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Bachelor of Science

# Algebras of Convolution Type Operators on Weighted Variable Lebesgue Spaces

Dissertation submitted in partial fulfillment
of the requirements for the degree of

Master of Science in
**Mathematics and Applications**

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CIÊNCIAS E TECNOLOGIA
UNIVERSIDADE NOVA DE LISBOA

**November, 2020**

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# Acknowledgements

I would like to thank my thesis adviser Oleksiy Karlovych for his unending patience and readiness in helping me with everything. I must highlight his encyclopedic knowledge of Functional Analysis and his extremely keen eye for proof-reading.

I would also like to thank my thesis co-adviser Cláudio Fernandes for his constant kindness, empathy and understanding. He was always a very strong source of support and made me feel comfortable in my work.

I must also thank all my friends, who every day bring joy and laughter to my life. Something as simple as their presence allowed me to get through this difficult year.

Finally, I give my eternal gratitude to my family for their material, emotional and spiritual support. They gave me my future.

# Abstract

We prove a version of the Riesz-Thorin interpolation theorem for some types of weighted variable Lebesgue spaces. In order to do this we use the theory developed by Calderón in his 1964 article, together with some Banach function space theory.

Using our version of the Riesz-Thorin theorem, we prove a version of the Stechkin inequality for weighted variable Lebesgue spaces, allowing us to define algebras of Fourier multipliers arising from functions of bounded variation.

After analyzing the invertibility of Fourier convolution operators with piecewise continuous symbols, we shift our attention to slowly oscillating Fourier multipliers, finishing with a proof that the image in the Calkin algebra of the algebra of convolution type operators with slowly oscillating data is commutative.

**Keywords:** Weighted variable Lebesgue spaces, Fourier multipliers, Interpolation, Functions of bounded variation, Slowly oscillating functions, Algebras of convolution type operators, Compactness of commutators

# Resumo

Provamos uma versão do teorema de interpolação de Riesz-Thorin para alguns tipos de espaços de Lebesgue com expoente variável e peso. De forma a atingir este objectivo, usamos a teoria desenvolvida por Calderón no seu artigo de 1964.

Usando a versão do teorema de Riesz-Thorin obtida, provamos uma versão da desigualdade de Stechkin para espaços de Lebesgue com expoente variável e peso. Isto permite-nos definir álgebras de multiplicadores de Fourier associados a funções de variação limitada.

Após analisada a invertibilidade dos operadores de convolução com símbolos contínuos por troços, deslocamos a nossa atenção para multiplicadores de Fourier fracamente oscilantes. Terminamos com a prova de que a imagem na álgebra de Calkin da álgebra de operadores tipo convolução com dados fracamente oscilantes é comutativa.

**Palavras-chave:** Espaços de Lebesgue com expoente variável e peso, Multiplicadores de Fourier, Interpolação, Funções de variação limitada, Funções fracamente oscilantes, Álgebras de operadores tipo convolução, Compacidade de comutadores

# Contents

# Chapter 1

# Introduction

Some of the key spaces in functional analysis are the *Lebesgue spaces* $L^p$. Given a real number $p \geq 1$, the $L^p$ space on $\mathbb{R}$ is the set of complex-valued measurable functions $f$ defined on the real line such that the $p$-norm

$$\|f\|_{L^p} := \left( \int_{\mathbb{R}} |f(x)|^p \, dx \right)^{1/p}$$

is finite.

One natural generalization of these spaces is obtained by adding a *weight*. This is a measurable function $w : \mathbb{R} \to [0, +\infty]$ such that $0 < w(x) < +\infty$ almost everywhere. The new space $L^p(w)$ is then the set of all measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that the norm

$$\|f\|_{L^p(w)} := \left( \int_{\mathbb{R}} |f(x)w(x)|^p \, dx \right)^{1/p}$$

is finite.

We will work in *weighted variable Lebesgue spaces*. These are a further generalization of Lebesgue spaces. We replace the constant $p$ by a measurable function $p(x) : \mathbb{R} \to ]1, \infty[$. Although we cannot just replace $p$ by $p(x)$ outside of the integral, we can define the $L^{p(\cdot)}(w)$ norm by

$$\|f\|_{L^{p(\cdot)}(w)} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}} \left| \frac{f(x)w(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

Under certain conditions on the weight, this is a Banach space.

Our main objects of study will be *multiplication operators* $aI$, which multiply some function by a bounded measurable function $a$, and *Fourier convolution operators* $W_b^0$, where $b$ is a suitable function, sometimes called *symbol*. These operators act on a function $g$ by

$$W_b^0(g) := F^{-1} b F g.$$

Here $F$ is the *Fourier transform*, defined for functions $f \in L^1(\mathbb{R})$ by

$$(Ff)(x) := \int_{\mathbb{R}} f(t)e^{itx}dt, \quad x \in \mathbb{R},$$

and $F^{-1}$ is its inverse, defined by

$$(F^{-1}g)(t) := \frac{1}{2\pi}\int_{\mathbb{R}} g(x)e^{-itx}dx, \quad t \in \mathbb{R}.$$

*Fourier multipliers* are precisely the functions $b \in L^\infty(\mathbb{R})$ that give rise to bounded operators $W_b^0$. Here $L^\infty(\mathbb{R})$ is the set of complex-valued functions that are bounded on $\mathbb{R}$. The set $M_{p(\cdot),w}$ of Fourier multipliers for a fixed $L^{p(\cdot)}(w)$ is a Banach algebra, with the norm given by

$$\|b\|_{M_{p(\cdot),w}} := \left\|W_b^0\right\|_{\mathcal{B}(L^{p(\cdot)}(w))},$$

where $\mathcal{B}(L^{p(\cdot)}(w))$ is the algebra of bounded linear operators in $L^{p(\cdot)}(w)$.

We will study properties of Fourier convolution operators $W_b^0$ with $b$ in several classes of symbols. Let $C(\dot{\mathbb{R}})$ be the set of all continuous functions $f$ on $\mathbb{R}$ with equal limits at $-\infty$ and $+\infty$, $PC(\mathbb{R})$ the set of piecewise continuous functions, $SO^\diamond$ the set of slowly oscillating functions, and $PSO^\diamond$ the set of piecewise slowly oscillating functions. These concepts will be rigorously defined later.

If $\mathcal{S}$ is one of the aforementioned spaces, let $\mathcal{S}_{p(\cdot),w}$ be the corresponding algebra of Fourier multipliers. The definition of $\mathcal{S}_{p(\cdot),w}$ varies according to the space $\mathcal{S}$, but it is generally the closure of some set with respect to the Fourier multiplier norm $\|\cdot\|_{M_{p(\cdot),w}}$ defined above.

Define the following algebras of operators:

$$\mathcal{C}_{p(\cdot),w} := \operatorname{alg}_{\mathcal{B}(L^{p(\cdot)}(w))}\{aI, W_b^0 : a \in C(\dot{\mathbb{R}}),\ b \in C_{p(\cdot),w}(\dot{\mathbb{R}})\},$$
$$\mathcal{SO}_{p(\cdot),w} := \operatorname{alg}_{\mathcal{B}(L^{p(\cdot)}(w))}\{aI, W_b^0 : a \in SO^\diamond,\ b \in SO^\diamond_{p(\cdot),w}\},$$
$$\mathcal{PSO}_{p(\cdot),w} := \operatorname{alg}_{\mathcal{B}(L^{p(\cdot)}(w))}\{aI, W_b^0 : a \in PSO^\diamond,\ b \in PSO^\diamond_{p(\cdot),w}\}.$$

These are the smallest closed subalgebras of $\mathcal{B}(L^{p(\cdot)}(w))$ that contain the operators indicated between brackets.

A compact operator is one that takes bounded sets to sets with compact closure. The set $\mathcal{K}(V)$ of all compact operators on a given Banach space $V$ is a closed ideal. This allows us to define the *Calkin algebra* as

$$\mathcal{B}^\pi(V) := \mathcal{B}(V)/\mathcal{K}(V).$$

A *Fredholm operator* is then an operator $F \in \mathcal{B}(V)$ such that $F + \mathcal{K}(V)$ is invertible in $\mathcal{B}^\pi(V)$.

After proving that the ideal of compact operators is a subset of each of the above algebras, we can define more quotient algebras. If $\mathcal{X}_{p(\cdot),w} \in \{\mathcal{C}_{p(\cdot),w}, \mathcal{SO}_{p(\cdot),w}, \mathcal{PSO}_{p(\cdot),w}\}$, define

$$\mathcal{X}^\pi := \mathcal{X}_{p(\cdot),w}/\mathcal{K}(L^{p(\cdot)}(w)).$$

A criterion for Fredholmness of an arbitrary operator $D \in \mathcal{PSO}_{p,w}$, in the case of constant exponent $p \in ]1,\infty[$ and so-called Muckenhoupt weight $w$, was obtained in [26] and [27]. The proof is based on Allan's local principle [1]. The latter result gives criteria for invertibility of an element of an algebra, if the algebra contains a suitable large commutative subalgebra.

The aim of this thesis is to begin studying the algebra $\mathcal{PSO}_{p(\cdot),w}$, with log-Hölder continuous exponents and suitable power weights

$$w(x) = |x-i|^{\lambda_\infty} \prod_{j=1}^{m} |x-x_j|^{\lambda_j}, \quad x \in \mathbb{R}.$$

We take the first steps in adapting the method followed in [26] and [27] to find a Fredholm criterion for $\mathcal{PSO}_{p(\cdot),w}$.

The thesis is organized as follows.

Chapter 2 is dedicated to some preliminaries. We define *algebras over a field* and *quotient algebras*. We then give the definitions needed for the statement of the *local principle* developed by Israel Gohberg and Nahum Krupnik in their 1973 paper [18]. This result will later allow us to say precisely when certain operators are invertible. We recall some basic concepts in operator theory, namely those of *bounded* and *compact* operators, along with some results. After presenting the concept of a *piecewise continuous function*, we repeat Roland Duduchava's proof that each piecewise continuous function can be approximated by a piecewise constant function. The set of *functions of bounded variation*, which will be an important concept in the following chapters, is also introduced. We finish this chapter with some basic facts about the convolution of two functions.

In Chapter 3, after first giving an intuitive explanation of interpolation and the original interpolation theorem by Marcel Riesz and Olof Thorin, we start a foray into Alberto Calderon's interpolation theory, developed in his 1964 paper [9]. We then finally define the basic spaces in this thesis, the *weighted variable Lebesgue spaces*. We then apply some of Calderon's results to obtain an analogue of the Riesz-Thorin interpolation theorem for weighted variable Lebesgue spaces. We conclude the chapter with a theorem on interpolation of compactness in the setting of weighted variable Lebesgue spaces, analogous to the Krasnosel'skii interpolation theorem. It seems that both interpolation theorems were not stated explicitly in the literature.

In Chapter 4, we define *Fourier convolution operators* and *Fourier multipliers* on weighted variable Lebesgue spaces. We introduce the *classical Stechkin inequality*, which states that every function of bounded variation is a Fourier multiplier on the standard Lebesgue spaces. We then devote a section of this chapter to obtain a new generalization of this result to weighted variable Lebesgue spaces. This generalization allows us to define the algebra of piecewise continuous Fourier multipliers and obtain new results about invertibility of Fourier convolution operators with piecewise continuous symbols on weighted variable Lebesgue spaces.

In Chapter 5, using the results of the previous chapters, we find a description of the algebra $C_{p(\cdot),w}(\dot{\mathbb{R}})$ of continuous Fourier multipliers. We then study three diferent algebras of convolution type operators, $\mathcal{C}_{p(\cdot),w} \subset \mathcal{SO}_{p(\cdot),w} \subset \mathcal{PSO}_{p(\cdot),w}$, with continuous, slowly oscillating and piecewise slowly oscillating symbols, respectively. We show that the ideal of compact operators $\mathcal{K}(L^{p(\cdot)}(w))$ is contained in the algebra $\mathcal{C}_{p(\cdot),w}$. The main result of this chapter says that the commutators $aW^0(b) - W^0(b)aI$ are compact for all $a \in SO^\diamond$ and $b \in SO_{p(\cdot),w}$. These results imply that the quotient algebras $\mathcal{SO}^\pi_{p(\cdot),w} \subset \mathcal{PSO}^\pi_{p(\cdot),w}$ are well defined and that the algebra $\mathcal{SO}^\pi_{p(\cdot),w}$ is a commutative subalgebra of the algebra $\mathcal{PSO}^\pi_{p(\cdot),w}$. If, in the future, a description of the maximal ideal space of $\mathcal{SO}^\pi_{p(\cdot),w}$ is obtained, this would allow the use of the Allan local principle to study invertibility in the algebra $\mathcal{PSO}^\pi_{p(\cdot),w}$.

# Chapter 2

# Preliminaries

## 2.1 Algebras

### 2.1.1 Basic Concepts

An *algebra* $\mathscr{A}$ over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is a vector space over $\mathbb{K}$, equipped with an additional binary operation (which we will denote by juxtaposition), satisfying the following properties:

- $(ab)c = a(bc)$;
- $(a+b)c = ac + bc, \quad a(b+c) = ab + ac$;
- $(\lambda a)b = a(\lambda b) = \lambda(ab)$

for all $a, b, c \in \mathscr{A},\ \lambda \in \mathbb{K}$.

If $\mathbb{K} = \mathbb{R}$, we will call the algebra *real*, and *complex* otherwise. We will call an algebra $\mathscr{A}$ *commutative* if it satisfies

$$ab = ba, \quad a, b \in \mathscr{A},$$

and *unital* if it has a unit: an element $e \in \mathscr{A}$ such that

$$ea = ae = a, \quad a \in \mathscr{A}.$$

An element $a \in \mathscr{A}$ is said to be invertible if there exists $b \in \mathscr{A}$ such that

$$ab = ba = e.$$

An algebra $\mathscr{A}$ becomes a *Banach algebra* if in addition, we define a norm $\|\cdot\|$ on $\mathscr{A}$ that makes the underlying vector space complete and satisfies

- $\|ab\| \leq \|a\|\,\|b\|$ for each $a, b \in \mathscr{A}$;

- $\|e\| = 1$, if $\mathcal{A}$ is unital.

Let $\mathcal{A}$ be a Banach algebra and $B$ be a subset of $\mathcal{A}$. We denote by $\operatorname{clos}_{\mathcal{A}} B$ the closure of $B$ with respect to the norm of $\mathcal{A}$. If $B$ is an algebra, then $\mathcal{B} = \operatorname{clos}_{\mathcal{A}} B$ becomes a Banach subalgebra of $\mathcal{A}$.

Furthermore, for a subset $E \subset \mathcal{A}$, we denote by $\operatorname{alg}_{\mathcal{A}} E$ the smallest closed subalgebra of $\mathcal{A}$ that contains $E$, i.e., the intersection of all closed subalgebras of $\mathcal{A}$ containing $E$. Equivalently,

$$\operatorname{alg}_{\mathcal{A}} E = \operatorname{clos}_{\mathcal{A}} \left\{ \sum_j \gamma_j \prod_k a_{jk} \ : \ \gamma_j \in \mathbb{C},\ a_{jk} \in E \right\}.$$

Here the sum and the (ordered) product are finite.

The following simple result can be found in many textbooks on Banach algebras.

**Theorem 2.1.1.** *Let $\mathcal{A}$ be a Banach algebra with identity $e$. If $u \in \mathcal{A}$ satisfies $\|u\| < 1$, then the element $e - u$ is invertible with the inverse*

$$(e-u)^{-1} = \sum_{n=0}^{\infty} u^n.$$

*Proof.* Note that $\sum_{n=0}^{\infty} \|u^n\| \le \sum_{n=0}^{\infty} \|u\|^n$. The latter is a geometric series, which is convergent since $\|u\| < 1$. This implies that the series $\sum_{n=0}^{\infty} \|u^n\|$ is absolutely convergent, hence the series $\sum_{n=0}^{\infty} u^n$ is convergent by completeness of $\mathcal{A}$. Denote by $v_N = \sum_{n=0}^{N} u^n$ the partial sums of this series. Then

$$(e-u)v_N = v_N(e-u) = \sum_{n=0}^{N} u^n - \sum_{n=0}^{N} u^{n+1} = e - u^{N+1},$$

Since $\left\|u^{N+1}\right\| \le \|u\|^{N+1} \to 0$ as $N \to \infty$, the above identity implies that $\sum_{n=0}^{\infty} u^n$ is the inverse of $e - u$. □

### 2.1.2 Quotient Algebras

Let $\mathcal{A}$ be an algebra over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A subset $I \subset \mathcal{A}$ is called a *right ideal* if

- $\lambda a \in I$;
- $a + b \in I$;
- $ac \in I$

for all $a, b \in I$, $c \in \mathcal{A}$, $\lambda \in \mathbb{K}$. By replacing $ac$ with $ca$, we would define a *left ideal*. A subset that is both a left and right ideal is called a *two-sided ideal*.

Given a two sided-ideal $I$ of an algebra $\mathcal{A}$, we can define an equivalence relation $\sim$ on $\mathcal{A}$ as

$$a \sim b \text{ if and only if } a - b \in I.$$

The equivalence class of an element $a \in \mathscr{A}$ is given by

$$a^\pi = a + I = \{a + i : i \in I\}.$$

The set of equivalence classes is then an algebra over $\mathbb{K}$, denoted by $\mathscr{A}/_I$, with operations defined by

- $a^\pi + b^\pi = (a+b)^\pi$;
- $a^\pi b^\pi = (ab)^\pi$;
- $\lambda a^\pi = (\lambda a)^\pi$.

The algebra $\mathscr{A}/_I$ is called the *quotient algebra of $\mathscr{A}$ over $I$*.

**Theorem 2.1.2** ([16, Chap. 1, §2, Theorem 3])**.** *If $\mathscr{A}$ is a Banach algebra and $I$ is a closed ideal, we can define a norm on $\mathscr{A}/_I$ by*

$$\left\|a^\pi\right\|_{\mathscr{A}/_I} = \inf\{\|a+i\|_{\mathscr{A}} : i \in I\}.$$

*This turns $\mathscr{A}/_I$ into a Banach algebra.*

### 2.1.3 Gohberg and Krupnik Local Principle

The *Gohberg and Krupnik local principle* was developed by Israel Gohberg and Nahum Krupnik in their article "On a Local Principle and Algebras Generated by Toeplitz Matrices", published in Russian in 1973. In this section we will cite results from its English translation [18]. The same results are also available in [17, Chap. 5].

Given a unital Banach algebra $\mathscr{A}$, we will call a subset $M \subset \mathscr{A}$ a *localizing class* if

- $0 \notin M$;
- for any $f_1, f_2 \in M$ there exists an element $f \in M$ such that $f_j f = f f_j = f, j = 1, 2$.

Two elements $a, b \in \mathscr{A}$ are said to be *M-equivalent*, and we write $a \overset{M}{\sim} b$ if

$$\inf_{g \in M} \|(a-b)g\| = \inf_{g \in M} \|g(a-b)\| = 0.$$

The following two lemmas are stated in [13, p. 21] without proof.

**Lemma 2.1.3.** *Let $\mathscr{A}$ be a unital Banach algebra and $M$ a localizing class. Suppose that*

$$\sup_{g \in M} \|g\| = K < \infty.$$

*If $a_1 \overset{M}{\sim} b_1$, $a_2 \overset{M}{\sim} b_2$ and $\lambda_1, \lambda_2 \in \mathbb{K}$ then*

$$\lambda_1 a_1 + \lambda_2 a_2 \overset{M}{\sim} \lambda_1 b_1 + \lambda_2 b_2.$$

*Proof.* Fix $\varepsilon > 0$. If $\lambda_1 = \lambda_2 = 0$ the result is obvious. Suppose otherwise and take $g_1, g_2 \in M$ such that

$$\|(a_1 - b_1)g_1\|, \ \|(a_2 - b_2)g_2\| < \frac{\varepsilon}{K(|\lambda_1| + |\lambda_2|)}.$$

By definition of localizing class, there exists $g_3 \in M$ such that $g_j g_3 = g_3 g_j = g_3$, $j = 1, 2$. Then

$$\begin{aligned}
\|(\lambda_1 a_1 + \lambda_2 a_2 - \lambda_1 b_1 - \lambda_2 b_2)g_3\| &= \|\lambda_1(a_1 - b_1)g_3 + \lambda_2(a_2 - b_2)g_3\| \\
&= \|\lambda_1(a_1 - b_1)g_1 g_3 + \lambda_2(a_2 - b_2)g_2 g_3\| \\
&\leq |\lambda_1| \, \|(a_1 - b_1)g_1 g_3\| + |\lambda_2| \, \|(a_2 - b_2)g_2 g_3\| \\
&\leq \|g_3\| (|\lambda_1| \, \|(a_1 - b_1)g_1\| + |\lambda_2| \, \|(a_2 - b_2)g_2\|) \\
&< K(|\lambda_1| + |\lambda_2|) \frac{\varepsilon}{K(|\lambda_1| + |\lambda_2|)} = \varepsilon.
\end{aligned}$$

This proves that

$$\inf_{g \in M} \|(\lambda_1 a_1 + \lambda_2 a_2 - \lambda_1 b_1 - \lambda_2 b_2)g\| = 0,$$

and the proof of the other equality is analogous. □

**Lemma 2.1.4.** *Let $\mathcal{A}$ be a unital Banach algebra and $M$ a localizing class. Suppose that*

$$\sup_{g \in M} \|g\| = K < \infty.$$

*If $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ are sequences in $\mathcal{A}$ with limits $a$ and $b$, respectively, such that $a_n \overset{M}{\sim} b_n$ for all $n \in \mathbb{N}$, then $a \overset{M}{\sim} b$.*

*Proof.* Fix $\varepsilon > 0$. For each $n \in \mathbb{N}$, take $g_n \in M$ such that

$$\|(a_n - b_n)g_n\| < \frac{\varepsilon}{2K + 1}.$$

By definition of the limit, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$\forall n \geq N_1, \quad \|a - a_n\| < \frac{\varepsilon}{2K + 1}, \quad \forall n \geq N_2, \quad \|b - b_n\| < \frac{\varepsilon}{2K + 1}.$$

Take $N = \max\{N_1, N_2\}$. Then

$$\begin{aligned}
\|(a - b)g_N\| = \|a g_N - b g_N\| &= \|a g_N + a_N g_N - a_N g_N + b_N g_N - b_N g_N - b g_N\| \\
&= \|(a - a_N)g_N - (b - b_N)g_N + (a_N - b_N)g_N\| \\
&\leq \|(a - a_N)g_N\| + \|(b - b_N)g_N\| + \|(a_N - b_N)g_N\| \\
&\leq \|a - a_N\| \, \|g_N\| + \|b - b_N\| \, \|g_N\| + \|(a_N - b_N)g_N\| \\
&< \frac{2\varepsilon K}{2K + 1} + \frac{\varepsilon}{2K + 1} = \varepsilon.
\end{aligned}$$

This proves that

$$\inf_{g \in M} \|(a - b)g\| = 0,$$

and the proof of the other equality is analogous. □

An element $a \in \mathcal{A}$ is said to be *M-invertible* if there exists $b \in \mathcal{A}$ and $g \in M$ such that $bag = gab = g$.

