Nonlinear Behaviors of Digital Filters

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Introduction

- Developments and applications of digital filters
- Motivation of the research
- Model of second-order digital filters with two's complement arithmetic
- Stability analysis of the corresponding linear systems
- Effects of different initial conditions when the eigenvalues of system matrix are on the unit circle

Introduction

- Developments and applications of digital filters
  - Developments of digital filters
    - Wagner and Campbell design electric wave filters in 1915.
    - Nyquist and Gabor in 1928 and 1946 point out that a continuous-time signal can be represented by a finite number of discrete points.

      $\Rightarrow$Continuous-time filters can be approximated via digital filters.

      $\Rightarrow$Digital filters are studied extensively and found many applications in industry.

Introduction

- Applications of digital filters
  - Filtering in digital telephone networks, denoising systems, detection systems, compression standards, etc.
  - Data processing, such as time series analysis, numerical analysis, etc.

Introduction

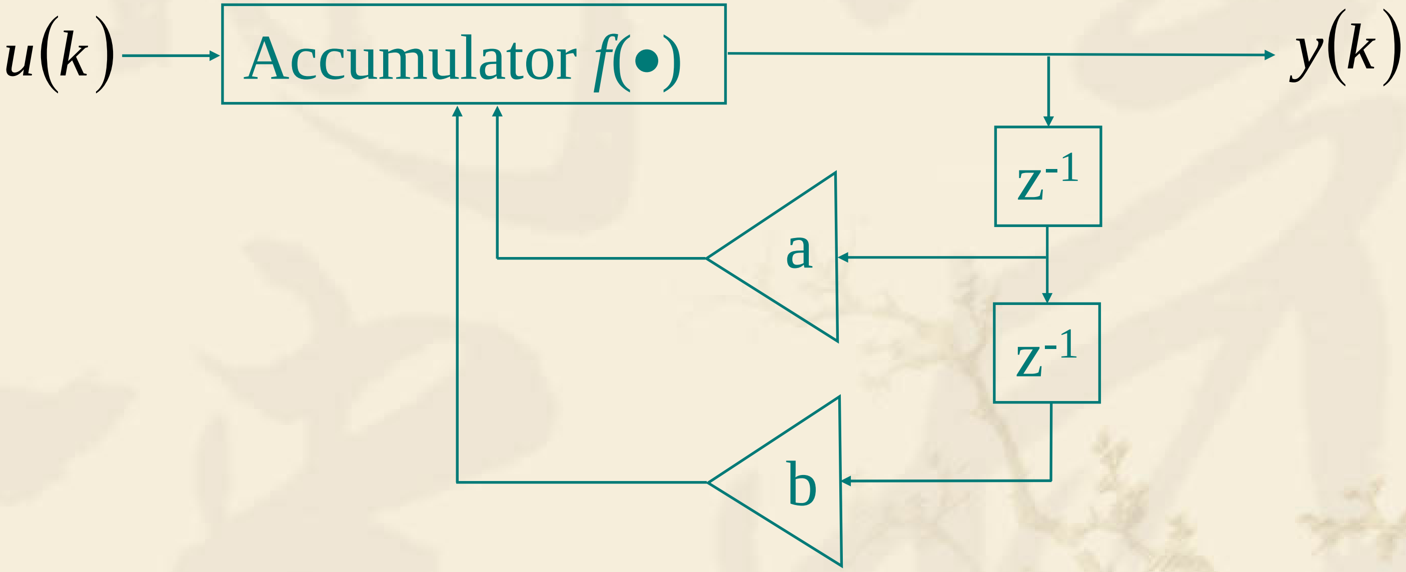
- Motivation of the research
  - Due to the two's complement arithmetic, the digital filters are nonlinear systems. Parker and Hess reported in 1971 that limit cycle behaviors would be exhibited in the digital filters. Chua reported in 1988 that a fractal geometry may occur on the phase portrait.
  - Since the second-order digital filters are widely applied in industry, we have to know the conditions for the occurrence of those nonlinear behaviors so that we can avoid the occurrence of those behaviors or make some useful applications using these nonlinear behaviors.

Introduction

- Model of second-order digital filters with two's complement arithmetic
  - Second-order digital filters
    - Fundamental building block of cascade and parallel realizations of arbitrary digital filters.
  - Two's complement arithmetic
    - Common in most of digital devices because subtraction of two numbers is equivalent to the addition of these two numbers in their two's complement forms.
  - Direct form
    - One of the simplest configuration for realizing the second-order digital filter which uses the least number of multipliers and adders.

Introduction

- Hardware schematic



Introduction

- When no input is present, the filter can be described by the following nonlinear state-space difference equation:

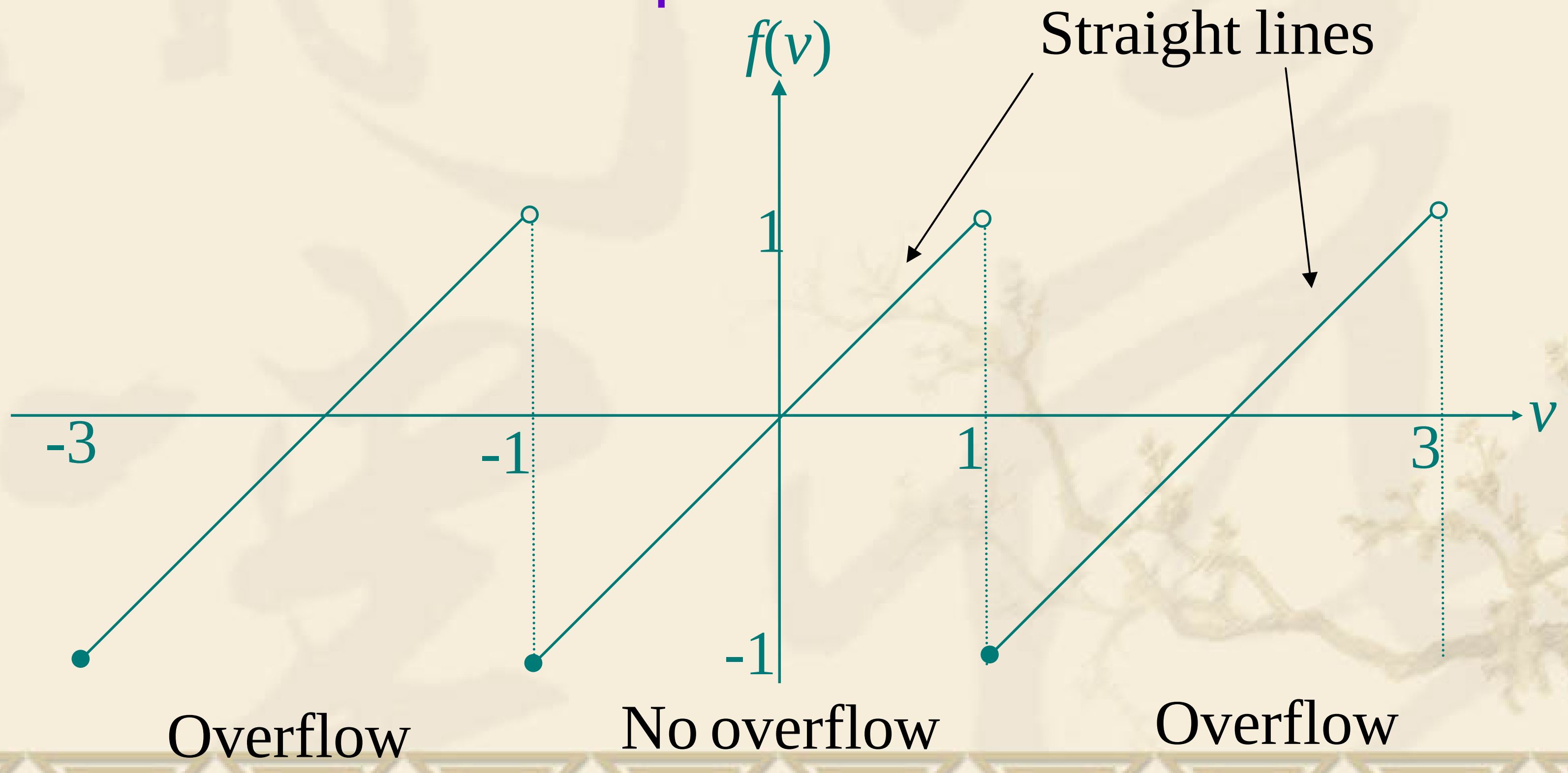
$$\mathbf{x}(k+1) \equiv \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \mathrm{F}(\mathbf{x}(k))$$

Direct form

$$= \left(\begin{bmatrix} x_2(k) \\ f\left(b \cdot x_1(k) + a \cdot x_2(k)\right) \end{bmatrix} \right)$$

Introduction

where $f(\bullet)$ is the nonlinear function associated with the two's complement arithmetic



Introduction

$$\mathbf{x}(k+1) = \mathbf{A} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cdot s(k)$$

where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix}$

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \in I^2 \equiv \{(x_1, x_2) : -1 \le x_1 < 1, -1 \le x_2 < 1\}$$

$$s(k) \in \{-m, \cdots, -1, 0, 1, \cdots, m\}$$

and *m* is the minimum integer satisfying

$$-2 \cdot m - 1 \le b \cdot x_1 + a \cdot x_2 \le 2 \cdot m + 1$$

Introduction

$s(k)$ is called symbolic sequences.

$$s(k)=1 \Rightarrow -3 \leq b \cdot x_1(k)+a \cdot x_2(k)<-1$$
$$s(k)=0 \Rightarrow -1 \leq b \cdot x_1(k)+a \cdot x_2(k)<1$$
$$s(k)=-1 \Rightarrow 1 \leq b \cdot x_1(k)+a \cdot x_2(k)<3$$

Introduction

For example: $\mathbf{x}(0) = [-0.6135 \quad 0.6135]^T, b = -1$ and $a = 0.5$

$$b \cdot x_1(0) + a \cdot x_2(0) = 0.9203 \Rightarrow s(0) = 0$$

$$b \cdot x_1(1) + a \cdot x_2(1) = -0.1534 \Rightarrow s(1) = 0$$

Two's complement value

$$\vdots$$

$$b \cdot x_1(14) + a \cdot x_2(14) = 1.0018 \Rightarrow s(14) = -1 \text{ and } x_2(15) = -0.9982$$

$$b \cdot x_1(15) + a \cdot x_2(15) = -0.7597 \Rightarrow s(15) = 0$$

$$b \cdot x_1(16) + a \cdot x_2(16) = 0.6184 \Rightarrow s(16) = 0$$

$$b \cdot x_1(17) + a \cdot x_2(17) = 1.0689 \Rightarrow s(17) = -1 \text{ and } x_2(18) = -0.9311$$

$$b \cdot x_1(18) + a \cdot x_2(18) = -1.0839 \Rightarrow s(18) = 1 \text{ and } x_2(19) = 0.9161$$

$$\vdots$$

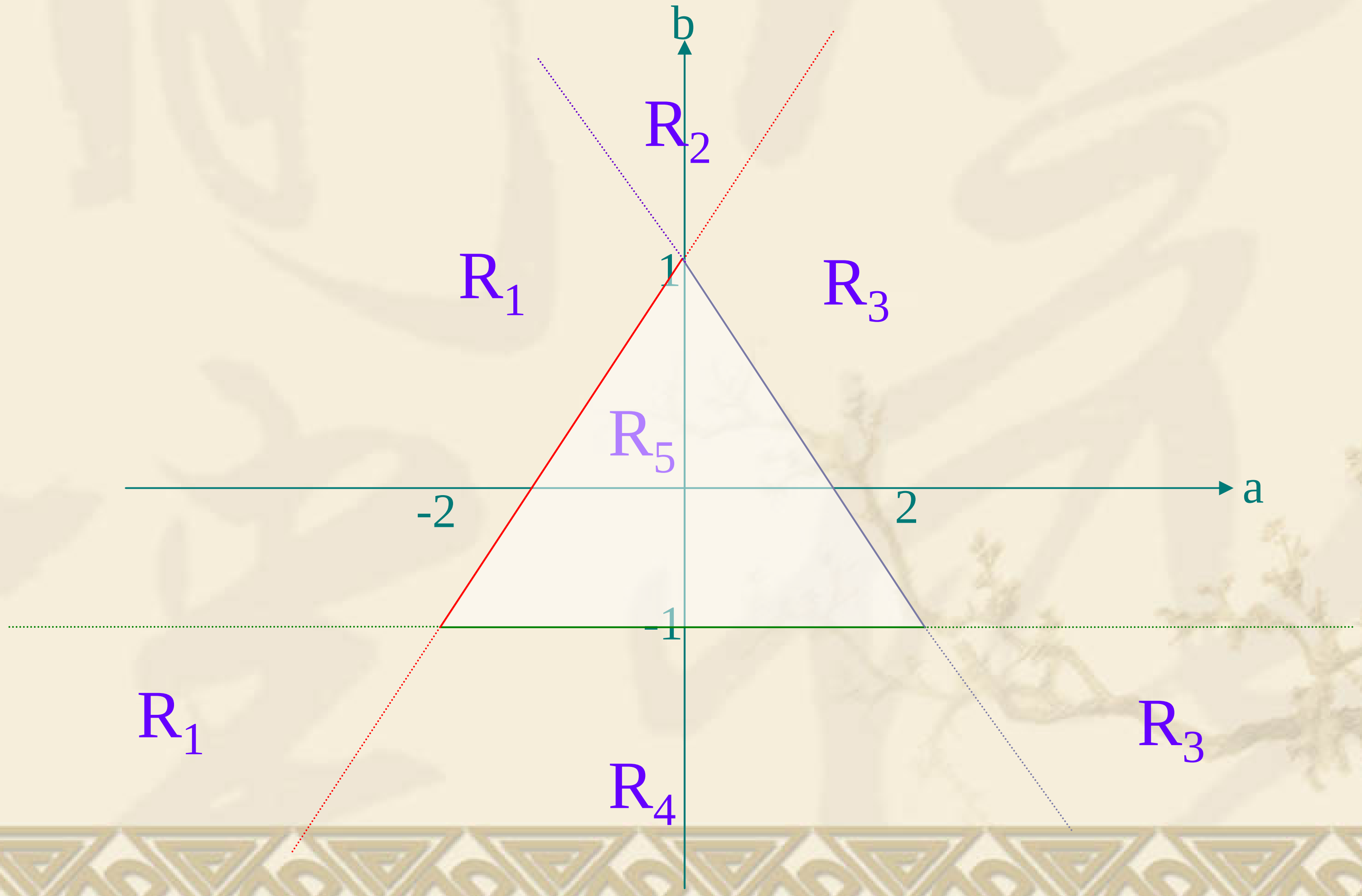
$$s = (0 \quad 0 \quad \cdots \quad 0 \quad -1 \quad 0 \quad 0 \quad -1 \quad 1 \quad \cdots)$$

