# Discrete-time Symmetric/Antisymmetric FIR Filter Design

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# Outline

- Introduction
- Filter Design via Semi-infinite Programming
- Conclusions
- References
- Q&A Session

# Introduction

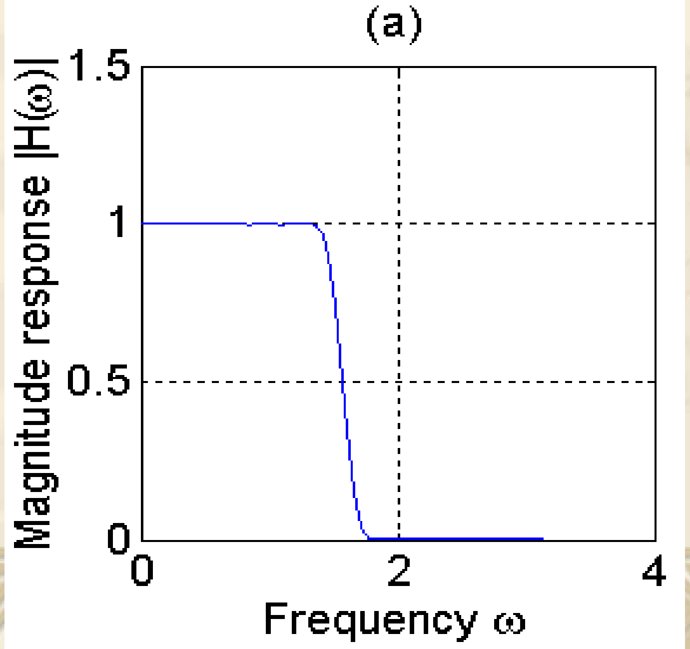
- Definition of discrete-time filters
- Types of discrete-time filters
- Common discrete-time filters
- Applications of discrete-time filters
- Common filter design techniques
- Challenges in filter design

# Introduction

- Definition of discrete-time filters
  - Complex exponent signals are eigenfunctions of discrete-time linear time invariant systems, that is $y(n)=H(\omega)e^{j\omega n}$ if $x(n)=e^{j\omega n}$ for $n\in Z$, $H(\omega)$ is called the frequency response.
  - A discrete-time filter is a discrete-time linear time invariant system characterized by its frequency response.

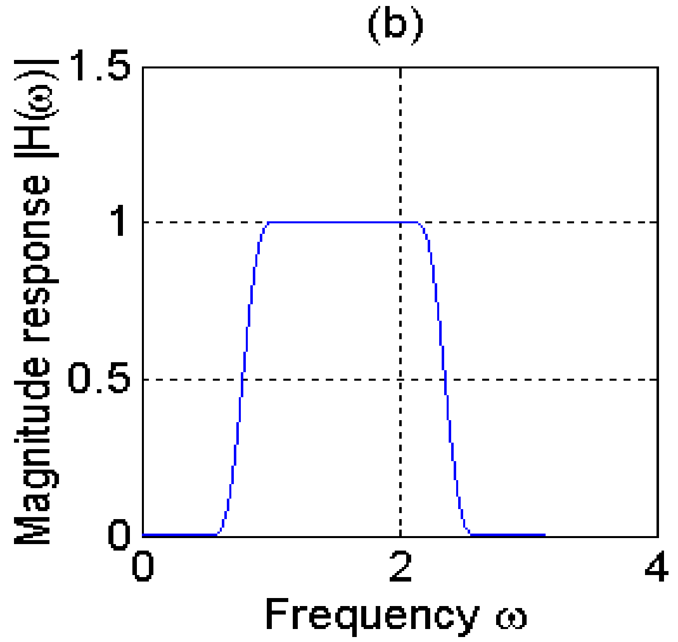
# Introduction

- Types of discrete-time filters
  - Lowpass filters
    - Allow a low frequency band of a signal to pass through and attenuate a high frequency band



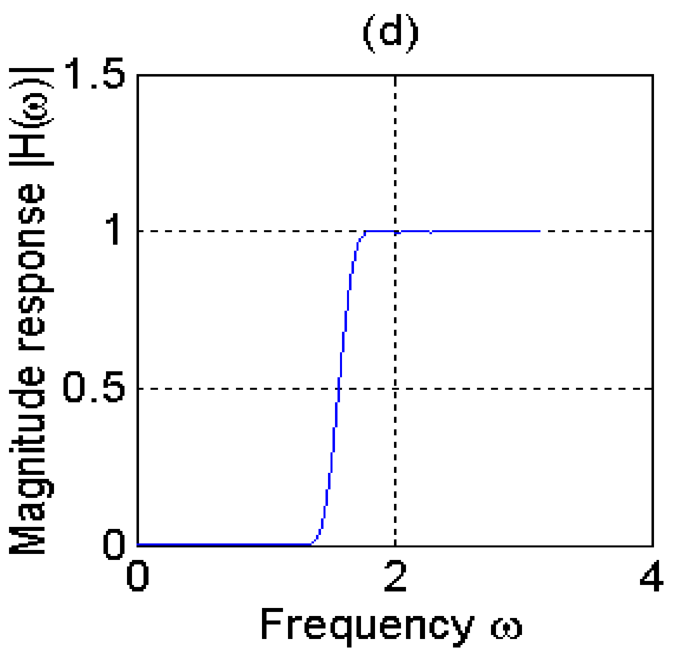
# Introduction

- Types of discrete-time filters
  - Bandpass filters
    - Allow intermediate frequency bands of a signal to pass through and attenuate both low and high frequency bands



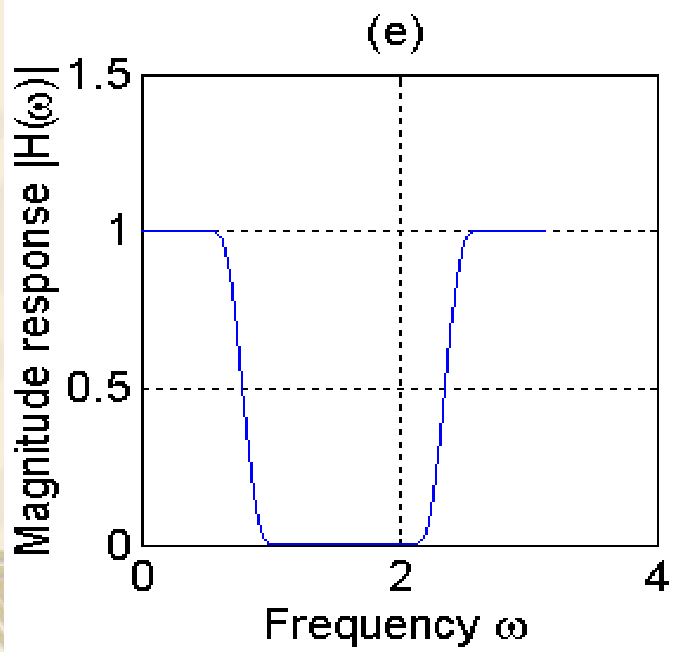
# Introduction

- Types of discrete-time filters
  - Highpass filters
    - Allow a high frequency band of a signal to pass through and attenuate a low frequency band



