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DSM of Newton type

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Abstract

This paper is a review of the authors' results on the DSM (Dynamical Systems Method) for solving operator equation (*) $F(u) = f$. It is assumed that (*) is solvable. The novel feature of the results is the minimal assumption on the smoothness of F. It is assumed that F is continuously Fréchet differentiable, but no smoothness assumptions on $F'(u)$ are imposed. The DSM for solving equation (*) is developed. Under weak assumptions global existence of the solution $u(t)$ is proved, the existence of $u(\infty)$ is established, and the relation $F(u(\infty)) = f$ is obtained. The DSM is developed for a stable solution of equation (*) when noisy data f_δ are given, $\|f - f_\delta\| \le \delta$.

MSC: 47J05; 47J06; 47J35.

Dynamical systems method (DSM); nonlinear operator equations; monotone operators; discrepancy principle.

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DSM of Newton type for solving operator equations $F(u) = f$ with minimal smoothness assumptions on F.

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1 Introduction

In this paper we study the DSM (Dynamical Systems Method) for solving operator equation

$$F(u) = f, \tag{1}$$

where $F : H \to H$ is a Fréchet differentiable operator and H is a Hilbert space. In Section 6 equation (1) is studied in Banach spaces. We assume that equation (1) has a solution, possibly nonunique.

The DSM for solving an operator equation $F(u) = f$ consists of finding a nonlinear operator $\Phi(t, u)$ such that the Cauchy problem

$$\dot{u} = \Phi(t, u), \qquad u(0) = u_0, \tag{2}$$

has a unique global solution $u = u(t; u_0)$, there exists $u(\infty) = \lim_{t\to\infty} u(t; u_0)$, and $F(u(\infty)) = f$:

$$\exists! u(t), \quad \forall t \geq 0; \quad \exists u(\infty); \quad F(u(\infty)) = f. \tag{3}$$

The problem is to find a Φ such that properties (3) hold. Various choices of Φ for which these properties hold are proposed in [25], where the DSM is justified for wide classes of operator equations, in particular, for wide classes of nonlinear ill-posed equations (i.e., equations $F(u) = f$ for which the linear operator $F'(u)$ is not boundedly invertible). By $F'(u)$ we denote the Fréchet derivative of the nonlinear map F at the element u.

Several versions of the DSM has been studied for solving (1). In [13] various versions of the DSM with stopping rules of Discrepancy Principle-type are proposed and justified.

This paper is a review of the recent results, published by the authors. It consists of six sections. In Section 2 the Newton-type DSM is discussed from various points of view. A version of an abstract inverse function theorem is used in a proof of the existence of the global solution to the Cauchy problem used in the Newton-type DSM. A novel feature of the proof is the lack of the assumption that $F'(u)$ is Lipschitz-continuous. A novel version of the convergence theorem for the classical Newton method is proved. In this theorem (Theorem 10) there is no assumption about smoothness of $F'(u)$, only continuity of $F'(u)$ is assumed. If $F'(u)$ is Hölder continuous, then a faster rate of convergence is established. In Section 3 a justification of the Newton-type DSM is given for the maps F which are global homeomorphisms. In Section 4 convergence of the DSM is proved for monotone operators, the case of noisy data is discussed and a discrepancy principle is justified for stable solution by the DSM of ill-posed problems with monotone operators. The operator $F'(u)$ is assumed continuous, and no Lipschitz continuity is assumed in this Section and in the paper. In Section 5 a version of the Newton-type DSM is studied under the assumption that $F'(u)$ is a smoothing injective operator, so that the operator $[F'(u)]^{-1}$ acts as a differential operator of finite order, so that the "loss of derivatives" phenomenon occurs (see [34]). In Section 6 the Newton-type DSM is studied in Banach spaces (see [37]). The cases when only continuity of $F'(u)$ is assumed and when $F'(u)$ is Hölder-continuous are considered and convergence of the DSM is proved.

2 DSM of Newton-type

Let us assume that equation (1) has a solution y and the Fréchet derivative $F'(y)$ exists and is boundedly invertible:

$$\|[F'(y)]^{-1}\| \leq m, \qquad m = const > 0, \qquad F(y) = f. \tag{4}$$

This assumption is relaxed in Remark 6, (see also Theorem 7), where it is assumed that the operator $[F'(y)]^{-1}$ is unbounded and causes loss of smoothness: it acts as a differential operator.

Let us also assume that $F'(u)$ exists in the ball $B(y, R) := \{u : \|u-y\| \leq R\}$, depends continuously on u, and $\omega(r), r \geq 0$, is its modulus of continuity in the ball $B(y, R)$:

$$\sup_{u,v \in B(y,R),\, \|u-v\| \leq r} \|F'(u) - F'(v)\| = \omega(r). \tag{5}$$

The function $\omega(r) \geq 0$ is assumed to be continuous on the interval $[0, 2R]$, strictly increasing, and $\omega(0) = 0$.

A widely used method for solving equation (1) is the Newton method:

$$u_{n+1} = u_n - [F'(u_n)]^{-1}[F(u_n) - f], \qquad u_0 = z, \tag{6}$$

where z is an initial approximation. Sufficient condition for the convergence of the iterative scheme (6) to the solution y of equation (1) are proposed in [3], [20], [21], [23], [25], and references therein. These conditions in most cases require a Lipschitz condition for $F'(u)$, a sufficient closeness of the initial approximation u_0 to the solution y, and other conditions (see, for example, [3], p.157).

Let us consider, instead of (1), the following equation

$$F(u) = h, \tag{7}$$

where $h \in H$ is "sufficiently close" to f. The meaning of "sufficiently close" is made precise in Assumption A1) in Section 2.1. Consider the following continuous analog of the Newton method:

$$\dot{u}(t) = -[F'(u(t))]^{-1}(F(u(t)) - h), \qquad u(0) = u_0; \qquad \dot{u}(t) = \frac{du(t)}{dt}. \tag{8}$$

The question of general interest is:

Under what assumptions on F, h and u_0, can one establish conclusions (3)*, that is, the global existence and uniqueness of the solution to problem* (8)*, the existence of $u(\infty)$, and the relation $F(u(\infty)) = h$?*

The usual condition, sufficient for the local existence and uniqueness of the solution to the Cauchy problem (8) is the local Lipschitz condition on the right-hand side of (8). Such condition can be satisfied, in general, only if $F'(u)$ satisfies a Lipschitz condition.

In [43] a *novel approach* was developed to a study of equation (8). The approach does not require a Lipschitz condition for $F'(u)$, and it leads to a justification of the conclusions (3) for the solution to problem (8) under natural assumptions on h and u_0.

Apparently for the first time a proof of convergence of the continuous analog (8) *of the Newton method* (6) *is given without any smoothness assumptions on $F'(u)$, only the local continuity of $F'(u)$ is assumed, see* (5).

This approach uses the special structure of equation (8), which corresponds to the Newton-type methods. The Newton-type methods are widely used in theoretical, numerical and applied research, and by this reason our results are of general interest.

Our results demonstrate the universality of the Newton-type methods in the following sense: we prove that any operator equation (7) can be solved by the DSM Newton method (8), provided that conditions (4)-(5) hold, the initial approximation u_0 is sufficiently close to y, where y is the solution of equation (1), and the right-hand side h in (7) is sufficiently close to f.

A generalization of the classical results on the Newton method is given in Theorem 10, where the usual assumption about Lipschitz condition for $F'(u)$ is replaced by the continuity of $F'(u)$ or by a Hölder condition for $F'(u)$. Under these conditions the rate of convergence of the Newton method depends on the properties of the modulus of continuity of $F'(u)$ with respect to u.

Precise formulations of the results are given in theorems 1, 3, 5, 7, 8 and 10.

The basic tool in this Section is a new version of the inverse function theorem. The novelty of this version is in a specification of the region in which the inverse function exists. This is done in terms of the modulus of continuity of the operator $F'(u)$ in the ball $B(y, R)$.

In Section 2.1 we formulate and prove this version of the inverse function theorem. The result is stated as Theorem 1.

In Section 2.2 we justify the DSM for equation (8). The result is stated in Theorem 3. Moreover, we generalize the result to the case when assumption (4) is not valid, and the operator $[F'(u)]^{-1}$ is unbounded, acting similar to a differential operator and causing the "loss of derivatives". The result is stated in Theorem 7.

In Section 2.3 we prove convergence of the usual Newton method (6). The result is stated in Theorem 8.

Results in this Section, except for Theorem 10, which is new, are taken from [43] and [34].

2.1 Inverse function theorem

Consider equation (7).

Let us make the following

Assumptions A):

1. *Equation* (1) *and estimates* (4), (5) *hold in* $B(y, R)$,
2. $h \in B(f, \rho), \qquad \rho = \frac{(1-q)R}{m}, \qquad q \in (0, 1)$,
3. $m\omega(R) = q, \qquad q \in (0, 1)$.

Assumption A3) defines R uniquely because $\omega(r)$ is assumed to be strictly increasing. We assume that equation $m\omega(R) = q$ has a solution. This assumption is always satisfied if $q \in (0, 1)$ is sufficiently small. The constant m is defined in (4).

Our first result, Theorem 1, says that under *Assumptions A)* equation (7) is uniquely solvable for any h in a sufficiently small neighborhood of f.

Theorem 1 *If Assumptions A) hold then equation* (7) *has a unique solution* u *for any* $h \in B(f, \rho)$, *and*

$$\|[F'(u)]^{-1}\| \le \frac{m}{1-q}, \qquad \forall u \in B(y, R). \tag{9}$$

Proof. Let us denote

$$Q := [F'(y)]^{-1}, \qquad \|Q\| \le m.$$

Then equation (7) is equivalent to

$$u = T(u), \qquad T(u) := u - Q(F(u) - h). \tag{10}$$

Let us check that T maps the ball $B(y, R)$ into itself:

$$TB(y, R) \subset B(y, R), \tag{11}$$

and that T is a contraction mapping in this ball:

$$\|T(u) - T(v)\| \le q\|u - v\|, \qquad \forall u, v \in B(y, R), \tag{12}$$

where $q \in (0,1)$ is defined in *Assumptions A)*.

If (10) and (11) are verified, then the contraction mapping principle guarantees existence and uniqueness of the solution to equation (10) in $B(y,R)$, where R is defined in *Assumptions A3)*.

Let us check the inclusion (11). One has

$$J_1 := \|u - y - Q(F(u) - h)\| = \|u - y - Q[F(u) - F(y) + f - h]\|, \tag{13}$$

and

$$\begin{aligned} F(u) - F(y) &= \int_0^1 F'(y + s(u-y))ds(u-y) \\ &= F'(y)(u-y) + \int_0^1 [F'(y + s(u-y)) - F'(y)]ds(u-y). \end{aligned} \tag{14}$$

Note that

$$\|Q(f-h)\| \le m\rho,$$

and

$$\sup_{s\in[0,1]} \|F'(y + s(u-y)) - F'(y)\| \le \omega(R).$$

Therefore, for any $u \in B(y,R)$ one gets from (4), (12) and (13) the following estimate:

$$J_1 \le m\rho + m\omega(R)R \le (1-q)R + qR = R, \tag{15}$$

where the inequalities

$$\|f - h\| \le \rho, \quad \|u - y\| \le R, \tag{16}$$

and Assumptions A2) and A3) in *Assumptions A)* were used.

Let us establish inequality (12):

$$J_2 := \|T(u) - T(v)\| = \|u - v - Q(F(u) - F(v))\|, \tag{17}$$

$$F(u) - F(v) = F'(y)(u-v) + \int_0^1 [F'(v + s(u-v)) - F'(y)]ds(u-v). \tag{18}$$

Note that

$$\|v + s(u-v) - y\| = \|(1-s)(v-y) + s(u-y)\| \le (1-s)R + sR = R.$$

Thus, from (17) and (18) one gets

$$J_2 \le m\omega(R)\|u - v\| \le q\|u - v\|, \qquad \forall u, v \in B(y,R). \tag{19}$$

Therefore, both conditions (11) and (12) are verified. Consequently, the existence of the unique solution to (7) in $B(y,R)$ is proved.

Let us prove estimate (9). One has

$$\begin{aligned} [F'(u)]^{-1} &= [F'(y) + F'(u) - F'(y)]^{-1} \\ &= [I + (F'(y))^{-1}(F'(u) - F'(y))]^{-1}[F'(y)]^{-1}, \end{aligned} \tag{20}$$

and

$$\|(F'(y))^{-1}(F'(u)-F'(y))\| \le m\omega(R) \le q, \qquad u \in B(y,R). \tag{21}$$

It is well known that if a linear operator A satisfies the estimate $\|A\| \le q$, where $q \in (0,1)$, then the inverse operator $(I+A)^{-1}$ does exist, and $\|(I+A)^{-1}\| \le \frac{1}{1-q}$. Thus, the operator $[I+(F'(y))^{-1}(F'(u)-F'(y))]^{-1}$ exists and its norm can be estimated as follows:

$$\|[I+(F'(y))^{-1}(F'(u)-F'(y))]^{-1}\| \le \frac{1}{1-q}. \tag{22}$$

Consequently, (20) and (22) imply (9).

Theorem 1 is proved. □

Remark 2 *If $h = h(t) \in C^1([0,T])$, then the solution $u = u(t)$ of equation* (7) *is in $C^1([0,T])$ provided that Assumptions A) hold.*

2.2 Convergence of the DSM (8)

Consider the following equation

$$F(u) = h + v(t), \qquad t \ge 0, \tag{23}$$

where

$$u = u(t), \quad v(t) = e^{-t}v_0, \quad v_0 := F(u_0) - h, \quad r = \|v_0\|. \tag{24}$$

At $t = 0$ equation (23) has a unique solution u_0.

Let us make the following assumptions:

Assumptions B):

1. *Assumptions A) hold,*
2. $h \in B(f,\delta), \quad \delta + r \le \rho := \frac{(1-q)R}{m}$.

Theorem 3 *If Assumptions B) hold, then conclusions* (3)*, with f replaced by h, hold for the solution of problem* (8)*.*

Proof. The proof is divided into 3 parts.

Part 1. Proof of the global existence and uniqueness of the solution to problem (8).

One has

$$\|h + v(t) - f\| \le \|h - f\| + \|v_0 e^{-t}\| \le \delta + r \le \rho, \qquad \forall t \ge 0.$$

Thus, it follows from Theorem 1 that equation (23) has a unique solution

$$u = u(t) \in B(y,R)$$

defined on the interval $t \in [0,\infty)$, and $u(t) \in C^1([0,\infty))$.

Differentiation of (23) with respect to t yields

$$F'(u)\dot{u} = \dot{v} = -v = -(F(u(t)) - h). \tag{25}$$

Since $u(t) \in B(y, R)$, the operator $F'(u(t))$ is boundedly invertible, so equation (25) is equivalent to (8). The initial condition $u(0) = u_0$ is satisfied, as was mentioned below (24). Therefore, the existence of the unique global solution to (8) is proved.

Part 2. Proof of the existence of $u(\infty)$.

From (23), (24), (9), and (8) it follows that

$$\|\dot{u}\| \leq \frac{mr}{1-q} e^{-t}, \qquad q \in (0, 1). \tag{26}$$

This and the Cauchy criterion for the existence of the limit $u(\infty)$ imply that $u(\infty)$ exists.

Integrating (26), one gets

$$\|u(t) - u_0\| \leq \frac{mr}{1-q}, \tag{27}$$

and

$$\|u(\infty) - u(t)\| \leq \frac{mr}{1-q} e^{-t}. \tag{28}$$

Part 3. Proof of the relation $F(u(\infty)) = h$.

Let us now prove that

$$F(u(\infty)) = h. \tag{29}$$

Relation (29) follows from (23) and (24) as $t \to \infty$, because $v(\infty) = 0$, $u(t) \in B(y, R)$, and F is continuous in $B(y, R)$.

Theorem 3 is proved. □

Remark 4 Let us explain why there is no assumption on the location of u_0 in Theorem 3. The reason is simple: in the proof of Theorem 3 it was established that $u(t) \in B(y, R)$ for all $t \geq 0$. Therefore, it follows that the assumptions of Theorem 3 imply the location of u_0, namely $u_0 = u(0) \in B(y, R)$.

From the proof of Theorem 3 we obtain Theorem 5.