Introduction

- Stability analysis of the corresponding linear system
  - Eigenvalues of **A**

$$\lambda = \frac{a \pm \sqrt{a^2 + 4 \cdot b}}{2}$$

Introduction



Introduction

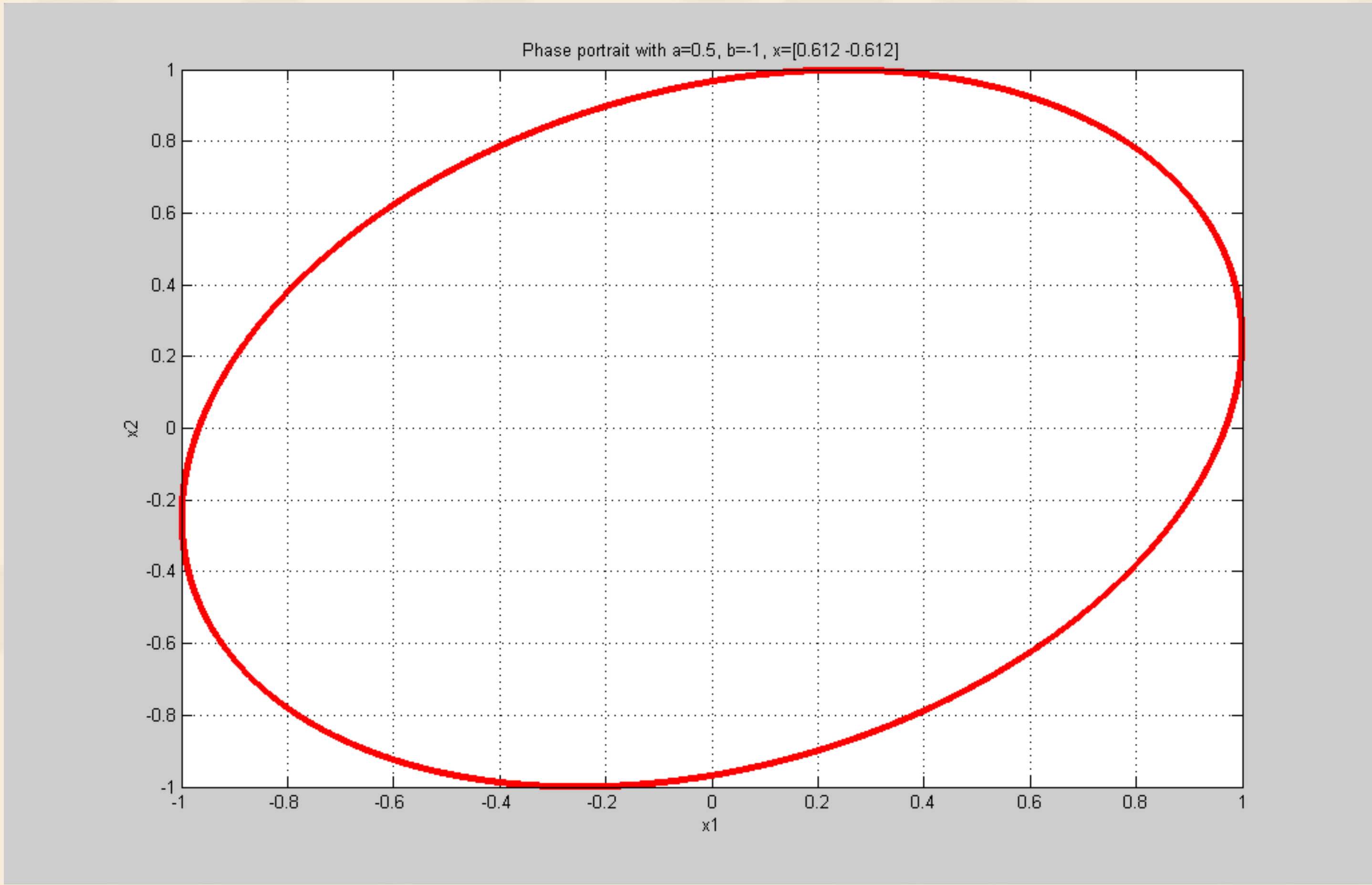
- **R_5** :Magnitudes of eigenvalues <1.
  - $\Rightarrow$The corresponding linear system is stable.
- **R_1** and **R_3** : One eigenvalue's magnitude < 1, while the other's magnitude > 1.
  - $\Rightarrow$The corresponding linear system is unstable.
- **R_2** and **R_4** :Magnitudes of eigenvalues are greater than 1.
  - $\Rightarrow$The corresponding linear system is unstable.

Introduction

- Effects of different initial conditions when the eigenvalues of system matrix are on the unit circle
  - Type I trajectory
    - There is a single rotated and translated ellipse in the phase portrait.

$$a = 0.5, b = -1, x(0) = \begin{bmatrix} 0.612 \\ -0.612 \end{bmatrix}$$

Introduction

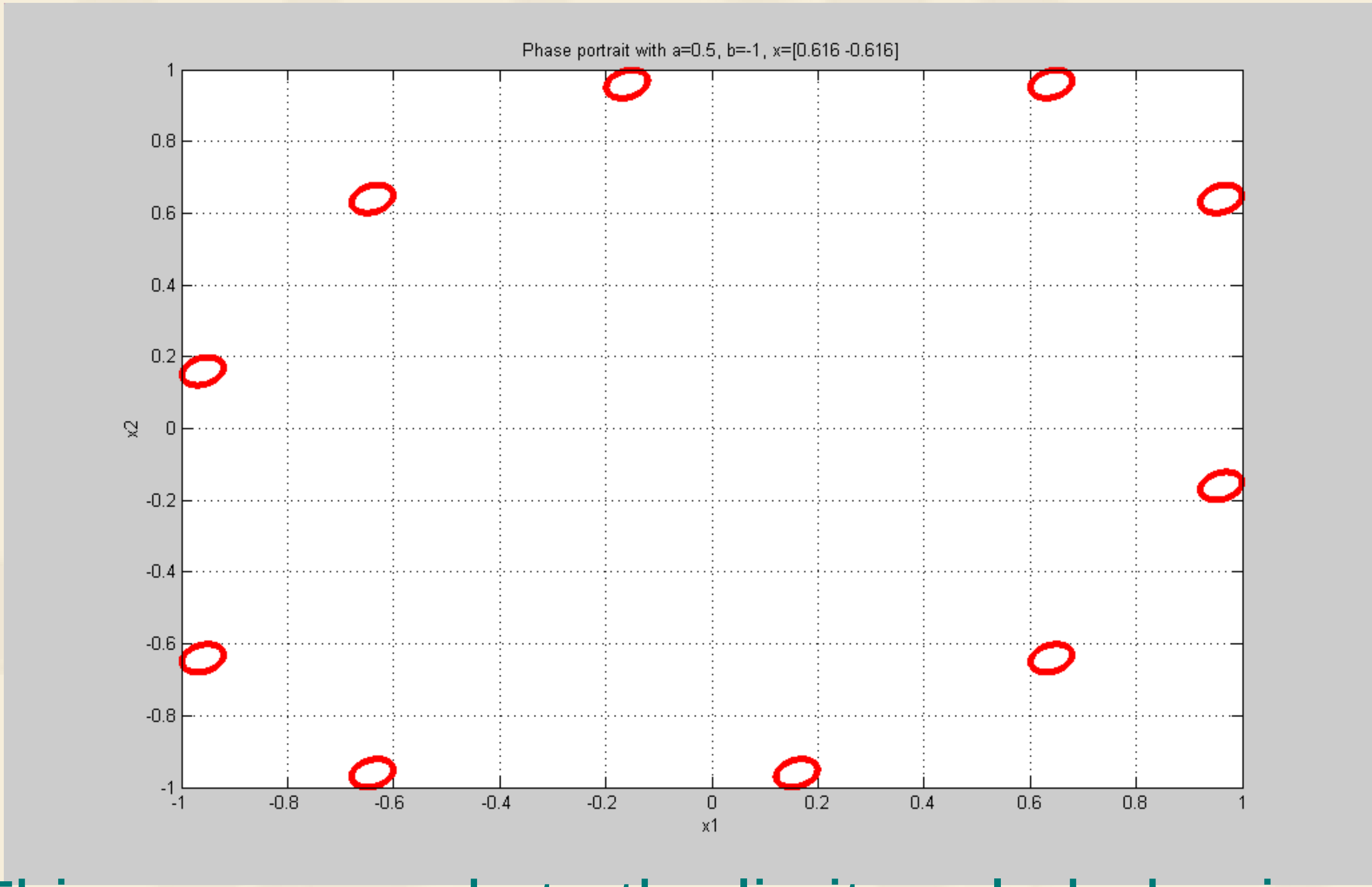


Introduction

- Type II trajectory
  - There are some rotated and translated ellipses in the phase portrait.

$$a = 0.5, b = -1, x(0) = \begin{bmatrix} 0.616 \\ -0.616 \end{bmatrix}$$

Introduction



This corresponds to the limit cycle behavior.

Introduction

- Type III trajectory
  - There is a fractal pattern on the phase portrait.

$$a = 0.5, b = -1, x(0) = \begin{bmatrix} 0.6135 \\ -0.6135 \end{bmatrix}$$

Introduction

- Conclusion: Very sensitive to initial conditions

Step response of second-order digital filters with two's complement arithmetic

- What are the behaviors of the digital filters with two's complement arithmetic when some step input is applied?
- What are the properties of the corresponding symbolic sequences?
- What are the corresponding sets of initial conditions?

Step response of second-order digital filters with two's complement arithmetic

The system can be represented as:

$$\mathbf{x}(k+1) = \left(\begin{bmatrix} x_2(k) \\ f\left(b \cdot x_1(k) + a \cdot x_2(k) + u(k)\right) \end{bmatrix} \right)$$

$$= \mathbf{A} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \mathbf{B} \cdot u(k) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cdot s(k)$$

where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Step response of second-order digital filters with two's complement arithmetic

Let $u(k) = c$ for $k \geq 0$

$$\cos\theta = \frac{a}{2}$$

$$\mathbf{T} \equiv \begin{bmatrix} 1 & 0 \\ \cos\theta & \sin\theta \end{bmatrix}$$

$$\hat{\mathbf{A}} \equiv \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{x}^* = \frac{c}{2-a} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\hat{\mathbf{x}}(k) = \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix} = \mathbf{T}^{-1} \cdot \left(\mathbf{x}(k) - \frac{c + 2 \cdot s_0}{c} \cdot \mathbf{x}^* \right)$$

Step response of second-order digital filters with two's complement arithmetic

where $s_0 \equiv s(0)$

then $\mathbf{A} = \mathbf{T} \cdot \hat{\mathbf{A}} \cdot \mathbf{T}^{-1}$ and $\mathbf{B} = \dfrac{(\mathbf{I} - \mathbf{A}) \cdot \mathbf{x}^*}{c}$

If $s(k) = s_0$ for $k \geq 0$

then $\mathbf{x}(k+1) = \mathbf{A} \cdot \mathbf{x}(k) + (c + 2 \cdot s_0) \cdot \mathbf{B}$