**Lemma 2.1.5** ([18, Lemma 1.1]). *Let $\mathcal{A}$ be a Banach algebra with identity e and M a localizing class. If $a, b \in \mathcal{A}$ are such that $a \overset{M}{\sim} b$, then a is M-invertible if and only if b is M-invertible.*

*Proof.* Suppose that $a$ is $M$-invertible. Then there exist $c \in \mathcal{A}$, $f \in M$ such that

$$caf = f.$$

Note that $c \neq 0$, otherwise $f = 0 \in M$. Since $a \overset{M}{\sim} b$, there exists $g \in M$ such that

$$\|(a-b)g\| < \|c\|^{-1}.$$

Since $M$ is a localizing class, there exists $h \in M$ such that

$$fh = gh = h.$$

Using the previous equalities, we deduce that

$$\begin{aligned} cbh &= cah - c(a-b)h \\ &= cafh - c(a-b)gh \\ &= fh - c(a-b)gh \\ &= h - c(a-b)gh. \end{aligned}$$

Defining $u = c(a-b)g$, we have

$$cbh = (e-u)h.$$

Since $\|u\| < 1$, the element $e - u$ is invertible by Theorem 2.1.1. Defining $d = (e-u)^{-1}c$, we obtain $dbh = h$. The proof of the equality $hbd = h$ is analogous. □

let $X$ be an index set. A collection $(M_x)_{x \in X}$ of localizing classes is said to be *covering* if for each choice of elements $a_x \in M_x$, one can select a finite number of elements $a_{x_1}, \dots a_{x_n}$ such that $a_{x_1} + \dots + a_{x_n}$ is invertible.

**Theorem 2.1.6** ([18, Theorem 1.1]). *Let $\mathcal{A}$ be a Banach algebra with identity e and X an index set. Let $(M_x)_{x \in X}$ be a covering collection of localizing classes and $a \in \mathcal{A}$ that satisfies the property:*

$$\text{for every } x \in X \text{ there exists } a_x \in \mathcal{A} \text{ such that } a \overset{M_x}{\sim} a_x.$$

*If a commutes with every element of $\bigcup_{x \in X} M_x$, then it is invertible in $\mathcal{A}$ if and only if each $a_x$ is $M_x$-invertible for every $x \in X$.*

*Proof.* If $a$ is invertible in $\mathcal{A}$, $a$ is evidently $M_x$-invertible for each $x \in X$. Since $a \overset{M_x}{\sim} a_x$, each $a_x$ is also $M_x$-invertible by Lemma 2.1.5. Now suppose that each $a_x$ is $M_x$-invertible and let us prove that $a$ is invertible. By Lemma 2.1.5, $a$ is $M_x$ invertible for each $x \in X$. This means that there exist $b_x \in \mathcal{A}$, $f_x \in M_x$ such that $b_x a f_x = f_x$. Since $(M_x)_{x \in X}$ is a covering collection, we can choose $f_{x_1}, \ldots, f_{x_m}$ such that $\sum_{j=1}^m f_{x_j}$ is invertible in $\mathcal{A}$. Now define

$$s := \sum_{j=1}^{m} b_{x_j} f_{x_j}.$$

Using the hypothesis that $a$ commutes with every element of $\bigcup_{x \in X} M_x$, we obtain

$$sa = \sum_{j=1}^{m} b_{x_j} f_{x_j} a = \sum_{j=1}^{m} b_{x_j} a f_{x_j} = \sum_{j=1}^{m} f_{x_j}.$$

This implies that

$$\left( \sum_{j=1}^{m} f_{x_j} \right)^{-1} sa = e.$$

In a similar manner we can prove that

$$a \left( \sum_{j=1}^{m} f_{x_j} \right)^{-1} s = e,$$

hence $a$ is invertible as desired. □

## 2.2 Operator Theory

### 2.2.1 Bounded Operators

Let $V$ and $W$ be normed vector spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A (not necessarily linear) operator $A : V \to W$ is said to be *bounded* if there exists a real number $K$ such that

$$\|Av\|_W \leq K \|v\|_V$$

for all $v \in V$.

The set of bounded linear operators $A : V \to W$, denoted by $\mathcal{B}(V, W)$, is a normed vector space if equipped with the *operator norm*

$$\|A\|_{\mathcal{B}(V,W)} := \inf\{K \geq 0 : \|Av\|_W \leq K \|v\|_V \text{ for all } v \in V\}.$$

If $W$ is a Banach space, so is $\mathcal{B}(V, W)$.

If $V = W$, the set of bounded linear operators is a Banach algebra (with multiplication given by composition of operators) which we denote by $\mathcal{B}(V)$.

### 2.2.2 Compact Operators

Let $V$ and $W$ be Banach spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A linear operator $T : V \to W$ is said to be *compact* if, for any bounded sequence $(x_n)$ in $V$, the sequence $(Tx_n)$ admits a convergent subsequence.

The set of compact operators from $V$ to $W$ will be denoted by $\mathcal{K}(V, W)$. If $V = W$, we abbreviate it to $\mathcal{K}(V)$.

**Theorem 2.2.1** ([34, Theorem 7.2])**.** *Any operator $T \in \mathcal{K}(V,W)$ is bounded.*

*Proof.* Assume $T$ is compact but not bounded. Then for each $n \in \mathbb{N}$ there exists $x_n \in V$ such that

$$\|x_n\| = 1 \text{ and } \|Tx_n\| \geq n.$$

The sequence $(x_n)$ is bounded, hence by compactness of $T$ there exists a convergent subsequence $(Tx_{n_k})$. But this sequence cannot simultaneously be convergent and satisfy $\left\|Tx_{n_k}\right\| \geq n_k$. Contradiction, so $T$ must be bounded. □

**Theorem 2.2.2** ([34, Theorem 7.9])**.** *If $(T_n)$ is a sequence of compact operators convergent to an operator $T$, then $T$ is also compact.*

*Proof.* Let $(x_n)$ be a bounded sequence in $V$. Since $T_1$ is compact, there exists a subsequence $(x_{n_{1,k}})_{k\in\mathbb{N}}$ such that the sequence $(T_1 x_{n_{1,k}})$ converges. Now the sequence $(x_{n_{1,k}})_{k\in\mathbb{N}}$ is also bounded, hence by compactness of $T_2$ it has a subsequence $(x_{n_{2,k}})_{k\in\mathbb{N}}$ such that the sequence $(T_2 x_{n_{2,k}})_{k\in\mathbb{N}}$ is convergent. In addition, $(T_1 x_{n_{2,k}})$ converges since it is a subsequence of the convergent sequence $(T_1 x_{n_{1,k}})$. Repeating this process, we obtain for each $j \in \mathbb{N}$ a subsequence $(x_{n_{j,k}})_{k\in\mathbb{N}}$ such that for every $r \leq j$ the sequence $(T_r x_{n_{j,k}})_{k\in\mathbb{N}}$ converges. Defining $n_k = n_{k,k}$, we obtain a single sequence $(x_{n_k})_{k\in\mathbb{N}}$ such that for each fixed $s \in \mathbb{N}$, the sequence $(T_s x_{n_k})$ is convergent. Let us show that the sequence $(Tx_{n_k})$ is Cauchy. Fix $\varepsilon > 0$. The sequence $(x_{n_k})_{k\in\mathbb{N}}$ is bounded, thus there exists $M > 0$ such that

$$\left\|x_{n_k}\right\| \leq M \text{ for all } k \in \mathbb{N}.$$

Because $T$ is the limit of the $T_n$, there exists $L \geq 1$ such that

$$\|T_L - T\| < \frac{\varepsilon}{3M}.$$

Furthermore, since the sequence $(T_L x_{n_k})$ is convergent, there exists $S \geq 1$ such that for each $r, s \geq S$,

$$\left\|T_L x_{n_r} - T_L x_{n_s}\right\| < \frac{\varepsilon}{3}.$$

Finally we can deduce for $r, s \geq S$,

$$\begin{aligned}\left\|Tx_{n_r} - Tx_{n_s}\right\| &\leq \left\|Tx_{n_r} - T_L x_{n_r}\right\| + \left\|T_L x_{n_r} - T_L x_{n_s}\right\| + \left\|T_L x_{n_s} - Tx_{n_s}\right\| \\ &< \|T_L - T\|\left\|x_{n_r}\right\| + \frac{\varepsilon}{3} + \|T_L - T\|\left\|x_{n_s}\right\| \\ &< \frac{\varepsilon}{3M}M + \frac{\varepsilon}{3} + \frac{\varepsilon}{3M}M = \varepsilon.\end{aligned}$$

Note that $W$ is complete, hence the sequence $(Tx_{s_k})_{k\in\mathbb{N}}$ is a convergent subsequence of $(Tx_n)_{n\in\mathbb{N}}$. We conclude that $T$ is a compact operator. □

**Theorem 2.2.3** ([34, Theorem 7.3])**.** *The set $\mathcal{K}(V)$ is a closed two-sided ideal of $\mathcal{B}(V)$.*

*Proof.* Let $\alpha, \beta \in \mathbb{K}$ and $S, T \in \mathcal{K}(V)$. Take a bounded sequence $(x_n)$. Since $S$ is compact, there exists a subsequence $(x_{n_k})$ such that $(Sx_{n_k})$ converges. Now by compactness of $T$, there exists a subsequence $(x_{n_{k_r}})$ of $(x_{n_k})$ such that the sequence $Tx_{n_{k_r}})$ converges. Now we can conclude that the sequence $(\alpha Sx_{n_{k_r}} + \beta Tx_{n_{k_r}})$ is convergent, hence $\alpha S + \beta T$ is compact.

Now assume only one of $S, T \in \mathcal{B}(V)$ is compact and let us prove that $TS$ is compact. Take a bounded sequence $(x_n)$. In the case that $S$ is compact, there exists a subsequence $(x_{n_k})$ such that $(Sx_{n_k})$ converges. Now $T$ is bounded, so the sequence $(TSx_{n_k})$ converges as well, hence $TS$ is compact. In the case that $S$ is bounded but not compact, the sequence $(Sx_n)$ is bounded. Since $T$ is compact, there exists a subsequence $(Sx_{n_k})$ such that $(TSx_{n_k})$ converges and we conclude that $TS$ is compact.

The fact that $\mathcal{K}(V)$ is closed follows from the previous theorem. □

This result allows us to define the *Calkin algebra* $\mathcal{B}^\pi(V)$ as

$$\mathcal{B}^\pi(V) := {}^{\mathcal{B}(V)}\!/_{\mathcal{K}(V)},$$

the quotient of the algebra of bounded operators by the ideal of compact operators. Since $V$ is a Banach space and $\mathcal{K}(V)$ is a closed ideal, we can use Theorem 2.1.2 to deduce that $\mathcal{B}^\pi(V)$ is a Banach algebra.

## 2.3 Piecewise Continuous Functions and Functions of Bounded Variation

### 2.3.1 Piecewise Continuous Functions

Let $PC(\mathbb{R})$ denote the set of functions $a : \mathbb{R} \to \mathbb{C}$ such that the limits

$$a(+\infty) := \lim_{t\to+\infty} a(t), \quad a(-\infty) := \lim_{t\to-\infty} a(t)$$

and

$$a(x+0) := \lim_{t\to x^+} a(t), \quad a(x-0) := \lim_{t\to x^-} a(t)$$

exist and are finite at each point of $\mathbb{R}$.

**Lemma 2.3.1** ([13, Lemma 2.9])**.** *The set $PC(\mathbb{R})$ is a Banach algebra with respect to the $L^\infty(\mathbb{R})$ norm.*

A function $a \in L^\infty(\mathbb{R})$ is called *piecewise constant* if there exist constants $\lambda_k \in \mathbb{C}$ and a partition of the real line $-\infty = t_0 < t_1 < \ldots < t_n = +\infty$ such that $a(t) = \sum_{k=0}^{n-1} \lambda_k \chi_k(t)$, where $\chi_k(t) = \chi_{]t_k, t_{k+1}[}$. The function can take any value at the points $t_k$.

**Lemma 2.3.2** ([13, Lemma 2.9])**.** *The set of piecewise constant functions is dense in $PC(\mathbb{R})$.*

*Proof.* We will prove that each function in $PC(\mathbb{R})$ can be approximated by a piecewise constant function. Let $a \in PC(\mathbb{R})$ and fix $n \in \mathbb{N}$. By definition of $PC(\mathbb{R})$, for each $t \in \mathbb{R}$ there exists a neighbourhood $U_t$ of $t$ such that

$$|a(t-0) - a(x)| < \frac{1}{2n} \text{ for all } x \in U_t \cap \,]{-\infty}, t[$$

and

$$|a(t+0) - a(x)| < \frac{1}{2n} \text{ for all } x \in U_t \cap \,]t, +\infty[.$$

For $t = \infty$, there exists a neighbourhood $U_\infty$ of the form $]{-\infty}, c[\, \cup \,]d, +\infty[$ such that

$$|a(+\infty) - a(x)| < \frac{1}{2n} \text{ for all } x \in ]d, +\infty[$$

and

$$|a(-\infty) - a(x)| < \frac{1}{2n} \text{ for all } x \in ]-\infty, c[.$$

By compactness of $\dot{\mathbb{R}}$, there exist a finite number of neighbourhoods $U_{t_0}, U_{t_1}, \ldots, U_{t_m}$, $t_0 = \infty$, such that $\dot{\mathbb{R}} = \bigcup_{j=0}^{m} U_{t_j}$. By taking set differences, we can assume that these neighbourhoods are disjoint, then define a function $a_n$ by

$$a_n(t) = \begin{cases} a(t_j - 0), & t \in U_{t_j} \cap \,]{-\infty}, t_j[ \text{ for some } j \geq 1, \\ a(t_j + 0), & t \in U_{t_j} \cap \,]t_j, +\infty[ \text{ for some } j \geq 1, \\ a(+\infty), & t \in ]d, +\infty[, \\ a(-\infty), & t \in ]-\infty, c[, \end{cases} \qquad t \in \mathbb{R}.$$

Finally, we assume that the function $a_n$ at each discontinuity point is defined to be continuous from the left.

This function is piecewise constant, and note that for almost every $t \in \mathbb{R}$ the following holds for some $j$,

$$|a_n(t) - a(t)| = |a(t_j \pm 0) - a(t)| < \frac{1}{2n},$$

hence

$$\|a_n - a\|_\infty \leq \frac{1}{2n} < \frac{1}{n}.$$

□

### 2.3.2 Functions of Bounded Variation

We will say that a function $a : \mathbb{R} \to \mathbb{C}$ has *bounded variation* if its total variation, defined by

$$V(a) := \sup \left\{ \sum_{k=1}^{n} |a(x_k) - a(x_{k-1})| : -\infty < x_0 < x_1 < \ldots < x_n < +\infty, n \in \mathbb{N} \right\}$$

is finite.

We denote the set of functions with bounded variation by $BV(\mathbb{R})$.

**Lemma 2.3.3** ([13, Lemma 2.10])**.** *Let $a \in BV(\mathbb{R})$. Then there exists a sequence $(a_n)_{n\in\mathbb{N}}$ of piecewise constant functions such that*

$$\lim_{n\to\infty} \|a - a_n\|_\infty = 0 \text{ and } V(a_n) \le V(a) \text{ for each } n \in \mathbb{N}.$$

*Proof.* Define the functions $a_n$ as Lemma 2.3.2. These functions are piecewise constant and $\lim_{n\to\infty}\|a_n - a\|_\infty = 0$. Furthermore, by simple computation we deduce that

$$V(a_n) = \sum_{j=1}^{m} \Big( |a(t_j - 0) - a(t_{j-1} + 0)| + |a(t_{j+1} + 0) - a(t_j - 0)| \Big) \le V(a),$$

where $t_{m+1} = t_0 = \infty$ and $a(\infty \pm 0) = a(\pm\infty)$. □

**Lemma 2.3.4.** *The set $BV(\mathbb{R})$ of functions with bounded variation is contained in $PC(\mathbb{R})$.*

*Proof.* Let $a : \mathbb{R} \to \mathbb{C}$ be a function of bounded variation. By [15, Theorem 3.27a], the real and imaginary parts of $a$ are also of bounded variation. By [15, Theorem 3.27b], they can be written as

$$\operatorname{Re} a = a_1 - a_2, \quad \operatorname{Im} a = a_3 - a_4,$$

where $a_1, a_2, a_3, a_4$ are bounded and increasing functions. Because these functions are bounded, they have finite limits at $-\infty, +\infty$, and because they are increasing, they have finite one-sided limits at every point of $\mathbb{R}$. We conclude that $a \in PC(\mathbb{R})$. □

## 2.4 Convolution

In the following, the *support* of a function $f$, denoted $\operatorname{supp} f$ is defined as

$$\operatorname{supp} f = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}.$$

The set $C_c^\infty(\mathbb{R})$ is the set of infinitely differentiable functions of compact support.

Given functions $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ for some $1 \le p \le \infty$, the *convolution* of $f$ and $g$ is a function denoted by $f * g$ and defined by

$$(f * g)(x) := \int_{\mathbb{R}} f(x - y) g(y) dy, \quad x \in \mathbb{R}.$$

The function $f * g$ is well-defined almost everywhere and $f * g \in L^p(\mathbb{R})$ [7, Theorem 4.15].

Denote by $C_c(\mathbb{R})$ the set of functions $\mathbb{R} \to \mathbb{C}$ with compact support. If $f \in C_c(\mathbb{R})$ and $g \in L^1_{\text{loc}}(\mathbb{R})$, then the convolution $f * g$ is always well-defined and furthermore, $f * g$ is continuous [7, Proposition 4.19].

For future use, we mention some properties of the convolution operation.

**Theorem 2.4.1** ([7, Proposition 4.18])**.** *Let $f \in L^1(\mathbb{R})$, $g \in L^p(\mathbb{R})$ with $1 \le p \le \infty$. Then*

$$\operatorname{supp}(f * g) \subset \overline{\operatorname{supp} f + \operatorname{supp} g}.$$

Denote by $C^k(\mathbb{R})$ the set of functions $f : \mathbb{R} \to \mathbb{C}$ that are $k$-times differentiable, with all derivatives continuous.

**Theorem 2.4.2** ([7, Proposition 4.20])**.** *Let $f \in C^k(\mathbb{R})$ with $k \geq 1$ and $g \in L^1_{\mathrm{loc}}(\mathbb{R})$. Then $f * g \in C^k(\mathbb{R})$ and*

$$(f * g)^{(n)} = (f^{(n)} * g), \quad n \leq k,$$

*where the superscript $(n)$ denotes the $n$-th derivative.*

A sequence of *mollifiers* is a sequence of non-negative functions $\rho_\delta$ on $\mathbb{R}$ such that

$$\rho_\delta \in C_c^\infty(\mathbb{R}), \quad \operatorname{supp} \rho_\delta \subset [-\delta, \delta], \quad \int_{\mathbb{R}} \rho_\delta = 1.$$

**Theorem 2.4.3** ([7, Proposition 4.21])**.** *Let $f \in C(\mathbb{R})$ and $(\rho_\delta)$ be a sequence of mollifiers. Then as $\delta \to 0$, the sequence $(\rho_\delta * f)$ converges uniformly to $f$ on compact subsets of $\mathbb{R}$.*

CHAPTER 3

# Interpolation in Weighted Variable Lebesgue Spaces

## 3.1 Interpolation

The general idea behind interpolation is this: we're interested in proving the boundedness of a certain operator $T$ defined on a Banach space $X$. We find spaces $X_0$ and $X_1$ such that $X$ is "between", in a sense that will be made precise later, the spaces $X_0$ and $X_1$. We prove the boundedness of $T$ on these spaces and conclude, by means of an interpolation theorem, that $T$ is bounded on $X$ also.

One of the first interpolation theorems was proven by Marcel Riesz in 1927 and improved by his student Olof Thorin in 1938.