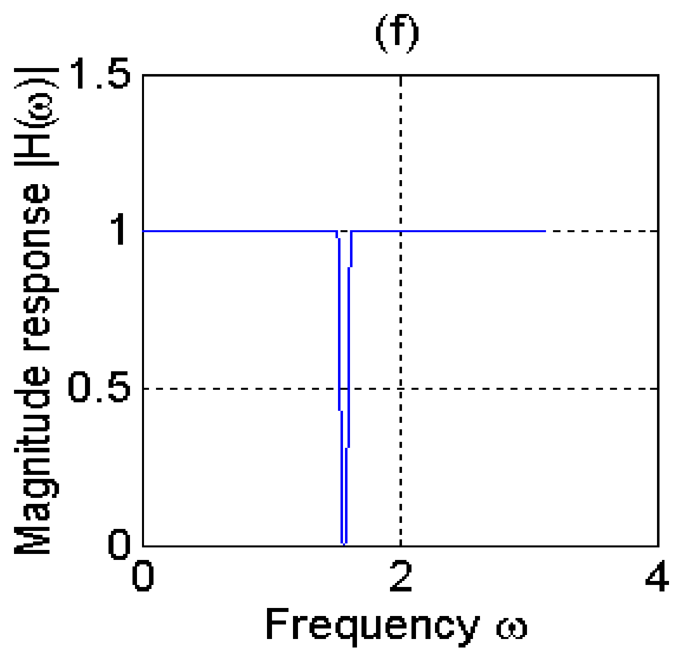
# Introduction

- Types of discrete-time filters
  - Band reject filters
    - Allow both low and high frequency bands of a signal to pass through and attenuate intermediate frequency bands



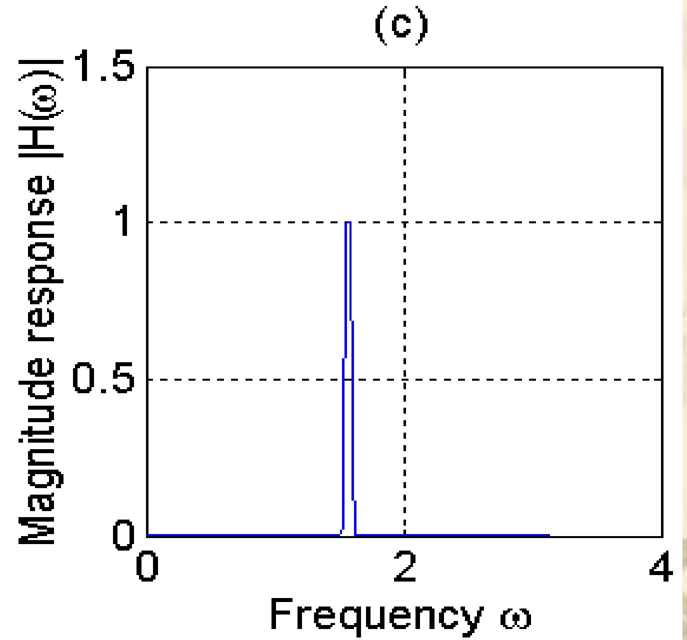
# Introduction

- Types of discrete-time filters
  - Notch filters
    - Allow almost all frequency components to pass through but attenuate particular frequencies



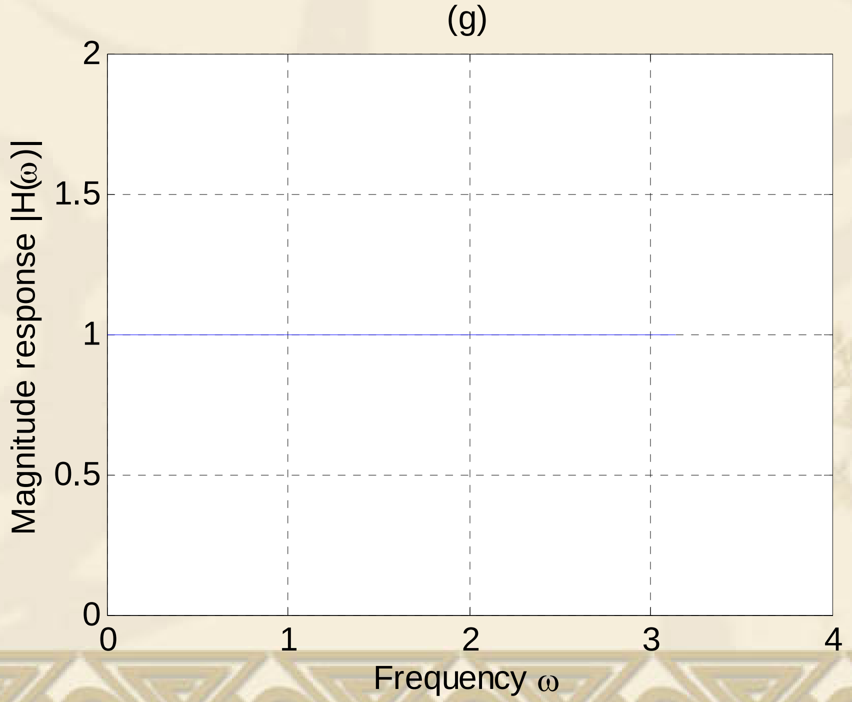
# Introduction

- Types of discrete-time filters
  - Oscillators
    - Allow particular frequency components to pass through and attenuate almost all frequency components



# Introduction

- Types of discrete-time filters
  - Allpass filters
    - Allow all frequency components to pass through



# Introduction

- Types of discrete-time filters
  - Finite impulse response (FIR) filters
    - The impulse response of the filters has finite time support.
    - Note that FIR filters are strictly bounded input bounded output stable.
  - Infinite impulse response (IIR) filters
    - The impulse response of the filters has infinite time support.
    - Note that rational IIR filters are particular types of IIR filters, but many IIR filters are irrational. For example, sinc filter is irrational IIR filter. However, rational IIR filters are easy to implement.

# Introduction

- Types of discrete-time filters
  - Linear phase filters
    - The phase response is linear.
    - Note that not all FIR filters are linear phase, but FIR filters can achieve linear phase easily.
  - Nonlinear phase filters
    - The phase response is nonlinear.
    - Note that not all IIR filters are nonlinear phase, but it is not easy to achieve good linear phase IIR filters.

# Introduction

- Common discrete-time filters
  - Differentiators
    - $H(\omega)=j\omega$ for $\omega\in(-\pi,\pi)$ and $2\pi$ periodic.
    - Note that $H(\omega)$ is discontinuous at odd multiples of $\pi$.
  - Hilbert transformers
    - $H(\omega)=\mathrm{sign}(\omega)$ for $\omega\in(-\pi,\pi)$ and $2\pi$ periodic.
    - Note that $H(\omega)$ is discontinuous at integer multiples of $\pi$.

