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A NONLINEAR INEQUALITY AND EVOLUTION PROBLEMS

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Abstract. Assume that $g(t) \geq 0$, and

$$\dot{g}(t) \leq -\gamma(t)g(t) + \alpha(t, g(t)) + \beta(t),\ t \geq 0; \quad g(0) = g_0; \quad \dot{g} := \frac{dg}{dt},$$

on any interval $[0, T)$ on which g exists and has bounded derivative from the right, $\dot{g}(t) := \lim_{s \to +0} \frac{g(t+s)-g(t)}{s}$. It is assumed that $\gamma(t)$, and $\beta(t)$ are nonnegative continuous functions of t defined on $\mathbb{R}_+ := [0, \infty)$, the function $\alpha(t, g)$ is defined for all $t \in \mathbb{R}_+$, locally Lipschitz with respect to g uniformly with respect to t on any compact subsets $[0, T]$, $T < \infty$, and non-decreasing with respect to g, $\alpha(t, g_1) \geq \alpha(t, g_2)$ if $g_1 \geq g_2$. If there exists a function $\mu(t) > 0$, $\mu(t) \in C^1(\mathbb{R}_+)$, such that

$$\alpha\left(t, \frac{1}{\mu(t)}\right) + \beta(t) \leq \frac{1}{\mu(t)}\left(\gamma(t) - \frac{\dot{\mu}(t)}{\mu(t)}\right), \quad \forall t \geq 0; \quad \mu(0)g(0) \leq 1,$$

then $g(t)$ exists on all of $\mathbb{R}_+$, that is $T = \infty$, and the following estimate holds:

$$0 \leq g(t) \leq \frac{1}{\mu(t)}, \quad \forall t \geq 0.$$

If $\mu(0)g(0) < 1$, then $0 \leq g(t) < \frac{1}{\mu(t)}, \quad \forall t \geq 0$.

A discrete version of this result is obtained.

The nonlinear inequality, obtained in this paper, is used in a study of the Lyapunov stability and asymptotic stability of solutions to differential equations in finite and infinite-dimensional spaces.

1. Introduction

The goal of this paper is to give a self-contained proof of an estimate for solutions of a nonlinear inequality

$$\dot{g}(t) \leq -\gamma(t)g(t) + \alpha(t, g(t)) + \beta(t),\ t \geq 0;\ g(0) = g_0;\ \dot{g} := \frac{dg}{dt}, \tag{1.1}$$

and to demonstrate some of its many possible applications.

Denote $\mathbb{R}_+ := [0, \infty)$. It is not assumed a priori that solutions $g(t)$ to inequality (1.1) are defined on all of $\mathbb{R}_+$, that is, that these solutions exist globally. We give sufficient conditions for the global existence of $g(t)$. Moreover, under these conditions a bound on $g(t)$ is given, see estimate (1.5) in Theorem 1. This bound yields the relation $\lim_{t\to\infty} g(t) = 0$ if $\lim_{t\to\infty} \mu(t) = \infty$ in (1.5).

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Let us formulate our assumptions.
Assumption A). We assume that the function $g(t) \geq 0$ is defined on some interval $[0, T)$, has a bounded derivative $\dot{g}(t) := \lim_{s\to+0} \frac{g(t+s)-g(t)}{s}$ from the right at any point of this interval, and $g(t)$ satisfies inequality (1.1) at all t at which $g(t)$ is defined. The functions $\gamma(t)$, and $\beta(t)$, are continuous, non-negative, defined on all of $\mathbb{R}_+$. The function $\alpha(t, g) \geq 0$ is continuous on $\mathbb{R}_+ \times \mathbb{R}_+$, nondecreasing with respect to g, and locally Lipschitz with respect to g. This means that $\alpha(t, g) \geq \alpha(t, h)$ if $g \geq h$, and

$$|\alpha(t, g) - \alpha(t, h)| \leq L(T, M)|g - h|, \tag{1.2}$$

if $t \in [0, T]$, $|g| \leq M$ and $|h| \leq M$, $M = const > 0$, where $L(T, M) > 0$ is a constant independent of g, h, and t.
Assumption B). There exists a $C^1(\mathbb{R}_+)$ function $\mu(t) > 0$, such that