Theorem 5 *Assume that F is a global homeomorphism, that $\|[F'(u)]^{-1}\| \leq m(u)$, where $m(u) > 0$ is a constant which depends on u, and that $F'(u)$ is continuous with respect to u. Then problem (8) has a unique global solution for any h and any u_0, there exists $u(\infty)$, and $F(u(\infty)) = h$.*

Proof. If F is a global homeomorphism, then equation (23) is uniquely solvable for any $v(t)$. Differentiation of this equation with respect to t yields equation (25), and this equation is equivalent to (8) because of the bounded invertibility of $F'(u)$ at any u. The existence of $u(\infty)$ and the equality $F(u(\infty)) = h$ follow from the relation $\lim_{t\to\infty}(h + v(t)) = h$ and from the assumption that F is a global homeomorphism. Theorem 5 is proved. □

A practically important example of equations (7) with a global homeomorphism F is the equation $F(u) := G(u) + bu = h$, where $b = const > 0$ and G is a monotone Fréchet differentiable operator. One has $F'(u) = G'(u) + bI$ and $\|[G'(u) + bI]^{-1}\| \leq \frac{1}{b}$, because the monotonicity of G implies $G'(u) \geq 0$. Recall that if a linear operator $A \geq 0$, and $b = const > 0$, then $\|(A + bI)^{-1}\| \leq \frac{1}{b}$.

It is known that such F are global homeomorphisms (see, e.g., [3]). For such F equation (7) can be solved for any h by the DSM Newton-type method (8) with any initial approximation u_0. In this sense convergence of the DSM Newton method (8) is *global* for the F, satisfying the above assumptions.

Remark 6 Our arguments can be generalized to the case when $F'(u)$ *is unbounded*. For example, let H_a be a Hilbert scale of spaces, $H_a \subset H_b$ if $a > b \geq 0$. A typical example is the case of Sobolev spaces H_a, $H_0 := H = L^2(D)$. Let us assume that the following assumptions hold:

Assumptions C:

(1) $F : H_a \to H_{a+b}$, $b > 0$;

the Fréchet derivative $A(u) := F'(u) : H_a \to H_{a+b}$ *exists and is continuous with respect to* u *in the following sense:*

$$||A(u) - A(v)||_{a+b} \leq \omega(||u - v||_a), \qquad \forall u, v \in B(y, R),$$

where y *solves the equation* $F(y) = f$, *the function* $\omega(r)$ *is continuous and strictly increasing for* $r \in (0, R_1)$, $\omega(r) \geq 0$, $\omega(0) = 0$, $R_1 > 0$ *is a sufficiently large constant, so that equation*

$$m\omega(R) = q, \qquad q \in (0, 1),$$

has a unique solution $R < R_1$, *and* $m > 0$ *is a constant in the following inequality:*

$$||[A(y)]^{-1}g||_a \leq m||g||_{a+b}, \qquad \forall g \in H_{a+b},$$

(2) $h \in B(f, \rho)$, *where* $\rho = \frac{(1-q)R}{m}$.

Theorem 7 *If Assumptions C) hold, then for any* $h \in B(f, \rho)$ *there exists a unique* $u \in B(y, R)$ *such that* $F(u) = h$. *The operator* F^{-1} *is continuous on* $B(f, \rho)$.

Proof. The proof is similar to the proof of Theorem 1. □

This Theorem deals with the case when the operator $A^{-1}(u)$ causes loss of $b > 0$ derivatives, it acts similarly to a differentiation operator of order $b > 0$.

One can prove that conclusions (3) hold for (8) if *Assumptions C)* hold and $h \in B(f, \delta)$, where $\delta + r \leq \rho$, $r = ||F(u_0) - h||_{a+b}$, and ρ is the same as in *Assumptions C)*. The proof is similar to the proof of Theorem 3.

2.3 The Newton method

The goal in this Section is to give a novel result on the convergence of the classical Newton method. We drop the usual assumption that $F'(u)$ satisfies a Lipschitz condition, and assume that $F'(u)$ is continuous. We also consider the case when $F'(u)$ satisfies a Hölder condition. The rate of the convergence is estimated. The result is formulated in Theorem 10. If $F'(u)$ satisfies the Hölder condition

$$\|F'(u) - F'(v)\| \leq c\|u - v\|^p, \qquad 0 < p \leq 1,$$

then the continuity modulus of $F'(u)$ is $\omega(r) = cr^p$, and condition (39) in Theorem 10 is satisfied.

We start with a proof of the convergence of the Newton method (6) to the solution y of equation (1) without any additional assumptions on the smoothness of $F'(u)$. Only the continuity of $F'(u)$ with respect to $u \in B(y, R)$ is assumed. The notations are the same as in *Assumptions A)* in Section 2.1.

Theorem 8 *Assume that $F(y) = f$, conditions* (1)–(5) *and Assumptions A) hold, and*

$$m\omega(R) = q \in (0, \frac{1}{2}), \qquad q_1\|z - y\| \leq R, \qquad q_1 := \frac{q}{1-q}. \tag{30}$$

Then process (6) *converges to y.*

Proof. Let $a_n := \|u_n - y\|$. From our assumptions one can derive that the sequence $(a_n)_{n=0}^{n}$ is decreasing at the rate not slower than that of a geometric sequence with ratio $r \in (0, 1)$. This implies the conclusion of Theorem 8. $\square$

Remark 9 In general, the global convergence of the Newton method (6) does not hold under the assumptions of Theorem 5. That is, there exists F satisfying assumptions of Theorem 5 so that the Newton method (6) does not converge for some f and z.

The following result is an extension of the standard result (see, e.g., Theorem 15.6 in [3, p. 157]) on the convergence of the Newton method. In [3] it is assumed that $F'(u)$ satisfies a Lipschitz condition. In Theorem 10 it is assumed only that $F'(u)$ is continuous, which is a much weaker assumption. Of course, the rate of convergence depends on the behavior of the modulus of continuity of $F'(u)$ in a neighborhood of zero. One cannot prove the quadratic rate of convergence, characteristic for the Newton method if $F'(u)$ satisfies the Lipschitz condition. In Theorem 10 it is proved that if $F'(u)$ is continuous, then convergence is at the rate of a geometric series, and if $F'(u)$ satisfies a Hölder condition, then the rate of convergence is superlinear and depends on the Hölder exponent p.

Theorem 10 *Let F be a Fréchet differentiable operator in a Banach space X, and $F'(u)$ be continuous with respect to u. Assume that*

$$\|F'(u) - F'(v)\| \le \omega(\|u-v\|), \qquad \forall u, v \in B(u_0, r), \tag{31}$$

$$K := \sup_{0<\xi\le\alpha} \frac{\int_0^1 \omega(s\xi)ds}{\omega(\xi)} < 1, \tag{32}$$

$$\frac{\alpha}{1-K} < r, \qquad \beta\omega(r) < 1, \tag{33}$$

$$\beta\omega(\alpha) < \min\left(\frac{1}{2}, 1 - \sup_{0<\xi\le\alpha} \frac{\omega(K\xi)}{\omega(\xi)}, 1 - K^p\right), \tag{34}$$

where

$$\alpha := \|F'(u_0)^{-1}(F(u_0) - f)\|, \tag{35}$$

$$\beta := \|F'(u_0)^{-1}\|, \tag{36}$$

$\omega(s) \ge 0$ is strictly increasing on the segment $[0, r]$, and $\omega(0) = 0$. Then equation (1) has a unique solution y in $B(u_0, r)$. Let u_n be defined by (6). Then

$$\lim_{n\to\infty} u_n = y, \tag{37}$$

and

$$\|u_n - y\| \le \frac{K^n}{1-K}, \qquad n \ge 0. \tag{38}$$

If in addition

$$\frac{\omega(t)}{\omega(t')} \le \left(\frac{t}{t'}\right)^p, \qquad 0 < t \le t', \qquad 0 < p \le 1, \tag{39}$$

then

$$\|u_n - y\| \le \frac{\alpha q^{\frac{(p+1)^n - 1}{p}}}{1-\kappa}, \qquad n > 0, \tag{40}$$

where

$$\kappa := (1 - \beta\omega(\alpha))^{\frac{1}{p}} < 1, \qquad q := \left(\frac{K^p}{1-\beta\omega(\alpha)}\right)^{\frac{1}{p}} \frac{\beta\omega(\alpha)}{1-\beta\omega(\alpha)} < 1. \tag{41}$$

Proof. The proof consists of 3 parts.

Part 1. Proof of the uniqueness of z.

Assume that y and $\bar{y}$ are two solutions to (1) in $B(u_0, r)$. Then one gets, using (31), the following inequalities

$$\begin{aligned} \|y - \bar{y}\| &\le \beta\|F(y) - F(\bar{y}) - F'(u_0)(y - \bar{y})\| \\ &\le \beta\|y - \bar{y}\| \int_0^1 \|F'(\bar{y} + t(y - \bar{y})) - F'(u_0)\|dt \\ &\le \|y - \bar{y}\|\beta\omega(r). \end{aligned} \tag{42}$$

This and the second inequality in (33) imply that $y = \bar{y}$.

Part 2. Proof of the relations (37) *and* (38).

Let

$$\alpha_n := \|u_{n+1} - u_n\|, \qquad \beta_n := \|F'(u_n)^{-1}\|, \qquad \gamma_n := \beta_n \omega(\alpha_n), \qquad n \ge 1, \tag{43}$$

and

$$\alpha_0 := \alpha, \qquad \beta_0 := \beta, \qquad \gamma_0 := \beta\omega(\alpha). \tag{44}$$

From (6) and (31) one gets

$$\begin{aligned} \alpha_n &\le \beta_n \|F(u_n) - f - \big[F(u_{n-1}) - f + F'(u_{n-1})(u_n - u_{n-1})\big]\| \\ &\le \beta_n \alpha_{n-1} \int_0^1 \|F'\big(u_{n-1} + t(u_n - u_{n-1})\big) - F'(u_{n-1})\| dt \\ &\le \beta_n \alpha_{n-1} \omega(\alpha_{n-1}) K(\alpha_{n-1}), \qquad n \ge 1, \end{aligned} \tag{45}$$

where

$$K(\xi) := \frac{\int_0^1 \omega(t\xi)dt}{\omega(\xi)}, \qquad \xi > 0. \tag{46}$$

From (6) one gets

$$F'(u_n) = F'(u_{n-1})\Big[I + F'(u_{n-1})^{-1}\big(F'(u_n) - F'(u_{n-1})\big)\Big], \qquad n \ge 1. \tag{47}$$

This and (43) imply

$$\beta_n \le \beta_{n-1}(1 - \beta_{n-1}\omega(\alpha_{n-1}))^{-1} = \beta_{n-1}(1 - \gamma_{n-1})^{-1}, \qquad \forall n \ge 1. \tag{48}$$

It follows from (43) and (48) that

$$\begin{aligned} \gamma_n = \beta_n \omega(\alpha_n) &\le \beta_{n-1}(1 - \gamma_{n-1})^{-1}\omega(\alpha_n) \\ &\le \frac{\gamma_{n-1}}{1 - \gamma_{n-1}} \frac{\omega(\alpha_n)}{\omega(\alpha_{n-1})}, \qquad n \ge 1. \end{aligned} \tag{49}$$

Inequalities (45) and (48) imply

$$\alpha_n \le \alpha_{n-1} \frac{\gamma_{n-1}}{1 - \gamma_{n-1}} K(\alpha_{n-1}), \qquad \forall n \ge 1. \tag{50}$$

From (34)–(32) one gets

$$\gamma_0 = \beta\omega(\alpha) < \frac{1}{2}, \qquad \frac{\gamma_0}{1 - \gamma_0} < 1, \qquad K(\xi) \le K < 1, \qquad \forall \xi \in (0, \alpha]. \tag{51}$$

It follows from (34) that

$$\frac{K^p}{1 - \beta\omega(\alpha)} < 1, \qquad \frac{\omega(K\xi)}{\omega(\xi)} < 1 - \gamma_0, \qquad \forall \xi \in (0, \alpha]. \tag{52}$$

From the first inequality in (52) and the second inequality in (51) one obtains

$$q := \left(\frac{K^p}{1-\beta\omega(\alpha)}\right)^{\frac{1}{p}} \frac{\beta\omega(\alpha)}{1-\beta\omega(\alpha)} < 1. \tag{53}$$

Thus, inequalities (41) hold.

Let us prove by induction that

$$\alpha_n \le \alpha_{n-1}K < \alpha_{n-1}, \qquad \gamma_n < \gamma_{n-1} \le \gamma_0, \tag{54}$$

for all $n \ge 1$

The first inequality in (54) for $n = 1$ follows from (50) and (51). The second inequality in (54) for $n = 1$ follows from (49), the second inequality in (52) and the first inequality in (54) for $n = 1$. Thus, (54) holds for $n = 1$. Assume that (54) holds for $n \ge 1$. From (50), (51), and the induction hypothesis, one gets

$$\alpha_{n+1} \le \alpha_n \frac{\gamma_n}{1-\gamma_n} K(\alpha_n) < \alpha_n \frac{\gamma_0}{1-\gamma_0} K \le \alpha_n K. \tag{55}$$

From (49), the induction hypothesis, and (55) one obtains

$$\gamma_{n+1} \le \frac{\gamma_n}{1-\gamma_n} \frac{\omega(\alpha_{n+1})}{\omega(\alpha_n)} \le \gamma_n \frac{1}{1-\gamma_0} \frac{\omega(K\alpha_n)}{\omega(\alpha_n)} < \gamma_n. \tag{56}$$

Here, we have used the second inequality in (52). From (55)–(56) one concludes that (54) holds for all $n \ge 1$.

It follows from the first inequality in (54) that

$$\alpha_n \le K^n \alpha_0 = K^n \alpha, \qquad n \ge 0. \tag{57}$$

This implies

$$\|u_{n+m} - u_n\| \le \sum_{i=n}^{n+m-1} \|u_{i+1} - u_i\| \le \sum_{i=n}^{n+m-1} \alpha K^i \le \alpha \frac{K^n}{1-K}, \qquad \forall m, n \ge 0. \tag{58}$$

This and the Cauchy criterion for convergence imply (37). Letting $m \to \infty$ in (58), one gets (38). It follows from inequality (38) with $n = 0$ that

$$\|y - u_0\| \le \alpha \frac{1}{1-K} < r, \tag{59}$$

where (33) was used. Therefore, $z \in B(u_0, r)$. Thus, (37) and (38) are proved.

Part 3. Proof of the estimate (40).

From (39) and (49)–(50), one obtains

$$\gamma_n \le \frac{\gamma_{n-1}}{1-\gamma_{n-1}} \frac{\omega(\alpha_n)}{\omega(\alpha_{n-1})} \le \frac{\gamma_{n-1}}{1-\gamma_{n-1}} \left(\frac{\gamma_{n-1}K}{1-\gamma_{n-1}}\right)^p, \qquad \forall n \ge 1. \tag{60}$$

This and (51)–(54) imply

$$\frac{\gamma_n K}{1-\gamma_n} \le \frac{\gamma_n K}{1-\gamma_0} \le \frac{1}{1-\gamma_0}\left(\frac{\gamma_{n-1}K}{1-\gamma_{n-1}}\right)^{p+1}, \qquad \forall n \ge 1. \tag{61}$$

From (61) one gets

$$\frac{\gamma_n K}{1-\gamma_n} \le \left(\frac{1}{1-\gamma_0}\right)^{\frac{(p+1)^n-1}{p}} \left(\frac{\gamma_0 K}{1-\gamma_0}\right)^{(p+1)^n} = \kappa q^{(p+1)^n}, \tag{62}$$

where

$$\kappa := (1-\gamma_0)^{\frac{1}{p}}, \qquad q := \left(\frac{1}{1-\gamma_0}\right)^{\frac{1}{p}} \frac{\gamma_0 K}{1-\gamma_0}. \tag{63}$$

From (50) and (62) one obtains, by induction, the following inequality

$$\alpha_n \le \kappa^n q^{\frac{(p+1)^n-1}{p}} \alpha, \qquad n \ge 1. \tag{64}$$

Therefore

$$\|u_n - y\| \le \sum_{i=n}^{\infty} \alpha_i \le q^{\frac{(p+1)^n-1}{p}} \alpha \sum_{i=n}^{\infty} \kappa^i \le \frac{\alpha q^{\frac{(p+1)^n-1}{p}} \kappa^n}{1-\kappa}, \qquad n > 0. \tag{65}$$

Thus, (40) holds and Theorem 10 is proved. □

3 A justification of the Dynamical Systems Method (DSM) for global homeomorphisms

We assume in this Section that F is a global homeomorphism.