**Theorem 3.1.1** ([4, Chap. 4, Theorem 2.2]). *Suppose* $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ *and* $0 \leq \theta \leq 1$. *Define*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

*Let* $T$ *be a linear operator defined on* $L^{p_0}(\mathbb{R}) + L^{p_1}(\mathbb{R})$. *Suppose that* $T \in \mathcal{B}(L^{p_i}(\mathbb{R}), L^{q_i}(\mathbb{R}))$, $i = 0, 1$. *Then* $T$ *is bounded as an operator from* $L^p(\mathbb{R})$ *to* $L^q(\mathbb{R})$. *In addition, its* $\mathcal{B}(L^p(\mathbb{R}), L^q(\mathbb{R}))$ *norm satisfies*

$$\|T\|_{\mathcal{B}(L^p(\mathbb{R}), L^q(\mathbb{R}))} \leq \|T\|^{1-\theta}_{\mathcal{B}(L^{p_0}(\mathbb{R}), L^{q_0}(\mathbb{R}))} \|T\|^{\theta}_{\mathcal{B}(L^{p_1}(\mathbb{R}), L^{q_1}(\mathbb{R}))}.$$

In his 1964 paper [9], Alberto Calderón generalized this theorem to more abstract spaces. We will use his results to prove a version of the Riesz-Thorin theorem for weighted variable Lebesgue spaces, to be defined later.

### 3.1.1 Intermediate Spaces

We will call a pair of complex Banach spaces $(B_0, B_1)$ a *compatible couple* if they are *continuously embedded* in some complex topological vector space $V$. This means that there exist continuous injections $f_i : B_i \to V$, $i = 0, 1$.

Denote the norm in $B_i$ by $\|\cdot\|_i, i = 0, 1$. We introduce the norm $\|\cdot\|_{B_0 \cap B_1}$, defined on $B_0 \cap B_1$ by

$$\|x\|_{B_0 \cap B_1} = \max\{\|x\|_0, \|x\|_1\}.$$

On the set $B_0 + B_1 = \{y + z : y \in B_0, z \in B_1\}$, we define the norm

$$\|x\|_{B_0+B_1} = \inf\{\|y\|_0 + \|z\|_1 : x = y + z,\ y \in B_0,\ z \in B_1\}.$$

Both $B_0 \cap B_1$ and $B_0 + B_1$ are Banach spaces with their respective norms (see [31, p. 9] or [4, p. 97, Theorem 1.3]).

If $(B_0, B_1)$ is a compatible couple, a Banach space $X$ is said to be an *intermediate space* between $B_0$ and $B_1$ if we have the following continuous inclusions

$$B_0 \cap B_1 \hookrightarrow X \hookrightarrow B_0 + B_1.$$

Given two compatible couples $(X_0, X_1)$, $(Y_0, Y_1)$, a linear operator $T : X_0 + X_1 \to Y_0 + Y_1$ is said to be *admissible* if for each $i = 0, 1$ the restriction of $T$ to $X_i$ takes values in $Y_i$ and, in addition, is bounded from $X_i$ to $Y_i$.

### 3.1.2 Lower Calderón Space

Given a compatible couple $B = (B_0, B_1)$ we consider the set $\mathcal{F}(B_0, B_1)$ of functions

$$f : \overline{\Omega} \to B_0 + B_1,$$

where $\overline{\Omega}$ is the closure of the strip

$$\Omega = \{z \in \mathbb{C} : 0 < Re(z) < 1\},$$

that satisfy:

- $f$ is continuous and bounded with respect to the norm of $B_0 + B_1$ in $\overline{\Omega}$;
- $f$ is analytic in $\Omega$;
- $f(it) \in B_0$ is $B_0$-continuous and tends to zero as $|t| \to \infty$;
- $f(1 + it) \in B_1$ is $B_1$-continuous and tends to zero as $|t| \to \infty$.

With the norm

$$\|f\|_{\mathcal{F}} = \max\left\{\sup_t \|f(it)\|_0,\ \sup_t \|f(1 + it)\|_1\right\},$$

$\mathcal{F}(B_0, B_1)$ becomes a Banach space [31, p. 217].

For a given $0 \leq \theta \leq 1$, the *lower Calderón space*, denoted by $[B_0, B_1]_\theta$, is the subset of $B_0 + B_1$ defined by

$$[B_0, B_1]_\theta = \{x : x = f(\theta) \text{ for some } f \in \mathcal{F}(B_0, B_1)\}.$$

This set becomes a Banach space continuously embedded in $B_0 + B_1$ if we introduce the norm

$$\|x\|_{[B_0,B_1]_\theta} = \inf\{\|f\|_{\mathcal{F}} : f(\theta) = x\}$$

[31, p. 221].

Calderón proved the following interpolation theorem (see [9, p. 115, par. 4] for the statement and [9, p. 129, par. 24] for the proof):

**Theorem 3.1.2.** *Let $0 \leq \theta \leq 1$ and $T$ be an admissible operator for the couples $(B_0, B_1)$ and $(C_0, C_1)$. Then the restriction of $T$ to $[B_0, B_1]_\theta$ takes values in $[C_0, C_1]_\theta$ and verifies*

$$\|T\|_{\mathcal{B}([B_0,B_1]_\theta,[C_0,C_1]_\theta)} \leq \|T\|^{1-\theta}_{\mathcal{B}(B_0,C_0)} \|T\|^{\theta}_{\mathcal{B}(B_1,C_1)}.$$

### 3.1.3 Upper Calderón Space

Similarly to above, we define the set $\overline{\mathcal{F}}(B_0, B_1)$ of functions $f : \overline{\Omega} \to B_0 + B_1$ that satisfy:

- $\|f(z)\|_{B_0+B_1} \leq c(1 + |z|)$;
- $f(z)$ is continuous in $\overline{\Omega}$;
- $f(z)$ is analytic in $\Omega$;
- $f(it_1) - f(it_2) \in B_0$ and $f(1 + it_1) - f(1 + it_2) \in B_1$ for any $t_1 < t_2$;
- $\max\left\{\sup_{t_1,t_2}\left\|\frac{f(it_2) - f(it_1)}{t_2 - t_1}\right\|_0, \sup_{t_1,t_2}\left\|\frac{f(1+it_2) - f(1+t_1)}{t_2 - t_1}\right\|_1\right\} = \|f\|_{\overline{\mathcal{F}}} < \infty.$

With the norm defined above, $\overline{\mathcal{F}}$ modulo the constant functions becomes a Banach space [9, p. 130, par. 25].

Given $0 < \theta < 1$, the *upper Calderón space*, denoted by $[B_0, B_1]^\theta$, is the subset of $B_0 + B_1$ defined by

$$[B_0, B_1]^\theta = \left\{x : x = \frac{df}{dz}(\theta) \text{ for some } f \in \overline{\mathcal{F}}(B_0, B_1)\right\}.$$

With the norm

$$\|x\|_{[B_0,B_1]^\theta} = \inf\left\{\|f\|_{\overline{\mathcal{F}}} : \frac{df}{dz}(\theta) = x\right\},$$

the set $[B_0, B_1]^\theta$ becomes a Banach space continuously embedded in $B_0 + B_1$ [9, p. 130, par. 26].

Note that the inclusion $[B_0, B_1]_\theta \subset [B_0, B_1]^\theta$ always holds [5, p. 93, Theorem 4.3.1].

### 3.1.4 Banach Lattices

In his original paper, Calderón defined the concept of a Banach lattice on an arbitrary measure space [9, p. 122, par. 13.1], but we will restrict our attention to $\mathbb{R}$ with the Lebesgue measure.

Consider the class $\mathcal{M}$ of complex valued, measurable, finite a.e. functions defined on $\mathbb{R}$. As usual, we identify functions which are equal almost everywhere. We call a subclass $X$ of measurable functions a *Banach lattice* if $X$ is a Banach space that satisfies

$$f \in X \text{ and } |g(x)| \leq |f(x)| \text{ a.e. implies } g \in X \text{ and } \|g\|_X \leq \|f\|_X.$$

Any positive integrable function $\mu(x)$ on $\mathbb{R}$ gives rise to a metric

$$d_\mu(f,g) = \int_{\mathbb{R}} \mu(x) \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|} dx,$$

where $f,g$ are measurable functions on $\mathbb{R}$.

With this metric, the space $\mathcal{M}$ of measurable functions on $\mathbb{R}$ becomes a complete metric vector space (with the topology independent of the choice of $\mu$) [8, p. 4, Theorem 1.2.1] in which $X$ is continuously embedded [31, p. 41, Theorem 1].

### 3.1.5 Calderón Product

Let $X_0, X_1$ be two Banach lattices on $\mathbb{R}$. By the above discussion, they can be continuously embedded in the space $\mathcal{M}$ of measurable functions on $\mathbb{R}$ and thus $(X_0, X_1)$ is a compatible couple. Fix $0 < \theta < 1$. The *Calderón product* $X_0^{1-\theta} X_1^\theta$ is the class of functions $f \in \mathcal{M}$ such that

$$|f(x)| \leq \lambda |g(x)|^{1-\theta} |h(x)|^\theta$$

for some $\lambda > 0$ and $g \in X_0, h \in X_1$ with $\|g\|_{X_0}, \|h\|_{X_1} \leq 1$. The norm in $X_0^{1-\theta} X_1^\theta$ is the infimum of all $\lambda$ for which the above inequality holds. With this norm $X_0^{1-\theta} X_1^\theta$ becomes a Banach lattice [9, p. 123]. In general we have $X_0^{1-\theta} X_1^\theta \subset [X_0, X_1]^\theta$ [9, p. 125].

### 3.1.6 Sufficient Condition for Equality of Calderón Spaces

**Theorem 3.1.3** ([9, p. 125])**.** *Let $0 < \theta < 1$ and $X_0, X_1$ be Banach lattices on $\mathbb{R}$. If $X_0$ is reflexive, then*

$$[X_0, X_1]_\theta = X_0^{1-\theta} X_1^\theta = [X_0, X_1]^\theta$$

*and the norms of these spaces coincide.*

## 3.2 Banach Function Spaces

In the following $\mathcal{M}$ denotes the set of Lebesgue measurable functions on $\mathbb{R}$, $\chi_E$ denotes the characteristic function of a measurable set $E$, and $|E|$ denotes the Lebesgue measure of $E$.

Given a norm $\|\cdot\|$ defined on $\mathcal{M}$, the set

$$X = \{f \in \mathcal{M} : \|f\| < \infty\},$$

where we identify functions differing only on a set of measure zero, is a *Banach function space* ([12, p. 72]) if the following properties are satisfied

- $\|f\| = \||f|\|$;
- $|g| \leq |f|$ a.e. implies $\|g\| \leq \|f\|$;
- $|f_n| \uparrow f$ a.e. implies $\|f_n\| \uparrow \|f\|$;
- $|E| < \infty$ implies $\|\chi_E\| < \infty$;
- $|E| < \infty$ implies $\int_E |f(x)|dx \leq C_E \|f\|$,

for all $f, g \in \mathcal{M}$, $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}$, $a \in \mathbb{R}$ and measurable subsets $E \subset \mathbb{R}$. The constant $C_E$ may depend on $E$ and $X$ but not on $f$. With this norm $X$ is a Banach space [4, Chap. 1, Theorem 1.6].

### 3.2.1 Associate Space

Given a Banach function space $X$, its *associate space* $X^*$ is the set of functions $g \in \mathcal{M}$ such that

$$\|g\|_{X^*} = \sup\left\{\int_{\mathbb{R}} |f(x)g(x)|dx \ : \ f \in \mathcal{M}, \|f\|_X \leq 1\right\} < \infty.$$

This turns out to be a Banach function space itself [4, Chap. 1, Theorem 2.2].

The following concept is useful in order to describe the associate space. We will say a Banach function space $X$ has *absolutely continuous norm* if for every $f \in X$ we have $\left\|f\chi_{E_n}\right\|_X \to 0$ for every sequence $\{E_n\}_{n=1}^{\infty}$ with $\chi_{E_n} \to 0$.

**Theorem 3.2.1** ([4, Chap. 1, Corollary 4.3])**.** *Let $X$ be a Banach function space. Its Banach space dual $X'$ is isometrically isomorphic to the associate space $X^*$ if and only if $X$ has absolutely continuous norm.*

### 3.2.2 Weighted Banach Function Spaces

Given a Banach function space $X$, define $X_{\text{loc}}$ as the set of all functions $f$ such that $f\chi_E \in X$ for every measurable set $E \subset \mathbb{R}$ with finite measure.

A measurable function $w : \mathbb{R} \to [0, +\infty]$ such that $0 < w(x) < \infty$ a.e. on $\mathbb{R}$, will be called a *weight*. Given a weight $w$, define a norm by

$$\|f\|_w := \|fw\|_X, \quad f \in \mathcal{M}, \tag{3.1}$$

and the set $X(w)$ by

$$X(w) := \{f \in \mathcal{M} \ : \ fw \in X\}.$$

**Lemma 3.2.2** ([25, Lemma 2.4 and Corollary 2.9])**.** *If a weight $w$ satisfies $w \in X_{\text{loc}}$, $1/w \in X^*_{\text{loc}}$, then $X(w)$ is a Banach function space with the norm $\|\cdot\|_w$. In addition, if $X$ is reflexive then $X(w)$ is also reflexive.*

## 3.3 Variable Lebesgue Spaces

Let $p : \mathbb{R} \to ]1, +\infty[$ be a measurable function. We consider the set $L^{p(\cdot)}(\mathbb{R})$ of measurable complex-valued functions $f$ defined on $\mathbb{R}$ such that the quantity

$$\Phi_{p(\cdot)}(f/\lambda) = \int_{\mathbb{R}} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx$$

is finite for some $\lambda > 0$ depending on $f$. This is a vector space [12, Theorem 2.15] and it becomes a Banach space [12, Theorem 2.71] with the norm

$$\|f\|_{p(\cdot)} = \inf\{\lambda > 0 : \Phi_{p(\cdot)}(f/\lambda) \le 1\}.$$

If $p$ is constant, this is the standard Lebesgue space $L^p(\mathbb{R})$.

Define

$$p_- := \operatorname*{ess\,inf}_{x \in \mathbb{R}} p(x), \quad p_+ := \operatorname*{ess\,sup}_{x \in \mathbb{R}} p(x).$$

We will only consider exponents $p$ that satisfy

$$1 < p_- \le p_+ < \infty. \tag{3.2}$$

In this case, $L^{p(\cdot)}(\mathbb{R})$ is a separable [12, Theorem 2.78] and reflexive [12, Corollary 2.81] space.

The dual space of $L^{p(\cdot)}(\mathbb{R})$ is isomorphic (with equivalent norms) to $L^{p'(\cdot)}(\mathbb{R})$ [12, Theorem 2.80], where $p'$ is defined by

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad x \in \mathbb{R}.$$

Note that $L^{p(\cdot)}(\mathbb{R})$ is a Banach function space [12, Section 2.10.3].

### 3.3.1 Weighted Variable Lebesgue Spaces

Let $p$ be as in above and $w$ be a weight. The space $L^{p(\cdot)}(w)$ is the set of all measurable complex-valued functions $f$ on $\mathbb{R}$ such that $wf \in L^{p(\cdot)}(\mathbb{R})$, equipped with the natural norm

$$\|f\|_{L^{p(\cdot)}(w)} := \|wf\|_{p(\cdot)}.$$

From the definition of the norm and definition (3.1) we see that, under the hypotheses of Lemma 3.2.2, $L^{p(\cdot)}(w)$ is a Banach function space.

**Theorem 3.3.1.** *Let $p$ be an exponent such that $1 < p_-, p_+ < \infty$ and $w$ a weight such that $w \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R})$ and $1/w \in L^{p'(\cdot)}_{\text{loc}}(\mathbb{R})$. Then $L^{p(\cdot)}(w)$ is a reflexive Banach function space.*

*Proof.* The condition on the exponent $p$ guarantees that $L^{p(\cdot)}(\mathbb{R})$ is reflexive. $L^{p(\cdot)}(\mathbb{R})$ has absolutely continuous norm by [12, p. 73], thus by Theorem 3.2.1 its associate space is isomorphic to the dual space $(L^{p(\cdot)}(\mathbb{R}))'$, which in turn is isomorphic to $L^{p'(\cdot)}(\mathbb{R})$. We've proven that the associate space $\left(L^{p(\cdot)}(\mathbb{R})\right)^*$ of $L^{p(\cdot)}(\mathbb{R})$ is isomorphic to $L^{p'(\cdot)}(\mathbb{R})$, hence

$\chi_E/w \in \left(L^{p(\cdot)}(\mathbb{R})\right)^*$ for all measurable sets $E \subset \mathbb{R}$ with finite measure. This means that $1/w \in \left(L^{p(\cdot)}(\mathbb{R})\right)^*_{\text{loc}}$, so we can now use Theorem 3.2.2 and the hypotheses on the weight $w$ to conclude that $L^{p(\cdot)}(w)$ is reflexive. □

**Theorem 3.3.2.** *Let $p$ be a variable exponent such that $1 < p_-, p_+ < \infty$ and $w$ be a weight satisfying $w \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R})$ and $1/w \in L^{p'(\cdot)}_{\text{loc}}(\mathbb{R})$, where*

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad x \in \mathbb{R}.$$

*Then the associate space of $L^{p(\cdot)}(w)$ is isomorphic to $L^{p'(\cdot)}(1/w)$.*

*Proof.* Apply [25, Lemma 2.4], keeping in mind that the associate space of $L^{p(\cdot)}(\mathbb{R})$ is isomorphic to $L^{p'(\cdot)}(\mathbb{R})$. □

## 3.4 Interpolation in Weighted Variable Lebesgue Spaces

Theorem 3.3.1 gives us conditions under which $L^{p(\cdot)}(w)$ is reflexive. This allows us to use Theorem 3.1.3 and deduce the following result:

**Theorem 3.4.1.** *Let $0 < \theta < 1$. For $i = 0, 1$, let $p_i$ be variable exponents satisfying $1 < (p_i)_-, (p_i)_+ < \infty$ and $w_i$ be weights satisfying $w_i \in L^{p_i(\cdot)}_{\text{loc}}(\mathbb{R})$ and $1/w \in L^{p_i'(\cdot)}_{\text{loc}}(\mathbb{R})$. Then*

$$\left[L^{p_0(\cdot)}(w_0), L^{p_1(\cdot)}(w_1)\right]_\theta = L^{p_0(\cdot)}(w_0)^{1-\theta} L^{p_1(\cdot)}(w_1)^\theta = \left[L^{p_0(\cdot)}(w_0), L^{p_1(\cdot)}(w_1)\right]^\theta,$$

*with equal norms.*

### 3.4.1 Calderón Product of Weighted Variable Lebesgue Spaces

**Theorem 3.4.2.** *Let $0 < \theta < 1$. For $i = 0, 1$, let $p_i$ be variable exponents satisfying $1 < (p_i)_-, (p_i)_+ < \infty$ and $w_i$ be weights satisfying $w_i \in L^{p_i(\cdot)}_{\text{loc}}(\mathbb{R})$ and $1/w_i \in L^{p_i'(\cdot)}_{\text{loc}}(\mathbb{R})$. Then the Calderón product of the spaces $L^{p_0(\cdot)}(w_0)$ and $L^{p_1(\cdot)}(w_1)$ satisfies*

$$(L^{p_0(\cdot)}(w_0))^{1-\theta}(L^{p_1(\cdot)}(w_1))^\theta = L^{p_\theta(\cdot)}(w_\theta)$$

*with norm equivalence*

$$\|f\|_{(L^{p_0(\cdot)}(w_0))^{1-\theta}(L^{p_1(\cdot)}(w_1))^\theta} \le \|f\|_{L^{p_\theta(\cdot)}(w_\theta)} \le 2^{1/(p_\theta)_-}\|f\|_{(L^{p_0(\cdot)}(w_0))^{1-\theta}(L^{p_1(\cdot)}(w_1))^\theta}, \tag{3.3}$$

*where*

$$\frac{1}{p_\theta(x)} = \frac{1-\theta}{p_0(x)} + \frac{\theta}{p_1(x)} \quad \text{and} \quad w_\theta(x) = w_0(x)^{1-\theta} w_1(x)^\theta,\ x \in \mathbb{R}.$$

*Proof.* The idea of the proof is borrowed from [32, Example 3, pp. 179-180].

Let $f \in (L^{p_0(\cdot)}(w_0))^{1-\theta}(L^{p_1(\cdot)}(w_1))^\theta$ satisfy $\|f\|_{(L^{p_0(\cdot)}(w_0))^{1-\theta}(L^{p_1(\cdot)}(w_1))^\theta} < \lambda$.

By definition of the norm in the Calderón product we have

$$|f(x)| \leq \lambda |f_0(x)|^{1-\theta} |f_1(x)|^{\theta} \text{ a.e.}$$

for some $f_0 \in L^{p_0(\cdot)}(w_0), f_1 \in L^{p_1(\cdot)}(w_1)$ with respective norms bounded by 1. This implies

$$\int_{\mathbb{R}} |f_0(x)w_0(x)|^{p_0(x)} dx, \int_{\mathbb{R}} |f_1(x)w_1(x)|^{p_1(x)} dx \leq 1.$$

We have

$$\begin{aligned}
\left(\frac{|f(x)|w_\theta(x)}{\lambda}\right)^{p_\theta(x)} &\leq (|f_0(x)|^{1-\theta}|f_1(x)|^{\theta} w_\theta(x))^{p_\theta(x)} \\
&= (|f_0(x)w_0(x)|^{1-\theta}|f_1(x)w_1(x)|^{\theta})^{p_\theta(x)} \\
&= \left[(|f_0(x)w_0(x)|^{p_0(x)})^{\frac{1-\theta}{p_0(x)}}(|f_1(x)w_1(x)|^{p_1(x)})^{\frac{\theta}{p_1(x)}}\right]^{p_\theta(x)} \\
&\leq (g(x)^{\frac{1-\theta}{p_0(x)}+\frac{\theta}{p_1(x)}})^{p_\theta(x)} = g(x),
\end{aligned}$$

where $g(x) = \max\{|f_0(x)w_0(x)|^{p_0(x)}, |f_1(x), w_1(x)|^{p_1(x)}\}$.