$$\alpha\left(t, \frac{1}{\mu(t)}\right) + \beta(t) \leq \frac{1}{\mu(t)}\left(\gamma(t) - \frac{\dot{\mu}(t)}{\mu(t)}\right), \quad \forall t \geq 0, \tag{1.3}$$

$$\mu(0)g(0) < 1. \tag{1.4}$$

If $\mu(0)g(0) \leq 1$, then the inequality sign $< \frac{1}{\mu(t)}$ in Theorem 1, in formula (1.5), is replaced by $\leq \frac{1}{\mu(t)}$.

Our results are formulated in Theorems 1 and 2, and *Propositions 1,2*. *Proposition 1* is related to Example 1, and *Proposition 2* is related to Example 2, see below.

Theorem 1. *If Assumptions A) and B) hold, then any solution $g(t) \geq 0$ to inequality* (1.1) *exists on all of $\mathbb{R}_+$, i.e., $T = \infty$, and satisfies the following estimate:*

$$0 \leq g(t) < \frac{1}{\mu(t)} \quad \forall t \in \mathbb{R}_+. \tag{1.5}$$

If $\mu(0)g(0) \leq 1$, then $0 \leq g(t) \leq \frac{1}{\mu(t)} \quad \forall t \in \mathbb{R}_+$.

Remark. *If $\lim_{t\to\infty} \mu(t) = \infty$, then $\lim_{t\to\infty} g(t) = 0$.*

Let us explain how one applies estimate (1.5) in various problems (see also papers [3], [4], and the monograph [5] for other applications of differential inequalities which are particular cases of inequality (1.1)).
Example 1. Consider the problem

$$\dot{u} = A(t)u + B(t)u, \quad u(0) := u_0, \tag{1.6}$$

where $A(t)$ is a linear bounded operator in a Hilbert space H and $B(t)$ is a bounded linear operator such that

$$\int_0^\infty \|B(t)\|dt := C < \infty.$$

Assume that

$$\mathrm{Re}(A(t)u, u) \leq 0 \quad \forall u \in H, \ \forall t \geq 0. \tag{1.7}$$

Operators satisfying inequality (1.7) are called *dissipative*. They arise in many applications, for example in a study of passive linear and nonlinear networks (e.g., see [6], and [7], Chapter 3).

One may consider some classes of unbounded linear operator using the scheme developed in the proofs of *Propositions 1,2*. For example, in *Proposition 1* the operator $A(t)$ can be a generator of C_0 semigroup $T(t)$ such that $\sup_{t\geq 0}\|T(t)\|\leq m$, where $m>0$ is a constant.

Let $A(t)$ be a linear closed, densely defined in H, dissipative operator, with domain of definition $D(A(t))$ independent of t, and I be the identity operator in H. Assume that the Cauchy problem

$$\dot{U}(t)=A(t)U(t), \quad U(0)=I,$$

for the operator-valued function $U(t)$ has a unique global solution and

$$\sup_{t\geq 0}\|U(t)\|\leq m,$$

where $m>0$ is a constant. Then such an unbounded operator $A(t)$ can be used in *Example 1.*

Proposition 1. If condition (1.7) *holds and* $C:=\int_0^\infty \|B(t)\|dt<\infty$*, then the solution to problem* (1.6) *exists on* $\mathbb{R}_+$*, is unique, and satisfies the following inequality:*

$$\sup_{t\geq 0}\|u(t)\|\leq e^C\|u_0\|. \tag{1.8}$$

Inequality (1.8) implies Lyapunov stability of the zero solution to equation (1.6).

Recall that the zero solution to equation (1.6) is called Lyapunov stable if for any $\epsilon>0$, however small, one can find a $\delta=\delta(\epsilon)>0$, such that if $\|u_0\|\leq\delta$, then the solution to Cauchy problem (1.6) satisfies the estimate $\sup_{t\geq 0}\|u(t)\|\leq\epsilon$. If, in addition, $\lim_{t\to\infty}\|u(t)\|=0$, then the zero solution to equation (1.6) is called asymptotically stable in the Lyapunov sense.

