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The global sliding mode tracking control for a class of variable order fractional differential systems

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ABSTRACT

In this paper, a novel variable order fractional control approach is proposed for tracking control of both of variable order fractional and constant order fractional order system with uncertain and external disturbance terms. In term of the global sliding mode control theory and terminal sliding mode control method, the system states are guaranteed to stay on the switching surface from the initial time, and then converge to the origin by the designed controllers which are continuous input signals. Such controllers avoid the undesirable chattering and remove the effects of uncertain and external disturbance completely. Finally, the comparison between the variable order fraction controller and the constant order fractional controller is given by numerical simulation, furthermore, numerical results on the effects of the tracking control are provided.

1. Introduction

Many complicate phenomena in practical problems can be described by the fractional order mathematical formulation, which stimulates the rapid development of the basic mathematical theory of fractional calculus, see [1–7]. In 1998, Lorenzo and Hartley [8] proposed a kind of definition of the variable order fractional derivative and gave the basic properties, whereafter, some kinds of definitions of variable order fractional derivative, such as the Riemann-Liouville type [9], Caputo type and Marchaud type [10], were obtained. It is mentioned that in Ref. [11], Ramirez and Coimbra illustrated how to select a proper variable order fractional derivative to describe physical problem. For more information on the application and distinction of variable order fractional (VOF) operator and constant order fractional (COF) operator, see [12]. Recent years, the nonlinear fractional differential equations have been employed to investigate the practical problems, fruitful results on the nonlinear phenomenon have been obtained, such as [13–18].

Study on chaos control is one of the most important problem in fractional dynamical system with uncertain and external terms. There exist many approaches to tackle the problem of chaos control for fractional chaotic systems, such as, back stepping methods, feedback control, adaptive control [19–22]. Sliding mode control (SMC) method [23] has been proved to be an advantageous robust approach for the tracking control of the nonlinear systems with uncertainties and external disturbances [24–26], which depends on the significant characteristics of SMC approach, such as less sensitivity and acceptable transient performance [27]. In practical applications, it often requires high precision asymptotic convergence, finite time convergence, better disturbance compensation and eliminated the reaching phase. But, the traditional sliding mode controller can not satisfy these characteristics, thus, the terminal sliding mode controller [28–30] and global sliding mode control(GSMC) approach [31,32], and composite learning adaptive sliding mode control with actuator faults [33], which largely meet the engineering requirements, are obtained.

The GSMC method can eliminate the reaching phase by designing a system structure, and the state trajectories can be guaranteed on the sliding mode surface from the initial instant. Based on double hidden layer recurrent neural network, an adaptive global sliding mode controller for a designed full regulated neural network structure was investigated in [34]. In addition, Mobayen [35], Mobayen et al. [36], Mobayen [37] considered the control problem of uncertain dynamical systems by applying GSMC. Chen et al. [38] designed a global fast terminal sliding mode controller

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and handled the slow convergence problem as well as frequency change problem of hydraulic turbine regulating system. In [39], the tracking problem was investigated for a class of nonlinear systems under parameter uncertainty and external disturbance, and a fast terminal sliding mode controller combined with global sliding mode method applied in [34–38] (GFTSMC) was derived at a predetermined convergence time to guarantee the fast convergence to the equilibrium of the system.

To our best knowledge, there hardly exist results on the global sliding mode tracking control for fractional chaotic system. Thus, this paper considers the tracking control problem for both COF system and VOF system with uncertain and external terms. For each system, the global sliding mode tracking controllers are proposed and the stability of the controllers are proved by Lyapunov direct method. The rest of paper is organized as follows. In Section 2, the basic theory of the fractional calculus and the system description are presented. The global sliding mode tracking control of the COF system is considered in Section 3 and the VOF case is presented is Section 4. Section 5 provides the numerical simulations to show the viability and efficiency of the proposed controllers.

2. Preliminaries

The following is the definition of VOF derivatives [9]:

$$ {}^{C}D_t^{q(t)}x(t) = \frac{1}{\Gamma(1-q(t))}\int_0^t (t-s)^{-q(t)}x'(s)ds, 0 < q(t) < 1, \quad (1) $$

when $q(t)$ is a constant, the above definition is the Caputo COF operators.

The n-dimensional nonlinear COF differential system with uncertainties and external disturbances described by

$$ \begin{cases} {}_0^C D_t^{\alpha} x_1(t) = f_1(t,X) + \Delta f_1(t,X) + d_1(t) + u_1(t), \\ {}_0^C D_t^{\alpha} x_2(t) = f_2(t,X) + \Delta f_2(t,X) + d_2(t) + u_2(t), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ {}_0^C D_t^{\alpha} x_n(t) = f_n(t,X) + \Delta f_n(t,X) + d_n(t) + u_n(t), \end{cases} \quad (2) $$

where $0 < \alpha \le 1$, the states vector is $X(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$, and $f_i(t,X) \in R, i = 1,2,\ldots,n$ represent the known nonlinear functions, $d_i(t) \in \mathbb{R}$ and $\Delta f_i(t,X) \in \mathbb{R}, i = 1,2,\ldots,n$ denote the uncertainty and the external disturbances of the system respectively, which is required to be bounded. $u_i(t) \in R, i = 1,2,\ldots,n$ denotes the control input.

