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Ramm, A. G. (2010). Slow manifolds for dissipative dynamical systems. Retrieved from http://krex.ksu.edu

## Published Version Information

**Citation**: Ramm, A.G. (2011). Slow manifolds for dissipative dynamical systems. **Journal of Mathematical Analysis and Applications**, 363 (2), 729-732.



**Digital Object Identifier (DOI)**: doi:10.1016/j.jmaa.2009.09.049

**Publisher's Link**:
http://www.elsevier.com/wps/find/journaldescription.cws_home/622886/description#description

A.G.Ramm, Slow manifolds for dissipative dynamical systems, J. Math. Anal.Appl., 363, (2010), 729-732.

# Slow manifolds for dissipative dynamical systems

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**Abstract**

A class of infinite-dimensional dissipative dynamical systems is defined for which the slow invariant manifolds can be calculated. Large-time behavior of the evolution of such systems is studied.

**Key words:** Dissipative systems; dynamical systems; attractors; invariant manifolds; nonlinear evolution.

**MSC**: 35Q30, 78A40, 80A30

**PACS**: 02.30.Jr; 02.30. Tb

## 1 Introduction and statement of the results

A dynamical system is described by the evolution problem:

$$\dot{u} = -F(u), \quad u(0) = u_0; \quad \dot{u} := \frac{du}{dt}, \tag{1}$$

where $u_0 \in D(F)$ is arbitrary, $D(F)$ is the domain of $F$. The system is called dissipative if $F : H \to H$ is a monotone, closed and hemicontinuous operator in a Hilbert space $H$: $(F(u) - F(v), u - v) \geq 0, \quad u, v \in D(F)$. Here $(u, v)$ is the inner product in $H$, $D(F)$ is assumed to be a linear set, dense in $H$, and $F$ is maximal monotone, $R(I + F) = H$, where $R(F)$ is the range of $F$. Under these assumptions problem (1) has a unique solution $u(t) := S(t)u_0$, defined for all $t \geq 0$, and the operator family $S(t)$ is a semigroup. A set $\mathcal{A}$ is called a global attractor for problem (1) if for any $u_0 \in H$ $\lim_{t\to\infty} d(u(t), \mathcal{A}) = 0$, where $d(u, \mathcal{A})$ is the distance between $u$ and the set $\mathcal{A}$, and $u(t)$ solves (1). A set $M$ is called an invariant set for problem (1) if $u_0 \in M$ implies $u(t) \in M$ for all $t > 0$. If an invariant set $M$ is a manifold, it is called an invariant manifold for problem (1). Attractors and invariant manifolds for dissipative dynamical systems are studied in [?], [?], and [?]. In [?], Chapter 3, and in [?] a class of dissipative nonlinear systems is studied. This class consists of passive nonlinear networks.

Assume that $A = A^* \geq m > 0$ is a selfadjoint operator in $H$, denote by $\sigma(A)$ its spectrum, and by $E_s$ its resolution of the identity. If $\Delta_\delta = [m, m+\delta]$, the $E(\Delta_\delta) = E_{m+\delta} - E_{m-0}$ is an orthogonal projection operator in $H$ and $E(\Delta_\delta)H$ is an invariant subspace of $A$. By definition, $E_{s+0} = E_s$. If $\lambda$ is an eigenvalue of $A$, then $[E_\lambda - E_{\lambda-0}]H$ is the corresponding eigenspace. By $\sigma(A)$ we denote the spectrum of $A$, $m = \inf \sigma(A)$.

One is often interested in finding "slow" invariant manifolds for problem (1). If $F = A$ is a linear operator, then its invariant manifold is called "slow", if it corresponds to the smallest (lowest) eigenvalue of $A$. The corresponding eigenspace of $A$ is a linear invariant manifold for problem (1). A method for finding "slow" invariant manifolds for problem (1) is proposed in [?]. It consists of solving the problem

$$\dot{u} = -Au + b(u(t))u, \quad u(0) = u_0; \quad b(t) := b(u(t)) := \frac{(Au, u)}{(u, u)}, \tag{2}$$

and studying the limit $\lim_{t\to\infty} u(t) := v$. The existence of this limit will be established in this paper under suitable assumptions. One hopes that this limit, if it exists, is an eigenvector of $A$, corresponding to the smallest eigenvalue $\Lambda$ of $A$. Our goal is to find sufficient conditions for the validity of such a conclusion.

In [?] no rigorous results have been established for the global existence of the solution to equation (2), for the existence of the limit $\lim_{t\to\infty} u(t)$, and for finding slow manifolds in an infinite-dimensional Hilbert space. Our aim is to establish such results in this paper.

Let us formulate our results. Their proofs are outlined in Section 2.

