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Linear stability of planar premixed flames: Reactive Navier-Stokes equations with finite activation energy and arbitrary Lewis number

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Abstract. A numerical shooting method for performing linear stability analyses of travelling waves is described and applied to the problem of freely propagating planar premixed flames. Previous linear stability analyses of premixed flames either employ high activation temperature asymptotics or have been performed numerically with finite activation temperature, but either for unit Lewis numbers (which ignores thermal-diffusive effects) or in the limit of small heat release (which ignores hydrodynamic effects). In this paper the full Reactive Navier-Stokes equations are used with arbitrary values of the parameters (activation temperature, Lewis number, heat of reaction, Prandtl number), for which both thermal-diffusive and hydrodynamic effects on the instability, and their interactions, are taken into account. Comparisons are made with previous asymptotic and numerical results. For Lewis numbers very close to or above unity, for which hydrodynamic effects caused by thermal expansion are the dominant destablizing mechanism, it is shown that slowly varying flame analyses give qualitatively good but quantitatively poor predictions, and also that the stability is insensitive to the activation temperature. However, for Lewis numbers sufficiently below unity for which thermal-diffusive effects play a major role, the stability of the flame becomes very sensitive to the activation temperature. Indeed, unphysically high activation temperatures are required for the high activation temperature analysis to give quantitatively good predictions at such low Lewis numbers. It is also shown that state-insensitive viscosity has a small destabilizing effect on the cellular instability at low Lewis numbers.

1. Introduction

A premixed flame is a slow (subsonic) combustion wave which propagates via conduction of heat and diffusion of chemical species between the hot burnt products and the cold unburnt fuel. While flames may propagate as planar and steady waves, experiments show that in many cases the flame is wrinkled and possibly time-dependent (Buckmaster & Ludford 1982; Sivashinsky 1983; Strehlow 1985), so called 'cellular' flames. A first step in understanding the origins of these multi-dimensional flames is a linear stability analysis of the underlying steady, planar wave. The linear stability of premixed flames dates back to the Landau-Darrieus analysis (Landau & Lifshitz 1959), which treats the flame as a discontinuity that separates inert hydrodynamic flows. This analysis predicts that the growth rate of perturbations is inversely proportional to the wavelength of the perturbation, contrary to experimental

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results. Later, high activation temperature asymptotic analyses were employed, in which the reaction zone is still treated as a discontinuity, the 'flame sheet', but the structure of the diffusive pre-heat zone is taken into account. Sivashinsky (1977) used the constant density approximation (CDA), which completely ignores hydrodynamic effects and is formally valid in the limit of small heat release. He showed that for Lewis numbers (ratio of thermal to species diffusivities) sufficiently less than unity, thermal-diffusive effects alone are sufficient to cause the cellular instability. He also found a pulsating instability for Lewis numbers sufficiently above unity. However this pulsating instability regime is outside the normal parameters, and hence not attainable, for adiabatic flames in gases (Joulin & Clavin 1979; Pelce & Clavin 1982; Sivashinsky 1983) (large enough heat losses can make premixed flames unstable to the pulsating instability (Joulin & Clavin 1979; Jackson & Kapila 1986; Buckmaster 1983), but such effects are not considered here). Frankel & Sivashinsky (1982), Matalon & Matkowsky (1982) and Pelce & Clavin (1982) performed slowly-varying flame (SVF) analyses in which the wavelength of the perturbation is assumed to be much longer than the flame thickness (note that in this paper we follow the notation used in Jackson & Kapila (1986), i.e. we use the term SVF to describe a flame varying over length and time scales much longer than the diffusive scales, a more formal definition of SVF's is given in Buckmaster & Ludford (1982)). They showed that for Lewis numbers sufficiently close to or above unity, the primary instability mechanism is due to hydrodynamic effects (thermal expansion caused by the heat release). The SVF analysis also provides higher order corrections to the Landau-Darrieus result, which show that there is wavelength for which the growth rate is maximum and that the flame is stable for sufficiently small perturbation wavelengths. Jackson & Kapila (1984, 1986) then solved numerically the leading order linearized equations in the infinite activation temperature limit, but made no assumptions about the size of the heat release or wavelength of the perturbation. They showed that even for Lewis numbers less than unity, hydrodynamic effects still play a major role.

More recently linear stability analyses have been performed using numerical methods for finite activation temperatures, in which the full structure, including that of the reaction zone, of the flame is taken into account. Mukunda & Drummond (1993) used a spectral collocation scheme with a simple model for hydrogen-oxygen involving four species and variable transport properties, but only give results for a couple of parameter sets. Liberman *et al.* (1994) examined the problem using a shooting method, ignoring viscosity and for unit Lewis number, and hence did not take into account thermal-diffusive effects. Lasseigne, Jackson & Jameson (1999) used the CDA model with finite activation temperature, which ignores the hydrodynamics.

In this paper the complete problem is solved, i.e. we investigate the linear stability of freely propagating planar, steady premixed flame solutions of the full Reactive Navier-Stokes equations, for arbitrary values of the parameters (activation temperature, Lewis number, heat release, Prandtl number and wavenumber of disturbance). There are several reasons for performing analyses with finite activation temperatures. Firstly, one does not know from an asymptotic analysis how high the activation temperature has to be for the results to be quantitatively or even qualitatively correct, only that the predictions of the analysis become better as the activation temperature tends to infinity. For the CDA model Lasseigne *et al.* (1999) found that the infinite activation temperature results of Sivashinsky (1977) were at best only qualitatively correct for realistic, but finite activation temperatures. Indeed, in other combustion problems, numerical results for moderately high, but

realistic, activation temperatures can give results which are qualitatively different than the predictions of an infinite activation temperature asymptotic analysis (e.g Shah, Thatcher & Dold (2000) for flame balls or Singh & Clarke (1992) for shock ignition of detonations). Hence it is important to check the validity of the asymptotic linear stability results, and also the validity of the other asymptotic approximations made (the limits of small heat release in the CDA analysis and small wavenumber in the SVF analysis), for realistic parameter values. Also, the high activation temperature asymptotic results mentioned above all assume that the Lewis number is asymptotically close to one, i.e. near-equidiffusional flames (NEFs). However, the Lewis number can vary between about 0.3 for hydrogen (Short, Buckmaster & Kochevets 2001) to 1.8 for propane (Pelce & Clavin 1982), and hence a method is required for determining the stability of flames with Lewis numbers $O(1)$ different from unity.

Secondly, a numerical method needs to be developed for determining the linear stability of flames, which can, in principal, be applied to more complex flame models for which an asymptotic stability analysis of the Reactive Navier-Stokes equations may not be straightforward or possible, e.g. for simple two- or three-step chain-branching models (Dold *et al.* 2002, Gasser & Szmolyan 1995), or even for complex chemistry, in which some of the activation temperatures of the reactions may be moderate or small. Indeed for chain-branching chemistry, the chain-termination steps, which release most of the heat (Short & Quirk 1997) usually have very weak temperature dependences (low activation temperatures) and hence a high activation temperature analysis is not appropriate for these reactions, so that for chain-branching models the exothermic reaction zones cannot be reduced to thin sheets (Buckmaster & Ludford 1982).

Finally, a major use of linear stability results is in guiding and validating numerical schemes for simulating the full non-linear problem. For instance, in detonation theory the linear stability analysis results are extensively used to validate numerical simulations (Bourlioux, Majda & Roytburd 1991; Short & Quirk 1997; Sharpe & Falle 2000a, b). Capturing the correct stability boundaries, as predicted by the linear analysis, is a severe and essential test problem for numerical codes (for examples see Lasseigne *et al.* (1999) for the CDA model of flames, or Sharpe & Falle (2000a, b) for detonations). However, such numerical schemes tend to employ moderate but realistic activation temperatures since the reaction zone, which must be resolved in the simulations, becomes extremely thin as the activation temperature increases and hence the simulations become difficult if the activation temperature is too high. For such finite activation temperatures, the high activation temperature asymptotic results may give only qualitative results, and hence cannot be used for quantitative tests for the numerical simulations. Instead, linear stability analysis using the same parameters, including finite activation temperature, are required to determine the exact (i.e. not asymptotic) linear dispersion relations. Rogg (1982) performed numerical simulations of the CDA model, and found that his results actually began to deviate from the high activation temperature asymptotic predictions as the activation temperature increased. On the contrary, Lasseigne *et al.* (1999) found that their results did tend to those of the asymptotic analysis as the activation temperature increased. However, they used a sophisticated adaptive scheme and found excellent quantitative agreement with their finite activation temperature linear stability predictions, validating their results and numerical scheme and invalidating those of Rogg (1982), whose scheme presumably did not resolve the reaction zone adequately as the activation temperature increased. Similarly, Denet & Haldenwang

(1995) found a different dependence on the activation temperature in their Reactive Navier-Stokes simulations of hydrodynamically dominated flame instabilities than Frölich & Peyret (1991). Denet & Haldenwang (1995) suggest this is because Frölich & Peyret (1991) use a numerical mesh that does not provide enough resolution in the reaction zone at higher activation temperatures. However, comparisons with a linear stability analysis for finite activation temperature will determine which, if either, of the results of Denet & Haldenwang (1995) or Frölich & Peyret (1991) are correct, and hence which numerical scheme is more appropriate for nonlinear flame stability calculations.