# Introduction

- Applications of discrete-time filters
  - Lowpass and notch filters are widely used in noise reduction applications.
  - Hilbert transformers are widely used in single sideband modulation systems.
  - Differentiators are widely used in measurement systems.
  - Oscillators are widely used as sinusoidal signal generators.
  - etc…

# Introduction

- Common filter design techniques
  - Notations:
- $|H(e^{j\omega})|$:

magnitude response of a lowpass discrete-time FIR filter

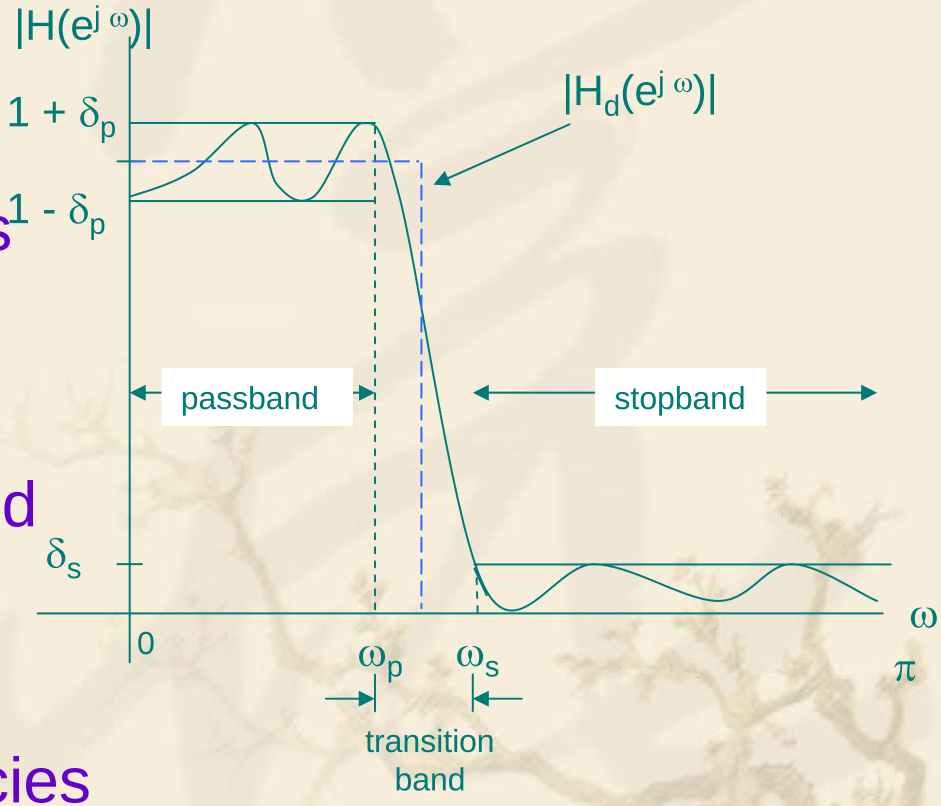
- $\delta_p$ and $\delta_s$:

maximum passband and stopband ripple magnitudes

- $\omega_p$ and $\omega_s$:

passband and stopband frequencies

- N: filter length



# Introduction

- Common filter design techniques
  - Antisymmetric impulse response

For N is odd, $h(k)=-h(N-1-k)$

for $k = 0,1,\ldots,(N-3)/2$ and $h((N-1)/2)=0$.

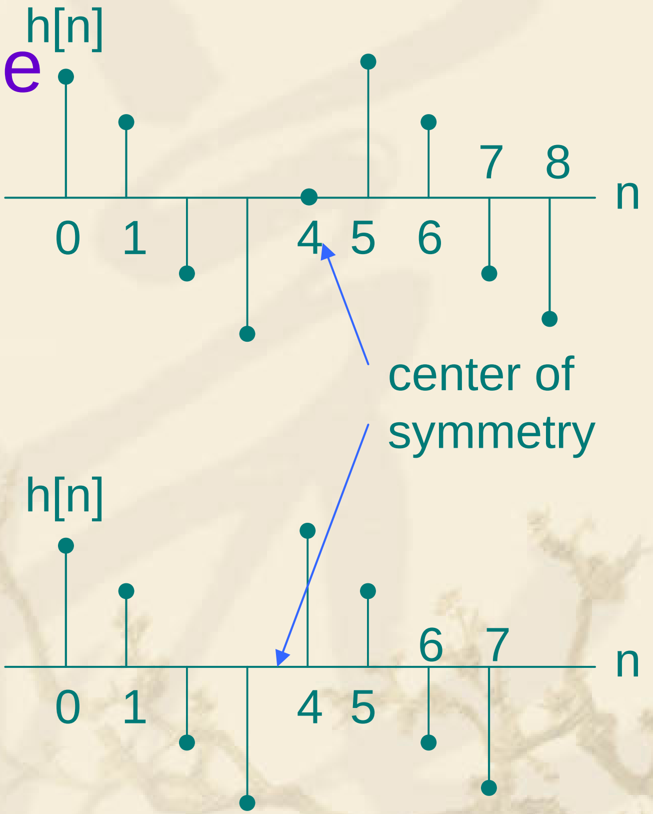
The frequency response is

$H(e^{j\omega})=je^{-j(N-1)\omega/2}H_0(\omega)$

For N is even, $h(k)=-h(N-1-k)$

for $k = 0,1,\ldots,N/2-1$.

The frequency response is $H(e^{j\omega})=je^{-j(N-1)\omega/2}H_0(\omega)$



# Introduction

- Common filter design techniques
  - Antisymmetric impulse response

$$H_o(\omega)=\begin{cases} 2\sum_{k=0}^{\frac{N-3}{2}} h[k]\sin\left(\frac{N-1-2k}{2}\omega\right) & N \text{ is odd} \\ 2\sum_{k=0}^{\frac{N}{2}-1} h[k]\sin\left(\frac{N-1-2k}{2}\omega\right) & N \text{ is even} \end{cases}$$

$$\mathbf{x}\equiv\begin{cases} \begin{bmatrix} h_0 & h_1 & \cdots & h_{\frac{N-3}{2}} \end{bmatrix}^T & N \text{ is odd} \\ \begin{bmatrix} h_0 & h_1 & \cdots & h_{\frac{N}{2}-1} \end{bmatrix}^T & N \text{ is even} \end{cases}$$

$$\boldsymbol{\eta}(\omega)\equiv\begin{cases} 2\begin{bmatrix} \sin\left(\left(\frac{N-1}{2}\right)\omega\right) & \cdots & \cdots & \sin\omega \end{bmatrix}^T & N \text{ is odd} \\ 2\begin{bmatrix} \sin\left(\left(\frac{N-1}{2}\right)\omega\right) & \cdots & \cdots & \sin\frac{\omega}{2} \end{bmatrix}^T & N \text{ is even} \end{cases}$$

$$\Rightarrow H_o(\omega)\equiv\left(\boldsymbol{\eta}(\omega)\right)^T\mathbf{x}$$

# Introduction

- Common filter design techniques
  - Symmetric impulse response

For N is odd, $h(k)=h(N-1-k)$

for $k = 0,1,\ldots,(N-3)/2$.

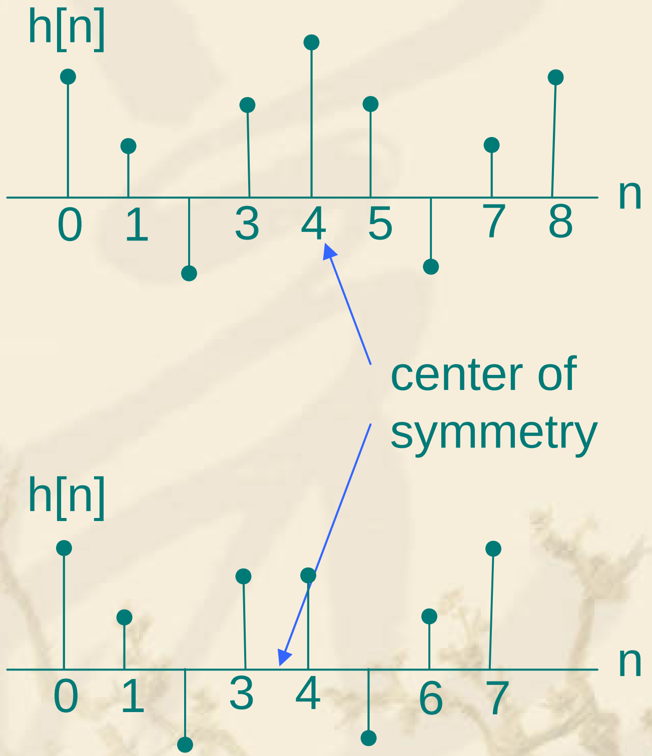
The frequency response is

$H(e^{j\omega})=e^{-j(N-1)\omega/2}H_0(\omega)$

For N is even, $h(k)=h(N-1-k)$

for $k = 0,1,\ldots,N/2-1$.