Since $\hat{\mathbf{x}}(k+1) = \mathbf{T}^{-1} \cdot \left(\mathbf{x}(k+1) - \dfrac{(c + 2 \cdot s_0)}{c} \cdot \mathbf{x}^* \right)$

$$\Rightarrow \hat{\mathbf{x}}(k+1) = \mathbf{T}^{-1} \cdot \left(\mathbf{A} \cdot \mathbf{x}(k) + (c + 2 \cdot s_0) \cdot \mathbf{B} - \frac{(c + 2 \cdot s_0)}{c} \cdot \mathbf{x}^* \right)$$

$$\hat{\mathbf{x}}(k+1) = \mathbf{T}^{-1} \cdot \mathbf{A} \cdot \mathbf{x}(k) + \mathbf{T}^{-1} \cdot (c + 2 \cdot s_0) \cdot \mathbf{B} - \mathbf{T}^{-1} \cdot \frac{(c + 2 \cdot s_0)}{c} \cdot \mathbf{x}^*$$

Step response of second-order digital filters with two's complement arithmetic

$$\hat{\mathbf{x}}(k+1)=\mathbf{T}^{-1}\cdot\mathbf{T}\cdot\hat{\mathbf{A}}\cdot\mathbf{T}^{-1}\cdot\mathbf{x}(k)+\mathbf{T}^{-1}\cdot(c+2\cdot s_0)\cdot\frac{(\mathbf{I}-\mathbf{A})\cdot\mathbf{x}^*}{c}-\mathbf{T}^{-1}\cdot\frac{(c+2\cdot s_0)}{c}\cdot\mathbf{x}^*$$

$$\hat{\mathbf{x}}(k+1)=\hat{\mathbf{A}}\cdot\mathbf{T}^{-1}\cdot\mathbf{x}(k)-\mathbf{T}^{-1}\cdot\frac{(c+2\cdot s_0)}{c}\cdot\mathbf{A}\cdot\mathbf{x}^*$$

$$\hat{\mathbf{x}}(k+1)=\hat{\mathbf{A}}\cdot\mathbf{T}^{-1}\cdot\left(\mathbf{T}\cdot\hat{\mathbf{x}}(k)+\frac{(c+2\cdot s_0)}{c}\cdot\mathbf{x}^*\right)-\mathbf{T}^{-1}\cdot\frac{(c+2\cdot s_0)}{c}\cdot\mathbf{T}\cdot\hat{\mathbf{A}}\cdot\mathbf{T}^{-1}\cdot\mathbf{x}^*$$

Hence, we have $\boxed{\hat{\mathbf{x}}(k+1)=\hat{\mathbf{A}}\cdot\hat{\mathbf{x}}(k)}$

Circle

Step response of second-order digital filters with two's complement arithmetic

Similarly, if $\hat{\mathbf{x}}(k+1)=\hat{\mathbf{A}}\cdot\hat{\mathbf{x}}(k)$

then $\mathbf{T}^{-1}\cdot\left(\mathbf{x}(k+1)-\dfrac{(c+2\cdot s_0)}{c}\cdot\mathbf{x}^*\right)=\hat{\mathbf{A}}\cdot\mathbf{T}^{-1}\cdot\left(\mathbf{x}(k)-\dfrac{(c+2\cdot s_0)}{c}\cdot\mathbf{x}^*\right)$

$$\mathbf{x}(k+1)-\frac{(c+2\cdot s_0)}{c}\cdot\mathbf{x}^*=\mathbf{T}\cdot\hat{\mathbf{A}}\cdot\mathbf{T}^{-1}\cdot\left(\mathbf{x}(k)-\frac{(c+2\cdot s_0)}{c}\cdot\mathbf{x}^*\right)$$

$$\mathbf{x}(k+1)=\mathbf{A}\cdot\left(\mathbf{x}(k)-\frac{(c+2\cdot s_0)}{c}\cdot\mathbf{x}^*\right)+\frac{(c+2\cdot s_0)}{c}\cdot\mathbf{x}^*$$

$$\mathbf{x}(k+1)=\mathbf{A}\cdot\mathbf{x}(k)+(c+2\cdot s_0)\cdot\frac{(\mathbf{I}-\mathbf{A})\cdot\mathbf{x}^*}{c}$$

$$\Rightarrow\mathbf{x}(k+1)=\mathbf{A}\cdot\mathbf{x}(k)+\mathbf{B}\cdot c+\begin{bmatrix}0\\2\end{bmatrix}\cdot s_0$$

Step response of second-order digital filters with two's complement arithmetic

Hence, we have $\boxed{s(k) = s_0 \text{ for } k \geq 0}$

Constant symbolic sequences

$$\boxed{\hat{\mathbf{x}}(k+1) = \hat{\mathbf{A}} \cdot \hat{\mathbf{x}}(k) \text{ if and only if } s(k) = s_0 \text{ for } k \geq 0}$$

If $s(k) = s_0 \neq 0$, overflow does occur. But there is still an ellipse exhibited on the phase portrait. Hence, we cannot conclude whether overflow occurs or not just by looking an ellipse on the phase portrait.

Step response of second-order digital filters with two's complement arithmetic

if $s(k)=s_0$ for $k\geq 0$, we have $\hat{\mathbf{x}}(k+1)=\hat{\mathbf{A}}\cdot\hat{\mathbf{x}}(k)$ for $k\geq 0$

Since the phase portrait of $\hat{\mathbf{x}}(k+1)=\hat{\mathbf{A}}\cdot\hat{\mathbf{x}}(k)$ is a circle with radius

$$\left\|\hat{\mathbf{x}}(k)\right\|_2=\left\|\mathbf{T}^{-1}\cdot\left(\mathbf{x}(k)-\frac{(c+2\cdot s_0)}{c}\cdot\mathbf{x}^*\right)\right\|_2$$

$$=\left\|\mathbf{T}^{-1}\cdot\left(\mathbf{x}(0)-\frac{(c+2\cdot s_0)}{c}\cdot\mathbf{x}^*\right)\right\|_2$$

We have $\hat{\mathbf{x}}(k)=\left\|\mathbf{T}^{-1}\cdot\left(\mathbf{x}(0)-\frac{(c+2\cdot s_0)}{c}\cdot\mathbf{x}^*\right)\right\|_2\cdot\begin{bmatrix}\cos(\phi(k))\\ \sin(\phi(k))\end{bmatrix}$

Step response of second-order digital filters with two's complement arithmetic

Since $\mathbf{x}(k)=\mathbf{T}\cdot\hat{\mathbf{x}}(k)+\frac{c+2\cdot s_0}{c}\cdot\mathbf{x}^*$

we have

$$\mathbf{x}(k)=\begin{bmatrix}1 & 0\\ \cos\theta & \sin\theta\end{bmatrix}\cdot\left\|\mathbf{T}^{-1}\cdot\left(\mathbf{x}(0)-\frac{(c+2\cdot s_0)}{c}\cdot\mathbf{x}^*\right)\right\|_2\cdot\begin{bmatrix}\cos(\phi(k))\\ \sin(\phi(k))\end{bmatrix}$$

$$+\frac{c+2\cdot s_0}{c}\cdot\frac{c}{2-a}\cdot\begin{bmatrix}1\\1\end{bmatrix}$$

$$\mathbf{x}(k)=\left\|\mathbf{T}^{-1}\cdot\left(\mathbf{x}(0)-\frac{(c+2\cdot s_0)}{c}\cdot\mathbf{x}^*\right)\right\|_2\cdot\begin{bmatrix}\cos(\phi(k))\\ \cos(\theta-\phi(k))\end{bmatrix}+\frac{c+2\cdot s_0}{2-a}\cdot\begin{bmatrix}1\\1\end{bmatrix}$$

Step response of second-order digital filters with two's complement arithmetic

Since $\mathbf{x}(k) \in I^2$, we have $-1 \le x_1(k) < 1$ and $-1 \le x_2(k) < 1$

Hence $-1 \le \left| \left\| \mathbf{T}^{-1} \cdot \left(\mathbf{x}(0) - \frac{(c + 2 \cdot s_0)}{c} \cdot \mathbf{x}^* \right) \right\|_2 \cdot \cos(\phi(k)) + \frac{c + 2 \cdot s_0}{2 - a} \right| < 1$

$$\Rightarrow \left\| \mathbf{T}^{-1} \cdot \left(\mathbf{x}(0) - \frac{(c + 2 \cdot s_0)}{c} \cdot \mathbf{x}^* \right) \right\|_2 \le 1 - \frac{|c + 2 \cdot s_0|}{2 - a}$$

$$\boxed{\mathbf{x}(0) \in \left\{ \mathbf{x}(0) : \left\| \mathbf{T}^{-1} \cdot \left(\mathbf{x}(0) - \frac{(c + 2 \cdot s_0)}{c} \cdot \mathbf{x}^* \right) \right\|_2 \le 1 - \frac{|c + 2 \cdot s_0|}{2 - a} \right\}}$$

Sets of initial conditions

Step response of second-order digital filters with two's complement arithmetic

$$\hat{\mathbf{x}}(k)=\mathbf{T}^{-1}\cdot\left(\mathbf{x}(k)-\frac{(c+2\cdot s_0)}{c}\cdot\mathbf{x}^*\right)$$

$$\Rightarrow\hat{\mathbf{x}}(k+1)=\mathbf{T}^{-1}\cdot\left(\mathbf{x}(k+1)-\frac{(c+2\cdot s_0)}{c}\cdot\mathbf{x}^*\right)$$

$$=\mathbf{T}^{-1}\cdot\left(\mathbf{A}\cdot\mathbf{x}(k)+\mathbf{B}\cdot c+s(k)\cdot\begin{bmatrix}0\\2\end{bmatrix}-\frac{(c+2\cdot s_0)}{c}\cdot\mathbf{x}^*\right)$$

$$=\mathbf{T}^{-1}\cdot\left[\mathbf{A}\cdot\left(\mathbf{T}\cdot\hat{\mathbf{x}}(k)+\frac{(c+2\cdot s_0)}{c}\cdot\mathbf{x}^*\right)+\mathbf{B}\cdot c+s(k)\cdot\begin{bmatrix}0\\2\end{bmatrix}-\frac{(c+2\cdot s_0)}{c}\cdot\mathbf{x}^*\right]$$

$$=\mathbf{T}^{-1}\cdot\mathbf{A}\cdot\mathbf{T}\cdot\hat{\mathbf{x}}(k)+\mathbf{T}^{-1}\cdot\left[(\mathbf{A}-\mathbf{I})\cdot\frac{(c+2\cdot s_0)}{c}\cdot\mathbf{x}^*+\mathbf{B}\cdot c+s(k)\cdot\begin{bmatrix}0\\2\end{bmatrix}\right]$$

Step response of second-order digital filters with two's complement arithmetic

$$= \mathbf{T}^{-1}\cdot\mathbf{T}\cdot\hat{\mathbf{A}}\cdot\mathbf{T}^{-1}\cdot\mathbf{T}\cdot\hat{\mathbf{x}}(k)+\mathbf{T}^{-1}\cdot\left[-\mathbf{B}\cdot c\cdot\frac{(c+2\cdot s_0)}{c}+\mathbf{B}\cdot c+s(k)\cdot\begin{bmatrix}0\\2\end{bmatrix}\right]$$

$$= \hat{\mathbf{A}}\cdot\hat{\mathbf{x}}(k)+\mathbf{T}^{-1}\cdot(s(k)-s_0)\cdot\begin{bmatrix}0\\2\end{bmatrix}$$

If $\mathbf{x}(0)\in\left\{\mathbf{x}(0):\left\|\mathbf{T}^{-1}\cdot\left(\mathbf{x}(0)-\frac{(c+2\cdot s_0)}{c}\cdot\mathbf{x}^*\right)\right\|_2\le 1-\frac{|c+2\cdot s_0|}{2-a}\right\}$

then $\|\hat{\mathbf{x}}(0)\|_2\le 1-\frac{|c+2\cdot s_0|}{2-a}$

Since $\|\hat{\mathbf{A}}\|_2=1$, we have $s(0)=s_0$ and

$$\|\hat{\mathbf{x}}(1)\|_2=\left\|\hat{\mathbf{A}}\cdot\hat{\mathbf{x}}(0)\right\|_2\le 1-\frac{|c+2\cdot s_0|}{2-a}$$

Step response of second-order digital filters with two's complement arithmetic

Assume $\mathbf{x}(k) \in \left\{ \mathbf{x}(k) : \left\| \mathbf{T}^{-1} \cdot \left(\mathbf{x}(k) - \frac{(c + 2 \cdot s_0)}{c} \cdot \mathbf{x}^* \right) \right\|_2 \le 1 - \frac{|c + 2 \cdot s_0|}{2 - a} \right\}$

then

$$\left\| \hat{\mathbf{x}}(k+1) \right\|_2 = \left\| \hat{\mathbf{A}} \cdot \hat{\mathbf{x}}(k) \right\|_2 \le 1 - \frac{|c + 2 \cdot s_0|}{2 - a}$$

and $\boxed{s(k) = s_0 \text{ for } k \ge 0}$

Hence

$$\boxed{\begin{gathered}\mathbf{x}(0) \in \left\{ \mathbf{x}(0) : \left\| \mathbf{T}^{-1} \cdot \left(\mathbf{x}(0) - \frac{(c + 2 \cdot s_0)}{c} \cdot \mathbf{x}^* \right) \right\|_2 \le 1 - \frac{|c + 2 \cdot s_0|}{2 - a} \right\} \\ \text{if and only if } s(k) = s_0 \text{ for } k \ge 0\end{gathered}}$$

Step response of second-order digital filters with two's complement arithmetic

Hence, for the type I trajectory, the following three statements are equivalent each others:

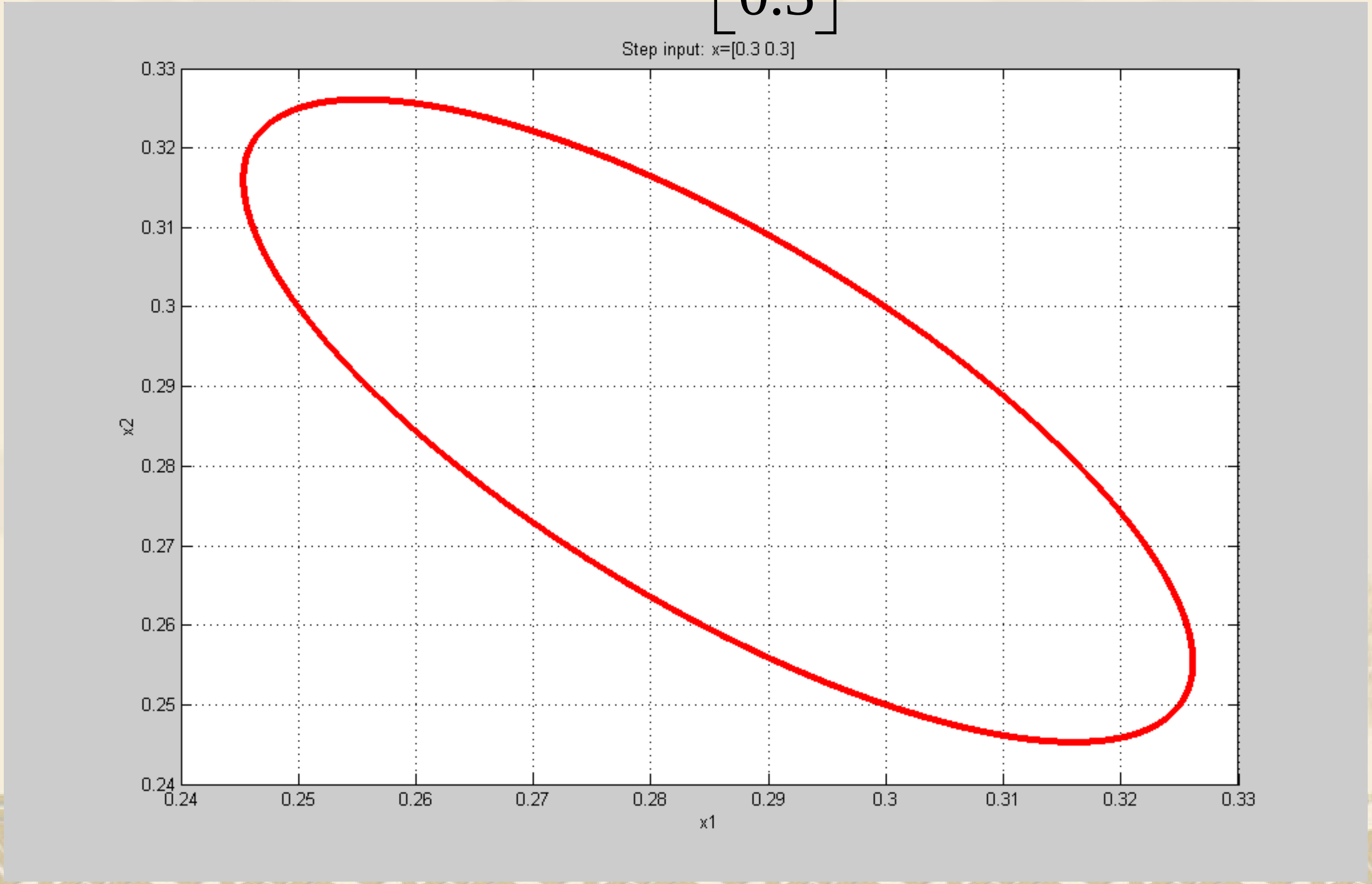
$$\hat{\mathbf{x}}(k+1) = \hat{\mathbf{A}} \cdot \hat{\mathbf{x}}(k)$$

$s(k) = s_0$ for $k \geq 0$

$$\mathbf{x}(0) \in \left\{ \mathbf{x}(0) : \left\| \mathbf{T}^{-1} \cdot \left(\mathbf{x}(0) - \frac{(c + 2 \cdot s_0)}{c} \cdot \mathbf{x}^* \right) \right\|_2 \leq 1 - \frac{|c + 2 \cdot s_0|}{2 - a} \right\}$$

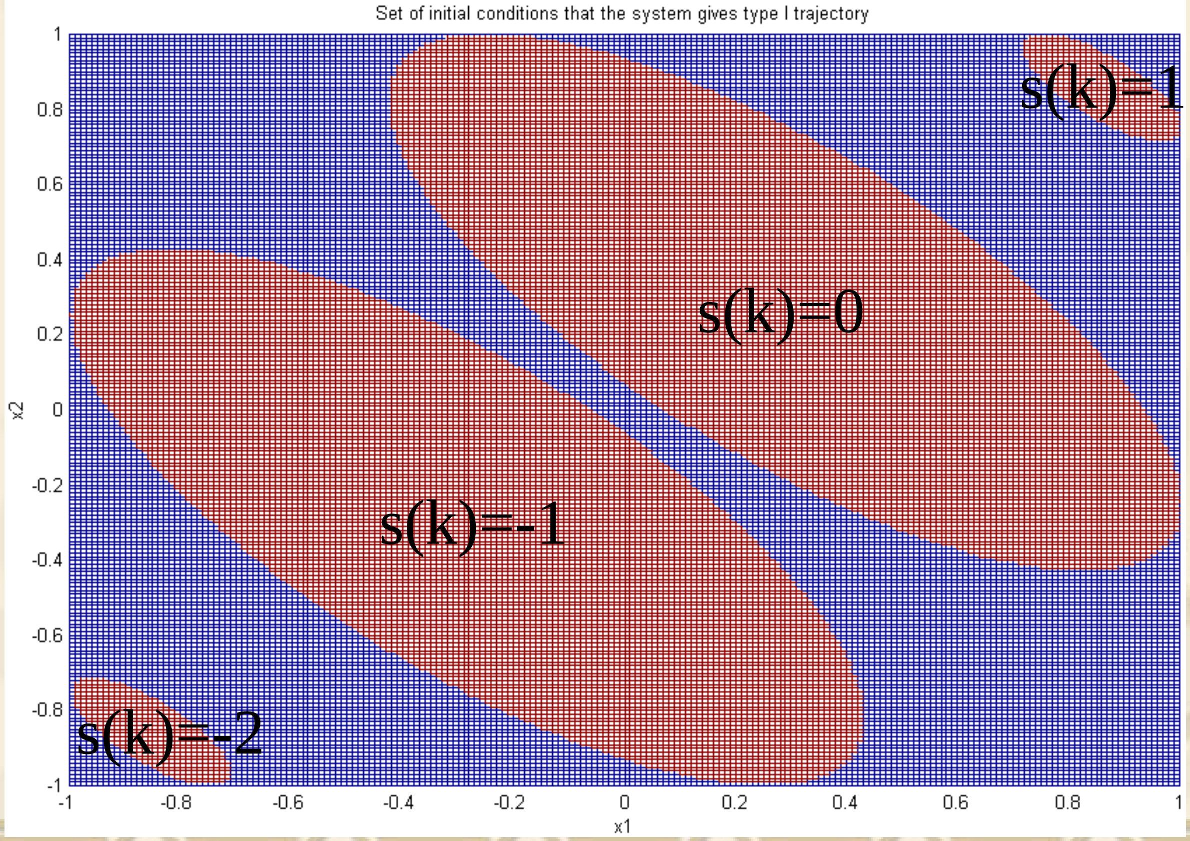
Step response of second-order digital filters with two's complement arithmetic

For $a = -1.5, b = -1, c = 1, \mathbf{x}(0) = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix}$

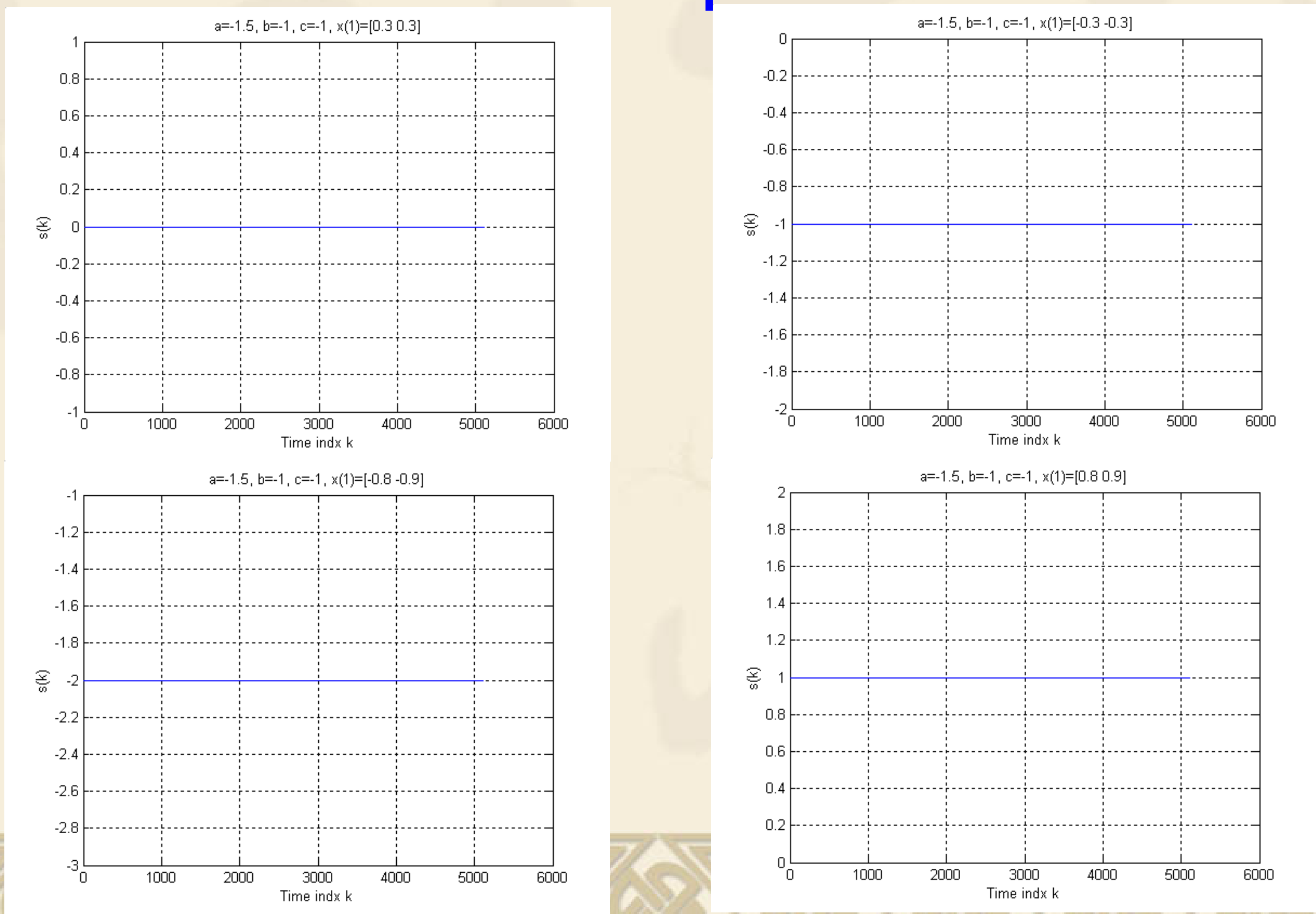


Step response of second-order digital filters with two's complement arithmetic

For $a = -1.5, b = -1$ and $c = 1$



Step response of second-order digital filters with two's complement arithmetic



Step response of second-order digital filters with two's complement arithmetic

Define $\mathbf{x}_0{}^* \equiv \left(\mathbf{I} - \mathbf{A}^M\right)^{-1} \cdot \left(\sum_{j=0}^{M-1} \mathbf{A}^j \cdot \mathbf{B} \cdot c + \sum_{j=0}^{M-1} \mathbf{A}^{M-1-j} \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cdot s(j) \right)$

$\mathbf{x}_{i+1}{}^* \equiv \mathbf{A} \cdot \mathbf{x}_i{}^* + \mathbf{B} \cdot c + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cdot s(i)$ for $i = 0, 1, \cdots, M-2$

$\hat{\mathbf{x}}_i(k) \equiv \mathbf{T}^{-1} \cdot \left(\mathbf{x}(k \cdot M + i) - \mathbf{x}_i{}^* \right)$ for $i = 0, 1, \cdots, M-1$ and $k \geq 0$

Step response of second-order digital filters with two's complement arithmetic

For the type II trajectory, the following three statements are equivalent each others:

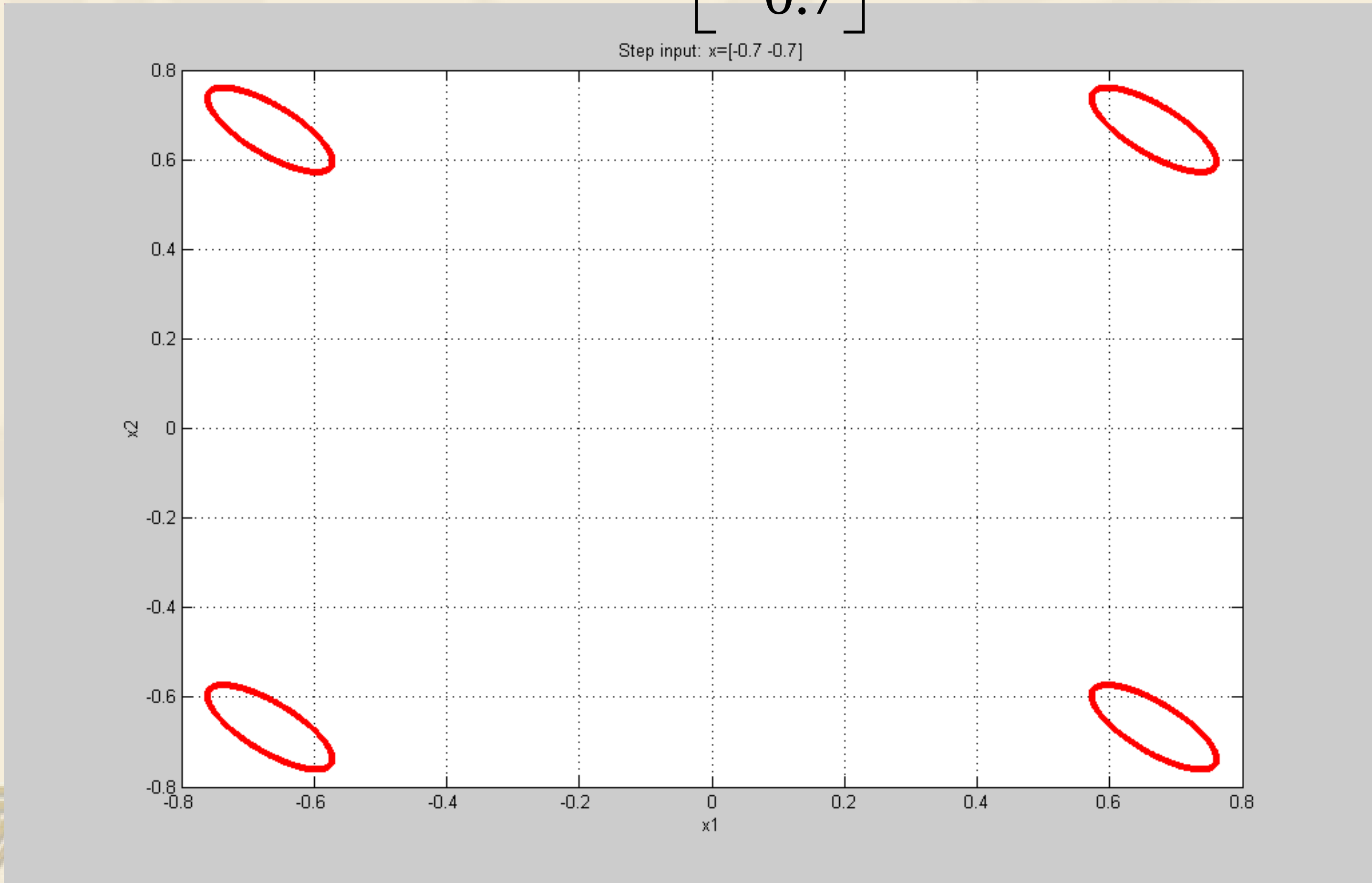
$\hat{\mathbf{x}}_i(k+1) = \hat{\mathbf{A}}^M \cdot \hat{\mathbf{x}}_i(k)$ for $k \geq 0$ and $i = 0,1,\cdots,M-1$

$\exists M$ such that $s(M \cdot k + i) = s(i)$ for $k \geq 0$ and $i = 0,1,\cdots,M-1$

$\mathbf{x}(0) \in \left\{ \mathbf{x}(0) : \left\| \mathbf{T}^{-1} \cdot \left(\mathbf{x}(i) - \mathbf{x}_i^{*} \right) \right\|_\infty \leq 1 - \left\| \mathbf{x}_i^{*} \right\|_\infty \right\}$ for $i = 0,1,\cdots,M-1$

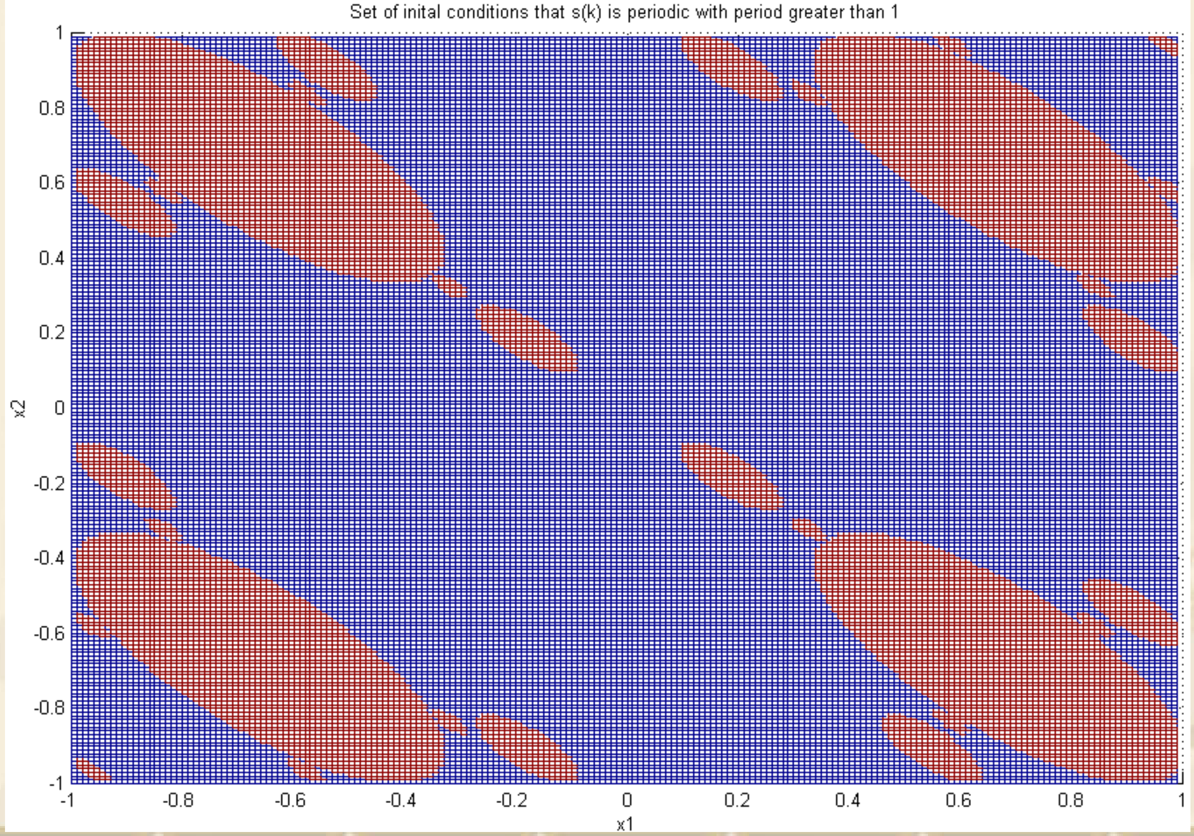
Step response of second-order digital filters with two's complement arithmetic

For $a=-1.5, b=-1, c=1, \mathbf{x}(0)=\begin{bmatrix}-0.7\\-0.7\end{bmatrix}, s=(0,-1,-1,0,\cdots)$



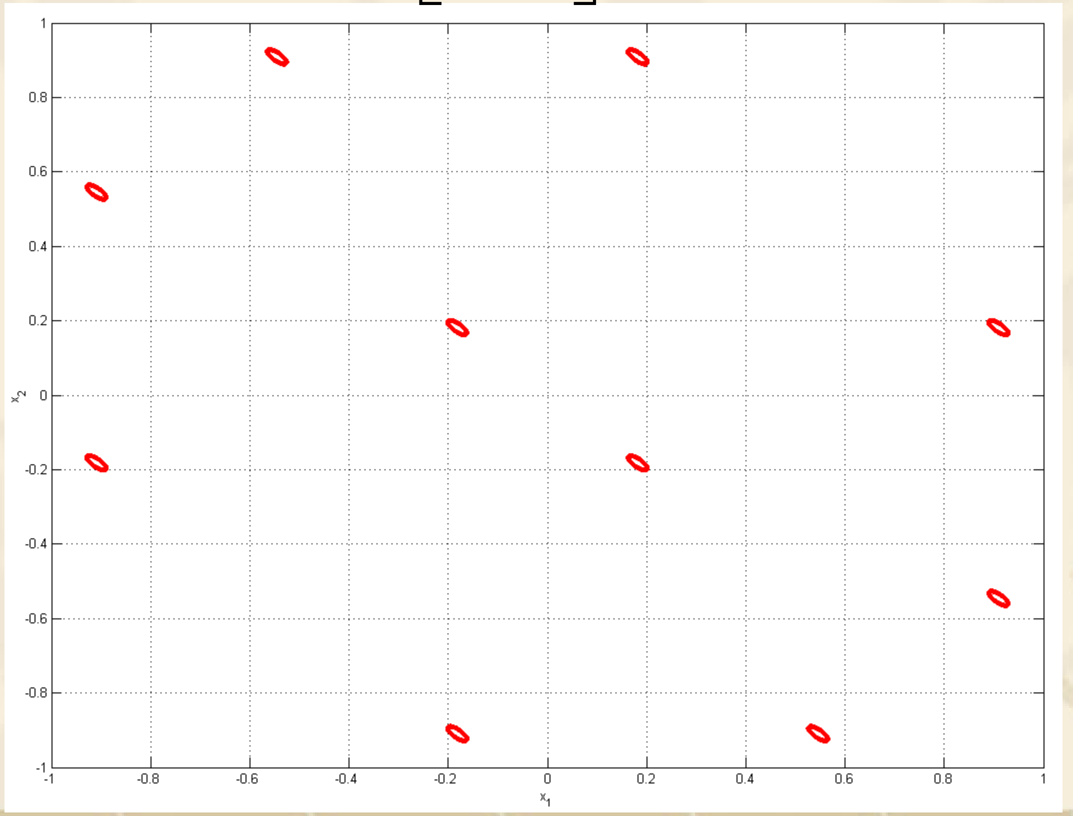
Step response of second-order digital filters with two's complement arithmetic

For $a = -1.5, b = -1$ and $c = 1$



Step response of second-order digital filters with two's complement arithmetic

For $a = -1.5, b = -1, c = 1, \mathbf{x}(0) = \begin{bmatrix} 0.2 \\ -0.2 \end{bmatrix}, s = (-1,-1,-1,-1,-1,0,0,0,0,0,\cdots)$

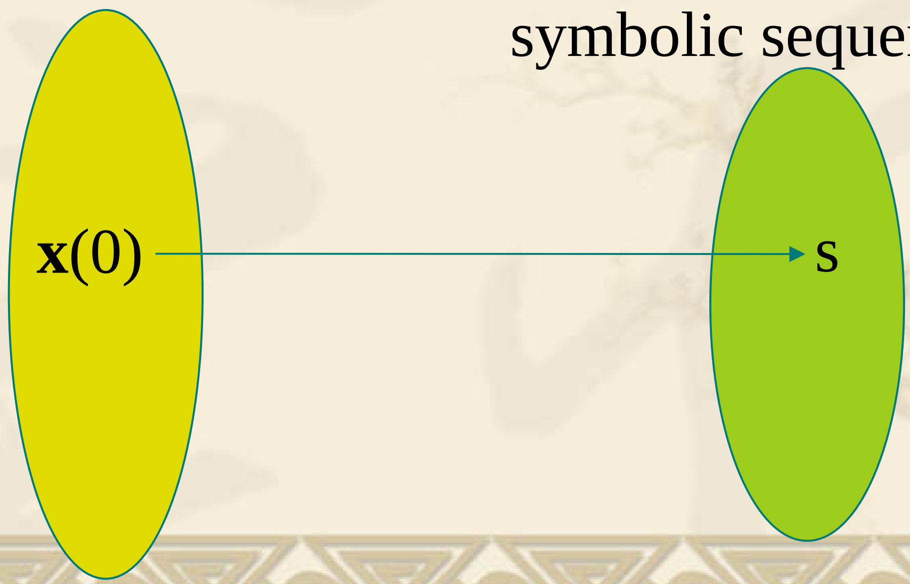


Step response of second-order digital filters with two's complement arithmetic

Admissible set of periodic symbolic sequences is defined as a set of periodic symbolic sequences such that there exists an initial condition that produces the symbolic sequences.