If the fractional order parameter in system (2) is a variable with respect to the time, it means that the order α in (2) is changed to be $\alpha(t)$, then the system (2) becomes the following n-dimensional uncertain VOF nonlinear system

$$ \begin{cases} {}_0^C D_t^{\alpha(t)} x_1(t) = f_1(t,X) + \Delta f_1(t,X) + d_1(t) + u_1(t), \\ {}_0^C D_t^{\alpha(t)} x_2(t) = f_2(t,X) + \Delta f_2(t,X) + d_2(t) + u_2(t), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ {}_0^C D_t^{\alpha(t)} x_n(t) = f_n(t,X) + \Delta f_n(t,X) + d_n(t) + u_n(t). \end{cases} \quad (3) $$

In addition, the following hypothesises are held,

Assumption 1. There exists a known positive constant $0 < M_i < \infty$ such that $|\Delta f_i(t,X) + d_i(t)| \le M_i$ for $i = 1,2,\ldots,n$.

Assumption 2. There exists a known positive constant $0 < N_i < \infty$ such that $|D^{\alpha(t)}(\Delta f_i + d_i)| \le N_i$ for $i = 1,2,\ldots,n$.

Remark 1. For most systems, the system uncertainty terms and external disturbances are always bounded. From an applied point of view, the amplitude of the designed controller is limited. Thus, the above requirements for Assumption 1 and 2 can be achieved and not restricting.

When $\alpha(t) = \alpha$, we might as well still use the constant N_i to represent the bound.

Consider the tracking control of the trajectories for the fractional system (2) and (3), and the error is chosen as follows:

$$ \begin{cases} e_1(t) = x_1(t) - x_{1d}(t), \\ e_2(t) = x_2(t) - x_{2d}(t), \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ e_n(t) = x_n(t) - x_{nd}(t). \end{cases} \quad (4) $$

Here $x_{id}, i = 1,2,\ldots,n$ are the desired reference trajectories.

3. The global sliding mode tracking control of the COF system

The goal of this section is to design a GSMC scheme such that the trajectories of the system (2) converge to the reference trajectories asymptotically in a finite time.

The global sliding mode surface (GSMS) is designed as follows

$$ s_i(t) = D^{\alpha}e_i(t) + c_i e_i(t) - r_i(0)e^{-p_i t}, i = 1,2,\ldots,n, \quad (5) $$

where $c_i > 0, i = 1,2,\ldots,n$, $p_i > 0, i = 1,2,\ldots,n$ and $r_i(t) = D^{\alpha}e_i(t) + c_i e_i(t), i = 1,2,\ldots,n$.

Lemma 3.1. *[1] Let $n > 0, r > 0, \varphi \in [-\pi,\pi]$ and $\lambda = rexp(i\varphi)$. Denote $y(t) = E_n(-\lambda t^n)$, then (a) $\lim_{t\to\infty} y(t) = 0$ if $|\varphi| < n\pi/2$;*
(b) $y(t)$ is unbounded as $t \to \infty$ if $|\varphi| > n\pi/2$.

According to the GSMS scheme (5), when $s_i(t) = 0$, then, one can obtain the sliding mode dynamics system:

$$ D^{\alpha}e_i(t) + c_i e_i(t) = r_i(0)e^{-p_i t}. \quad (6) $$

On the other hand, for the Eq. (6), one has

$$ \begin{aligned} e_i(t) &= e_i(0)E_{\alpha}(-c_i t^{\alpha}) \\ &\quad + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-c_i(t-\tau)^{\alpha})r_i(0)e^{-p_i\tau}d\tau. \\ &\equiv I + II, \end{aligned} \quad (7) $$

where $E_{\alpha,\beta} = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}(z, \beta \in CR(\alpha) > 0)$.

As for II in (7), we have the following result according to Lemma 3.1:

$$ \begin{aligned} &\left|\left|\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-c_i(t-\tau)^{\alpha})r_i(0)e^{-p_i\tau}d\tau\right|\right| \\ &\le |r_i(0)|||E_{\alpha,\alpha}(-c_i t^{\alpha})|||\left|\int_0^t (t-\tau)^{\alpha-1}e^{-p_i\tau}d\tau\right| \\ &\le \frac{|r_i(0)|}{t^{1-\alpha}}||E_{\alpha,\alpha}(-c_i t^{\alpha})|||\frac{1}{p_i} - \frac{e^{-p_i t}}{p_i}| \to 0(t\to\infty). \end{aligned} \quad (8) $$

Let $t \to \infty$ in both sides of (8), we have the result that II in (7) trends to 0. Thus, $||e_i(t)|| \to 0$ with $t \to \infty$, which implies that the sliding mode error dynamics (7) is asymptotically stable, alternatively, error states converge to the origin.

From the above global sliding mode surface, we know that once the error states are driven to the sliding mode surface, then, the error states can be made to converge to zero. Thus, in order to drive the system to the sliding mode surface, the following control law is designed with the initial conditions $u_i(0) = 0$:

$$ \begin{aligned} &D^{\alpha}u_i + c_i u_i \\ &= -[D^{\alpha}f_i + N_i sgn(s_i) - D^{\alpha}[D^{\alpha}x_{id}] + c_i f_i + c_i M_i sgn(s_i) - c_i D^{\alpha}x_{id} \\ &\quad + k_i sgn(s_i)|s_i|^{\eta_i} + r_i s_i + \delta_i sgn(s_i)], i = 1,2,\ldots,n. \end{aligned} \quad (9) $$

Remark 2. Under the Assumption 1 and 2, the form of the above controller is proper. Moreover, the control law proposed above combines the fractional order operator and the traditional sliding mode control approach. And it guarantees the convergence of the error signals to the origin from the initial time. Additionally, it is a continuous signal which is free of chattering.