While the numerical shooting method is developed here is described in the context of the stability of freely propagating planar flames, it can be implemented to determine the linear stability of many other travelling waves solutions, such as reaction-diffusion fronts (Zhang & Falle 1994; Gubernov *et al.* 2001). A version of it has already been applied to the stability of detonation waves (Sharpe 1997*a*, 1999). This method is an alternative to compound matrix methods for eigenvalue problems of systems of ordinary differential equations (Ng & Reid 1985; Gubernov *et al.* 2001; Lasseigne *et al.* 1999).

In §2 we give the governing equations and non-dimensionalization. The steady, one-dimensional waves are then considered in §3. The linearized equations are derived in §4 and the numerical shooting method described in §5. The results and conclusions, together with suggestion for future work, are given in §6 and §7, respectively.

2. Governing equations

The governing equations of the model are the Reactive Navier-Stokes equations for a single reaction A→B. The dimensionless versions of these equations are, in two-dimensions,

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0 \tag{1}$$

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \frac{\partial P}{\partial x} = Pr\left(\frac{4}{3}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{1}{3}\frac{\partial^2 v}{\partial x \partial y}\right) \tag{2}$$

$$\rho \frac{\partial v}{\partial t} + \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} + \frac{\partial P}{\partial y} = Pr\left(\frac{4}{3}\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} + \frac{1}{3}\frac{\partial^2 u}{\partial x \partial y}\right) \tag{3}$$

$$\rho \frac{\partial T}{\partial t} + \rho u \frac{\partial T}{\partial x} + \rho v \frac{\partial T}{\partial y} + Q\left(\rho \frac{\partial Y}{\partial t} + \rho u \frac{\partial Y}{\partial x} + \rho v \frac{\partial Y}{\partial y}\right) =$$

$$\frac{Q}{Le}\left(\frac{\partial^2 Y}{\partial x^2} + \frac{\partial^2 Y}{\partial y^2}\right) + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \tag{4}$$

$$\rho \frac{\partial Y}{\partial t} + \rho u \frac{\partial Y}{\partial x} + \rho v \frac{\partial Y}{\partial y} = \frac{1}{Le}\left(\frac{\partial^2 Y}{\partial x^2} + \frac{\partial^2 Y}{\partial y^2}\right) + W \tag{5}$$

$$\rho T = 1. \tag{6}$$

where ρ is the density, u and v are the x and y components of the fluid velocity, respectively, p the pressure, T the temperature, Y the mass fraction of the fuel and

W the reaction rate. These equations have been non-dimensionalize in the following way:

$$\rho = \frac{\bar{\rho}}{\bar{\rho}_f}, \quad u = \frac{\bar{u}}{\bar{V}_f}, \quad v = \frac{\bar{v}}{\bar{V}_f}, \quad p = \frac{\bar{p}}{\bar{p}_f}, \quad T = \frac{p}{\rho} = \frac{\bar{T}}{\bar{T}_f},$$

$$x = \frac{\bar{\rho}_f \bar{V}_f \bar{c}_p}{\bar{\kappa}} \bar{x}, \quad y = \frac{\bar{\rho}_f \bar{V}_f \bar{c}_p}{\bar{\kappa}} \bar{y}, \quad t = \frac{\bar{\rho}_f \bar{V}_f^2 \bar{c}_p}{\bar{\kappa}} \bar{t},$$

where a bar ($^-$) denotes dimensional quantities, a zero (0) subscript denotes quantities in the steady, planar flame, an f subscript denotes quantities in the fresh, unburnt gas upstream of the flame (and a b subscript will be used to denote quantities in the completely burnt state downstream of the flame). Here $\bar{V}_f$ is the speed of the steady, planar flame, $\bar{c}_p$ is the specific heat at constant pressure, $\bar{\kappa}$ is the co-efficient of thermal conductivity and $\bar{\rho}_f \bar{V}_f \bar{c}_p/\bar{\kappa}$ is the lengthscale of the preheat zone in the steady, planar flame (Strehlow 1985).

The reaction rate is assumed to have an Arrhenius form, i.e.

$$W = -\Lambda \rho Y \mathrm{e}^{-\theta/T} H(T - T_i), \tag{7}$$

where H is the Heaviside function. Here an ignition temperature T_i is specified, below which the reaction is switched off, to avoid the cold boundary difficulty (Williams 1985). The dimensionless parameters appearing in (2)-(5) and (7) are the Prandtl number, $Pr = \bar{\mu}\bar{c}_p/\bar{\kappa}$ (ratio of viscous to thermal diffusivities), Lewis number, $Le = \bar{\kappa}/(\bar{c}_p\bar{\lambda})$ (ratio of thermal to mass diffusivities), dimensionless activation temperature, $\theta = \bar{\theta}/\bar{T}_f$, and heat release, $Q = \bar{Q}/(\bar{c}_p\bar{T}_f)$, and $\Lambda = Da/M_f^2$ is the eigenvalue for the steady, planar flame speed, where Da is the Damköhler number, $Da = \bar{k}\bar{\kappa}/(\gamma\bar{p}_f\bar{c}_p)$ (ratio of diffusion time to reaction time) and M_f is the Mach number of the flame, $M_f = \bar{V}_f(\bar{\rho}_f/(\gamma\bar{p}_f))^{\frac{1}{2}}$, with γ the ratio of specific heats. Here $\bar{\mu}$ and $\bar{\lambda}$ are the co-efficients of viscosity and species diffusion.

Note that θ is the dimensionless activation temperature scaled with the temperature in the fresh fuel. High activation temperature asymptotic analyses employ an alternative scaling for the activation temperature, namely the Zeldovich number, β, defined by

$$\beta = \frac{\bar{\theta}(\bar{T}_b - \bar{T}_f)}{\bar{T}_b^2} = \frac{Q\theta}{(1+Q)^2}, \tag{8}$$

the asymptotic analyses then assume β is large.

Most flames travel at speeds from 1 to 100 cm s^{-1} (Williams 1985), so that they propagate highly subsonically, $M_f << 1$. The process is then nearly isobaric. The quantity P appearing in (2) and (3) is the $O(M_f^2)$ deviation of the pressure from the upstream value, i.e. $p = 1 + \gamma M_f^2 P$. Equations (1) to (6) are thus the leading order equations in an expansion in M_f^2 (note that the viscous terms in the energy equation (4) are $O(M_f^2)$ and hence have been neglected). The system (1)-(6) are the standard model equations for premixed flames (e.g. Buckmaster & Ludford 1982).

Representative parameter regimes for normal gases are: $5 \leq \beta \leq 15$ (Williams 1985); $0.3 \leq Le \leq 1.8$ (see above); $4 \leq Q \leq 9$ (Sivashinsky 1983; Zeldovich *et al.* 1985); $0.6 \leq Pr \leq 1$ (Strehlow 1985) (although Pr can be much greater than 1 for liquids). In this paper we will only be concerned with these normal gaseous parameter regimes.

3. Steady, planar flames

In the laboratory frame the steady, planar flame is assumed to travel in the negative x-direction at unit speed in dimensionless variables, so that the fresh, unburnt fuel is approached as $x \to -\infty$ and the completely burnt state approached as $x \to \infty$. However, the reactive Navier-Stokes equations are Galilean invariant. Hence we will consider them to be written in a frame moving with the steady flame. In this frame the flame is stationary, the flow is steady (independent of t) and the upstream fuel is moving at unit speed. After integrating once with respect to x and employing the boundary conditions $T_0 = \rho_0 = u_0 = Y_0 = 1$, $P_0 = 0$ and $\mathrm{d}T_0/\mathrm{d}x = \mathrm{d}Y_0/\mathrm{d}x = 0$ as $x \to -\infty$, (1)-(5) can be reduced to

$$\frac{\mathrm{d}T_0}{\mathrm{d}x} = T_0 - 1 + Q(Z_0 - 1), \quad \frac{\mathrm{d}Y_0}{\mathrm{d}x} = Le(Y_0 - Z_0),$$

$$\frac{\mathrm{d}Z_0}{\mathrm{d}x} = -\frac{\Lambda Y_0}{T_0} \mathrm{e}^{-\theta/T_0} H(T_0 - T_i), \tag{9}$$

where Z_0 is a reaction progress variable defined by the second of (9) (Gasser & Szmolyan 1993), and

$$u_0 = \frac{1}{\rho_0} = T_0, \qquad P_0 = \frac{4Pr}{3}(T_0 - 1 + Q(Z_0 - 1)) - (T_0 - 1). \tag{10}$$

In the fully burnt state $Y_0 = \mathrm{d}Y_0/\mathrm{d}x = \mathrm{d}T_0/\mathrm{d}x = 0$ so that (9)-(10) give the burnt state as $T_{0b} = u_{0b} = 1/\rho_{0b} = 1 + Q$, $P_{0b} = -Q$ as $x \to \infty$. In order to satisfy both the boundary conditions as $x \to -\infty$ and $x \to \infty$, Λ, which is related to the flame speed, must have a specific value. Hence Λ is the eigenvalue of (9) which needs to be determined numerically.

