Introduction to Linear Dynamical Systems and Linear Control Strategies

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Outline

- Linear Time Invariant Systems
- Linear Time Invariant Feedback Controls
  - Pole Placement Approach
  - State Feedback
  - Output Feedback
- Linear Multirate Systems
- Control of Linear Multirate Systems via Filter Banks Approach
- Conclusions
- Questions and Answers

Linear Time Invariant Systems

- Definition of linear systems

$$\sum_{i=0}^{N} a_i y_i(k) = T\left(\sum_{i=0}^{N} a_i u_i(k)\right)$$

- Definition of time invariant systems

$$y(k-1) = T(u(k-1))$$

- Definition of linear time invariant systems
  - A system is both linear and time invariant.

Linear Time Invariant Systems

- Definition of an impulse response

$$h(k) \equiv T(\delta(k))$$

where $\delta(k) \equiv \begin{cases} 1 & k = 0 \\ 0 & otherwise \end{cases}$

- Definition of a frequency response

$$y(k) \equiv T\left(e^{j\omega k}\right)$$

Linear Time Invariant Systems

- Properties of linear time invariant systems
  - A system is linear and time invariant if and only if

    $$y(n) = \sum_{\forall k \in Z} h(k) u(n-k)$$

  - A system is linear and time invariant if and only if

    $$Y(z) = H(z) U(z)$$

    where

    $$Y(z) = \sum_{\forall n \in Z} y(n) z^{-n}$$

    $$H(z) = \sum_{\forall n \in Z} h(n) z^{-n}$$

    $$U(z) = \sum_{\forall n \in Z} u(n) z^{-n}$$

Linear Time Invariant Systems

- Characterization of linear time invariant systems
  - Constant linear coefficients difference equations

$$\sum_{i=0}^{N} a_i y(k-i) = \sum_{j=0}^{M} b_j u(k-j)$$

  - Transfer function

$$H(z) = \frac{\sum_{j=0}^{M} b_j z^{-j}}{\sum_{i=0}^{N} a_i z^{-i}}$$

  - State space representation

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$$
$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k)$$

Linear Time Invariant Systems

- Responses

$$\mathbf{x}(k) = \mathbf{A}^k\mathbf{x}(0) + \sum_{j=0}^{k-1}\mathbf{A}^{k-1-j}\mathbf{B}\mathbf{u}(j) \quad \forall k \geq 1$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{A}^k\mathbf{x}(0) + \mathbf{C}\sum_{j=0}^{k-1}\mathbf{A}^{k-1-j}\mathbf{B}\mathbf{u}(j) + \mathbf{D}\mathbf{u}(k) \quad \forall k \geq 1$$

zero input response — zero state response

$$y(n) = \sum_{\forall k\in Z} h(k)u(n-k)$$

Linear Time Invariant Systems

- Similarity transforms
  - Define

$$\tilde{\mathbf{x}}(k) \equiv \mathbf{T}^{-1}\mathbf{x}(k)$$

$$\tilde{\mathbf{A}} \equiv \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$$

$$\tilde{\mathbf{B}} \equiv \mathbf{T}^{-1}\mathbf{B}$$

$$\tilde{\mathbf{C}} \equiv \mathbf{C}\mathbf{T}$$

  - then

$$\tilde{\mathbf{x}}(k+1) = \tilde{\mathbf{A}}\tilde{\mathbf{x}}(k) + \tilde{\mathbf{B}}\mathbf{u}(k)$$

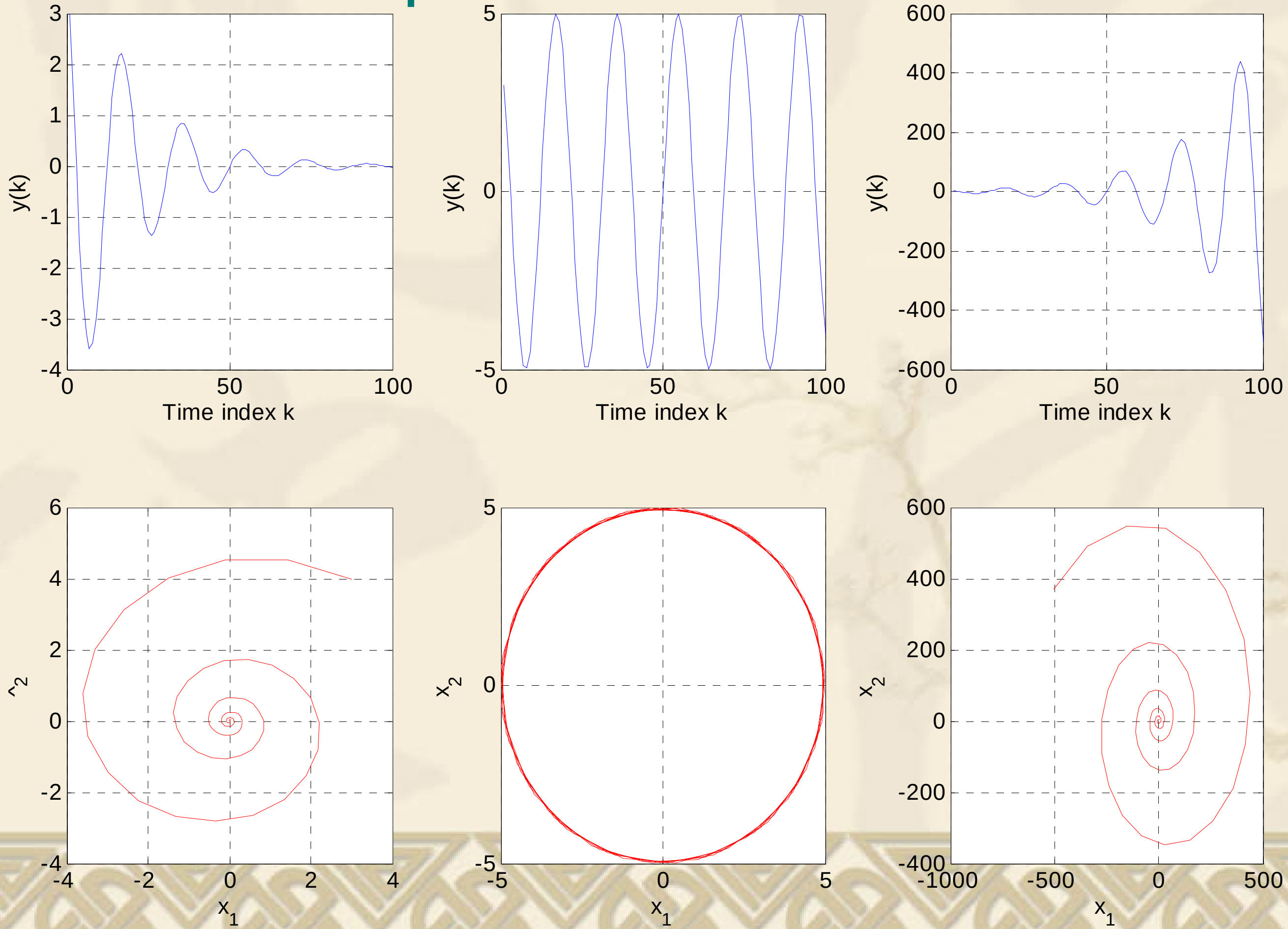
$$\mathbf{y}(k) = \tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) + \mathbf{D}\mathbf{u}(k)$$

Linear Time Invariant Systems

- Only three types of behaviors for autonomous response
  - converge to zero (all system poles are strictly inside the unit circle.)
  - oscillates (Some system poles are on the unit circle, while all other system poles are strictly inside the unit circle.)
  - diverge to infinity (Some system poles are outside the unit circle.)