For instance, F may be a hemicontinuous monotone operator operator such that a coercivity condition is satisfied:

$$\lim_{||u||\to\infty} \frac{\langle F(u), u\rangle}{||u||} = \infty, \tag{66}$$

where $\langle .,. \rangle$ denotes the inner product in H. We assume that $F \in C^1_{loc}$, i.e., the Fréchet derivative $F'(u)$ exists for every u and depends continuously on u. Furthermore, we assume that

$$||[F'(u)]^{-1}|| \le m(u), \tag{67}$$

where $m(u)$ is a constant depending on u. This assumption implies that F is a local homeomorphism. If $m(u) < m$, where $m > 0$ is a constant independent of u, then it was proved in [36] that F is a global homeomorphism.

In Remark 13, at the end of this Section, the following condition is mentioned:

$$\|F(u)\| < c \Rightarrow \|u\| < c_1, \qquad c, c_1 = const > 0, \tag{68}$$

which means that the preimages of bounded sets under the map F are bounded sets. This condition does not hold for the operator $F(u) := e^u$, $u \in \mathbb{R}$, $H = \mathbb{R}$, and that is why this monotone operator F is not surjective: equation $e^u = 0$ does not have a solution in H.

By $c > 0$ we denote various constants.

Our main result, Theorem 11, says that if $F \in C^1_{loc}$ is a global homeomorphism and condition (67) holds, then the DSM Newton-type method (69) converges globally, that is, it converges for any initial approximation $u_0 \in H$ and any right-hand side $f \in H$. One of the novel features of our result is the absence of any smoothness assumptions on $F'(u)$: only the continuity of $F'(u)$ with respect to u is assumed. In the earlier work (see [25] and references therein) it was often assumed that $F'(u)$ is Lipschitz continuous, or, at least, Hölder-continuous.

Results in this Section are taken from [42].

Let us formulate the result:

Theorem 11 *If $F \in C^1_{loc}$ is a global homeomorphism and condition* (67) *holds, then the problem*

$$\dot{u} = -[F'(u)]^{-1}(F(u) - f), \quad u(0) = u_0; \qquad \dot{u} = \frac{du}{dt}, \tag{69}$$

is solvable for any f and u_0 in H, the solution $u(t)$ exists for all $t \geq 0$, there exists the limit $u(\infty) = \lim_{t\to\infty} u(t)$, and $F(u(\infty)) = f$.

Proof. Denote

$$v := F(u(t)) - f. \tag{70}$$

If $u(t)$ solves (69), then

$$\dot{v} = F'(u(t))\dot{u} = -v.$$

Thus, problem (69) is reduced to the following problem:

$$\dot{v} = -v, \quad v(0) = F(u_0) - f. \tag{71}$$

Problem (71) obviously has a unique global solution:

$$v(t) = (F(u_0) - f)e^{-t}, \qquad \lim_{t\to\infty} v(t) := v(\infty) = 0. \tag{72}$$

Therefore, problem (69) has a unique global solution.

Indeed, consider an interval $[0, T]$, where $T > 0$ is arbitrarily large. The equation

$$F(u(t)) - f = v(t) \qquad 0 \leq t \leq T, \tag{73}$$

is uniquely solvable for $u(t)$ for any $v(t)$ because F is a global homeomorphism. The assumption (67), the continuity of $F'(u)$ with respect to u, and the inverse function theorem imply that the solution $u(t)$ to equation (73) is continuously differentiable with respect to t, because v and F are. Differentiating (73) and using (71) and (70), one gets

$$F'(u(t))\dot{u} = \dot{v} = -v = -(F(u(t)) - f). \tag{74}$$

Using assumption (67), one concludes from (74) that $u = u(t)$ solves (69) in the interval $t \in [0, T]$. Since $T > 0$ is arbitrary, $u = u(t)$ is a global solution to (69).

Since $\lim_{t\to\infty} v(t) := v(\infty)$ exists, and F is a global homeomorphism, one concludes that $\lim_{t\to\infty} u(t) := u(\infty)$ exists.

Since $v(\infty) = 0$, it follows that $F(u(\infty)) = f$.

Theorem 11 is proved. □

Remark 12 Theorem 11 implies that any equation (1) with F being a global homeomorphism and $F \in C^1_{loc}$, such that (67) holds, can be solved by the DSM method (69).

Remark 13 The equation $e^u = 0$, $u \in \mathbb{R}$, $H = \mathbb{R}$, does not have a solution, although $F(u) = e^u$ is monotone, i.e., $F'(u) \geq 0$, $F'(u) = e^u > 0$ is boundedly invertible for every $u \in \mathbb{R}$ and $\|[e^u]^{-1}\| = e^{-u} \leq m_u < \infty$ for every $u \in \mathbb{R}$. The assumption (68) is not satisfied in this example, and this is the reason for the unsolvability of the equation $e^x = 0$. Note that $e^x \leq c$ as $x \to -\infty$, so assumption (68) does not hold.

4 DSM of Newton-type for solving nonlinear equations with monotone operators

In this section we study a version of the Dynamical Systems Method (DSM) (see [25]) for solving the equation (1) where F is a nonlinear Fréchet differentiable monotone operator in a real Hilbert space H, and equation (1) is assumed solvable. Monotonicity means that

$$\langle F(u) - F(v), u - v\rangle \geq 0, \quad \forall u, v \in H. \tag{75}$$

It is known (see, e.g., [25]), that the set $\mathcal{N} := \{u : F(u) = f\}$ is closed and convex if F is monotone and continuous. A closed and convex set in a Hilbert space has a unique minimal-norm element. This element in $\mathcal{N}$ we denote y, $F(y) = f$. We assumed in earlier works that $F'(u)$ is locally Lipschitz. This assumption is considerably weakened in this work: we assume now only the continuity of $F'(u)$. Since F is monotone, one has $F'(u) \geq 0$, so $||[F'(u) + a(t)I]^{-1}|| \leq \frac{1}{a(t)}$ if $a(t) > 0$. The local and global existence and uniqueness of the solution to the Cauchy problem (109) (see below) was proved under these weak assumptions ([14], [43]).

The theory of monotone operators is presented in many books, e.g., in [3], [24], [47]. Many of the results of the theory of monotone operators, used in this Section, can be found in [25]. In [23] methods for solving well-posed nonlinear equations in a finite-dimensional space are discussed.

Methods for solving equation (1) with monotone operators are quite important in many applications. It is proved in [25] that solving any solvable linear operator equation $Au = f$ with a closed densely defined linear operator A can be reduced to solving equation (1) with a monotone operator. Equations (1) with monotone operators arise often when the physical system is dissipative. In the earlier papers and in monograph [25] it was assumed that F is locally twice Fréchet differentiable, and a nonlinear differential inequality ([25], p.97) was used in a study of the behavior of the solution to the DSM (109). The smoothness assumptions on F are weakened in this Section, the method of our proofs is new, and, as a result, the proofs are shorter and simpler than the earlier ones. The assumptions on the "regularizing function" $a(t)$ are also weakened.

In this section we propose and justify a stopping rule for solving ill-posed equation (1) based on a discrepancy principle (DP) for the DSM (109). The main result of this Section is Theorem 28 in which a DP is formulated, the existence of the stopping time t_δ is proved, and the convergence of the DSM (109) with the proposed DP is justified under some natural assumptions for a wide class of nonlinear equations with monotone operators.

Our result is novel because the convergence of the DSM is justified under less restrictive assumptions on F than in [25] and [12], where twice Fréchet differentiability was assumed and the discrepancy principle was not established for problem (109). Moreover, the rate of decay of the function $a(t)$ as $t \to \infty$ can be arbitrary in the power scale, while in [25] $a(t)$ was often assumed to satisfy the condition $\int_0^\infty a(t)dt = \infty$, which implies the rate of decay in the power scale not faster than $O(\frac{1}{t})$ as $t \to \infty$.

A few remarks about the history of the method (109) may be useful for the reader. Probably the first Section in which a continuous analog of the Newton's method was proposed for solving well-posed operator equation (1) was the paper [5]. Method (109) has been studied in the literature earlier by several authors, (see, e.g., [25] and references therein) usually under the assumption that $F'(u)$ satisfies a Lipschitz condition. Iterative versions of the method (109) were also studied, e.g., in [19], [25]. and in some of the cited papers by the authors, also under some smoothness assumptions on $F'(u)$. A discrepancy principle for linear ill-posed problems was proposed by V. A. Morozov (see [22]).

To the authors' knowledge it is for the first time a justification of the convergence of the method (109) is proved in this Section under the minimal assumption of the continuity of $F'(u)$. The method of the proof is novel and can be used can be used in a study of other problems. the justification of the discrepancy principle for stable solution of (1) with noisy data by the method (109) is also given under the minimal assumption of the continuity of $F'(u)$.

Results in this Section have been published in [14] and [15].

4.1 Existence of solution to the DSM and a justification for exact data

One of the versions of the DSM for solving nonlinear operator equation (1) with monotone continuously Fréchet differentiable operator F in a Hilbert space is based on a regularized DSM of Newton-type method, which consists of solving the following Cauchy problem

$$\dot{u} = -\big(F'(u) + a(t)I\big)^{-1}\big(F(u) + a(t)u - f\big), \quad u(0) = u_0. \tag{76}$$

Here $F : H \to H$ is a monotone continuously Fréchet differentiable operator in a Hilbert space H, $u_0, f \in H$ are arbitrary, and $a(t) > 0$ is a continuously differentiable function, defined for all $t \geq 0$ and monotonically decaying to zero as $t \to \infty$. This function is a regularizing function: if $F'(u)$ is not a boundedly invertible operator, and f is monotone, then $F'(u) \geq 0$ and the operator $F'(u) + a(t)I$ is boundedly invertible if $a(t) > 0$.

Throughout this section we denote by I the identity operator, by y the minimal-norm solution to (1), and by $c > 0$ various estimation constants.

If F is monotone and continuous, then the minimal-norm solution to (1) exists and is unique (see, e.g., [25]). Monotonicity of F is understood as follows

$$\langle F(u) - F(w), u - w \rangle \geq 0, \qquad \forall u, w \in H. \tag{77}$$

The DSM is a basis for developing efficient numerical methods for solving operator equations, both linear and nonlinear, especially when the problems are ill-posed, when $F'(u)$ is not a boundedly invertible operator (see [25], [7], [12]).

If one has a general evolution problem with a nonlinear operator in a Hilbert (or Banach) space

$$\dot{u} = B(u), \qquad u(0) = u_0, \tag{78}$$

then the local existence of the solution to this problem is usually established by assuming that $B(u)$ satisfies a Lipschitz condition, and the global existence is usually established by proving a uniform bound on the solution:

$$\sup_{t \geq 0} ||u(t)|| < c, \tag{79}$$

where $c > 0$ is a constant.

In (76) the operator

$$B(u) = -\big(F'(u) + a(t)I\big)^{-1}\big(F(u) + a(t)u - f\big)$$

is Lipschitz if one assumes that

$$\sup_{\{u:||u-u_0|| \leq R\}} ||F^{(j)}(u)|| \leq M_j(R), \quad 0 \leq j \leq 2.$$

This assumption was used in many cases in [25] and a bound (79) was established under suitable assumptions in [25].

There are many results (see, e.g., [3],[22] and references therein) concerning the properties and global existence of the solution to (78) if $-B(u)$ is a maximal monotone operator. However, even when F is a monotone operator, the operator $-B$ in the right-hand side of (76) is not monotone. *Therefore these known results are not applicable. Even the proof of local existence is an open problem if one makes only the following assumption*:

Assumption D):

F is monotone and $F'(u)$ is continuous with respect to u.

The main result of this Section is a proof that under Assumption D) problem (76) has a unique local solution $u(t)$, and that under assumptions (82) on $a(t)$

(see below) this local solution exists for all $t \geq 0$, so it is a global solution. These results are formulated in Theorems 14 and 15.

Moreover, if the equation $F(y) = f$ has a solution and y is its (unique) minimal-norm solution, and if $\lim_{t\to\infty} a(t) = 0$ and $\lim_{t\to\infty} \frac{\dot{a}(t)}{a(t)} = 0$, then there exists $u(\infty)$, and $u(\infty) = y$. This justifies the DSM for solving the equation $F(u) = 0$ with a monotone continuously Fréchet differentiable operator F, for the first time under the weak *Assumption D)*. The result is formulated in Theorem 17.

4.1.1 Local existence

Let us prove the following theorem

Theorem 14 *If Assumption D) holds, then problem* (76) *has a unique local solution.*

Proof. Let us prove the local existence of the solution to (76).

Let

$$\psi(t) = F(u) + a(t)u - f := \Psi(u,t) := \Psi(u). \tag{80}$$

If $a(t) > 0$ and F is monotone and hemicontinuous, then it is known (see, e.g., [3], p. 100) that the operator $F(u) + a(t)u$ is surjective. If $F'(u)$ is continuous, then, clearly, F is hemicontinuous. If F is monotone and $a(t) > 0$ then, clearly, the operator $F(u) + a(t)u$ is injective. Thus, Assumption D) implies that the operator $F(u)+a(t)u$ is injective and surjective, it is continuously Fréchet differentiable, as well as its inverse, so the map $u \mapsto F(u)+a(t)u$ is a diffeomorphism. Therefore equation (80) is uniquely solvable for u for any ψ at any $t \geq 0$, and the inverse map $\psi = \Psi(u)$ is a diffeomorphism. The inverse map $u = U(\psi)$, is continuously differentiable by the inverse function theorem since the operator $\Psi'_u = F'(u) + a(t)I$ is boundedly invertible if $a(t) > 0$, $\|(\psi'_u)^{-1}\| \leq \frac{1}{a(t)}$. Recall that $F'(u) \geq 0$, because F is monotone. If $a(t) \in C^1([0,\infty))$ then the solution $u = u(t)$ of equation (80) is continuously differentiable with respect to t (see [25], p. 260-261), and if $u = u(t)$ is continuously differentiable with respect to t, then so is $\psi(t) = \Psi(u(t))$. The differentiability of $u(t) = U(\psi(t))$ also follows from a consequence of the classical inverse function theorem (see, e.g., [3], Corollary 15.1, p. 150). Therefore, equation (76) can be written in an equivalent form as

$$\dot{\psi}(t) = \dot{a}(t)u(t) - \psi(t) := Q(t,\psi), \qquad \psi(0) := \psi(u_0), \tag{81}$$

where $u(t) = U(\psi(t))$ is continuously differentiable with respect to t, and $\psi(t)$ is continuously differentiable with respect to t. The map $Q(t,\psi)$ is Lipschitz with respect to ψ, and the local existence of the solution to problem (81) follows from the standard result (see, e.g., [25], p.247). Since the map $U(\psi)$ is continuously differentiable and $\dot{\psi}$ is a continuous function of t, the function $\dot{u}$ is a continuous function of t, and problem (81) is equivalent to problem (76). Thus, problem (76) has a unique local solution. Theorem 14 is proved. □

4.1.2 Global existence

Assuming that $0 < a(t) \in C^1(0, \infty)$ satisfying the following condition

$$\limsup_{t\to\infty} \frac{|\dot{a}(t)|}{a(t)} < q < 1. \tag{82}$$

We have the following result:

Theorem 15 *If Assumption D) and* (82) *hold, then problem* (76) *has a unique global solution.*

Proof. Since $G(t, \psi)$ is Lipschitz with respect to ψ and continuously differentiable with respect to t, the solution to (81) exists globally, i.e., for all $t \geq 0$, if

$$\sup_{t\geq 0} \|\psi(t)\| \leq c < \infty. \tag{83}$$

If the solution ψ to problem (81) exists globally, then the solution $u(t)$ to the equivalent problem (76) exists globally because the map $\psi \mapsto u$ is a diffeomorphism for $t \in [0, T]$, where $T > 0$ is an arbitrary large number.