Equations (9) are autonomous and hence we can replace the independent variable x with one of the dependent variables. Here we choose the temperature T_0 as the new independent variable, so that the system to be solved is reduced to 2 equations:

$$\frac{\mathrm{d}Y_0}{\mathrm{d}T_0} = \frac{Le(Y_0 - Z_0)}{T_0 - 1 + Q(Z_0 - 1)}, \qquad \frac{\mathrm{d}Z_0}{\mathrm{d}T_0} = -\frac{\Lambda Y_0 \mathrm{e}^{-\theta/T_0} H(T_0 - T_i)}{T_0(T_0 - 1 + Q(Z_0 - 1))}. \tag{11}$$

From a numerical perspective, this also has the advantage that an infinite domain $x \in (-\infty, \infty)$ is mapped onto a finite domain $T_0 \in [1, 1 + Q]$.

Note that for $Le = 1$, the first of (11) has the analytical solution $Y_0 = [(1 + Q) - T_0]/Q$. Note also that for general Lewis numbers, in the region $T_0 \leq T_i$, $Z_0 = 1$ and hence $Y_0 = 1 - A(T_0 - 1)^{Le}$, where A is a constant to be determined. Now consider the solution of (11) close to the burnt state $T_0 = 1 + Q$. Defining $w_b = 1 + Q - T_0$, then $w_b << 1$, $Y << 1$ and $Z << 1$ sufficiently near the burnt state, and (11) linearize to

$$\frac{\mathrm{d}Y_0}{\mathrm{d}w_b} = \frac{Le(Y_0 - Z_0)}{w_b - QZ_0}, \qquad \frac{\mathrm{d}Z_0}{\mathrm{d}w_b} = \frac{BY_0}{(w_b - QZ_0)},$$

$$B = -\frac{\Lambda \mathrm{e}^{-\theta/(1+Q)}}{1 + Q}. \tag{12}$$

The solution to (12) which is bounded at $w_b = 0$ is

$$Y_0 = \frac{h_0(1 - h_0)}{QB} w_b, \qquad Z_0 = \frac{(1 - h_0)}{Q} w_b,$$

$$h_0 = \frac{Le - (Le^2 - 4LeB)^{\frac{1}{2}}}{2}. \tag{13}$$

The numerical shooting method used to determine the eigenvalue Λ is as follows: for a given value of Λ, the asymptotic solutions (13) are used as initial conditions to start the integration of (11) near the burnt state, equations (11) are then integrated in the direction of decreasing T_0 until $T_0 = T_i$. If $Z_0 > 1$ at $T_0 = T_i$ then Λ is too high, whereas Λ is too low if $Z_0 < 1$ there. Hence one can iterate for Λ using bisection until the required condition $Z_0 = 1$ is satisfied at $T_0 = T_i$. As a good initial guess for Λ, the high activation temperature asymptotic result of Bush & Fendell (1970) is used. For moderate to high values of θ, the Arrhenius term in the reaction rate is exponentially small near $T_0 = T_i$ provided T_i is close to one and hence Λ is insensitive to the value of T_i chosen (see §6.5) . Here we use $T_i = 1.01$. Once the eigenvalue Λ has been found, the constant A can be determined using the value of Y_0 found at $T_0 = T_i$ from the numerical integration. The spatial profiles can then also be determine by integrating $\mathrm{d}x/\mathrm{d}T_0$ as an auxiliary equation. Note that the spatial origin is arbitrary. Here we choose $x = 0$ to correspond to $T_0 = 1 + Q/2$. We use a fourth-order Runge-Kutta routine with adaptive step doubling to perform all the integrations in this paper.

Figure 1 shows the steady, planar flame solutions for $Q = 4$, $\theta = 70$ and various Lewis numbers, as well as for $\theta = 90$ when $Le = 1$. Note that the reaction occurs over a relatively thin zone. As the activation temperature increases the reaction zone becomes narrower and the maximum reaction rate increases (in the limit $\beta \to \infty$ the reaction rate tends to a delta function). Note that increasing the activation temperature only effects the temperature profiles in the reaction zone, but not in the pre-heat zone. Note also the linear dependence of Y_0 on T_0 for $Le = 1$ in figure 1(a).

4. Linearized equations

We now suppose that the steady, planar flame is slightly perturbed such that the perturbations have the normal mode form

$$q(x, y, t) = q_0(x) + \epsilon q_1(x)e^{\sigma t}e^{iky}, \qquad \epsilon << 1, \tag{14}$$

where q is one of T, u, v, P or Y (note that $\rho = 1/T$ from (6) so that the density can be eliminated from (1)-(5)), σ is the (complex) growth rate and k is the wavenumber of the disturbance in the y-direction. Note that out choice of a two-dimensional disturbance also covers the case of three-dimensional perturbations, since if we have a three-dimensional perturbation of the form $\epsilon q_1(x)e^{\sigma t}e^{ik_1 y}e^{ik_2 z}$, we can choose a new transverse direction y' with wavenumber $k = (k_1^2 + k_2^2)^{\frac{1}{2}}$.

The reaction rate W is then expanded as

$$W = W_0(x) + \epsilon W_{0,T}T_1(x)e^{\sigma t}e^{iky} + \epsilon W_{0,Y}Y_1(x)e^{\sigma t}e^{iky} + \dots, \tag{15}$$

where $W_{0,T} = \partial W_0/\partial T_0$, etc. We then define the following quantities

$$\tau_1 = \frac{\mathrm{d}T_1}{\mathrm{d}x}, \qquad U_1 = \frac{\mathrm{d}u_1}{\mathrm{d}x}, \qquad V_1 = \frac{\mathrm{d}v_1}{\mathrm{d}x}, \qquad Z_1 = Y_1 - \frac{1}{Le}\frac{\mathrm{d}Y_1}{\mathrm{d}x}. \tag{16}$$

Equations (14)-(16) are then substituted into the governing equations (1)-(5) and the result linearized in ϵ. Note first that (1) contains only first x-derivatives and hence its linearized version can be used to eliminate τ_1 in terms of the other perturbed quantities:

$$\tau_1 = \frac{(T_0' - \sigma)T_1}{T_0} - \frac{T_0'u_1}{T_0} + ikv_1 + U_1,$$
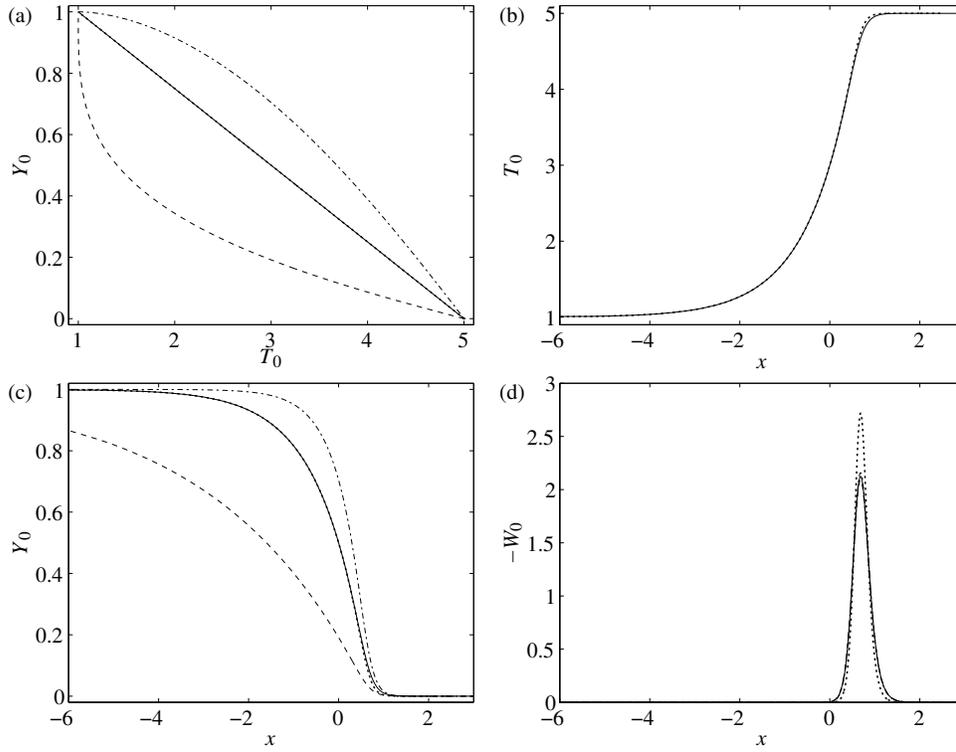
Figure 1. (a) Steady solutions in the (T_0, Y_0)-plane and spatial profiles of (b) temperature, (c) fuel mass fraction and (d) reaction rate, for $Q = 4$, $\theta = 70$ ($\beta = 11.2$) and Lewis numbers 0.3 (dashed lines), 1.0 (solid lines) and 1.8 (dot-dashed lines). Also shown are the profiles for $\theta = 90$ ($\beta = 14.4$), $Le = 1.0$ (dotted lines).

