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How to cite this manuscript

If you make reference to this version of the manuscript, use the following information:

Ramm, A.G. (2011). Scattering of scalar waves by many small particles. Retrieved from http://krex.ksu.edu

Published Version Information

Publisher's Link: http://aipadvances.aip.org/resource/1/aaidbi/v1/i2?&page=2

Scattering of scalar waves by many small particles

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Formulas are derived for solutions of many-body wave scattering problems by small particles in the case of acoustically soft, hard, and impedance particles embedded in an inhomogeneous medium. The limiting case is considered, when the size a of small particles tends to zero while their number tends to infinity at a suitable rate. Equations for the limiting effective (self-consistent) field in the medium are derived. *© 2011 Author(s). This article is distributed under a Creative Commons Attribution Non-Commercial Share Alike 3.0 Unported License.* [doi:10.1063/1.3600704]

I. INTRODUCTION

There is a large literature on wave scattering by small bodies, starting from Rayleigh's work (1871).[1,3,25] For the problem of wave scattering by one body an analytical solution was found only for the bodies of special shapes, for example, for balls and ellipsoids. If the scatterer is small then the scattered field can be calculated analytically for bodies of arbitrary shapes, see Ref. 7, where this theory is presented.

The many-body wave scattering problem was discussed in the literature mostly numerically, if the number of scatterers is small, or under the assumption that the influence of the waves, scattered by other particles on a particular particle is negligible (see Ref. 5, where one finds a large bibliography, 1386 entries). This corresponds to the case when the distance d between neighboring particles is much larger than the wavelength λ, and the characteristic size a of a small body (particle) is much smaller than λ. By $k = \frac{2\pi}{\lambda}$ the wave number is denoted.

The basic results of our paper consist of:

i) Derivation of formulas for the scattering amplitude for the wave scattering problem by one small ($ka \ll 1$) body of an arbitrary shape under the Dirichlet, impedance, or Neumann boundary condition (acoustically soft, impedance, or hard particle),

ii) Solution to many-body wave scattering problem by many such particles under the assumptions $a \ll d$ and $a \ll \lambda$, where d is the minimal distance between neighboring particles,

iii) Derivation of the equations for the limiting effective (self-consistent) field in the medium when $a \to 0$ and the number $M = M(a)$ of the small particles tends to infinity at an appropriate rate,

iv) Derivation of linear algebraic systems for solving many-body wave scattering problems; these system are not obtained by a discretization of boundary integral equations.

Let us formulate the wave scattering problems we deal with. First, let us consider a one-body scattering problem. Let D_1 be a bounded domain in $\mathbb{R}^3$ with a sufficiently smooth boundary S_1. The scattering problem consists of finding the solution to the problem:

$$(\nabla^2 + k^2)u = 0 \text{ in } D_1' := \mathbb{R}^3 \setminus D_1, \tag{1}$$

$$\Gamma u = 0 \text{ on } S_1, \tag{2}$$

$$u = u_0 + v, \tag{3}$$

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where

$$u_0 = e^{ik\alpha \cdot x}, \quad \alpha \in S^2, \tag{4}$$

S^2 is the unit sphere in $\mathbb{R}^3$, u_0 is the incident field, v is the scattered field satisfying the radiation condition

$$v_r - ikv = o\left(\frac{1}{r}\right), \quad r := |x| \to \infty, \; v_r := \frac{\partial v}{\partial r}, \tag{5}$$

Γu is the boundary condition (bc) of one of the following types

$$\Gamma u = \Gamma_1 u = u \quad \text{(Dirichlet bc)}, \tag{6}$$

$$\Gamma u = \Gamma_2 u = u_N - \zeta_1 u, \quad \text{Im}\zeta_1 \leq 0, \; \text{(impedance bc)}, \tag{7}$$

where ζ_1 is a constant, N is the unit normal to S_1, pointing out of D_1, and

$$\Gamma u = \Gamma_3 u = u_N, \; \text{(Neumann bc)}. \tag{8}$$

It is well known (see, e.g., Ref. 6) that problem (1)–(3) has a unique solution. We now assume that

$$a := 0.5 \, \text{diam} D_1, \quad ka \ll 1, \tag{9}$$

and look for the solution to problem (1)–(3) of the form

$$u(x) = u_0(x) + \int_{S_1} g(x,t)\sigma_1(t)dt, \quad g(x,y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}, \tag{10}$$

where dt is the element of the surface area of S_1. One can prove that the unique solution to the scattering problem (1)–(3) with any of the boundary conditions (6)–(8) can be found in the form (10), and the function σ_1 in equation (10) is uniquely defined from the boundary condition (2). The scattering amplitude $A(\beta, \alpha) = A(\beta, \alpha, k)$ is defined by the formula

$$v = \frac{e^{ikr}}{r} A(\beta, \alpha, k) + o\left(\frac{1}{r}\right), \quad r \to \infty, \; \beta := \frac{x}{r}. \tag{11}$$