**Theorem 1.** *Problem (2) has a unique global solution $u(t)$ which is given by the formula*

$$u(t) = \frac{e^{-tA}u_0}{(1 - 2\int_0^t h(\tau)d\tau)^{1/2}}, \quad h(t) := (Ae^{-2tA}u_0, u_0), \tag{3}$$

*and $||u(t)|| = ||u_0||$ for all $t > 0$.*

**Remark 1.** The last statement of Theorem 1 allows one to assume without loss of generality that $||u_0|| = 1$. Everywhere below we make this assumption. The closed form solution of the nonlinear evolution problem (2) is quite useful: among other things, it yields existence and uniqueness of the global solution to problem (2) in an infinite-dimensional Hilbert space.

**Theorem 2.** *Assume that $A$ has a discrete spectrum. Let $\Lambda = m = \inf \sigma(A)$ be the smallest eigenvalue of $A$. Assume that $m$ is an isolated point of spectrum, $P_m$ is the orthoprojector in $H$ onto the corresponding eigenspace, and $P_m u_0 \neq 0$.*

*Under these assumptions there exists strong limit* $\lim_{t\to\infty} u(t) = v$, $v \in H_m$, *and* $||v|| = ||u_0|| = 1$.

**Remark 2.** Suppose that the spectrum of $A$ in the interval $[m, m+\epsilon)$ is arbitrary, containing, possibly, countably many eigenvalues $\lambda_j$, which possibly form a set dense in the interval $[m, m+\epsilon)$, so that the spectrum of $A$ in $[m, m+\epsilon)$ contains a singular component. Here $\epsilon > 0$ is a small fixed number. Assume that $m$ is an eigenvalue of $A$, possibly of infinite multiplicity, $H_m$ is the corresponding eigenspace, $P_m$ is the orthogonal projector onto $H_m$, and $P_m u_0 \neq 0$. Let $H_1 := H_m$ and $H_2 := H_1^{\perp}$. Then the conclusion of Theorem 2 remains valid.

The idea of the proof is the same as in the proof of Theorem 2. We leave the details of the proof to the reader.

**Theorem 3.** *If the spectrum* $\sigma(A)$ *is absolutely continuous on the interval* $[m, m+\delta] := \Delta_\delta$, $\delta > 0$, *and* $E(\Delta_\delta)u_0 \neq 0$, *then there does not exist strong limit* $\lim_{t\to\infty} u(t) = v$, $v \in H$.

**Theorem 4.** *If* $E(\Delta_\delta)u_0 \neq 0$, $m = \inf \sigma(A)$ *is an eigenvalue of* $A$, *and* $m$ *is an isolated eigenvalue embedded into absolutely continuous spectrum of* $A$, *then there exists strong limit* $\lim_{t\to\infty} u(t) = v$, $v \in H_m$, *and* $||v|| = ||u_0|| = 1$.

## 2. **Proofs**

*Proof of Theorem 1.* If a solution to (2) exists, then $||u(t)|| = ||u_0||$. Indeed, multiply (2) by $u(t)$ and get $\frac{d||u(t)||^2}{dt} = 0$. This implies the desired conclusion. Therefore, without loss of generality we will assume below that $||u(t)|| = ||u_0|| = 1$.

Denote $z(t) := \int_0^t (Au(\tau), u(\tau))d\tau$, so $\dot{z} = (Au(t), u(t))$. From (2) one gets

$$u(t) = e^{z(t)}e^{-tA}u_0, \qquad z(t) := \int_0^t (Au(\tau), u(\tau))d\tau. \tag{4}$$

Apply the operator $A$ to (4) and then multiply by $u$ to get

$$\dot{z} = e^{2z(t)}h(t), \qquad h(t) = (Ae^{-2tA}u_0, u_0). \tag{5}$$

From (5) one gets $e^z = (1 - 2\int_0^t h(\tau)d\tau)^{-\frac{1}{2}}$. This and (4) yield (3). Theorem 1 is proved. □

**Corollary 1.** Formula (3) for $||u_0|| = 1$ can be rewritten as

$$u(t) = \frac{\int_m^\infty e^{-st}dE_s u_0}{(\int_m^\infty e^{-2st}d\rho)^{\frac{1}{2}}}, \quad d\rho := d(E_s u_0, u_0). \tag{6}$$

To derive (6) one uses formula (3) and the spectral theorem, in particular, the relation $\int_m^\infty d\rho = ||u_0||^2 = 1$.