Denote $h(t) := \|\psi(t)\|$. The function $\psi(t)$ is continuously differentiable with respect to $t > 0$. The function $h(t)$ is continuously differentiable with respect to t at any point at which $h(t) > 0$. If $h(t) = 0$ on an open interval, then $\dot{h}(t) = 0$ on this interval. If $h(s) = 0$, then we understand by $\dot{h}(s)$ the one-sided derivative from the right,

$$\dot{h}(s) = \lim_{\tau\to+0} \frac{h(s+\tau)}{\tau}.$$

This limit does exist if $\psi(t)$ is continuously differentiable. Indeed,

$$\lim_{\tau\to+0} \frac{h(s+\tau)}{\tau} = \lim_{\tau\to+0} \frac{\|\dot{\psi}(s)\tau + o(\tau)\|}{\tau} = \|\dot{\psi}(s)\|.$$

The left-sided derivative of $h(t)$ also exists and is equal to $-\|\dot{\psi}(s)\|$.

Multiply both sides of (81) by $\psi(t)$ and get

$$h\dot{h} = -h^2 + \langle \dot{a}(t)u(t), \psi \rangle. \tag{84}$$

Let $w(t)$ solve the equation:

$$F(w(t)) + a(t)w(t) - f = 0, \qquad t \geq 0. \tag{85}$$

It is known (see [25], p.112) that if F is monotone, continuous, and equation (1) is solvable and $\lim_{t\to\infty} a(t) = 0$, then there exists $w(\infty)$, and $w(\infty) = y$, where y is the unique minimal-norm solution to (1). So

$$\sup_{t\geq 0} ||w(t)|| < c,$$

where $c > 0$ denotes various constants. Equation (84) implies

$$\dot{h} \le -h + \|\dot{a}|\|u(t) - w(t)\| + |\dot{a}(t)|\|w(t)\|. \tag{86}$$

Let us prove the following estimate

$$\|u(t) - w(t)\| \le \frac{h(t)}{a(t)}, \qquad \forall t \ge 0. \tag{87}$$

Using (77) one gets:

$$\langle F(u) - F(w) + a(u-w), u-w \rangle \ge a\|u-w\|^2. \tag{88}$$

Thus,

$$\|u(t) - w(t)\| \le \frac{\|F(u(t)) - F(w(t)) + a(t)(u(t) - w(t)\|}{a(t)} = \frac{h(t)}{a(t)}. \tag{89}$$

From (87) and (86) one obtains

$$\dot{h} \le -h\Big(1 - \frac{|\dot{a}(t)|}{a(t)}\Big) + |\dot{a}|\|w(t)\|. \tag{90}$$

From (82) there exists $T > 0$ such that

$$\frac{|\dot{a}(t)|}{a(t)} \le q, \qquad \forall t \ge T. \tag{91}$$

Thus, inequality (90) holds if

$$\dot{h} \le -(1-q)h + |\dot{a}|\|w(t)\|.$$

Since $\sup_{t\ge0}\|w(t)\| < c$, the above inequality implies by the usual argument the following inequality

$$h(t) \le h(T)e^{-(t-T)(1-q)} + ce^{-t(1-q)}\int_T^t e^{s(1-q)}|\dot{a}(s)|ds, \qquad \forall t \ge T. \tag{92}$$

From (92) and (91) one gets

$$h(t) \le h(T)e^{-(t-T)(1-q)} + qce^{-t(1-q)}\int_T^t e^{\frac{s}{2}}a(s)ds, \qquad \forall t \ge T. \tag{93}$$

Since we have assumed that $a(t) > 0$ is a $C^1([0,\infty))$ function, such that $a(t) \to 0$ as $t \to \infty$, we have $\sup_{t\ge0} a(t) < c$. Thus, from inequality (93) one gets

$$h(t) \le h(T)e^{-(t-T)(1-q)} + c(1 - e^{-(t-T)(1-q)}), \qquad \forall t \ge T. \tag{94}$$

One the other hand, one gets from (90) and the Gronwall inequality the following estimate

$$h(t) \le h(0)e^{-\phi(t)} + e^{-\phi(t)}c\int_0^t e^{\phi(s)}|a(s)|ds, \qquad 0 \le t \le T, \tag{95}$$

where

$$\phi(t) := \int_0^t \left(1 - \frac{|\dot{a}(s)|}{a(s)}\right) ds.$$

Estimate (83) follows from (94) and (95).

Theorem 15 is proved. □

4.1.3 Justification of the DSM for exact data

By the justification of the DSM for solving equation (1) we mean the statements (3).

In Theorem 15 the first of these statements is proved. Let us assume

$$\lim_{t\to\infty} a(t) = 0, \qquad \lim_{t\to\infty} \frac{\dot{a}(t)}{a(t)} = 0, \tag{96}$$

and prove the remaining two statements from (3).

Remark 16 Actually our argument allows for the following generalization of the results: assumption (82) can be weakened to $\frac{|\dot{a}(t)|}{a(t)} \le q, \forall t \ge 0, q \in (0,1)$ and the second assumption (96) can be weakened to $\limsup_{t\to\infty} \frac{|\dot{a}(t)|}{a(t)} \le q'$, where $q + q' < 1$.

Theorem 17 *If Assumption D) and* (96) *hold, and equation* (1) *has a solution, then* (3) *hold, and* $u(\infty) = y$, *where* y *is the unique minimal-norm solution to* (1).

Proof. It is known (see, [25], p.112) that

$$\lim_{t\to\infty} w(t) = y, \tag{97}$$

so $\limsup_{t\to\infty} ||w(t)|| < c$. Inequality (87) implies

$$a(t)||u(t)|| \le a(t)||w(t)|| + h(t) \le ca(t) + h(t). \tag{98}$$

Inequalities (98) and (84) imply

$$\dot{h} \le -h + \frac{|\dot{a}(t)|}{a(t)}[ca(t) + h(t)]. \tag{99}$$

Assumptions (82) and (96) imply that

$$\lim_{t\to\infty} |\dot{a}(t)| = 0. \tag{100}$$

From the second assumption (96) it follows that

$$\frac{|\dot{a}(t)|}{a(t)} < \delta, \qquad \forall t > t_\delta, \tag{101}$$

where $\delta > 0$ is an arbitrary small fixed number. From (99)–(101) it follows that

$$\lim_{t\to\infty} h(t) = 0. \tag{102}$$

Indeed, (99) implies

$$\dot{h} \le -(1-\delta)h + c|\dot{a}(t)|, \qquad t > t_\delta. \tag{103}$$

Thus

$$h(t) \le h(t_\delta)e^{-(1-\delta)t} + ce^{-(1-\delta)t}\int_{t_\delta}^{t} e^{(1-\delta)s}|\dot{a}(s)|ds, \qquad t \ge t_\delta. \tag{104}$$

Clearly $\lim_{t\to\infty} h(t_\delta)e^{-(1-\delta)t} = 0$. The L'Hospital rule yields

$$0 \le \lim_{t\to\infty} \frac{\int_{t_\delta}^{t} e^{(1-\delta)s}|\dot{a}(s)|ds}{e^{(1-\delta)t}} = \lim_{t\to\infty}(1-\delta)^{-1}|\dot{a}(t)| = 0. \tag{105}$$

Thus, (102) is proved.

Let us prove that (102) implies the existence of the limit $u(\infty) := \lim_{t\to\infty} u(t)$, the relation

$$F(u(\infty)) = f, \tag{106}$$

and the relation $u(\infty) = y$, where y is the minimal-norm solution of the equation $F(u) = f$.

It is proved in [25], p.112, that the limit $w(\infty)$, as $a = a(t) \to 0$, i.e., $t \to \infty$, of the solution w_a to the following equation:

$$F(w_a) + aw_a - f = 0, \qquad a > 0, \tag{107}$$

with a hemicontinuous monotone operator F, exists, and $w(\infty) = y$, provided that equation (1) is solvable.

Thus, the existence of $u(\infty)$ follows from (87) if one proves that

$$\lim_{t\to\infty} \frac{h(t)}{a(t)} = 0, \tag{108}$$

To verify (108), we claim that the second assumption (82) implies that

$$\lim_{t\to\infty} \frac{e^{-(1-\delta)t}}{a(t)} = 0.$$

Indeed, the inequality $\dot{a}(t) \ge -0.5a(t)$ implies $a(t) \ge ce^{-0.5t}$, where $c > 0$ is a constant. Thus, the claim follows if $\delta < 0.5$.

Let us now prove (108). Divide both side of (104) by $a(t)$ and let $t \to \infty$. The first term on the right tends to zero, and the second term by the L'Hospital's rule tends also to zero because of the second assumption (96). Thus, (108) is established.

Since the limit $w(\infty) = y$ exists, it follows from (87) and (108) that $u(\infty)$ exists and $u(\infty) = y$.

Theorem 17 is proved. □

4.2 Solving equations with monotone operators when the data are noisy

Assume that f is not known but f_δ, the noisy data, are known, and $\|f_\delta - f\| \le \delta$. If $F'(u)$ is not boundedly invertible, then solving equation (1) for u, given noisy data f_δ, is often (but not always) an ill-posed problem. When F is a linear bounded operator many methods for stable solution of (1) were proposed (see [6], [18], [22], [25], [33], and references therein). However, when F is nonlinear then the theory is less complete.

The DSM for solving equation (1) was studied extensively in [25]–[32], [8]-[10], where also numerical examples, illustrating efficiency of the algorithms, based on the DSM methods, were given. In [25] the following version of the DSM for solving equation (1) was studied:

$$\dot{u}_\delta = -\big(F'(u_\delta) + a(t)I\big)^{-1}\big(F(u_\delta) + a(t)u_\delta - f_\delta\big), \quad u_\delta(0) = u_0. \tag{109}$$

Here F *is a monotone operator, and* $a(t) > 0$ *is a continuous function, defined for all* $t \ge 0$*, strictly monotonically decaying,* $\lim_{t\to\infty} a(t) = 0$. These assumptions on $a(t)$ hold throughout the Section and are not repeated. Additional assumptions on $a(t)$ will appear in Theorem 28. Convergence of the above DSM was proved in [25] for any initial value u_0 with an *a priori* choice of stopping time t_δ, provided that $a(t)$ is suitably chosen. In this Section an a posteriori choice of t_δ is formulated and justified.

4.2.1 Auxiliary results

Let us consider the following equation

$$F(V_{\delta,a}) + aV_{\delta,a} - f_\delta = 0, \qquad a > 0, \tag{110}$$

where $a = const$. It is known (see, e.g., [25]) that equation (110) with monotone continuous operator F has a unique solution for any $f_\delta \in H$.

Let us recall the following result (see [25, p.112]):

Lemma 18 *Assume that equation* (1) *is solvable,* y *is its minimal-norm solution, and* F *is monotone and continuous. Then*

$$\lim_{a\to 0} \|V_{0,a} - y\| = 0,$$

where $V_{0,a}$ *solves* (110) *with* $\delta = 0$.

Lemma 19 (Lemma 3, [7]) *If* (75) *holds and* F *is continuous, then* $\|V_{\delta,a}\| = O(\frac{1}{a})$ *as* $a \to \infty$*, and*

$$\lim_{a\to\infty} \|F(V_{\delta,a}) - f_\delta\| = \|F(0) - f_\delta\|. \tag{111}$$

Let $a = a(t)$, $0 < a(t) \searrow 0$, and assume $a \in C^1[0, \infty)$. Then the solution $V_\delta(t) := V_{\delta,a(t)}$ of (110) is a function of t. From the triangle inequality one gets:

$$\|F(V_\delta(0)) - f_\delta\| \geq \|F(0) - f_\delta\| - \|F(V_\delta(0)) - F(0)\|.$$

From Lemma 19 it follows that for large $a(0)$ one has:

$$\|F(V_\delta(0)) - F(0)\| \leq M_1 \|V_\delta(0)\| = O\biggl(\frac{1}{a(0)}\biggr), \quad M_1 = \max_{||u|| \leq ||V_\delta(0)||} ||F'(u)||.$$

Therefore, if $\|F(0) - f_\delta\| > C\delta$, then $\|F(V_\delta(0)) - f_\delta\| \geq (C - \epsilon)\delta$, where $\epsilon > 0$ is arbitrarily small, for sufficiently large $a(0) > 0$.

Below the words decreasing and increasing mean strictly decreasing and strictly increasing.

Lemma 20 (Lemma 2, [7]) *Assume* $\|F(0) - f_\delta\| > 0$. *Let* $0 < a(t) \searrow 0$, *and* F *be monotone. Denote*

$$\phi(t) := \|F(V_\delta(t)) - f_\delta\|, \quad \psi(t) := \|V_\delta(t)\|,$$

where $V_\delta(t)$ *solves* (110) *with* $a = a(t)$. *Then* $\phi(t)$ *is decreasing, and* $\psi(t)$ *is increasing.*

Lemma 21 (cf. Lemma 4, [7]) *Assume* $0 < a(t) \searrow 0$. *Then the following inequality holds*

$$\lim_{t\to\infty} \|F(V_\delta(t)) - f_\delta\| \leq \delta. \tag{112}$$

Remark 22 Let $V := V_\delta(t)|_{\delta=0}$, so $F(V) + a(t)V - f = 0$. Let y be the minimal-norm solution to the equation $F(u) = f$. We claim that

$$\|V_\delta - V\| \leq \frac{\delta}{a}. \tag{113}$$

Indeed, from (110) one gets

$$F(V_\delta) - F(V) + a(V_\delta - V) = f_\delta - f.$$

Multiply this equality by $V_\delta - V$ and use (75) to obtain

$$\begin{aligned}
\delta \|V_\delta - V\| &\geq \langle f_\delta - f, V_\delta - V\rangle \\
&= \langle F(V_\delta) - F(V) + a(V_\delta - V), V_\delta - V\rangle \\
&\geq a\|V_\delta - V\|^2.
\end{aligned}$$

This implies (113).

Similarly, from the equation

$$F(V) + aV - F(y) = 0$$

one can derive that

$$\|V\| \leq \|y\|. \tag{114}$$

From (113) and (114), one gets the following estimate:

$$\|V_\delta\| \leq \|V\| + \frac{\delta}{a} \leq \|y\| + \frac{\delta}{a}. \tag{115}$$

Lemma 23 *Let $a(t)$ satisfy* (161). *Then one has*

$$e^{-\frac{t}{2}}\int_0^t e^{\frac{s}{2}}|\dot a(s)|\|V_\delta(s)\|ds \le \frac{1}{2}a(t)\|V_\delta(t)\|, \qquad t \ge 0. \tag{116}$$

Proof. Let us check that

$$e^{\frac{t}{2}}|\dot a(t)| \le \frac{d}{dt}\left(\frac{1}{2}a(t)e^{\frac{t}{2}}\right), \qquad t > 0. \tag{117}$$

One has

$$\frac{d}{dt}\left(\frac{1}{2}a(t)e^{\frac{t}{2}}\right) = \frac{a(t)e^{\frac{t}{2}}}{4} + \frac{\dot a(t)e^{\frac{t}{2}}}{2} = \frac{a(t)e^{\frac{t}{2}}}{4} - \frac{|\dot a(t)|e^{\frac{t}{2}}}{2}. \tag{118}$$

Thus, inequality (117) is equivalent to

$$\frac{3}{2}|\dot a(t)| \le \frac{1}{4}a(t), \qquad \forall t > 0. \tag{119}$$

Inequality (119) holds because by our assumptions the function $a(t)$ satisfies (161). Integrating both sides of (117) from 0 to t, one gets

$$\int_0^t e^{\frac{s}{2}}|\dot a(s)|ds \le \frac{1}{2}a(t)e^{\frac{t}{2}} - \frac{1}{2}a(0)e^0 < \frac{1}{2}a(t)e^{\frac{t}{2}}, \qquad t \ge 0. \tag{120}$$

Multiplying (120) by $e^{-\frac{t}{2}}\|V_\delta(t)\|$, and using the fact that $\|V_\delta(t)\|$ is increasing (see Lemma 20), one gets (116). Lemma 23 is proved. □

Lemma 24 *Let $0 < a(t)$ satisfy the following relations*

$$0 < a(t) \searrow 0, \qquad \frac{|\dot a(t)|}{a(t)} \searrow 0. \tag{121}$$

Define $\phi(t) := \int_0^t \left(1 - \frac{|\dot a(s)|}{a(s)}\right)ds$. *Then*

$$\lim_{t\to\infty} e^{\frac{t}{2}}a(t) = \infty, \tag{122}$$

$$\lim_{t\to\infty} \frac{\int_0^t e^{\frac{s}{2}}\frac{|\dot a(s)|}{a(s)}ds}{e^{\frac{t}{2}}} = 0, \tag{123}$$

$$\lim_{t\to\infty} \frac{\int_0^t e^{\frac{s}{2}}|\dot a(s)|ds}{e^{\frac{t}{2}}a(t)} = 0, \tag{124}$$

and

$$\lim_{t\to\infty} \phi(t) = \infty, \tag{125}$$

$$\lim_{t\to\infty} e^{-\phi(t)}\int_0^t e^{\phi(s)}\frac{|\dot a(s)|}{a(s)}ds = 0, \tag{126}$$

$$\lim_{t\to\infty} e^{\phi(t)}a(t) = \infty, \tag{127}$$

$$\lim_{t\to\infty} \frac{\int_0^t e^{\phi(s)}|\dot a(s)|ds}{e^{\phi}(t)a(t)} = 0. \tag{128}$$

Proof. Let us prove (122)–(124). Relations (125)–(128) are obtained similarly.

