in T} B_i)$ for each $T \subseteq M$. That is, $(M, v_B)$ is the game induced by $(B, v)$ by considering the coalitions of $B$ as players. Notice that for the trivial coalition structure $B^n$ we have $(M, v_{B^n}) \equiv (N, v)$. We say that, for each $\{k, l\} \subseteq M$, $B_k$ and $B_l$ are *symmetric coalitions* in $(B, v)$ if the players $k$ and $l$ are symmetric in the game $(M, v_B)$. We say that $B_k \in B$ is a *null coalition* if player $k \in M$ is a null player in the game $(M, v_B)$.

The evaluation of players' expectations in the game with a coalition structure is given by a *coalitional value*, which takes into account the fact that the interaction among agents is now played on two levels: firstly, among the unions as players, and secondly, among the players within each union. Formally, a *coalitional value* is a function $\Phi$ that assigns a vector in $\mathbb{R}^N$ to each game with coalition structure $(B, v) \in \mathcal{CSG}^N$. One of the most important coalitional values is the *Owen value* (Owen, 1977). We denote the Owen value of a game $(B, v)$ as $Ow(B, v)$.

Similar to the Shapley value, the Owen value can also be defined by orders. Let $B$ be a coalition structure over $N$ and $\omega \in \Omega(N)$. We say that $\omega$ is *admissible with respect to* $B$ if for each $\{i, j, k\} \subseteq N$ and each $l \in M$, such that $\{i, k\} \subseteq B_l$ and $\omega(i) < \omega(j) < \omega(k)$, then it holds that $j \in B_l$. In other words, $\omega$ is admissible with respect to

[1] It can be assumed that either the remaining players in the union leave the game, or will break apart into individuals (singletons), or into a new union. In all three cases, the payoffs obtained in this auxiliary game are the same. See Hart and Kurz (1983) for more details.

$B$ if players in the same component of $B$ appear successively in $\omega$. We denote by $\Omega(B, N)$ the set of all admissible orders (on $N$) with respect to $B$. The Owen value of a game $(B, v) \in \mathcal{CSG}^N$ is given by the formula

$$Ow_i(B, v) = \frac{1}{|\Omega(B, N)|} \sum_{\omega \in \Omega(B,N)} [v(P_i^\omega(N) \cup i) - v(P_i^\omega(N))]$$

for each $i \in N$, that is, the Owen value assigns to each player his expected marginal contribution with respect to a uniform distribution over all orders on $N$ that are admissible with respect to the coalition structure.

For each coalitional value $\Phi$ and each $S \subseteq N$, let $\Phi(B, v)[S] := \sum_{i \in s} \Phi_i(B, v)$. We now present the axioms that characterize the Owen value in $\mathcal{CSG}^N$.

(E) *Efficiency*. For each $(B, v) \in \mathcal{CSG}^N$: $\Phi(B, v)[N] = v(N)$.

(A) *Additivity*. For each $(B, v)$ and $(B', w) \in \mathcal{CSG}^N$ with $B = B'$: $\Phi(B, v + w) = \Phi(B, v) + \Phi(B, w)$.

(NP) *Null player*. For each $(B, v) \in \mathcal{CSG}^N$ and each $i \in N$, if $i$ is a null player in $(N, v)$, then $\Phi_i(B, v) = 0$.

(ISy) *Intracoalitional symmetry*. For each $(B, v) \in \mathcal{CSG}^N$, each $k \in M$, and each $\{i, j\} \subseteq B_k$, if $i$ and $j$ are symmetric players in $(N, v)$, then $\Phi_i(B, v) = \Phi_j(B, v)$.

(CSy) *Coalitional symmetry*.[2] For each $(B, v) \in \mathcal{CSG}^N$, and each $\{k, l\} \subseteq M$, if $B_k$ and $B_l$ are symmetric coalitions, then $\Phi(B, v)[B_k] = \Phi(B, v)[B_l]$.

The following theorem is due to Owen (1977).

**Theorem 2** (Owen, 1977). *A value $\Phi$ on $\mathcal{CSG}^N$ satisfies* efficiency, additivity, null player axiom, intracoalitional symmetry *and* coalitional symmetry *if, and only if, $\Phi$ is the Owen value.*

Note that for the trivial coalition structures $B^n$ and $B^N$, $Ow(B^N, v) = Ow(B^n, v) = Sh(N, v)$.

Kamijo (2009) considered the following coalitional value, named the two-step Shapley value, and defined two new axioms about null players and symmetric players in order to axiomatize it.

**Definition 1.** For each $(B, v) \in \mathcal{CSG}^N$, the two-step Shapley value of $(B, v)$ is given by the formula:

$$\Psi_i(B, v) = Sh_i(B_k, v) + \frac{1}{|B_k|}[Sh_k(M, v_B) - v(B_k)],$$
$$\text{for each } k \in M \text{ and each } i \in B_k. \quad (1)$$

The first term of $\Psi$ is a sort of "competitive component", since it rewards each player on the basis of his strategic strength in the restricted game $(B_k, v)$, whereas the second one, which is common to all members of $B_k$, represents the "solidarity component" of the value.

Note that, since the Shapley value satisfies *efficiency*, $\Psi$ satisfies the following relationship

$$\sum_{i \in B_k} \Psi_i(B, v) = Sh_k(M, v_B), \quad \text{for each } k \in M.$$

We now present the axioms that characterize $\Psi$ in $\mathcal{CSG}^N$.