Example 2. Consider an abstract nonlinear evolution problem

$$\dot{u}=A(t)u+F(t,u)+b(t), \quad u(0)=u_0, \tag{1.9}$$

where $u(t)$ is a function with values in a Hilbert space H, $A(t)$ is a linear bounded operator in H which satisfies inequality

$$\mathrm{Re}(Au,u)\leq -\gamma(t)\|u\|^2, \quad t\geq 0; \qquad \gamma=\frac{r}{1+t}, \tag{1.10}$$

$r>0$ is a constant, $F(t,u)$ is a nonlinear map in H, and the following estimates hold:

$$\|F(t,u)\|\leq\alpha(t,g), \quad g:=g(t):=\|u(t)\|; \quad \|b(t)\|\leq\beta(t), \tag{1.11}$$

where $\beta(t)\geq 0$ and $\alpha(t,g)\geq 0$ satisfy the conditions in *Assumption A)*.

Let us assume that

$$\alpha(t,g)\leq c_0 g^p, \quad p>1; \quad \beta(t)\leq\frac{c_1}{(1+t)^\omega}, \tag{1.12}$$

where c_0, p, ω and c_1 are positive constants.

Proposition 2. If conditions (1.9)-(1.12) *hold, and inequalities* (2.7),(2.8) *and* (2.10) *are satisfied (see these inequalities in the proof of Proposition 2), then the solution to the evolution problem* (1.9) *exists on all of* $\mathbb{R}_+$ *and satisfies the following estimate:*

$$0\leq\|u(t)\|\leq\frac{1}{\lambda(1+t)^q}, \qquad \forall t\geq 0, \tag{1.13}$$

where λ *and* q *are some positive constants the choice of which is specified by inequalities* (2.7),(2.8) *and* (2.10).

The choice of λ and q is motivated and explained in the proof of *Proposition 2* (see inequalities (2.7), (2.8) and (2.10) in Section 2).

Inequality (1.13) implies asymptotic stability of the solution to problem (1.9) in the sense of Lyapunov and, additionally, gives a rate of convergence of $\|u(t)\|$ to zero as $t\to\infty$.

The results in *Examples 1,2* can be obtained in Banach space, but we do not go into detail.

Proofs of Theorem 1 and *Propositions 1,2* are given in Section 2. Theorem 2, which is a discrete analog of Theorem 1, is formulated and proved in Section 3.

2. Proofs

Proof of Proposition 1. Local existence of the solution $u(t)$ to problem (1.6) is known (see, e.g., [1]). Uniqueness of this solution follows from the linearity of the problem and from estimate (1.8). Let us prove this estimate.

Multiply (1.6) by $u(t)$, let $g(t):=\|u(t)\|$, take real part, use (1.7), and get

$$\frac{1}{2}\frac{dg^2(t)}{dt}\le \mathrm{Re}(B(t)u(t),u(t))\le \|B(t)\|g^2(t).$$

This implies $g^2(t)\le g^2(0)e^{2C}$, so (1.8) follows. *Proposition 1* is proved. □

Proof of Proposition 2. The local existence and uniqueness of the solution $u(t)$ to problem (1.9) follow from *Assumption A* (see, e.g., [1]). The existence of $u(t)$ for all $t\ge 0$, that is, the global existence of $u(t)$, follows from estimate (1.13) (see, e.g., [5], pp.167-168).

Let us derive estimate (1.13). Multiply (1.9) by $u(t)$, let $g(t):=\|u(t)\|$, take real part, use (1.10)-(1.12) and get

$$g\dot g\le -\gamma(t)g^2(t)+\alpha(t,g(t))g(t)+\beta(t)g(t),\ t\ge 0. \tag{2.1}$$

Since $g\ge 0$, one obtains from this inequality inequality (1.1). However, first we would like to explain in detail the meaning of the derivative $\dot g$ in our proof.