The frequency response is $H(e^{j\omega})=e^{-j(N-1)\omega/2}H_0(\omega)$



# Introduction

- Common filter design techniques
  - Symmetric impulse response

$$H_o(\omega)=\begin{cases} h\left[\dfrac{N-1}{2}\right]+2\displaystyle\sum_{k=0}^{\frac{N-3}{2}} h[k]\cos\left(\dfrac{N-1-2k}{2}\omega\right) & N \text{ is odd} \\ 2\displaystyle\sum_{k=0}^{\frac{N}{2}-1} h[k]\cos\left(\dfrac{N-1-2k}{2}\omega\right) & N \text{ is even} \end{cases}$$

$$\mathbf{x}\equiv\begin{cases} \begin{bmatrix} h_0 & h_1 & \cdots & h_{\frac{N-1}{2}} \end{bmatrix}^T & N \text{ is odd} \\ \begin{bmatrix} h_0 & h_1 & \cdots & h_{\frac{N}{2}-1} \end{bmatrix}^T & N \text{ is even} \end{cases}$$

$$\boldsymbol{\eta}(\omega)\equiv\begin{cases} \begin{bmatrix} 2\cos\left(\left(\dfrac{N-1}{2}\right)\omega\right) & \cdots & 2\cos\omega & 1 \end{bmatrix}^T & N \text{ is odd} \\ \begin{bmatrix} 2\cos\left(\left(\dfrac{N-1}{2}\right)\omega\right) & \cdots & \cdots & 2\cos\dfrac{\omega}{2} \end{bmatrix}^T & N \text{ is even} \end{cases}$$

$$\Rightarrow H_o(\omega)\equiv\left(\boldsymbol{\eta}(\omega)\right)^T\mathbf{x}$$

# Introduction

- Common filter design techniques
  - Total Weighted Ripple Energy

$$J(\mathbf{x}) \equiv \int_{B_p \cup B_s} W(\omega)\left|H_o(\omega) - D(\omega)\right|^2 d\omega$$

$$= \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{b}^T\mathbf{x} + p$$

$$\mathbf{Q} \equiv 2\int_{B_p \cup B_s} W(\omega)\boldsymbol{\eta}(\omega)\left(\boldsymbol{\eta}(\omega)\right)^T d\omega \qquad \mathbf{b} \equiv -2\int_{B_p \cup B_s} W(\omega)D(\omega)\boldsymbol{\eta}(\omega)d\omega \qquad p \equiv \int_{B_p \cup B_s} W(\omega)\left(D(\omega)\right)^2 d\omega$$

  - $J(\mathbf{x})$ : total weighted ripple energy
  - $B_p$ : $\{\omega: |\omega| \le \omega_p\}$, passband
  - $B_s$ : $\{\omega: \omega_s \le |\omega| \le \pi\}$, stopband
  - $W(\omega)$ : weighted function, $W(\omega)>0$
  - $D(\omega)$ : desired magnitude response

# Introduction

- Common filter design techniques
  - Maximum Ripple Magnitude

$$\left|H_o(\omega) - D(\omega)\right| \le \delta$$

$$\mathbf{A}(\omega) \equiv \begin{bmatrix}\boldsymbol{\eta}(\omega) & -\boldsymbol{\eta}(\omega)\end{bmatrix}^T$$

$$\mathbf{c}_\delta(\omega) \equiv -\begin{bmatrix}D(\omega)+\delta & \delta - D(\omega)\end{bmatrix}^T$$

$$\mathbf{A}(\omega)\mathbf{x} + \mathbf{c}_\delta(\omega) \le \mathbf{0}$$

  - δ: the acceptable bound of the maximum ripple magnitude of filters

# Introduction

- Common filter design techniques
  - **Problem (P**2**) :**

$$\min_{\mathbf{x}} \quad J(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{b}^T\mathbf{x} + p$$

  - $\mathbf{x}^*_2$ : optimal solution of problem (**P**2)
  - $\delta_2$ : minimum value that $\left|(\boldsymbol{\eta}(\omega))^T \mathbf{x}_2^* - D(\omega)\right| \le \delta_2 \quad \forall \omega \in B_p \cup B_s$
  - $F_{\delta 2}$: $\left\{\mathbf{x} : \left|(\boldsymbol{\eta}(\omega))^T \mathbf{x} - D(\omega)\right| \le \delta_2, \forall \omega \in B_p \cup B_s\right\}$

# Introduction

- Common filter design techniques
  - **Problem (P$^\infty$)** :

$$\min_{\mathbf{x}} \quad J_\infty(\mathbf{x}) \equiv \delta$$

Subject to: $\mathbf{g}_\delta(\mathbf{x},\omega) \equiv \mathbf{A}(\omega)\mathbf{x} + \mathbf{c}_\delta(\omega) \leq \mathbf{0} \quad \forall \omega \in B_p \bigcup B_s$

  - **x***$_\infty$ : optimal solution of problem (**P**$^\infty$)
  - $\delta_\infty$ : minimum value that $\left|(\boldsymbol{\eta}(\omega))^T \mathbf{x}^*_\infty - D(\omega)\right| \leq \delta_\infty \quad \forall \omega \in B_p \bigcup B_s$
  - $F_{\delta\infty}$ : $\left\{\mathbf{x} : \left|(\boldsymbol{\eta}(\omega))^T \mathbf{x} - D(\omega)\right| \leq \delta_\infty, \forall \omega \in B_p \bigcup B_s\right\}$

# Introduction

- Challenges in filter design
  - Although $H_2$ approach minimizes the total ripple energy, maximum ripple magnitude may be very large.
  - Although $H_\infty$ approach minimizes the maximum ripple magnitude, total ripple energy may be very large.
  - How to tradeoff between the $H_2$ approach and $H_\infty$ approach?