We have $p_\theta(x) \geq (p_\theta)_-$ and so $2^{-\frac{p_\theta(x)}{(p_\theta)_-}} \leq \frac{1}{2}$. We can now estimate the norm of $f$ in $L^{p_\theta(\cdot)}(w_\theta)$:

$$\begin{aligned}
\int_{\mathbb{R}} \left|\frac{f(x)w_\theta(x)}{2^{1/(p_\theta)_-}\lambda}\right|^{p_\theta(x)} dx &\leq \frac{1}{2}\int \left|\frac{f(x)w_\theta(x)}{\lambda}\right|^{p_\theta(x)} dx \\
&\leq \frac{1}{2}\int_{\mathbb{R}} g(x)dx \leq \frac{1}{2}\int_{\mathbb{R}} |f_0(x)w_0(x)|^{p_0(x)}dx + \frac{1}{2}\int_{\mathbb{R}} |f_1(x)w_1(x)|^{p_1(x)}dx \\
&\leq \frac{1}{2} + \frac{1}{2} = 1.
\end{aligned}$$

This shows that $f \in L^{p_\theta(\cdot)}(w_\theta)$ and

$$\|f\|_{L^{p_\theta(\cdot)}(w_\theta)} \leq 2^{1/(p_\theta)_-} \|f\|_{(L^{p_0(\cdot)}(w_0))^{1-\theta}(L^{p_1(\cdot)}(w_1))^{\theta}}. \tag{3.4}$$

On the other hand, if $f \in L^{p_\theta(\cdot)}(w_\theta)$ and $\|f\|_{L^{p_\theta(\cdot)}(w_\theta)} > 0$ (the case of the zero norm is trivial) then

$$\int_{\mathbb{R}} \left|\frac{f(x)w_\theta(x)}{\|f\|_{L^{p_\theta(\cdot)}(w_\theta)}}\right|^{p_\theta(x)} dx \leq 1.$$

Define

$$h_i(x) := \left(\frac{|f(x)|w_\theta(x)}{\|f\|_{L^{p_\theta(\cdot)}(w_\theta)}}\right)^{\frac{p_\theta(x)}{p_i(x)}} \frac{1}{w_i(x)}, \quad i = 0, 1, \quad x \in \mathbb{R}.$$

An easy calculation shows that

$$|f(x)| = \|f\|_{L^{p_\theta(\cdot)}(w_\theta)}(h_0(x))^{1-\theta}(h_1(x))^{\theta}, \quad x \in \mathbb{R}. \tag{3.5}$$

By hypothesis and from the definition of $h_i$, we have for $i = 0, 1$,

$$\int_{\mathbb{R}} (h_i(x)w_i(x))^{p_i(x)} dx = \int_{\mathbb{R}} \left(\frac{|f(x)|w_\theta(x)}{\|f\|_{L^{p_\theta(\cdot)}(w_\theta)}}\right)^{p_\theta(x)} dx \leq 1.$$

Thus, taking in account the definition of the norm in $L^{p_i(\cdot)}(w_i)$ we see that

$$\|h_i\|_{L^{p_i(\cdot)}(w_i)} \leq 1, \quad i = 0, 1. \tag{3.6}$$

From (3.5) and (3.6) we conclude that $f \in (L^{p_0(\cdot)}(w_0))^{1-\theta}(L^{p_1(\cdot)}(w_1))^{\theta}$ and

$$\|f\|_{(L^{p_0(\cdot)}(w_0))^{1-\theta}(L^{p_1(\cdot)}(w_1))^{\theta}} \leq \|f\|_{L^{p_\theta(\cdot)}(w_\theta)}. \tag{3.7}$$

Combining inequalities (3.4) and (3.7), we immediately arrive at (3.3). □

### 3.4.2 Riesz-Thorin Interpolation Theorem for Weighted Variable Lebesgue Spaces

Combining the previous results, we arrive at the following version of the Riesz-Thorin theorem for weighted variable Lebesgue spaces

**Theorem 3.4.3.** *Let $0 < \theta < 1$. For $i = 0, 1$, let $p_i$ be variable exponents satisfying $1 < (p_i)_-, (p_i)_+ < \infty$ and $w_i$ be weights satisfying $w_i \in L^{p_i(\cdot)}_{\text{loc}}(\mathbb{R})$ and $1/w_i \in L^{p_i'(\cdot)}_{\text{loc}}(\mathbb{R})$. Let*

$$T : L^{p_0(\cdot)}(w_0) + L^{p_1(\cdot)}(w_1) \to L^{p_0(\cdot)}(w_0) + L^{p_1(\cdot)}(w_1)$$

*be an admissible operator. Then the restriction of $T$ to $L^{p_\theta(\cdot)}(w_\theta)$ takes values in $L^{p_\theta(\cdot)}(w_\theta)$ and*

$$\|T\|_{\mathcal{B}(L^{p_\theta(\cdot)}(w_\theta))} \leq 2^{1/(p_\theta)_-} \|T\|^{1-\theta}_{\mathcal{B}(L^{p_0(\cdot)}(w_0))} \|T\|^{\theta}_{\mathcal{B}(L^{p_1(\cdot)}(w_1))},$$

*where*

$$\frac{1}{p_\theta(x)} = \frac{1-\theta}{p_0(x)} + \frac{\theta}{p_1(x)} \quad \text{and} \quad w_\theta(x) = w_0(x)^{1-\theta} w_1(x)^{\theta}, \ x \in \mathbb{R}.$$

*Proof.* Setting both couples equal to $(L^{p_0(\cdot)}(w_0), L^{p_1(\cdot)}(w_1))$ in Theorem 3.1.2, we have that the restriction of $T$ to $\left[L^{p_0(\cdot)}(w_0), L^{p_1(\cdot)}(w_1)\right]_\theta$ takes values in the same space and the inequality

$$\|Tf\|_{\left[L^{p_0(\cdot)}(w_0), L^{p_1(\cdot)}(w_1)\right]_\theta} \leq \|T\|^{1-\theta}_{\mathcal{B}(L^{p_0(\cdot)}(w_0))} \|T\|^{\theta}_{\mathcal{B}(L^{p_1(\cdot)}(w_1))} \|f\|_{\left[L^{p_0(\cdot)}(w_0), L^{p_1(\cdot)}(w_1)\right]_\theta} \tag{3.8}$$

holds for every $f \in \left[L^{p_0(\cdot)}(w_0), L^{p_1(\cdot)}(w_1)\right]_\theta$. By Theorem 3.4.1 we have

$$L^{p_0(\cdot)}(w_0)^{1-\theta} L^{p_1(\cdot)}(w_1)^{\theta} = \left[L^{p_0(\cdot)}(w_0), L^{p_1(\cdot)}(w_1)\right]_\theta$$

and

$$\|f\|_{L^{p_0(\cdot)}(w_0)^{1-\theta} L^{p_1(\cdot)}(w_1)^{\theta}} = \|f\|_{\left[L^{p_0(\cdot)}(w_0), L^{p_1(\cdot)}(w_1)\right]_\theta} \tag{3.9}$$

for every $f \in \left[L^{p_0(\cdot)}(w_0), L^{p_1(\cdot)}(w_1)\right]_\theta$. Theorem 3.4.2 then gives us

$$L^{p_\theta(\cdot)}(w_\theta) = \left[L^{p_0(\cdot)}(w_0), L^{p_1(\cdot)}(w_1)\right]_\theta,$$

together with the equivalence

$$\|f\|_{L^{p_\theta(\cdot)}(w_\theta)} \leq 2^{1/(p_\theta)_-} \|f\|_{(L^{p_0(\cdot)}(w_0))^{1-\theta}(L^{p_1(\cdot)}(w_1))^{\theta}}, \tag{3.10}$$

$$\|f\|_{(L^{p_0(\cdot)}(w_0))^{1-\theta}(L^{p_1(\cdot)}(w_1))^{\theta}} \leq \|f\|_{L^{p_\theta(\cdot)}(w_\theta)}. \tag{3.11}$$

To establish the desired inequality, note that

$$\begin{aligned}
\frac{1}{2^{1/(p_\theta)_-}}\|Tf\|_{L^{p_\theta(\cdot)}(w_\theta)} &\leq \|Tf\|_{(L^{p_0(\cdot)}(w_0))^{1-\theta}(L^{p_1(\cdot)}(w_1))^{\theta}} = \|Tf\|_{[L^{p_0(\cdot)}(w_0),L^{p_1(\cdot)}(w_1)]_\theta} \\
&\leq \|T\|^{1-\theta}_{\mathscr{B}(L^{p_0(\cdot)}(w_0))}\|T\|^{\theta}_{\mathscr{B}(L^{p_1(\cdot)}(w_1))}\|f\|_{[L^{p_0(\cdot)}(w_0),L^{p_1(\cdot)}(w_1)]_\theta} \\
&= \|T\|^{1-\theta}_{\mathscr{B}(L^{p_0(\cdot)}(w_0))}\|T\|^{\theta}_{\mathscr{B}(L^{p_1(\cdot)}(w_1))}\|f\|_{(L^{p_0(\cdot)}(w_0))^{1-\theta}(L^{p_1(\cdot)}(w_1))^{\theta}} \\
&\leq \|T\|^{1-\theta}_{\mathscr{B}(L^{p_0(\cdot)}(w_0))}\|T\|^{\theta}_{\mathscr{B}(L^{p_1(\cdot)}(w_1))}\|f\|_{L^{p_\theta(\cdot)}(w_\theta)},
\end{aligned}$$

where we used, in order, inequality (3.10), equality of norms as in (3.9), inequality (3.8), equality of norms again and finally inequality (3.11). We obtain the desired result by taking suprema over all $f \in L^{p_\theta(\cdot)}(w_\theta)$ with unit norm. □

### 3.4.3 Interpolation of Compactness in Weighted Variable Lebesgue Spaces

To conclude this chapter, we will study a different kind of interpolation. Given a compact operator defined on some Banach spaces $B_0$ and $B_1$, we want to deduce compactness of the operator on some intermediate space. It turns out we will only need compactness on one of the spaces.

The first theorem of this type was proven by Mark Krasnoesel'skii for standard Lebesgue spaces.

**Theorem 3.4.4** ([30, Theorem 3.10]). *Suppose* $1 \leq p_0, p_1, q_0 \leq \infty$, $1 \leq q_1 < \infty$ *and* $0 \leq \theta \leq 1$. *Define*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

*Let* $T$ *be a linear operator defined on* $L^{p_0}(\mathbb{R}) + L^{p_1}(\mathbb{R})$. *Suppose that* $T \in \mathscr{B}(L^{p_0}(\mathbb{R}), L^{q_0}(\mathbb{R}))$ *and is compact as an operator from* $L^{p_1}(\mathbb{R})$ *to* $L^{q_1}(\mathbb{R})$. *Then* $T$ *is compact as an operator from* $L^p(\mathbb{R})$ *to* $L^q(\mathbb{R})$.

It is an open problem whether Krasnoesel'skii's theorem applies to the spaces defined by Calderón. This question has an affirmative answer in the case of weighted variable Lebesgue spaces.

To that end, note that the Fatou property mentioned in [10, Theorem 3.2] is the third axiom of Banach function spaces as in Section 3.2. We obtain the following result.

**Theorem 3.4.5** ([10, Theorem 3.2]). *Let* $0 < \theta < 1$. *Let* $(B_0, B_1)$ *and* $(C_0, C_1)$ *be two compatible couples of Banach function spaces and let* $T : B_0 + B_1 \to C_0 + C_1$ *be an admissible operator such that the restriction of* $T$ *to* $B_0$ *is compact from the space* $B_0$ *to the space* $C_0$. *Then the restriction of* $T$ *to* $[B_0, B_1]_\theta$ *is a compact operator from the space* $[B_0, B_1]_\theta$ *to the space* $[C_0, C_1]_\theta$.

We can now apply the previous result to weighted variable Lebesgue spaces.

**Theorem 3.4.6.** *Let* $0 < \theta < 1$*. For* $i = 0,1$*, let* $p_i$ *be a variable exponent satisfying* $1 < (p_i)_-, (p_i)_+ < \infty$ *and* $w_i$ *be a weight satisfying* $w_i \in L^{p_i(\cdot)}_{\text{loc}}(\mathbb{R})$ *and* $1/w \in L^{p_i'(\cdot)}_{\text{loc}}(\mathbb{R})$*. Let*

$$T : L^{p_0(\cdot)}(w_0) + L^{p_1(\cdot)}(w_1) \to L^{p_0(\cdot)}(w_0) + L^{p_1(\cdot)}(w_1)$$

*be an admissible operator. Suppose that the restriction of* $T$ *to* $L^{p_0(\cdot)}(w_0)$ *is a compact operator. Then the restriction of* $T$ *to* $L^{p_\theta(\cdot)}(p_\theta)$ *is also a compact operator, where*

$$\frac{1}{p_\theta(x)} = \frac{1-\theta}{p_0(x)} + \frac{\theta}{p_1(x)} \quad \text{and} \quad w_\theta(x) = w_0(x)^{1-\theta} w_1(x)^\theta,\ x \in \mathbb{R}.$$

*Proof.* From Theorem 3.4.1 we know that the lower Calderón space of the spaces $L^{p_0(\cdot)}(w_0)$ and $L^{p_1(\cdot)}(w_1)$ coincides with the Calderón product of the same spaces. Theorem 3.4.2 now asserts that

$$L^{p_0(\cdot)}(w_0)^{1-\theta} L^{p_1(\cdot)}(w_1)^\theta = L^{p_\theta(\cdot)}(w_\theta).$$

Hence from Theorem 3.4.5 we conclude that the operator $T : L^{p_\theta(\cdot)}(w_\theta) \to L^{p_\theta(\cdot)}(w_\theta)$ is compact. □

CHAPTER 4

# Piecewise Continuous Fourier Multipliers on Weighted Variable Lebesgue Spaces

## 4.1 Fourier Convolution Operators on $L^2(\mathbb{R})$

For $f \in L^1(\mathbb{R})$, we define the *Fourier transform* of $f$ by

$$(Ff)(x) := \int_{\mathbb{R}} f(t)e^{itx}dt, \quad x \in \mathbb{R}.$$

**Theorem 4.1.1** ([2, Theorem 11.82]). *If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then $Ff \in L^2(\mathbb{R})$ and*

$$\|Ff\|_{L^2} = \sqrt{2\pi}\|f\|_{L^2}.$$

Since $F$ is bounded and $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, the Fourier transform extends by continuity to a bounded linear operator $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

The inverse of $F$ is defined by

$$(F^{-1}g)(t) := \frac{1}{2\pi}\int_{\mathbb{R}} g(x)e^{-itx}dx, \quad t \in \mathbb{R}.$$

Given a function $a \in L^\infty(\mathbb{R})$, we can define an operator $W_a^0$ on $L^2(\mathbb{R})$ by

$$W_a^0(f) := F^{-1}aFf.$$

Lars Hörmander proved that every function $a \in L^\infty(\mathbb{R})$ gives rise to a bounded operator $W_a^0$.

**Theorem 4.1.2** ([20, Theorem 1.5]). *Let $a \in L^\infty(\mathbb{R})$. Then the operator $W_a^0$, as defined above, is bounded on $L^2(\mathbb{R})$. Furthermore its norm satisfies*

$$\left\|W_a^0\right\|_{\mathcal{B}(L^2(\mathbb{R}))} = \|a\|_\infty. \tag{4.1}$$

## 4.2 Fourier Multipliers

It is natural to ask whether we can define the operators $W_a^0$ for spaces other than $L^2(\mathbb{R})$.

A function $a \in L^\infty(\mathbb{R})$ is said to be a *Fourier multiplier on* $L^{p(\cdot)}(w)$ if the map

$$f \mapsto F^{-1}aFf$$

maps $L^2(\mathbb{R}) \cap L^{p(\cdot)}(w)$ into $L^{p(\cdot)}(w)$ and satisfies

$$\sup_{f \in L^2(\mathbb{R}) \cap L^{p(\cdot)}(w)} \frac{\|F^{-1}aFf\|_{L^{p(\cdot)}(w)}}{\|f\|_{L^{p(\cdot)}(w)}} < \infty.$$

Since $L^2(\mathbb{R}) \cap L^{p(\cdot)}(w)$ is dense in $L^{p(\cdot)}(w)$ (because the set of bounded functions with compact support is dense in both spaces [25, Lemma 2.12]), we can extend this map by continuity to the whole of $L^{p(\cdot)}(w)$, denoting this extension by $W_a^0$. Let $M_{p(\cdot),w}$ denote the set of Fourier multipliers on $L^{p(\cdot)}(w)$.

If $a, b \in M_{p(\cdot),w}$ and $c \in \mathbb{C}$, the operator $W_a^0$ satisfies $W_{a+b}^0 = W_a^0 + W_b^0$, $W_{ab}^0 = W_a^0 W_b^0$ and $W_c^0 = cI$. This implies that $M_{p(\cdot),w}$ is an algebra, and we can turn it into a normed algebra by defining

$$\|a\|_{M_{p(\cdot),w}} = \left\|W_a^0\right\|_{\mathcal{B}(L^{p(\cdot)}(w))}.$$

It is interesting to ask what are the Fourier multipliers on a given space $L^{p(\cdot)}(w)$. Hörmander proved that the Fourier multipliers on $L^2(\mathbb{R})$ are precisely the bounded functions. The Fourier multipliers on $L^1(\mathbb{R})$ are also fully characterized (see [19, Theorem 2.5.8]), but we have only sufficient conditions for $L^p(\mathbb{R})$ with $1 < p < \infty$. One of these conditions is the *classical Stechkin inequality*, which states that any function of bounded variation is a Fourier multiplier for $L^p(\mathbb{R})$, $1 < p < \infty$.

**Theorem 4.2.1** ([13, Theorem 2.11])**.** *Let $p \in ]1, +\infty[$. If the function $a$ has finite total variation $V(a)$, then the operator $W_a^0$ is bounded on $L^p(\mathbb{R})$ and satisfies*

$$\left\|W_a^0\right\|_{\mathcal{B}(L^p)} \le K\left[\|a\|_\infty + V(a)\right],$$

*where $K$ is a positive constant independent of $a$.*

Despite bearing his name, this result wasn't obtained by Sergey Stechkin in this form. He proved a similar result for an analogous operator on the unit circle.

Our goal in this chapter is to give a version of this theorem for weighted variable Lebesgue spaces $L^{p(\cdot)}(w)$.

## 4.3 Stechkin Inequality for Weighted Variable Lebesgue Spaces

### 4.3.1 Preliminary Definitions

For every $f \in C_c^\infty(\mathbb{R})$, define the *Cauchy singular integral operator $S$* by the principal value integral

$$(Sf)(x) := \frac{1}{\pi i} \lim_{\varepsilon \to 0} \int_{\mathbb{R} \setminus ]x-\varepsilon, x+\varepsilon[} \frac{f(\tau)}{\tau - x} d\tau, \quad x \in \mathbb{R},$$

where $C_c^\infty(\mathbb{R})$ is the set of all infinitely differentiable functions $f : \mathbb{R} \to \mathbb{C}$ with compact support.

It's not obvious that this principal value integral exists even for relatively regular functions. Nonetheless, the Cauchy singular operator is well defined and bounded on $L^p(\mathbb{R})$ for every $1 < p < \infty$ [4, Chap. 3, Theorem 4.9].

We can in fact relate the Cauchy singular integral operator to the concepts in the previous section.

For every $\lambda \in \mathbb{R}$, define the function $e_\lambda$ by

$$e_\lambda(x) := e^{i\lambda x}, \quad x \in \mathbb{R}.$$

**Lemma 4.3.1** ([13, Lemma 1.35])**.** *Let $u \in L^2(\mathbb{R})$, $c \in \mathbb{R}$ and define*

$$v(t) := e_{-c}(t)[S(e_c u)](t) = \frac{1}{\pi i} \lim_{\varepsilon \to 0} \int_{\mathbb{R} \setminus ]t-\varepsilon, t+\varepsilon[} \frac{e^{ic(x-t)}}{x-t} u(x)\, dx.$$

*Then the Fourier transform $Fv$ exists and is given by*

$$Fv(t) = -\operatorname{sgn}(t-c) Fu(t),$$

*where* $\operatorname{sgn} t = -1$ *if* $t \leq 0$ *and* $\operatorname{sgn} t = 1$ *if* $t > 0$.

Taking $c = 0$, this theorem says that the function $-\operatorname{sgn}(t)$ is a Fourier multiplier corresponding to the Cauchy singular integral operator. We can write this fact as

$$W^0_{-\operatorname{sgn}} = S.$$

This means we can consider the operators $W^0_a$ arising from Fourier multipliers as generalizations of the Cauchy singular integral operator.

We will solely use power weights: functions of the form

$$w(x) = |x - i|^{\lambda_\infty} \prod_{j=1}^{m} |x - x_j|^{\lambda_j}, \quad x \in \mathbb{R}, \tag{4.2}$$

where $i$ is the imaginary unit and $x_1 < \ldots < x_m$, $\lambda_1, \ldots, \lambda_m, \lambda_\infty \in \mathbb{R}$.

These weights were studied by Boris Khvedelidze, who proved the boundedness of the Cauchy singular operator on $L^{p(\cdot)}(w)$, with $p$ constant and $w$ a power weight (see [17, Chap. 1, Theorem 5.1] for the proof).

It is possible to generalize this result to certain variable exponents $p$, called *log-Hölder continuous*.

Given a function $p : \mathbb{R} \to [1, +\infty[$, we say that $p$ is *locally log-Hölder continuous* if

$$|p(x) - p(y)| \leq \frac{C_0}{-\log(|x-y|)} \tag{4.3}$$

for some constant $C_0 > 0$ and for all $x, y \in \mathbb{R}$ with $|x - y| < 1/2$.