By $\dot g$ the right derivatives is understood:

$$\dot g(t):=\lim_{s\to+0}\frac{g(t+s)-g(t)}{s}.$$

If $g(t)=\|u(t)\|$ and $u(t)$ is continuously differentiable, then $(t):=g^2(t)=(u(t),u(t))$ is continuously differentiable, and its derivative at the point t at which $g(t)>0$ can be computed by the formula:

$$\dot g=Re(\dot u(t),u^0(t)),$$

where $u^0(t):=\frac{u(t)}{\|u(t)\|}$. Thus, the function $g(t)=\sqrt{\ (t)}$ is continuously differentiable at any point at which $g(t)\ne 0$. At a point t at which $g(t)=0$, the vector $u^0(t)$ is not defined, the derivative of $g(t)$ does not exist in the usual sense, but the right derivative of $g(t)$ still exists and can be calculated explicitly:

$$\dot g(t)=\lim_{s\to+0}\frac{\|u(t+s)\|-\|u(t)\|}{s}=\lim_{s\to+0}\frac{\|u(t)+s\dot u(t)+o(s)\|}{s}$$
$$=\lim_{s\to 0}\|\dot u(t)+o(1)\|=\|\dot u(t)\|.$$

If $u(t)$ is continuously differentiable at some point t, and $u(t)\ne 0$, then

$$\dot g=\|u(t)\|^{\cdot}\le\|\dot u(t)\|.$$

Indeed,

$$2g(t)\dot g(t)=(\dot u(t),u(t))+(u(t),\dot u(t))\le 2\|\dot u\|\|u\|=2\|\dot u(t)\|g(t).$$

If $g(t) \neq 0$, then the above inequality implies $\dot g(t) \leq \|\dot u(t)\|$, as claimed. One can also derive this inequality from the formula $\dot g = Re(\dot u(t), u^0(t))$, since $|Re(\dot u(t), u^0(t))| \leq \|\dot u(t)\|$.

If $g(t) > 0$, then from (2.1) one obtains

$$\dot g(t) \leq -\gamma(t)g(t) + \alpha(t, g(t)) + \beta(t), \quad t \geq 0. \tag{2.2}$$

If $g(t) = 0$ on an open set, then inequality (2.2) holds on this set also, because $\dot g = 0$ on this set while the right-hand side of (2.2) is non-negative at $g = 0$. If $g(t) = 0$ at some point $t = t_0$, then (2.2) holds at $t = t_0$ because, as we have proved above, $\dot g(t_0) = 0$, while the right-hand side of (2.2) is equal to $\beta(t) \geq 0$ if $g(t_0) = 0$, and is, therefore, non-negative if $g(t_0) = 0$.

If assumptions (1.12) hold, then inequality (2.2) can be rewritten as

$$\dot g \leq -\frac{1}{(1+t)^\nu} g + c_0 g^p + \frac{c_1}{(1+t)^\omega}, \quad p > 1. \tag{2.3}$$

Let us look for $\mu(t)$ of the form

$$\mu(t) = \lambda(1+t)^q, \quad q = const > 0, \quad \lambda = const > 0. \tag{2.4}$$

Inequality (1.3) takes the form

$$\frac{c_0}{[\lambda(1+t)^q]^p} + \frac{c_1}{(1+t)^\omega} \leq \frac{1}{\lambda(1+t)^q}\left(\frac{r}{(1+t)^\nu} - \frac{q}{1+t}\right), \quad t > 0, \tag{2.5}$$

or

$$\frac{c_0}{\lambda^{p-1}(1+t)^{q(p-1)}} + \frac{c_1\lambda}{(1+t)^{\omega-q}} + \frac{q}{1+t} \leq \frac{r}{(1+t)^\nu}, \quad t > 0 \tag{2.6}$$

Assume that the following inequalities (2.7)-(2.8) hold:

$$q(p-1) \geq \nu, \quad \omega - q \geq \nu, \quad 1 \geq \nu, \tag{2.7}$$

and

$$\frac{c_0}{\lambda^{p-1}} + c_1\lambda + q \leq r. \tag{2.8}$$

Then inequality (2.6) holds, and Theorem 1 yields

$$g(t) = \|u(t)\| < \frac{1}{\lambda(1+t)^q}, \quad \forall t \geq 0, \tag{2.9}$$

provided that

$$\|u_0\| < \frac{1}{\lambda}. \tag{2.10}$$

Note that for any $\|u_0\|$ inequality (2.10) holds if λ is sufficiently large. For a fixed λ, however large, inequality (2.8) holds if r is sufficiently large.