Set of initial conditions

Admissible set of periodic symbolic sequences



Step response of second-order digital filters with two's complement arithmetic

For example, when $a=0.5, b=-1, c=0,$

M	Admissible s	Not admissible s
1	$s=(0,\cdots)$	$s=(1,\cdots)$ $s=(-1,\cdots)$
2	$s=(-1,1,\cdots)$ $s=(1,-1,\cdots)$	$s=(1,0,\cdots)$ $s=(0,1,\cdots)$
		$s=(-1,0,\cdots)$ $s=(0,-1,\cdots)$
3	$s=(0,0,1,\cdots)$ $s=(0,1,0,\cdots)$	$s=(0,1,1,\cdots)$ $s=(0,1,-1,\cdots)$
	$s=(1,0,0,\cdots)$ $s=(-1,0,0,\cdots)$	$s=(0,-1,1,\cdots)$ $s=(0,-1,-1,\cdots)$
	$s=(0,0,-1,\cdots)$ $s=(0,-1,0,\cdots)$	$s=(1,0,1,\cdots)$ $s=(1,0,-1,\cdots)$
		$s=(1,-1,0,\cdots)$ $s=(1,-1,0,\cdots)$
		$\cdots$
15	Not found	$\cdots$

Step response of second-order digital filters with two's complement arithmetic

A periodic sequence s with period M is admissible if and only if for $i = 0, 1, \cdots, M-1$

$$-1 \le \frac{\sum_{j=0}^{M-1} s(\mathrm{mod}(i+j, M)) \cdot \cos\left(\left(\frac{M}{2} - j - 1\right) \cdot \theta\right)}{\sin\left(\frac{M \cdot \theta}{2}\right) \cdot \sin\theta} < 1$$

Step response of second-order digital filters with two's complement arithmetic

For the type III trajectory, the following three statements are equivalent each others:

There is an elliptical fractal pattern exhibited on the phase plane.

The symbolic sequences are aperiodic.

The set of initial conditions is $I^2 \setminus \bigcap_{\forall M} D_M$

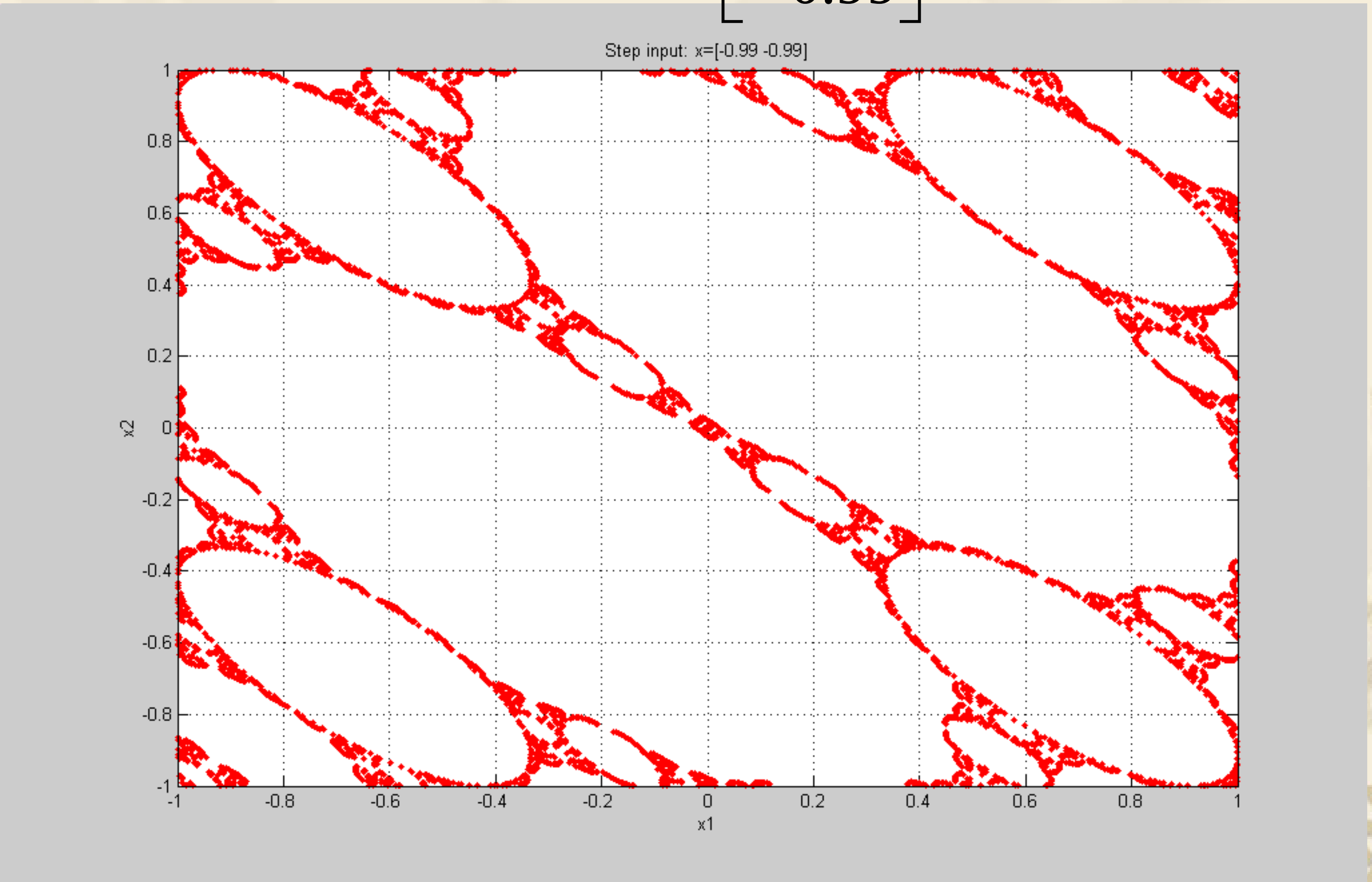
where

$$D_M = \left\{ \mathbf{x}(0) : \left\| \mathbf{T}^{-1} \cdot \left(\mathbf{x}(i) - \mathbf{x}_i^{\ *} \right) \right\| \leq 1 - \left\| \mathbf{x}_i^{\ *} \right\|_\infty \text{ and } s(i) = s(i + M \cdot k) \right\}$$

which is also an elliptical fractal set.

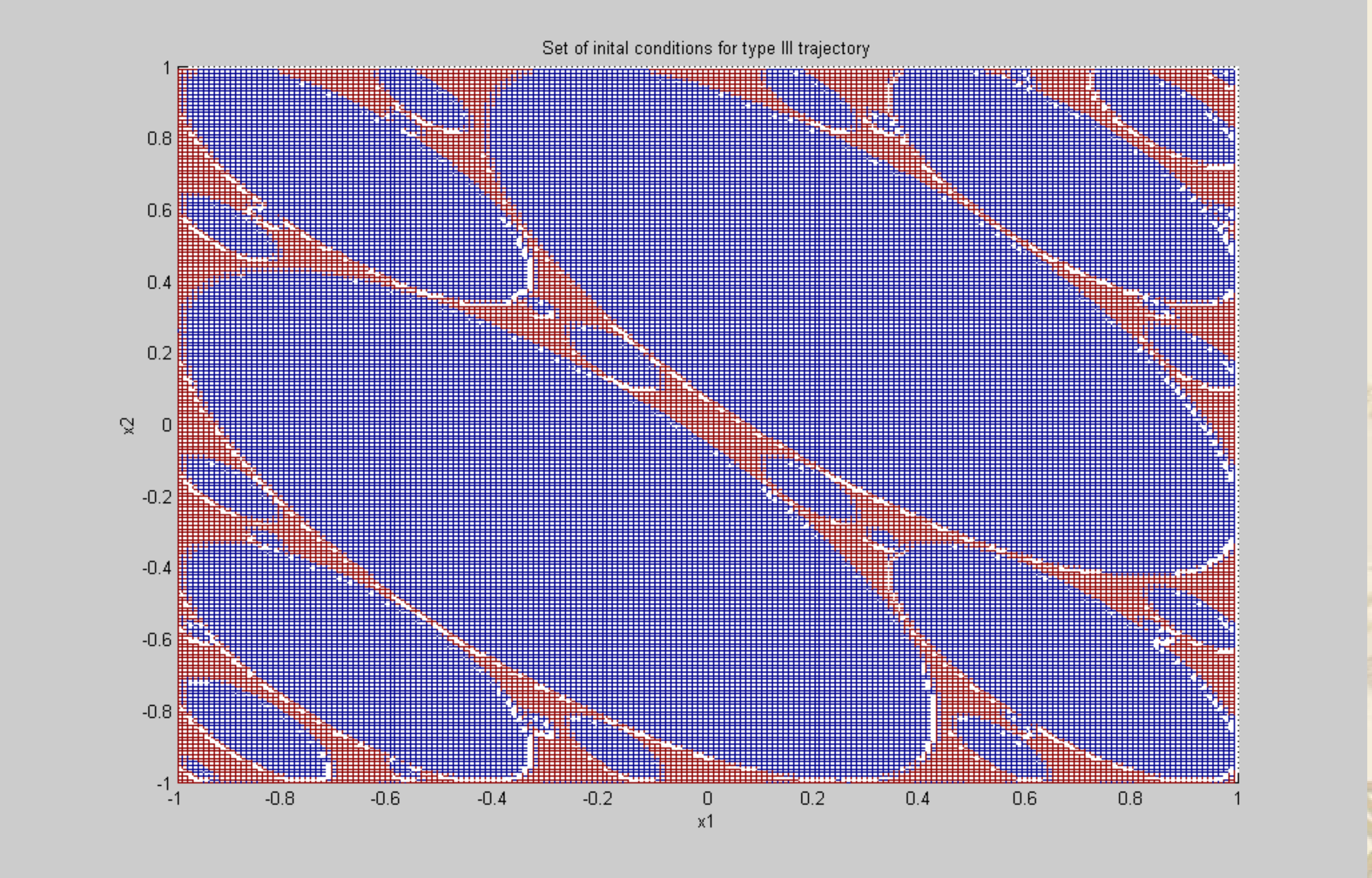
Step response of second-order digital filters with two's complement arithmetic

For $a = -1.5, b = -1, c = 1, \mathbf{x}(0) = \begin{bmatrix} -0.99 \\ -0.99 \end{bmatrix}$



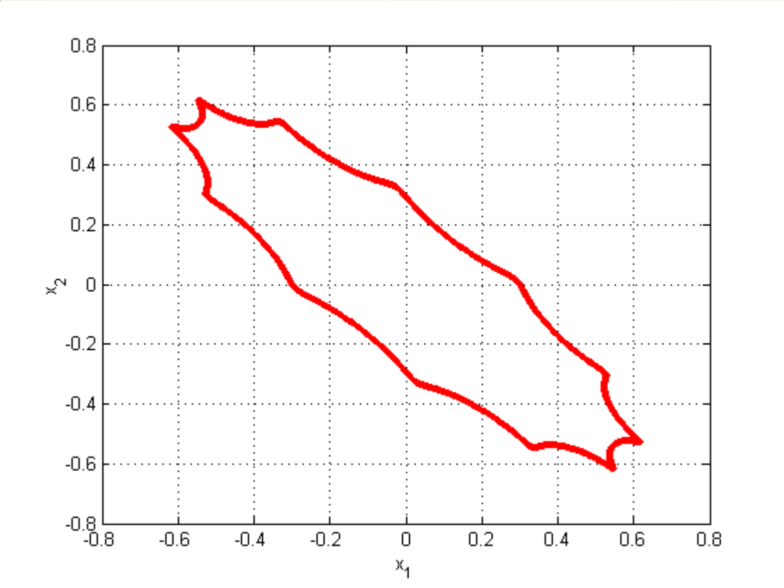
Step response of second-order digital filters with two's complement arithmetic