where the prime denotes differentiation with respect to x. The linearized versions of equations (2)-(5), together with (16), can then be written in the form

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}x} = \mathbf{A}\mathbf{u}, \tag{17}$$

where $\mathbf{u} = (T_1, u_1, v_1, P_1, Y_1, U_1, V_1, Z_1)^T$ and

$$\mathbf{A} =$$

$$
\left(
\begin{array}{cccc}
\dfrac{(T'_0-\sigma)}{T_0} & \dfrac{-T'_0}{T_0} & ik & 0 \\[2mm]
0 & 0 & 0 & 0 \\[1mm]
0 & 0 & 0 & 0 \\[1mm]
A_{41} & A_{42} & \dfrac{4ikPr(\sigma+T_0-T'_0)}{3T_0} & 0 \\[3mm]
0 & 0 & 0 & 0 \\[1mm]
A_{61} & \dfrac{-\sigma T'_0+T_0 T''_0}{T_0^2} & \dfrac{ik(\sigma+T_0-T'_0)}{T_0} & 0 \\[3mm]
0 & 0 & \dfrac{3\sigma+4Prk^2T_0}{3PrT_0} & \dfrac{ik}{Pr} \\[3mm]
\dfrac{Y'_0+W_{0,T}T_0}{T_0} & \dfrac{-Y'_0}{T_0} & 0 & 0
\end{array}
\right.
$$

$$
\left.
\begin{array}{cccc}
0 & 1 & 0 & 0 \\[1mm]
0 & 1 & 0 & 0 \\[1mm]
0 & 0 & 1 & 0 \\[1mm]
\dfrac{4PrQW_{0,Y}}{3} & \dfrac{4Pr(\sigma+T_0)-3T_0}{3T_0} & -ikPr & 0 \\[3mm]
Le & 0 & 0 & -Le \\[1mm]
QW_{0,Y} & \dfrac{\sigma+T_0}{T_0} & -ik & 0 \\[3mm]
0 & \dfrac{ik}{3} & \dfrac{1}{Pr} & 0 \\[3mm]
\dfrac{-Le\sigma-k^2T_0+LeW_{0,Y}T_0}{LeT_0} & 0 & 0 & 0
\end{array}
\right),
$$

where

$$
A_{41} = \frac{4Pr(-\sigma^2+T'_0\sigma+(k^2+QW_{0,T})T_0^2-T_0T''_0)}{3T_0^2}+\frac{T'_0}{T_0},
$$

$$
A_{42} = \frac{-(4PrT'_0+3T_0)\sigma}{3T_0^2}+\frac{Pr(4T''_0-3T_0k^2)}{3T_0}-\frac{T'_0}{T_0},
$$

$$
A_{61} = \frac{(-\sigma^2+T'_0\sigma+(k^2+QW_{0,T})T_0^2-T_0T''_0)}{T_0^2}.
$$

Note that the $Pr=0$ case investigated by Liberman *et al.* (1994) is a singular limit of (17). For $Pr=0$, u''_1 and v''_1 do not appear in the problem, and the corresponding version of (17) is reduced to a 6×6 problem.

Since the steady structure is infinite in length it is again beneficial to use T_0 as the independent variable. Equation (17) then becomes

$$
T'_0\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}T_0}=(T_0-1+Q(Z_0-1))\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}T_0}=\mathbf{A}\mathbf{u}. \tag{18}
$$

The boundary conditions are that the solutions of (18) are bounded as $T_0\to1$ ($x\to-\infty$) and as $T_0\to1+Q$ ($x\to\infty$). Only for certain, discrete values of the (possibly complex) growth rate σ can the boundedness conditions in both the fresh and burnt states be satisfied.

5. Determining the growth rates

In this section we describe the numerical method for determining the eigenvalues of the growth rate σ. We first need to determine asymptotic solutions to (18) valid as the fresh state is approached ($T_0 \to 1$) and as the burnt state is approached ($T_0 \to 1+Q$). Note that $T_0' = 0$ at $T_0 = 1$ and at $T_0 = 1+Q$ and hence these are both regular singular points of (18).

Consider first the solutions near the fresh state, $T_0 - 1 << 1$. Defining $w_f = T_0 - 1$, we can expand the steady variables and hence $\mathbf{A}$ in terms of w_f, recalling that $Y_0 = 1 - A w_f^{Le}$, $Z_0 = 1$ for $T_0 < T_i$ (i.e. for sufficiently small w_f), to give.

$$w_f \frac{d\mathbf{u}}{dw_f} = (\mathbf{A}_0 + \mathbf{A}_1 w_f + \mathbf{A}_2 w_f^{Le} + \ldots)\mathbf{u}, \tag{19}$$

where the co-efficient matrices, $\mathbf{A}_0$, etc., depend only on Pr, Le, k and σ. Note that the ordering of the higher order terms in the expansion depends on whether Le is greater or less than unity (for $Le = 1$ the expansion for $\mathbf{A}$ is simply of the form $\mathbf{A}_0 + \mathbf{A}_1 w_f + \ldots$). Equation (19) has 8 independent solutions of the form

$$\mathbf{u}_i = w_f^{h_i}(\mathbf{a}_{i0} + \mathbf{a}_{i1} w_f + \mathbf{a}_{i2} w_f^{Le} + \ldots), \qquad i = 1, \ldots, 8 \tag{20}$$

where h_i are the eigenvalues of $\mathbf{A}_0$ and $\mathbf{a}_{i0}$ are the corresponding eigenvectors. The $\mathbf{a}_{i1}$, $\mathbf{a}_{i2}$, etc., are found by substituting (20) into (19) and equating powers of w_f. The eigenvalues of $\mathbf{A}_0$ are

$$\frac{1 \pm [1 + 4(\sigma + k^2)]^{\frac{1}{2}}}{2}, \quad \frac{Le \pm [Le^2 + 4(\sigma Le + k^2)]^{\frac{1}{2}}}{2},$$

$$\frac{1 \pm [1 + 4Pr(\sigma + Prk^2)]^{\frac{1}{2}}}{2Pr}, \quad \pm k. \tag{21}$$

For $\mathrm{Re}(\sigma) \geq 0$ the eigenvalues with the negative signs have negative real part and correspond to unbounded solutions as $w_f \to 0$ and hence we must discard these solutions. We are thus left with 4 independent solutions corresponding to the positive signs in (21).

Now consider the solutions of (18) near the burnt state ($T_0 \to 1+Q$). Expanding the steady variables, and hence $\mathbf{A}$ in terms of $w_b = 1 + Q - T_0 << 1$ (see (13)) gives

$$h_0 w_b \frac{d\mathbf{u}}{dw_b} = (\mathbf{A}_0^* + \mathbf{A}_1^* w_b + \ldots)\mathbf{u}, \tag{22}$$

where h_0 is defined in (13), and now $\mathbf{A}_0^*$, etc., depend on Pr, Le, Q, θ, k and σ. Equation (22) then has 8 independent solutions of the form

$$\mathbf{u}_i = w_b^{(h_i^*/h_0)}(\mathbf{a}_{i0}^* + \mathbf{a}_{i1}^* w_b + \ldots), \qquad i = 1, \ldots, 8 \tag{23}$$

where h_i^* are the eigenvalues of $\mathbf{A}_0^*$ and $\mathbf{a}_{i0}^*$ are the corresponding eigenvectors. The eigenvalues of $\mathbf{A}_0^*$ are

$$\frac{1 \pm [1 + 4(\sigma/(1+Q) + k^2)]^{\frac{1}{2}}}{2},$$

$$\frac{Le \pm [Le^2 + 4(\sigma Le/(1+Q) + k^2 - BLe)]^{\frac{1}{2}}}{2},$$

$$\frac{1 \pm [1 + 4Pr(\sigma/(1+Q) + k^2 Pr)]^{\frac{1}{2}}}{2Pr}, \quad \pm k, \tag{24}$$

where B is defined in (12). Since $h_0 < 0$, for $\mathrm{Re}(\sigma) \geq 0$, the eigenvalues with positive signs in (24) give negative values of h_i^*/h_0 and hence correspond to unbounded solutions of (22) as $w_b \to 0$, and these solutions must be discarded. Again, we are left with 4 independent solutions corresponding to the negative signs in (24).