Proposition 2 is proved. □

The proof of *Proposition 2* provides a flexible general scheme for obtaining estimates of the behavior of the solution to evolution problem (1.9) for $t \to \infty$.

Proof of Theorem 1. Let

$$g(t) = \frac{v(t)}{a(t)}, \quad a(t) := e^{\int_0^t \gamma(s)ds}, \tag{2.11}$$

$$\eta(t) := \frac{a(t)}{\mu(t)}, \quad \eta(0) = \frac{1}{\mu(0)} > g(0). \tag{2.12}$$

Then inequality (1.1) reduces to

$$\dot v(t) \le a(t)\alpha\left(t, \frac{v(t)}{a(t)}\right) + a(t)\beta(t), \quad t \ge 0; \quad v(0) = g(0). \tag{2.13}$$

One has

$$\dot\eta(t) = \frac{\gamma(t)a(t)}{\mu(t)} - \frac{\dot\mu(t)a(t)}{\mu^2(t)} = \frac{a(t)}{\mu(t)}\left(\gamma(t) - \frac{\dot\mu(t)}{\mu(t)}\right). \tag{2.14}$$

From (1.3), (2.11)-(2.14), one gets

$$v(0) < \eta(0), \quad \dot v(0) \le \dot\eta(0). \tag{2.15}$$

Therefore there exists a $T > 0$ such that

$$0 \le v(t) < \eta(t), \quad \forall t \in [0, T). \tag{2.16}$$

Let us prove that $T = \infty$.

First, note that if inequality (2.16) holds for $t \in [0, T)$, or, equivalently, if

$$0 \le g(t) < \frac{1}{\mu(t)}, \qquad \forall t \in [0, T), \tag{2.17}$$

then

$$\dot v(t) \le \dot\eta(t), \qquad \forall t \in [0, T). \tag{2.18}$$

One can pass to the limit $t \to T - 0$ in this inequality and get

$$\dot v(T) \le \dot\eta(T). \tag{2.19}$$

Indeed, from inequality (2.17) it follows that

$$\alpha\left(t, \frac{v}{a}\right) + \beta = \alpha(t, g) + \beta \le \alpha(t, \frac{1}{\mu}) + \beta,$$

because $\alpha(t, g) \le \alpha(t, \frac{1}{\mu})$.

Furthermore, from inequality (1.3) one derives:

$$\alpha\left(t, \frac{1}{\mu}\right) + \beta \le \frac{1}{\mu(t)}\left(\gamma(t) - \frac{\dot\mu(t)}{\mu(t)}\right).$$

Consequently, from inequalities (2.13)-(2.14) one obtains

$$\dot v(t) \le \frac{a(t)}{\mu(t)}\left(\gamma(t) - \frac{\dot\mu(t)}{\mu(t)}\right) = \dot\eta(t), \qquad t \in [0, T),$$

and inequality (2.18) is proved.

Let $t \to T - 0$ in (2.18). The function $\eta(t)$ is defined for all $t \in \mathbb{R}_+$ and $\dot\eta(t)$ is continuous on $\mathbb{R}_+$. Thus, there exists the limit

$$\lim_{t \to T-0} \dot\eta(t) = \dot\eta(T).$$

By $\dot v(T)$ in inequality (2.19) one may understand $\limsup_{t \to T-0} \dot v(t)$, which does exist because $\dot v(t)$ is bounded for all $t < T$ by a constant independent of $t \in [0, T]$, due to the estimate (2.18).

To prove that $T = \infty$ we prove that the "upper" solution $w(t)$ to the inequality (2.13) exists for all $t \in \mathbb{R}_+$.

Define $w(t)$ as the solution to the problem

$$\dot w(t) = a(t)\alpha\left(t, \frac{w(t)}{a(t)}\right) + a(t)\beta(t), \quad w(0) = v_0. \tag{2.20}$$

The unique solution to problem (2.20) exists locally, on $[0, T)$, because $\alpha(t, g)$ is assumed locally Lipschitz. On the interval $[0, T)$ one obtains inequality

$$0 \le v(t) \le w(t), \qquad t \in [0, T),$$

by the standard comparison lemma (see, e.g., [5], p.99, or [2]). Thus, inequality

$$0 \le v(t) \le w(t) \le \eta(t), \qquad t \in [0, T), \tag{2.21}$$

holds.