(CNP) *Coalitional null player*. For each $(B, v) \in \mathcal{CSG}^N$, each $k \in M$ and each $i \in B_k$, if $i$ is a null player in $(N, v)$, and $k$ is a dummy player in $(M, v_B)$, then $\Phi_i(B, v) = 0$.

(IE) *Internal equity*. For each $(B, v) \in \mathcal{CSG}^N$, each $k \in M$ and each $\{i, j\} \subseteq B_k$, if $i$ and $j$ are symmetric players in $(B_k, v)$, then $\Phi_i(B, v) = \Phi_j(B, v)$.

In the statement of the *coalitional null player axiom*, the usual requirement that a null player obtains nothing in any situation is weakened so that he could obtain a positive reward because of the mutual assistance between the members of the coalition to which he belongs. *Internal equity* requires that two distinct players who are symmetric in the internal game $(B_k, v)$ should be treated equally, and thus receive equal payoffs. It is clear that *intracoalitional symmetry* is weaker than *internal equity*, and that *null player axiom* is stronger than *coalitional null player axiom*.

The following theorem holds.

**Theorem 3** (Kamijo, 2009; Theorem 1). *A value $\Phi$ on $\mathcal{CSG}^N$ satisfies* efficiency, additivity, coalitional symmetry, coalitional null player axiom *and* internal equity *if, and only if, $\Phi \equiv \Psi$.*

Kamijo also showed that $\Psi$ can be computed by means of orders. Let $(B, v) \in \mathcal{CSG}^N$ be a game. For each $\omega \in \Omega(B, N)$ and each $\{k, l\} \subseteq M$, we write $\omega(B_k) < \omega(B_l)$ when players of $B_k$ appear before players of $B_l$ at $\omega$. For each $k \in M$, and each $i \in B_k$, define:

$$P_i^\omega(B_k) = P_i^\omega(N) \cap B_k; \qquad T_k^\omega = \{l \in M : \omega(B_l) < \omega(B_k)\};$$

$$d_i^\omega(v) = \begin{cases} v(P_i^\omega(N) \cup i) - v(B_k \setminus i) - v\left(\bigcup_{k \in T_k^\omega} B_k\right) \\ \quad \text{if } \omega \in \Omega_i^1(B, N), \\ v(P_i^\omega(B_k) \cup i) - v(P_i^\omega(B_k)) \\ \quad \text{if } \omega \in \Omega_i^2(B, N), \end{cases}$$

where

$$\Omega_i^1(B, N) = \left\{\omega \in \Omega(B, N) : P_i^\omega(B_k) = B_k \setminus i\right\},$$
$$\Omega_i^2(B, N) = \left\{\omega \in \Omega(B, N) : P_i^\omega(B_k) \subsetneq B_k \setminus i\right\}.$$

That is, for all orders in $\Omega_i^1(B, N)$, $i$ is the last player completing the union $B_k$.

**Proposition 1** (Kamijo, 2009; Theorem 3). *For each $(B, v) \in \mathcal{CSG}^N$,*

$$\Psi_i(B, v) = \frac{1}{|\Omega(B, N)|} \sum_{\omega \in \Omega(B,N)} d_i^\omega(v), \quad \textit{for each } i \in N.$$

## 3. The solidarity axiom

We now want to express the solidarity principle that guides union formation in the sense that when the game changes due to addition or deletion of players outside the union, all members of the union will experience the same gains/losses. For each $l \in M$, and each $h \in B_l$, define $B_{-h} := (B_1, \ldots, B_l \setminus h, \ldots, B_m)$.

(PS) *Population solidarity within unions*. For each $(B, v) \in \mathcal{CSG}^N$, each $\{k, l\} \subseteq M (k \neq l)$, and each $\{i, j, h\} \subseteq N$, where $\{i, j\} \subseteq B_k$ and $h \in B_l$:

$$\Phi_i(B, v) - \Phi_i(B_{-h}, v) = \Phi_j(B, v) - \Phi_j(B_{-h}, v).$$

This axiom makes sense for coalitional values defined on $\mathcal{CSG}$, given that in this axiom the value must be applied on $N$ and also on $N \setminus h$, for each $h \in N$.

**Remark 1.** The idea that variations in population should affect all agents equally has a long standing tradition. As such, it is a strengthening of the idea of population monotonicity, introduced by Thomson (1983a,b) in the context of bargaining and applied by Sprumont (1990) to standard coalitional games. A value $\gamma$ satisfies

[2] Axioms *ISy* and *CSy* are often called in the literature *symmetry within unions* and *symmetry in the quotient game*, respectively.

*population monotonicity* if, for each game $(N, v) \in G^N$, each $S \subseteq N$ and each $i \in S$, we have $\gamma_i(N, v) \geq \gamma_i(S, v)$. That is, if new agents join a society, no initial agent should be worse off. A direct adaptation of this idea to games with a coalition structure could be expressed as follows:

*(PM) Population monotonicity within unions*: For each $(B, v) \in \mathcal{CSG}^N$, each $\{k, l\} \subseteq M$ $(k \neq l)$, and each $\{i, h\} \subseteq N$, where $i \in B_k$ and $h \in B_l$:

$$\Phi_i(B, v) \geq \Phi_i(B_{-h}, v).$$

It must be noted that, on the one hand, *PM* is an *ordinal* requirement, because it only says that changes in payoffs inside the union have the same sign, whereas in *PS* the changes are of the *same* magnitude. On the other hand, *PM* requires that the effect of drawing a player $h$ out of union $B_l$ must be always *negative* for players in $B_k$, but in *PS* the sign of this effect is not imposed; it is a consequence of the characteristics of the game.