# Filter Design via Semi-infinite Programming

- Definition of peak constrained least square filter design
- Computer numerical simulation results of peak constrained least square filter design
- Open problems in peak constrained least square filter design
- Properties of peak constrained least square filter design
- Dual parameterization approach for solving peak constrained least square filter design problem

# Filter Design via Semi-infinite Programming

- Definition of peak constrained least square filter design

**Problem (P)** :

$$\min_{\mathbf{x}} \quad J(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{b}^T\mathbf{x} + p$$

subject to $\quad \mathbf{g}_\delta(\mathbf{x},\omega) \le \mathbf{0} \quad \forall \omega \in B_p \cup B_s$

$\mathbf{x}^*_\delta$: optimal solution of problem (**P**)

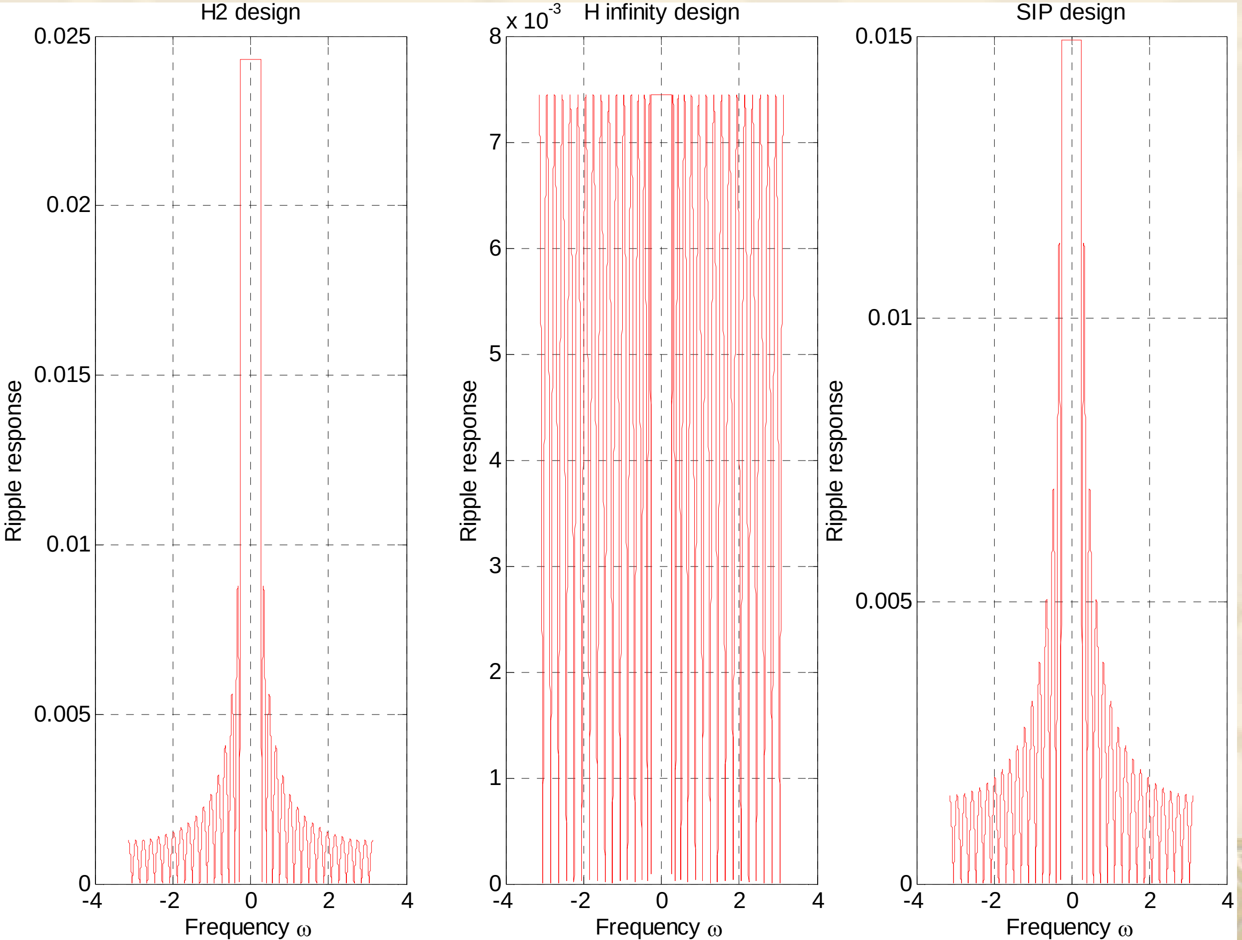
$\delta$ : the acceptable bound of the maximum ripple magnitude of filters



$F_\delta$ : $\left\{\mathbf{x} : \left|(\boldsymbol{\eta}(\omega))^T\mathbf{x} - D(\omega)\right| \le \delta, \forall \omega \in B_p \cup B_s\right\}$

# Filter Design via Semi-infinite Programming

- Computer numerical simulation results of peak constrained least square filter design



# Filter Design via Semi-infinite Programming

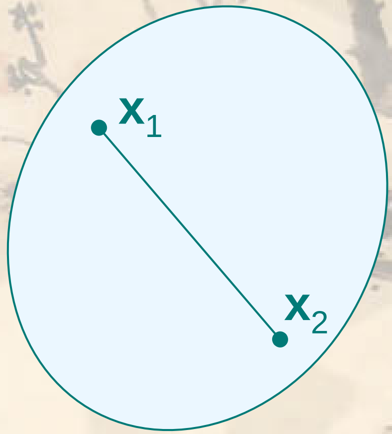
- Open problems in peak constrained least square filter design
  - How to determine the specification for peak constrained least square filter design? In particular, how to determine the value of the acceptable maximum ripple magnitude?

# Filter Design via Semi-infinite Programming

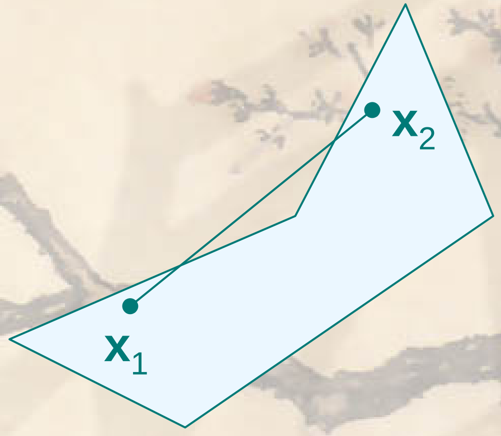
- Open problems in peak constrained least square filter design
  - $\omega$ is a continuous function, so for each frequency, say $\omega_0$, it corresponds to a single constraint. In fact, a continuous function consists of infinite number of discrete frequencies, so the problem is actually an infinite constrained optimization problem.
  - How to guarantee that these infinite number of constraints are satisfied?

# Filter Design via Semi-infinite Programming

- Properties of peak constrained least square filter design
  - Definition of convex set
    - If $\mathbf{x}_1$ and $\mathbf{x}_2$ are in $S$, then $\lambda\mathbf{x}_1+(1-\lambda)\mathbf{x}_2$ also belongs to $S$ $\forall\lambda\in[0,1]$.