First, let us prove (122). We claim that, for sufficiently large $t > 0$, the following inequality holds:

$$\frac{t}{2} > \ln\frac{1}{a^2(t)}. \tag{129}$$

By L'Hospital's rule and (121), one obtains

$$\lim_{t\to\infty}\frac{t}{2\ln\frac{1}{a^2(t)}} = \lim_{t\to\infty}\frac{1}{2a^2(t)\frac{-2\dot{a}(t)}{a^3(t)}} = \lim_{t\to\infty}\frac{a(t)}{4|\dot{a}(t)|} = \infty. \tag{130}$$

This implies that (122) holds for $t > 0$ sufficiently large. From (129) one concludes

$$\lim_{t\to\infty} e^{\frac{t}{2}}a(t) \geq \lim_{t\to\infty} e^{\ln\frac{1}{a^2(t)}}a(t) = \lim_{t\to\infty}\frac{1}{a(t)} = \infty. \tag{131}$$

Thus, relation (122) is proved.

Let us prove (123). If $I := \int_0^\infty e^{\frac{s}{2}}\frac{|\dot{a}(s)|}{a(s)}ds < \infty$ then (123) is obvious. If $I = \infty$, then (123) follows from L'Hospital's rule.

Let us prove (124). The denominator of (124) tends to ∞ as $\delta \to 0$ by (122). Thus, if the numerator of (124) is bounded then (124) holds. Otherwise, relation (164) and L'Hospital's rule yield

$$\lim_{\delta\to 0}\frac{\int_0^{t_\delta} e^{\frac{s}{2}}|\dot{a}(s)|ds}{e^{\frac{t_\delta}{2}}a(t_\delta)} = \lim_{t\to\infty}\frac{e^{\frac{t}{2}}|\dot{a}(t)|}{\frac{1}{2}e^{\frac{t}{2}}a(t) - e^{\frac{t}{2}}|\dot{a}(t)|} = 0. \tag{132}$$

Lemma 24 is proved. □

4.2.2 Main results

Denote

$$A := F'(u_\delta(t)), \quad A_a := A + aI,$$

where I is the identity operator, and $u_\delta(t)$ solves the following Cauchy problem:

$$\dot{u}_\delta = -A_{a(t)}^{-1}[F(u_\delta) + a(t)u_\delta - f_\delta], \quad u_\delta(0) = u_0, \tag{133}$$

where $u_0 \in H$.

Theorem 25 *Let F be a Fréchet differentiable monotone operator. Assume that F' is continuous. Let $0 < a(t)$ satisfy conditions* (121). *Then problem* (133) *has a unique global solution.*

Proof. The uniqueness and local existence of u_δ follows from similar arguments as in Theorem 14. Let us prove that $u_\delta(t)$ is defined globally.

Let us prove the existence of t_δ. From (133), one obtains:

$$\frac{d}{dt}\big(F(u_\delta) + au_\delta - f_\delta\big) = A_a\dot{u}_\delta + \dot{a}u_\delta = -\big(F(u_\delta) + au_\delta - f_\delta\big) + \dot{a}u_\delta.$$

This and (110) imply:

$$\frac{d}{dt}\big[F(u_\delta)-F(V_\delta)+a(u_\delta-V_\delta)\big]=-\big[F(u_\delta)-F(V_\delta)+a(u_\delta-V_\delta)\big]+\dot a u_\delta. \quad (134)$$

Denote

$$v:=v(t):=F(u_\delta(t))-F(V_\delta(t))+a(t)(u_\delta(t)-V_\delta(t)),\qquad h:=h(t):=\|v(t)\|.$$

Multiply (134) by v and get

$$h\dot h=-h^2+\langle v,\dot a(u_\delta-V_\delta)\rangle+\dot a\langle v,V_\delta\rangle\le -h^2+h|\dot a|\|u_\delta-V_\delta\|+|\dot a|h\|V_\delta\|. \quad (135)$$

This implies

$$\dot h\le -h+|\dot a|\|u_\delta-V_\delta\|+|\dot a|\|V_\delta\|. \quad (136)$$

Since $\langle F(u_\delta)-F(V_\delta),u_\delta-V_\delta\rangle\ge 0$, one obtains from two equations

$$\langle v,u_\delta-V_\delta\rangle=\langle F(u_\delta)-F(V_\delta)+a(t)(u_\delta-V_\delta),u_\delta-V_\delta\rangle,$$

and

$$\langle v,F(u_\delta)-F(V_\delta)\rangle=\|F(u_\delta)-F(V_\delta)\|^2+a(t)\langle u_\delta-V_\delta,F(u_\delta)-F(V_\delta)\rangle,$$

the following two inequalities:

$$a\|u_\delta-V_\delta\|^2\le\langle v,u_\delta-V_\delta\rangle\le\|u_\delta-V_\delta\|h, \quad (137)$$

and

$$\|F(u_\delta)-F(V_\delta)\|^2\le\langle v,F(u_\delta)-F(V_\delta)\rangle\le h\|F(u_\delta)-F(V_\delta)\|. \quad (138)$$

Inequalities (137) and (138) imply:

$$a\|u_\delta-V_\delta\|\le h,\quad \|F(u_\delta)-F(V_\delta)\|\le h. \quad (139)$$

Inequalities (136) and (139) imply

$$\dot h\le -h\bigg(1-\frac{|\dot a|}{a}\bigg)+|\dot a|\|V_\delta\|. \quad (140)$$

From (140) and the Gronwall inequality one obtains

$$h(t)\le h(0)e^{-\phi(t)}+e^{-\phi(t)}\int_0^t e^{\phi(s)}|\dot a(s)|\|V_\delta(s)\|ds, \quad (141)$$

where

$$\phi(t):=\int_0^t\bigg(1-\frac{|\dot a(s)|}{a(s)}\bigg)ds. \quad (142)$$

Thus, $v(t)=F(u_\delta)+a(t)u_\delta-f_\delta$ is defined globally. As we have mentioned in the previous section, this implies that $u_\delta(t)$ exists globally.

Theorem 25 is proved. □

Assume that equation (1) has a solution, possibly nonunique, and y is the minimal norm solution to this equation. Let f be unknown but f_δ be given, $\|f_\delta - f\| \le \delta$.

Theorem 26 *Let $a(t)$ satisfy (121). Let $C > 0$ and $\zeta \in (0,1]$ be constants such that $C\delta^\zeta > \delta$. Assume that $F : H \to H$ is a Fréchet differentiable monotone operator, and u_0 is an element of H, satisfying the following inequality*

$$\|F(u_0) - f_\delta\| > C\delta^\zeta. \tag{143}$$

Then there exists a unique t_δ, such that

$$\|F(u_\delta(t_\delta)) - f_\delta\| = C\delta^\zeta, \quad \|F(u_\delta(t)) - f_\delta\| > C\delta^\zeta, \quad \forall t \in [0, t_\delta). \tag{144}$$

If $\zeta \in (0,1)$ and

$$\lim_{\delta\to 0} t_\delta = \infty, \tag{145}$$

then

$$\lim_{\delta\to 0} \|u_\delta(t_\delta) - y\| = 0. \tag{146}$$

Remark 27 Inequality (143) is a natural assumption because if this inequality does not hold and $\|u_0\|$ is not "too large", then u_0 can be considered as an approximate solution to (1).

In Theorem 28 the existence of t_δ satisfying (144) is guaranteed for any $\zeta \in (0,1]$. However, we prove relation (146) for $\zeta \in (0,1)$. If $\zeta = 1$ it is possible to prove that $u_\delta(t_\delta)$ converges to a solution to (1), but it is not known whether this solution is the minimal-norm solution of (1) if (1) has more than one solution.

Further results on the choices of ζ require extra assumptions on F and y. Since the minimal-norm solution y satisfies the relation $\|F(y) - f_\delta\| = \|f - f_\delta\| \le \delta$, it is natural to choose $C > 0$ and $\zeta \in (0,1)$ so that $C\delta^\zeta$ be close to δ.

Proof. The uniqueness of t_δ follows from (144). Indeed, if t_δ and $\tau_\delta > t_\delta$ both satisfy (144), then the second inequality in (144) does not hold on the interval $[0, \tau_\delta)$.

From (139) and (141) one obtains

$$\|F(u_\delta) - F(V_\delta)\| \le h(0)e^{-\phi(t)} + e^{-\phi(t)} \int_0^t e^{\phi(s)} |\dot a(s)| \|V_\delta(s)\| ds, \tag{147}$$

This and the triangle inequality imply

$$\|F(u_\delta(t)) - f_\delta\| \le \|F(V_\delta(t)) - f_\delta\| + h(0)e^{-\phi(t)} + e^{-\phi(t)} \int_0^t e^{\phi(s)} |\dot a| \|V_\delta\| ds. \tag{148}$$

Since $a(s)\|V_\delta(s)\| = \|F(V_\delta(s)) - f_\delta\|$ is decreasing, by Lemma 20, one obtains:

$$\lim_{t\to\infty} e^{-\phi(t)} \int_0^t e^{\phi(s)} |\dot a| \|V_\delta\| ds \le \lim_{t\to\infty} e^{-\phi(t)} \int_0^t e^{\phi(s)} \frac{|\dot a(s)|}{a(s)} a(0) \|V_\delta(0)\| ds. \tag{149}$$

It follows from (112), (148)–(149) and (125)–(126) that

$$\lim_{t\to\infty}\|F(u_\delta(t))-f_\delta\|\le\lim_{t\to\infty}\|F(V_\delta(t))-f_\delta\|+\lim_{t\to\infty}e^{-\phi(t)}\int_0^t e^{\phi(s)}|\dot a|\|V_\delta\|ds\le\delta. \tag{150}$$

The assumption $\|F(u_0)-f_\delta\|>C\delta^\zeta>\delta$ and inequality (150) imply the existence of a $t_\delta>0$ such that (144) holds because $\|F(u_\delta(t))-f_\delta\|$ is a continuous function of t.

From (127), (145), and the fact that the function $\|V_\delta(t)\|$ is increasing, one gets the following inequality for all sufficiently small $\epsilon>0$

$$h(0)e^{-\phi(t_\delta)}\le a(t_\delta)\|V_\delta(t_\delta)\|. \tag{151}$$

Similarly, from (128) and (145) one obtains

$$e^{-\phi(t_\delta)}\int_0^{t_\delta}e^{\phi(s)}|\dot a(s)|\|V_\delta(s)\|ds\le a(t_\delta)\|V_\delta(t_\delta)\|, \tag{152}$$

for all sufficiently small $\delta>0$.

From (144), (148), (151)–(152), and (115) one gets

$$C\delta^\zeta=\|F(u_\delta(t_\delta))-f_\delta\|\le a(t_\delta)\|V_\delta(t_\delta)\|\big(1+1+1\big)\le 3\Big(a(t_\delta)\|y\|+\delta\Big). \tag{153}$$

This and the relation $\lim_{\delta\to0}\frac{\delta}{\delta^\zeta}=0$, for a fixed $\zeta\in(0,1)$, imply

$$\lim_{\delta\to0}\frac{\delta^\zeta}{a(t_\delta)}\le\frac{3\|y\|}{C}. \tag{154}$$

It follows from inequality (141) and the first inequality in (139) that

$$a(t)\|u_\delta(t)-V_\delta(t)\|\le h(0)e^{-\phi(t)}+e^{-\phi(t)}\int_0^t e^{\phi(s)}|\dot a(s)|\|V_\delta(s)\|ds. \tag{155}$$

From (154) and the first inequality in (115) one gets, for sufficiently small δ, the following inequality

$$\|V_\delta(t)\|\le\|y\|+\frac{\delta}{a(t)}<\|y\|+\frac{C\delta^\zeta}{a(t)}<4\|y\|,\qquad 0\le t\le t_\delta. \tag{156}$$

Therefore,

$$\lim_{\delta\to0}\frac{\int_0^{t_\delta}e^{\phi(s)}|\dot a(s)|\|V_\delta(s)\|ds}{e^{\phi(t)}a(t_\delta)}\le4\|y\|\lim_{\delta\to0}\frac{\int_0^{t_\delta}e^{\phi(s)}|\dot a(s)|ds}{e^{\phi(t)}a(t_\delta)}. \tag{157}$$

It follows from (157) and (128) that

$$\lim_{\delta\to0}\frac{\int_0^{t_\delta}e^{\phi(s)}|\dot a(s)|\|V_\delta(s)\|ds}{e^{\phi(t)}a(t_\delta)}=0. \tag{158}$$

From (158), (155), and (145), one gets

$$0 \le \lim_{\delta\to 0} \|u_\delta(t_\delta) - V_\delta(t_\delta)\| = \lim_{\delta\to 0} \frac{h(t_\delta)}{a(t_\delta)} = 0. \tag{159}$$

It is now easy to finish the proof of Theorem 26.

From the triangle inequality and inequality (113) one obtains

$$\begin{aligned} \|u_\delta(t_\delta) - y\| &\le \|u_\delta(t_\delta) - V_\delta(t_\delta)\| + \|V(t_\delta) - V_\delta(t_\delta)\| + \|V(t_\delta) - y\| \\ &\le \|u_\delta(t_\delta) - V_\delta(t_\delta)\| + \frac{\delta}{a(t_\delta)} + \|V(t_\delta) - y\|, \end{aligned} \tag{160}$$

where $V(t_\delta) = V_{0,a(t_\delta)}$ (see equation (110)). From (145), (159), inequality (160), and Lemma 18 one obtains (146).

Theorem 26 is proved. □

The following result gives sufficient conditions for (145) to hold:

Theorem 28 *Let $a(t)$ satisfy* (121) *and*

$$\frac{1}{6} \ge \frac{|\dot a(t)|}{a(t)}, \qquad t \ge 0. \tag{161}$$

Assume that u_0 satisfies either inequality

$$\|F(u_0) + a(0)u_0 - f_\delta\| \le \frac{1}{4} a(0)\|V_\delta(0)\|, \tag{162}$$

or inequality

$$\|F(u_0) + a(0)u_0 - f_\delta\| \le \theta\delta^\zeta, \qquad 0 < \theta < C, \tag{163}$$

where $V_\delta(t) := V_{\delta,a(t)}$ solves (110) *with $a = a(t)$. Then*

$$\lim_{\delta\to 0} t_\delta = \infty. \tag{164}$$

Remark 29 One can choose u_0 satisfying inequality (162). Indeed, if u_0 approximates $V_\delta(0)$, the solution to equation (110), with a small error, then the first inequality in (162) is satisfied. Inequality (162) is a sufficient condition for the following inequality

$$e^{-\frac{t}{2}}\|F(u_0) + a(0)u_0 - f_\delta\| \le \frac{1}{4} a(t)\|V_\delta(t)\|, \quad t \ge 0, \tag{165}$$

to hold. In our proof inequality (165) is used at $t = t_\delta$. The stopping time t_δ is often sufficiently large for the quantity $e^{\frac{t_\delta}{2}} a(t_\delta)$ to be large. This follows from the fact that $\lim_{t\to\infty} e^{\frac{t}{2}} a(t) = \infty$ (see (122)). In this case inequality (165) with $t = t_\delta$ is satisfied for a wide range of u_0.