**Claim 1.** *The Owen value does not satisfy* population solidarity within unions.

**Proof.** Consider the following game.[3] The set of players is $N = \{1, 2, 3, 4\}$, and there are two commodities, say $x_l$ is the number of "left-gloves" and $x_r$ is the number of "right-gloves". Each player $i \in N$ has an endowment of goods, $\omega^i = (\omega^i_l, \omega^i_r)$, and the worth of each coalition $S \subseteq N$ is given by $v(S) := \min\left\{\sum_{i\in S}\omega^i_l, \sum_{i\in S}\omega^i_r\right\}$. In our example, let $\omega^1 = (1-\epsilon, 0)$, $\omega^2 = (0, 1-\epsilon)$, $\omega^3 = (\epsilon, \epsilon)$, and $\omega^4 = (\epsilon, 0)$.[4] Assume initially that only players 1, 2, 3 are in the game, and that they act as singletons: $B^n_{-4} = \{\{1\}, \{2\}, \{3\}\}$. The Owen value is

$$Ow_1(B^n_{-4}, v) = Ow_2(B^n_{-4}, v) = \frac{1}{2} - \frac{\epsilon}{2}, \qquad Ow_3(B^n_{-4}, v) = \epsilon.$$

If players 1 and 2 form a union $\{1, 2\}$, we have the coalition structure $B \equiv \{B_k, B_l\}$, where $B_k \equiv \{1, 2\}$ and $B_l \equiv \{3\}$, but the payoffs remain unchanged:

$$Ow(B^n_{-4}, v) = Ow(B, v).$$

Suppose now that player 4 enters the game as a singleton. The resulting coalition structure is $B' \equiv \{B_k, B_l, B_t\}$, where $B_k \equiv \{1, 2\}$, $B_l \equiv \{3\}$ and $B_t \equiv \{4\}$. It can be checked that $B_t$ is a null coalition in the game $(B', v)$. And note that $B'_{-4} = \{\{1, 2\}, \{3\}\} = B$. Nevertheless, easy computations yield the following payoffs:

$$Ow_1(B', v) = \frac{1}{2} - \frac{3}{4}\epsilon < Ow_1(B'_{-4}, v) = \frac{1}{2} - \frac{\epsilon}{2},$$
$$Ow_2(B', v) = \frac{1}{2} - \frac{1}{4}\epsilon > Ow_2(B'_{-4}, v) = \frac{1}{2} - \frac{\epsilon}{2},$$
$$Ow_3(B', v) = Ow_3(B'_{-4}, v) = \epsilon,$$
$$Ow_4(B', v) = 0.$$

Alternatively, we can consider $\bar{B} \equiv \{B_k, B_l\}$, where $B_k \equiv \{1, 2\}$ and $B_l \equiv \{3, 4\}$. But again $\bar{B}_{-4} = \{\{1, 2\}, \{3\}\} = B$ and the Owen value also yields the same payoffs, i.e., $Ow(B', v) = Ow(\bar{B}, v)$. In both cases, there is a redistribution in favour of player 2, although the total worth of that union $\{1, 2\}$ is $1 - \epsilon$, independently of whether or not player 4 is in the game. $\square$

[3] This is a variation of Shafer's Example 2, in Shafer (1980).

[4] It is assumed that $\epsilon$ is small enough. For our example, a value of $\epsilon$ less than 1/5 suffices.

**Remark 2.** Casajus (2009) introduced a new coalitional value that satisfies a rather similar solidarity property called *splitting*. The main difference is that the two-step Shapley value is efficient within the grand coalition, i.e. $\sum_{i\in N}\Psi_i(B, v) = v(N)$, and the value introduced by Casajus satisfies efficiency within each union, i.e. $\sum_{i\in B_k}\Psi_i(B, v) = v(B_k)$ for each $k \in M$.

We now define two additional axioms:

(NC) *Null coalition*. For each $(B, v) \in \mathcal{CSG}^N$ and each $k \in M$, if $B_k$ is a null coalition, then $\Phi(B, v)[B_k] = 0$.

(Coh) *Coherence*. For each $(N, v) \in \mathcal{G}^N$: $\Phi(B^N, v) = \Phi(B^n, v)$.

Coherence means that games in which all players belong to only one union and when all of them act as singletons are indistinguishable.

**Remark 3.** The Owen value also satisfies NC and Coh.

We wish to stress the independence of the *null player* and *null coalition* axioms. There is no relation between *null coalition axiom* and *coalitional null player axiom* either. We illustrate these aspects with the following propositions.