(a) convex

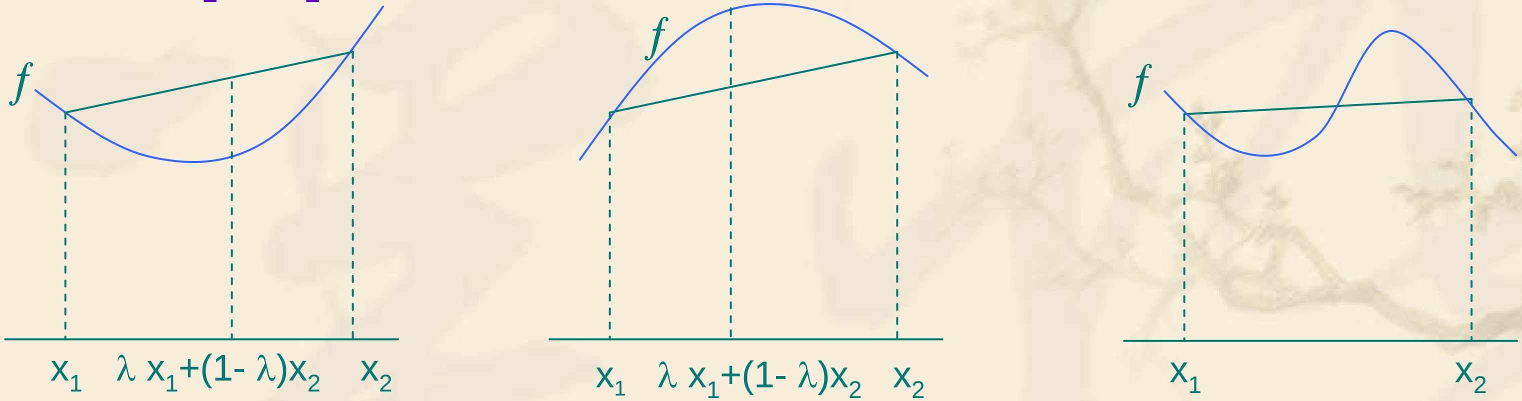



(b) not convex

# Filter Design via Semi-infinite Programming

❖Properties of Peak constrained least square filter design

- Definition of convex function:
  - ❖Let $f$: $S \rightarrow E_1$, where $S$ is a nonempty convex set in $E_n$. The function $f$ is said to be convex on $S$ if $f(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \leq \lambda f(\mathbf{x}_1) + (1-\lambda) f(\mathbf{x}_2)$ for $\forall \mathbf{x}_1, \mathbf{x}_2 \in S$ and $\forall \lambda \in [0,1]$.




convex function | concave function | neither convex nor concave

# Filter Design via Semi-infinite Programming

- Properties of peak constrained least square filter design
  - Property 1
    - The feasible set $F_\delta$ is convex.
    - Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two distinct elements of $F_\delta$, which means that $\mathrm{A}(\omega)\mathbf{x}_1+\mathbf{c}_\delta(\omega)\leq \mathbf{0}$ and $\mathrm{A}(\omega)\mathbf{x}_2+\mathbf{c}_\delta(\omega)\leq \mathbf{0}$. $\forall\lambda\in[0,1]$, since $\mathrm{A}(\omega)(\lambda\mathbf{x}_1+(1-\lambda)\mathbf{x}_2)+\mathbf{c}_\delta(\omega)\leq \mathbf{0}$, this implies that $\lambda\mathbf{x}_1+(1-\lambda)\mathbf{x}_2\in F_\delta$.

# Filter Design via Semi-infinite Programming

- Properties of peak constrained least square filter design
  - Property 2
    - The matrix **Q** is positive definite.
    - $\mathbf{x}^T\mathbf{Q}\mathbf{x} = 2\int_{B_p \cup B_s} W(\omega)\left|(\boldsymbol{\eta}(\omega))^T\mathbf{x}\right|^2 d\omega$ is nonnegative. Suppose that $\mathbf{x}^T\mathbf{Q}\mathbf{x}=0$, which implies that $(\boldsymbol{\eta}(\omega))^T\mathbf{x}=0 \ \forall\omega\in B_p \cup B_s$.
    - In particular, we have $[\boldsymbol{\eta}(\omega_1) \quad \boldsymbol{\eta}(\omega_2) \quad \cdots \quad \boldsymbol{\eta}(\omega_{N'})]^T\mathbf{x}=\mathbf{0}$ such that $rank([\boldsymbol{\eta}(\omega_1) \quad \boldsymbol{\eta}(\omega_2) \quad \cdots \quad \boldsymbol{\eta}(\omega_{N'})]) = N'$. This implies that $\mathbf{x}=0$.
    - Since $\mathbf{x}^T\mathbf{Q}\mathbf{x}>0$ for $\mathbf{x}\neq 0$, the result follows directly.

# Filter Design via Semi-infinite Programming

- Properties of peak constrained least square filter design
  - Property 3
    - The cost function $J(\mathbf{x})$ is strictly convex.
    - $J(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{b}^T\mathbf{x} + p$ is twice differentiable with respect to **x**, and its Hessian matrix is equal to **Q** which is positive definite. This implies that $J(\mathbf{x})$ is strictly convex.

# Filter Design via Semi-infinite Programming

- Properties of peak constrained least square filter design
  - Property 4
    - $\mathbf{x}^*_\delta$ is uniquely defined.
    - Let $\mathbf{x}^*_a$ and $\mathbf{x}^*_b$ be optimal solutions of the SIP problem, that is $J(\mathbf{x}^*_a)=J(\mathbf{x}^*_b)$.
    - Suppose that $\mathbf{x}^*_a \neq \mathbf{x}^*_b$, since the feasible set $F_\delta$ is convex and $J(\mathbf{x})$ is strictly convex, this implies that $\exists \lambda \mathbf{x}^*_a+(1-\lambda)\mathbf{x}^*_b \in F_\delta$ such that $J(\lambda \mathbf{x}^*_a+(1-\lambda)\mathbf{x}^*_b)<\lambda J(\mathbf{x}^*_a)+(1-\lambda)J(\mathbf{x}^*_b)=J(\mathbf{x}^*_a)=J(\mathbf{x}^*_b)$. This contradicts to the hypothesis that $\mathbf{x}^*_a$ and $\mathbf{x}^*_b$ are the optimal solutions of the SIP problem because $\lambda \mathbf{x}^*_a+(1-\lambda)\mathbf{x}^*_b$ is the optimal solution. Hence, $\mathbf{x}^*_a=\mathbf{x}^*_b$.

# Filter Design via Semi-infinite Programming

- Properties of peak constrained least square filter design
  - Property 5
    - $\mathbf{x}^*_2$ is uniquely defined.
    - Since **Q** is positive definite, all eigenvalues of **Q** are positive and $\mathbf{Q}^{-1}$exists. Consequently, $\mathbf{x}^*_2 = -\mathbf{Q}^{-1}\mathbf{b}$.

# Filter Design via Semi-infinite Programming

- Properties of peak constrained least square filter design
  - Property 6
    - $\mathbf{x}^*_\infty$ is uniquely defined.
    - By alternation theorem, $\mathbf{x}^*_\infty$ is uniquely defined.

# Filter Design via Semi-infinite Programming

- Properties of peak constrained least square filter design
  - Property 7
    - Suppose that $\mathbf{x}^*_2 \neq \mathbf{x}^*_\delta$, then $\exists \omega_0 \in B_p \cup B_s$ such that $\left| (\boldsymbol{\eta}(\omega_0))^T \mathbf{x}^*_\delta - D(\omega_0) \right| = \delta$ .
    - Since $\mathbf{x}^*_2 \neq \mathbf{x}^*_\delta$, $\mathbf{x}^*_2 \notin F_\delta$ and $F_\delta \subset F_{\delta 2}$. Otherwise $\mathbf{x}^*_2 \in F_\delta$ implies that $J(\mathbf{x}^*_2) = J(\mathbf{x}^*_\delta)$, which contradicts the uniqueness property of the solution. For $F_\delta \subset F_{\delta 2}$, $J(\mathbf{x}^*_2) \neq J(\mathbf{x}^*_\delta)$. Otherwise $\exists \lambda \in (0,1)$ such that $J(\lambda \mathbf{x}^*_2 + (1-\lambda)\mathbf{x}^*_\delta) < J(\mathbf{x}^*_\delta)$ and $J(\lambda \mathbf{x}^*_2 + (1-\lambda)\mathbf{x}^*_\delta) < J(\mathbf{x}^*_2)$, which contradicts the fact that $\mathbf{x}^*_2$ and $\mathbf{x}^*_\delta$ are the optimal solutions. Hence, $J(\mathbf{x}^*_2) < J(\mathbf{x}^*_\delta)$.