Proof. [Proof of Theorem 28]

From (140) and the assumption $1-\frac{|\dot a|}{a}\geq\frac{1}{2}$ one gets

$$\dot h \leq -\frac{1}{2}h + |\dot a|\|V_\delta\|. \tag{166}$$

Inequality (166) implies:

$$h(t)\leq h(0)e^{-\frac{t}{2}} + e^{-\frac{t}{2}}\int_0^t e^{\frac{s}{2}}|\dot a(s)|\|V_\delta(s)\|ds. \tag{167}$$

From (167) and (139), one gets

$$\|F(u_\delta(t)) - F(V_\delta(t))\| \leq h(0)e^{-\frac{t}{2}} + e^{-\frac{t}{2}}\int_0^t e^{\frac{s}{2}}|\dot a|\|V_\delta\|ds. \tag{168}$$

From the triangle inequality and (168) one gets

$$\begin{aligned}\|F(u_\delta(t)) - f_\delta\| &\geq \|F(V_\delta(t)) - f_\delta\| - \|F(V_\delta(t)) - F(u_\delta(t))\| \\ &\geq a(t)\|V_\delta(t)\| - h(0)e^{-\frac{t}{2}} - e^{-\frac{t}{2}}\int_0^t e^{\frac{s}{2}}|\dot a|\|V_\delta\|ds.\end{aligned} \tag{169}$$

Recall that $a(t)$ satisfies (161) by our assumptions. From (161) and Lemma 23 one obtains

$$\frac{1}{2}a(t)\|V_\delta(t)\| \geq e^{-\frac{t}{2}}\int_0^t e^{\frac{s}{2}}|\dot a|\|V_\delta(s)\|ds. \tag{170}$$

From (162) we have

$$h(0)e^{-\frac{t}{2}} \leq \frac{1}{4}a(0)\|V_\delta(0)\|e^{-\frac{t}{2}},\qquad t\geq 0. \tag{171}$$

It follows from (161) that

$$e^{-\frac{t}{2}}a(0)\leq a(t). \tag{172}$$

Specifically, inequality (172) is obviously true for $t=0$, and

$$\left(a(t)e^{\frac{t}{2}}\right)'_t = a(t)e^{\frac{t}{2}}\left(\frac{1}{2} - \frac{|\dot a(t)|}{a(t)}\right) > 0,$$

by (161). Therefore, one gets from (172) and (171) the following inequality:

$$e^{-\frac{t}{2}}h(0) \leq \frac{1}{4}a(t)\|V_\delta(0)\| \leq \frac{1}{4}a(t)\|V_\delta(t)\|,\quad t\geq 0, \tag{173}$$

Here, we have used the inequality $\|V_\delta(t')\|\leq\|V_\delta(t)\|$ for $t'<t$, established in Lemma 20 in Section 4.2.1. From (144) and (169)–(173), one gets

$$C\delta^\zeta = \|F(u_\delta(t_\delta)) - f_\delta\| \geq \frac{1}{4}a(t_\delta)\|V_\delta(t_\delta)\|. \tag{174}$$

From (113) and the triangle inequality one derives

$$a(t)\|V(t)\| \le a(t)\|V(t)-V_\delta(t)\| + a(t)\|V_\delta(t)\| \le \delta + a(t)\|V_\delta(t)\|, \qquad \forall t \ge 0. \tag{175}$$

It follows from (174) and (175) that

$$0 \le \lim_{\delta\to 0} a(t_\delta)\|V(t_\delta)\| \le \lim_{\delta\to 0} \left(\delta + 4C\delta^\zeta\right) = 0. \tag{176}$$

Since $\|V(t)\|$ increases (see Lemma 20), the above formula implies $\lim_{\delta\to 0} a(t_\delta) = 0$. Since $0 < a(t) \searrow 0$, it follows that $\lim_{\delta\to 0} t_\delta = \infty$, i.e., (164) holds.

Theorem 28 is proved. □

5 Implicit Function Theorem via the DSM

The aim of this Section is to demonstrate the power of the Dynamical Systems Method (DSM) as a tool for proving theoretical results. The DSM was systematically developed in [25] and applied to solving nonlinear operator equations in [25] (see also [35]), where the emphasis was on convergence and stability of the DSM-based algorithms for solving operator equations, especially nonlinear and ill-posed equations. In this Section the DSM is used as a tool for proving a "hard" implicit function theorem.

Results in this Section are published in [34].

Let us first recall the usual implicit function theorem. Let U solve the equation $F(U) = f$.

Proposition 30 *If $F(U) = f$, F is a C^1-map in a Hilbert space H, and $F'(U)$ is a boundedly invertible operator, i.e., $\|[F'(U)]^{-1}]\| \le m$, then the equation*

$$F(u) = h \tag{177}$$

is uniquely solvable for every h sufficiently close to f.

For convenience of the reader we include a proof of this known result.

Proof. First, one can reduce the problem to the case $u = 0$ and $h = 0$. This is done as follows. Let $u = U + z$, $h - f = p$, $F(U + z) - F(U) := \phi(z)$. Then $\phi(0) = 0$, $\phi'(0) = F'(U)$, and equation 177 is equivalent to the equation

$$\phi(z) = p, \tag{178}$$

with the assumptions

$$\phi(0) = 0, \quad \lim_{z\to 0} \|\phi'(z) - \phi'(0)\| = 0, \quad \|[\phi'(0)]^{-1}\| \le m. \tag{179}$$

We want to prove that equation (178) under the assumptions (179) has a unique solution $z = z(p)$, such that $z(0) = 0$, and $\lim_{p\to 0} z(p) = 0$. To prove this, consider the equation

$$z = z - [\phi'(0)]^{-1}(\phi(z) - p) := B(z), \tag{180}$$

and check that the operator B is a contraction in a ball $\mathcal{B}_\epsilon := \{z : \|z\| \leq \epsilon\}$ if $\epsilon > 0$ is sufficiently small, and B maps $\mathcal{B}_\epsilon$ into itself. If this is proved, then the desired result follows from the contraction mapping principle.

One has

$$\|B(z)\| = \|z - [\phi'(0)]^{-1}(\phi'(0)z + \eta - p)\| \leq m\|\eta\| + m\|p\|, \tag{181}$$

where $\|\eta\| = o(\|z\|)$. If ϵ is so small that $m\|\eta\| < \frac{\epsilon}{2}$ and p is so small that $m\|p\| < \frac{\epsilon}{2}$, then $\|B(z)\| < \epsilon$, so $B : \mathcal{B}_\epsilon \to \mathcal{B}_\epsilon$.

Let us check that B is a contraction mapping in $\mathcal{B}_\epsilon$. One has:

$$\begin{aligned} \|Bz - By\| &= \|z - y - [\phi'(0)]^{-1}(\phi(z) - \phi(y))\| \\ &= \|z - y - [\phi'(0)]^{-1}\int_0^1 \phi'(y + t(z-y))dt(z-y)\| \\ &\leq m\int_0^1 \|\phi'(y + t(z-y)) - \phi'(0)\|dt\|z-y\|. \end{aligned} \tag{182}$$

If $y, z \in \mathcal{B}_\epsilon$, then

$$\sup_{0\leq t\leq 1} \|\phi'(y + t(z-y)) - \phi'(0)\| = o(1), \qquad \epsilon \to 0.$$

Therefore, if ϵ is so small that $mo(1) < 1$, then B is a contraction mapping in $\mathcal{B}_\epsilon$, and equation (178) has a unique solution $z = z(p)$ in $\mathcal{B}_\epsilon$, such that $z(0) = 0$. The proof is complete. □

The crucial assumptions, on which this proof is based, are assumptions (179).

Suppose now that $\phi'(0)$ *is not boundedly invertible, so that the last assumption in* (179) *is not valid*. Then a theorem which still guarantees the existence of a solution to equation (178) for some set of p is called a "hard" implicit function theorem. Examples of such theorems one may find, e.g., in [1], [2], [3], and [17].

Our goal in this Section is to establish a new theorem of this type using a new method of proof, based on the Dynamical Systems Method (DSM). In [36] we have demonstrated a theoretical application of the DSM by establishing some surjectivity results for nonlinear operators.

The result, presented in this Section, is a new illustration of the applicability of the DSM as a tool for proving theoretical results.

To formulate the result, let us introduce the notion of a scale of Hilbert spaces H_a (see, e.g., [25], p.256). Let $H_a \subset H_b$ and $\|u\|_b \leq \|u\|_a$ if $a \geq b$. Example of spaces H_a is the scale of Sobolev spaces $H_a = W^{a,2}(D)$, where $D \subset \mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary.

Consider equation (177). Assume that

$$F(U) = f; \qquad F : H_a \to H_{a+\delta}, \qquad u \in B(U,R) := B_a(U,R), \tag{183}$$

where $B_a(U,R) := \{u : \|u - U\|_a \leq R\}$, $\delta = const > 0$, and the operator $F : H_a \to H_{a+\delta}$ is continuous. Furthermore, assume that $A := A(u) := F'(u)$ exists and is an isomorphism of H_a onto $H_{a+\delta}$:

$$c_0\|v\|_a \leq \|A(u)v\|_{a+\delta} \leq c_0'\|v\|_a, \qquad u, v \in B(U,R). \tag{184}$$

Assume also that

$$\|A^{-1}(v)A(w)\|_a \le c, \qquad v, w \in B(U,R), \tag{185}$$

and

$$\|A^{-1}(u)[A(u) - A(v)]\|_a \le c\|u - v\|_a, \qquad u, v \in B(U,R). \tag{186}$$

By $c > 0$ we denote various constants. Note that (184) implies

$$\|A^{-1}(u)\psi\|_a \le c_0^{-1}\|\psi\|_{a+\delta}, \qquad \psi = A(u)[F(v) - h], \quad v \in B(U,R).$$

Assumption (184) implies that $A(u)$ is a smoothing operator similar to a smoothing integral operator, and its inverse is similar to the differentiation operator of order $\delta > 0$. Therefore, the operator $A^{-1}(u) = [F'(u)]^{-1}$ causes the "loss of the derivatives". In general, this may lead to a breakdown of the Newton process (method) (6) in a finitely many steps. Our assumptions (183)-(186) guarantee that this will not happen.

Assume that

$$u_0 \in B_a(U,\rho), \qquad h \in B_{a+\delta}(f,\rho), \tag{187}$$

where $\rho > 0$ is a sufficiently small number:

$$\rho \le \rho_0 := \frac{R}{1 + c_0^{-1}(1 + c_0')},$$

and c_0, c_0' are the constants from (184). Then $F(u_0) \in B_{a+\delta}(f, c_0'\rho)$, because

$$\|F(u_0) - F(U)\| \le c_0'\|u_0 - U\| \le c_0'\rho.$$

Consider the problem

$$\dot{u} = -[F'(u)]^{-1}(F(u) - h), \quad u(0) = u_0. \tag{188}$$

Our basic result is:

Theorem 31 *If the assumptions* (183)-(187) *hold, and* $0 < \rho \le \rho_0 := \frac{R}{1+c_0^{-1}(1+c_0')}$, *where* c_0, c_0' *are the constants from* (184)*, then problem* (188) *has a unique global solution* $u(t)$*, there exists* $V := u(\infty)$*,*

$$\lim_{t\to\infty} \|u(t) - V\|_a = 0, \tag{189}$$

and

$$F(V) = h. \tag{190}$$

Theorem 31 says that if $F(U) = f$ and $\rho \le \rho_0$, then for any $h \in B_{a+\delta}(f,\rho)$ equation (177) is solvable and a solution to (177) is $u(\infty)$, where $u(t)$ solves problem (188).

Let us prove Theorem 31.

Proof. Let us outline the ideas of the proof. The local existence and uniqueness of the solution to (188) will be established if one verifies that the operator $A^{-1}(u)[F(u)-h]$ is locally Lipschitz in H_a. The global existence of this solution $u(t)$ will be established if one proves the uniform boundedness of $u(t)$:

$$\sup_{t\geq 0} \|u(t)\|_a \leq c. \tag{191}$$

Let us first prove (in paragraph a) below) estimate (191), the existence of $u(\infty)$, and the relation (190), *assuming the local existence* of the solution to (188).

In paragraph b) below the local existence of the solution to (188) is proved.

a) *Proof of* (190), (191), *and the existence of* $u(\infty)$.

If $u(t)$ exists locally, then the function

$$g(t) := \|\phi\|_{a+\delta} := \|F(u(t)) - h\|_{a+\delta} \tag{192}$$

satisfies the relation

$$g\dot{g} = \langle F'(u(t))\dot{u}, \phi\rangle_{a+\delta} = -g^2, \tag{193}$$

where equation (188) was used. Since $g \geq 0$, it follows from (193) that

$$g(t) \leq g(0)e^{-t}, \qquad g(0) = \|F(u_0) - h\|_{a+\delta}. \tag{194}$$

From (188), (193) and (184) one gets:

$$\|\dot{u}\|_a \leq \frac{1}{c_0}\|\phi\|_{a+\delta} = \frac{g(0)}{c_0}e^{-t} := re^{-t}, \qquad r := \frac{\|F(u_0) - h\|_{a+\delta}}{c_0}. \tag{195}$$

Therefore,

$$\lim_{t\to\infty} \|\dot{u}(t)\|_a = 0, \tag{196}$$

and

$$\int_0^\infty \|\dot{u}(t)\|_a dt < \infty. \tag{197}$$

This inequality implies

$$\|u(\tau) - u(s)\| \leq \int_s^\tau \|\dot{u}(t)\|_a dt < \epsilon, \qquad \tau > s > s(\epsilon),$$

where $\epsilon > 0$ is an arbitrary small fixed number, and $s(\epsilon)$ is a sufficiently large number. Thus, the limit $V := \lim_{t\to\infty} u(t) := u(\infty)$ exists by the Cauchy criterion, and (189) holds.

Assumptions (183) and (184) and relations (188), (189), and (196) imply (190).

Integrating inequality (195) yields

$$\|u(t) - u_0\|_a \leq r, \tag{198}$$

and

$$\|u(t) - u(\infty)\|_a \le re^{-t}. \tag{199}$$

Inequality (198) implies (191).

b) *Let us now prove the local existence of the solution to* (188).

We prove that the operator in (188) $A^{-1}(u)[F(u)-h]$ is locally Lipschitz in H_a. This implies the local existence of the solution to (188).

One has

$$\|A^{-1}(u)(F(u)-h) - A^{-1}(v)(F(v)-h)\|_a \le \|[A^{-1}(u) - A^{-1}(v)](F(u)-h)\|_a + \|A^{-1}(v)(F(u)-F(v))\|_a := I_1 + I_2. \tag{200}$$

Write

$$F(u) - F(v) = \int_0^1 A(v + t(u-v))(u-v)dt, \tag{201}$$

and use assumption (185) with $w = v + t(u-v)$ to conclude that

$$I_2 \le c\|u - v\|_a. \tag{202}$$

Write

$$A^{-1}(u) - A^{-1}(v) = A^{-1}(u)[A(v) - A(u)]A^{-1}(v), \tag{203}$$

and use the estimate

$$\|A^{-1}(v)[F(u) - h]\|_a \le c, \tag{204}$$

which is a consequence of assumptions (183) and (184). Then use assumption (186) to conclude that

$$I_1 \le c\|u - v\|_a. \tag{205}$$

From (200), (202) and (205) it follows that the operator $A^{-1}(u)[F(u)-h]$ is locally Lipschitz.

Note that

$$\|u(t) - U\|_a \le \|u(t) - u_0\|_a + \|u_0 - U\|_a \le r + \rho, \tag{206}$$

and

$$\begin{aligned}\|F(u(t)) - h\|_{a+\delta} &\le \|F(u_0) - h\|_{a+\delta} \\ &\le \|F(u_0) - f\|_{a+\delta} + \|f - h\|_{a+\delta} \le (1 + c_0')\rho,\end{aligned} \tag{207}$$

so, from (195) one gets

$$r \le \frac{(1 + c_0')\rho}{c_0}. \tag{208}$$

Choose

$$R \ge r + \rho. \tag{209}$$

Then the trajectory $u(t)$ stays in the ball $B(U, R)$ for all $t \ge 0$, and, therefore, assumptions (183)-(186) hold in this ball for all $t \ge 0$.