The equations for finding σ_1 are:

$$\int_{S_1} g(s,t)\sigma_1(t)dt = -u_0(s), \tag{12}$$

$$u_{0N} - \zeta_1 u_0 + \frac{A\sigma_1 - \sigma_1}{2} - \zeta_1 \int_{S_1} g(s,t)\sigma_1(t)dt = 0, \tag{13}$$

$$u_{0N} + \frac{A\sigma_1 - \sigma_1}{2} = 0, \tag{14}$$

respectively, for conditions (6)–(8). The operator A is defined as follows:

$$A\sigma := 2\int_{S_1} \frac{\partial}{\partial N_s} g(s,t)\sigma_1(t)dt. \tag{15}$$

Equations (12)–(14) are uniquely solvable, but there are no analytic formulas for their solutions for bodies of arbitrary shapes. However, if the body D_1 is small, $ka \ll 1$, one can rewrite (10) as

$$u(x) = u_0(x) + g(x,0)Q_1 + \int_{S_1} [g(x,t) - g(x,0)]\sigma_1(t)dt, \tag{16}$$

where

$$Q_1 := \int_{S_1} \sigma_1(t)dt, \tag{17}$$

and $0 \in D_1$ is the origin.

If $ka \ll 1$, then we prove that

$$|g(x,0)Q_1| \gg \left| \int_{S_1} [g(x,t) - g(x,0)]\sigma_1(t)dt \right|, \quad |x| > a. \tag{18}$$

Therefore, the scattered field is determined outside D_1 by a single number Q_1. This number can be obtained analytically without solving equations (12) and (13). The case (14) requires a special approach by the reason discussed in detail later.

Let us give the results for equations (12) and (13) first. For equation (12) one has

$$Q_1 = \int_{S_1} \sigma_1(t)dt = -Cu_0(0)[1 + o(1)], \quad a \to 0, \tag{19}$$

where C is the electric capacitance of a perfect conductor with the shape D_1. For equation (13) one has

$$Q_1 = -\zeta|S_1|u_0(0)[1 + o(1)], \quad a \to 0, \tag{20}$$

where $|S_1|$ is the surface area of S_1. The scattering amplitude for problem (1)–(3) with $\Gamma = \Gamma_1$ (acoustically soft particle) is

$$A_1(\beta, \alpha) = -\frac{C}{4\pi}[1 + o(1)], \tag{21}$$

since

$$u_0(0) = e^{ik\alpha \cdot x}|_{x=0} = 1.$$

Therefore, in this case the scattering is isotropic and of the order $O(a)$, because the capacitance $C = O(a)$.

The scattering amplitude for problem (1)–(3) with $\Gamma = \Gamma_2$ (small impedance particles) is:

$$A_2(\alpha, \beta) = -\frac{\zeta_1|S_1|}{4\pi}[1 + o(1)], \tag{22}$$

since $u_0(0) = 1$.

In this case the scattering is also isotropic, and of the order $O(\zeta|S_1|)$.

If $\zeta_1 = O(1)$, then $A_2 = O(a^2)$, because $|S_1| = O(a^2)$. If $\zeta_1 = O\left(\frac{1}{a^\kappa}\right)$, $\kappa \in (0, 1)$, then $A_2 = O(a^{2-\kappa})$. The case $\kappa = 1$ was considered in Ref. 9.

The scattering amplitude for problem (1)–(3) with $\Gamma = \Gamma_3$ (acoustically hard particles) is

$$A_3(\beta, \alpha) = -\frac{k^2|D_1|}{4\pi}(1 + \beta_{pq}\beta_p\alpha_q), \quad \text{if } u_0 = e^{ik\alpha \cdot x}. \tag{23}$$

Here and below summation is understood over the repeated indices, $\alpha_q = \alpha \cdot e_q$, $\alpha \cdot e_q$ denotes the dot product of two vectors in $\mathbb{R}^3$, $p, q = 1, 2, 3$, $\{e_p\}$ is an orthonormal Cartesian basis of $\mathbb{R}^3$, $|D_1|$ is the volume of D_1, β_{pq} is the magnetic polarizability tensor defined as follows (Ref. 7, p.62):

$$\beta_{pq} := \frac{1}{|D_1|} \int_{S_1} t_p \sigma_{1q}(t)dt, \tag{24}$$

σ_{1q} is the solution to the equation

$$\sigma_{1q}(s) = A_0\sigma_{1q} - 2N_q(s), \tag{25}$$

$N_q(s) = N(s) \cdot e_q$, $N = N(s)$ is the unit outer normal to S_1 at the point s, i.e., the normal pointing out of D_1, and A_0 is the operator A at $k = 0$. For small bodies $\|A - A_0\| = o(ka)$.

If $u_0(x)$ is an arbitrary field satisfying equation (1), not necessarily the plane wave $e^{ik\alpha \cdot x}$, then

$$A_3(\beta, \alpha) = \frac{|D_1|}{4\pi} \left(ik\beta_{pq}\frac{\partial u_0}{\partial x_q}\beta_p + \triangle u_0 \right). \tag{26}$$

The above formulas are derived in Section II. In Section III we develop a theory for many-body wave scattering problem and derive the equations for effective field in the medium, in which many small particles are embedded, as $a \to 0$.