Let

$$V_1(s_i) = \frac{1}{2}\sum_{i=1}^{n} s_i^2, \tag{10}$$

obviously, it is a Lyapunov function. Moreover

$$\begin{aligned} D^\alpha V_1(s_i) &\le \sum_{i=1}^{n} s_i D^\alpha s_i \\ &= \sum s_i D^\alpha [D^\alpha x_i - D^\alpha x_{id} + c_i x_i - c_i x_{id} - r_i(0)e^{-p_i t}] \\ &= \sum_{i=1}^{n} s_i [D^\alpha f_i + D^\alpha(\Delta f_i + d_i) + D^\alpha u_i - D^\alpha[D^\alpha x_{id}] \\ &\quad + c_i D^\alpha x_i - c_i D^\alpha x_{id} + r_i(0) p_i t^{1-\alpha} E_{1,2-\alpha}(-p_i t)] \\ &= \sum_{i=1}^{n} s_i [D^\alpha f_i + D^\alpha(\Delta f_i + d_i) + D^\alpha u_i + c_i u_i - D^\alpha[D^\alpha x_{id}] \\ &\quad + c_i f_i + c_i(\Delta f_i + d_i) - c_i D^\alpha x_{id} + r_i(0) p_i t^{1-\alpha} E_{1,2-\alpha}(-p_i t)]. \end{aligned}$$

By applying the proposed control law (9) into the above system, thus, the lyapunov function $V_1(s)$ can be simplified into the following form

$$\begin{aligned} D^\alpha V_1(s_i) &\le \sum_{i=1}^{n} s_i(-k_i sgn(s_i)|s_i|^{\eta_i} - r_i s_i - \delta_i sgn(s_i) + r_i(0) p_i t^{1-\alpha} \\ &\quad \times E_{1,2-\alpha}(-p_i t)) \\ &= \sum_{i=1}^{n} (-k_i|s_i|^{\eta_i+1} - r_i s_i^2 - \delta_i|s_i| + s_i r_i(0) p_i t^{1-\alpha} E_{1,2-\alpha}(-p_i t)). \end{aligned} \tag{11}$$

Let $t \to \infty$ in both sides of (11), we have $\lim_{t\to\infty} t^{1-\alpha} e^{-p_i t} = 0$, then,

$$\begin{aligned} D^\alpha V_1(s_i) &\le \sum_{i=1}^{n} (-k_i|s_i|^{\eta_i+1} - r_i s_i^2 - \delta_i|s_i|) \\ &\le -\alpha_1 V_1(s_i), \end{aligned}$$

where $\alpha_1 = min\{r_1, r_2, \cdots, r_n\}$. Thus, the following theorem can be obtained.

Theorem 3.2. *For the COF nonlinear system (2), if the GSMS scheme (5) is applied, then, the error $e_i(t), i = 1, 2, \ldots, n$ of system (2) are firstly driven to the sliding mode surface (5) under the proposed control law (9), then converge to zero asymptotically.*

Remark 3. For this nonlinear system, it needs to know the relative bound of the uncertain and external disturbances for this control problem. And, the proposed controller law is a global variable order fractional sliding mode control approach, and it has the global robustness property.

4. The global sliding mode tracking control of the variable order fractional system

For the systems (3), the variable order fractional sliding mode surface is designed as follows:

$$\bar{s}_i(t) = D^{\alpha(t)} e_i(t) + \bar{c}_i e_i(t) - \bar{r}_i(0) e^{-\bar{p}_i t}, \tag{12}$$

where $\bar{r}_i(t) = D^{\alpha(t)} e_i(t) + \bar{c}_i e_i(t), i = 1, 2, \ldots, n, \bar{p}_i > 0$. When the states of the system reach the sliding manifold, the following must be satisfied:

$$\bar{s}_i(t) = 0, \ \ i = 1, 2, \cdots, n.$$

Then, it can be obtained that

$$D^{\alpha(t)} e_i(t) = -\bar{c}_i e_i(t) + \bar{r}_i(0) e^{-\bar{p}_i t}. \tag{13}$$

Theorem 4.1. *For the variable order fractional sliding mode dynamics (3), if it's applied the global sliding mode surface (12) into the system, then, its error state trajectories converge to zero asymptotically.*

Proof. Choose the Lyapunov function as following

$$V_2(t) = \frac{1}{2}\sum_{i=1}^{n} e_i^2(t). \tag{14}$$

Applying the variable order fractional operator $D^{\alpha(t)}$ to both the sides of the Eq. (14), we have that

$$\begin{aligned} D^{\alpha(t)} V_2(t) &\le \sum_{i=1}^{n} e_i(t) D^{\alpha(t)} e_i(t) \\ &= \sum_{i=1}^{n} e_i(t)(-\bar{c}_i e_i(t) + r_i(0) e^{-\bar{p}_i t}) \\ &= -\sum_{i=1}^{n} \bar{c}_i e_i^2(t) + \sum_{i=1}^{n} r_i(0) e_i(t) e^{-\bar{p}_i t}, \end{aligned} \tag{15}$$

let $t \to \infty$, we have

$$D^{\alpha(t)} V_2(t) \le -\sum_{i=1}^{n} \bar{c}_i e_i^2(t) \le 2\bar{c} V_2(t), \tag{16}$$

where $\bar{c} = \min\{\bar{c}_1, \bar{c}_2, \cdots, \bar{c}_n\}$. Thus, when the system is driven to the sliding mode surface, then, the error states converge to the origin asymptotically. □