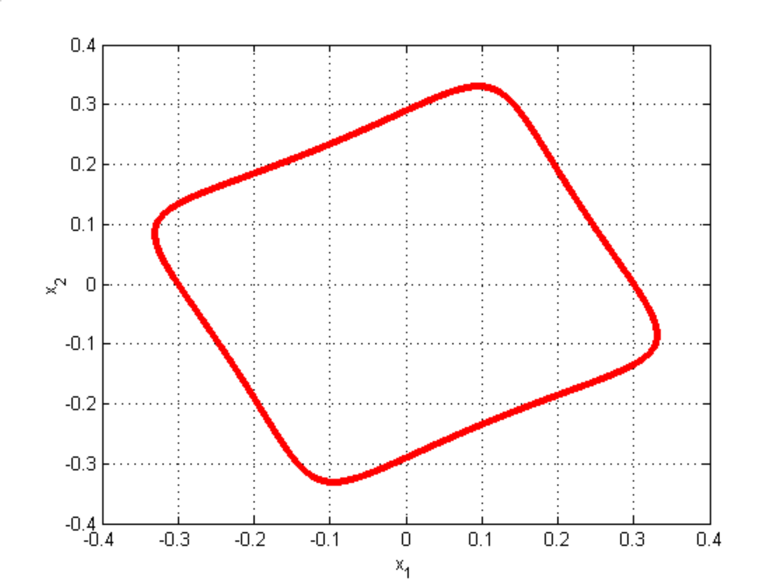
For $a = -1.5, b = -1$ and $c = 1$



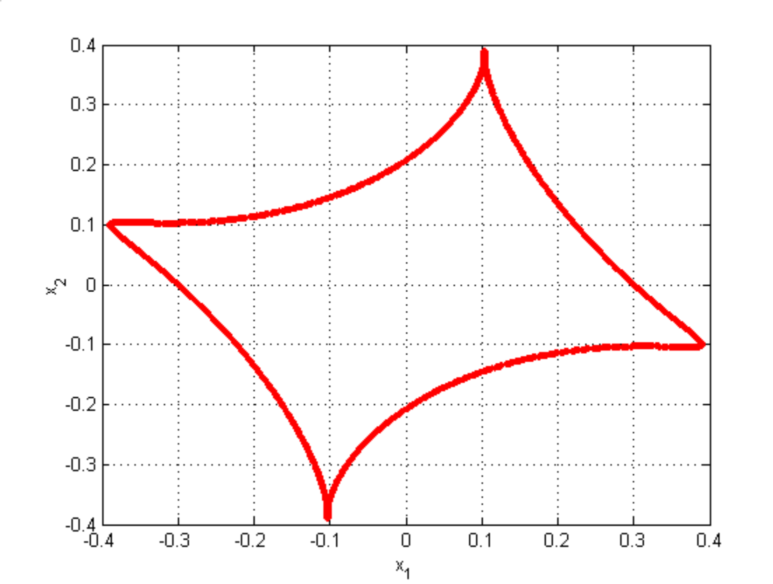
Other simulation results

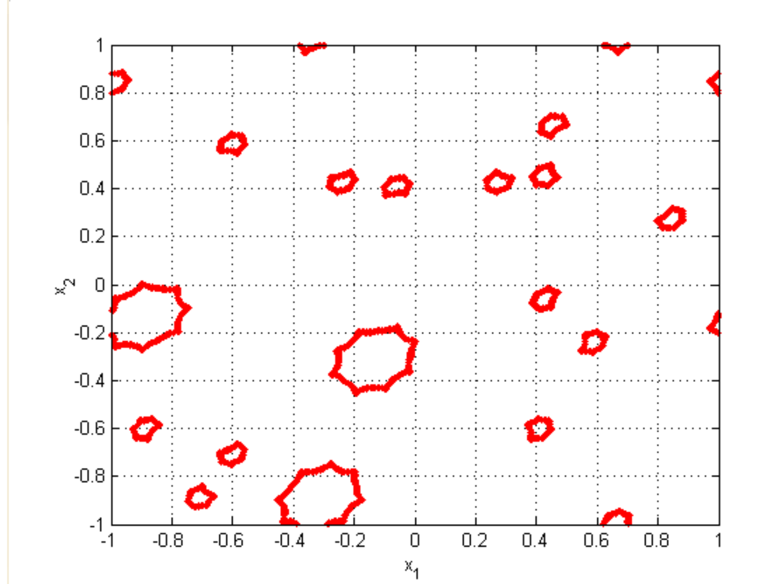
- Sinusoidal case for second-order digital filters with two’s complement arithmetic