**Proposition 2.** *The* null coalition axiom *does not imply the* null player axiom.

**Proof.** Consider the following coalitional value $\digamma$ defined by

$$\digamma_i(B, v) = \frac{Sh_k(M, v_B)}{|B_k|}, \quad \text{for each } k \in M \text{ and each } i \in B_k.$$

$\digamma$ satisfies *null coalition axiom* since the Shapley value satisfies *null player axiom*. Let $(B, v)$ be the game defined by $N = \{1, 2, 3\}$, $B_k = \{1, 2\}$, $B_l = \{3\}$ and $B = \{B_k, B_l\}$, and $v$ is given by $v(1) = 0$, $v(2) = 1$, $v(3) = 2$, $v(1, 2) = 1$, $v(1, 3) = 2$, $v(2, 3) = 4$ and $v(N) = 4$. Player 1 is a null player, however

$$\digamma_1(B, v) = \digamma_2(B, v) = 3/4, \qquad \digamma_3(B, v) = 5/2. \quad \square$$

Note that, although *NC* does not imply *NP*, it is true that *null coalition axiom* implies the following weaker version of the *null player axiom*:

(NP*) *Null player in singletons*. For each $(B, v) \in \mathcal{CSG}^N$ and each $i \in N$, if $i$ is a null player in $(N, v)$ and $\{i\} \in B$, then $\Phi_i(B, v) = 0$.

**Proposition 3.** *The* null player axiom *does not imply the* null coalition axiom.

**Proof.** Define the following coalitional value:

$$\Gamma(B, v) = Sh(N, v), \quad \text{for each } (B, v) \in \mathcal{CSG}^N.$$

Taking into account the properties of the Shapley value, $\Gamma$ satisfies the *null player axiom*. Let $(B, v)$ be the game defined by $N = \{1, 2, 3\}$, $B_k = \{1, 2\}$, $B_l = \{3\}$ and $B = \{B_k, B_l\}$, and $v$ is given by $v(1) = v(2) = 0$, $v(3) = 3$, $v(1, 2) = 0$, $v(1, 3) = 4$, $v(2, 3) = 3$ and $v(N) = 3$. Union $\{1, 2\}$ is a null coalition, but $\Gamma(B, v) = (1/6, -2/6, 19/6)$. Since $\Gamma_1(B, v) + \Gamma_2(B, v) = -1/6 \neq 0$, $\Gamma$ does not satisfy the *null coalition axiom*. $\square$

**Proposition 4.** *The* null coalition axiom *does not imply the* coalitional null player axiom.

**Proof.** The coalitional value $\digamma$ defined in Proposition 2 does not satisfy *coalitional null player axiom* because $N$ is always a dummy coalition in $(B^N, v)$ and $\digamma_i\left(B^N, v\right) = v(N)/|N|$, for each $i \in N$. So, with the same $v$ as in Proposition 2, player 1 is a null player and $\digamma_1(B^N, v) \neq 0$. $\square$

**Corollary 1.** *The* coalitional null player axiom *does not imply the* null coalition axiom.

**Proof.** If *coalitional null player axiom* implied *null coalition axiom*, since *NP* implies *CNP*, then *null player axiom* would imply *null coalition axiom*. □

We are now ready to present the new axiomatic characterization of the value $\Psi$.

**Theorem 4.** *A value $\Phi$ on $\mathcal{CSG}$ satisfies* efficiency, additivity, coalitional symmetry, null coalition axiom, coherence *and* population solidarity within unions *if, and only if, $\Phi \equiv \Psi$.*

**Proof.** *Existence*. Let $(B, v) \in \mathcal{CSG}^N$. Since the Shapley value satisfies *efficiency*, for each $k \in M$, we have that $\sum_{i \in B_k} \Psi_i(B, v) = Sh_k(M, v_B)$, and then $\sum_{i \in N} \Psi_i(B, v) = \sum_{k \in M} \sum_{i \in B_k} \Psi_i(B, v) = \sum_{k \in M} Sh_k(M, v_B) = v(N)$. Thus $\Psi$ satisfies *efficiency*. Moreover, since the Shapley value satisfies *additivity*, $\Psi$ also satisfies *additivity*.

It is straightforward to verify that $\Psi$ satisfies *coalitional symmetry* because the Shapley value satisfies *symmetry* and $\sum_{i \in B_k} \Psi_i(B, v) = Sh_k(M, v_B)$, for each $k \in M$.

In order to prove that $\Psi$ satisfies *null coalition axiom*, note that, for each $k \in M$, if $B_k$ is a null coalition, then $k \in M$ is a null player in $(M, v_B)$, hence, as the Shapley value satisfies *null player axiom*, then $Sh_k(M, v_B) = 0$ and therefore $\sum_{i \in B_k} \Psi_i(B, v) = Sh_k(M, v_B) = 0$.

In order to prove *coherence*, if $B = B^N$, then $M = \{1\}$, $Sh_1(M, v_B) = v(N)$, and then $\Psi_i(B^N, v) = Sh_i(N, v) + \frac{1}{|N|}[v(N) - v(N)] = Sh_i(N, v)$ for each $i \in N$. On the other hand, if $B = B^n$, then $(M, v_B) = (N, v)$, so $\Psi_i(B^n, v) = v(i) + Sh_i(N, v) - v(i) = Sh_i(N, v)$. Therefore $\Psi(B^N, v) = \Psi(B^n, v) = Sh(N, v)$.