The results, presented in this paper, are based on the earlier works of the author.[8–23] Our presentation and some of the results are novel. These results and methods of their derivation differ much from those in the homogenization theory.[2,4] The differences are:

i) no periodic structure in the problems is assumed,

ii) the operators in our problems are non-selfadjoint and have continuous spectrum,

iii) the limiting medium is not homogeneous and its parameters are not periodic,

iv) the technique for passing to the limit is different from one used in homogenization theory.

II. DERIVATION OF THE FORMULAS FOR ONE-BODY WAVE SCATTERING PROBLEMS

Let us recall the known result (see e.g., Ref. 6)

$$\frac{\partial}{\partial N_s^-} \int_{S_1} g(x,t)\sigma_1(t)dt = \frac{A\sigma_1 - \sigma_1}{2} \tag{27}$$

concerning the limiting value of the normal derivative of single-layer potential from outside. Let $x_m \in D_m, t \in S_m$, S_m is the surface of D_m, $a = 0.5 \,\mathrm{diam}\, D_m$.

In this Section $m = 1$, and $x_m = 0$ is the origin.

We assume that $ka \ll 1$, $ad^{-1} \ll 1$, so $|x - x_m| = d \gg a$. Then

$$\frac{e^{ik|x-t|}}{4\pi|x-t|} = \frac{e^{ik|x-x_m|}}{4\pi|x-x_m|}e^{-ik(x-x_m)^o\cdot(t-x_m)}\left(1 + O(ka + \frac{a}{d})\right), \tag{28}$$

$$k|x-t| = k|x-x_m| - k(x-x_m)^o \cdot (t-x_m) + O\left(\frac{ka^2}{d}\right), \tag{29}$$

where

$$d = |x - x_m|, \quad (x-x_m)^o := \frac{x-x_m}{|x-x_m|},$$

and

$$\frac{|x-t|}{|x-x_m|} = 1 + O\left(\frac{a}{d}\right). \tag{30}$$

Let us derive estimate (19). Since $|t| \le a$ on S_1, one has

$$g(s,t) = g_0(s,t)(1 + O(ka)),$$

where $g_0(s,t) = \frac{1}{4\pi|s-t|}$. Since $u_0(s)$ is a smooth function, one has $|u_0(s) - u_0(0)| = O(a)$. Consequently, equation (12) can be considered as an equation for electrostatic charge distribution $\sigma_1(t)$ on the surface S_1 of a perfect conductor D_1, charged to the constant potential $-u_0(0)$ (up to a small term of the order $O(ka)$). It is known that the total charge $Q_1 = \int_{S_1} \sigma_1(t)dt$ of this conductor is equal to

$$Q_1 = -Cu_0(0)(1 + O(ka)), \tag{31}$$

where C is the electric capacitance of the perfect conductor with the shape D_1.

Analytic formulas for electric capacitance C of a perfect conductor of an arbitrary shape, which allow to calculate C with a desired accuracy, are derived in Ref. 7. For example, the zeroth approximation formula is

$$C^{(0)} = \frac{4\pi|S_1|^2}{\int_{S_1}\int_{S_1}\frac{dsdt}{r_{st}}}, \quad r_{st} = |t - s|, \tag{32}$$

and we assume in (32) that $\epsilon_0 = 1$, where ϵ_0 is the dielectric constant of the homogeneous medium in which the perfect conductor is placed. Formula (31) is formula (19). If $u_0(x) = e^{ik\alpha \cdot x}$, then $u_0(0) = 1$, and $Q_1 = -C(1 + O(ka))$. In this case

$$A_1(\beta, \alpha) = \frac{Q_1}{4\pi} = -\frac{C}{4\pi}[1 + O(ka)],$$

which is formula (21).

Consider now wave scattering by an impedance particle.

Let us derive formula (20). Integrate equation (13) over S_1, use the divergence formula

$$\int_{S_1} u_{0N}ds = \int_{D_1} \nabla^2 u_0 dx = -k^2 \int_{D_1} u_0 dx = k^2|D_1|u_0(0)[1+o(1)], \qquad (33)$$

where $|D_1| = O(a^3)$, and the formula

$$-\zeta_1 \int_{S_1} u_0 ds = -\zeta_1|S_1|u_0(0)[1+o(1)]. \qquad (34)$$

Futhermore $|\int_{S_1} g(s,t)ds| = O(a)$, so

$$\zeta_1 \int_{S_1} ds \int_{S_1} g(s,t)\sigma_1(t)dt = O(aQ_1). \qquad (35)$$

Therefore, the term (35) is negligible compared with Q_1 as $a \to 0$. Finally, if $ka \ll 1$, then $g(s,t) = g_0(s,t)(1+ik|s-t|+\ldots)$, and

$$\frac{\partial}{\partial N_s} g(s,t) = \frac{\partial}{\partial N_s} g_0(s,t)[1+O(ka)]. \qquad (36)$$

Denote by A_0 the operator

$$A_0\sigma = 2\int_{S_1} \frac{\partial g_0(s,t)}{\partial N_s} \sigma_1(t)dt. \qquad (37)$$

It is known from the potential theory that

$$\int_{S_1} A_0\sigma_1 ds = -\int_{S_1} \sigma_1(t)dt, \quad 2\int_{S_1} \frac{\partial g_0(s,t)}{\partial N_s} ds = -1, \quad t \in S_1. \qquad (38)$$

Therefore,

$$\int_{S_1} ds \frac{A\sigma_1 - \sigma_1}{2} = -Q_1[1+O(ka)]. \qquad (39)$$

Consequently, from fromulas (33)–(39) one gets formula (22).