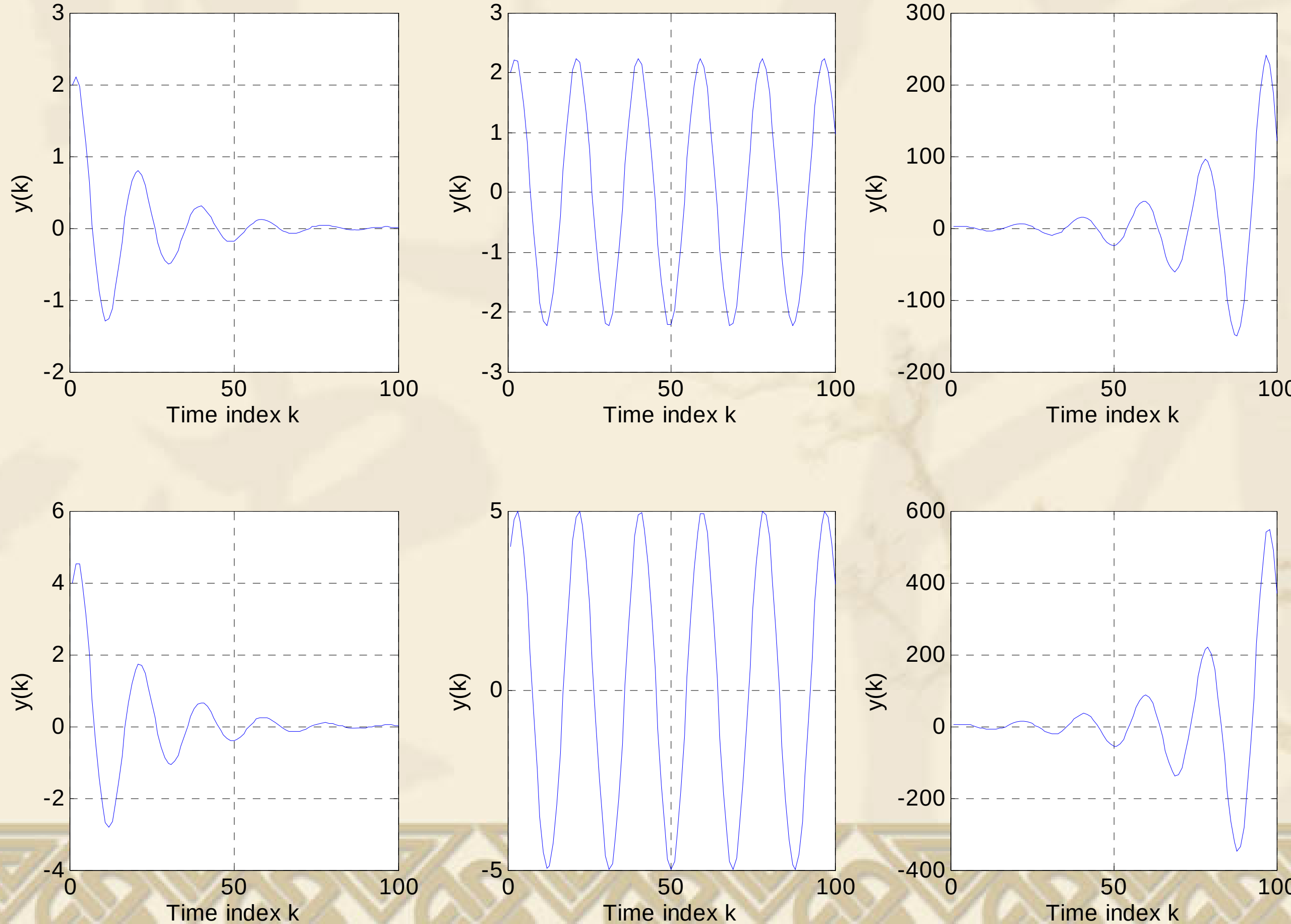
Linear Time Invariant Systems

- Autonomous responses



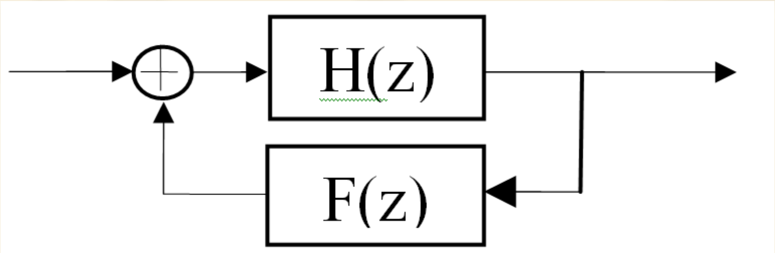
Linear Time Invariant Systems

- Effects on initial conditions
  - Behaviors only depend on the system poles, not on initial conditions.



Linear Time Invariant Feedback Controls

- Pole placement



Plant transfer function $H(z)=\dfrac{N_H(z)}{D_H(z)}$

Controller transfer function $F(z)=\dfrac{N_F(z)}{D_F(z)}$

$$T(z)=\frac{H(z)}{1+H(z)F(z)}=\frac{N_H(z)D_F(z)}{N_H(z)N_F(z)+D_H(z)D_F(z)}$$

$N_H(z)N_F(z)+D_H(z)D_F(z)$ is stable.