We say that $p$ is *log-Hölder continuous at infinity* if

$$|p(x) - p_\infty| \le \frac{C_\infty}{\log(e + |x|)} \tag{4.4}$$

for some constants $p_\infty \ge 1, C_\infty > 0$ and for all $x \in \mathbb{R}$.

We will denote by $LH$ the set of all functions $p : \mathbb{R} \to [1, +\infty[$ that are log-Hölder continuous locally and at infinity, and by $LH^*$ the set of all functions $p \in LH$ such that there exist constants $L, K > 0$ such that

$$\left|\frac{1}{x} - \frac{1}{y}\right| \le \frac{1}{2} \text{ and } |x|, |y| > L$$

implies

$$|p(x) - p(y)| \le \frac{K}{-\log\left(\left|\frac{1}{x} - \frac{1}{y}\right|\right)}. \tag{4.5}$$

Note that this condition implies condition (4.4). In particular, there exists

$$p(\infty) := \lim_{t \to \infty} p(t).$$

### 4.3.2 Preliminary Results

We will need several preliminary results in order to prove the Stechkin inequality for $L^{p(\cdot)}(w)$ spaces.

**Theorem 4.3.2** ([29, Theorem A]). *Let $p$ be a variable exponent satisfying $1 < p_-, p_+ < \infty$, $p \in LH^*$ and $w$ be a power weight as in (4.2). The Cauchy singular integral operator is bounded on $L^{p(\cdot)}(w)$ if and only if*

$$0 < \frac{1}{p(x_j)} + \lambda_j < 1 \textit{ for } j = 1, \ldots, m, \quad 0 < \frac{1}{p(\infty)} + \lambda_\infty + \sum_{j=1}^{m} \lambda_j < 1.$$

We will denote by $CW^{p(\cdot)}$ the set of power weights that satisfy the inequalities of Theorem 4.3.2.

**Theorem 4.3.3.** *Let $p \in LH^*$ be a variable exponent satisfying $1 < p_-, p_+ < \infty$ and $w \in CW^{p(\cdot)}$ be a weight. Then $L^{p(\cdot)}(w)$ is a reflexive Banach function space.*

*Proof.* Since $w \in CW^{p(\cdot)}$, the Cauchy singular integral operator $S$ is bounded on $L^{p(\cdot)}(w)$ by Theorem 4.3.2. Then [25, Theorem 1.3] asserts that the weight $w$ belongs to the class $\mathcal{A}_{p(\cdot)}(\mathbb{R})$. This means that

$$\sup_{-\infty < a < b < \infty} \frac{1}{b-a} \left\|\chi_{]a,b[}\right\|_{L^{p(\cdot)}(w)} \left\|\chi_{]a,b[}\right\|_{L^{p'(\cdot)}(1/w)} < \infty, \tag{4.6}$$

where

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad x \in \mathbb{R}.$$

From condition (4.6), we can see that the norm of any characteristic function of a bounded interval is finite in both $L^{p(\cdot)}(w)$ and $L^{p'(\cdot)}(1/w)$. This implies that $w \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R})$ and $1/w \in L^{p'(\cdot)}_{\text{loc}}(\mathbb{R})$. From Lemma 3.3.1, we deduce that $L^{p(\cdot)}(w)$ is a reflexive Banach function space. □

**Theorem 4.3.4.** *Let $p \in LH^*$ be a variable exponent satisfying $1 < p_-, p_+ < \infty$ and $w \in CW^{p(\cdot)}$ be a weight. Then the norm of $a \in M_{p(\cdot),w}$ satisfies*

$$\|a\|_\infty \leq \|a\|_{M_{p(\cdot),w}}.$$

*The constant 1 on the right-hand side is optimal.*

*Proof.* From the previous Theorem, we know that $L^{p(\cdot)}(w)$ is a Banach function space. This property allows to use [24, Lemma 3.3] to conclude that $L^{p(\cdot)}(w)$ satisfies the *doubling property*, that is, there exist constants $\tau > 1, C_\tau > 0$ such that for all $R > 0$ and $y \in \mathbb{R}$,

$$\frac{\left\|\chi_{]y-\tau R, y+\tau R[}\right\|_{L^{p(\cdot)}(w)}}{\left\|\chi_{]y-R, y+R[}\right\|_{L^{p(\cdot)}(w)}} \leq C_\tau.$$

This implies that $L^{p(\cdot)}(w)$ satisfies the *weak doubling property*, which means that there exists a constant $\lambda > 1$ such that

$$\liminf_{R\to\infty}\left(\inf_{y\in\mathbb{R}} \frac{\left\|\chi_{]y-\lambda R, y+\lambda R[}\right\|_{L^{p(\cdot)}(w)}}{\left\|\chi_{]y-R, y+R[}\right\|_{L^{p(\cdot)}(w)}}\right) < \infty.$$

We then directly obtain the desired result using [24, Theorem 1.3], together with inequality (1.2) of the same article. □

**Corollary 4.3.5** (adapted from the proof of [21, Corollary 1])**.** *Let $p \in LH^*$ be a variable exponent satisfying $1 < p_-, p_+ < \infty$ and $w \in CW^{p(\cdot)}$ be a weight. Then $M_{p(\cdot),w}$ is a Banach algebra.*

*Proof.* Let $(a_n)$ be a Cauchy sequence in $M_{p(\cdot),w}$. By definition of the $M_{p(\cdot),w}$ norm, this means that $\left(W^0_{a_n}\right)$ is a Cauchy sequence in $\mathcal{B}(L^{p(\cdot)}(w))$. Additionally, by Theorem 4.3.4, the following holds for each $n, m \in \mathbb{N}$:

$$\|a_n - a_m\|_\infty \leq \|a_n - a_m\|_{M_{p(\cdot),w}}.$$

This implies that $(a_n)$ is a Cauchy sequence in $L^\infty(\mathbb{R})$. Since the latter space is complete, the sequence $(a_n)$ converges in the $L^\infty(\mathbb{R})$ norm to a function $a \in L^\infty(\mathbb{R})$. Because $(a_n)$ is a bounded sequence, the quantity $k := \sup_{n\in\mathbb{N}} \|a_n\|_\infty$ is finite.

Now fix $f \in C^\infty_c(\mathbb{R})$ and notice that, denoting the Fourier transform by $F$, we have

$$|a_n(x) Ff(x) e^{-itx}| \leq kF|f|(x)$$

for each $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Since $kF|f|$ is integrable, this allows the use of the dominated convergence theorem [2, Theorem 3.31] to deduce that

$$(W^0_{a_n} f)(t) = \frac{1}{2\pi}\int_{\mathbb{R}} a_n(x)Ff(x)e^{-itx}dx \to \frac{1}{2\pi}\int_{\mathbb{R}} a(x)Ff(x)e^{-itx}dx = (W^0_a f)(t)$$

for almost every $t \in \mathbb{R}$.

Since the sequence $(a_n)$ is bounded in $M_{p(\cdot),w}$, we have

$$M := \sup_{n\in\mathbb{N}} \left\|W^0_{a_n}\right\|_{\mathcal{B}\left(L^{p(\cdot)}(w)\right)} < \infty \text{ and } \sup_{n\in\mathbb{N}} \left\|W^0_{a_n} f\right\|_{L^{p(\cdot)}(w)} \le M\|f\|_{L^{p(\cdot)}(w)}.$$

We know from Theorem 4.3.3 that $L^{p(\cdot)}(w)$ is a Banach function space. Applying [4, Chap. 1, Lemma 1.5(ii)] to the a.e.-convergent sequence $(W^0_{a_n})$, we get

$$\left\|W^0_a f\right\|_{L^{p(\cdot)}(w)} \le \liminf_{n\to\infty} \left\|W^0_{a_n} f\right\|_{L^{p(\cdot)}(w)} \le \liminf_{n\to\infty} \left\|W^0_{a_n}\right\|_{\mathcal{B}\left(L^{p(\cdot)}(w)\right)} \|f\|_{L^{p(\cdot)}(w)},$$

for each $f \in C^\infty_c(\mathbb{R})$. Now [25, Lemma 2.12(a)] asserts that $C^\infty_c(\mathbb{R})$ is dense in $L^{p(\cdot)}(w)$, and it follows from the previous inequality that $W^0_a \in \mathcal{B}\left(L^{p(\cdot)}(w)\right)$, which by definition means that $a \in M_{p(\cdot),w}$. This shows that if $a_n \in M_{p(\cdot),w}$ and $\|a_n - a\|_\infty \to 0$, then $a \in M_{p(\cdot),w}$ and

$$\|a\|_{M_{p(\cdot),w}} \le \liminf_{n\to\infty} \|a_n\|_{M_{p(\cdot),w}}.$$

For each $k \in \mathbb{N}$, apply the previous inequality to $a_k - a$ and $a_k - a_n$ to obtain

$$\|a_k - a\|_{M_{p(\cdot),w}} \le \liminf_{n\to\infty} \|a_k - a_n\|_{M_{p(\cdot),w}}, \qquad k \in \mathbb{N}.$$

Fix $\varepsilon > 0$. Since $(a_n)$ is Cauchy in $M_{p(\cdot),w}$, there exists $N \in \mathbb{N}$ such that $n,m > N$ implies $\|a_k - a_n\|_{M_{p(\cdot),w}} < \frac{\varepsilon}{2}$. From the previous inequality, we deduce that $\|a_k - a\|_{M_{p(\cdot),w}} < \varepsilon$. We conclude that

$$\|a_k - a\|_{M_{p(\cdot),w}} \to 0,$$

thus $M_{p(\cdot),w}$ is complete. □

We can first prove the boundedness of the operator $W^0_a$ when $a$ is a characteristic function.

**Lemma 4.3.6.** *Let $p \in LH^*$ be a variable exponent satisfying $1 < p_-, p_+ < +\infty$ and $w \in CW^{p(\cdot)}$ be a weight. Denote by $\chi$ the characteristic function of an interval $]c,+\infty[$, $c \in \mathbb{R}$. Then $W^0_\chi$ is bounded on $L^{p(\cdot)}(w)$ and its norm satisfies*

$$\left\|W^0_\chi\right\|_{\mathcal{B}(L^{p(\cdot)}(w))} \le \frac{1}{2}\left(1 + \|S\|_{\mathcal{B}(L^{p(\cdot)}(w))}\right).$$

*Proof.* Taking $a(t) = \operatorname{sgn}(t-c)$ and $v$ defined as in Lemma 4.3.1 we have

$$Fv(t) = -\operatorname{sgn}(t-c)Fu(t) = -a(t)Fu(t),$$

or equivalently

$$v = -F^{-1}aFu = -W^0_a u.$$

Therefore

$$W_a^0 u(t) = -v(t) = -e_{-c}(t)[S(e_c u)](t).$$

Now notice that $\chi$ can be written as

$$\chi(t) = \frac{1}{2}[1 + a(t)],$$

hence we have for $u \in L^{p(\cdot)}(w) \cap L^2(\mathbb{R})$,

$$W_\chi^0 u(t) = \frac{1}{2}[u(t) - e_{-c}(t)(S(e_c u))(t)].$$

Applying norms, keeping in mind that $|e_c(t)| = 1$, we see that

$$\left\|W_\chi^0\right\|_{\mathcal{B}(L^{p(\cdot)}(w))} \leq \frac{1}{2}\left(1 + \|S\|_{\mathcal{B}(L^{p(\cdot)}(w))}\right).$$ □

We will need the following lemma in order to apply interpolation theory:

**Lemma 4.3.7** ([33, Corollary 2.3])**.** *If $p$ is an exponent such that $1 < p_-, p_+ < \infty$, let*

$$\theta_p := \min\{1,\ 2/p_+,\ 2 - 2/p_-\}.$$

*Then for all $\theta \in ]0, \theta_p[$, the exponent $p_0$ defined by*

$$\frac{1}{p(x)} = \frac{1-\theta}{p_0(x)} + \frac{\theta}{2} \tag{4.7}$$

*satisfies $1 < (p_0)_-, (p_0)_+ < \infty$.*

**Lemma 4.3.8.** *In the conditions of the previous Lemma, if $p$ belongs to the class $LH^*$ then so does $p_0$.*

*Proof.* Let $p \in LH^*$ and $p_0$ defined as in above. Then

$$p_0(x) = \frac{2(1-\theta)p(x)}{2 - \theta p(x)}$$

Taking into account the definition of $\theta_p$ in Lemma 4.3.7, we have

$$2 - \theta p(x) \geq 2 - \theta p_+ > 0.$$

This gives us the estimate

$$|p_0(x) - p_0(y)| = \left|\frac{4(1-\theta)(p(x) - p(y))}{(2 - \theta p(x))(2 - \theta p(y))}\right| \leq \frac{4(1-\theta)}{(2 - \theta p_+)^2}|p(x) - p(y)|.$$

From this we can see that conditions (4.3) and (4.5) hold for $p_0$. We conclude that $p_0 \in LH^*$. □

**Lemma 4.3.9.** *Let $p \in LH^*$ be a variable exponent satisfying $1 < p_-, p_+ < \infty$ and $w$ be a power weight. Take $\theta_p$ as in Lemma 4.3.7. If the Cauchy singular operator $S$ is bounded on $L^{p(\cdot)}(w)$, it is also bounded on $L^{p_0(\cdot)}(w^{1/(1-\theta)})$ for some $\theta \in ]0, \theta_p[$ and $p_0$ defined as in (4.7).*

*Proof.* The weight $w$ is of the form

$$w(x) = |x-i|^{\lambda_\infty} \prod_{j=1}^{m} |x-x_j|^{\lambda_j}, \quad x \in \mathbb{R},$$

with $-\infty < x_1 < \ldots < x_m < +\infty$ and $\lambda_1, \ldots, \lambda_m, \lambda_\infty \in \mathbb{R}$. Then

$$w(x)^{1/(1-\theta)} = |x-i|^{\lambda_\infty/(1-\theta)} \prod_{j=1}^{m} |x-x_j|^{\lambda_j/(1-\theta)}, \quad x \in \mathbb{R}.$$

Since $S$ is bounded on $L^{p(\cdot)}(w)$, by Theorem 4.3.2 the exponents satisfy

$$0 < \frac{1}{p(x_j)} + \lambda_j < 1,\ j = 1, \ldots, m, \quad 0 < \frac{1}{p(\infty)} + \lambda_\infty + \sum_{j=1}^{m} \lambda_j < 1.$$

By the same Theorem, we want to prove that

$$0 < \frac{1}{p_0(x_j)} + \frac{\lambda_j}{1-\theta} < 1,\ j = 1, \ldots, m, \quad 0 < \frac{1}{p_0(\infty)} + \frac{\lambda_\infty}{1-\theta} + \sum_{j=1}^{m} \frac{\lambda_j}{1-\theta} < 1$$

for some $\theta \in ]0, \theta_p[$ and $p_0$ defined as in (4.7).

It is easy to see that these inequalities are equivalent to

$$\frac{\theta}{2} < \frac{1}{p(x_j)} + \lambda_j < 1 - \frac{\theta}{2},\ j = 1, \ldots, m, \quad \frac{\theta}{2} < \frac{1}{p(x_j)} + \lambda_\infty + \sum_{j=1}^{m} \lambda_j < 1 - \frac{\theta}{2}. \tag{4.8}$$

Inequalities (4.8) are satisfied if we choose $\theta$ such that

$$0 < \theta < \min\left\{\theta_p, 2c_j, 2(1-c_j)\right\},\ j = 1, \ldots, m, \quad 0 < \theta < \min\left\{\theta_p, 2c_\infty, 2(1-c_\infty)\right\},$$

where

$$c_j = \frac{1}{p(x_j)} + \lambda_j,\ j = 1, \ldots, m, \quad c_\infty = \frac{1}{p(\infty)} + \lambda_\infty + \sum_{j=1}^{m} \lambda_j.$$

In view of Theorem 4.3.2, the operator $S$ is bounded on the space $L^{p_0(\cdot)}(w^{1/(1-\theta)})$. □

### 4.3.3 Proof of the Stechkin Inequality for Weighted Variable Lebesgue Spaces

**Theorem 4.3.10.** *Let $p \in LH^*$ be a variable exponent satisfying $1 < p_-, p_+ < \infty$ and $w \in CW^{p(\cdot)}$ be a weight. If $a : \mathbb{R} \to \mathbb{C}$ has finite total variation $V(a)$, then $a \in M_{p(\cdot),w}$ and*

$$\|a\|_{M_{p(\cdot),w}} \leq \|S\|_{\mathcal{B}(L^{p(\cdot)}(w))} \left[\|a\|_\infty + V(a)\right],$$

*where $S$ is the Cauchy singular integral operator.*

*Proof.* Suppose $a$ is a piecewise constant function. Then there exist constants $\lambda_k \in \mathbb{C}$ and a partition of the real line $-\infty = t_0 < t_1 < \ldots < t_n = +\infty$ such that $a(t) = \sum_{k=0}^{n-1} \lambda_k \chi_k(t)$, where $\chi_k(t) = \chi_{]t_k, t_{k+1}[}$. Rewrite $a$ in the form

$$a(t) = b_1 + \sum_{k=2}^{n} (b_k - b_{k-1}) \chi_k(t),$$

where $\chi_k$ is now the characteristic function of the interval $]c_k, +\infty[$.

Lemma 4.3.1 tells us that $S = W^0_{-\mathrm{sgn}}$, and by Theorem 4.3.4 we have

$$1 = \|-\mathrm{sgn}\|_\infty \leq \|-\mathrm{sgn}\|_{M_{p(\cdot),w}} = \left\|W^0_{-\mathrm{sgn}}\right\|_{\mathcal{B}(L^{p(\cdot)}(w))} = \|S\|_{\mathcal{B}(L^{p(\cdot)}(w))}.$$

We estimate the norm of $W^0_a$ using Lemma 4.3.6 and the previous inequality

$$\begin{aligned}
\left\|W^0_a\right\|_{\mathcal{B}(L^{p(\cdot)}(w))} &\leq \left\|W^0_{b_1}\right\|_{\mathcal{B}(L^{p(\cdot)}(w))} + \sum_{k=2}^{n} \left\|W^0_{(b_k - b_{k-1})\chi_k}\right\|_{\mathcal{B}(L^{p(\cdot)}(w))} \\
&\leq |b_1| + \sum_{k=2}^{n} \left\|W^0_{b_k - b_{k-1}}\right\|_{\mathcal{B}(L^{p(\cdot)}(w))} \left\|W^0_{\chi_k}\right\|_{\mathcal{B}(L^{p(\cdot)}(w))} \\
&\leq |b_1| + \frac{1}{2} \sum_{k=2}^{n} |b_k - b_{k-1}| \left(1 + \|S\|_{\mathcal{B}(L^{p(\cdot)}(w))}\right) \\
&\leq |b_1| + \sum_{k=2}^{n} |b_k - b_{k-1}| \|S\|_{\mathcal{B}(L^{p(\cdot)}(w))} \\
&\leq \|S\|_{\mathcal{B}(L^{p(\cdot)}(w))} \left[\|a\|_\infty + V(a)\right].
\end{aligned} \tag{4.9}$$

Now suppose $a$ is any function of bounded variation. By Lemma 2.3.3, there exists a sequence $(a_n)$ of piecewise constant functions such that

$$\|a_n - a\|_\infty \to 0 \text{ and } V(a_n) \leq V(a). \tag{4.10}$$

By Lemma 4.3.9, there exists some $\theta \in ]0, 1[$ such that the Cauchy singular operator $S$ is bounded on $L^{p_0(\cdot)}(w_0)$, where $p_0$ and $w_0$ are defined by

$$\frac{1}{p(x)} = \frac{1-\theta}{p_0(x)} + \frac{\theta}{2}, \quad w_0(x) = w(x)^{1/(1-\theta)}, \quad x \in \mathbb{R}.$$

If we define $w_1(x) = 1$ for all $x \in \mathbb{R}$, the following decomposition holds:

$$w(x) = w_0(x)^{1-\theta} w_1(x)^\theta, \quad x \in \mathbb{R}.$$

By Lemma 4.3.7, $p_0$ satisfies $1 < (p_0)_-, (p_0)_+ < \infty$. We have the following chain of inequalities, where we use Theorem 3.4.3, inequality (4.9), Theorem 4.1.2 and the inequality in

(4.10).

$$\begin{aligned}\left\|W^0_{a_n} - W^0_{a_m}\right\|_{\mathscr{B}(L^{p(\cdot)}(w))} &= \left\|W^0_{a_n-a_m}\right\|_{\mathscr{B}(L^{p(\cdot)}(w))} \\ &\leq 2^{1/p_-}\left\|W^0_{a_n-a_m}\right\|^{1-\theta}_{\mathscr{B}(L^{p_0(\cdot)}(w_0))}\left\|W^0_{a_n-a_m}\right\|^{\theta}_{\mathscr{B}(L^2)} \\ &\leq 2^{1/p_-}\|a_n - a_m\|^{\theta}_{\infty}\|S\|^{1-\theta}_{\mathscr{B}(L^{p_0(\cdot)}(w_0))}\left[\|a_n - a_m\|_\infty + V(a_n - a_m)\right]^{1-\theta} \\ &\leq 2^{1/p_-}\|a_n - a_m\|^{\theta}_{\infty}\|S\|^{1-\theta}_{\mathscr{B}(L^{p_0(\cdot)}(w_0))}\left[\|a_n - a_m\|_\infty + V(a_n) + V(a_m)\right]^{1-\theta} \\ &\leq 2^{1/p_-}\|a_n - a_m\|^{\theta}_{\infty}\|S\|^{1-\theta}_{\mathscr{B}(L^{p_0(\cdot)}(w_0))}\left[\|a_n - a\|_\infty + 2V(a)\right]^{1-\theta}.\end{aligned}$$

From this inequality and the $L^\infty$ convergence in (4.10) it follows that the sequence $(W^0_{a_n})$ is Cauchy, hence by completeness of $\mathscr{B}(L^{p(\cdot)}(w))$ it is convergent to an operator $A \in \mathscr{B}(L^{p(\cdot)}(w))$. Using again Theorem 4.1.2, we have

$$\left\|W^0_{a_n} - W^0_{a}\right\|_{\mathscr{B}(L^2)} = \left\|W^0_{a_n-a}\right\|_{\mathscr{B}(L^2)} = \|a_n - a\|_\infty \to 0.$$

Uniqueness of the limit then allows us to conclude $Au = W^0_a u$ for $u \in L^2(\mathbb{R}) \cap L^{p(\cdot)}(w)$. Since this set is dense in $L^{p(\cdot)}(w)$, we conclude that $A = W^0_a$.

