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Published Version Information

Citation: Ramm, A. G. (2010). How to prepare materials with a desired refraction coefficient? AIP Conference Proceedings, 1233, 165-168. doi: 10.1063/1.3452159

Digital Object Identifier (DOI): doi:10.1063/1.3452159

Publisher's Link: http://link.aip.org/link/?APCPCS/1233/165/1

How To Prepare Materials With a Desired Refraction Coefficient?

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Abstract

Abstract. In this talk a method is described for preparing materials with a desired refraction coefficient. The method consists of embedding into a material with known refraction coefficient many small particles of size a. The number of particles per unit volume around any point is prescribed, the distance between neighboring particles is $O(a^{\frac{2-\kappa}{3}})$ as $a \to 0$, $0 < \kappa < 1$ is a fixed parameter. The total number of the embedded particle is $O(a^{\kappa-2})$. The physical properties of the particles are described by the boundary impedance ζ_m of the $m-th$ particle, $\zeta_m = O(a^{-\kappa})$ as $a \to 0$. The refraction coefficient is the coefficient $n^2(x)$ in the wave equation $[\nabla^2 + k^2n^2(x)]u = 0$.

Keywords: mesarteritis, refraction coefficient, wave scattering, small particles.
PACS: 03.40.Kf, 03.50 De, 41.20.Jb, 71.36.+c

Introduction

The problem we are concerned with is the following one:

How does one create in a given bounded domain $D \subset \mathbb{R}^3$ a material with a desired refraction coefficient $n^2(x)$?

The domain D originally is assumed to be filled with a material with a known refraction coefficient $n_0^2(x)$. We assume that Im $n_0^2(x) \geq 0$ and $n_0^2(x) = 1$ in $D' := \mathbb{R}^3 \setminus D$. Originally the wave equation is:

$$L_0u_0 := [\nabla^2 + k^2n_0^2(x)]u_0 = 0 \quad \text{in } \mathbb{R}^3, \quad k = const > 0, \tag{1}$$

$$u_0 = e^{ik\alpha\cdot x} + v_0, \quad \alpha \in S^2, \tag{2}$$

$$v_0 = A_0(\beta, \alpha, k)\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r = |x| \to \infty, \quad \beta := \frac{x}{r}. \tag{3}$$

The function v_0 is the scattered field, $A_0(\beta, \alpha, k)$ is the scattering amplitude, $u(x, \alpha, k)$ is the scattering solution, S^2 is the unit sphere in $\mathbb{R}^3$.

We embed M small particles D_m, $S_m := \partial D_m$, $1 \leq m \leq M$, into D, so that in any subdomain $\Delta \subset D$ there are

$$\mathcal{N}(\Delta) = \frac{1}{a^{2-\kappa}} \int_\Delta N(x)dx[1 + o(1)], \quad a \to 0 \tag{4}$$

small particles. Here $N(x) \geq 0$ is a continuous (or piecewise-continuous) function which we can choose as we wish, $0 < \kappa < 1$ is a parameter which is our disposal. For simplicity we assume that particles D_m are balls centered at the points x_m and of radius a independent of m. The distance d between neighboring particles is assumed to be

$$d = O(a^{\frac{2-\kappa}{3}}) \quad \text{as } a \to 0. \tag{5}$$

The properties of a particle are described by the boundary impedance

$$\zeta_m = \frac{h(x_m)}{a^\kappa}, \tag{6}$$

where $h(x)$ is a continuous function on D, Im $h(x) \leq 0$. The function $h(x)$, as $N(x)$, we can choose as we wish. The scattering solution $u(x, \alpha, k)$ in the presence of the embedded particles solves the problem:

$$L_0u = 0 \quad \text{in } \mathbb{R}^3 \setminus \cup_{m=1}^M D_m, \tag{7}$$

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$$u_N + \zeta_m u = 0 \quad \text{on } S_m, \quad 1 \le m \le M, \tag{8}$$

$$u = u_0(x, \alpha, k) + v, \tag{9}$$

$$v = A_M(\beta, \alpha, k)\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad |x| = r \to \infty, \quad \beta := \frac{x}{r}. \tag{10}$$

Let us now describe our results. We prove that problem (7)-(10) has a unique solution $u(x, \alpha, k) := u_M(x, \alpha, k)$. We prove that given an arbitrary function $n^2(x)$ such that $n^2(x) = 1$ in D', $n^2(x)$ is continuous or piecewise-continuous in D (with the set of discontinuities of Lebesgue measure zero in $\mathbb{R}^3$), one can choose $N(x)$ and $h(x)$ so that the limit

$$\psi := \psi(x, \alpha, k) = \lim_{M\to\infty} u_M(x, \alpha, k) \tag{11}$$

exists and satisfies the equation

$$[\nabla^2 + k^2 n^2(x)]\psi = 0 \quad \text{in } \mathbb{R}^3, \tag{12}$$

$$\psi = u_0(x, \alpha, k) + w(x, \alpha, k), \tag{13}$$

$$\psi = e^{ik\alpha\cdot x} + A(\beta, \alpha, k)\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r = |x| \to \infty, \quad \beta := \frac{x}{r}. \tag{14}$$