Given the definition of $\Psi$, for each $\{k, l\} \subseteq M (k \neq l)$, each $h \in B_l$ and each $\{i, j\} \subseteq B_k$:

$$\begin{aligned} \Psi_i(B, v) - \Psi_i(B_{-h}, v) &= \frac{1}{|B_k|} \left[ Sh_k(M, v_B) - Sh_k(M', v_{B_{-h}}) \right] \\ &= \Psi_j(B, v) - \Psi_j(B_{-h}, v), \end{aligned}$$

where $M' = M \setminus l$ if $|B_l| = 1$, $M = M'$ otherwise. So $\Psi$ satisfies *population solidarity within unions*.

*Uniqueness*. Let $\Phi$ be a coalitional value satisfying the above axioms and let $(B, v) \in \mathcal{CSG}^N$. We define the following value $\gamma$ on $\mathcal{G}^M$ by:

$$\gamma_k(M, v_B) = \Phi(B, v)[B_k], \quad \text{for each } k \in M.$$

In order to see that $\gamma$ is well-defined, suppose there exist two games $u$ and $v$ such that $u_B = v_B$. We prove that $\Phi(B, u)[B_k] = \Phi(B, v)[B_k]$ for each $k \in M$. Indeed, since $u_B = v_B$, for each $k \in M$, $B_k$ is null in $(B, N, u - v)$.[5] Then, by the *null coalition axiom*, $\Phi(B, u - v)[B_k] = 0$, for each $k \in M$, and by *additivity*,[6]

$$\begin{aligned} 0 &= \Phi(B, u - v)[B_k] = \Phi(B, u)[B_k] + \Phi(B, -v)[B_k] \\ &= \Phi(B, u)[B_k] - \Phi(B, v)[B_k], \quad \text{for each } k \in M. \end{aligned}$$

Now, since $\Phi$ satisfies *efficiency, additivity, coalitional symmetry* and *null coalition axiom*, then by Theorem 1, $\gamma_k(M, v_B) = \Phi(B, v)[B_k] = Sh_k(M, v_B)$ for each $k \in M$. In particular, this expression jointly with *coherence* implies that $\Phi(B^n, v) = Sh(N, v) = \Phi(B^N, v)$. Thus $\Phi$ is uniquely determined for the two trivial coalition structures.

Suppose now that $|B| \geq 2$ and let $k \in M$. By *population solidarity within unions*, for each $\{k, l\} \subseteq M (k \neq l)$, each $i \in B_k$ and each $h \in B_l$, it holds that

$$\Phi_i(B, v) - \Phi_i(B_{-h}, v) = d_k,$$

and then

$$\begin{aligned} \Phi(B, v)[B_k] - \Phi(B_{-h}, v)[B_k] &= Sh_k(M, v_B) - Sh_k(M', v_{B_{-h}}) \\ &= |B_k| d_k, \end{aligned}$$

hence

$$\Phi_i(B, v) = \Phi_i(B_{-h}, v) + \frac{1}{|B_k|}[Sh_k(M, v_B) - Sh_k(M', v_{B_{-h}})],$$

for each $i \in B_k$.

Applying *population solidarity within unions* repeatedly so that all coalitions except $B_k$ leave the game, we finally obtain

$$\begin{aligned} \Phi_i(B, v) &= \Phi_i(B^{B_k}, v) + \frac{1}{|B_k|}[Sh_k(M, v_B) - Sh_k(\{k\}, v_{\{B_k\}})] \\ &= Sh_i(B_k, v) + \frac{1}{|B_k|}[Sh_k(M, v_B) - v(B_k)] \\ &= \Psi_i(B, v), \quad \text{for each } i \in B_k. \quad \square \end{aligned}$$

**Remark 4.** The coalitional value $\Psi$ also satisfies *intracoalitional symmetry*. *This follows from the fact that, for each $k \in M$, and each $\{i, j\} \in B_k$, if* $i$ and $j$ are symmetric players in $(N, v)$ they are also symmetric in $(B_k, v)$. By the *symmetry* of the Shapley value and the definition of $\Psi$ it follows that $\Psi_i(B, v) = \Psi_j(B, v)$.

An important fact which distinguishes the coalitional value $\Psi$ from the Owen value is the *null player axiom*. The Owen value satisfies the *null player axiom* and $\Psi$ does not. For each $k \in M$ and each $i \in B_k$, if $i$ is a null player in $(N, v)$, he obtains $\frac{1}{|B_k|}[Sh_k(M, v_B) - v(B_k)]$ which in general is different from zero. This fact is in accordance with the solidarity principle: as soon as a null player is accepted in the union, he gets the same benefits as any other member in the union (all the members "are in the same boat").[7] A different question altogether arises when the coalition structure is not given a priori, and we want to predict which coalition structure will emerge and be stable (as done in Hart and Kurz, 1983). In that case, if we use the coalitional value $\Psi$, for each game $(B, v)$, for which $Sh_k(M, v_B) - v(B_k) > 0$ for some $k \in M$, the coalition structure $B$ will not be stable if some null player $h$ belongs to $B_k$, because by excluding $h$ from the union and forming the new coalition structure $B' = \{(B_l)_{l \neq k}, B_k \setminus h, \{h\}\}$, all the members in $B_k \setminus h$ increase their payoffs: as $h$ is a null player in $(N, v)$, then $v(B_k) = v(B_k \setminus h)$, and in the new quotient game $(M', v_{B'})$, where $M' = M \cup \{h\}$, we have that $Sh_k(M', v_{B'}) = Sh_k(M, v_B)$, and then

$$\Psi_i(B', v) - \Psi_i(B, v) = \frac{1}{|B_k| \cdot |B_k \setminus h|}[Sh_k(M, v_B) - v(B_k)],$$

for each $i \in B_k \setminus h$.