From inequality (4.9), together with the property $V(a_n) \leq V(a)$ we deduce

$$\left\|W^0_{a_n}\right\|_{\mathscr{B}(L^{p(\cdot)}(w))} \leq \|S\|_{\mathscr{B}(L^{p(\cdot)}(w))}\left[\|a_n\|_\infty + V(a)\right].$$

Passing to the limit as $n \to \infty$ we have the desired inequality. □

## 4.4 Algebra of Piecewise Continuous Fourier Multipliers on $L^{p(\cdot)}(w)$

Given a variable exponent $p \in LH^*$ satisfying $1 < p_-, p_+ < \infty$ and a weight $w \in CW^{p(\cdot)}$, define the algebra

$$PC_{p(\cdot),w} := \operatorname{clos}_{M_{p(\cdot),w}} BV(\mathbb{R}).$$

To be clear, $PC_{p(\cdot),w}$ is the closure of the set of functions with bounded variation with respect to the Fourier multiplier norm

$$\|a\|_{M_{p(\cdot),w}} = \left\|W^0_a\right\|_{\mathscr{B}(L^{p(\cdot)}(w))}.$$

This algebra is well defined by Theorem 4.3.10.

**Theorem 4.4.1.** *Let $p \in LH^*$ be a variable exponent satisfying $1 < p_-, p_+ < \infty$, and $w \in CW^{p(\cdot)}$ be a weight. Then $PC_{p(\cdot),w}$ is a commutative, unital Banach algebra. Furthermore, $PC_{p(\cdot),w}$ is a subset of $PC(\mathbb{R})$.*

*Proof.* The function $f(x) := 1$ is the identity, with $\|f\|_{M_{p(\cdot),w}} = \|I\|_{\mathscr{B}(L^{p(\cdot)}(w))} = 1$. The algebra $PC_{p(\cdot),w}$ is commutative, since it is an algebra of complex-valued functions with the usual product.

To prove that $PC_{p(\cdot),w}$ is a Banach algebra, note that

$$\begin{aligned}\|ab\|_{M_{p(\cdot),w}} &= \left\|W^0_{ab}\right\|_{\mathcal{B}(L^{p(\cdot)}(w))} = \left\|W^0_a W^0_b\right\|_{\mathcal{B}(L^{p(\cdot)}(w))} \\ &\le \left\|W^0_a\right\|_{\mathcal{B}(L^{p(\cdot)}(w))}\left\|W^0_b\right\|_{\mathcal{B}(L^{p(\cdot)}(w))} = \|a\|_{M_{p(\cdot),w}}\|b\|_{M_{p(\cdot),w}}.\end{aligned}$$

By Lemma 2.3.4, $BV(\mathbb{R}) \subset PC(\mathbb{R})$ and $PC(\mathbb{R})$ is complete with respect to the $L^\infty$ norm. Using this and Theorem 4.3.4 we can conclude

$$PC_{p(\cdot),w} = \operatorname{clos}_{M_{p(\cdot),w}} BV(\mathbb{R}) \subset \operatorname{clos}_\infty BV(\mathbb{R}) \subset \operatorname{clos}_\infty PC(\mathbb{R}) = PC(\mathbb{R}). \qquad \square$$

Examining the proof of Theorem 4.3.10, we get the following result:

**Theorem 4.4.2.** *Let $p \in LH^*$ be a variable exponent satisfying $1 < p_-, p_+ < \infty$ and $w \in CW^{p(\cdot)}$ be a weight. Then the algebra $PC_{p(\cdot),w}$ is the closure of the algebra of piecewise constant functions with respect to the norm $\|\cdot\|_{M_{p(\cdot),w}}$.*

*Proof.* Let $a$ be a function of bounded variation. By Theorem 4.3.10, we know that the operator $W^0_a$ is bounded. In addition, Lemma 2.3.3 gives us a sequence $(a_n)$ of piecewise constant functions such that

$$\|a_n - a\|_\infty \to 0 \text{ and } V(a_n) \le V(a). \tag{4.11}$$

By Lemma 4.3.9, there exists some $\theta \in ]0,1[$ such that the Cauchy singular operator $S$ is bounded on $L^{p_0(\cdot)}(w_0)$, where $p_0$ and $w_0$ are defined by

$$\frac{1}{p(x)} = \frac{1-\theta}{p_0(x)} + \frac{\theta}{2}, \quad w_0(x) = w(x)^{1/(1-\theta)}, \quad x \in \mathbb{R}.$$

If we define $w_1(x) = 1$ for all $x \in \mathbb{R}$, the following decomposition holds

$$w(x) = w_0(x)^{1-\theta} w_1(x)^\theta, \quad x \in \mathbb{R}.$$

By Lemma 4.3.7, $p_0$ satisfies $1 < (p_0)_-, (p_0)_+ < \infty$. Then, using Theorem 3.4.3, inequality (4.9) and Theorem 4.1.2, we get

$$\begin{aligned}\|a_n - a\|_{M_{p(\cdot),w}} &= \left\|W^0_{a_n-a}\right\|_{\mathcal{B}(L^{p(\cdot)}(w))} \\ &\le 2^{1/p_-}\left\|W^0_{a_n-a}\right\|^\theta_{\mathcal{B}(L^2)}\left\|W^0_{a_n-a}\right\|^{1-\theta}_{\mathcal{B}(L^{p_0(\cdot)}(w_0))} \\ &\le 2^{1/p_-}\|a_n-a\|^\theta_\infty \|S\|^{1-\theta}_{\mathcal{B}(L^{p_0(\cdot)}(w_0))}\left[\|a_n-a\|_\infty + V(a_n-a)\right]^{1-\theta} \\ &\le 2^{1/p_-}\|a_n-a\|^\theta_\infty \|S\|^{1-\theta}_{\mathcal{B}(L^{p_0(\cdot)}(w_0))}\left[\|a_n-a\|_\infty + V(a_n)+V(a)\right]^{1-\theta} \\ &\le 2^{1/p_-}\|a_n-a\|^\theta_\infty \|S\|^{1-\theta}_{\mathcal{B}(L^{p_0(\cdot)}(w_0))}\left[\|a_n-a\|_\infty + 2V(a)\right]^{1-\theta}.\end{aligned}$$

From this inequality and the $L^\infty(\mathbb{R})$ convergence in (4.11), it follows that the sequence $(a_n)$ is convergent in the $M_{p(\cdot),w}$ norm to the function $a$.

This proves that $BV(\mathbb{R}) = \operatorname{clos}_{M_{p(\cdot),w}}\{f : \mathbb{R} \to \mathbb{C} : f \text{ is piecewise constant}\}$. Taking the closure in both sides we get the desired result. $\square$

### 4.4.1 Invertibility of Convolution Operators with Piecewise Continuous Symbols

We proceed to investigate invertibility in the algebra $PC_{p(\cdot),w}$.

The following fact is used in [13] without proof.

**Lemma 4.4.3.** *Let $a \in PC(\mathbb{R})$. Then*

$$\operatorname*{ess\,inf}_{t\in\mathbb{R}} |a(t)| > 0$$

*if and only if*

$$a(+\infty) \neq 0, a(-\infty) \neq 0 \text{ and } a(x-0) \neq 0, a(x+0) \neq 0$$

*for all $x \in \mathbb{R}$.*

*Proof.* To prove the direct implication, suppose $a(y+0) = 0$ for some $y \in \mathbb{R}$ and take $r \in \mathbb{R}$ such that

$$r \leq |a(t)| \text{ a.e.}$$

Assume that $r > 0$, then there exists $\delta > 0$ such that

$$|a(t)| < r \text{ when } t \in \,]y, y+\delta[,$$

by definition of right sided limit. Since the interval $]y, y+\delta[$ has positive measure, this is a contradiction. We conclude that $r \leq 0$, hence

$$\operatorname*{ess\,inf}_{t\in\mathbb{R}} |a(t)| = 0.$$

The three other proofs are analogous.

To prove the opposite implication, let $D$ be the set of points of discontinuity of $a$ and assume that

$$\inf_{t\in\mathbb{R}\setminus D} |a(t)| = 0.$$

Then for each $n \in \mathbb{N}$, there exists $t_n \in \mathbb{R} \setminus D$ such that $|a(t_n)| < 1/n$, which implies

$$\lim_{n\to\infty} |a(t_n)| = 0.$$

This is a contradiction since $a$ is continuous on $\mathbb{R}\setminus D$ and $a(x) = a(x \pm 0) \neq 0$ for all $x \in \dot{\mathbb{R}}\setminus D$. Now notice that the set of points of discontinuity of a $PC(\mathbb{R})$ function is countable [6, Chap. 2, Theorem 3], thus it has measure zero. We conclude that

$$0 < \inf_{t\in\mathbb{R}\setminus D} |a(t)| \leq \operatorname*{ess\,inf}_{t\in\mathbb{R}\setminus D} |a(t)| = \operatorname*{ess\,inf}_{t\in\mathbb{R}} |a(t)|. \qquad \square$$

**Theorem 4.4.4.** *Let $p \in LH^*$ be a variable exponent satisfying $1 < p_-, p_+ < \infty$ and $w \in CW^{p(\cdot)}$ be a weight. If $a \in PC_{p(\cdot),w}$ is such that*

$$\operatorname*{ess\,inf}_{t\in\mathbb{R}} |a(t)| > 0,$$

*then $1/a \in PC_{p(\cdot),w}$.*

*Proof.* For every $x \in \mathbb{R}$, let

$$M_x^- := \{\chi_{]c,x[} : c < x\}, \quad M_x^+ := \{\chi_{]x,d[} : x < d\}.$$

For $x = \infty$, let

$$M_\infty^- := \{\chi_{]-\infty,-c[} : c \in \mathbb{R}\}, \quad M_\infty^+ := \{\chi_{]d,+\infty[} : d \in \mathbb{R}\}.$$

Take $f_1 = \chi_{]c_1,x[}$, $f_2 = \chi_{]c_2,x[} \in M_x^+$. Defining $c_3 = \min\{c_1, c_2\}$ and $f = \chi_{]c_3,x[}$ we have $f f_1 = f f_2 = f$. This proves $M_x^-$ is a localizing class and the other three cases are analogous.

For every $x \in \dot{\mathbb{R}}$, choose functions $\chi_x^- \in M_x^-$, $\chi_x^+ \in M_x^+$. The respective intervals form an open covering of $\dot{\mathbb{R}}$, and by compactness there exist a finite number of functions $\chi_{x_1}^-, \dots, \chi_{x_n}^-, \chi_{x_1}^+, \dots \chi_{x_n}^+$ such that

$$g(t) = \sum_{k=1}^{n} \left[\chi_{x_k}^-(t) + \chi_{x_k}^+(t)\right] \geq 1.$$

Then $g$ is evidently of bounded variation and we have

$$\sum_{k=1}^{N} \left|\frac{1}{g(y_k)} - \frac{1}{g(y_{k-1})}\right| = \sum_{k=1}^{N} \left|\frac{g(y_k) - g(y_{k-1})}{g(y_k)g(y_{k-1})}\right| \leq \sum_{k=1}^{N} \left|g(y_k) - g(y_{k-1})\right| \leq V(g) < \infty,$$

for any $N \in \mathbb{N}$ and $-\infty < y_0 < \dots < y_N < +\infty$, hence $1/g$ is also of bounded variation. By the Stechkin inequality, $1/g$ is in $PC_{p(\cdot),w}$. We conclude that $(M_x^\pm)_{x\in\dot{\mathbb{R}}}$ is a covering system of localizing classes in $PC_{p(\cdot),w}$.

Take $a \in PC_{p(\cdot),w}$. By Theorem 4.4.2, there exists a sequence of piecewise constant functions $(a_n)_{n\in\mathbb{N}}$ convergent to $a$ in $PC_{p(\cdot),w}$. Each $a_n$ is piecewise constant, thus of the form

$$a_n = \sum_{k=1}^{N} b_k \chi_{]c_k,c_{k+1}[},$$

where $b_k \in \mathbb{C}$ and $-\infty = c_1 < c_2 < \dots < c_{N+1} = +\infty$. If $x \in ]c_j, c_{j+1}[$, then $a_n(x+0) = a_n(x) = b_j$, hence

$$[a_n - a_n(x+0)]\,\chi_{]x,c_{j+1}[} = \left[\sum_{k=1}^{N} b_k \chi_{]c_k,c_{k+1}[} - b_j\right] \chi_{]x,c_{j+1}[} = \left[\sum_{\substack{k=1\\k\neq j}}^{N} b_k \chi_{]c_k,c_{k+1}[} + b_j\left(\chi_{]c_j,c_{j+1}[} - 1\right)\right] \chi_{]x,c_{j+1}[} = 0.$$

This proves $a_n \overset{M_x^+}{\sim} a_n(x+0)$ for each $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

Now notice that for any characteristic function $\chi$,

$$\|\chi\|_{M_{p(\cdot),w}} \leq \|S\|_{\mathcal{B}(L^{p(\cdot)}(w))}(\|\chi\|_\infty + V(\chi)) = 3\|S\|_{\mathcal{B}(L^{p(\cdot)}(w))},$$

by Theorem 4.3.10 and calculation of the total variation $V(\chi)$ of any characteristic function. This allows us to use Lemma 2.1.4 to conclude that $a \overset{M_x^+}{\sim} a(x+0)$. The other three cases can be proved analogously. Using the hypothesis and Lemma 4.4.3 we deduce

$$a(x \pm 0),\ a(+\infty),\ a(-\infty) \neq 0,$$

hence these constants are invertible in the algebra $PC_{p(\cdot),w}$. Since $(M_x^{\pm})_{x\in\dot{\mathbb{R}}}$ is a covering system of localizing classes, we can use Theorem 2.1.6 to conclude that $1/a$ is in $PC_{p(\cdot),w}$. □

The following theorem generalizes a result [13, Theorem 2.18], proven by Roland Duduchava for standard Lebesgue spaces.

**Theorem 4.4.5.** *Let $p \in LH^*$ be a variable exponent satisfying $1 < p_-, p_+ < \infty$ and $w \in CW^{p(\cdot)}$ be a weight. If $a \in PC_{p(\cdot),w}$, the operator $W_a^0$ is invertible in $\mathcal{B}(L^{p(\cdot)}(w))$ if and only if*

$$\operatorname*{ess\,inf}_{t\in\mathbb{R}} |a(t)| > 0.$$

*Proof.* Suppose that the essential infimum is positive. Then, by the previous proposition, we have that $1/a \in PC_{p(\cdot),w}$. Then

$$W_a^0 W_{1/a}^0 = W_{1/a}^0 W_a^0 = W_1^0 = I,$$

hence $W_{1/a}^0$ is the inverse of $W_a^0$.

Now suppose $\operatorname*{ess\,inf}_{t\in\mathbb{R}} |a(t)| = 0$. For each $x \in \dot{\mathbb{R}}$, consider the sets $M_x^{0-}, M_x^{0+}$ formed by the operators $W_g^0$ with $g \in M_x^+, M_x^-$, respectively. Then $(M_x^{0\pm})_{x\in\dot{\mathbb{R}}}$ is a covering system of localizing classes and

$$W_a^0 \overset{M_x^{0\pm}}{\sim} W_{a(x\pm 0)}^0 = a(x \pm 0)I.$$

Here we use the convention that $a(\infty \pm 0) = a(\pm\infty)$ (cf. the proof of Lemma 2.3.3). By hypothesis and Lemma 4.4.3 there exists some $x_0 \in \dot{\mathbb{R}}$ such that $a(x_0 + 0) = 0$ or $a(x_0 - 0) = 0$. Then

$$W_a^0 \overset{M_x^{0\pm}}{\sim} 0$$

and by Theorem 2.1.6 the operator $W_a^0$ cannot be invertible. □

CHAPTER 5

# Algebras of Convolution Type Operators and Their Images in the Calkin Algebra

## 5.1 Algebra of Continuous Fourier Multipliers

Let $p \in LH^*$ be a variable exponent satisfying $1 < p_-, p_+ < \infty$ and $w \in CW^{p(\cdot)}$ be a weight. Recall that $LH^*$ is the set of functions $p : \mathbb{R} \to [1, +\infty[$ that satisfy conditions (4.3), (4.4) and (4.5) and that $CW^{p(\cdot)}$ is the set of power weights that satisfy the inequalities of Theorem 4.3.2.

We are then in the conditions of Theorem 4.3.10, which allows us to define the algebra

$$C_{p(\cdot),w}(\dot{\mathbb{R}}) := \mathrm{clos}_{M_{p(\cdot),w}}\left(C(\dot{\mathbb{R}}) \cap BV(\mathbb{R})\right),$$

where $C(\dot{\mathbb{R}})$ is the set of continuous functions $f : \mathbb{R} \to \mathbb{C}$ such that the limits $\lim_{t\to+\infty} f(t)$ and $\lim_{t\to-\infty} f(t)$ are equal and finite.

Define the Schwartz space $S(\mathbb{R})$ as the set of infinitely differentiable functions $f : \mathbb{R} \to \mathbb{C}$ that satisfy

$$\sup_{x\in\mathbb{R}} |x^n f^{(m)}(x)| < \infty$$

for all $n, m \in \mathbb{N}_0$. The interested reader can consult properties of this space in, for example, [19, Section 2.2].

**Lemma 5.1.1.** *Let $\varphi \in C_c^\infty(\mathbb{R})$ be a non-negative even function that satisfies*

$$\int_{\mathbb{R}} \varphi(x)dx = 1.$$

*For each $\delta > 0$, define $\varphi_\delta$ by*

$$\varphi_\delta(x) := \delta^{-1}\varphi(x/\delta), \quad x \in \mathbb{R}.$$

*If $p(\cdot) \in LH^*$ satisfies $1 < p_-, p_+ < \infty$ and $w \in CW^{p(\cdot)}$, then for every $a \in M_{p(\cdot),w}$ and $\delta > 0$,*

$$\|a * \varphi_\delta\|_{M_{p(\cdot),w}} \leq \|a\|_{M_{p(\cdot),w}}.$$

*Proof.* For each $a \in M_{p(\cdot),w}$, define

$$\|a\|_{M^0_{p(\cdot),w}} := \sup\left\{ \frac{\left\|F^{-1} aFf\right\|_{L^{p(\cdot)}(w)}}{\|f\|_{L^{p(\cdot)}(w)}} : f \in S(\mathbb{R}) \cap L^{p(\cdot)}(w) \setminus \{0\} \right\}.$$

According to the proof of Theorem 4.3.4, the space $L^{p(\cdot)}(w)$ satisfies the hypothesis of [24, Theorem 1.3]. In the proof of the latter theorem, it is shown that for every $\delta > 0$ and $f \in S(\mathbb{R}) \cap L^{p(\cdot)}(w)$,

$$\left\|F^{-1}(a * \varphi_\delta)Ff\right\|_{L^{p(\cdot)}(w)} \leq \|a\|_{M^0_{p(\cdot),w}} \|f\|_{L^{p(\cdot)}(w)}.$$

Taking suprema, this implies that

$$\|a * \varphi_\delta\|_{M^0_{p(\cdot),w}} \leq \|a\|_{M^0_{p(\cdot),w}}. \tag{5.1}$$

The space $L^{p(\cdot)}(w)$ is separable under our hypothesis that $p_+ < \infty$. Then $L^{p(\cdot)}(w)$ has absolutely continuous norm (as in the definition of Section 3.2.2) by [4, Chap.1, Corollary 5.6]. Using [24, Theorem 2.3], we deduce that $L^{p(\cdot)}(w)$ has the *bounded $L^2$-approximation property*. This means that for every function $u \in L^2(\mathbb{R}) \cap L^{p(\cdot)}(w)$, there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ such that

$$\lim_{n\to\infty} \|u - u_n\|_{L^2} = 0, \quad \limsup_{n\to\infty} \|u_n\|_{L^{p(\cdot)}(w)} \leq \|u\|_{L^{p(\cdot)}(w)}.$$

This property allows us to use [24, Theorem 6.1], together with inequality (5.1), to conclude that

$$\|a * \varphi_\delta\|_{M_{p(\cdot),w}} \leq \|a\|_{M_{p(\cdot),w}}.$$ □

Denote by $C_0(\mathbb{R})$ the set of continuous functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$f(+\infty) = f(-\infty) = 0.$$

Our version of the Riesz-Thorin theorem for weighted variable Lebesgue spaces (Theorem 3.4.3) allows us to follow ideas of Hörmander from his proof of [20, Theorem 1.16] to prove the following result. The case of $w = 1$ and more general variable exponents was proved in [23, Theorem 3.1 b)].