Theorem 4.2. *Consider the variable order fractional nonlinear system (3). Employing the following variable order fractional sliding mode controller with the initial conditions $u_i(0) = 0$:*

$$\begin{aligned} D^{\alpha(t)} u_i + \bar{c}_i u_i = &-[D^{\alpha(t)} f_i + N_i sgn(s_i) - D^{\alpha(t)}[D^{\alpha(t)} x_{id}] + \bar{c}_i f_i \\ &+ \bar{c}_i M_i sgn(s_i) - \bar{c}_i D^{\alpha(t)} x_{id} + m_i sgn(s_i)|s_i|^{\bar{n}_i} \\ &+ \bar{r}_i s_i + \bar{\delta}_i sgn(s_i)]. \end{aligned} \tag{17}$$

Then, its state trajectories converge to zero asymptotically under the controllers.

Proof. Let

$$V_3(t) = \frac{1}{2}\sum_{i=1}^{n} s_i^2. \tag{18}$$

Applying the fractional order operator to both sides of (18), we have

$$\begin{aligned} D^{\alpha(t)} V_3(t) &\le \sum_{i=1}^{n} s_i D^{\alpha(t)} s_i \\ &= \sum_{i=1}^{n} s_i D^{\alpha(t)} [f_i(t, X) + \Delta f_i + d_i(t) + u_i(t) - D^{\alpha(t)} x_{id} \\ &\quad + \bar{c}_i x_i - \bar{c}_i x_{id} - \bar{r}_i(0) e^{-\bar{p}_i t}] \\ &= \sum_{i=1}^{n} s_i [D^{\alpha(t)} f_i(t, X) + D^{\alpha(t)} \Delta f_i + D^{\alpha(t)} d_i(t) \\ &\quad + D^{\alpha(t)} u_i(t) - D^{\alpha(t)}(D^{\alpha(t)} x_{id}) + \bar{c}_i f_i + \bar{c}_i \Delta f_i + \bar{c}_i d_i(t) \\ &\quad + \bar{c}_i u_i - \bar{c}_i D^{\alpha(t)} x_{id} - \bar{r}_i(0) D^{\alpha(t)}(e^{-\bar{p}_i t})]. \end{aligned}$$

Since

$$D^{\alpha(t)}(e^{-\bar{p}_i t}) \le \frac{\Gamma(1-q_1)}{\Gamma(1-q_2)} D^{q_1}(e^{-\bar{p}_i t}), \tag{19}$$

let $t \to \infty$ of (19), we get that $\lim_{t\to\infty} D^{\alpha(t)}(e^{-\bar{p}_i t}) = 0$.

According to the control law (17), as $t \to \infty$, we have

$$D^{\alpha(t)} V_3(t) \le \sum_{i=1}^{n} s_i[-m_i sgn(s_i)|s_i|^{\bar{n}_i} - \bar{r}_i s_i - \bar{\delta}_i sgn(s_i)] \tag{20}$$

$$= -\sum_{i=1}^{n} m_i|s_i|^{1+\bar{n}_i} - \sum_{i=1}^{n} \bar{r}_i s_i^2 - \sum_{i=1}^{n} \bar{\delta}_i|s_i|$$
$$\leq \xi V_3(t)$$

where $\xi = \min\{\bar{r}_1, \bar{r}_2, \cdots, \bar{r}_n\}$, which completes the theorem. □

Remark 4. In the presentation of the control law (17), the variable order fractional operator is applied such that it can be used to realize the tracking control of the variable order fractional system. It extends the sliding mode control to the variable order fractional type, produces a continuous control signal, and then obtains a good control performance.

5. Numerical simulation

This section is used to give two examples to demonstrate the validity of the theoretical results obtained in Section 3 and Section 4. It is mentioned that, as a special case of predictor-corrector methods, Adams-Bashforth-Moulton is a natural and common numerical approximation method for VOF differential equations.

5.1. *Example 1: Sliding mode control of COF system*

Consider the following nonlinear COF differential system

$$\begin{cases} D^\alpha x_1 = x_2 + u_1(t) \\ D^\alpha x_2 = x_3 + 0.2cos(2t) + u_2(t) \\ D^\alpha x_3 = -(7 - 0.3sin(t))x_1(t) - 4x_2(t) - x_3(t) \\ \quad +(1 + 0.35cos(0.6t))x_1^2(t) \\ +1.4cos(2t) + u_3(t) \end{cases} \tag{21}$$

where the model external disturbances, uncertainties and the nonlinear terms are given as

$$\begin{cases} \Delta f_3 = 0.3sin(t)x_1(t) + 0.35cos(0.6t)x_1^2(t), \\ f_1 = x_2(t), f_2 = x_3(t), f_3 = x_1^2(t) - 7x_1(t) - 4x_2(t) - x_3(t), \\ d_2 = 0.2cos(2t), d_3 = 1.4cos(2t). \end{cases} \tag{22}$$