We summarize this section with a table of properties that $\Psi$ and the Owen value satisfy (* means that the property is used in the characterization of the value).

| | E | A | CSy | Coh | NC | CNP | ISy | IE | PS | NP |
|---|---|---|---|---|---|---|---|---|---|---|
| $\Psi$ (on $\mathcal{CSG}$) | yes* | yes* | yes* | yes* | yes* | yes | yes | yes | yes* | no |
| $\Psi$ (Kamijo) | yes* | yes* | yes* | yes | yes | yes* | yes | yes* | yes | no |
| *Ow* | yes* | yes* | yes* | yes | yes | yes | yes* | no | no | yes* |

[5] The game $u - v$ is defined as $(u - v)(S) = u(S) - v(S)$, for each $S \subset N$.

[6] In particular, $0 = \Phi(B, v - v)[B_k] = \Phi(B, v)[B_k] + \Phi(B, -v)[B_k]$, then $\Phi(B, -v)[B_k] = -\Phi(B, v)[B_k]$.

[7] Returning to the marriage example, if the couple has a child, he can be considered as a null member of the family during his childhood.

**Remark 5.** Note that the characterizations of Owen (1977) and Kamijo (2009) hold true on $\mathcal{CSG}^N$, for each finite set $N$ and, therefore, they also hold when viewing the two coalitional values as values on $\mathcal{CSG}$. Nevertheless, our characterization of $\Psi$ by means of the population solidarity axiom only holds on $\mathcal{CSG}$, because in this axiom the value must be applied on $N$ and $N \setminus h$.

## 4. Complementary results

### 4.1. Independence of the axiomatic system

The axiom system in Theorem 4 is independent. Indeed:

(i) *Population solidarity within unions*: The Owen value satisfies all axioms, except *population solidarity within unions*. See Claim 1 of Section 3.

(ii) *Coherence:* Let the coalitional value $F^1$ be defined as

$$F_i^1(B,v) = \frac{Sh_k(M, v_B)}{|B_k|}, \quad \text{for each } k \in M \text{ and each } i \in B_k.$$

$F^1$ satisfies all the axioms except *coherence*, because $F_i^1(B^N, v) = v(N)/|N|$ and $F_i^1(B^n, v) = Sh_i(N, v)$, for each $i \in N$.

(iii) *Null coalition axiom*: Let $F^2$ be defined as

$$F_i^2(B,v) = \frac{v(N)}{|B_k|\,|M|}, \quad \text{for each } k \in M \text{ and each } i \in B_k.$$

$F^2$ satisfies all the axioms except *null coalition axiom*.

(iv) *Coalitional symmetry:* Define the following coalitional value $F^3$ by

(a) If $N = \{1,2,3\}$ and $B = \{\{1,2\},\{3\}\}$, then:

$$\begin{cases} F_1^3(B,v) = \dfrac{v(\{1,2\}) + v(1) - v(2)}{2} \\ F_2^3(B,v) = \dfrac{v(\{1,2\}) + v(2) - v(1)}{2} \\ F_3^3(B,v) = v(N) - v(\{1,2\}). \end{cases}$$

(b) Otherwise, $F_i^3(B, v) = \Psi_i(B, v)$, for each $i \in N$.

It is easy to see that $F^3$ satisfies all the axioms except *coalitional symmetry*, because in (a), $F_1^3(B, v) + F_2^3(B, v) = v(\{1, 2\})$ is, in general, different from $F_3^3(B, v) = v(N) - v(\{1, 2\})$, although coalitions $\{1, 2\}$ and $\{3\}$ are symmetric.

(v) *Efficiency*: Let $\pi$ be any semivalue other than the Shapley value (see Dubey et al., 1981). We define the following coalitional value $\Phi^\pi$ by

(a) If $B$ is different from $B^n$ or $B^N$, then:

$$\Phi_i^\pi(B,v) = Sh_i(B_k, v) + \frac{1}{|B_k|}[\pi_k(M, v_B) - v(B_k)],$$

for each $k \in M$ and each $i \in B_k$.

(b) $\Phi^\pi(B^n, v) = \Phi^\pi(B^N, v) = Sh(N, v)$.

The coalitional value $\Phi^\pi$ satisfies all the axioms except *efficiency*. When $\pi$ is the Shapley value, we obtain our coalitional value $\Psi$ which is the only one that satisfies *efficiency*.

(vi) *Additivity*: Define the coalitional value $F^4$ by

(a) If $(B, v) \in \mathcal{CSG}^N$ verifies that $B \neq B^n, B^N$, and $\sum_{k\in M} v(B_k) \neq 0$, then:

$$F_i^4(B,v) = Sh_i(B_k, v) + \frac{1}{|B_k|}\left[\frac{v(B_k)}{\sum\limits_{k\in M} v(B_k)} v(N) - v(B_k)\right],$$

for each $k \in M$ and each $i \in B_k$.