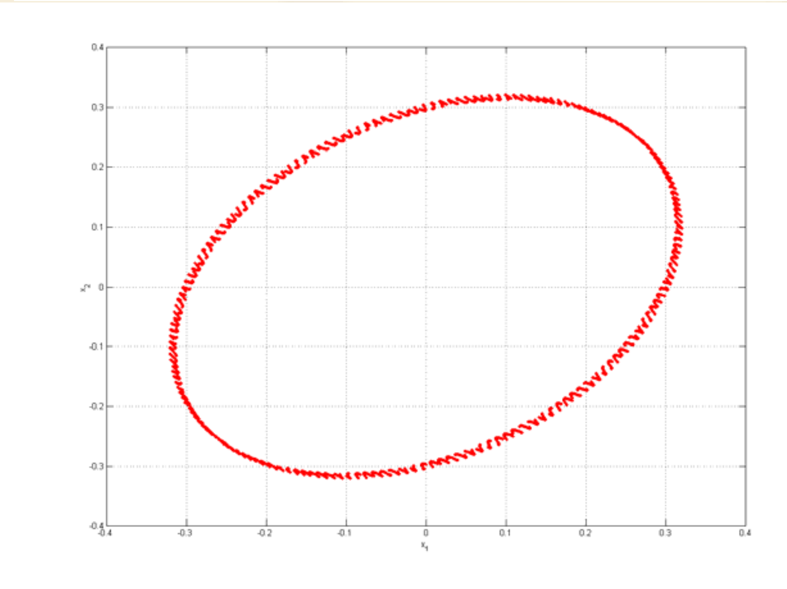


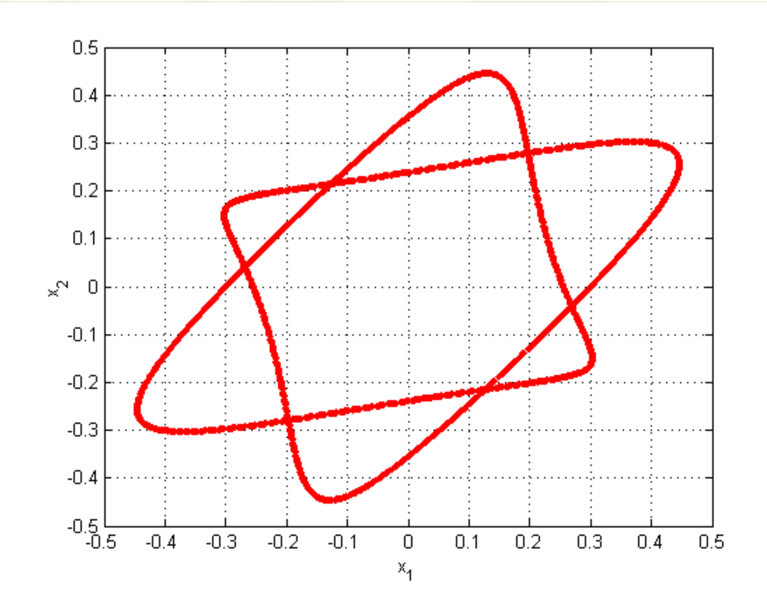


Other simulation results



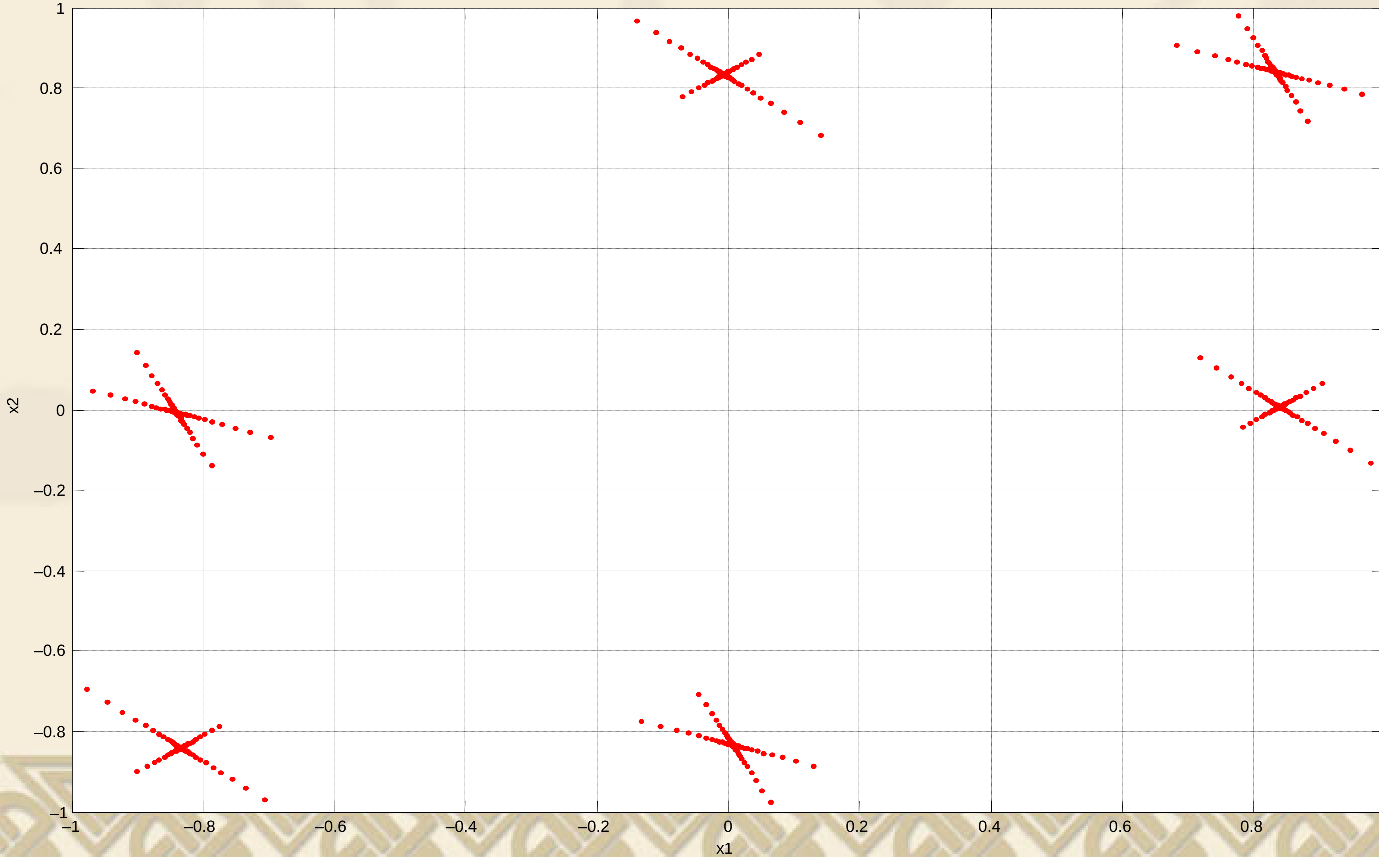




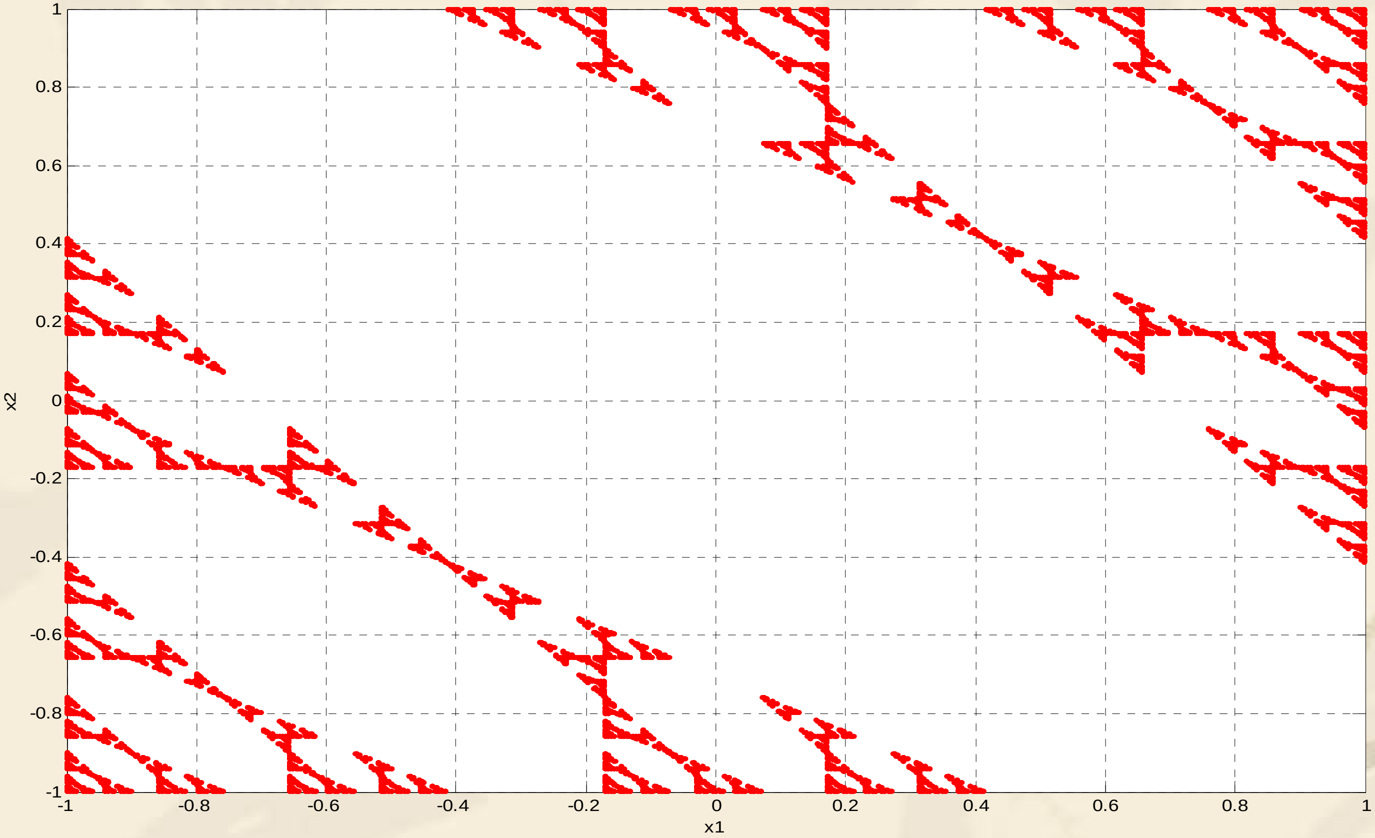


Other simulation results

- Autonomous case when the eigenvalues of **A** are inside the unit circle

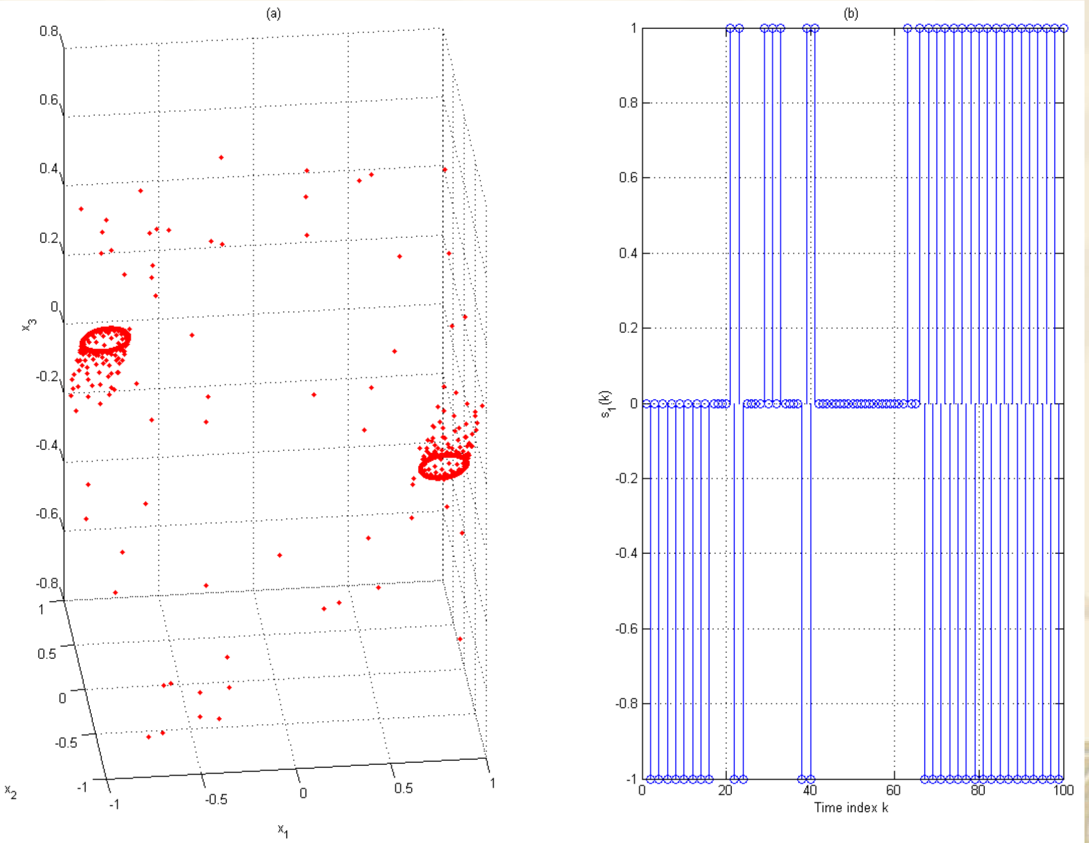


Other simulation results

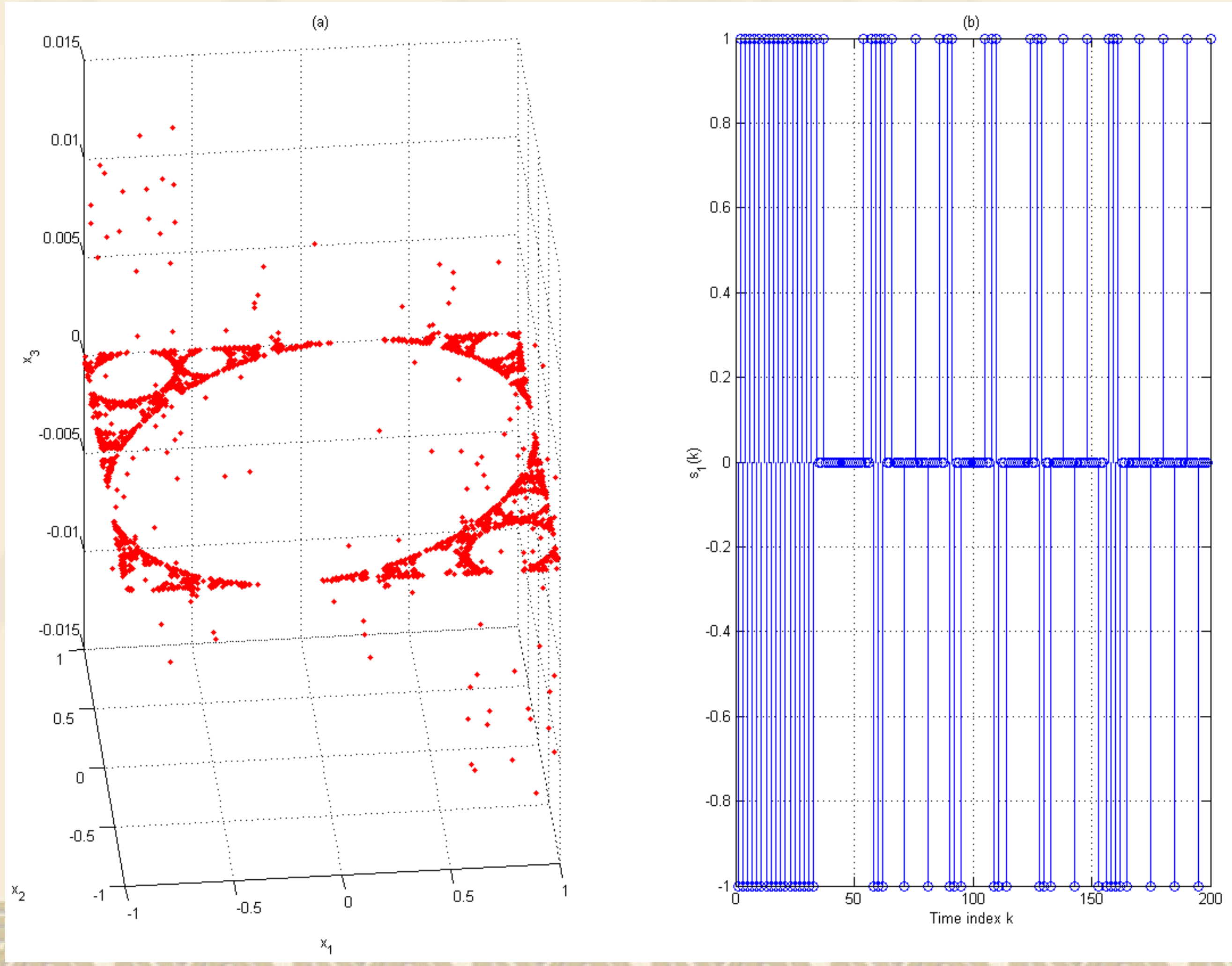


Other simulation results

- Autonomous case for third-order digital filters with two's complement arithmetic realized in cascade form

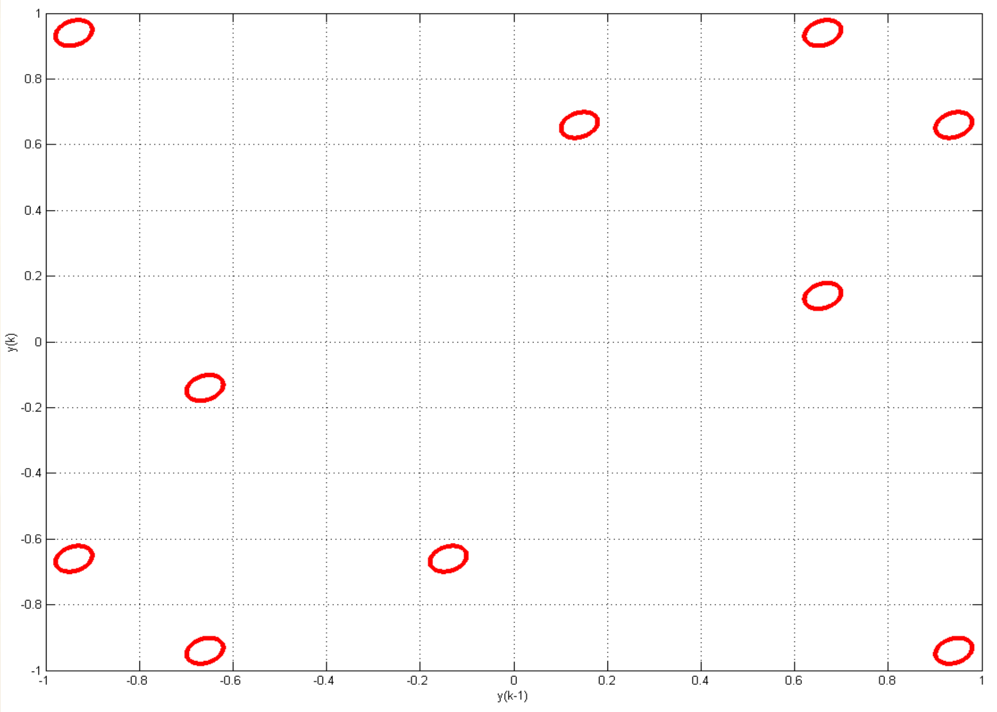


Other simulation results

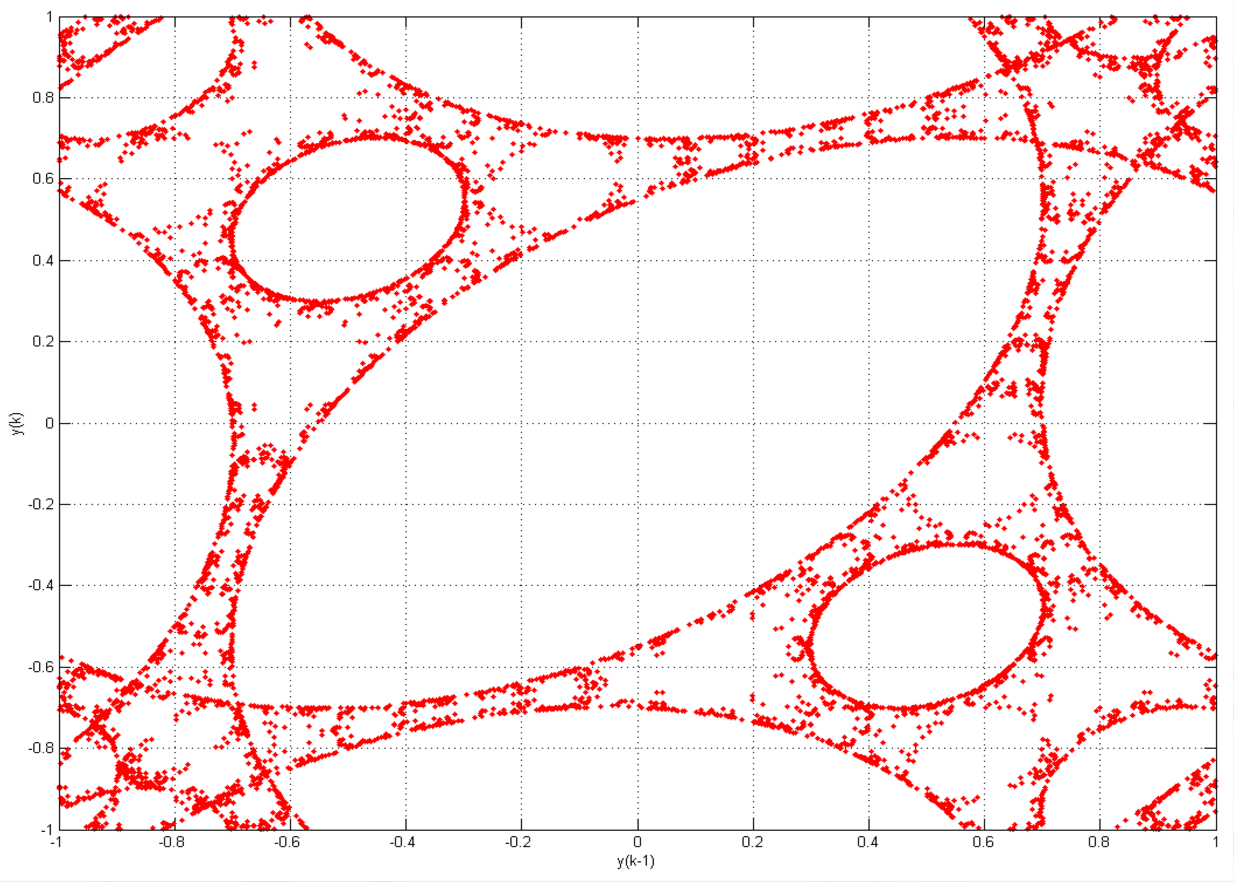


Other simulation results

- Autonomous case for third-order digital filters with two’s complement arithmetic realized in parallel form



Other simulation results



Conclusions

- The trajectory equations and the sets of initial conditions for various types of trajectory are derived.
- The admissible set of periodic symbolic sequences are discussed.
- Simulation results for other systems are shown.

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Q&A Session

Thank you!

Let me think…