We are now in a position to determine the eigenvalue growth rates σ using a numerical shooting method. For a given value of σ, the four bounded asymptotic solutions (20) valid as $T_0 \to 1$ are used as initial conditions to start the integration of (18) away from the fresh state to the middle of the domain, $T_0 = 1 + Q/2$. We then have a general solution for $\mathbf{u}$ at $T_0 = 1 + Q/2$,

$$\mathbf{u} = \alpha_1 \mathbf{u}_1^f + \alpha_2 \mathbf{u}_2^f + \alpha_3 \mathbf{u}_3^f + \alpha_4 \mathbf{u}_4^f,$$

where $\mathbf{u}_i^f$, $i = 1, \ldots, 4$, are the four solutions at $T_0 = 1 + Q/2$ found from using each of the 4 bounded asymptotic solutions near $T_0 = 1$ as initial conditions for the integration and the α_i are the corresponding (complex) constants of integration. Next, the four bounded asymptotic solutions (23) valid as $T_0 \to 1 + Q$ are used as initial conditions to start the integration of (18) away from the burnt state to $T_0 = 1 + Q/2$. We then have a second general solution for $\mathbf{u}$ at $T_0 = 1 + Q/2$

$$\mathbf{u} = \alpha_5 \mathbf{u}_1^b + \alpha_6 \mathbf{u}_2^b + \alpha_7 \mathbf{u}_3^b + \alpha_8 \mathbf{u}_4^b,$$

where $\mathbf{u}_i^b$, $i = 1, \ldots, 4$, are the four solutions at $T_0 = 1 + Q/2$ found from using each of the 4 bounded asymptotic solutions near $T_0 = 1 + Q$ as initial conditions for the integration and $\alpha_5, \ldots, \alpha_8$ are (complex) constants of integration.

If σ is an eigenvalue then the constants $\alpha_1, \ldots, \alpha_8$ can be chosen so that the two general solutions at $T_0 = 1 + Q/2$ match, i.e.

$$\alpha_1 \mathbf{u}_1^f + \alpha_2 \mathbf{u}_2^f + \alpha_3 \mathbf{u}_3^f + \alpha_4 \mathbf{u}_4^f = \alpha_5 \mathbf{u}_1^b + \alpha_6 \mathbf{u}_2^b + \alpha_7 \mathbf{u}_3^b + \alpha_8 \mathbf{u}_4^b. \quad (25)$$

Since we are interested in non-trivial solutions to (25), i.e. those for which not all the $\alpha_i = 0$, we can divide through by one of the α_i (α_8, say) to give

$$a_1 \mathbf{u}_1^f + a_2 \mathbf{u}_2^f + a_3 \mathbf{u}_3^f + a_4 \mathbf{u}_4^f = a_5 \mathbf{u}_1^b + a_6 \mathbf{u}_2^b + a_7 \mathbf{u}_3^b + \mathbf{u}_4^b,$$

where $a_i = \alpha_i/\alpha_8$, $i = 1, \ldots, 7$. Let $a_i = b_i + \mathrm{i}c_i$, where b_i and c_i are real, and consider the quantity

$$m = |(b_1 + \mathrm{i}c_1)\mathbf{u}_1^f + (b_2 + \mathrm{i}c_2)\mathbf{u}_2^f + (b_3 + \mathrm{i}c_3)\mathbf{u}_3^f + (b_4 + \mathrm{i}c_4)\mathbf{u}_4^f$$

$$-(b_5 + \mathrm{i}c_5)\mathbf{u}_1^b - (b_6 + \mathrm{i}c_6)\mathbf{u}_2^b - (b_7 + \mathrm{i}c_7)\mathbf{u}_3^b - \mathbf{u}_4^b|^2, \quad (26)$$

(where $|\mathbf{q}|^2 = \mathbf{q} \cdot \bar{\mathbf{q}}$). Then if σ is an eigenvalue we can choose the b_i and c_i such that $m = 0$. For any given σ we can minimize m by partially differentiating (26) with respect to each of the b_i and c_i and setting these quantities to zero, which clearly corresponds to the minimum m given each of the other constants. This gives a 14×14 system of linear equations, $\mathbf{Cv} = \mathbf{r}$ say, where $\mathbf{v} = (b_1, c_1, \ldots, b_7, c_7)^T$ and $\mathbf{C}$ and $\mathbf{r}$ are a constant matrix and vector respectively. Solving these for $\mathbf{v}$ gives the b_i and c_i such that m is a minimum.

Thorough iterative searches of the complex σ-space were performed and $\min(m)$ determined at each point in order to find the eigenvalues where $\min(m)$ has local minima of zero. However, in agreement with the asymptotic analyses, only one eigenvalue was found for any reasonable parameter set, and for this eigenvalue σ is real (note that the flame is found to be stable to the pulsating instability for the normal gaseous parameter regimes which are considered here, in agreement with previous asymptotic results). These real eigenvalues can then be quickly found for fixed values

of Pr, Le, θ, Q and k, using Newton-Raphson iteration to determine the value of σ where $\min(m)=0$. The wavenumber can then be stepwise increased and the process repeated to determine the whole dispersion relation.

The novel aspect of the above method is the use of one of the steady variables as the independent variable. For linear stability of travelling wave solutions, x is usually kept as the independent variable (e.g. Zhang & Falle 1994; Liberman *et al.* 1994; Lasseigne *et al.* 1999; Gubernov *et al.* 2001). In this case, one finds solutions to the linearized equations of the form $e^{\lambda_i x}\mathbf{r}_i$ as $x \to \pm\infty$, where λ_i are the eigenvalues of $\lim_{x\to\pm\infty} \mathbf{A}$ (where $\mathbf{A}$ is the co-efficient matrix of the linearized problem) and $\mathbf{r}_i$ the corresponding eigenvectors (the h_i or h_i^* and $\mathbf{a}_{i0}$ or $\mathbf{a}_{i0}^*$ in our case, hence such solutions correspond to the leading order terms in (20) and (23) for the premixed flame problem). Gubernov *et al.* (2001) state that a straightforward shooting method using these asymptotic exponential solutions for initial conditions (after discarding the solutions which are unbounded as $x \to \pm\infty$) cannot be used because only the solutions corresponding to the maximum $|\lambda_i|$ (λ_{max}, say) can be found numerically, since even when starting with the other solutions corresponding to the lower eigenvalues, the faster growing solution corresponding λ_{max} will still be excited due to numerical errors. Neither Zhang & Falle (1994) for reaction-diffusion waves or Liberman *et al.* (1994) for premixed flames with $Pr = 0$, $Le = 1$ reported any such problems and did manage to calculate the dispersion relations using a straightforward shooting method with these asymptotic exponential solutions as initial conditions, at least for certain parameter sets. However, in the present case and also for the linear stability of detonations (Sharpe 1997*b*) we did find that when only the leading order term in the asymptotic expansions (20) or (23) was used, for some parameters sets some of the solutions rapidly diverged from the asymptotic solutions as (18) was integrated away from the boundaries, and the value of these solutions found at $T_0 = 1 + Q/2$ did not converge as the starting value of $w \to 0$ (where w represents either w_f or w_b).

Such numerical difficulties have lead to the use of compound matrix methods, as described in Ng & Reid (1985), for example Lasseigne *et al.* (1999) and Gubernov *et al.* (2001). However, such methods are impractical for systems of order higher than six (Ng & Reid 1985), and even for lower order systems a straightforward shooting method is in some sense preferable. The advantage of using one of the steady variables as the independent variable is that the boundary conditions for the linearized equations then become regular singular point problems. The solutions then only grow algebraically instead of exponentially, which alleviates the problem somewhat, but more importantly it is then easy to determine higher order terms in the asymptotic expansions near the boundaries, as in (20) and (23). Provided one retains enough terms in the expansions when these asymptotic solutions are used as the initial conditions for the numerical integration, the numerical problems do not occur for any of the solutions, i.e. the numerical solutions agree with the asymptotic solutions for small values of w, and the solutions obtained at $T_0 = 1 + Q/2$ converge as the starting value of $w \to 0$.

Indeed, we have found that the numerical difficulty is due to the fact that some entries of the eigenvectors $\mathbf{a}_{0i}$ or $\mathbf{a}_{0i}^*$ (corresponding to the first term in the asymptotic expansions) are zero. Hence these first terms do not give the leading order approximations for every component of $\mathbf{u}$. The higher order terms show that while these components of $\mathbf{u}$ are smaller than the other components, they are not necessarily zero. If the integration is started with only this first term in the expansion, the smaller components of $\mathbf{u}$ are initialized as zero instead of their correct w dependent values.
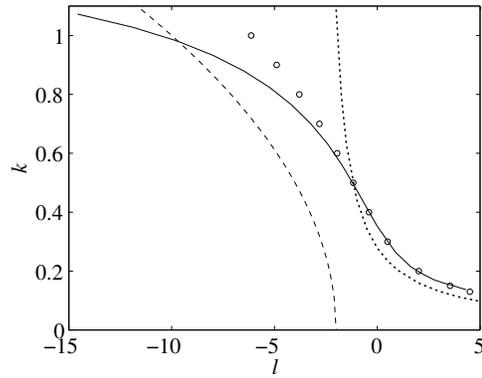
Figure 2. Neutral stability boundary in the (l, k)-plane for $Pr = 0.75$, $Q = 5$ and $\theta = 70$ ($\beta = 9.72$). Also shown are the infinite activation temperature results from Jackson & Kapila (1984) (open circles), the SVF results from Matalon & Matkowsky (1982) (dotted line) and the CDA results of Sivashinsky (1977) (dashed line).

These components then do not move away from zero in the correct direction, but in a way dictated by the numerical stepsize and starting value of w. These incorrect values of the initially smaller components of **u** can then feedback and quickly pollute the numerical solution for all components of **u**. However, provided one uses enough terms in the asymptotic expansions for the initial conditions of the numerical integration, such that the correct leading order w dependency is retained for every component of **u**, then the numerical solution follows the correct trajectory. Usually, this only involves retaining the first two or three terms in the asymptotic expansions. The only difficult case we have encountered using this method is for the stability of Chapman-Jouguet detonations, for which the singular point at the burnt boundary is an irregular singular point (Sharpe 1997*b*).