Linear Time Invariant Feedback Controls

- State feedback
  - Plant state space matrices $(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})$
  - Controller state space matrices $(\tilde{\mathbf{A}},\tilde{\mathbf{B}},\tilde{\mathbf{C}},\tilde{\mathbf{D}})$

$$\mathbf{x}(k+1)=\mathbf{A}\mathbf{x}(k)+\mathbf{B}(\mathbf{u}(k)-\tilde{\mathbf{y}}(k))$$

$$\mathbf{y}(k)=\mathbf{C}\mathbf{x}(k)+\mathbf{D}(\mathbf{u}(k)-\tilde{\mathbf{y}}(k))$$

$$\tilde{\mathbf{x}}(k+1)=\tilde{\mathbf{A}}\tilde{\mathbf{x}}(k)+\tilde{\mathbf{B}}\mathbf{x}(k)$$

$$\tilde{\mathbf{y}}(k)=\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k)+\tilde{\mathbf{D}}\mathbf{x}(k)$$

$$\mathbf{y}(k)=\mathbf{C}\mathbf{x}(k)+\mathbf{D}\left(\mathbf{u}(k)-\left(\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k)+\tilde{\mathbf{D}}\mathbf{x}(k)\right)\right)$$

$$=\left(\mathbf{C}-\mathbf{D}\tilde{\mathbf{D}}\right)\mathbf{x}(k)-\mathbf{D}\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k)+\mathbf{D}\mathbf{u}(k)$$

$$\mathbf{x}(k+1)=\mathbf{A}\mathbf{x}(k)+\mathbf{B}\left(\mathbf{u}(k)-\left(\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k)+\tilde{\mathbf{D}}\mathbf{x}(k)\right)\right)$$

$$=\left(\mathbf{A}-\mathbf{B}\tilde{\mathbf{D}}\right)\mathbf{x}(k)-\mathbf{B}\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k)+\mathbf{B}\mathbf{u}(k)$$

Linear Time Invariant Feedback Controls

- State feedback

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \tilde{\mathbf{x}}(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{A}-\mathbf{B}\tilde{\mathbf{D}} & -\mathbf{B}\tilde{\mathbf{C}} \\ \tilde{\mathbf{B}} & \tilde{\mathbf{A}} \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \tilde{\mathbf{x}}(k) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{u}(k)$$

$$\mathbf{y}(k) = \begin{bmatrix} \mathrm{C}-\mathbf{D}\tilde{\mathbf{D}} & -\mathbf{D}\tilde{\mathrm{C}} \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \tilde{\mathbf{x}}(k) \end{bmatrix} + \mathbf{D}\mathbf{u}(k)$$

$\begin{bmatrix} (\mathbf{A}-\mathbf{B}\tilde{\mathbf{D}}) & -\mathbf{B}\tilde{\mathbf{C}} \\ \tilde{\mathbf{B}} & \tilde{\mathbf{A}} \end{bmatrix}$ is stable.

Linear Time Invariant Feedback Controls

- Output feedback
  - Plant state space matrices $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$
  - Controller state space matrices $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}, \tilde{\mathbf{D}})$

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}(\mathbf{u}(k) - \tilde{\mathbf{y}}(k))$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}(\mathbf{u}(k) - \tilde{\mathbf{y}}(k))$$

$$\tilde{\mathbf{x}}(k+1) = \tilde{\mathbf{A}}\tilde{\mathbf{x}}(k) + \tilde{\mathbf{B}}\mathbf{y}(k)$$

$$\tilde{\mathbf{y}}(k) = \tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) + \tilde{\mathbf{D}}\mathbf{y}(k)$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\left(\mathbf{u}(k) - \left(\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) + \tilde{\mathbf{D}}\mathbf{y}(k)\right)\right)$$

$$= \mathbf{C}\mathbf{x}(k) - \mathbf{D}\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) + \mathbf{D}\mathbf{u}(k) - \mathbf{D}\tilde{\mathbf{D}}\mathbf{y}(k)$$

$$\mathbf{y}(k) = \left(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\left(\mathbf{C}\mathbf{x}(k) - \mathbf{D}\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) + \mathbf{D}\mathbf{u}(k)\right)$$

$$= \left(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{C}\mathbf{x}(k) - \left(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) + \left(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\mathbf{u}(k)$$

Linear Time Invariant Feedback Controls

- Output feedback

$$\tilde{\mathbf{y}}(k)=\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k)+\tilde{\mathbf{D}}\left(\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{C}\mathbf{x}(k)-\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k)+\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\mathbf{u}(k)\right)$$

$$=\tilde{\mathbf{D}}\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{C}\mathbf{x}(k)+\left(\mathbf{I}-\tilde{\mathbf{D}}\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\right)\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k)+\tilde{\mathbf{D}}\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\mathbf{u}(k)$$

$$\mathbf{x}(k+1)=\mathbf{A}\mathbf{x}(k)+\mathbf{B}\left(\mathbf{u}(k)-\left(\tilde{\mathbf{D}}\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{C}\mathbf{x}(k)+\left(\mathbf{I}-\tilde{\mathbf{D}}\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\right)\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k)+\tilde{\mathbf{D}}\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\mathbf{u}(k)\right)\right)$$

$$=\left(\mathbf{A}-\mathbf{B}\tilde{\mathbf{D}}\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{C}\right)\mathbf{x}(k)-\mathbf{B}\left(\mathbf{I}-\tilde{\mathbf{D}}\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\right)\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k)+\left(\mathbf{B}-\mathbf{B}\tilde{\mathbf{D}}\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\right)\mathbf{u}(k)$$

$$\tilde{\mathbf{x}}(k+1)=\tilde{\mathbf{A}}\tilde{\mathbf{x}}(k)+\tilde{\mathbf{B}}\left(\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{C}\mathbf{x}(k)-\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k)+\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\mathbf{u}(k)\right)$$

$$=\tilde{\mathbf{B}}\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{C}\mathbf{x}(k)+\left(\tilde{\mathbf{A}}-\tilde{\mathbf{B}}\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\tilde{\mathbf{C}}\right)\tilde{\mathbf{x}}(k)+\tilde{\mathbf{B}}\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\mathbf{u}(k)$$

Linear Time Invariant Feedback Controls

- Output feedback

$$\begin{bmatrix}\mathbf{x}(k+1)\\ \tilde{\mathbf{x}}(k+1)\end{bmatrix}=\begin{bmatrix}\mathbf{A}-\mathbf{B}\tilde{\mathbf{D}}\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{C} & -\mathbf{B}\left(\mathbf{I}-\tilde{\mathbf{D}}\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\right)\tilde{\mathbf{C}}\\ \tilde{\mathbf{B}}\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{C} & \left(\tilde{\mathbf{A}}-\tilde{\mathbf{B}}\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\tilde{\mathbf{C}}\right)\end{bmatrix}\begin{bmatrix}\mathbf{x}(k)\\ \tilde{\mathbf{x}}(k)\end{bmatrix}+\begin{bmatrix}\left(\mathbf{B}-\mathbf{B}\tilde{\mathbf{D}}\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\right)\\ \tilde{\mathbf{B}}\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\end{bmatrix}\mathbf{u}(k)$$

$$\tilde{\mathbf{y}}(k)=\begin{bmatrix}\tilde{\mathbf{D}}\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{C} & \left(\mathbf{I}-\tilde{\mathbf{D}}\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\right)\tilde{\mathbf{C}}\end{bmatrix}\begin{bmatrix}\mathbf{x}(k)\\ \tilde{\mathbf{x}}(k)\end{bmatrix}+\tilde{\mathbf{D}}\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\mathbf{u}(k)$$

$$\begin{bmatrix}\mathbf{A}-\mathbf{B}\tilde{\mathbf{D}}\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{C} & -\mathbf{B}\left(\mathbf{I}-\tilde{\mathbf{D}}\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\right)\tilde{\mathbf{C}}\\ \tilde{\mathbf{B}}\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{C} & \left(\tilde{\mathbf{A}}-\tilde{\mathbf{B}}\left(\mathbf{I}+\mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\tilde{\mathbf{C}}\right)\end{bmatrix}$$ is stable.