Condition (209) and inequality (208) imply

$$\rho \le \rho_0 = \frac{R}{1 + c_0^{-1}(1 + c_0')}. \tag{210}$$

This is the "smallness" condition on ρ.

Theorem 31 is proved. $\square$

5.1 Example

Let

$$F(u) = \int_0^x u^2(s)ds, \qquad x \in [0,1].$$

Then

$$A(u)q = 2\int_0^x u(s)q(s)ds.$$

Let $f = x$ and $U = 1$. Then $F(U) = x$. Choose $a = 1$ and $\delta = 1$. Denote by $H_a = H_a(0,1)$ the usual Sobolev space. Assume that

$$h \in B_2(x,\rho) := \{h \ : \ \|h - x\|_2 \le \rho\},$$

and $\rho > 0$ is sufficiently small. One can verify that

$$A^{-1}(u)\psi = \frac{\psi'(x)}{2u(x)}$$

for any $\psi \in H_1$.

Let us check conditions (183)-(187) for this example.

Condition (183) holds, because if $u_n \to u$ in H_1, then

$$\int_0^x u_n^2(s)ds \to \int_0^x u^2(s)ds$$

in H_2. To verify this, it is sufficient to check that

$$\frac{d^2}{dx^2}\int_0^x u_n^2(s)ds \to 2uu',$$

where $\to$ means the convergence in $H := H_0 := L^2(0,1)$. In turn, this is verified if one checks that $u_n'u_n \to u'u$ in $L^2(0,1)$, provided that $u_n' \to u'$ in $L^2(0,1)$.

One has

$$I_n := \|u_n'u_n - u'u\|_0 \le \|(u_n' - u')u_n\|_0 + \|u'(u_n - u)\|_0.$$

Since $\|u_n'\|_0 \le c$, one concludes that $\|u_n\|_{L^\infty(0,1)} \le c_1$ and $\lim_{n\to\infty}\|u_n - u\|_{L^\infty} = 0$. Thus,

$$\lim_{n\to\infty} I_n = 0.$$

Condition (184) holds because $\|u\|_{L^\infty(0,1)} \leq c\|u\|_1$, and

$$\| \int_0^x u(s)q(s)ds\|_2 \leq c\|u'q + uq'\|_0 \leq c(\|q\|_{L^\infty(0,1)}\|u\|_1 + \|u\|_{L^\infty(0,1)}\|q\|_1),$$

so

$$\| \int_0^x u(s)q(s)ds\|_2 \leq c_0'\|u\|_1\|q\|_1,$$

and

$$\| \int_0^x uqds\|_2 \geq \|uq\|_1 \geq c_0\|q\|_1,$$

provided that $u \in B_1(1,\rho)$ and $\rho > 0$ is sufficiently small.

Condition (185) holds because

$$\|A^{-1}(v)A(w)q\|_1 = \|\frac{1}{v(x)}w(x)q\|_1 \leq c\|q\|_1,$$

provided that $u, w \in B_1(1,\rho)$ and $\rho > 0$ is sufficiently small.

Condition (186) holds because

$$\|A^{-1}(u)\int_0^x (u-v)qds\|_1 = \|\frac{u-v}{2u}q\|_1 \leq c\|u-v\|_1\|q\|_1,$$

provided that $u, v \in B_1(1,\rho)$ and $\rho > 0$ is sufficiently small.

By Theorem 31 the equation

$$F(u) := \int_0^x u^2(s)ds = h,$$

where $\|h - x\|_2 \leq \rho$ and $\rho > 0$ is sufficiently small, has a solution V,

$$F(V) = h.$$

This solution can be obtained as $u(\infty)$, where $u(t)$ solves problem (188) and conditions (187) and (210) hold.

6 DSM for solving nonlinear operator equations in Banach spaces

Consider an operator equation

$$F(u) = f, \tag{211}$$

where F is an operator in a Banach space X. By X^* denote the dual space of bounded linear functionals on X.

Assume that F is continuously Fréchet differentiable, $F'(u) := A(u)$, and

$$\|A(u) - A(v)\| \leq \omega(\|u - v\|). \tag{212}$$

The function $\omega(r)$, defined on $[0,\infty)$, is continuous, strictly growing, and $\omega(0)=0$.

Assume that

$$\|A_a^{-1}(u)\| \leq \frac{c_1}{|a|^b};\ |a|>0,\ A_a := A+aI,\ c_1 = const > 0,\ b>0. \tag{213}$$

Here a may be a complex number, $|a|>0$, and there exists a smooth path L on the complex plane $\mathbb{C}$, such that for any $a \in L$, $|a| < \epsilon_0$, where $\epsilon_0 > 0$ is a small fixed number independent of u, estimate (213) holds, and L joins the origin and some point a_0, $0 < |a_0| < \epsilon_0$. Assumption (213) holds if there is a smooth path L on a complex a-plane, consisting of regular points of the operator $A(u)$, such that the norm of the resolvent $A_a^{-1}(u)$ grows, as $a \to 0$, not faster than a power $|a|^{-b}$. Thus, assumption (213) is a weak assumption. For example, assumption (213) is satisfied for the class of linear operators A, satisfying the spectral assumption, introduced in [25], Chapter 8. This spectral assumption says, that the set $\{a : |\arg a - \pi| \leq \phi_0,\ 0 < |a| < \epsilon_0\}$ consists of the regular points of the operator A. This assumption implies the estimate $||A_a^{-1}|| \leq \frac{c_1}{a},\ 0 < a < \epsilon_0$, similar to estimate (213).

Assume additionally that the equation

$$F(w_a) + aw_a - f = 0, \quad a \in L, \tag{214}$$

is uniquely solvable for any $f \in X$, and

$$\lim_{a\to 0, a\in L} \|w_a - y\| = 0, \quad F(y) = f. \tag{215}$$

All the above assumptions are standing and are not repeated in the formulation of Theorem 36 and Theorem 38, which are our main results.

These assumptions are satisfied, e.g., if F is a monotone operator in a Hilbert space H and L is a segment $[0,\epsilon_0]$, in which case $c_1 = 1$ and $b = 1$ (see [25]). Sufficient conditions for (215) to hold are given in [45] (see also [25]).

Every equation (211) with a linear, closed, densely defined in a Hilbert space H operator $F = A$ can be reduced to an equation with a monotone operator A^*A, where A^* is the adjoint to A. The operator $T := A^*A$ is selfadjoint and densely defined in H. If $f \in D(A^*)$, where $D(A^*)$ is the domain of A^*, then the equation $Au = f$ is equivalent to $Tu = A^*f$, provided that $Au = f$ has a solution, i.e., $f \in R(A)$, where $R(A)$ is the range of A. Recall that $D(A^*)$ is dense in H if A is closed and densely defined in H. If $f \in R(A)$ but $f \notin D(A^*)$, then equation $Tu = A^*f$ still makes sense and its normal solution y, i.e., the solution with minimal norm, can be defined as

$$y = \lim_{a\to 0} T_a^{-1}A^*f. \tag{216}$$

One proves that $Ay = f$, and $y \perp N(A)$, where $N(A)$ is the null-space of A. These results are proved in [38]- [39].

Our aim is to prove convergence of the DSM (Dynamical Systems Method) for solving equation (211):

$$\dot{u} = -A_{a(t)}^{-1}[F(u(t)) + a(t)u(t) - f], \quad u(0) = u_0; \quad \dot{u} := \frac{du}{dt}, \tag{217}$$

where $u_0 \in X$ is an initial element, $a(t) \in C^1[0,\infty)$, $a(t) \in L$. Our main results are formulated in Theorem 36 in Section 6.1 and Theorem 38 in Section 6.2.

The DSM for solving operator equations has been developed in the monograph [25] and in a series of papers [38]-[40]. It was used as an efficient computational tool in [8]-[12]. One of the earliest papers on the continuous analog of Newton's method for solving well-posed nonlinear operator equations was [5].

Results in this Section, except for Theorem 38, are taken from [44].

6.1 The case of Hölder continuous $F'(u)$

In this Section we assume that $F'(u)$ is Hölder continuous, i.e.,

$$\|A(u) - A(v)\| \le \omega(\|u - v\|), \quad \omega(r) = c_0 r^\kappa, \quad \kappa \in (0,1], \tag{218}$$

$c_0 > 0$ is a constant.

The novel points in this Section include the larger class of the operator equations than earlier considered, and the weakened assumptions on the smoothness of the nonlinear operator F. While in [25] it was often assumed that $F''(u)$ is locally bounded, in this Section a weaker assumption (218) is used.

Our proof of Theorem 36 uses the following result from [11].

Lemma 32 *Assume that $g(t) \ge 0$ is continuously differentiable on any interval $[0,T)$, on which it is defined, and satisfies the following inequality:*

$$\dot{g}(t) \le -\gamma(t)g(t) + \alpha(t)g^p(t) + \beta(t), \quad t \in [0,T), \tag{219}$$

where $p > 1$ is a constant, $\alpha(t) > 0$, $\gamma(t)$ and $\beta(t)$ are three continuous on $[0,\infty)$ functions. Suppose that there exists a $\mu(t) > 0$, $\mu(t) \in C^1[0,\infty)$, such that

$$\alpha(t)\mu^{-p}(t) + \beta(t) \le \mu^{-1}(t)[\gamma(t) - \dot{\mu}(t)\mu^{-1}(t)], \qquad t \ge 0, \tag{220}$$

and

$$\mu(0)g(0) < 1. \tag{221}$$

Then $T = \infty$, i.e., g exists on $[0,\infty)$, and

$$0 \le g(t) < \mu^{-1}(t), \quad t \ge 0. \tag{222}$$

This lemma generalizes a similar result for $p = 2$ proved in [25].

In Section 6.1 a method is given for a proof of the following conclusions:

There exists a unique solution $u(t)$ to problem (217) *for all $t \ge 0$, there exists $u(\infty) := \lim_{t\to\infty} u(t)$, and $F(u(\infty)) = f$, that is conditions* (3) *hold.*

The assumptions on u_0 and $a(t)$ under which conclusions (3) hold for the solution to problem (217) are formulated in Theorem 36 in Section 6.1. Theorem 36 in Section 6.1 is our main result. Roughly speaking, this result says that conclusions (3) hold for the solution to problem (217), provided that $a(t)$ is suitably chosen.

Let $|a(t)| := r(t) > 0$. If $a(t) = a_1(t) + ia_2(t)$, where $a_1(t) = \operatorname{Re} a(t)$, $a_2(t) = \operatorname{Im} a(t)$, then

$$|\dot{r}(t)| \le |\dot{a}(t)|. \tag{223}$$

Indeed,

$$|\dot{r}(t)| = \frac{|a_1\dot{a}_1 + a_2\dot{a}_2|}{r(t)} \le \frac{r(t)|\dot{a}(t)|}{r(t)}, \tag{224}$$

and (224) implies (223).

Let $h \in X^*$ be arbitrary with $\|h\| = 1$, and

$$g(t) := (z(t), h), \qquad z(t) := u(t) - w_a(t), \tag{225}$$

where $u(t)$ solves (217) and $w_a(t)$ solves (214) with $a = a(t)$. By the assumption, $w_a(t)$ exists for every $t \ge 0$. The local existence of $u(t)$, the solution to (217), is the conclusion of Lemma 33. Let $\psi(t) \in C^1([0,\infty); X)$. In the following lemma a proof of the local existence of the solution to problem (217) is given by a novel argument. The right-hand side of (217) is a nonlinear function of u, which does not satisfy the Lipschitz condition, which is the standard condition in the usual proofs of the local existence of the solution to an evolution problem. Our argument uses an abstract inverse function theorem. This argument is valid under the assumption that $F'(u)$ depends continuously on u.

Lemma 33 *If assumption* (213) *holds and* (214) *is uniquely solvable for any $f \in X$, then the solution $u(t)$ to* (217) *exists locally.*

Proof. Differentiate equation (214) with $a = a(t)$ with respect to t. The result is

$$A_{a(t)}(w_a(t))\dot{w}_a(t) = -\dot{a}(t)w_a(t), \tag{226}$$

or

$$\dot{w}_a(t) = -\dot{a}(t)A^{-1}_{a(t)}(w_a(t))w_a(t). \tag{227}$$

Denote

$$\psi(t) := F(u(t)) + a(t)u(t) - f. \tag{228}$$

For any $\psi \in H$ equation (228) is uniquely solvable for $u(t)$ by our assumption (214), which is used with $f + \psi(t)$ in place of f in (214). By the inverse function theorem, which holds due to our assumption (213), and by assumption (218), the solution $u(t)$ to (228) is continuously differentiable with respect to t if $\psi(t)$ is. One may solve (228) for u and write $u = G(\psi)$, where the map G is continuously Fréchet differentiable because F is.

Differentiate (228) and get

$$\dot{\psi}(t) = A_{a(t)}(u(t))\dot{u}(t) + \dot{a}(t)u. \tag{229}$$

If one wants the solution to (228) to be a solution to (217), then one has to require that

$$A_{a(t)}(u(t))\dot{u} = -\psi(t). \tag{230}$$

If (230) holds, then (229) can be written as

$$\dot{\psi}(t) = -\psi + \dot{a}(t)G(\psi), \quad G(\psi) := u(t), \tag{231}$$

where $G(\psi)$ is continuously Fréchet differentiable. Thus, equation (231) is equivalent to (217) at all $t \geq 0$ if

$$\psi(0) = F(u_0) + a(0)u_0 - f. \tag{232}$$

Indeed, if u solves (217) then ψ, defined in (228), solves the Cauchy problem (231)-(232). Conversely, if ψ solves (231)-(232), then $u(t)$, defined as the unique solution to (228), solves (217). Since the right-hand side of (231) is Fréchet differentiable, it satisfies a local Lipschitz condition. Thus, problem (231)-(232) is locally, solvable. Therefore, problem (217) is locally solvable.

Lemma 33 is proved. □

To prove that the solution $u(t)$ to (217) exists globally, it is sufficient to prove the following estimate

$$\sup_{t\geq 0} \|u(t)\| < \infty. \tag{233}$$

Lemma 34 *Estimate* (233) *holds.*

Proof. Denote

$$z(t) := u(t) - w(t), \tag{234}$$

where $u(t)$ solves (217) and $w(t)$ solves (214) with $a = a(t)$. If one proves that

$$\lim_{t\to\infty} \|z(t)\| = 0, \tag{235}$$

then (233) follows from (235) and (215):

$$\sup_{t\geq 0} \|u(t)\| \leq \sup_{t\geq 0} \|z(t)\| + \sup_{t\geq 0} \|w(t)\| < \infty. \tag{236}$$

□

To prove (235) *we use Lemma 32.*

Let

$$g(t) := \|z(t)\|. \tag{237}$$

Rewrite (217) as

$$\dot{z} = -\dot{w} - A^{-1}_{a(t)}(u(t))[F(u(t)) - F(w(t)) + a(t)z(t)]. \tag{238}$$

Note that:

$$\sup_{h\in X^*, \|h\|=1} (\dot{w}(t), h) = \|\dot{w}(t)\|. \tag{239}$$

Lemma 35 *If the norm $\|w(t)\|$ in X is differentiable, then*

$$\frac{d\|w(t)\|}{dt} \leq \|\dot{w}(t)\|. \tag{240}$$

Proof. The triangle inequality implies:

$$\frac{\|w(t+s)\| - \|w(t)\|}{s} \leq \frac{\|w(t+s) - w(t)\|}{s}, \quad s > 0. \tag{241}$$

Passing to the limit $s \searrow 0$ and using the assumption concerning the differentiabilty of the norm in X, one gets (240).

Lemma 35 is proved. □

The norm is differentiable if X is strictly convex (see, e.g, [3]). A Banach space X is called strictly convex if $||u + v|| < 2$ for any $u \neq v \in X$ such that $||u|| = ||v|| = 1$. A Banach space X is called uniformly convex if for any $\epsilon > 0$ there is a $\delta > 0$ such that for all $u, v \in B(0, 1)$ with $||u - v|| = \epsilon$ one has $||u + v|| \leq 2(1 - \delta)$. Here $B(0, 1)$ is the closed ball in X, centered at the origin and of radius one.

Various necessary and sufficient conditions for the Fréchet differentiability of the norm in Banach spaces are known in the literature (see, e.g., [3] and [4]), starting with Shmulian's paper of 1940, see [46].