(b) Otherwise $F^4(B, v) = \Psi(B, v)$.

The coalitional value $F^4$ satisfies all the axioms except *additivity*.

### 4.2. Properties of balanced contributions

This section provides another characterization for the coalitional value $\Psi$ based on the principle of balanced contributions.

Myerson (1980) introduced this principle to characterize the Shapley value jointly with efficiency. Consider the following property of a value $\gamma$ on $\mathcal{G}^N$:

(BC) *Balanced contributions.* For each $v \in \mathcal{G}^N$ and each $\{i, j\} \subseteq N$:

$$\gamma_i(N, v) - \gamma_i(N \setminus j, v) = \gamma_j(N, v) - \gamma_j(N \setminus i, v).$$

This property states that for any two players, the amount that each player would gain or lose by the other player's withdrawal from the game should be equal.

Calvo et al. (1996) used the same principle to axiomatize the *level structure value*. This value was considered in Winter (1989) and is an extension of the Owen value for several levels of cooperation (union of players, union of union of players, and so on). In the particular case of games with a coalition structure (a single level), Calvo et al. (1996) proved that the Owen value is the only efficient coalitional value that satisfies the two following properties:

(IBC) *Intracoalitional balanced contributions*. For each $(B, v) \in \mathcal{CSG}^N$, each $k \in M$ and each $\{i, j\} \subseteq B_k$:

$$\Phi_i(B, v) - \Phi_i(B_{-j}, v) = \Phi_j(B, v) - \Phi_j(B_{-i}, v).$$

(CBC) *Coalitional balanced contributions*. For each $(B, v) \in \mathcal{CSG}^N$ and each $\{k, l\} \subseteq M$:

$$\Phi(B, v)[B_k] - \Phi(B \setminus B_l, v)[B_k] = \Phi(B, v)[B_l] - \Phi(B \setminus B_k, v)[B_l].$$

In the *IBC* property, the principle of balanced contributions is applied inside a union. The *CBC* property states that, for each $k, l \in M$, the contribution of $B_k$ to the total payoff of the members in $B_l$ must be equal to the contribution of $B_l$ to the total payoff of the members in $B_k$, hence balanced contributions is applied between unions.

We now show that the coalitional value $\Psi$ can also be characterized with the *CBC* property.

**Theorem 5.** *The coalitional value $\Psi$ is the only one that satisfies* efficiency, coalitional balanced contributions, population solida - rity within unions *and* coherence.

**Proof.** To prove *existence*, we only need to show that $\Psi$ satisfies *CBC*. For each $(B, v) \in \mathcal{CSG}^N$, and each $\{k, l\} \subseteq M$, we have that $\Psi(B, v)[B_k] = Sh_k(M, v_B)$, and $\Psi(B, v)[B_l] = Sh_l(M, v_B)$, then $\Psi$ satisfies *CBC* if and only if

$$Sh_k(M, v_B) - Sh_k(M \setminus l, v_B) = Sh_l(M, v_B) - Sh_l(M \setminus k, v_B).$$

And this is true because the Shapley value satisfies *BC*.

In order to prove *uniqueness*, let $\Phi$ be a coalitional value satisfying the above axioms. Let $(N, v) \in \mathcal{G}^N$ be a game, applying *CBC* for $B = B^n$, we have:

$$\Phi_i(B^n, v) - \Phi_i(B^n_{-j}, v) = \Phi_j(B^n, v) - \Phi_j(B^n_{-i}, v),$$

for each $\{i, j\} \subseteq N$.

And due to the characterization of Myerson (1980), this expression jointly with *efficiency* implies that $\Phi(B^n, v) = Sh(N, v)$ for each game $(N, v) \in \mathcal{G}^N$. By *coherence*, we have that $\Phi(B^N, v) = \Phi(B^n, v) = Sh(N, v)$ for each game $(N, v) \in \mathcal{G}^N$. Thus, $\Phi$ is uniquely determined when $|B| = 1$.

We now use induction on $|B|$. Let us assume that the uniqueness is established for $|B| \leq m$ and let $(B, v) \in \mathcal{CSG}^N$ be a game such that $|B| = m + 1$. By *CBC*, for each $\{k, l\} \subseteq M (k \neq l)$:

$$\Phi(B, v)[B_k] - \Phi(B, v)[B_l] = \Phi(B \setminus B_l, v)[B_k] - \Phi(B \setminus B_k, v)[B_l]. \quad (2)$$

The induction hypothesis yields

$$\begin{cases} \Phi(B \setminus B_l, v)[B_k] = \Psi(B \setminus B_l, v)[B_k] \\ \Phi(B \setminus B_k, v)[B_l] = \Psi(B \setminus B_k, v)[B_l]. \end{cases}$$

And, because $\Psi$ satisfies *CBC*, we have

$$\Psi(B \setminus B_l, v)[B_k] - \Psi(B \setminus B_k, v)[B_l] = \Psi(B, v)[B_k] - \Psi(B, v)[B_l].$$