Linear Multirate Systems

- Definition

$$y(k) = \sum_{\forall l \in Z} g(k,l)u(l) \quad \forall k \in Z$$

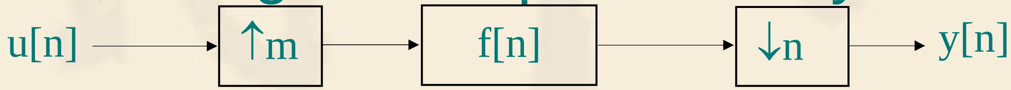
where

$$g(k,l) = g(k+m, l+n) \quad \forall k,l \in Z$$

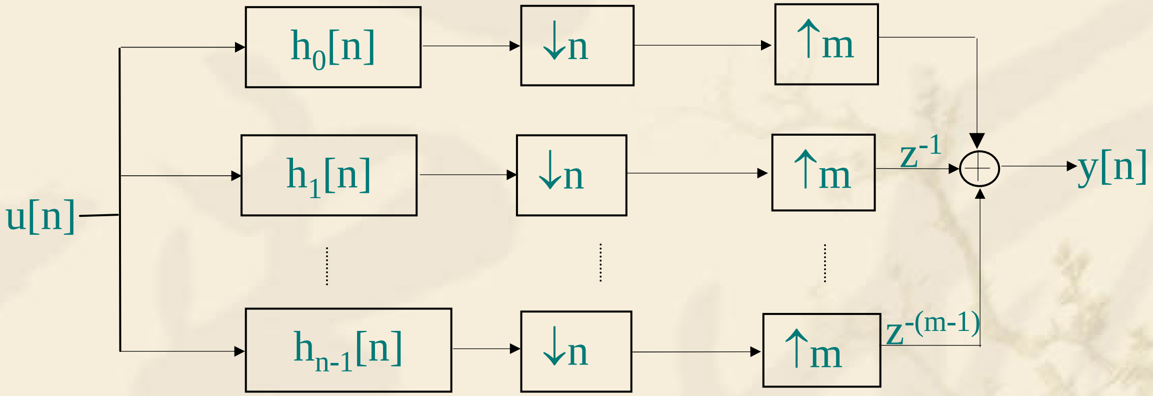
  - Input shifts by n samples, output shifts by m samples.

Linear Multirate Systems

- Examples:
  - Rate changers/sampled data systems



  - Filter banks

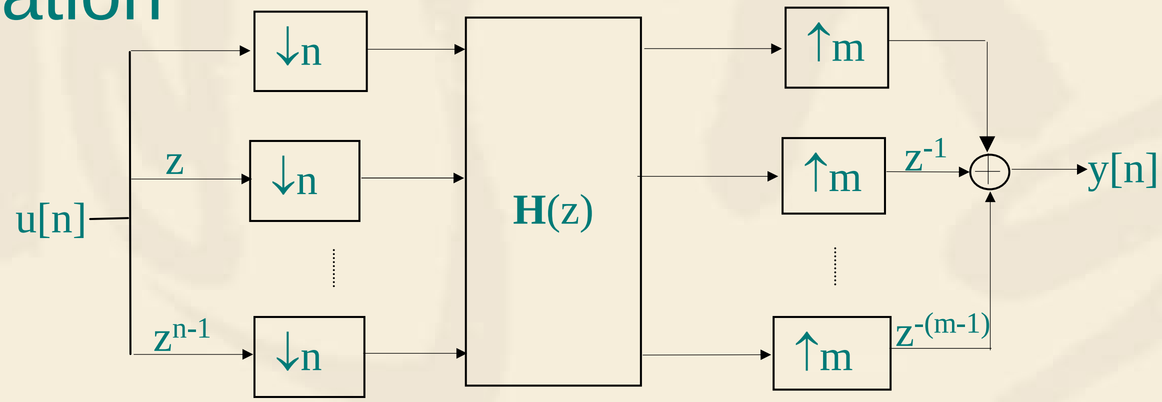


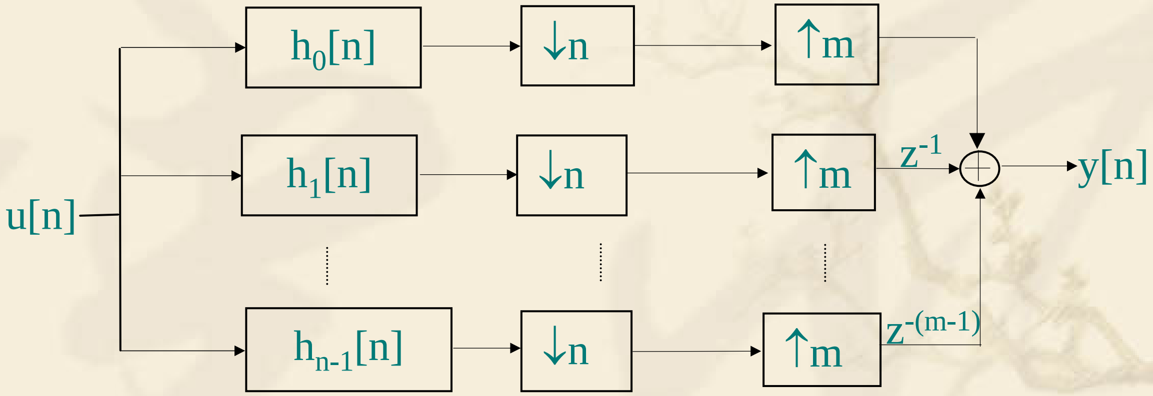
Linear Multirate Systems

- Realization
  - A linear multirate system can be realized by a filter bank system.
  - Define a blocked input signal as
    $$\mathbf{u}(k) \equiv \left[u(nk) \quad \cdots \quad u(nk+n-1)\right]^T$$
  - Define a block output signal as
    $$\mathbf{y}(k) \equiv \left[y(mk) \quad \cdots \quad y(mk+m-1)\right]^T$$
  - Input shifts by n samples, the blocked input signal shifts by 1 sample. Output shifts by m samples, the blocked output signal shifts by 1 sample.
  - Hence, there exists an $m \times n$ transfer matrix $\mathbf{H}(z)$ such that
    $$\mathbf{Y}(z) = \mathbf{H}(z)\mathbf{X}(z)$$

Linear Multirate Systems

- Realization





Linear Multirate Systems

- Realization
  - Denote $f[kn - lm] = g[k,l] \quad \forall k,l \in Z$
  - Define the map $I:\{0,1,\cdots,m-1\}\times Z \to Z$ such that $I(k,l) = kn - lm$
  - I is bijective if and only if m and n is co-prime. Or in other words, I is bijective if and only if the highest common factor of m and n is 1.