According to the proposed sliding mode surface (12), the suitable sliding mode surface for this model is

$$\begin{cases} s_1(t) = D^\alpha e_1(t) + c_1 e_1(t) - r_1(0)e^{-p_1 t}, \\ s_2(t) = D^\alpha e_2(t) + c_2 e_2(t) - r_2(0)e^{-p_2 t}, \\ s_3(t) = D^\alpha e_3(t) + c_3 e_3(t) - r_3(0)e^{-p_3 t}, \end{cases} \tag{23}$$

where $r_i(t) = D^\alpha e_i(t) + c_i e_i(t), i = 1, 2, 3$. And the tracking errors term $e_i(t)$ is chosen as

$$\begin{cases} e_1(t) = x_1(t) - x_{1d}(t), \\ e_2(t) = x_2(t) - x_{2d}(t), \\ e_3(t) = x_3(t) - x_{3d}(t), \end{cases} \tag{24}$$

with the reference signal $x_{id}(t), i = 1, 2, 3$ supposed as

$$x_{1d} = sint, x_{2d} = sin(t + \alpha\pi/2), x_{3d} = sin(t + \alpha\pi). \tag{25}$$

Based on the proposed control scheme, the control input for this model is as following with the initial conditions $u_1(0) = 0, u_2(0) = 0, u_3(0) = 0$:

$$\begin{cases} D^\alpha u_1 + c_1 u_1 = -D^\alpha x_2 + D^{2\alpha}(sint) - c_1 x_2 + c_1 D^\alpha(sin(t)) \\ -k_1 sgn(s_1)|s_1|^{\eta_1} - r_1 s_1 - \delta_1 sgn(s_1), \\ D^\alpha u_2 + c_2 u_2 = -D^\alpha x_3 - N_2 sgn(s_2) + D^{2\alpha}(cost) \\ \quad -c_2 x_3 - c_2 M_2 sgn(s_2) \\ +c_2 D^\alpha(cos(t)) - k_2 sgn(s_2)|s_2|^{\eta_2} - r_2 s_2 - \delta_2 sgn(s_2), \\ D^\alpha u_3 + c_3 u_3 = -D^\alpha(x_1^2 - 7x_1 - 4x_2 - x_3) - N_3 sgn(s_3) \\ \quad +D^{2\alpha}(-sint) - c_3(x_1^2 \\ -7x_1 - 4x_2 - x_3) - c_3 M_3 sgn(s_3) + c_3 D^\alpha(-sint) \\ \quad -k_3 sgn(s_3)|s_3|^{\eta_3} - r_3 s_3 - \delta_3 sgn(s_3). \end{cases} \tag{26}$$

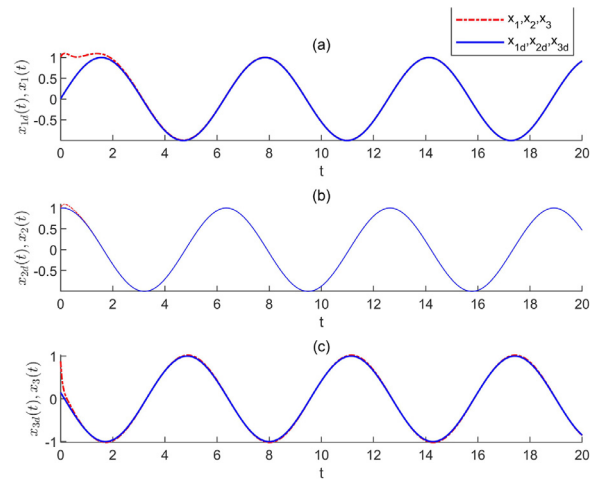

Fig. 1. The state trajectories $x_i(t), x_{id}(t)$ of the system (21) for $\alpha = 0.9$; (a)the $x_1, x_{1d} - t$ space; (b) the $x_2, x_{2d} - t$ space; (c) the $x_3, x_{3d} - t$ space.

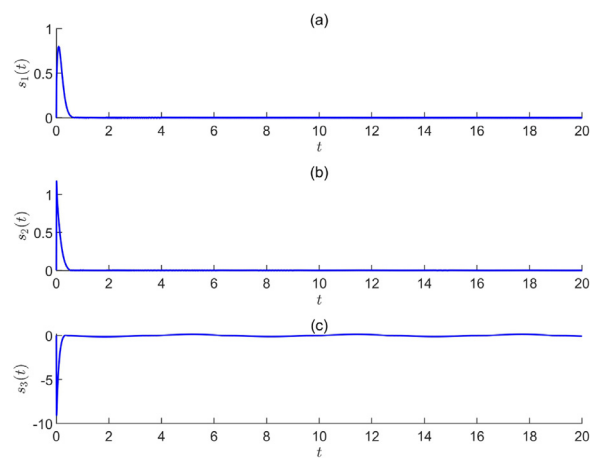

Fig. 2. The sliding mode surface of the system (21) for $\alpha = 0.9$; (a)the $s_1 - t$ space; (b) the $s_2 - t$ space; (c) the $s_3 - t$ space.