The desired conclusion $T = \infty$ one derives from the following result.

Proposition 3. The solution $w(t)$ to problem (2.20) *exists on every interval* $[0, T]$ *on which it is a priori bounded by a constant depending only on T.*

We prove this result later. Assuming that *Proposition 3.* is established, one concludes that $T = \infty$.

Let us finish the proof of Theorem 1 using *Proposition 3.* Since $\eta(t)$ is bounded on any interval $[0, T]$ (by a constant depending only on T) one concludes from *Proposition 3* that $w(t)$ (and, therefore, $v(t)$) exists on all of $\mathbb{R}_+$. If $v(t) \le \eta(t)$ $\forall t \in \mathbb{R}_+$, then inequality (1.5) holds (see (2.11) and (2.12)), and Theorem 1 is proved. □

Let us prove *Proposition 3.*

Proof of Proposition 3. We prove a more general statement, namely, *Proposition 4*, from which *Proposition 3* follows.

Proposition 4. Assume that

$$\dot{u} = f(t, u), \quad u(0) = u_0, \tag{2.22}$$

where $f(t, u)$ is an operator in a Banach space X, locally Lipschitz with respect to u for every t, i.e., $\|f(t, u) - f(t, v)\| \le L(t, M)\|u - v\|$, $\forall v, v \in \{u : \|u\| \le M\}$. The unique solution to problem (2.22) *exists for all $t \ge 0$ if and only if*

$$\|u(t)\| \le c(t), \quad t \ge 0, \tag{2.23}$$

where $c(t)$ is a continuous function defined for all $t \ge 0$, and inequality (2.23) *holds for all t for which $u(t)$ exists.*

Proof of Proposition 4. The necessity of condition (2.23) is obvious: one may take $c(t) = \|u(t)\|$.

To prove its sufficiency, recall a known local existence theorem, see, e.g., [1].

Proposition 5. If $\|f(t, u)\| \le M_1$ and $\|f(t, u) - f(t, v)\| \le L\|u - v\|$, $\forall t \in [t_0, t_0 + T_1]$, $\|u - u_0\| \le R$, $u_0 = u(t_0)$, then there exists a $\delta > 0$, $\delta = \min(\frac{R}{M_1}, \frac{1}{L}, T_1 - T)$, such that for every $\tau_0 \in [t_0, T]$, $T < T_1$, there exists a unique solution to equation (2.22) *in the interval $(\tau_0 - \delta, \tau + \delta)$ and $\|u(t) - u(t_0)\| \le R$.*

Using *Proposition 5*, let us prove the sufficiency of the assumption (2.23) for the global existence of $u(t)$, i.e., for the existence of $u(t)$ for all $t \ge t_0$.

Assume that condition (2.23) holds and the solution to problem (2.22) exists on $[t_0, T)$ but does not exist on $[t_0, T_1)$ for any $T_1 > T$. Let us derive a contradiction from this assumption.

Proposition 5 guarantees the existence and uniqueness of the solution to problem (2.22) with $t_0 = T$ and the initial value $u_0 = u(T - 0)$. The value $u(T - 0)$ exists if inequality (2.23) holds, as we prove below. The solution $u(t)$ exists on the interval $[T - \delta, T + \delta]$ and, by the uniqueness theorem, coincides with the solution $u(t)$ of the problem (2.22) on the interval $(T - \delta, T)$. Therefore, the solution to (2.22) can be uniquely extended to the interval $[0, T+\delta)$, contrary to the assumption that it does not exist on the interval $[0, T_1)$ with any $T_1 > T$. This contradiction proves that $T = \infty$, i.e., the solution to problem (2.22) exists for all $t \geq t_0$ if estimate (2.23) holds and $c(t)$ is defined and continuous $\forall t \geq t_0$.