Linear Multirate Systems

- Realization

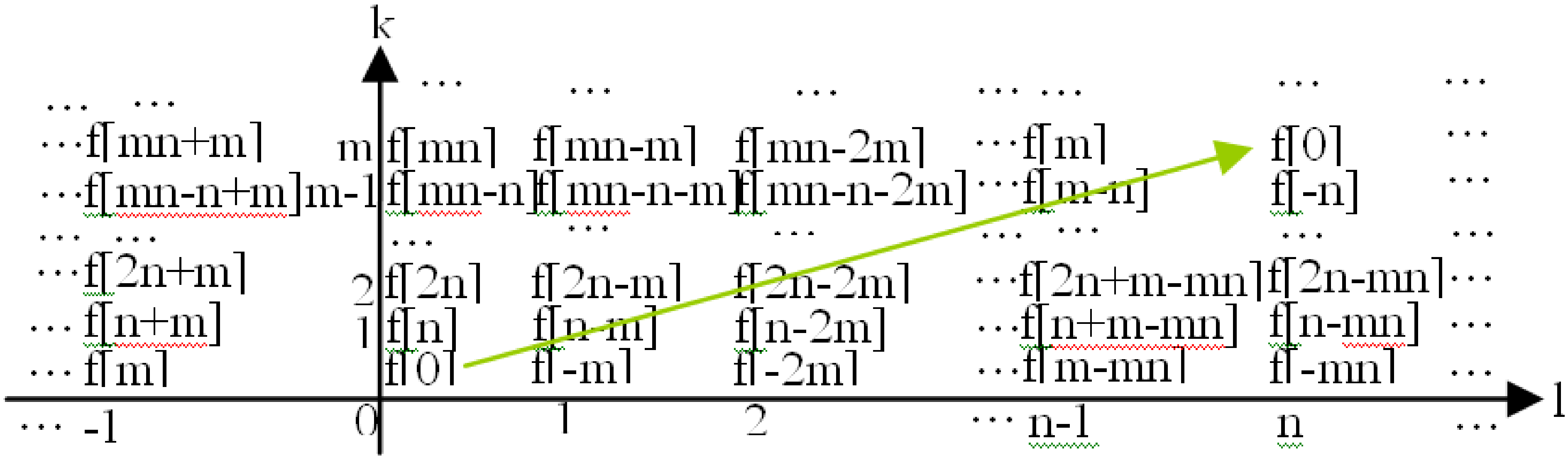


Figure 1a. Mapping from $g[n,k]$ to $f[k]$ when m and n are co-prime.

Linear Multirate Systems

- Realization

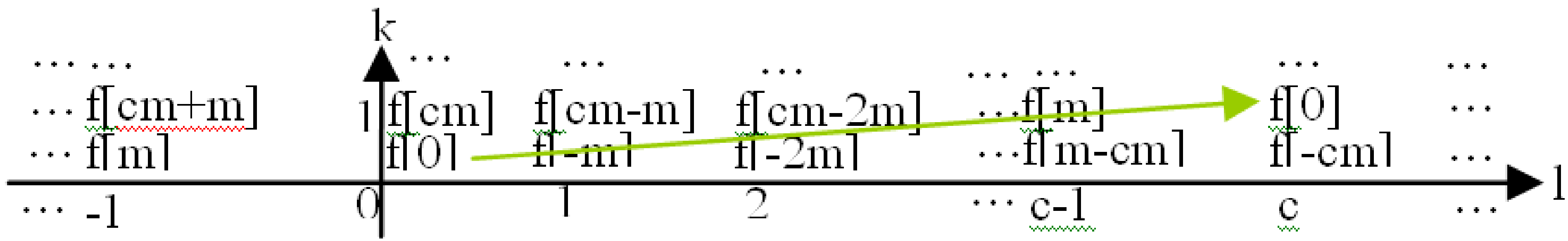


Figure 1b. Mapping from $g[n,k]$ to $f[k]$ when $n=cm$.

Linear Multirate Systems

- Realization

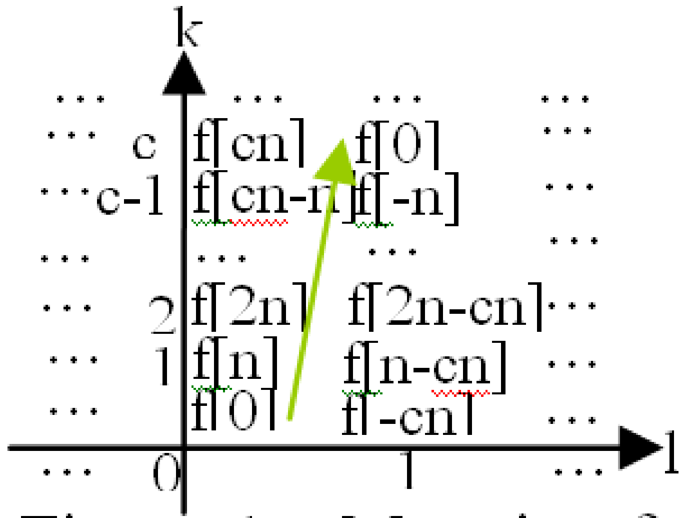


Figure 1c. Mapping from $g[n,k]$ to $f[k]$ when $m=cn$.

Linear Multirate Systems

❖ Realization

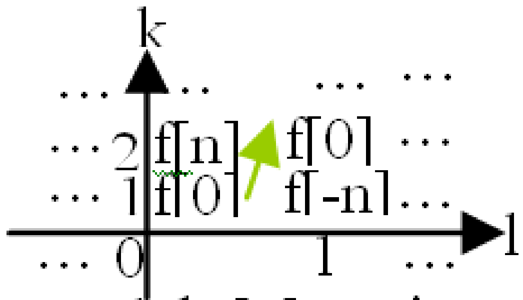
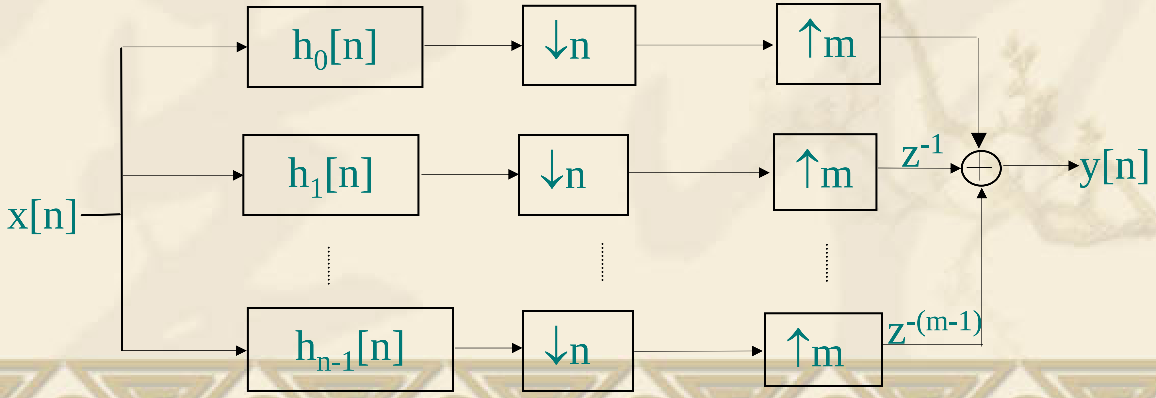