*Proof of Theorem 2.* Assume for simplicity that $\Lambda = m$ is an isolated eigenvalue of $A$ and $H_m$ is the corresponding eigenspace. Decompose $H$ into an orthogonal sum of two subspaces, invariant with respect to $A$, one of which is $H_m$. Then the solution to (2) can be written as $u = u_1 + u_2$, where $u_1 \in H_m$ and $u_2$ is orthogonal to $u_1$. One has $||u(t)||^2 = ||u_1(t)||^2 + ||u_2(t)||^2 = 1$, and $||u_2(t)|| = o(||u_1(t)||)$ as $t \to \infty$. Therefore, $\lim_{t\to\infty} ||u_1(t)|| = \lim_{t\to\infty} ||u(t)|| = 1$. If $\dim H_m = 1$, and the corresponding eigenvector is $\phi$, $||\phi|| = 1$, then there exists strong limit $\lim_{t\to\infty} u(t) = \phi$. In the general case, equation (**??**) is equivalent to the system of equations:

$$\dot{u}_1 = -Au_1 + bu_1, \quad \dot{u}_2 = -Au_2 + bu_2, \quad b := b(u(t)), \tag{7}$$

$$u_1(0) = u_{01} = E(\Delta_\delta)u_0, \qquad u_2 \perp u_1. \tag{8}$$

One has

$$u_j(t) = e^{z(t)-tA}u_{0j}, \quad j = 1, 2, \tag{9}$$

where $z(t)$ is defined in (4). Therefore, $\lim_{t\to\infty} \frac{\|u_2(t)\|^2}{\|u_1(t)\|^2} = 0$, and

$$\lim_{t\to\infty} e^{-mt+z(t)} = \frac{\|u_0\|}{\|u_{01}\|}. \tag{10}$$

Consequently, there exists the strong limit:

$$v := \lim_{t\to\infty} u(t) = u_{01}\frac{\|u_0\|}{\|u_{01}\|}, \quad v \in H_m, \tag{11}$$

and $||v|| = ||u_0||$. Theorem 2 is proved. □

*Proof of Theorem 3.* Suppose to the contrary that there exists strong limit $\lim_{t\to\infty} u(t) = v$. Clearly, $v \neq 0$, because $||u(t)|| = 1$ for all $t \geq 0$, so $||v|| = 1$. Without loss of generality assume that $u(t) \in E(\Delta_\delta)H$ and $A$ is bounded, because the part $A_1$ of $A$ in the invariant subspace $E(\Delta_\delta)H$ is bounded. Then the limit

$$\lim_{t\to\infty} (Au(t), u(t)) = (Av, v) := \lambda$$

exists, and

$$\lim_{t\to\infty} Au(t) = Av.$$

Therefore $\lim_{t\to\infty} \dot{u}(t) := w$ exists, and

$$w = -Av + \lambda v.$$

*We claim that* $w = 0$.

Indeed, if $w \neq 0$, then

$$u(t+h) - u(t) = \int_t^{t+h} \dot{u}d\tau = wh[1 + o(1)], \quad t \to \infty.$$

This contradicts the Cauchy criterion for the existence of the limit $\lim_{t\to\infty} u(t) = v$, unless $w = 0$. Thus, $w = 0$ and $Av = \lambda v$, $||v|| = 1$. Therefore, $\lambda \in \Delta_\delta$ is an eigenvalue of $A$, contrary to our assumption. Theorem 3 is proved. □

**Remark 3.** If the interval $\Delta = [m, m + \delta)$ consists of the points of absolutely continuous spectrum of $A$, and the projection of the initial data $u_0$ onto the invariant subspace $E(\Delta)H$ of $A$ is non-zero, then there does not exist strong limit of the solution $u(t)$ to problem (2) as $t \to \infty$; the trajectory of the solution $u(t)$ does not stay in any fixed finite-dimensional subspace of $H$, and does not stay in any fixed compact subset of $H$. It stays on an infinite-dimensional sphere $||u(t)|| = ||u_0||$ in $H$. *In this sense the trajectory of the solution $u(t)$ is chaotic.*

*Proof of Theorem 4.* This proof is briefly sketched. If the spectrum of $A$ in the interval $(m, m + \delta)$ is absolutely continuous, then the solution $u(t)$ to (2) can be written as $u = u_m + u'$, where $u_m \in H_m$ and $u'$ is orthogonal to $H_m$, and $||u'(t)|| = o(||u_m(t)||)$ as $t \to \infty$. If the spectrum of $A$ is absolutely continuous on $(m, m + \delta)$, then the function $d\rho = \mu(s)ds$, where $\mu \in L^1(\Delta_\delta)$, and the following estimate holds: $\int_m^{m+\delta} e^{-ts}\mu(s)ds = o(e^{-mt})$ as $t \to \infty$. On the other hand, the part of the solution, which lies in $H_m$ is $e^{-mt}\psi$, where $\psi \in H_m$, $\psi \neq 0$. This part is the main part of the solution as $t \to \infty$. Dividing this solution by the normalizing factor as in formula (6), one gets in the limit $t \to \infty$ a normalized element $v$ of $H_m$. The outline of the proof of Theorem 4 is completed. $\square$

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