# Filter Design via Semi-infinite Programming

- Properties of peak constrained least square filter design
  - Property 7
    - Suppose that $|(\boldsymbol{\eta}(\omega_0))^T\mathbf{x}^*_\delta - D(\omega_0)| < \delta$, then $\exists\lambda\in(0,1)$ and $\exists\Delta\mathbf{x}=(1-\lambda)(\mathbf{x}^*_2-\mathbf{x}^*_\delta)$ such that $|(\boldsymbol{\eta}(\omega_0))^T(\mathbf{x}^*_\delta+\Delta\mathbf{x})-D(\omega_0)|=\delta$.
    - Since $J(\mathbf{x}^*_\delta+\Delta\mathbf{x})=J(\lambda\mathbf{x}^*_\delta+(1-\lambda)\mathbf{x}^*_2)$ and $J(\mathbf{x})$ is strictly convex, we have $J(\mathbf{x}^*_\delta+\Delta\mathbf{x})<\lambda J(\mathbf{x}^*_\delta)+(1-\lambda)J(\mathbf{x}^*_2)$. As $J(\mathbf{x}^*_2)<J(\mathbf{x}^*_\delta)$, we have $J(\mathbf{x}^*_\delta+\Delta\mathbf{x})<J(\mathbf{x}^*_\delta)$. However, this contradicts to the assumption that $\mathbf{x}^*_\delta$ is the optimal solution of the SIP problem. Hence the result follows directly.

# Filter Design via Semi-infinite Programming

- Properties of peak constrained least square filter design
  - Property 8
    - Denote $\mathbf{x}^*_a$ and $\mathbf{x}^*_b$ as the solutions of the SIP problems for $\delta=\delta_a$ and $\delta=\delta_b$, respectively. Denote $F_{\delta b}$ and $F_{\delta a}$ as the corresponding feasible sets, respectively. If $\delta_\infty<\delta_b<\delta_a<\delta_2$, then $J(\mathbf{x}^*_2)<J(\mathbf{x}^*_a)<J(\mathbf{x}^*_b)<J(\mathbf{x}^*_{\delta\infty})$ and $F_{\delta\infty}\subset F_{\delta b}\subset F_{\delta a}\subset F_{\delta 2}$.
    - $\mathbf{x}\in F_{\delta\infty}$ implies $\mathbf{A}(\omega)\mathbf{x}+D(\omega)\begin{bmatrix}-1\\1\end{bmatrix}\le\delta_\infty\begin{bmatrix}1\\1\end{bmatrix}<\delta_b\begin{bmatrix}1\\1\end{bmatrix}<\delta_a\begin{bmatrix}1\\1\end{bmatrix}<\delta_2\begin{bmatrix}1\\1\end{bmatrix}\quad\forall\omega\in B_p\bigcup B_s$.
    - This implies that $F_{\delta\infty}\subseteq F_{\delta b}\subseteq F_{\delta a}\subseteq F_{\delta 2}$ and $J(\mathbf{x}^*_2)\le J(\mathbf{x}^*_a)\le J(\mathbf{x}^*_b)\le J(\mathbf{x}^*_{\delta\infty})$.
    - Suppose that $F_{\delta\infty}=F_{\delta b}$, then $\mathbf{x}^*_b\in F_{\delta b}=F_{\delta\infty}$. $\exists\omega_0\in B_p\bigcup B_s$ such that $\left|(\boldsymbol{\eta}(\omega_0))^T\mathbf{x}_b^*-D(\omega_0)\right|=\delta_b>\delta_\infty$.
    - But this contradicts to the fact that $\mathbf{x}^*_b\in F_{\delta\infty}$. Hence, $\mathbf{x}^*_b\notin F_{\delta\infty}$ and $F_{\delta\infty}\subset F_{\delta b}$. Since the solution is uniquely defined and J($\mathbf{x}$) is strictly convex, $J(\mathbf{x}^*_b)<J(\mathbf{x}^*_\infty)$. Similarly, the result follows directly.

# Filter Design via Semi-infinite Programming

- Properties of peak constrained least square filter design
  - Property 9
    - Denote F as a map from the set of the maximum ripple magnitudes to the set of the total ripple energy of the filters. Then F(δ) is convex with respect to δ for $\delta_\infty<\delta<\delta_2$.
    - Let $\delta_a$ and $\delta_b$ be the maximum ripple magnitude such that $\delta_\infty<\delta_a<\delta_b<\delta_2$. Let $\mathbf{x}^*_a$ and $\mathbf{x}^*_b$ be the solutions of the SIP problems corresponding to $\delta_a$ and $\delta_b$, respectively. Also, let $F_{\delta a}$ and $F_{\delta b}$ be the corresponding feasible sets, respectively. Since, $\mathbf{x}^*_a \in F_{\delta a}$ and $\mathbf{x}^*_b \in F_{\delta b}$, we have $\left|(\eta(\omega))^T x_a^* - D(\omega)\right| \le \delta_a \ \forall \omega \in B_p \cup B_s$ and $\left|(\boldsymbol{\eta}(\omega))^T \mathbf{x}_b^* - D(\omega)\right| \le \delta_b \ \forall \omega \in B_p \cup B_s$.
    - Hence $\forall \lambda \in (0,1) \ \lambda A(\omega) x_a^* + \lambda D(\omega)\begin{bmatrix}-1\\1\end{bmatrix} \le \lambda\delta_a\begin{bmatrix}1\\1\end{bmatrix}$ and

# Filter Design via Semi-infinite Programming

- Properties of peak constrained least square filter design
  - Property 9

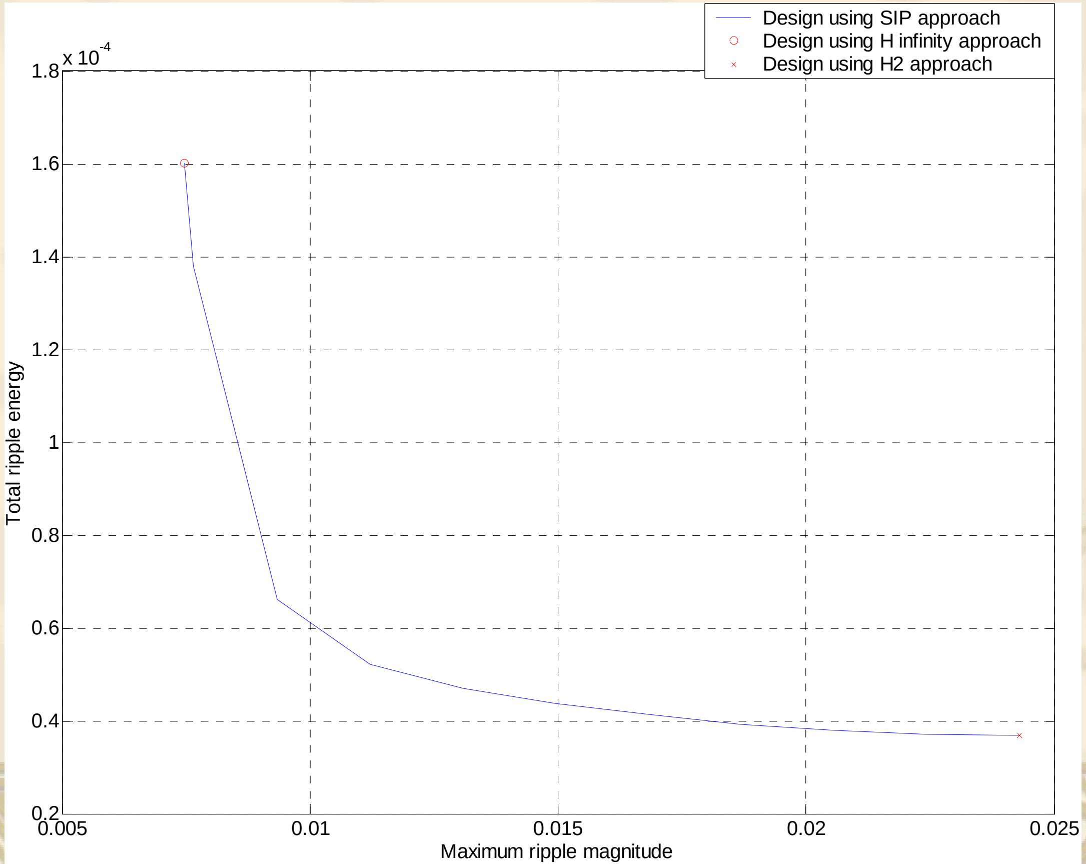
$$(1-\lambda)A(\omega)x_b^* + (1-\lambda)D(\omega)\begin{bmatrix}-1\\1\end{bmatrix} \le (1-\lambda)\delta_b\begin{bmatrix}1\\1\end{bmatrix} \quad \forall\omega\in B_p\cup B_s$$