**Theorem 5.1.2.** *Let $p \in LH^*$ be a variable exponent satisfying $1 < p_-, p_+ < \infty$ and $w \in CW^{p(\cdot)}$ be a weight. Then*

$$C_0(\mathbb{R}) \cap BV(\mathbb{R}) \subset \operatorname{clos}_{M_{p(\cdot),w}} C_c^\infty(\mathbb{R}).$$

*Proof.* Let $a \in C_0(\mathbb{R}) \cap BV(\mathbb{R})$ and fix $\varepsilon > 0$. By Lemma 4.3.9, there exists some $\theta \in \,]0,1[$ such that the Cauchy singular operator $S$ is bounded on $L^{p_0(\cdot)}(w_0)$, where $p_0$ and $w_0$ are defined by

$$\frac{1}{p(x)} = \frac{1-\theta}{p_0(x)} + \frac{\theta}{2}, \quad w_0(x) = w(x)^{1/(1-\theta)}, \quad x \in \mathbb{R}. \tag{5.2}$$

If we define $w_1(x) = 1$ for all $x \in \mathbb{R}$, the following decomposition holds

$$w(x) = w_0(x)^{1-\theta} w_1(x)^{\theta}, \quad x \in \mathbb{R}. \tag{5.3}$$

By Lemma 4.3.7, $p_0$ satisfies $1 < (p_0)_-, (p_0)_+ < \infty$. For each $n \in \mathbb{N}$, define the function $\psi_n$ as

$$\psi_n(x) = \begin{cases} 1, & |x| \leq n \\ n+1-|x|, & n < |x| < n+1 \\ 0, & |x| \geq n+1. \end{cases}$$

It is evident that these functions have compact support. Furthermore we have, for each $n \in \mathbb{N}$,

$$\left\|\psi_n\right\|_\infty + V(\psi_n) = 3.$$

By Theorem 4.3.10, we have

$$\left\|\psi_n\right\|_{M_{p_0(\cdot),w_0}} \leq \|S\|_{\mathcal{B}(L^{p_0(\cdot)}(w_0))} \left[\left\|\psi_n\right\|_\infty + V(\psi_n)\right] = 3\|S\|_{\mathcal{B}(L^{p_0(\cdot)}(w_0))} =: k_0.$$

Define $a_n := a\psi_n$. Then $a_n \in C_0(\mathbb{R})$ has compact support and

$$\lim_{n\to\infty} \|a - a_n\|_\infty = 0. \tag{5.4}$$

Using Theorem 4.3.10 again, we deduce the inequalities

$$\begin{aligned} \|a - a_n\|_{M_{p_0(\cdot),w_0}} &= \left\|a(1-\psi_n)\right\|_{M_{p_0(\cdot),w_0}} \leq \|a\|_{M_{p_0(\cdot),w_0}} \left(1 + \left\|\psi_n\right\|_{M_{p_0(\cdot),w_0}}\right) \\ &\leq (1+k_0)\|a\|_{M_{p_0(\cdot),w_0}} \leq k_0(1+k_0)\left[\|a\|_\infty + V(a)\right] \end{aligned} \tag{5.5}$$

and

$$\|a_n\|_{M_{p_0(\cdot),w_0}} \leq \left\|\psi_n\right\|_{M_{p_0(\cdot),w_0}} \|a\|_{M_{p_0(\cdot),w_0}} \leq k_0\|a\|_{M_{p_0(\cdot),w_0}} \leq (k_0)^2\left[\|a\|_\infty + V(a)\right]. \tag{5.6}$$

Now using Theorem 3.4.3 (with the decompositions in (5.2) and (5.3)), Theorem 4.1.2 and inequality (5.5), we obtain

$$\begin{aligned} \|a - a_n\|_{M_{p(\cdot),w}} &= \left\|W^0_{a-a_n}\right\|_{\mathcal{B}(L^{p(\cdot)}(w))} \\ &\leq 2^{1/p_-}\left\|W^0_{a-a_n}\right\|^{\theta}_{\mathcal{B}(L^2)} \left\|W^0_{a-a_n}\right\|^{1-\theta}_{\mathcal{B}(L^{p_0(\cdot)}(w_0))} \\ &= 2^{1/p_-}\|a-a_n\|^{\theta}_\infty \|a-a_n\|^{1-\theta}_{M_{p_0(\cdot),w_0}} \\ &\leq 2^{1/p_-} k_0^{1-\theta}(1+k_0)^{1-\theta}\|a-a_n\|^{\theta}_\infty \left[\|a\|_\infty + V(a)\right]^{1-\theta}. \end{aligned}$$

This inequality and equality (5.4) imply that

$$\lim_{n\to\infty} \|a - a_n\|_{M_{p(\cdot),w}} = 0,$$

so there exists $n_0 \in \mathbb{N}$ such that

$$\left\|a - a_{n_0}\right\|_{M_{p(\cdot),w}} < \frac{\varepsilon}{2}. \tag{5.7}$$

Let $\varphi$ be defined by

$$\varphi(x) := \begin{cases} Ke^{1/(x^2-1)}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

where $K = \left(\int_{-1}^{1} e^{1/(x^2-1)} dx\right)^{-1}$. Then $\varphi$ is in the conditions of Lemma 5.1.1 and the following holds for all $\delta > 0$, where the sequence $(\varphi_\delta)$ is defined in the same Lemma:

$$\left\|a_{n_0} * \varphi_\delta\right\|_{M_{p_0(\cdot),w_0}} \leq \left\|a_{n_0}\right\|_{M_{p_0(\cdot),w_0}}. \tag{5.8}$$

Now Theorems 2.4.1 and 2.4.2 tell us that $a_{n_0} * \varphi_\delta \in C_c^\infty(\mathbb{R})$. Theorem 2.4.3 implies that

$$\lim_{\delta \to 0} \left\|a_{n_0} * \varphi_\delta - a_{n_0}\right\|_\infty = 0. \tag{5.9}$$

Using Theorem 3.4.3 (with the decompositions in (5.2) and (5.3)) and Theorem 4.1.2, we obtain

$$\begin{aligned} \left\|a_{n_0} * \varphi_\delta - a_{n_0}\right\|_{M_{p(\cdot),w}} &= \left\|W^0_{a_{n_0} * \varphi_\delta - a_{n_0}}\right\|_{\mathcal{B}(L^{p(\cdot)}(w))} \\ &\leq 2^{1/p_-} \left\|W^0_{a_{n_0} * \varphi_\delta - a_{n_0}}\right\|^\theta_{\mathcal{B}(L^2)} \left\|W^0_{a_{n_0} * \varphi_\delta - a_{n_0}}\right\|^{1-\theta}_{\mathcal{B}(L^{p_0(\cdot)}(w_0))} \\ &= 2^{1/p_-} \left\|a_{n_0} * \varphi_\delta - a_{n_0}\right\|^\theta_\infty \left\|a_{n_0} * \varphi_\delta - a_{n_0}\right\|^{1-\theta}_{M_{p_0(\cdot),w_0}}. \end{aligned} \tag{5.10}$$

Now using the triangle inequality, (5.8) and (5.6), we have

$$\begin{aligned} \left\|a_{n_0} * \varphi_\delta - a_{n_0}\right\|_{M_{p_0(\cdot),w_0}} &\leq \left\|a_{n_0} * \varphi_\delta\right\|_{M_{p_0(\cdot),w_0}} + \left\|a_{n_0}\right\|_{M_{p_0(\cdot),w_0}} \\ &\leq 2\left\|a_{n_0}\right\|_{M_{p_0(\cdot),w_0}} \\ &\leq 2(k_0)^2 \left[\|a\|_\infty + V(a)\right]. \end{aligned}$$

Combining this with (5.10), we conclude

$$\left\|a_{n_0} * \varphi_\delta - a_{n_0}\right\|_{M_{p(\cdot),w}} \leq 2^{1-\theta+1/p_-} k_0^{2(1-\theta)} \left\|a_{n_0} * \varphi_\delta - a_{n_0}\right\|^\theta_\infty \left[\|a\|_\infty + V(a)\right]^{1-\theta},$$

which, together with equality (5.9), implies that there exists $\delta_0 > 0$ such that

$$\left\|a_{n_0} * \varphi_{\delta_0} - a_{n_0}\right\|_{M_{p_0(\cdot),w_0}} < \frac{\varepsilon}{2}. \tag{5.11}$$

From (5.7) and (5.11) it follows that for each $\varepsilon > 0$ there exists a function $a_{n_0} * \varphi_{\delta_0} \in C_c^\infty(\mathbb{R})$ such that

$$\left\|a - a_{n_0} * \varphi_{\delta_0}\right\|_{M_{p(\cdot),w}} < \varepsilon,$$

which means that $a \in \operatorname{clos}_{M_{p(\cdot),w}} C_c^\infty(\mathbb{R})$. □

**Theorem 5.1.3** (adapted from [23, Theorem 1.1]). *Let $p \in LH^*$ be a variable exponent satisfying $1 < p_-, p_+ < \infty$ and $w \in CW^{p(\cdot)}$ be a weight. Then*

$$C_{p(\cdot),w}(\dot{\mathbb{R}}) = \operatorname{clos}_{M_{p(\cdot),w}} (\mathbb{C} + C_c^\infty(\mathbb{R})),$$

*where $\mathbb{C} + C_c^\infty(\mathbb{R})$ is the set of functions of the form $f = \lambda + c$, where $\lambda \in \mathbb{C}$ and $c \in C_c^\infty(\mathbb{R})$.*

*Proof.* Any function $f \in C_c^\infty(\mathbb{R})$ satisfies

$$f(+\infty) = f(-\infty) = 0,$$

hence $\mathbb{C} + C_c^\infty(\mathbb{R}) \subset C(\dot{\mathbb{R}})$. Furthermore, any function $\mathbb{C} + C_c^\infty(\mathbb{R})$ has bounded variation, so we infer that

$$\mathbb{C} + C_c^\infty(\mathbb{R}) \subset C(\dot{\mathbb{R}}) \cap BV(\mathbb{R}).$$

Now

$$\operatorname{clos}_{M_{p(\cdot),w}}(\mathbb{C} + C_c^\infty(\mathbb{R})) \subset \operatorname{clos}_{M_{p(\cdot),w}}\left(C(\dot{\mathbb{R}}) \cap BV(\mathbb{R})\right) = C_{p(\cdot),w}(\dot{\mathbb{R}}).$$

To prove the opposite inclusion, take $a \in C_{p(\cdot),w}(\dot{\mathbb{R}})$ and fix $\varepsilon > 0$. By definition of $C_{p(\cdot),w}(\dot{\mathbb{R}})$, there exists $b \in C(\dot{\mathbb{R}}) \cap BV(\mathbb{R})$ such that

$$\|a - b\|_{M_{p(\cdot),w}} < \frac{\varepsilon}{2}.$$

Evidently $b - b(+\infty) \in C_0(\mathbb{R}) \cap BV(\mathbb{R})$, so by Theorem 5.1.2 there exists $c \in C_c^\infty(\mathbb{R})$ such that

$$\|b - b(+\infty) - c\|_{M_{p(\cdot),w}} < \frac{\varepsilon}{2}.$$

From these inequalities it follows that

$$\|a - (b(+\infty) + c)\|_{M_{p(\cdot),w}} < \varepsilon.$$

Since $b(+\infty) + c \in \mathbb{C} + C_c^\infty(\mathbb{R})$, this implies that $a \in \operatorname{clos}_{M_{p(\cdot),w}}(\mathbb{C} + C_c^\infty(\mathbb{R}))$. □

## 5.2 Slowly Oscillating Fourier Multipliers

### 5.2.1 Algebra $SO^\diamond$ of Slowly Oscillating Functions

Given a set $E \subset \dot{\mathbb{R}}$ and a function $f : \dot{\mathbb{R}} \to \mathbb{R}$ in $L^\infty(\mathbb{R})$, define the *oscillation* of $f$ over $E$ by

$$\operatorname{osc}(f, E) := \operatorname*{ess\,sup}_{s,t \in E} |f(s) - f(t)|.$$

We'll say a function $f \in L^\infty(\mathbb{R})$ is *slowly oscillating* at a point $\lambda \in \mathbb{R}$ if for some $r \in ]0,1[$, we have

$$\lim_{x \to 0^+} \operatorname{osc}(f, \lambda + ([-x, -rx] \cup [rx, x])) = 0.$$

The function $f$ is said to be *slowly oscillating at infinity* if for some $r \in ]0,1[$,

$$\lim_{x \to +\infty} \operatorname{osc}(f, [-x, -rx] \cup [rx, x]) = 0$$

For every $\lambda \in \dot{\mathbb{R}}$, denote by $SO_\lambda$ the subalgebra of $L^\infty(\mathbb{R})$ defined by

$$SO_\lambda := \left\{ f \in C(\dot{\mathbb{R}} \setminus \{\lambda\}) \cap L^\infty(\mathbb{R}) : f \text{ is slowly oscillating at } \lambda \right\}.$$

Let $SO^\diamond$ be the smallest Banach subalgebra of $L^\infty(\mathbb{R})$ that contains all the algebras $SO_\lambda$ for $\lambda \in \dot{\mathbb{R}}$.

The algebras $SO_\lambda$ and $SO^\diamond$ were defined in [3] in the case of the unit circle and in [28] in the case of the real line.

### 5.2.2 Algebra $SO^{\diamond}_{p(\cdot),w}$ of Slowly Oscillating Fourier Multipliers

For every $\lambda \in \dot{\mathbb{R}}$, denote by $SO^3_\lambda$ the algebra

$$SO^3_\lambda := \left\{ a \in SO_\lambda \cap C^3(\mathbb{R} \setminus \{\lambda\}) : \lim_{x \to \lambda} (D^k_\lambda a)(x) = 0,\ k = 1,2,3 \right\},$$

where $C^3(\mathbb{R} \setminus \{\lambda\})$ is the set of three times differentiable functions on $\mathbb{R} \setminus \{\lambda\}$ and $D_\lambda$ is the operator defined by $(D_\lambda a)(x) = (x-\lambda)\frac{da}{dx}(x)$ if $\lambda \in \mathbb{R}$ and $(D_\infty a)(x) = x\frac{da}{dx}(x)$.

Define a norm in $SO^3_\lambda$ by

$$\|a\|_{SO^3_\lambda} := \sum_{k=0}^{3} \frac{1}{k!} \left\| D^k_\lambda a \right\|_\infty .$$

To formulate sufficient conditions guaranteeing that a function in $SO^3_\lambda$ is a Fourier multiplier on a Banach function space $X(\mathbb{R})$, we will need the notion of the Hardy-Littlewood maximal operator. This operator is defined by

$$(Mf)(x) := \sup_{]a,b[\ni x} \frac{1}{b-a} \int_a^b |f(y)| dy, \quad f \in L^1_{\mathrm{loc}}(\mathbb{R}).$$

This operator is *sublinear*, which means that

$$M(f+g)(x) \le (Mf)(x) + (Mg)(x), \quad f,g \in L^1_{\mathrm{loc}}(\mathbb{R}),\ x \in \mathbb{R},$$

and it is bounded on $L^p$ when $1 < p \le +\infty$.

**Theorem 5.2.1** ([22, Theorem 2.5])**.** *Let $X(\mathbb{R})$ be a separable Banach function space such that the Hardy-Littlewood maximal operator is bounded on $X(\mathbb{R})$ and on its associate space $X^*(\mathbb{R})$. If $\lambda \in \dot{\mathbb{R}}$ and $a \in SO^3_\lambda$, then the convolution operator $W^0_a$ is bounded on the space $X(\mathbb{R})$ and*

$$\left\| W^0_a \right\|_{\mathcal{B}(X(\mathbb{R}))} \le K \|a\|_{SO^3_\lambda},$$

*where $K > 0$ depends only on $X(\mathbb{R})$.*

We know precisely when the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}(w)$ if $p \in LH$.

**Theorem 5.2.2** ([11, Theorem 1.1])**.** *Let $p \in LH$. Then the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}(w)$ if and only if the weight $w$ satisfies*

$$\sup_{-\infty < a < b < \infty} \frac{1}{b-a} \left\| \chi_{]a,b[} \right\|_{L^{p(\cdot)}(w)} \left\| \chi_{]a,b[} \right\|_{L^{p'(\cdot)}(w^{-1})} < \infty, \tag{5.12}$$

*where*

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad x \in \mathbb{R}.$$

The set of weights satisfying the above condition is denoted by $\mathcal{A}_{p(\cdot)}(\mathbb{R})$.

The following theorem states that every function in $SO^3_\lambda$ for some $\lambda \in \dot{\mathbb{R}}$ is a Fourier multiplier on the weighted variable Lebesgue space $L^{p(\cdot)}(w)$.

**Theorem 5.2.3.** *Let $p \in LH^*$ be a variable exponent satisfying $1 < p_-, p_+ < \infty$ and $w \in CW^{p(\cdot)}$ be a weight. If $\lambda \in \dot{\mathbb{R}}$ and $a \in SO^3_\lambda$, then $a \in M_{p(\cdot),w}$ and*

$$\|a\|_{M_{p(\cdot),w}} \le K\|a\|_{SO^3_\lambda},$$

*where $K$ is a positive constant independent of $a$.*

*Proof.* Since $w \in CW^{p(\cdot)}$, the Cauchy singular integral operator $S$ is bounded on $L^{p(\cdot)}(w)$ by Theorem 4.3.2. Then [25, Theorem 1.3] asserts that the weight $w$ belongs to the class $\mathcal{A}_{p(\cdot)}(\mathbb{R})$. This means that

$$\sup_{-\infty<a<b<\infty} \frac{1}{b-a} \left\|\chi_{]a,b[}\right\|_{L^{p(\cdot)}(w)} \left\|\chi_{]a,b[}\right\|_{L^{p'(\cdot)}(w^{-1})} < \infty, \tag{5.13}$$

where

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad x \in \mathbb{R}.$$

Substituting $p'$ for $p$ and $1/w$ for $w$ in condition (5.13) (noting that $(p')' = p$), we obtain $1/w \in \mathcal{A}_{p'(\cdot)}(\mathbb{R})$. We have proved that

$$w \in \mathcal{A}_{p(\cdot)}(\mathbb{R}) \text{ and } 1/w \in \mathcal{A}_{p'(\cdot)}(\mathbb{R}).$$

By Theorem 5.2.2 we deduce that the operator $M$ is bounded both on $L^{p(\cdot)}(w)$ and $L^{p'(\cdot)}(1/w)$. Now Theorem 3.3.2 tells us that the associate space of $L^{p(\cdot)}(w)$ is isomorphic to $L^{p'(\cdot)}(1/w)$, hence $M$ is bounded both on $L^{p(\cdot)}(w)$ and on its associate space. Then Theorem 5.2.1 gives us the desired result. □

The last result allows us to define $SO_{\lambda,p(\cdot),w}$ as

$$SO_{\lambda,p(\cdot),w} := \operatorname{clos}_{M_{p(\cdot),w}} SO^3_\lambda.$$

Furthermore, we denote by $SO^\diamond_{p(\cdot),w}$ the smallest Banach subalgebra of $M_{p(\cdot),w}$ that contains all the algebras $SO_{\lambda,p(\cdot),w}$ for $\lambda \in \dot{\mathbb{R}}$. The functions in $SO^\diamond_{p(\cdot),w}$ are called *slowly oscillating Fourier multipliers*.

**Theorem 5.2.4.** *Let $p \in LH^*$ be a variable exponent satisfying $1 < p_-, p_+ < \infty$ and $w \in CW^{p(\cdot)}$. Then*

$$C_{p(\cdot),w}(\dot{\mathbb{R}}) \subset SO^\diamond_{p(\cdot),w}.$$

*Proof.* Fix $a \in \mathbb{C} + C_c^\infty(\mathbb{R})$. Then $a = \alpha + c$, where $\alpha \in \mathbb{C}$ and $c \in C_c^\infty(\mathbb{R})$. Since $c$ has compact support, it vanishes for large enough $x$ and we have

$$\operatorname{osc}(a, [-x,-x/2] \cup [x/2,x]) = \operatorname*{ess\,sup}_{s,t\in[-x,-x/2]\cup[x/2,x]} |a(s)-a(t)| = \operatorname*{ess\,sup}_{s,t\in[-x,-x/2]\cup[x/2,x]} |\alpha-\alpha| = 0,$$

hence $a \in SO_\infty$ by definition. Again because $c$ has compact support, for large enough $x$ its derivative vanishes and we have

$$(D_\infty a)(x) = x\frac{dc}{dx}(x) = 0.$$

This proves that $\lim_{x\to\infty}(D_\infty a)(x) = 0$, and analogously that

$$\lim_{x\to\infty}(D_\infty^2 a)(x) = \lim_{x\to\infty}(D_\infty^3 a)(x) = 0.$$

Hence $a \in SO_\infty^3$ We have proved that $\mathbb{C}+C_c^\infty(\mathbb{R}) \subset SO_\infty^3$, and by Theorem 5.1.3 we conclude that

$$C_{p(\cdot),w}(\dot{\mathbb{R}}) = \operatorname{clos}_{M_{p(\cdot),w}}(\mathbb{C}+C_c^\infty(\mathbb{R})) \subset \operatorname{clos}_{M_{p(\cdot),w}} SO_\infty^3 = SO_{\infty,p(\cdot),w} \subset SO^\diamond_{p(\cdot),w}. \qquad \square$$

## 5.3 Algebra $\mathcal{SO}_{p(\cdot),w}$ of Convolution Type Operators with Slowly Oscillating Data

Now consider the algebra

$$\mathcal{C}_{p(\cdot),w} := \operatorname{alg}_{\mathcal{B}(L^{p(\cdot)}(w))}\{aI, W_b^0 : a \in C(\dot{\mathbb{R}}),\ b \in C_{p(\cdot),w}(\dot{\mathbb{R}})\}.$$

This is the smallest closed subalgebra of $\mathcal{B}(L^{p(\cdot)}(w))$ that contains the operators $aI$ of multiplication by functions $a \in C(\dot{\mathbb{R}})$ and the operators $W_b^0$ with $b \in C_{p(\cdot),w}(\dot{\mathbb{R}})$.