One can see that the wave scattering by an impedance particle is isotropic, and the scattered field is of the order $O(\zeta_1|S_1|)$. Since $|S_1| = O(a^2)$, one would have $O(\zeta_1|S_1|) = O(a^{2-\kappa})$ if $\zeta_1 = O\left(\frac{1}{a^\kappa}\right)$, $\kappa \in (0,1)$.

Consider now wave scattering by an acoustically hard small particle, i.e., the problem with the Neumann boundary condition.

In this case we will prove that:

i) The scattering is anisotropic,

ii) It is defined not by a single number, as in the previous two cases, but by a tensor,
and

iii) The order of the scattered field is $O(a^3)$ as $a \to 0$, for a fixed $k > 0$, i.e., the scattered field is much smaller than in the previous two cases.

When one integrates over S_1 equation (13), one gets

$$Q_1 = \int_{D_1} \nabla^2 u_0 dx = \nabla^2 u_0(0)|D_1|[1+o(1)], \quad a \to 0. \qquad (40)$$

Thus, $Q_1 = O(a^3)$. Therefore, the contribution of the term $e^{-ikx^o \cdot t}$ in formula (28) with $x_m = 0$ will be also of the order $O(a^3)$ and should be taken into account, *in contrast to the previous two cases.* Namely,

$$u(x) = u_0(x) + g(x,0)\int_{S_1} e^{-ik\beta \cdot t}\sigma_1(t)dt, \quad \beta := \frac{x}{|x|} = x^o. \qquad (41)$$

One has

$$\int_{S_1} e^{-ik\beta \cdot t}\sigma_1(t)dt = Q_1 - ik\beta_p \int_{S_1} t_p\sigma_1(t)dt, \qquad (42)$$

where the terms of higher order of smallness are neglected and summation over index p is understood. The function σ_1 solves equation (14):

$$\sigma_1 = A\sigma_1 + 2u_{0N} = A\sigma_1 + 2ik\alpha_q N_q u_0(s), \quad s \in S_1 \tag{43}$$

if $u_0(x) = e^{ik\alpha \cdot x}$.

Comparing (43) with (25), using (24), and taking into account that $ka \ll 1$, one gets

$$-ik\beta_p \int_{S_1} t_p \sigma_1(t)dt = -ik\beta_p|D_1|\beta_{pq}(-ik\alpha_q)u_0(0)[1 + O(ka)]$$

$$= -k^2|D_1|\beta_{pq}\beta_p\alpha_q u_0(0)[1 + O(ka)]. \tag{44}$$

From (40), (42) and (44) one gets formula (23), because $\nabla^2 u_0 = -k^2 u_0$.

If $u_0(x)$ is an arbitrary function, satisfying equation (1), then $ik\alpha_q$ in (43) is replaced by $\frac{\partial u_0}{\partial x_q}$, and $-k^2 u_0 = \triangle u_0$, which yields formula (26).

This completes the derivation of the formulas for the solution of scalar wave scattering problem by one small body on the boundary of which the Dirichlet, or the impedance, or the Neumann boundary condition is imposed.

III. MANY-BODY SCATTERING PROBLEM

In this Section we assume that there are $M = M(a)$ small bodies (particles) D_m, $1 \leq m \leq M$, $a = 0.5 \max \operatorname{diam} D_m$, $ka \ll 1$. The distance $d = d(a)$ between neighboring bodies is much larger than a, $d \gg a$, but we do not assume that $d \gg \lambda$, so *there may be many small particles on the distances of the order of the wavelength* λ. This means that our medium with the embedded particles is not necessarily diluted.

We assume that the small bodies are embedded in an arbitrary large but finite domain D, $D \subset \mathbb{R}^3$, so $D_m \subset D$. Denote $D' := \mathbb{R}^3 \setminus D$ and $\Omega := \cup_{m=1}^M D_m, S_m := \partial D_m$, $\partial\Omega = \cup_{m=1}^M S_m$. By N we denote a unit normal to $\partial\Omega$, pointing out of Ω, by $|D_m|$ the volume of the body D_m is denoted.

The scattering problem consists of finding the solution to the following problem

$$(\nabla^2 + k^2)u = 0 \text{ in } \mathbb{R}^3 \setminus \Omega, \tag{45}$$

$$\Gamma u = 0 \text{ on } \partial\Omega, \tag{46}$$

$$u = u_0 + v, \tag{47}$$

where u_0 is the incident field, satisfying equation (45) in $\mathbb{R}^3$, for example, $u_0 = e^{ik\alpha \cdot x}$, $\alpha \in S^2$, and v is the scattered field, satisfying the radiation condition (5). The boundary condition (46) can be of the types (6)–(8).