It follows that $A(\omega)\left[\lambda x_a^* + (1-\lambda)x_b^*\right] + D(\omega)\begin{bmatrix}-1\\1\end{bmatrix} \le \left(\lambda\delta_a + (1-\lambda)\delta_b\right)\begin{bmatrix}1\\1\end{bmatrix} \forall\omega\in B_p\cup B_s$

- Denote $F_{\lambda\delta a+(1-\lambda)\delta b}$ as the feasible set corresponding to the maximum ripple magnitude equal to $\lambda\delta_a+(1-\lambda)\delta_b$. Then $\lambda\mathbf{x}^*_a+(1-\lambda)\mathbf{x}^*_b\in F_{\lambda\delta a+(1-\lambda)\delta b}$. Denote $x^*_{\lambda\delta a+(1-\lambda)\delta b}$ as the solution of the SIP problem corresponding to the maximum ripple magnitude equal to $\lambda\delta_a+(1-\lambda)\delta_b$. Then $J(x^*_{\lambda\delta a+(1-\lambda)\delta b})\le J(\lambda\mathbf{x}^*_a+(1-\lambda)\mathbf{x}^*_b)$. Since $J$(x) is strictly convex, we have $J(x^*_{\lambda\delta a+(1-\lambda)\delta b})<\lambda\mathbf{J}(\mathbf{x}^*_a)+(1-\lambda)\mathbf{J}(\mathbf{x}^*_b)$. Hence, F(δ) is convex with respect to δ.

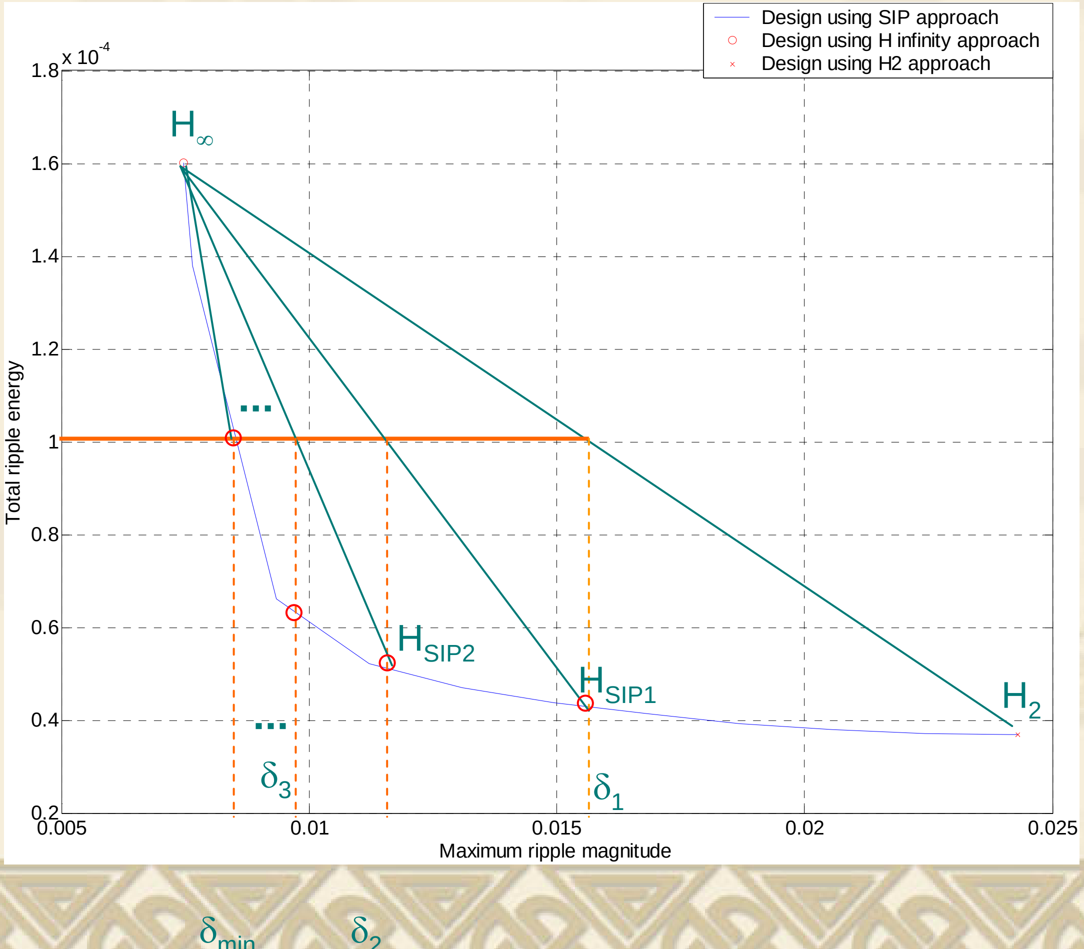
# Filter Design via Semi-infinite Programming

- Properties of peak constrained least square filter design



# Filter Design via Semi-infinite Programming

- Properties of peak constrained least square filter design




- monotonic decreasing
- convex

# Filter Design via Semi-infinite Programming

- Dual parameterization approach for solving peak constrained least square filter design problem
  - The magnitude response contains finite number of maxima and minima.
  - If the constraints are satisfied in these extrema, then all constraints are satisfied.

# Filter Design via Semi-infinite Programming

- Dual parameterization approach for solving peak constrained least square filter design problem
  - However, the locations of these extrema are unknown. Hence, we optimize both the filter coefficients and finite number of frequencies so that the cost function is minimized and the constraints are satisfied.

# Conclusions

- Filters are designed via peak constrained least square approach and the problem can be solved via a dual parameterization approach.
- The plot of the total ripple energy against the maximum ripple magnitude is monotonic decreasing and convex, this information helps to determine the specifications for filter design.

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# Q&A Session

Thank you!

Let me think…