The initial value condition is chosen as $x_1(0) = 1, x_2(0) = 1, x_3(0) = 1$, and $c_1 = 2, c_2 = 3, c_3 = 6, k_1 = 8, k_2 = 5, k_3 = 6, \eta_1 = 0.8, \eta_2 = 0.8, \eta_3 = 0.8, r_1 = 0.3, r_2 = 0.3, r_3 = 0.4, \delta_1 = 0.3, \delta_2 = 0.3, \delta_3 = 0.4, p_1 = 10, p_2 = 10, p_3 = 10, N_2 = 0.4, N_3 = 3, M_2 = 0.2, M_3 = 2$ the fractional order parameter is 0.9. Then, under the designed controller, the tracking performances are illustrated in Figs. 1–3 respectively.

From Fig. 1, we can see that the proposed control scheme makes the desired good tracking for the trajectories realized, and the tracking errors are very small. Figs. 2 and 3 show that the manifold responses of the sliding mode surface of the system (21) converge to zero from the beginning of the trajectory, and the control signal is smooth and converges to zero in a finite time.

When the FO parameter is changed to time-varied function $\alpha(t)$, then, the system (21) becomes a VOF system. Let $q(t) = 0.95 + 0.02t/T, t \in [0, T]$, the comparison of the control effect between the COF system and the VOF system is depicted in Fig. 4, where x_i, x_{id}, $i = 1, 2, 3$ are the state trajectories and the reference signals of the VOF system respectively, and x_{ci}, x_{icd}, $i = 1, 2, 3$ are the state trajectories and the reference signals of the COF system with $\alpha = 0.7$ respectively. We find that both constant and variable order fractional systems can achieve the desired good tracking performance. And with the changing of FO parameters, the tracking states are changing accordingly which implies that the VOF system is the generalization of the COF system.

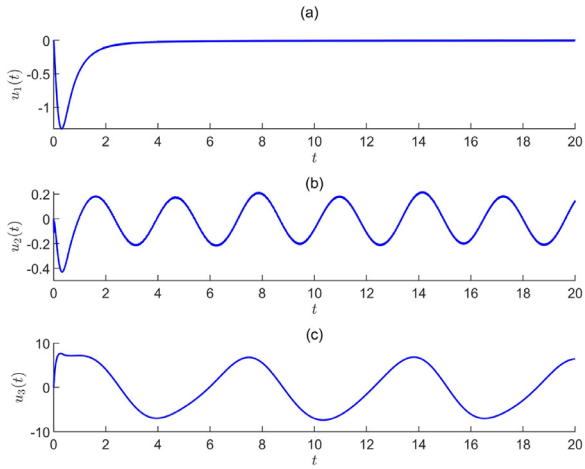

Fig. 3. The control input signals of the system (21) for $\alpha = 0.9$; (a)the $u_1 - t$ space; (b) the $u_2 - t$ space; (c) the $u_3 - t$ space.

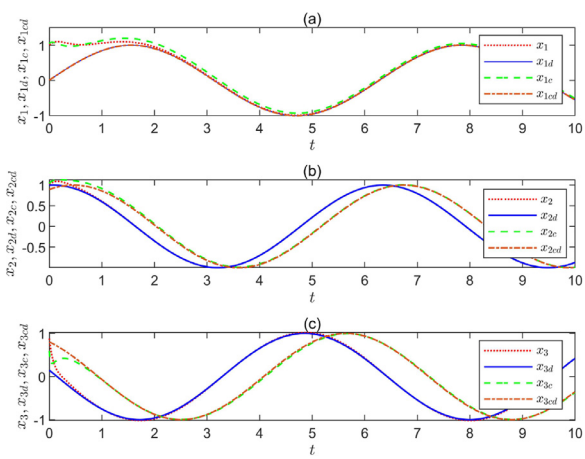

Fig. 4. The comparison of the state trajectories between the VOF system for $\alpha(t) = 0.95 + 0.02t/T, t \in [0, T]$ and the COF system for $\alpha = 0.7$; (a)the $x_1, x_{1d}, x_{1c}, x_{1cd} - t$ space; (b) the $x_2, x_{2d}, x_{2c}, x_{2cd} - t$ space; (c) the $x_3, x_{3d}, x_{3c}, x_{3cd} - t$ space.

5.2. *Example 2: Sliding mode control of VOF*

Genesios chaotic system The following is the variable order fractional Genesio's chaotic system with additional input

$$\begin{cases} D^{\alpha(t)}x_1 = x_2 + u_1(t), \\ D^{\alpha(t)}x_2 = x_3 + u_2(t), \\ D^{\alpha(t)}x_3 = -6x_1(t) - 2.92x_2(t) - 1.2x_3(t) + x_1^2(t) - 0.1sint + u_3(t). \end{cases} \tag{27}$$

Let

$$\begin{cases} f_1 = x_2(t),\ f_2 = x_3(t),\ f_3 = -6x_1(t) - 2.92x_2(t) - 1.2x_3(t) + x_1^2(t), \\ d_3 = -0.1sint, \end{cases} \tag{28}$$

based on the sliding mode surface (23) and the proposed control scheme, the variable order fractional sliding mode surface and control input are given as

$$\begin{cases} s_1(t) = D^{\alpha(t)}e_1(t) + \bar{c}_1 e_1(t) - r_1(0)e^{-\bar{p}_1 t}, \\ s_2(t) = D^{\alpha(t)}e_2(t) + \bar{c}_2 e_2(t) - r_2(0)e^{-\bar{p}_2 t}, \\ s_3(t) = D^{\alpha(t)}e_3(t) + \bar{c}_3 e_3(t) - r_3(0)e^{-\bar{p}_3 t}. \end{cases} \tag{29}$$