Therefore the medium with embedded particles in the limit $M \to \infty$, or, which is the same by (4), in the limit $a \to 0$, has a desired refraction coefficient $n^2(x)$.
In Section 2 we formulate the recipe for choosing $N(x)$ and $h(x)$ which guarantees the existence of the limit (11) which solves problem (12)-(14). We do not assume that the small particles are embedded periodically.
The aim of this paper is to make clear for a wide audience of engineers and physicists our recipe for creating material with any desired refraction coefficient and to formulate two technological problems which must be solved in order that our theory can be immediately implemented experimentally. Theoretical justification of our results are given in [1]-[4], see also [5]- [12].

The recipe for creating material with a desired refraction coefficient

The problem we are interested in is the following one:

One is given a refraction coefficient $n_0^2(x)$ and wants to create a refraction coefficient $n^2(x)$.

Here is our recipe for doing this.

Step 1. Calculate the function

$$p(x) := k^2[n_0^2(x) - n^2(x)] := p_1(x) + ip_2(x), \tag{15}$$

where $p_1(x) = \text{Re } p(x)$, $p_2 = \text{Im } p(x)$.
Step 2. Find two functions $N(x) \ge 0$ and $h(x) = h_1(x) + ih_2(x)$ from the relation

$$4\pi h(x)N(x) = p(x). \tag{16}$$

This can be done by infinitely many ways. For example, one may fix $N(x) > 0$ and define

$$h_1 = \frac{p_1(x)}{4\pi N(x)}, \quad h_2 = \frac{p_2(x)}{4\pi N(x)}. \tag{17}$$

If one wishes to deal only with passive materials, then one requires Im $n^2(x) \ge 0$, Im $h(x) \le 0$. If Im $n_0^2(x) \le$ Im $n^2(x)$, then Im $p(x) \le 0$.
Step 3. Partition the domain D into a union of small cubes Δ_p, $1 \le p \le P$, without common interior points, $D = \cup_{p=1}^P \Delta_p$, the center of Δ_p is denoted by y_p, the side of Δ_p is of the order $O(a^{\frac{2-\kappa}{6}})$. In each cube Δ_p embed $\mathcal{N}(\Delta_p)$ small particles, where $\mathcal{N}(\Delta_p)$ is defined in (4). The distance d between neighboring particles should be $d = O(a^{\frac{2-\kappa}{3}})$. The given order of the smallness of d as $a \to 0$ is important, but the distance need not be exactly the same. The boundary impedance of each of the small particle embedded in Δ_p make equal to $\frac{h(y_p)}{a^\kappa}$, where $h(x) = h_1(x) + ih_2(x)$ is the function found in Step 2 of the recipe.

Theorem 0.1. *After the completion of Step 3, the material, obtained from the original one with the refraction coefficient $n_0^2(x)$, will have the refraction coefficient $n_M^2(x)$, and $\lim_{M\to\infty} n_M^2(x) = n^2(x)$, where $n^2(x)$ is the desired refraction coefficient.*

Proof of this theorem one finds in papers [1] and [4].

A discussion of the recipe

Step 1 of the recipe is trivial. Step 2 is also trivial. One may choose $N(x) > 0$ to satisfy some practical requirements. For example, if one chooses $N(x)$ small, then the total number of particles will be smaller. Practically one cannot take the limit $M \to \infty$, i.e., in the limit $a \to 0$, and one stops at some finite value of M, or of $a > 0$. The two technological problems, that have to be solved in order that our recipe can be implemented experimentally, are:

1) How does one embed a small particle at a given point into the given material in D?
2) How does one prepare a small particle, a ball of radius a centered at a point x_m, with the prescribed boundary impedance $\zeta_m = \frac{h(x_m)}{a^\kappa}$?
Here $h(x)$ is the function, found at Step 2 of the recipe

Possibly, the first technological problem can be solved by the stereolitography process.

One should be able to solve the second technological problem because its limiting cases $\zeta = 0$ (acoustically hard particles, particles from insulating material) and $\zeta_m = \infty$ (acoustically soft particles, perfectly conducting particles) are easy to solve in practice, so the intermediate values of the boundary impedance should be also possible to prepare.
A similar theory has been developed in paper [3] for electromagnetic wave scattering by many small dielectric and conducting particles embedded in an inhomogeneous medium.