Figure 1d. Mapping from $g[n,k]$ to $f[k]$ when $m=n$.

Linear Multirate Systems

- Realization
  - A linear multirate system is equivalent to a rate changer if and only if m and n is co-prime. That is:

is equivalent to



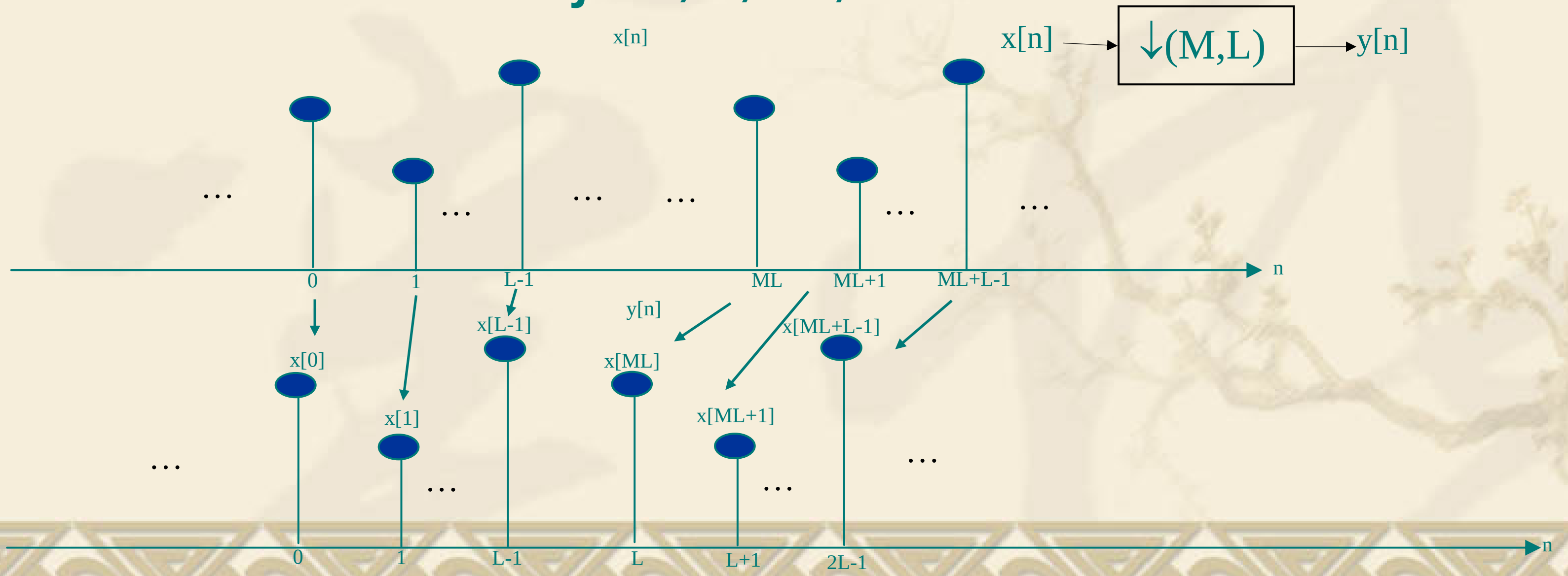
Linear Multirate Systems

- Properties
  - A linear multirate system is stable if and only if $h_i[n]$ for $i=0,1,\dots,n-1$ are all stable.
  - A linear multirate system is finite impulse response if and only if $h_i[n]$ for $i=0,1,\dots,n-1$ are all finite impulse response.

Linear Multirate Systems

- Realization
  - Block decimators (decimation ratio M and block length L)

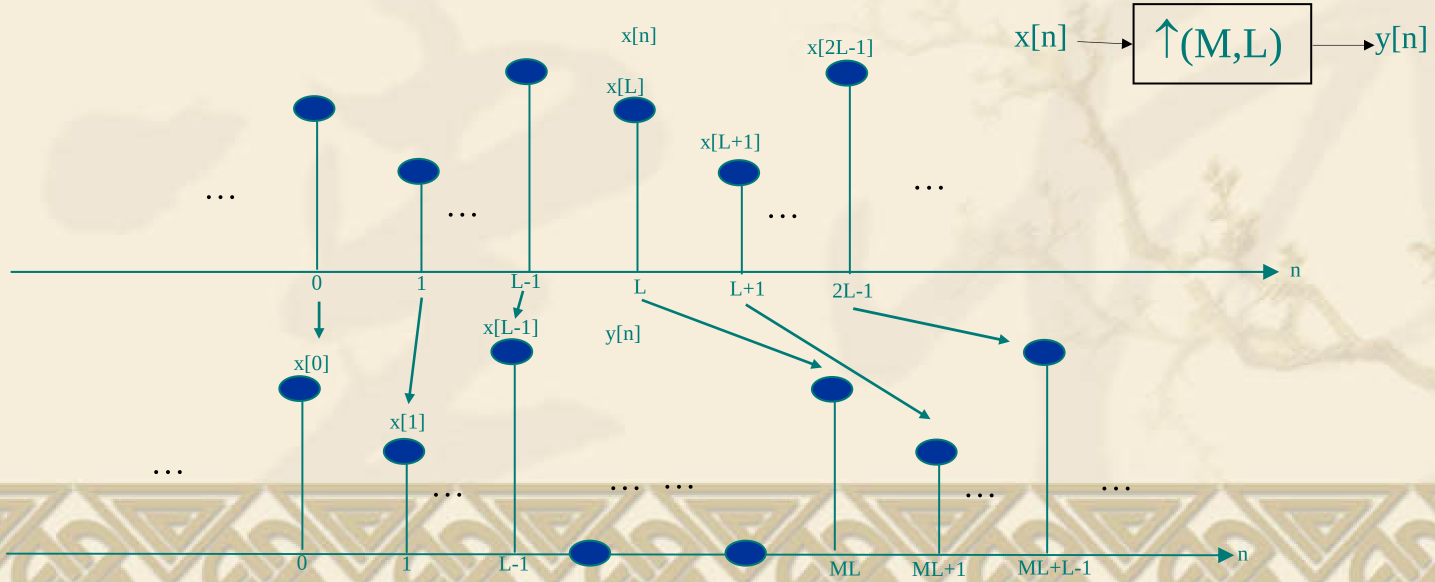
$y[Lk+j]=x[kML+j]$ for j=0,1,…,L-1 and k∈Z.