Hilbert spaces, $L^p(D)$ and ℓ^p-spaces, $p \in (1, \infty)$, and Sobolev spaces $W^{\ell,p}(D)$, $p \in (1, \infty)$, $D \in \mathbb{R}^n$ is a bounded domain, have Fréchet differentiable norms. These spaces are uniformly convex and they have the $E-$property, i.e., if $u_n \rightharpoonup u$ and $||u_n|| \to ||u||$ as $n \to \infty$, then $\lim_{n\to\infty} ||u_n - u|| = 0$, where $\rightharpoonup$ denotes weak convergence.

Let us state the main result of this Section.

Theorem 36 *If $r(t) = |a(t)|$ is defined in (266), and inequalities (271) and (275) hold, then*

$$\|z(t)\| < r^k(t)^{-1}, \qquad \lim_{t\to\infty} \|z(t)\| = 0, \tag{242}$$

where $k > 0$ and > 0 are some constants. Thus, problem (217) has a unique global solution $u(t)$ and

$$\lim_{t\to\infty} \|u(t) - y\| = 0, \tag{243}$$

where $F(y) = f$.

Proof. From (227), (239) and (213) one gets

$$\|\dot{w}\| \leq c_1 |\dot{a}(t)| r^{-b}(t) \|w(t)\|, \quad r(t) = |a(t)|, \tag{244}$$

where $w(t) := w_a(t)$. Since we assume that $\lim_{t\to\infty} |a(t)| = 0$, one concludes that (215) and (244) imply the following inequality:

$$\|\dot{w}\| \leq c_2 |\dot{a}(t)| r^{-b}(t), \quad c_2 = const > 0, \tag{245}$$

because (215) implies the following estimate:

$$c_1\|w(t)\| \le c_2, \quad t \ge 0. \tag{246}$$

Inequality (223) implies that inequality (245) holds if

$$\|\dot{w}\| \le c_2|\dot{r}(t)|r^{-b}(t), \quad t \ge 0. \tag{247}$$

Recall that $F'(u) := A(u)$ and note that

$$F(u) - F(w) = \int_0^1 F'(w+sz)dsz = A(u)z + \int_0^1 [A(w+sz) - A(u)]dsz, \tag{248}$$

where $z := z(t) = u(t) - w(t)$. Apply h to (238), take $\sup_{h\in X^*,\|h\|=1}$, and use Lemma 35, relation (248), inequality (247), estimate (213), and inequality (218), to get:

$$\dot{g}(t) \le \|\dot{z}(t)\| \le c_2|\dot{r}(t)|r^{-b}(t) + c_3r^{-b}(t)g^p - g, \tag{249}$$

where $g(t)$ is defined in (237),

$$p = 1+\kappa, \quad c_3 := c_0c_1. \tag{250}$$

Inequality (249) is of the form (219) with

$$\gamma(t) = 1, \quad \alpha(t) = c_3r^{-b}(t), \quad \beta(t) = c_2|\dot{r}(t)|r^{-b}(t). \tag{251}$$

Choose

$$\mu(t) = r^{-k}(t), \quad = const > 0, \quad k = const > 0. \tag{252}$$

Then

$$\dot{\mu}\mu^{-1} = -k\dot{r}r^{-1}. \tag{253}$$

Let us assume that

$$r(t) \searrow 0, \quad \dot{r} < 0, \quad |\dot{r}| \searrow 0. \tag{254}$$

Assumption (221) implies

$$g(0)\frac{}{r^k(0)} < 1, \tag{255}$$

and inequality (220) holds if

$$\frac{c_3r^{-b}(t)r^{kp}}{p} + c_2|\dot{r}(t)|r^{-b}(t) \le \frac{r^k(t)}{}\big(1 - k|\dot{r}(t)|r^{-1}(t)\big), \qquad t \ge 0. \tag{256}$$

Inequality (256) can be written as

$$\frac{c_3r^{k(p-1)-b}(t)}{p-1} + \frac{c_2|\dot{r}(t)|}{r^{k+b}(t)} + \frac{k|\dot{r}(t)|}{r(t)} \le 1, \qquad t \ge 0. \tag{257}$$

Let us choose k so that

$$k(p-1) - b = 1,$$

that is,

$$k = \frac{b+1}{p-1}. \tag{258}$$

Choose , for example, as follows:

$$:= \frac{r^k(0)}{2g(0)}. \tag{259}$$

Then inequality (255) holds, and inequality (257) can be written as:

$$c_3 \frac{r(t)[2g(0)]^{p-1}}{[r^k(0)]^{p-1}} + c_2 \frac{r^k(0)}{2g(0)} \frac{|\dot{r}(t)|}{r^{k+b}(t)} + k\frac{|\dot{r}(t)|}{r(t)} \leq 1, \qquad t \geq 0. \tag{260}$$

Note that (258) implies:

$$k + b = kp - 1. \tag{261}$$

Choose $r(t)$ so that relations (254) hold and

$$k\frac{|\dot{r}(t)|}{r(t)} \leq \frac{1}{2}, \qquad t \geq 0. \tag{262}$$

Since $r(0) \geq r(t)$ and (262) holds, then inequality (260) holds if

$$c_3 \frac{[2g(0)]^{p-1}}{r^b(0)} + c_2 \frac{r^k(0)}{2g(0)} \frac{|\dot{r}(t)|}{r^{kp-1}} \leq \frac{1}{2}, \qquad t \geq 0. \tag{263}$$

Denote

$$c_2 \frac{r^k(0)}{2g(0)} = c_2 := c_4. \tag{264}$$

Let

$$c_4 \frac{|\dot{r}(t)|}{r^{kp-1}} = \frac{1}{4}, \qquad t \geq 0, \tag{265}$$

and $kp > 2$. Then equation (265) implies

$$r(t) = \left[1 + t\frac{4c_4}{kp-2} 4c_4\right]^{-\frac{1}{kp-2}}. \tag{266}$$

This $r(t)$ satisfies conditions (254), and equation (265) implies:

$$k\frac{|\dot{r}(t)|}{r(t)} = \frac{kr^{kp-2}(t)}{4c_4}, \quad t \geq 0. \tag{267}$$

Recall that $r(t)$ decays monotonically. Therefore, inequality (262) holds if

$$\frac{kr^{kp-2}(0)}{4c_4} \leq \frac{1}{2}. \tag{268}$$

Inequality (268) holds if

$$\frac{kg(0)}{c_2} r^{k(p-1)-2}(0) = \frac{kg(0)}{c_2} r^{b-1}(0) \leq 1, \tag{269}$$

because (258) implies:

$$k(p-1)-2=b-1. \tag{270}$$

Condition (269) holds if $g(0)$ is sufficiently small or $r^{b-1}(0)$ is sufficiently large:

$$g(0)\le \frac{c_2}{k}r^{b-1}(0). \tag{271}$$

If $b>1$, then condition (271) holds for any fixed $g(0)$ if $r(0)$ is sufficiently large. If $b=1$, then (271) holds if $g(0)\le \frac{c_2}{k}$. If $b\in(0,1)$ then (271) holds either if $g(0)$ is sufficiently small or $r(0)$ is sufficiently small.

Consequently, if (266) and (271) hold, then (265) holds. Therefore, (263) holds if

$$c_3\frac{[2g(0)]^{p-1}}{r^b(0)}\le \frac{1}{4}. \tag{272}$$

It follows from (271) that (272) holds if

$$c_3 2^{p-1}\left(\frac{c_2}{k}\right)^{p-1}\frac{1}{r^{-1+p+2b-bp}(0)}\le\frac{1}{4}. \tag{273}$$

One has $p=1+\kappa$, and $\kappa\in(0,1]$. If $b>0$ and $\kappa\in(0,1]$, then

$$-1+p-pb+2b=\kappa+(1-\kappa)b>0. \tag{274}$$

Thus, (273) always holds if $r(0)$ is sufficiently large, specifically, if

$$r(0)\ge [4c_3\left(2c_2k^{-1}\right)^{p-1}]^{\frac{1}{\kappa+(1-\kappa)b}}. \tag{275}$$

Theorem 36 is proved. □

6.2 The case of continuous $F'(u)$

In this Section the Hölder continuity of ω is replaced by a weaker assumption: we only assume that $0\le\omega(r)$ is a strictly increasing continuous function and $\omega(0)=0$.

Assume that F is continuously Fréchet differentiable, $F'(u):=A(u)$, and inequality (212) holds.

Let us consider the following inequality

$$\dot g(t)\le-\gamma(t)g(t)+\alpha(t,g(t))+\beta(t),\qquad t\ge t_0,\quad \dot g=\frac{dg}{dt},\quad g\ge 0, \tag{276}$$

where $\beta(t)$ and $\gamma(t)$ are continuous functions, defined on $[t_0,\infty)$, and $0\le\alpha(t,x)$ is a nondecreasing function of x on $[0,1]$ and is continuous with respect to t on $[t_0,\infty)$.

We have the following result (see [16])

Lemma 37 *Let $\beta(t)$ and $\gamma(t)$ be continuous functions on $[t_0, \infty)$, and $0 \leq \alpha(t, x)$ be a nondecreasing function of x on $[0, 1]$ continuous with respect to t on $[t_0, \infty)$. Assume that there exists a function $\mu(t) > 0$, $\mu \in C^1[t_0, \infty)$, such that*

$$\alpha\left(t, \frac{1}{\mu(t)}\right) + \beta(t) \leq \frac{1}{\mu(t)}\left[\gamma - \frac{\dot{\mu}(t)}{\mu(t)}\right], \qquad t \geq t_0. \tag{277}$$

Let $g(t) \geq 0$ be a solution to inequality (276) such that

$$\mu(t_0)g(t_0) < 1. \tag{278}$$

Then $g(t)$ exists globally and the following estimate holds:

$$0 \leq g(t) < \frac{1}{\mu(t)}, \qquad \forall t \geq t_0. \tag{279}$$

Consequently, if $\lim_{t\to\infty} \mu(t) = \infty$, then

$$\lim_{t\to\infty} g(t) = 0. \tag{280}$$

Assume

$$0 < \rho(t) \searrow 0, \qquad \int_0^\infty \rho(s)ds = \infty, \qquad \frac{|\dot{\rho}|}{\rho} \leq C < 1, \qquad \forall t \geq 0. \tag{281}$$

Theorem 38 *Assume that conditions (213)–(215) and (281) hold. Let*

$$r(t) = \begin{cases} r_0 e^{-C_2 \int_0^t \rho(s)ds} & if \quad 0 < b \leq 1 \\ \left(r_0^{1-b} + (b-1)\int_0^t C_2\rho(s)ds\right)^{\frac{1}{1-b}} & if \quad b > 1 \end{cases}, \qquad t \geq 0, \tag{282}$$

where C_2 and r_0 are positive constants. Assume that the function $\omega(r)$ satisfies the following condition

$$\omega(\rho(t)) \leq C_1 r^b(t), \qquad t \geq 0, \qquad C_1 = const > 0. \tag{283}$$

Assume

$$c_2C_2 + c_1C_1 + C \leq 1, \qquad if \qquad b > 1, \tag{284}$$
$$c_2C_2r_0^{1-b} + c_1C_1 + C \leq 1, \qquad if \qquad 0 < b \leq 1. \tag{285}$$

Let $u(t)$ solve (217) with $a(t)$ chosen so that $r(t) = |a(t)|$. Let u_0 satisfy

$$\|u_0 - w(0)\| < \rho(0). \tag{286}$$

Then

$$\lim_{t\to\infty} u(t) = y. \tag{287}$$

Proof. It follows from (282) that

$$\dot r(t) = \begin{cases} -C_2 r(t)\rho(t) & \text{if} \quad 0 < b \le 1 \\ -C_2 r^b(t)\rho(t) & \text{if} \quad b > 1 \end{cases}, \qquad t \ge 0. \tag{288}$$

From equation (238) and (248) one gets

$$\dot z = \dot w - z + A^{-1}_{a(t)}(u(t)) \int_0^1 \big[A(tz+w) - A(u)\big] z dt, \tag{289}$$

where $z := z(t) = u(t) - w(t)$. Let $g(t) := \|z(t)\|$. From (212)–(213), (247), (289), and Lemma 35 one obtains

$$\dot g \le -g(t) + \|\dot w\| + \frac{c_1\omega(g)}{r^b(t)} g(t) \le -g(t) + \frac{c_2|\dot r(t)|}{r^b(t)} + \frac{c_1\omega(g)}{r^b(t)} g(t), \qquad t \ge 0. \tag{290}$$

Let

$$\mu(t) := \frac{1}{\rho(t)}, \qquad t \ge 0. \tag{291}$$

We claim that the following inequality holds

$$\frac{1}{\mu(t)} \frac{c_1\omega(\frac{1}{\mu(t)})}{r^b(t)} + \frac{c_2|\dot r(t)|}{r^b(t)} \le \frac{1}{\mu(t)}\left(1 - \frac{\dot\mu(t)}{\mu(t)}\right). \tag{292}$$

Let us prove this claim. From (288) and (291) one gets

$$\frac{|\dot r(t)|}{r^b(t)} = C_2 r^{1-b}(t)\rho(t) \le \frac{C_2 r_0^{1-b}}{\mu(t)}, \qquad 0 < b \le 1, \tag{293}$$

$$\frac{|\dot r(t)|}{r^b(t)} = C_2\rho(t) = \frac{C_2}{\mu(t)}, \qquad b > 1. \tag{294}$$

From (291) and (283) one obtains

$$\frac{\omega(\frac{1}{\mu(t)})}{r^b(t)} = \frac{\omega(\rho(t))}{r^b(t)} \le \frac{C_1 r^b(t)}{r^b(t)} = C_1, \qquad b > 0. \tag{295}$$

From (291) and (281) one gets

$$\frac{|\dot\mu(t)|}{\mu(t)} = \frac{|\dot\rho(t)|}{\rho(t)} \le C. \tag{296}$$

It follows from (284), (285), and (294)–(296) that

$$\frac{1}{\mu}\frac{c_1\omega(\frac{1}{\mu})}{r^b(t)} + \frac{c_2|\dot r(t)|}{r^b(t)} \le \frac{1}{\mu}(1-C) \le \frac{1}{\mu}\left(1 - \frac{\dot\mu}{\mu}\right), \qquad b > 0. \tag{297}$$

Thus, inequality (292) holds.

From Lemma 37, (286) and (292), one obtains

$$\|u(t)-w(t)\| = g(t) < \frac{1}{\mu(t)} = \rho(t), \qquad t \geq 0. \tag{298}$$

Since $\lim_{t\to\infty} \rho(t) = 0$, it follows from (298) that $\lim_{t\to\infty} \|u(t)-w(t)\| = 0$. This and the triangle inequality imply

$$\lim_{t\to\infty} \|u(t)-y\| \leq \lim_{t\to\infty} \|u(t)-w(t)\| + \lim_{t\to\infty} \|w(t)-y\| = 0. \tag{299}$$

Thus, Theorem 38 is proved. □

Remark 39 In this remark we give an example which shows that the continuity modulus $\omega(r)$ may satisfy inequality (301) but does not satisfy (218).

From (282) it follows that $r(t)$ exists and is unique for all $t > 0$ and

$$\lim_{t\to\infty} r(t) = 0. \tag{300}$$

If $b = 1$, then (282) implies that $r(t) = r(0)e^{-C_2\int_0^t \rho(s)ds}$. In this case inequality (283) becomes

$$\omega(\rho(t)) \leq r_0 e^{-C_2\int_0^t \rho(s)ds}. \tag{301}$$

Let

$$\rho(t) = \frac{1}{(e+t)\ln(e+t)}, \qquad t \geq 0, \tag{302}$$

then (281) is satisfied. One has

$$e^{-C_2\int_0^t \rho(s)ds} = e^{-C_2 \ln\ln(e+t)} = \frac{1}{\ln^{C_2}(e+t)}. \tag{303}$$

Thus, if

$$\omega\left(\frac{1}{(e+t)\ln(e+t)}\right) = \frac{C_3}{\ln^{C_2}(e+t)}, \qquad C_3 > 0, \qquad t \geq 0,$$

then inequality (301) is satisfied. One can see that for this $\omega(r)$ there does not exist $C_4 > 0$ and $\kappa > 0$ such that

$$\omega(r) \leq C_4 r^\kappa, \qquad 0 \leq r \ll 1.$$

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