Let us now prove the existence of the limit

$$\lim_{t \to T-0} u(t) := u(T-0).$$

Let $t_n \to T$, $t_n < T$. Then

$$\|u(t_n) - u(t_{n+m})\| \leq \int_{t_n}^{t_{n+m}} \|f(t, u(s))\| ds \leq (t_{n+m} - t_n) M_1 \to 0 \text{ as } n \to \infty.$$

Therefore, by the Cauchy criterion, there exists the limit

$$\lim_{t_n \to T-0} u(t) = u(T-0).$$

Estimate (2.23) guarantees the existence of the constant M_1.

Proposition 4 is proved □

Therefore *Proposition 3* is also proved and, consequently, the statement of Theorem 1, corresponding to the assumption (1.5), is proved. In our case $t_0 = 0$, but one may replace the initial moment $t_0 = 0$ in (1.1) by an arbitrary $t_0 \in \mathbb{R}_+$.

Finally, if $g(0) \leq \frac{1}{\mu(0)}$, then one proves the inequality

$$0 \leq g(t) \leq \frac{1}{\mu(t)}, \qquad \forall t \in \mathbb{R}_+$$

using the argument similar to the above. This argument is left to the reader.

Theorem 1 is proved. □

3. Discrete version of Theorem 1

Theorem 2. *Assume that* $g_n \geq 0$, $\alpha(n, g_n) \geq 0$,

$$g_{n+1} \leq (1 - h_n \gamma_n) g_n + h_n \alpha(n, g_n) + h_n \beta_n, \quad h_n > 0,\ 0 < h_n \gamma_n < 1, \tag{3.1}$$

and $\alpha(n, g_n) \geq \alpha(n, q_n)$ *if* $g_n \geq q_n$. *If there exists a sequence* $\mu_n > 0$ *such that*

$$\alpha(n, \frac{1}{\mu_n}) + \beta_n \leq \frac{1}{\mu_n}(\gamma_n - \frac{\mu_{n+1} - \mu_n}{h_n \mu_n}), \tag{3.2}$$

and

$$g_0 \leq \frac{1}{\mu_0}, \tag{3.3}$$

then

$$0 \leq g_n \leq \frac{1}{\mu_n} \qquad \forall n \geq 0. \tag{3.4}$$

Proof. For $n = 0$ inequality (3.4) holds because of (3.3). Assume that it holds for all $n \leq m$ and let us check that then it holds for $n = m + 1$. If this is done, Theorem 2 is proved. Using the inductive assumption, one gets:

$$g_{m+1} \leq (1 - h_m \gamma_m) \frac{1}{\mu_m} + h_m \alpha(m, \frac{1}{\mu_m}) + h_m \beta_m.$$

This and inequality (3.2) imply:

$$
\begin{aligned}
g_{m+1} &\le (1 - h_m\gamma_m)\frac{1}{\mu_m} + h_m\frac{1}{\mu_m}(\gamma_m - \frac{\mu_{m+1} - \mu_m}{h_m\mu_m}) \\
&= \frac{\mu_m h_m - \mu_m h_m^2\gamma_m + h_m^2\gamma_m\mu_m - h_m\mu_{m+1} + h_m\mu_m}{\mu_m^2 h_m} \\
&= \frac{2\mu_m h_m - h_m\mu_{m+1}}{\mu_m^2 h_m} = \frac{2\mu_m - \mu_{m+1}}{\mu_m^2} = \frac{1}{\mu_{m+1}} + \frac{2\mu_m - \mu_{m+1}}{\mu_m^2} - \frac{1}{\mu_{m+1}}.
\end{aligned}
$$

The proof is completed if one checks that

$$\frac{2\mu_m - \mu_{m+1}}{\mu_m^2} \le \frac{1}{\mu_{m+1}},$$

or, equivalently, that

$$2\mu_m\mu_{m+1} - \mu_{m+1}^2 - \mu_m^2 \le 0.$$

The last inequality is obvious since it can be written as

$$-(\mu_m - \mu_{m+1})^2 \le 0.$$

Theorem 2 is proved. □

Theorem 2 was formulated in [3] and proved in [4]. We included for completeness a proof, which is different from the one in [4] only slightly.

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