6. Results

In this section we compare our results with the asymptotic and numerical results of previous workers and examine the effect of each of the parameters on the stability of the steady, planar flame. The high activation temperature asymptotics show that the stability depends on the Lewis number through the parameter

$$l = \beta(1 - 1/Le),$$

where β is the Zeldovich number defined in (8) and $l = O(1)$. Hence as $\beta \to \infty$, these analyses assume that the Lewis number is within $O(1/\beta)$ of unity, i.e. near-equidiffusional. Results of stability analyses are usually displayed as neutrally stable wavenumbers in a (l, k)-plane. For the arbitrary Lewis number results determined in this paper, we also show the neutral stability boundaries in a (Le, k)-plane as well as the dispersion relations.
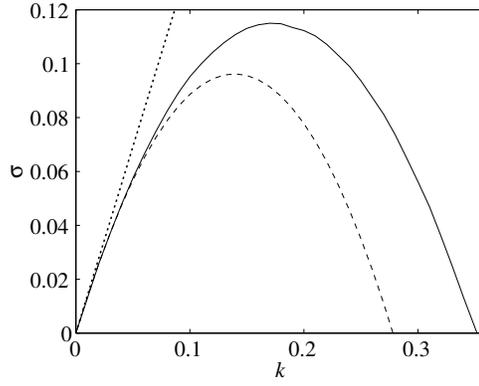
Figure 3. Dispersion relation for $Pr = 0.75$, $Q = 5$, $Le = 1$ and $\theta = 70$ ($\beta = 9.72$). Also shown are the SVF (dashed line) and Landau-Darrieus (dotted line) dispersion relations.

6.1. Comparison with previous results

We first compare our exact linear dispersion relation results to those of high-activation temperature studies, $\beta \to \infty$. Sivashinsky (1977) determined the dispersion relation for the CDA model, which is formally valid in the distinguished limit $\beta \to \infty$, $Q \to 0$, to be

$$16\sigma^3 + (48k^2 + 8 + 2l - l^2/4)\sigma^2 + (1 + 12k^2)(1 + 4k^2 + l/2)\sigma +$$

$$k^2(1 + 4k^2 + l/2)^2.$$

This gives a neutral stability boundary on which $\sigma = 0$ given by

$$k = \frac{(-4 - 2l)^{\frac{1}{2}}}{4}$$

for $l \leq -2$. Frankel & Sivashinsky (1982), Pelce & Clavin (1982) and Matalon & Matkowsky (1982) performed an SVF analysis, valid in the limit $k \to 0$ (i.e. valid for large wavelength disturbances) and determined the asymptotic expansion in k for the dispersion relation, up to $O(k^2)$. Matalon & Matkowsky (1982) give this dispersion relation in the form

$$\sigma = \sigma_0 k + \sigma_1 k^2, \tag{27}$$

where

$$\sigma_0 = \frac{1 + Q}{2 + Q}\left[\left(1 + \frac{(2 + Q)Q}{1 + Q}\right)^{\frac{1}{2}} - 1\right],$$

$$\sigma_1 =$$

$$-\frac{(1 + Q)[lI(1 + \sigma_0)(1 + Q + \sigma_0) + Q^2 + (1 + Q)\ln(1 + Q)(2(1 + \sigma_0) + Q)]}{2Q[(1 + Q) + (2 + Q)\sigma_0]}$$

and

$$I = \int_{-\infty}^{0} \ln(1 + Qe^x)dx.$$

The $O(k)$ term in (27) is the Landau-Darrieus result. Jackson & Kapila (1984) then solved the leading order version of (17) in the limit $\beta \to \infty$ numerically for arbitrary Q and k.

Figure 2 shows the neutral stability boundary in the (l, k)-plane for $Q = 5$, $Pr = 0.75$ and a finite activation temperature of $\theta = 70$ (corresponding to $\beta = 9.72$), together with the boundaries predicted from the CDA and SVF analyses as well as the infinite activation temperature results of Jackson & Kapila (1984). The flame is predicted to be stable to perturbations with wavenumbers above and to the left of the curves, and unstable for wavenumbers below and to the right of them. Note that our results and those of Jackson & Kapila (1984) show that the flame is always unstable to a band of wavenumbers between zero and the neutrally stable wavenumber. For Lewis numbers sufficiently close to or above unity (l close to or above zero), the flame is only unstable to relatively small wavenumbers (large wavelengths). This is usually termed the 'hydrodynamic' instability. As the Lewis numbers decreases below unity (l negative), the flame becomes unstable to $O(1)$ wavenumbers, i.e. to wavelengths comparable to the flame length. Here thermal-diffusive effects become important, and this is usually termed the 'cellular' instability. Note, however, that there is no clear distinction between the two regimes, the unstable band of wavenumbers continuously widens as Le decreases, and both hydrodynamic and thermal-diffusive (non-unity Lewis number) effects have a role in each regime.

Figure 2 shows that for $l \geq 0$ the results for finite activation temperature of $\theta = 70$ are in excellent quantitative agreement with the infinite activation temperature results of Jackson & Kapila (1984). However, for $l < 0$, as l decreases and the flame becomes unstable to higher wavenumbers, the finite activation temperature results begin to diverge from those of Jackson & Kapila (1984).

Figure 2 also shows that for $l \geq 0$, the SVF results from (27), which assume small k, significantly underestimate the neutrally stable wavenumber. Figure 3 shows the dispersion relation for $Le = 1$ ($l = 0$) with $\theta = 70$, together with the SVF dispersion relation given by (27) (and also the $O(k)$ Landau-Darrieus result). For low wavenumbers (k less than about 0.1) the SVF analysis is in good agreement with the exact dispersion relation, but starts to give poorer predictions of the growth rates as the wavenumber increases. Importantly, it can be seen from figure 3 that the SVF result underpredicts both the maximum growth rate and the corresponding wavenumber, as well as the neutrally stable wavenumber. This agrees with the numerical simulation results of Denet & Haldenwang (1995), who found that their measured growth rates agreed well with those of the SVF analysis at sufficiently low wavenumbers, but that the SVF underpredicted the growth rates as the wavenumber increased. As l decreases below zero and the flame becomes unstable to larger wavenumbers, the SVF stability boundary results rapidly become invalid. Since the SVF analysis only gives qualitative results, such an analysis cannot be used for quantitative validation of numerical codes for simulating the full non-linear problem.

Finally, figure 2 shows that the high activation temperature CDA results give quantitatively poor predictions of the stability boundary as compared to the boundary for realistic $O(1)$ values of the heat of reaction and activation temperatures in normal gases ($Q \geq 4$, $\beta \leq 15$). Jackson & Kapila (1984) showed that their infinite activation temperature results tend to those of the CDA analysis as $Q \to 0$, i.e. the CDA results are recovered under the appropriate limits, but that for realistic $O(1)$ values of Q the neutral stability boundary lies above and to the right of the CDA boundary (as can be seen in figure 2), although the CDA analysis still gives the correct qualitative trends.

β	Le	Q	Pr	σ	σ_{DH}	σ_{FP}
10	1	4	0.7	0.086	0.081	-
10	0.9	4	0.7	0.145	0.143	-
15	0.9333	4	0.7	0.142	0.140	-
20	0.95	4	0.7	0.140	0.139	-
10	0.9	2	0.71	0.074	-	0.077
10	0.9	4	0.71	0.145	-	0.14
10	0.9	8	0.71	0.260	-	0.26
10	0.9	10	0.71	0.311	-	0.31

Table 1. Comparison of growth rates with those of Denet & Haldenwang (1995) and Frölich & Peyret (1991) for $k = 0.20944$.

However, as discussed in §6.2, the infinite activation temperature results do not give good predictions for the instability at low Lewis numbers unless β is extremely large. Indeed, Lasseigne *et al.* (1999) found, using the CDA limit $Q \to 0$ but with finite activation temperature, that the high activation temperature results of Sivashinsky (1977) do not give quantitatively good results for realistic values of β. Hence neither of the limits $Q \to 0$ or $\beta \to \infty$ give accurate results for low Lewis number instabilities as compared to the results using realistic flame parameters. Of course, both asymptotic limits are useful for determining the qualitative trends and, more importantly, for revealing the physical mechanisms of the instability. Note that the stability boundary for finite activation temperature intersects with the high activation temperature CDA results (in this case at about $l = -9$). Hence as l decreases from zero, the CDA results initially underestimate the neutrally stable wavenumber, but for sufficiently large and negative l the CDA results overpredict this wavenumber.