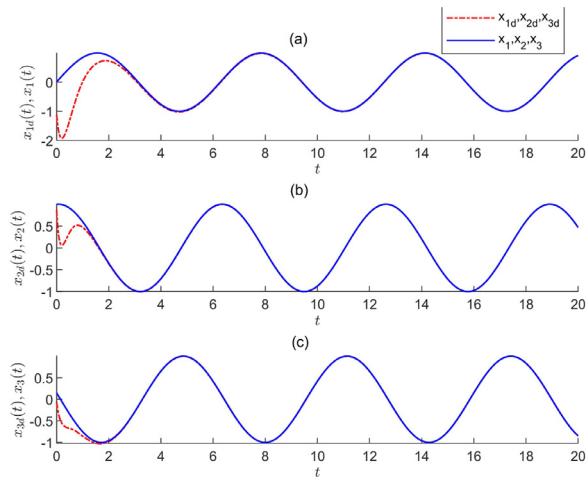

Fig. 5. The state trajectories $x_i(t), x_{id}(t)$ of the system (27); (a)the $x_1, x_{1d} - t$ space; (b) the $x_2, x_{2d} - t$ space; (c) the $x_3, x_{3d} - t$ space.

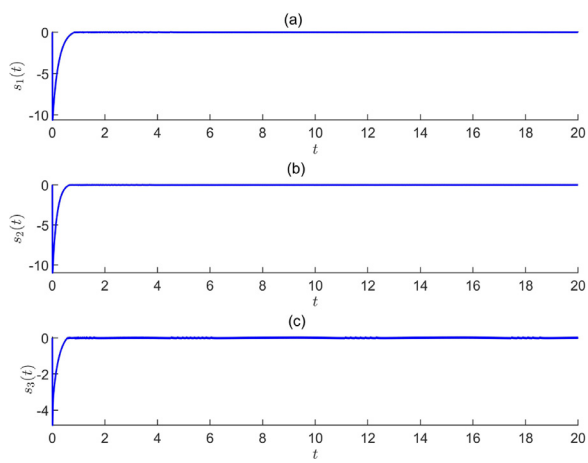

Fig. 6. The sliding mode surface of the system (27); (a)the $s_1 - t$ space; (b) the $s_2 - t$ space; (c) the $s_3 - t$ space.

Then

$$\begin{cases} D^{\alpha(t)}u_1 + \bar{c}_1 u_1 = -D^{\alpha(t)}x_2 + D^{\alpha(t)}[D^{\alpha(t)}(sint)] \\ \quad -\bar{c}_1 x_2 + \bar{c}_1 D^{\alpha(t)}(sin(t)) - \\ \bar{k}_1 sgn(s_1)|s_1|^{\bar{\eta}_1} - \bar{r}_1 s_1 - \bar{\delta}_1 sgn(s_1), \\ D^{\alpha(t)}u_2 + \bar{c}_2 u_2 = -D^{\alpha(t)}x_3 + D^{\alpha(t)}[D^{\alpha(t)}x_{2d}] \\ \quad -\bar{c}_2 x_3 + \bar{c}_2 D^{\alpha(t)}x_{2d} - \\ \bar{k}_2 sgn(s_2)|s_2|^{\bar{\eta}_2} - \bar{r}_2 s_2 - \bar{\delta}_2 sgn(s_2), \\ D^{\alpha(t)}u_3 + \bar{c}_3 u_3 = -D^{\alpha(t)}(-6x_1(t) - 2.92x_2(t) - 1.2x_3(t) \\ \quad + x_1^2(t)) + D^{\alpha(t)}[D^{\alpha(t)}x_{3d}] \\ -\bar{c}_3(-6x_1(t) - 2.92x_2(t) - 1.2x_3(t) + x_1^2(t)) + \bar{c}_3 D^{\alpha(t)}x_{3d} \\ \quad -\bar{k}_3 sgn(s_3)|s_3|^{\bar{\eta}_3} - \bar{r}_3 s_3 \\ -\bar{\delta}_3 sgn(s_3) - N_3 sgn(s_3) - c_3 M_3 sgn(s_3), \end{cases} \tag{30}$$

with the initial conditions $u_1(0) = 0, u_2(0) = 0, u_3(0) = 0$.

With the same reference signal (25) and the initial value condition $x_1(0) = -1, x_2(0) = 1, x_3(0) = 0$, let $\bar{c}_1 = 2, \bar{c}_2 = 3, \bar{c}_3 = 3, \bar{k}_1 = 3, \bar{k}_2 = 3, \bar{k}_3 = 3, \bar{\eta}_1 = 0.8, \bar{\eta}_2 = 0.8, \bar{\eta}_3 = 0.8, \bar{r}_1 = 2, \bar{r}_2 = 3, \bar{r}_3 = 3, \bar{\delta}_1 = 3, \bar{\delta}_2 = 2, \bar{\delta}_3 = 3, \bar{p}_1 = 2, \bar{p}_2 = 2, \bar{p}_3 = 2, N_3 = M_3 = 0.1$, the variable order fractional parameter is $q(t) = 0.95 + 0.02t/T, t \in [0, T]$, then, the numerical results are demonstrated by Figs. 5–7.