Linear Multirate Systems

- Realization
  - Block expanders (expansion ratio M and block length L)

$$y[k]=\begin{cases} x\left[\dfrac{k-\mathrm{mod}(k,ML)}{M}+\mathrm{mod}(k,ML)\right] & k-\mathrm{mod}(k,ML)\le k<k-\mathrm{mod}(k,ML)+L \\ 0 & k-\mathrm{mod}(k,ML)+L\le k<k+ML-\mathrm{mod}(k,ML) \end{cases}$$



Linear Multirate Systems

- Realization
  - $\forall m,n \in Z^+$ (no matter m and n are co-prime or not), all linear multirate systems (shifting input by n samples resulting to shifting an output by m samples) can be represented via a series cascade of $\uparrow m$, followed by an LTI filter with an impulse response f[k], and then followed by $\downarrow (n,m)$.

Linear Multirate Systems

- Realization
  - The input output relationship of all linear multirate rate systems is $y[km+i] = \sum_{l \to -\infty}^{+\infty} g[i, l-kn]\, u[l]$, $\forall$k,l$\in$Z, $\forall$m,n$\in$$Z^+$ and i=0,1,…,m-1.
  - The input output relationship of the system with block sampler is $y[km+i] = \sum_{\forall l} f[kmn-ml+i]\, u[l]$, $\forall$k,l$\in$Z, $\forall$m,n$\in$$Z^+$ and i=0,1,…,m-1.
  - $\forall$k,l$\in$Z, $\forall$m,n$\in$$Z^+$ and i=0,1,…,m-1, the mapping from {0,1,…,m-1}xZ to Z, where [i,l-kn]$\in${0,1,…,m-1}xZ and kmn-ml+i$\in$Z is bijective.

Linear Multirate Systems

- Realization
  - Hence, $\forall k,l \in Z$, $\forall m,n \in Z^+$ and $i=0,1,\ldots,m-1$, there exists a unique time index $kmn-ml+i$ corresponding to the time index $[i,l-kn]$.
  - As a result, there exists an LTI filter with an impulse response $f[k]$ satisfying $f[kmn-ml+i]=g[i,l-kn]$, $\forall k,l \in Z$, $\forall m,n \in Z^+$ and $i=0,1,\ldots,m-1$, that the linear multirate rate systems and the system with block sampler are input output equivalent.

Linear Multirate Systems

- Realization
  - $\forall m,n \in Z^{+}$ (no matter m and n are co-prime or not), all linear multirate rate systems (with shifting input by n samples resulting to shifting an output by m samples) can be represented via a series cascade of ↑(m,n), followed by an LTI filter with an impulse response f[k], and then followed by ↓n.

Linear Multirate Systems

- Realization
  - The input output relationship of all linear multirate rate systems is $y[k]=\sum_{l\to-\infty}^{+\infty}\sum_{i=0}^{n-1} g[k,nl+i]u[nl+i]$, $\forall$k,l$\in$Z, $\forall$m,n$\in Z^+$ and i=0,1,…,n-1.
  - The input output relationship of the system with block sampler is $y[k]=\sum_{l\to-\infty}^{+\infty}\sum_{i=0}^{n-1} f[kn-mnl-i]u[nl+i]$, $\forall$k,l$\in$Z, $\forall$m,n$\in Z^+$ and i=0,1,…,n-1.
  - $\forall$l$\in$Z, $\forall$m,n$\in Z^+$, k$\in${0,1,…,m-1} and i$\in${0,1,…,n-1}, the mapping from {0,1,…,m-1}xZ to Z, where [k,nl+i]$\in${0,1,…,m-1}xZ and kn-mnl-i$\in$Z is bijective.

Linear Multirate Systems

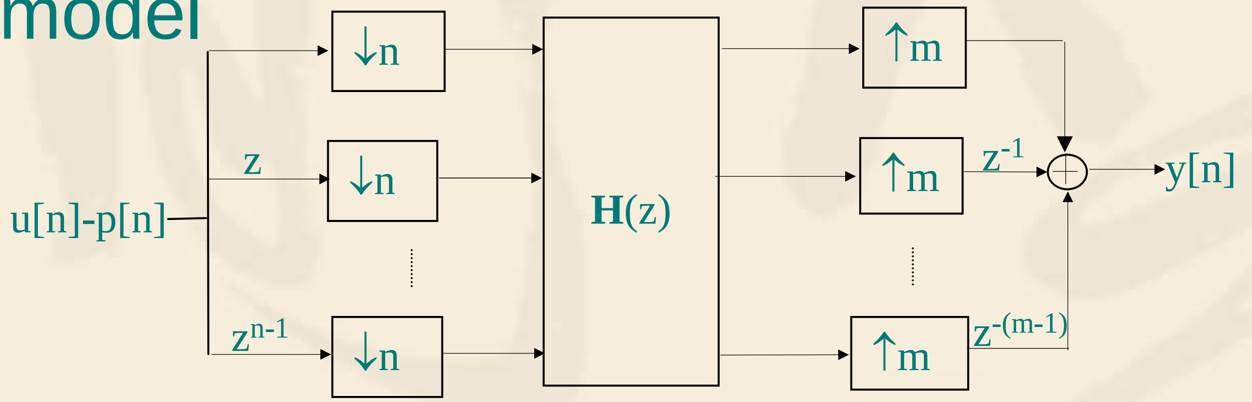
- Realization
  - Hence, $\forall l \in Z$, $\forall m,n \in Z^+$, $k \in \{0,1,\ldots,m-1\}$ and $i \in \{0,1,\ldots,n-1\}$, there exists a unique time index $kn-mnl-i$ corresponding to the time index $[k,nl+i]$.
  - As a result, there exists an LTI filter with an impulse response $f[k]$ satisfying $f[kn-mnl-i]=g[k,nl+i]$, $\forall k,l \in Z$, $\forall m,n \in Z^+$ and $i=0,1,\ldots,n-1$, that the linear multirate rate systems and the system with block sampler are input output equivalent.