It is also worth comparing our results with those of the finite activation temperature numerical simulations of Denet & Haldenwang (1995) and Frölich & Peyret (1991), who both tabulated growth rates for various parameter sets for a wavenumber of 0.20944. Table 1 shows that the growth rates as measured from these non-linear simulations for the parameter sets given are in good agreement with the exact linear stability results. In the case of Denet & Haldenwang (1995) this includes the results for a relatively high Zeldovich number of $\beta = 20$. This attests to the careful implementation of the numerical method in Denet & Haldenwang (1995). The results of Frölich & Peyret (1991) also compare well with the linear growth rates, which shows that their numerical scheme gives good results for relatively moderate activation temperatures ($\beta = 10$). However, in figure $3(d)$ of Frölich & Peyret (1991) they show results for $\beta = 5$, $\beta = 10$ and $\beta = 15$ when $l = -1.11111$. Their results for $\beta = 5$ and $\beta = 10$ are in good agreement with each other, but the results for $\beta = 15$ are quite different, which shows their results become sensitive to the activation temperature as β increases. This is in disagreement with the the linear stability analysis, which predicts the stability is insensitive to β for such values of l, and hence shows that the method of Frölich & Peyret (1991) fails to give good results even for higher values of the activation temperature. Frölich & Peyret (1991) admit themselves that they had difficulties in obtaining satisfactory spatial resolution for larger β.
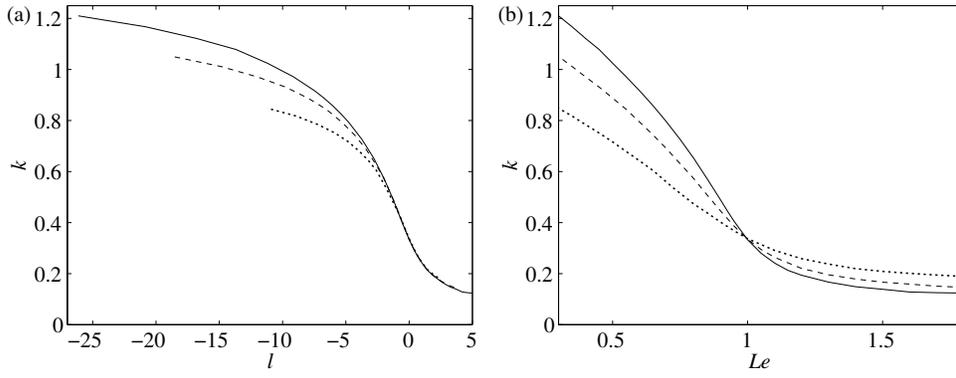
Figure 4. Neutral stability boundaries in (a) the (l, k)-plane and (b) the (Le, k)-plane, for $Pr = 0.75$, $Q = 4$ and dotted lines: $\theta = 30$ ($\beta = 4.8$), dashed lines: $\theta = 50$ ($\beta = 8$) and solid lines: $\theta = 70$ ($\beta = 11.2$).
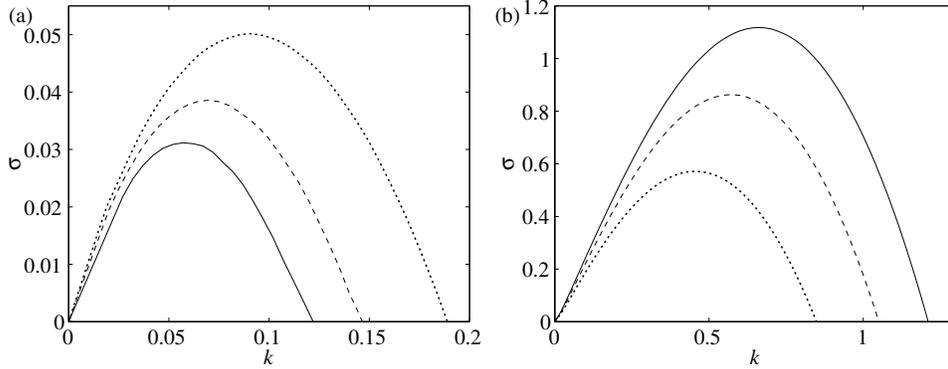
Figure 5. Dispersion relations for $Pr = 0.75$, $Q = 4$, (a) $Le = 1.8$ and (b) $Le = 0.3$, and dotted lines: $\theta = 30$ ($\beta = 4.8$), dashed lines: $\theta = 50$ ($\beta = 8$) and solid lines: $\theta = 70$ ($\beta = 11.2$).

6.2. Effect of activation temperature

We now explore the effect of the activation temperature in more detail. Figure 4 shows the neutrally stable wavenumber in (l, k)- and (Le, k)-planes, for $Pr = 0.75$, $Q = 4$ and activation temperatures of $\theta = 70$ (corresponding to $\beta = 11.2$), $\theta = 50$ ($\beta = 8$) and $\theta = 30$ ($\beta = 4.8$). For $Le = 1$ ($l = 0$) we find that the stability is independent of the activation temperature. The curves in figure 4(b) all cross and meet at $Le = 1$. Figure 5 shows the dispersion relations when $Le = 1.8$ and $Le = 0.3$ for various activation temperatures. Figures 4(b) and 5 also show that, for fixed $Le > 1$, increasing the activation temperature stabilizes the flame somewhat (the maximum growth rate decreases and the band of unstable wavenumbers narrows), while increasing activation temperature destabilizes the flame for fixed $Le < 1$ (the maximum growth rate increases and the unstable band widens as θ increases).

Figure 4(a) shows that for the hydrodynamic instability with fixed l close to or above zero (Le close to or above one), the neutral stability boundary has converged to the infinite activation temperature results even for the moderate value of $\theta = 30$
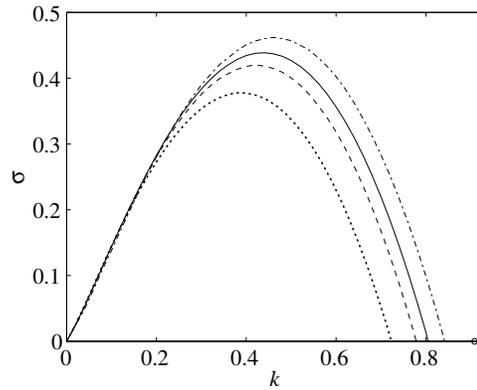
Figure 6. Dispersion relations for $Pr = 0.75$, $Q = 4$, $l = -5$ and dotted line: $\theta = 30$ ($\beta = 4.8$, $Le = 0.4898$), dashed line: $\theta = 50$ ($\beta = 8$, $Le = 0.6154$), solid line: $\theta = 70$ ($\beta = 11.2$, $Le = 0.6914$) and dot-dashed line $\theta = 140$ ($\beta = 22.4$, $Le = 0.8175$). Also shown as an open circle is the infinite activation temperature neutrally stable wavenumber.

($\beta = 4.8$). Hence these results show that for the hydrodynamic instability, high activation temperature asymptotics give quantitatively good predictions even for not particularly high values of the activation temperature and even when the Lewis number is $O(1)$ different from unity. This is also in agreement with the results of the numerical simulations of Denet & Haldenwang (1995), who found that the measured growth rates did not depend very much on β for fixed l.

For $l < 0$, figure $4(a)$ shows that as l decreases, the neutral stability boundaries for different activation temperatures begin to rapidly diverge. Figure 6 shows the dispersion relations when $l = -5$ for $\theta = 30$, 50, 70 and 140 (corresponding to $\beta = 4.8$, 8, 11.2 and 22.4, respectively). Note from figure 6 that at small wavenumbers, the growth rates are insensitive to the activation temperature, but that the dispersion relations for different activation temperatures diverge as the wavenumber increases. For comparison the neutrally stable wavenumber for infinite activation is also shown as the open circle. Even for a physically very large Zeldovich number of $\beta = 22.4$, the results are not well converged. The lower l, the slower the convergence to the infinite activation temperature results. Indeed, for low Lewis numbers the activation temperature must be very high for the infinite activation temperature results to give quantitatively good predictions. This sensitivity to the activation temperature for the cellular instability regime agrees with the finite activation temperature CDA results of Lasseigne *et al.* (1999) and the CDA numerical simulations of Denet & Haldenwang (1992), who both found that for the cellular instability the results for finite β were rather different from those for $\beta \to \infty$ unless the activation temperature was extremely high.