Electromagnetic waves

Assume now that the governing equations are the Maxwell equations

$$\nabla \times E = i\omega\mu H, \quad \nabla \times H = -i\omega\epsilon'(x)E \qquad in \quad \mathbb{R}^3, \tag{18}$$

$\mu = const$, $\epsilon'(x) = \epsilon = const$ in D', $\omega > 0$ is frequency, $\epsilon'(x) = \epsilon(x) + i\frac{\sigma(x)}{\omega}$, $\sigma(x) \geq 0$ is the conductivity, $\sigma(x) = 0$ in D'. We assume that $\epsilon'(x) \in C^2(\mathbb{R}^3)$, $\epsilon'(x) \neq 0$, is a twice continuously differentiable function. Let $k = \frac{\omega}{c}$, $c = \omega\sqrt{\epsilon\mu}$ is the wave velocity in D'. The incident plane wave is $\mathcal{E}e^{ik\alpha\cdot x}$, $\alpha \in S^2$, $\alpha \cdot \mathcal{E} = 0$, $\mathcal{E}$ is a constant vector.

Under the above assumptions the electrical field $E(x)$ is the unique solution to the equation (see [3]):

$$\begin{aligned} E_0(x) &= \mathcal{E}e^{ik\alpha\cdot x} + \int_D g(x,y)p(y)E_0(y)dy \\ &+ \nabla_x \int_D g(x,y)q(y)\cdot E_0(y)dy, \quad g(x,y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}, \end{aligned} \tag{19}$$

where

$$p(x) := K^2(x) - k^2, \quad K^2(x) := \omega^2\epsilon'(x)\mu; \quad q(x) := \frac{\nabla K^2(x)}{K^2(x)}. \tag{20}$$

If M small particles D_m, $1 \leq m \leq M$, are embedded in D, then the basic equation (19) becomes

$$E_M(x) = E_0(x) + \sum_{m=1}^{M} \int_{D_m} g(x,y)p(y)E_M(y)dy + \nabla_x \sum_{m=1}^{M} \int_{D_m} g(x,y)q(y)\cdot E_M(y)dy. \tag{21}$$

It is proved in [3] that if the size a of small particles tends to zero, if the number of these particles in any open subset Δ of D is

$$\mathcal{N}(\Delta) = \frac{1}{a^{3-\kappa}} \int_\Delta N(x)dx[1 + o(1)], \quad a \to 0, \tag{22}$$

and if the distance d between neighboring particles is $d = O(a^{\frac{3-\kappa}{3}})$, then there exists the limit $\lim_{M\to 0} E_M(x) = E_e(x)$.

The limiting field $E_e(x)$, i.e., the effective field in the medium, solves the equation:

$$E_e(x) = E_0(x) + \int_D g(x,y)C(y)E_e(y)dy, \quad C(y) = N(y)c(y), \tag{23}$$

where

$$c(y) = \lim_{a\to 0} a^{\frac{1}{3-\kappa}} \int_{|y-x|\leq a} p(x)dx. \tag{24}$$

If, e.g., the small particle D_m is a ball of radius a centered at a point y, and

$$p(x) = \begin{cases} \frac{\gamma(y)}{4\pi a^{\kappa}} \left(1 - \frac{|x|}{a}\right)^2, & |x| \leq a; \\ 0, & |x| > a, \end{cases}$$

in the coordinate system with the origin at the point y, and $\gamma(y)$ is a number we can choose as we wish, then $c(y)$ in (24) can be calculated: $c(y) = \gamma(y)/30$. Equation (23) implies:

$$[\nabla^2 + \mathcal{K}^2(x)]E_e = 0 \quad \text{in } \mathbb{R}^3, \quad \mathcal{K}^2(x) := K^2 + C(x), \tag{25}$$

where $C(x)$ is defined in (23). This equation can be rewritten as

$$\nabla \times \nabla \times E_e = \mathcal{K}^2(x)E_e + \nabla\nabla \cdot E_e. \tag{26}$$

The term $\nabla\nabla \cdot E_e$ plays the role of the current $i\omega\mu J$.

This term can also be interpreted as the term due to a non-local susceptibility χ: if

$$D_e(x) = \tilde{\epsilon}(x)E_e - i\omega \int_D \chi(x,y)E_e(y)dy,$$

then the Maxwell's equations

$$\nabla \times E_e = i\omega\mu H_e, \quad \nabla \times H_e = -i\omega\epsilon\tilde{(x)}E_e - i\omega \int_D \chi(x,y)E_e(y)dy$$

imply

$$\nabla \times \nabla \times E_e = \omega^2\tilde{\epsilon}(x)\mu E_e(x) + \omega^2\mu \int_D \chi(x,y)E_e(y)dy.$$

This equation is of the form (26) if $\tilde{\epsilon}(x) = \frac{\mathcal{K}^2(x)}{\omega^2\mu}$, and

$$\chi(x,y) = (\omega^2\mu)^{-1}\nabla_x(\delta(x-y)\nabla_y),$$

where $\delta(x-y)$ is the delta-function.

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