In the case of impedance boundary condition (7) we assume that

$$u_N = \zeta_m u \text{ on } S_m, \quad 1 \leq m \leq M, \tag{48}$$

so the impedance may vary from one particle to another. We assume that

$$\zeta_m = \frac{h(x_m)}{a^\kappa}, \quad \kappa \in (0, 1), \tag{49}$$

where $x_m \in D_m$ is a point in D_m, and $h(x)$, $x \in D$, is a given function, which we can choose as we wish, subject to the condition $\operatorname{Im} h(x) \leq 0$. For simplicity we assume that $h(x)$ is a continuous function.

Let us make the following assumption about the distribution of small particles: if $\Delta \subset D$ is an arbitrary open subset of D, then the number $\mathcal{N}(\Delta)$ of small particles in Δ, assuming the impedance boundary condition, is:

$$\mathcal{N}_\zeta(\Delta) = \frac{1}{a^{2-\kappa}} \int_\Delta N(x)dx[1 + o(1)], \quad a \to 0, \tag{50}$$

where $N(x) \geq 0$ is a given function. If the Dirichlet boundary condition is assumed, then

$$\mathcal{N}_D(\Delta) = \frac{1}{a} \int_\Delta N(x) dx [1 + o(1)], \quad a \to 0. \tag{51}$$

The case of the Neumann boundary condition will be considered later.

We look for the solution to problem (45)–(47) with the Dirichlet boundary condition of the form

$$u = u_0 + \sum_{m=1}^M \int_{S_m} g(x, t) \sigma_m(t) dt, \tag{52}$$

where $\sigma_m(t)$ are some functions to be determined from the boundary condition (46). It is proved in Ref. 9 that problem (45)–(47) has a unique solution of the form (52). For any $\sigma_m(t)$ function (52) solves equation (45) and satisfies condition (47). The boundary condition (46) determines σ_m uniquely. However, if $M \gg 1$, then numerical solution of the system of integral equations for σ_m, $1 \leq m \leq M$, which one gets from the boundary condition (46), is practically not feasible.

To avoid this principal difficulty we prove that the solution to scattering problem (45)–(47) *is determined by M numbers*

$$Q_m := \int_{S_m} \sigma_m(t) dt, \tag{53}$$

rather than M functions $\sigma_m(t)$.

This is possible to prove if the particles D_m are small. We derive analytical formulas for Q_m as $a \to 0$.

Let us define the effective (self-consistent) field $u_e(x) = u_e^{(j)}(x)$, acting on the $j-$th particle, by the formula

$$u_e(x) := u(x) - \int_{S_j} g(x, t) \sigma_j(t) dt, \quad |x - x_j| \sim a. \tag{54}$$

Physically this field acts on the $j-$th particle and is a sum of the incident field and the fields acting from all other particles:

$$u_e(x) = u_e^{(j)}(x) := u_0(x) + \sum_{m \neq j} \int_{S_m} g(x, t) \sigma_m(t) dt. \tag{55}$$

Let us rewrite (55) as follows:

$$u_e(x) = u_0(x) + \sum_{m \neq j}^M g(x, x_m) Q_m + \sum_{m \neq j}^M \int_{S_m} [g(x, t) - g(x, x_m)] \sigma_m(t) dt. \tag{56}$$

We want to prove that the last sum is negligible compared with the first one as $a \to 0$. To prove this, let us give some estimates. One has $|t - x_m| \leq a$, $d = |x - x_m|$,

$$|g(x, t) - g(x, x_m)| = \max \left\{ O\left(\frac{a}{d^2}\right), O\left(\frac{ka}{d}\right) \right\}, \quad |g(x, x_m)| = O(1/d). \tag{57}$$

Therefore, if $|x - x_j| = O(a)$, then

$$\frac{\left| \int_{S_m} [g(x, t) - g(x, x_m)] \sigma_m(t) dt \right|}{|g(x, x_m) Q_m|} \leq O(ad^{-1} + ka). \tag{58}$$

One can also prove that

$$J_1 / J_2 = O(ka + ad^{-1}), \tag{59}$$

where J_1 is the first sum in (56) and J_2 is the second sum in (56). Therefore, at any point $x \in \Omega' = \mathbb{R}^3 \setminus \Omega$ one has

$$u_e(x) = u_0(x) + \sum_{m=1}^M g(x, x_m) Q_m, \quad x \in \Omega', \tag{60}$$

where the terms of higher order of smallness are omitted.

A. The case of acoustically soft particles

If (46) is the Dirichlet condition, then, as we have proved in Section II (see formula (31)), one has

$$Q_m = -C_m u_e(x_m). \tag{61}$$

Thus,

$$u_e(x) = u_0(x) - \sum_{m=1}^{M} g(x, x_m) C_m u_e(x_m), \quad x \in \Omega'. \tag{62}$$

One has

$$u(x) = u_e(x) + o(1), \quad a \to 0, \tag{63}$$

so the full field and effective field are practically the same.

Let us write a linear algebraic system (LAS) for finding unknown quantities $u_e(x_m)$:

$$u_e(x_j) = u_0(x_j) - \sum_{m \neq j}^{M} g(x_j, x_m) C_m u_e(x_m). \tag{64}$$

If M is not very large, say $M = O(10^3)$, then LAS (64) can be solved numerically, and formula (62) can be used for calculation of $u_e(x)$.