Similarly, define the algebra $\mathcal{SO}_{p(\cdot),w}$ as

$$\mathcal{SO}_{p(\cdot),w} := \operatorname{alg}_{\mathcal{B}(L^{p(\cdot)}(w))}\{aI, W_b^0 : a \in SO^\diamond,\ b \in SO^\diamond_{p(\cdot),w}\}.$$

The algebras $\mathcal{C}_{p(\cdot),w}$ and $\mathcal{SO}_{p(\cdot),w}$ are called the algebras of convolution type operators with continuous and slowly oscillating data, respectively.

### 5.3.1 Algebras of Convolution Type Operators with Continuous and Slowly Oscillating Data Contain the Ideal of Compact Operators

**Theorem 5.3.1.** *Let $p \in LH^*$ be a variable exponent satisfying $1 < p_-, p_+ < \infty$ and $w \in CW^{p(\cdot)}$ be a weight. Then*

$$\mathcal{K}(L^{p(\cdot)}(w)) \subset \mathcal{C}_{p(\cdot),w} \subset \mathcal{SO}_{p(\cdot),w}.$$

*Proof.* Since $w \in CW^{p(\cdot)}$, the Cauchy singular integral operator $S$ is bounded on $L^{p(\cdot)}(w)$ by Theorem 4.3.2. Then [25, Theorem 1.3] asserts that the weight $w$ belongs to the class $\mathcal{A}_{p(\cdot)}(\mathbb{R})$. This means that

$$\sup_{-\infty<a<b<\infty} \frac{1}{b-a}\left\|\chi_{]a,b[}\right\|_{L^{p(\cdot)}(w)}\left\|\chi_{]a,b[}\right\|_{L^{p'(\cdot)}(w^{-1})} < \infty, \tag{5.14}$$

where

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad x \in \mathbb{R}.$$

Substituting $p'$ for $p$ and $1/w$ for $w$ in condition (5.13) (noting that $(p')' = p$), we obtain $1/w \in \mathcal{A}_{p'(\cdot)}(\mathbb{R})$. We have proved that

$$w \in \mathcal{A}_{p(\cdot)}(\mathbb{R}) \text{ and } 1/w \in \mathcal{A}_{p'(\cdot)}(\mathbb{R}).$$

By Theorem 5.2.2 we deduce that the Hardy-Littlewood maximal operator is bounded both on $L^{p(\cdot)}(w)$ and $L^{p'(\cdot)}(1/w)$. Now Theorem 3.3.2 tells us that the associate space of $L^{p(\cdot)}(w)$ is isomorphic to $L^{p'(\cdot)}(1/w)$, hence the Hardy-Littlewood maximal operator is bounded both on $L^{p(\cdot)}(w)$ and on its associate space and we can use [14, Theorem 1.1] to conclude that

$$\mathcal{K}(L^{p(\cdot)}(w)) \subset \mathrm{alg}_{\mathcal{B}(L^{p(\cdot)}(w))}\{aI, W_b^0 : a \in C(\dot{\mathbb{R}}),\ b \in C_{p(\cdot),w}(\dot{\mathbb{R}})\} = \mathcal{C}_{p(\cdot),w}.$$

Now let us prove that $C(\dot{\mathbb{R}}) \subset SO_\infty$. Note that any function $a \in C(\dot{\mathbb{R}})$ is continuous on the compact set $\dot{\mathbb{R}}$, hence it is bounded. Let us prove that

$$\lim_{x\to\infty} \mathrm{osc}(a, [-x, -x/2] \cup [x/2, x]) = 0.$$

Fix $\varepsilon > 0$. Denote by $a(\infty)$ the finite limits $\lim\limits_{t\to+\infty} a(t) = \lim\limits_{t\to-\infty} a(t)$. By definition there exist constants $N$ and $M$ such that

$$|a(x) - a(\infty)| < \frac{\varepsilon}{4} \text{ for all } x < N \text{ or } x > M.$$

Take $K = 2\max\{|M|, |N|\}$, then for $x > K$ and $s, t \in [-x, -x/2] \cup [x/2, x]$ we have

$$|a(s) - a(t)| = |a(s) - a(\infty) + a(\infty) - a(t)| \le |a(s) - a(\infty)| + |a(t) - a(\infty)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

This proves that

$$\mathrm{osc}(a, [-x, -x/2] \cup [x/2, x]) \le \frac{\varepsilon}{2} < \varepsilon,$$

hence $a \in SO_\infty$.

Since $C(\dot{\mathbb{R}}) \subset SO_\infty \subset SO^\diamond$ and $C_{p(\cdot),w}(\dot{\mathbb{R}}) \subset SO^\diamond_{p(\cdot),w}$ by Theorem 5.2.4, we conclude that

$$\mathcal{C}_{p(\cdot),w} \subset \mathcal{SO}_{p(\cdot),w}.$$ □

This result ensures that the quotient algebras

$$\mathcal{C}^\pi_{p(\cdot),w} := \mathcal{C}_{p(\cdot),w} / \mathcal{K}(L^{p(\cdot)}(w)),$$
$$\mathcal{SO}^\pi_{p(\cdot),w} := \mathcal{SO}_{p(\cdot),w} / \mathcal{K}(L^{p(\cdot)}(w)),$$

are well defined and that $\mathcal{C}^\pi_{p(\cdot),w} \subset \mathcal{SO}^\pi_{p(\cdot),w}$.

### 5.3.2 Commutativity of the Image of $\mathcal{SO}_{p(\cdot),w}$ in the Calkin Algebra

From [28, Theorem 4.6] we extract the following result.

**Theorem 5.3.2.** *Let $1 < p < +\infty$ be a constant exponent. If $a \in SO^\diamond$ and $b \in SO^\diamond_p$, then the commutator $aW_b^0 - W_b^0 aI$ is compact on the standard Lebesgue space $L^p(\mathbb{R})$.*

Now we are in a position to prove the main result of this chapter.

**Theorem 5.3.3.** *Let $p \in LH^*$ be a variable exponent satisfying $1 < p_-, p_+ < \infty$ and $w \in CW^{p(\cdot)}$ be a weight. Let $a \in SO^\diamond$ and $b \in SO^\diamond_{p(\cdot),w}$. Then the commutator $aW_b^0 - W_b^0 aI$ is compact as an operator $L^{p(\cdot)}(w) \to L^{p(\cdot)}(w)$.*

*Proof.* First of all, we use Theorem 4.3.7 to obtain the decomposition

$$\frac{1}{p(x)} = \frac{1-\theta}{p_0(x)} + \frac{\theta}{2}, \quad w(x) = w_0(x)^{1-\theta} w_1(x)^{\theta},$$

where $0 < \theta < 1$, $w_0(x) = w(x)^{1/(1-\theta)}$, $w_1(x) = 1$ and $p_0 \in LH^*$ satisfies $1 < (p_0)_-, (p_0)_+ < \infty$. By definition of the algebra $SO^{\diamond}_{p(\cdot),w}$, there exists a sequence $(b_n)$, where each $b_n$ is of the form

$$b_n = \sum_{i=1}^{k} \prod_{j=1}^{m} b_{ij},$$

with $b_{ij} \in \bigcup_{\lambda \in \dot{\mathbb{R}}} SO^3_\lambda$, such that

$$\|b_n - b\|_{M_{p(\cdot),w}} = \left\| W^0_{b_n} - W^0_b \right\|_{\mathcal{B}(L^{p(\cdot)}(w))} \to 0.$$

Note that the set $\bigcup_{\lambda \in \dot{\mathbb{R}}} SO^3_\lambda$ is independent of the exponent $p$, which implies that the functions $b_n$ are in $SO^{\diamond}_{2,w}$. By the previous theorem, the operators $aW^0_{b_n} - W^0_{b_n} aI$ are compact as operators $L^2(\mathbb{R}) \to L^2(\mathbb{R})$, thus by Theorem 3.4.6 they are also compact as operators $L^{p(\cdot)}(w) \to L^{p(\cdot)}(w)$. This sequence is convergent in the $M_{p(\cdot),w}$ norm to the operator $aW^0_b - W^0 aI$, which is compact as a consequence of Theorem 2.2.2. □

**Theorem 5.3.4.** *Let $p \in LH^*$ be a variable exponent satisfying $1 < p_-, p_+ < \infty$ and $w \in CW^{p(\cdot)}$ be a weight. Then the quotient algebra*

$$\mathcal{SO}^\pi_{p(\cdot),w} = \mathcal{SO}_{p(\cdot),w} / \mathcal{K}(L^{p(\cdot)}(w))$$

*is commutative.*

*Proof.* Take $R + \mathcal{K}(L^{p(\cdot)}(w))$, $T + \mathcal{K}(L^{p(\cdot)}(w)) \in \mathcal{SO}^\pi_{p(\cdot),w}$. Then $R, T$ are limits (in the $\mathcal{B}\left(L^{p(\cdot)}(w)\right)$ norm) of finite sums of finite products of elements of the set $\{aI, W^0_b : a \in SO^\diamond,\ b \in SO^\diamond_{p(\cdot),w}\}$. Since limits, sums and products of commuting elements also commute, we can assume $R$ and $T$ are elements of the latter set. If $R = a_1 I, T = a_2 I, a_1, a_2 \in SO^\diamond$ then

$$RT = a_1 I a_2 I = (a_1 a_2) I = a_2 I a_1 I = TR.$$

If $R = W^0_{b_1}, T = W^0_{b_2}, b_1, b_2 \in SO^\diamond_{p(\cdot),w}$ then

$$RT = W^0_{b_1} W^0_{b_2} = W^0_{b_1 b_2} = W^0_{b_2} W^0_{b_1} = TR.$$

Now if $R = aI, T = W^0_b, a \in SO^\diamond, b \in SO^\diamond_{p(\cdot),w}$, by Theorem 5.3.3 we have $RT - TR \in \mathcal{K}(L^{p(\cdot)}(w))$, which is equivalent to

$$RT + \mathcal{K}(L^{p(\cdot)}(w)) = TR + \mathcal{K}(L^{p(\cdot)}(w))$$

in the quotient algebra. □

Together with the inclusion $\mathcal{C}^\pi_{p(\cdot),w} \subset \mathcal{SO}^\pi_{p(\cdot),w}$, this result implies the following corollary.

**Corollary 5.3.5.** *Let $p \in LH^*$ be a variable exponent satisfying $1 < p_-, p_+ < \infty$ and $w \in CW^{p(\cdot)}$ be a weight. Then the quotient algebra*

$$\mathcal{C}^{\pi}_{p(\cdot),w} := \mathcal{C}_{p(\cdot),w}/\mathcal{K}(^{p(\cdot)}(w))$$

*is commutative.*

## 5.4 Algebra $\mathcal{PSO}^{\pi}_{p(\cdot),w}$ and its Commutative Subalgebra $\mathcal{SO}^{\pi}_{p(\cdot),w}$

Given a variable exponent $p \in LH^*$ satisfying $1 < p_-, p_+ < \infty$ and a weight $w \in CW^{p(\cdot)}$, recall that the algebra $PC_{p(\cdot),w}$ is defined as the closure of the set of functions with bounded variation with respect to the Fourier multiplier norm

$$\|a\|_{M_{p(\cdot),w}} = \left\|W^0_a\right\|_{\mathcal{B}(L^{p(\cdot)}(w))}.$$

This algebra is well defined by Theorem 4.3.10.

Denote by $PSO^{\diamond}$ the smallest Banach subalgebra of $L^{\infty}(\mathbb{R})$ generated by the algebras $SO^{\diamond}$ and $PC(\mathbb{R})$. The latter algebra is defined in Section 2.3.1.

Now we define the algebra of piecewise slowly oscillating Fourier multipliers as

$$PSO^{\diamond}_{p(\cdot),w} := \operatorname{alg}_{M_{p(\cdot),w}}\left\{PC_{p(\cdot),w}, SO^{\diamond}_{p(\cdot),w}\right\}.$$

Consider as well the algebra $\mathcal{PSO}_{p(\cdot),w}$ of convolution type operators with piecewise slowly oscillating data defined by

$$\mathcal{PSO}_{p(\cdot),w} := \operatorname{alg}_{\mathcal{B}(L^{p(\cdot)}(w))}\{aI, W^0_b : a \in PSO^{\diamond},\ b \in PSO^{\diamond}_{p(\cdot),w}\},$$

and the quotient algebra $\mathcal{PSO}^{\pi}_{p(\cdot),w}$ defined by

$$\mathcal{PSO}^{\pi}_{p(\cdot),w} = {}^{\mathcal{PSO}_{p(\cdot),w}}\!/\!_{\mathcal{K}(L^{p(\cdot)}(w))}.$$

It is evident that $SO^{\diamond}_{p(\cdot),w} \subset PSO^{\diamond}_{p(\cdot),w}$, and by Theorem 5.3.1 we have $\mathcal{K}(L^{p(\cdot)}(w)) \subset \mathcal{SO}_{p(\cdot),w}$. These two facts imply that

$$\mathcal{K}(L^{p(\cdot)}(w)) \subset \mathcal{SO}_{p(\cdot),w} \subset \mathcal{PSO}_{p(\cdot),w},$$

hence the quotient algebras are well defined and

$$\mathcal{SO}^{\pi}_{p(\cdot),w} \subset \mathcal{PSO}^{\pi}_{p(\cdot),w}.$$

Using this inclusion and Theorem 5.3.4, we obtain the following result.

**Theorem 5.4.1.** *Let $p \in LH^*$ be a variable exponent satisfying $1 < p_-, p_+ < \infty$ and $w \in CW^{p(\cdot)}$ be a weight. Then $\mathcal{SO}^{\pi}_{p(\cdot),w}$ is a commutative subalgebra of $\mathcal{PSO}^{\pi}_{p(\cdot),w}$.*

We believe that this result will serve as an important step in the further study of invertibility in the algebra $\mathcal{PSO}^{\pi}_{p(\cdot),w}$ by means of the Allan local principle [1] analogously to [26], [27], [28], where the invertibility problem in the algebra $\mathcal{PSO}^{\pi}_{p,w}$ was solved for constant $p \in ]1,\infty[$ and so-called Muckenhoupt weights $w$. The further study will require a description of the maximal ideal space (see, e.g., [16] for the definition of this concept) of the commutative Banach algebra $\mathcal{SO}^{\pi}_{p(\cdot),w}$, which is still not available.

# Bibliography


[1] G. R. Allan. "Ideals of Vector-Valued Functions". In: *Proceedings of the London Mathematical Society* s3-18.2 (1968), pp. 193–216. DOI: 10.1112/plms/s3-18.2.193.

[2] Sheldon Axler. *Measure, Integration & Real Analysis*. Springer, 2020. DOI: 10.1007/978-3-030-33143-6.

[3] M. A. Bastos, C. A. Fernandes, and Yu. I. Karlovich. "$C^*$-algebras of integral operators with piecewise slowly oscillating coefficients and shifts acting freely". In: *Integral Equations Operator Theory* 55.1 (2006), pp. 19–67. DOI: 10.1007/s00020-005-1377-1.

[4] Colin Bennett and Robert Sharpley. *Interpolation of Operators*. Academic Press, Inc, 1988. URL: https://www.elsevier.com/books/interpolation-of-operators/bennett/978-0-12-088730-9.

[5] Jöran Bergh and Jörgen Löfström. *Interpolation Spaces*. Springer, 1976. DOI: 10.1007/978-3-642-66451-9.

[6] Nicolas Bourbaki. *Functions of a Real Variable*. Springer, 2004. DOI: 10.1007/978-3-642-59315-4.

[7] Haim Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, 2011. DOI: 10.1007/978-0-387-70914-7.

[8] Yu. A. Brudnyi and N. Ya. Krugljak. *Interpolation Functors and Interpolation Spaces*. Vol. 1. Elsevier Science Publishers, 1991. URL: https://www.elsevier.com/books/interpolation-functors-and-interpolation-spaces/brudnyi/978-0-444-88001-7.

[9] Alberto P. Calderón. "Intermediate spaces and interpolation, the complex method". In: *Studia Mathematica* 24 (1964), pp. 113–190. URL: http://eudml.org/doc/217085.

[10] Fernando Cobos, Thomas Kühn, and Tomás Schonbek. "One-sided compactness results for Aronszajn-Gagliardo functors". In: *Journal of Functional Analysis* 106 (1992), pp. 274–313. DOI: 10.1016/0022-1236(92)90049-0.

[11] David Cruz-Uribe, Lars Diening, and Peter Hästo. “The maximal operator on weighted variable Lebesgue spaces”. In: *Fractional Calculus and Applied Analysis* 14 (2011), pp. 361–374. DOI: 10.2478/s13540-011-0023-7.

[12] David V. Cruz-Uribe and Alberto Fiorenza. *Variable Lebesgue Spaces*. Birkhäuser, 2013. DOI: 10.1007/978-3-0348-0548-3.

[13] Roland Duduchava. *Integral Equations with Fixed Singularities*. Teubner, 1979.

[14] Cláudio A. Fernandes, Alexei Yu. Karlovich, and Yuri I. Karlovich. “Algebra of convolution type operators with continuous data on Banach function spaces”. In: *Function spaces XII. Selected papers based on the presentations at the 12th conference, Krakow, Poland, July 9–14, 2018*. Warsaw: Polish Academy of Sciences, Institute of Mathematics, 2019, pp. 157–171. DOI: 10.4064/bc119-8.

[15] Gerald B. Folland. *Real Analysis*. John Wiley & Sons, Inc., New York, 1999.

[16] Israel Gelfand, Dmitri Raikov, and George Shilov. *Commutative Normed Rings*. Chelsea Publishing Company, 1964.

[17] Israel Gohberg and Nahum Krupnik. *One-Dimensional Linear Singular Integral Equations*. Vol. I. Birkhäuser, 1992. DOI: 10.1007/978-3-0348-8647-5.

[18] Israel Gohberg and Nahum Krupnik. “On a local principle and algebras generated by Toeplitz matrices”. In: *Operator Theory: Advances and Applications*. Vol. 206. Birkhäuser, 2010, pp. 157–184. DOI: 10.1007/978-3-7643-8956-7_11.

[19] Loukas Grafakos. *Classical Fourier Analysis*. Springer, 2014. DOI: 10.1007/978-1-4939-1194-3.

[20] Lars Hörmander. “Estimates for translation invariant operators in $L^p$ spaces”. In: *Acta Mathematica* 104 (1960), pp. 93–140. DOI: 10.1007/BF02547187.

[21] Alexei Karlovich. “Banach algebra of the Fourier multipliers on weighted Banach function spaces”. In: *Concrete Operators* 2 (2015), pp. 27–36. DOI: 10.1515/conop-2015-0001.

[22] Alexei Karlovich. “Commutators of convolution type operators on some Banach function spaces”. In: *Annals of Functional Analysis* 6 (2015), pp. 191–205. DOI: 10.15352/afa/06-4-191.

[23] Alexei Karlovich. “Algebras of continuous Fourier multipliers on variable Lebesgue spaces”. In: *Mediterranean Journal of Mathematics* 17.102 (2020). DOI: 10.1007/s00009-020-01537-z.

[24] Alexei Karlovich and Eugene Shargorodsky. “When does the norm of a Fourier multiplier dominate its $L^\infty$ norm?” In: *Proceedings of the London Mathematical Society* 118.4 (2018), pp. 901–941. DOI: 10.1112/plms.12206.

[25] Alexei Karlovich and Ilya Spitkovsky. “The Cauchy singular integral operator on weighted variable Lebesgue spaces”. In: *Operator Theory: Advances and Applications*. Vol. 236. Birkhäuser, 2014, pp. 275–291. DOI: 10.1007/978-3-0348-0648-0_17.

[26] Yuri Karlovich. "Algebras of convolution-type operators with piecewise slowly oscillating data on weighted Lebesgue spaces". In: *Mediterranean Journal of Mathematics* 14.182 (2017). DOI: 10.1007/s00009-017-0979-6.

[27] Yuri Karlovich and Iván Loreto Hernández. "Algebras of convolution type operators with piecewise slowly oscillating data. II: Local spectra and Fredholmness". In: *Integral Equations and Operator Theory* 75 (2013), pp. 49–86. DOI: 10.1007/s00020-012-2003-7.

[28] Yuri Karlovich and Iván Loreto Hernández. "On convolution type operators with piecewise slowly oscillating data". In: *Operator Theory: Advances and Applications*. Vol. 228. Birkhäuser, 2013, pp. 185–207. DOI: 10.1007/978-3-0348-0537-7_10.

[29] Vakhtang Kokilashvili, Vakhtang Paatashvili, and Stefan Samko. "Boundedness in Lebesgue spaces with variable exponent of the Cauchy singular Operator on Carleson curves". In: *Operator Theory: Advances and Applications*. Vol. 170. Birkhäuser, 2007, pp. 167–186. DOI: 10.1007/978-3-7643-7737-3_10.

[30] M.A. Krasnosel'skii et al. *Integral Operators in Spaces of Summable Functions*. Noordhoff International Publishing, 1976. URL: springer.com/gp/book/9789401015448.

[31] S. G. Krein, J. I. Petunin, and E. M. Semenov. *Interpolation of Linear Operators*. American Mathematical Society, 1982. URL: https://bookstore.ams.org/mmono-54.

[32] Lech Maligranda. *Orlicz Spaces and Interpolation*. Vol. 5. Seminários de Matemática [Seminars in Mathematics]. Universidade Estadual de Campinas, Departamento de Matemática, Campinas, 1989.

[33] Vladimir Rabinovich and Stefan Samko. "Boundedness and Fredholmness of pseudodifferential operators in variable exponent spaces". In: *Integral Equations and Operator Theory* 60.4 (2008), pp. 507–537. DOI: 10.1007/s00020-008-1566-9.

[34] Bryan Rynne and Martin Youngson. *Linear Functional Analysis*. Springer, 2008. DOI: 10.1007/978-1-84800-005-6.