Fig. 5 shows that the designed control scheme realizes the tracking effects with small error, and the states converge to the original within a finite time. From Figs. 6 and 7. We have that the

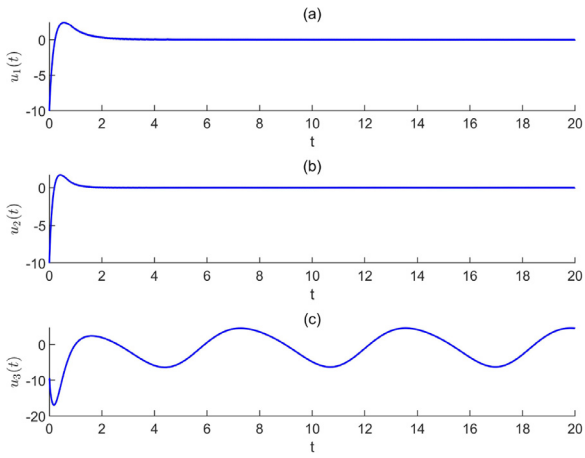

Fig. 7. The control input signals of the system (27); (a)the $u_1 - t$ space; (b) the $u_2 - t$ space; (c) the $u_3 - t$ space.

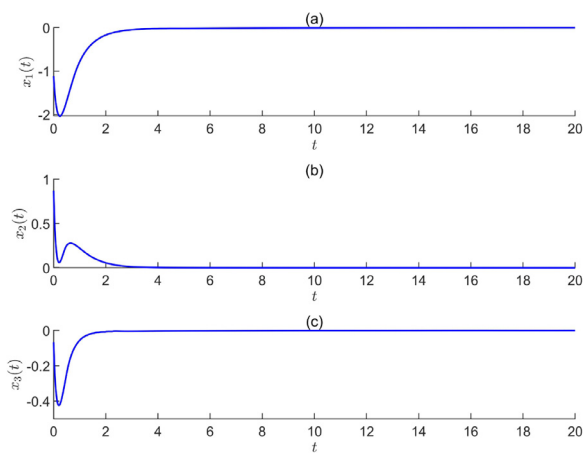

Fig. 8. The states of the Genesio's chaotic system (27); (a)$x_1 - t$ space; (b)$x_2 - t$ space; (c) $x_3 - t$ space.

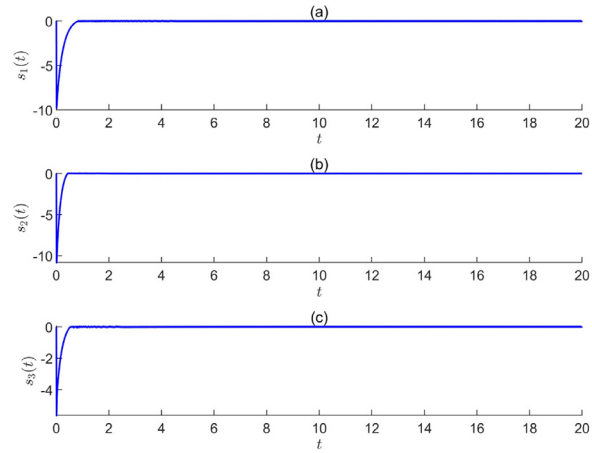

Fig. 9. The sliding mode surface of the system (27); (a)the $s_1 - t$ space; (b) the $s_2 - t$ space; (c) the $s_3 - t$ space.

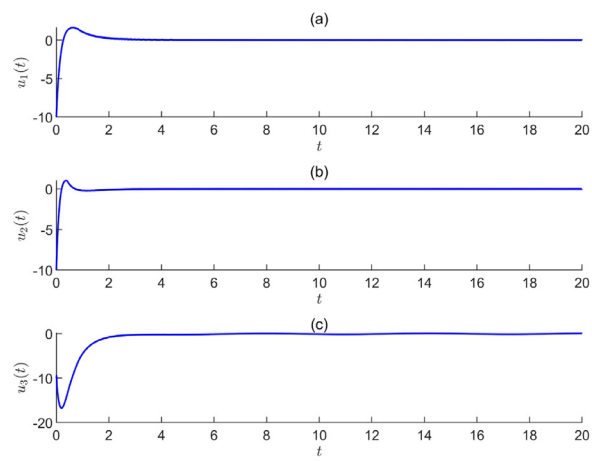

Fig. 10. The control input signals of the system (27); (a)the $u_1 - t$ space; (b) the $u_2 - t$ space; (c) the $u_3 - t$ space.

responses converge to the origin from the start of the trajectory and the global robustness is guaranteed, which implies that the proposed controller has a good performance of the tracking control and is feasible.

In addition, we investigate the chaos control of the VOF Genesio's chaotic system without the reference signal under the same controller. With the same value of the system parameters and the variable order fractional parameters, the time responses of the system states, the sliding mode surface and the control input are depicted in Figs. 8–10, it can be obtained that the proposed continuous controller can drive the system states to the origin in a finite time from the trajectories start, which means that the proposed control is effective and successful.

6. Conclusion

The present study is concerned with the tracking control problem of a class of fractional order differential systems disturbed by uncertain and external disturbances, which include the variable order fractional system and constant order fractional system. Based on the global sliding mode control and terminal control approach, a novel control schemes have been proposed to realize the global robustness control and finite time convergence. Moreover, the control signals are continuous free of chattering. The stability of the controllers have been proved by variable order fractional Lyapunov stability approach. The efficiency of the proposed controllers have been provided by simulation.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper

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