Consider the limiting case, when $a \to 0$. One can rewrite (64) as follows:

$$u_e(\xi_q) = u_0(\xi_q) - \sum_{p \neq q}^{P} g(\xi_q, \xi_p) u_e(\xi_p) \sum_{x_m \in \Delta_p} C_m, \tag{65}$$

where $\{\Delta_p\}_{p=1}^{P}$ is a union of cubes which forms a covering of D,

$$\max_{p} diam \Delta_p := b = b(a) \gg a,$$

$$\lim_{a \to 0} b(a) = 0. \tag{66}$$

By $|\Delta_p|$ we denote the volume (measure) of Δ_p, and ξ_p is the center of Δ_p, or a point x_p in an arbitrary small body D_p, located in Δ_p. Let us assume that there exists the limit

$$\lim_{a \to 0} \frac{\sum_{x_m \in \Delta_p} C_m}{|\Delta_p|} = C(\xi_p), \quad \xi_p \in \Delta_p. \tag{67}$$

For example, one may have

$$C_m = c(\xi_p)a \tag{68}$$

for all m such that $x_m \in \Delta_p$, where $c(x)$ is some function in D. If all D_m are balls of radius a, then $c(x) = 4\pi$. We have

$$\sum_{x_m \in \Delta_p} C_m = C_p a \mathcal{N}(\Delta_p) = C_p N(\xi_p) |\Delta_p| [1 + o(1)], \quad a \to 0, \tag{69}$$

so limit (67) exists, and

$$C(\xi_p) = c(\xi_p)N(\xi_p). \tag{70}$$

From (65) and (68)–(70) one gets

$$u_e(\xi_q) = u_0(\xi_q) - \sum_{p \neq q} g(\xi_q, \xi_p)c(\xi_p)N(\xi_p)u_e(\xi_p)|\Delta_p|, \quad 1 \leq p \leq P. \tag{71}$$

Linear algebraic system (71) can be considered as the *collocation method for solving integral equation*

$$u(x) = u_0(x) - \int_D g(x, y)c(y)N(y)u(y)dy. \tag{72}$$

It is proved in Ref. 24 that system (71) is uniquely solvable for all sufficiently small $b(a)$, and the function

$$u_P(x) := \sum_{p=1}^{P} \chi_p(x)u_e(\xi_p) \tag{73}$$

converges in $L^\infty(D)$ to the unique solution of equation (72). The function $\chi_p(x)$ in (73) is the characteristic function of the cube Δ_p: it is equal to 1 in Δ_p and vanishes outside Δ_p. Thus, if $a \to 0$, the solution to the many-body wave scattering problem in the case of the Dirichlet boundary condition is well approximated by the unique solution of the integral equation (72).

Applying the operator $L_0 := \nabla^2 + k^2$ to (72), and using the formula $L_0 g(x, y) = -\delta(x - y)$, where $\delta(x)$ is the delta-function, one gets

$$\nabla^2 u + k^2 u - q(x)u = 0 \text{ in } \mathbb{R}^3, \quad q(x) := c(x)N(x). \tag{74}$$

The physical conclusion is:

If one embeds $M(a) = O(1/a)$ small acoustically soft particles, which are distributed as in (51), *then one creats, as $a \to 0$, a limiting medium, which is inhomogeneous, and has a refraction coefficient $n^2(x) = 1 - k^{-2}q(x)$.*

It is interesting from the physical point of view to note that *the limit, as $a \to 0$, of the total volume of the embedded particles is zero.*

Indeed, the volume of one particle is $O(a^3)$, the total number M of the embedded particles is $O(a^3 M) = O(a^2)$, and $\lim_{a\to 0} O(a^2) = 0$.

The second observation is: if (51) holds, then on a unit length straight line there are $O(\frac{1}{a^{1/3}})$ particles, so the distance between neighboring particles is $d = O(a^{1/3})$. If $d = O(a^\gamma)$ with $\gamma > \frac{1}{3}$, then the number of the embedded particles in a subdomain Δ_p is $O(\frac{1}{d^3}) = O(a^{-3\gamma})$. In this case, for $3\gamma > 1$, the limit in (69) is $C(\xi_p) = \lim_{a\to 0} c_p a O(a^{-3\gamma}) = \infty$. Therefore, the product of this limit by u remains finite only if $u = 0$ in D. Physically this means that if the distances between neighboring perfectly soft particles are smaller than $O(a^{1/3})$, namely, they are $O(a^\gamma)$ with any $\gamma > \frac{1}{3}$, then $u = 0$ in D.

On the other hand, if $\gamma < \frac{1}{3}$, then the limit $C(\xi_p) = 0$, and $u = u_0$ in D, so that the embedded particles do not change, in the limit $a \to 0$, properties of the medium.

This concludes our discussion of the scattering problem for many acoustically soft particles.