Therefore, using (2):

$$\Phi(B, v)[B_k] - \Phi(B, v)[B_l] = \Psi(B, v)[B_k] - \Psi(B, v)[B_l]$$
$$\Rightarrow \Phi(B, v)[B_k] - \Psi(B, v)[B_k] = \Phi(B, v)[B_l] - \Psi(B, v)[B_l],$$

for each $\{k, l\} \subseteq M$. And then, by *efficiency*,

$$\Phi(B, v)[B_r] = \Psi(B, v)[B_r], \quad \text{for each } r \in M. \quad (3)$$

We now prove that $\Phi(B, v) = \Psi(B, v)$. Let $l \in M$. If $|B_l| = 1$, expression (3) means that $\Phi_i(B, v) = \Psi_i(B, v)$ for $\{i\} = B_l$. Suppose that $|B_l| \geq 2$. By *population solidarity within unions* we have, for each $\{k, l\} \subseteq M$ $(k \neq l)$, and each $\{i, j\} \subseteq B_l$:

$$\Phi_i(B, v) - \Phi_i(B \setminus B_k, v) = \Phi_j(B, v) - \Phi_j(B \setminus B_k, v). \quad (4)$$

By the induction assumption:

$$\begin{cases} \Phi_i(B \setminus B_k, v) = \Psi_i(B \setminus B_k, v) \\ \Phi_j(B \setminus B_k, v) = \Psi_j(B \setminus B_k, v). \end{cases}$$

Hence, using (4):

$$\Phi_i(B, v) - \Phi_j(B, v) = \Psi_i(B, v) - \Psi_j(B, v) \Rightarrow$$
$$\Phi_i(B, v) - \Psi_i(B, v) = \Phi_j(B, v) - \Psi_j(B, v),$$

for each $\{i, j\} \subseteq B_l$.

And taking (3) into account, we conclude that $\Phi_i(B, v) = \Psi_i(B, v)$, for each $i \in B_l$. □

**Remark 6.** The advantage of the characterization given in Theorem 5 is that, since the additivity property is not used, it can be applied to any subdomain of games with a coalition structure without violating uniqueness. A paradigmatic example is the case of *simple games*, a subdomain that is not closed under addition. This domain has been important in applications to political sciences. The two-step Shapley value seems an interesting alternative to the Owen value for the computation of the power that political parties have in parliaments under different coalition configurations.

In the proof of Theorem 5, *coherence* is necessary only to prove that if $B = B^N$ the solution coincides with the Shapley value. But this is induced by *IBC* and *efficiency*. Therefore, the difference between the Owen value and $\Psi$ is based on the difference between *IBC* and *population solidarity within unions*.

Kamijo (2006, 2007) considered the following variation of the *coalitional Balanced contributions*:

(GBC) *Group Balanced Contributions*.[8] For each $(B, v) \in \mathcal{CSG}^N$ with $|B| \geq 2$, each $\{k, h\} \subseteq M(k \neq h)$, each $i \in B_k$ and each $j \in B_h$:

$$\Phi_i(B, v) - \Phi_i(B \setminus B_h, v) = \Phi_j(B, v) - \Phi_j(B \setminus B_k, v).$$

According to this axiom, two players in distinct unions are affected equally by the deletion of the union associated with the other player. With the properties of *efficiency*, *coherence* and *group balanced contributions* he characterized the *collective value*. The collective value $\Omega$ is a weighted version of $\Psi$, where the weights are proportional to the sizes of the unions which define the coalition structure. Formally:

$$\Omega_i(B, v) = Sh_i(B_k, v) + \frac{1}{|B_k|}\left[Sh_k^w(M, v_B) - v(B_k)\right],$$

for each $k \in M$ and each $i \in B_k$,

where $Sh^w$ is the weighted Shapley value (Kalai and Samet, 1987), with weights $w_k$ proportional to $|B_k|$, for each $k \in M$.[9]

In the proof of Theorem 5, *population solidarity within unions* is used only when a whole union is deleted (see expression (4) in the proof), but this is induced by *GBC* (applying *GBC* twice: Firstly, for each $i \in B_k$ and $j \in B_h$, and secondly, for each $l \in B_k$ and $j \in B_h$). Thus, the difference between the collective value and $\Psi$ is based on the difference between *GBC* and *CBC*. In fact, *CBC* and *population solidarity within unions* could be replaced in Theorem 5[10] by the following property:

(ABC) *Aggregate Balanced Contributions*. For each $(B, v) \in \mathcal{CSG}^N$ with $|B| \geq 2$, each $\{k, h\} \subseteq M(k \neq h)$, each $i \in B_k$ and each $j \in B_h$:

$$|B_k|[\Phi_i(B, v) - \Phi_i(B \setminus B_h, v)] = |B_h|[\Phi_j(B, v) - \Phi_j(B \setminus B_k, v)].$$

We summarize this section with a table of properties used in the characterization of these values.

| | E | IBC | CBC | Coh | PS | ABC | GBC |
|---|---|---|---|---|---|---|---|
| *Ow* | yes* | yes* | yes* | yes | no | no | no |
| $\Psi$ | yes* | no | yes* | yes* | yes* | yes | no |
| $\Psi$ | yes* | no | yes | yes* | yes | yes* | no |
| $\Omega$ | yes* | no | no | yes* | yes | no | yes* |


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8 In Kamijo (2007) this property changes its name to *collective balanced contributions*.

9 Two weighted versions of the Owen coalitional value can be found in Levy and McLean (1989) and Vidal-Puga (2006).

10 The proof is left to the reader.

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