6.3. *Effect of heat release*

Figure 7 shows the neutral stability boundary for $Pr = 0.75$, $Q = 4$ and $\theta = 70$ ($\beta = 11.2$) as well as those when Q is increased to 8, with θ kept fixed (so that $\beta = 6.91$) and also with β kept fixed (giving $\theta = 113.4$). Note first that for fixed Zeldovich number, β, increasing Q destabilizes the flame for all Lewis numbers (the
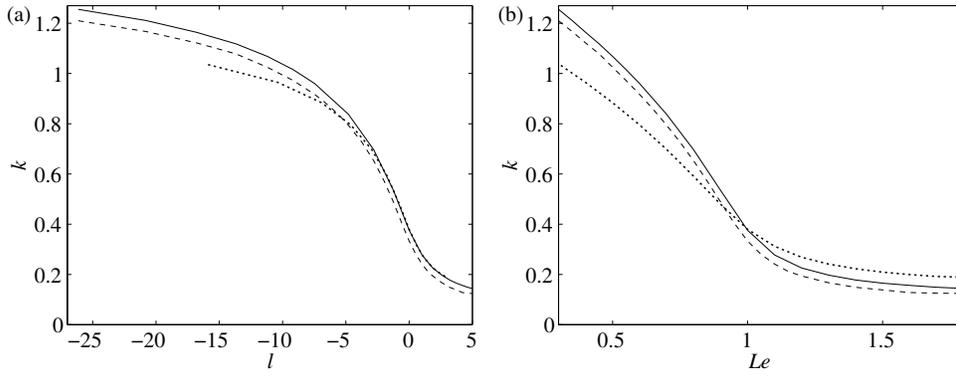
Figure 7. Neutral stability boundaries in (a) the (l,k)-plane and (b) the (Le,k)-plane, for $Pr = 0.75$ and dotted lines: $Q = 8$, $\theta = 70$ ($\beta = 6.91$), dashed lines: $Q = 4$, $\theta = 70$ ($\beta = 11.2$) and solid line: $Q = 8$, $\theta = 113.4$ ($\beta = 11.2$).
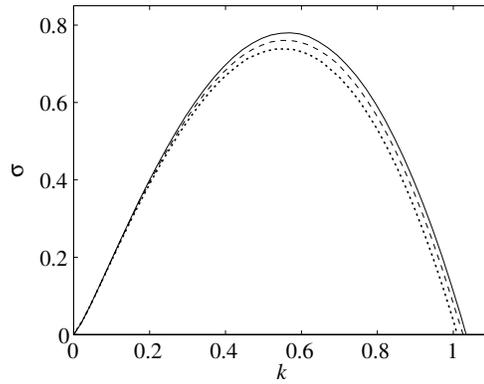
Figure 8. Dispersion relation for $Q = 4$, $Le = 0.5$, $\theta = 70$ and Prandtl numbers 0.5 (dotted line), 0.75 (dashed line) and 1.0 (solid line).

neutral stability boundary for $Q = 8$ lies above and to the right of that for $Q = 4$). For fixed activation temperature, θ, the stability boundary for $Q = 8$ is in agreement with that for fixed β in the (l,k)-plane for $l \geq 0$, and hence increasing Q destabilizes the wave. However, the fixed θ and fixed β boundaries diverge as l decreases below zero and the stability becomes sensitive to the activation temperature. At about $l = -5$ the neutral stability boundary for $Q = 8$ crosses that for $Q = 4$ when θ is kept fixed, and hence for $l < -5$ increasing Q stabilizes the flame.

6.4. Effect of viscosity

Note that the dispersion relation (27) from the SVF analysis is independent of the Prandtl number. This suggests that the stability of the flame is insensitive to the viscosity at low wavenumbers, for which the SVF analysis is valid (Frankel & Sivashinsky (1982); Matalon & Matkowsky 1982; Pelce & Clavin 1982). Indeed, it was found that changing Pr had very little effect on the dispersion relations calculated here for the hydrodynamic instability when the Lewis number is close to or above

T_i	Λ	k_{n1}	k_{n2}	σ_{max}	k_{max}
1.00001	24154.2	0.0	0.3375	0.166	0.0927
1.01	24154.2	0.0	0.3375	0.166	0.0927
1.1	24154.2	0.0	0.3375	0.167	0.0927
1.5	24154.5	0.0	0.3374	0.167	0.0926
2.0	24181.0	0.0093	0.3278	0.165	0.083057

Table 2. Linear stability results for $Q = 4$, $\theta = 30$ ($\beta = 4.8$), $Pr = 0.75$, $Le = 1$ and various ignition temperatures. Λ is the steady flame speed eigenvalue, k_{n1} and k_{n2} are the neutrally stable wavenumbers, σ_{max} is the maximum growth rate and k_{max} the corresponding wavenumber.

unity. However, it remains to check whether the $O(1)$ wavenumber unstable cellular instability at lower Lewis numbers, for which the SVF analysis is not valid, is affected by viscosity. Note that in the CDA approximation, hydrodynamic effects are ignored and hence the CDA analysis cannot reveal anything about the effect of viscosity. Figure 8, which shows the dispersion relations for $Q = 4$, $Le = 0.5$, $\theta = 70$ and various Prandtl numbers, reveals that for the cellular instability at lower Lewis numbers, viscosity has a slight destabilizing effect. Figure 8 shows that at low wavenumbers, the growth rates are very insensitive to the Prandtl number, in agreement with the SVF analysis. However, as the wavenumber increases and the SVF analysis becomes invalid, viscosity begins to play more of a role. At fixed higher wavenumbers, increasing Pr increases the growth rates. Increasing the Prandtl number also increases the maximum growth rate and shifts the corresponding wavenumber to slightly higher values, and the band of unstable wavenumbers is also somewhat widened.

However, Addabbo, Bechtold & Matalon (2002) recently found that in the realistic case where the viscosity is allowed to vary with temperature, increasing Pr has a stabilizing effect on spherical flames. Hence in real flames, viscosity is likely to be stabilizing.

6.5. Effect of ignition temperature

As can be seen from figure 1(d), in the steady flame the reaction rate is exponentially small outside of a relatively thin reaction zone region. Indeed, for high activation temperature, the reaction rate is exponentially small whenever $(1 + Q) - T_0$ is larger than $O(1/\beta)$ and hence the reaction term does not appear at leading order in the region outside of the flame sheet (Buckmaster & Ludford 1982). The requirement for an ignition temperature in finite activation temperature calculations is due to the use of an infinitely long domain. Hence without an ignition temperature, even though the reaction is exponentially small ahead of the reaction zone, a particle will have been reacting at this very small rate for an infinite time (and hence will be fully reacted) before it gets to the reaction zone, so that one cannot obtain a steady solution (Williams 1985). By supplying an ignition temperature, the exponentially small reaction is only switched on at a finite distance ahead of the reaction zone, and hence the degree of reaction outside of the reaction zone is negligible. The steady solution should not then be sensitive to the value of the ignition temperature used provided it corresponds to a temperature outside the reaction zone region (i.e. T_i sufficiently close to one) and that it is not exponentially close to one (which would correspond to the ignition point being located at an exponentially long distance ahead

of the reaction zone, again resulting in a significant degree of reaction before the reaction zone is reached).

However, we should check that the stability results are also not sensitive to the ignition temperature. Table 2 shows the results for various ignition temperatures for a physically low Zeldovich number of $\beta = 4.8$ (for which the results will be most sensitive to T_i). As can be seen both the steady flame solution (through the eigenvalue Λ) and the linear dispersion relation are very insensitive to T_i provided T_i is sufficiently close to one. Only when $T_i - 1$ becomes $O(1)$ does its value begin to have an effect. One point to note is that if too high a value of T_i is chosen (e.g. $T_i = 2$ in table 2), corresponding to the ignition point being inside the reaction zone, the lower neutrally stable wavenumber becomes positive and hence $k = 0$ becomes stable.

7. Conclusions

In this paper we have investigated the linear stability of freely propagating planar premixed flames for the Reactive Navier-Stokes equations with arbitrary values of the parameters, including finite activation temperature, using a numerical shooting method.

The exact linear stability results were compared to previous high activation temperature asymptotics. For Lewis numbers close enough to or above one, the finite activation temperature results agree with the infinite activation temperature results even for only moderate activation temperatures. Hence for these hydrodynamically dominated instabilities, the results are insensitive to the activation temperature for fixed $l = \beta(1 - 1/Le)$. However, as the Lewis number decreases below unity and thermal-diffusive effects become important, the stability becomes more and more sensitive to the activation temperature, and the results for fixed finite activation temperature diverge from the infinite activation activation temperature results. At low Lewis numbers, very high activation temperatures are required for quantitative agreement with the asymptotic predictions. Slowly varying flame analyses, which are based on small wavenumber of the perturbation, give qualitatively good results, but underpredict the wavenumber with the maximum growth rate and the neutrally stable wavenumber. Neither of the limits assumed in the constant density approximation model of Sivashinsky (1977), $\beta \to \infty$ and $Q \to 0$, give accurate results for the cellular instability when realistic values of the Zeldovich number and the heat release are used, and hence weakly nonlinear theories based on this model (Sivashinsky 1983) will also only give, at best, qualitative results.

The results were also compared with previous numerical simulations of the full non-linear problem, which demonstrated the role of exact linear stability analyses in validating numerical schemes for such simulations and determining in which parameter regimes a numerical method gives accurate results or fails.

A new result is that state-insensitive viscosity has a small destabilizing effect on the cellular instability at low Lewis numbers. However, in reality one must consider temperature dependent viscosity (Addabbo *et al.* 2002).

The numerical shooting method described here for determining the linear stability of premixed flames can in principal be extended to take into account such effects as heat loss, buoyancy (gravity) or variable transport properties on the stability of the flame, or to more complex chemical kinetic models. We intend to investigate these issues in the future. We also intend to perform numerical simulations of the Reactive Navier-Stokes equations in order to compare with the exact linear stability results

and investigate the full non-linear evolution of unstable premixed flames, extending the results of Denet & Haldenwang (1995) and Kadowaki (1997, 1999, 2000), such as investigating the role of wall boundaries on unstable flames in tubes.

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