B. Wave scattering by many impedance particles

We assume now that (49) and (50) hold, use the exact boundary condition (46) with $\Gamma = \Gamma_2$, that is,

$$u_{eN} - \zeta_m u_e + \frac{A_m \sigma_m - \sigma_m}{2} - \zeta_m \int_{S_m} g(s, t)\sigma_m(t)dt = 0, \tag{75}$$

and integrate (75) over S_m in order to derive an analytical asymptotic formula for $Q_m = \int_{S_m} \sigma_m(t)dt$.

We have

$$\int_{S_m} u_{eN} ds = \int_{D_m} \nabla^2 u_e dx = O(a^3), \tag{76}$$

$$\int_{S_m} \zeta_m u_e(s) ds = h(x_m) a^{-\kappa} |S_m| u_e(x_m)[1 + o(1)], \quad a \to 0, \tag{77}$$

$$\int_{S_m} \frac{A_m \sigma_m - \sigma_m}{2} ds = -Q_m[1 + o(1)], \quad a \to 0, \tag{78}$$

and

$$\zeta_m \int_{S_m} \int_{S_m} g(s, t) \sigma_m(t) dt = h(x_m) a^{1-\kappa} Q_m = o(Q_m), \quad 0 < \kappa < 1. \tag{79}$$

From (75)–(79) one finds

$$Q_m = -h(x_m) a^{2-\kappa} |S_m| a^{-2} u_e(x_m)[1 + o(1)]. \tag{80}$$

This yields the formula for the approximate solution to the wave scattering problem for many impedance particles:

$$u(x) = u_0(x) - a^{2-\kappa} \sum_{m=1}^{M} g(x, x_m) b_m h(x_m) u_e(x_m)[1 + o(1)], \tag{81}$$

where

$$b_m := |S_m| a^{-2}$$

are some positive numbers which depend on the geometry of S_m and are independent of a. For example, if all D_m are balls of radius a, then $b_m = 4\pi$.

A linear algebraic system for $u_e(x_m)$, analogous to (64), is

$$u_e(x_j) = u_0(x_j) - a^{2-\kappa} \sum_{m=1, m \neq j}^{M} g(x_j, x_m) b_m h(x_m) u_e(x_m). \tag{82}$$

The integral equation for the limiting effective field in the medium with embedded small particles, as $a \to 0$, is

$$u(x) = u_0(x) - b \int_D g(x, y) N(y) h(y) u(y) dy, \tag{83}$$

where

$$u(x) = \lim_{a \to 0} u_e(x), \tag{84}$$

and we have assumed in (83) for simplicity that $b_m = b$ for all m, that is, all small particles are of the same shape and size.

Applying operator $L_0 = \nabla^2 + k^2$ to equation (83), one finds the differential equation for the limiting effective field $u(x)$:

$$(\nabla^2 + k^2 - bN(x)h(x))u = 0 \text{ in } \mathbb{R}^3, \tag{85}$$

and u satisfies condition (47).

The conclusion is: the limiting medium is inhomogeneous, and its properties are described by the function

$$q(x) := bN(x)h(x). \tag{86}$$

Since the choice of the functions $N(x) \geq 0$ and $h(x)$, $\text{Im} h(x) \leq 0$, is at our disposal, we can create the medium with desired properties by embedding many small impedance particles, with suitable impedances, according to the distribution law (50) with a suitable $N(x)$. The function

$$1 - k^{-2} q(x) = n^2(x) \tag{87}$$

is the refraction coefficient of the limiting medium. Given a desired refraction coefficient $n^2(x)$, $\text{Im} n^2(x) \geq 0$, one can find $N(x)$ and $h(x)$ so that (87) holds, that is, one can create a material with a desired refraction coefficient by embedding into a given material many small particles with suitable boundary impedances.

This concludes our discussion of the wave scattering problem with many small impedance particles.

C. Wave scattering by many acoustically hard particles

Consider now the case of acoustically hard particles, i.e., the case of Neumann boundary condition. The exact boundary integral equation for the function σ_m in this case is:

$$u_{eN} + \frac{A_m \sigma_m - \sigma_m}{2} = 0. \tag{88}$$

Arguing as in Section II, see formulas (40)–(44), one obtains

$$u_e(x) = u_0(x) + \sum_{m=1}^{M} g(x, x_m) \left[\triangle u_e(x_m) + i k \beta_{pq}^{(m)} \frac{(x_p - (x_m)_p)}{|x - x_m|} \frac{\partial u_e(x_m)}{\partial(x)_q} \right] |D_m|. \tag{89}$$

Here we took into account that the unit vector β in (44) is now the vector $\frac{x - x_m}{|x - x_m|}$, and $\beta_p = \frac{(x)_p - (x_m)_p}{|x - x_m|}$, where $(x)_p := x \cdot e_p$ is the p−th component of vector x in the Euclidean orthonormal basis $\{e_p\}_{p=1}^3$.