Linear Multirate Systems

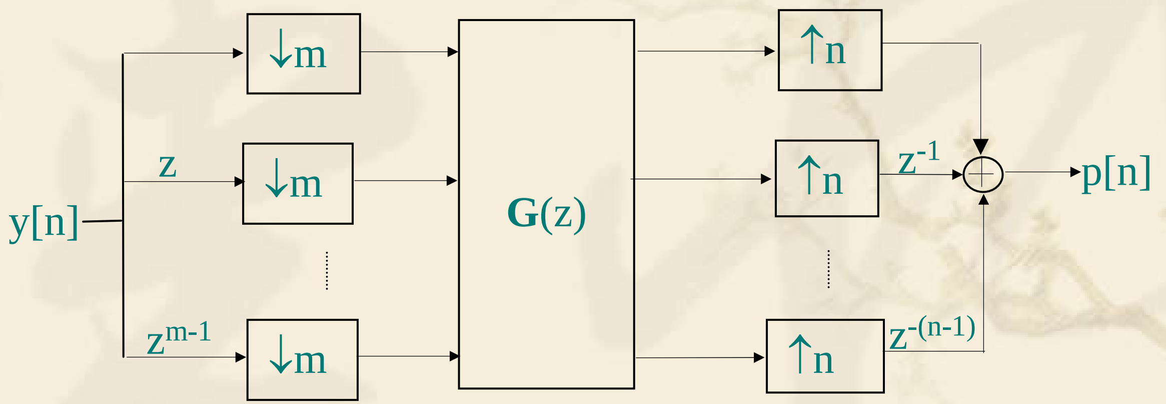
- Properties
  - A linear multirate system is stable if and only if f[n] is stable.
  - A linear multirate system is finite impulse response if and only if f[n] is finite impulse response.

Control of Linear Multirate Systems via Filter Banks Approach

- Plant model



- Controller model



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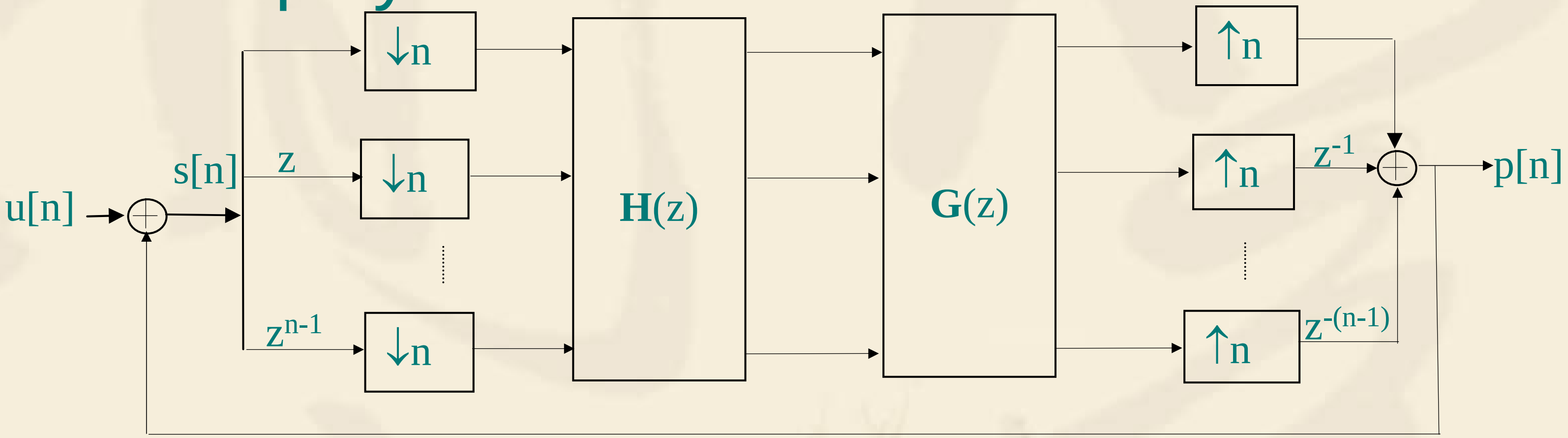
❖ Closed loop system model



$$U(z) \equiv \sum_{\forall k} u(k) z^{-k} \qquad P(z) \equiv \sum_{\forall k} p(k) z^{-k}$$

$$U_i(z) \equiv \sum_{\forall k} u(nk+i) z^{-k} \qquad P_i(z) \equiv \sum_{\forall k} p(nk+i) z^{-k}$$

$$U(z) = \sum_{i=0}^{n-1} z^{-i} U_i(z^n) \qquad P(z) = \sum_{i=0}^{n-1} z^{-i} P_i(z^n)$$

$$\mathbf{U}_p(z) \equiv \begin{bmatrix} U_0(z) & \cdots & U_{n-1}(z) \end{bmatrix}^T \qquad \mathbf{P}_p(z) \equiv \begin{bmatrix} P_0(z) & \cdots & P_{n-1}(z) \end{bmatrix}^T$$

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- Closed loop system model

$$Y(z) \equiv \sum_{\forall k} y(k) z^{-k}$$

$$Y_i(z) \equiv \sum_{\forall k} y(mk+i) z^{-k}$$

$$Y(z) = \sum_{i=0}^{m-1} z^{-i} Y_i(z^m)$$

$$\mathbf{Y}_p(z) \equiv \begin{bmatrix} Y_0(z) & \cdots & Y_{m-1}(z) \end{bmatrix}^T$$

$$\mathbf{G}(z)\mathbf{H}(z)\left(\mathbf{U}_p(z) - \mathbf{P}_p(z)\right) = \mathbf{P}_p(z)$$

$$\mathbf{P}_p(z) = \left(\mathbf{I} + \mathbf{G}(z)\mathbf{H}(z)\right)^{-1} \mathbf{G}(z)\mathbf{H}(z)\mathbf{U}_p(z)$$

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$$\mathbf{Y}_p(z)=\mathbf{H}(z)\left(\mathbf{U}_p(z)-\mathbf{P}_p(z)\right)$$

$$=\mathbf{H}(z)\left(\mathbf{I}-\left(\mathbf{I}+\mathbf{G}(z)\mathbf{H}(z)\right)^{-1}\mathbf{G}(z)\mathbf{H}(z)\right)\mathbf{U}_p(z)$$

$\mathbf{H}(z)\left(\mathbf{I}-\left(\mathbf{I}+\mathbf{G}(z)\mathbf{H}(z)\right)^{-1}\mathbf{G}(z)\mathbf{H}(z)\right)$ is stable.

Conclusions

- Only three types of behaviors for autonomous response of linear time invariant systems.
- Behaviors of linear time invariant systems only depend on the system poles, not on initial conditions.
- Stability conditions based on pole placement, state feedback and output feedback of linear time invariant systems are derived.
- Linear multirate systems can be realized via a filter bank.
- When the input rate and the output rate is co-prime, then linear multirate systems can be realized via linear rate changers. Otherwise, they can be realized via block samplers.
- Stability conditions for linear multirate feedback systems are derived based on filter bank approach.

Questions and Answers

Thank you!

Let me think…