There are three sets of unknowns in (89): $u_e(x_m)$, $\frac{\partial u_e(x_m)}{\partial(x)_q}$, and $\triangle u_e(x_m)$, $1 \leq m \leq M$, $1 \leq q \leq 3$. To obtain linear algebraic system for $u_e(x_m)$ and $\frac{\partial u_e(x_m)}{\partial(x)_q}$ one sets $x = x_j$ in (89), takes the sum in (89) with $m \neq j$. This yields the first set of equations for finding these unknowns. Then one takes derivative of equation (89) with respect to $(x)_q$, sets $x = x_j$, and takes the sum in (89) with $m \neq j$. This yields the second set of equations for finding these unknowns. Finally, one takes Laplacian of equation (89), sets $x = x_j$, and takes the sum in (89) with $m \neq j$. This yields the third set of linear algebraic equations for finding $u_e(x_m)$, $\frac{\partial u_e(x_m)}{\partial(x)_q}$, and $\triangle u_e(x_m)$.

Passing to the limit $a \to 0$ in equation (89), yields the equation for the limiting field

$$u(x) = u_0(x) + \int_D g(x, y) \left(\rho(y) \nabla^2 u(y) + i k \frac{\partial u(y)}{\partial y_q} \frac{x_p - y_p}{|x - y|} B_{pq}(y) \right) dy, \tag{90}$$

where $\rho(y)$ and $B_{pq}(y)$ are defined below, see formulas (92) and (93).

Let us derive equation (90). We start by transforming the sum in (89). Let $\{\Delta_l\}_{l=1}^L$ be a covering of D by cubes Δ_l, $\max_l \text{diam} \Delta_l = b = b(a)$. We assume that

$$b(a) \gg d \gg a, \qquad \lim_{a \to 0} b(a) = 0.$$

Thus, there are many small particles D_m in Δ_l. Let x_l be a point in Δ_l. One has

$$\sum_{m=1}^{M} g(x, x_m) \left[\triangle u_e(x_m) + i k \frac{\partial u_e(x_m)}{\partial(x)_q} \beta_{pq}^{(m)} \frac{((x)_p - (x_m)_p)}{|x - x_m|} \right] |D_m|$$

$$= \sum_{l=1}^{L} g(x, x_l) \left[\triangle u_e(x_l) \sum_{x_m \in \Delta_l} |D_m| + i k \frac{\partial u_e(x_l)}{\partial(x)_q} \frac{((x)_p - (x_l)_p)}{|x - x_l|} \sum_{x_m \in \Delta_l} \beta_{pq}^{(m)} |D_m| \right]. \tag{91}$$

Assume that the following limit exist:

$$\lim_{a \to 0, y \in \Delta_l} \frac{\sum_{x_m \in \Delta_l} |D_m|}{|\Delta_l|} = \rho(y), \tag{92}$$

$$\lim_{a \to 0, y \in \Delta_l} \frac{\sum_{x_m \in \Delta_l} \beta_{pq}^{(m)} |D_m|}{|\Delta_l|} = B_{pq}(y), \tag{93}$$

and

$$\lim_{a \to 0} u_e(y) = u(y), \quad \lim_{a \to 0} \frac{\partial u_e(y)}{\partial (y)_q} = \frac{\partial u(y)}{\partial y_q}, \quad \lim_{a \to 0} \nabla^2 u_e(y) = \nabla^2 u(y). \tag{94}$$

Then, the sum in (91) converges to

$$\int_D g(x, y) \left(\rho(y) \nabla^2 u(y) + ik \frac{\partial u(y)}{\partial y_q} \frac{x_p - y_p}{|x - y|} B_{pq}(y) \right) dy. \tag{95}$$

Consequently, (89) yields in the limit $a \to 0$ equation (90). Equation (90) cannot be reduced to a differential equation for $u(x)$, because (90) is an integrodifferential equation whose integrand depends on x and y.

IV. SCATTERING BY SMALL PARTICLES EMBEDDED IN AN INHOMOGENEOUS MEDIUM

Suppose that the operator $\nabla^2 + k^2$ in (1) and in (45) is replaced by the operator $L_0 = \nabla^2 + k^2 n_0^2(x)$, where $n_0^2(x)$ is a known function,

$$\text{Im } n_0^2(x) \geq 0. \tag{96}$$

The function $n_0^2(x)$ is the refraction coefficient of an inhomogeneous medium in which many small particles are embedded. The results, presented in Section I–III remain valid if one replaces function $g(x, y)$ by the Green's function $G(x, y)$,

$$[\nabla^2 + k^2 n_0^2(x)] G(x, y) = -\delta(x - y), \tag{97}$$

satisfying the radiation condition. We assume that

$$n_0^2(x) = 1 \text{ in } D' := \mathbb{R}^3 \setminus D. \tag{98}$$

The function $G(x, y)$ is uniquely defined (see, e.g., Ref. 9). The derivations of the results remain essentially the same because

$$G(x, y) = g_0(x, y)[1 + O(|x - y|)], \quad |x - y| \to 0, \tag{99}$$

where $g_0(x, y) = \frac{1}{4\pi |x-y|}$. Estimates of $G(x, y)$ as $|x - y| \to 0$ and as $|x - y| \to \infty$ are obtained in Ref. 9. Smallness of particles in an inhomogeneous medium with refraction coefficient $n_0^2(x)$ is described by the relation $k n_0 a \ll 1$, where $n_0 := \max_{x \in D} |n_0(x)|$, and $a = \max_{1 \leq m \leq M} \text{diam} D_m$.

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