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SOME DOUBLE DIFFUSION MODELS FOR STOCK PRICES

PEDRO P. MOTA, NADEZHDA P. KRASII, AND MANUEL L. ESQUÍVEL

Abstract. Regime switching diffusion processes with one or two thresholds and regime switching occurring by a change in the diffusion drift and/or volatility functions parameters of a stochastic differential equation, whose solution defines a continuous time diffusion process, were defined in previous works; the change in regime occurring whenever the trajectory of the process crosses a threshold, possibly with some delay. In this paper we generalise the previous results by allowing the underlying diffusion process to change from one family of diffusions in one regime to an entirely different one in the other regime; these families of diffusions are characterised by specific functional forms for drift and volatility coefficients depending on parameters. We propose an estimation procedure for all the parameters, namely the thresholds, the delay and, for both regimes, diffusion's parameters and we apply the introduced estimation procedure to both simulated and real data.

1. Introduction

The theory of general stochastic differential equations with regime switching has had many very interesting developments both in the case of externally induced regime switching — by means of an independent Markov process inducing the regime switching — and in the case of auto-induced regime switching where the regime switches when the trajectories hit some threshold and possibly after some delay (see [EKM20] for a recent partial synthesis essay).

In this paper, on auto-induced regime switching diffusions, we consider two families of stochastic processes, each one modelled by a double stochastic differential equation defining, in each case, a continuous time stochastic process with two regimes. For the first double diffusion process definition, we hypothesise that the regimes are induced by two thresholds m and M (with $m < M$) belonging to the state space of both processes. If, for instance, the initial process is a submartingale (with continuous trajectories) starting at x_0 below the upper threshold ($x_0 < M$), then in almost sure (a.s.) finite time it will hit the (upper) threshold M. Let's suppose that at that time the overall process dynamics changes, for instance to a

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supermartingale (again with continuous trajectories and starting at M) and again in a.s. finite time this process will hit the lower threshold m. If the process dynamics change again to the initial submartingale process (now with initial value m) and everything starts all over again, we get our first double diffusion process. For the second double diffusion process, we only need to hypothesise one given threshold m together with a positive delay d. We will also consider that both regimes have some kind of opposite trend and that , at each time t, the process is in the first regime if, at time $t-d$ the process is below the threshold or that the process is in the second regime if at time $t-d$ the process is above the threshold. Both double diffusion process were already presented and discussed in some way in [EM14] and [ME14], but in the much more restrictive situation where, apart from parameter changes, the functional form of the coefficients of the diffusion process were the same in both regimes. In the present work, we allow the underlying diffusion processes to be from different families, for instance, in one regime we can consider a Brownian motion with drift (BMD) and in the other regime a geometric Brownian motion (GBM), meaning that in each regime the process follows the dynamics of a simple continuous time process defined by a stochastic differential equation (SDE). The change in regime happens when the process hits a threshold — crosses a threshold with delay in the second case — and when that happens the process dynamics changes from one to the other SDE with different *drift* and *volatility* functions.

2. The double diffusion processes

Consider a parameter set $\mathbf{\Theta} = \mathbf{\Theta}_1 \times \mathbf{\Theta}_2$, some delay parameter $d > 0$ and two thresholds m and M (with $m < M$), defined in the state space of two real valued stochastic processes $(X_{1,t})_{t\geq 0}$ and $(X_{2,t})_{t\geq 0}$, with the diffusions processes driven in $[0,T]$ by the stochastic differential equations:

$$\begin{cases} dX_{1,t} = \mu_1(t, X_{1,t}; \theta_1)dt + \sigma_1(t, X_{1,t}; \theta_1)dW_t, \quad t \in [0,T], \quad \theta_1 \in \mathbf{\Theta}_1 \\ X_{1,0} = x_{1,0} \end{cases} \tag{2.1}$$

and

$$\begin{cases} dX_{2,t} = \mu_2(t, X_{2,t}; \theta_2)dt + \sigma_2(t, X_{2,t}; \theta_2)dW_t, \quad t \in [0,T], \quad \theta_2 \in \mathbf{\Theta}_2 \\ X_{2,0} = x_{2,0} \end{cases} \tag{2.2}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion process defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We can ensure the existence and uniqueness of $(X_{1,t})_{t\geq 0}$ and $(X_{2,t})_{t\geq 0}$ as solutions of the SDE's in (2.1) and (2.2), if the usual regularity conditions are verified, see for instance [KS98, p. 289], [Øks03, p. 68] or [LS01, p. 134]. Other more general conditions for existence and unicity of solutions may be considered (see, for instance [EKM20]) but are not needed in this work.

2.1. Double diffusion process with two thresholds. It is possible to define a new process $(X_t)_{t\geq 0}$ on the interval $[0,T]$, by joining the trajectories of the diffusions $X_{1,t}$ and $X_{2,t}$, defined by the sequence of the stopping times corresponding to the hitting of the thresholds m and M by the process trajectories. In a more detailed way and without loss of generality, let us assume that the process is initiated in the first regime (corresponding to the diffusion solution of equation (2.1)), that is, $(X_t)_{t\geq 0} \equiv (X_{1,t})_{t\geq 0}$ and we suppose that $x_{1,0} < M$. Consider that there exists a hitting time for the upper threshold M (the first hitting time) defined by:

$$\tau_1 = \inf\{t > 0\ : X_{1,t} = M\} \wedge T\ . \tag{2.3}$$

Suppose that $\tau_1 < T$ with non-zero probability (if not, the process is just an a.s. single regime process), then the process $(X_t)_{t\geq 0}$ is given for $t \in [0, \tau_1]$ by $X_t = X_{1,t}$. Next, for $t \geq \tau_1$, assume that the process dynamics is the one of the SDE (2.2) with initial condition $X_{2,\tau_1} = M$, that is, $X_t = X_{2,t}$ for $t \geq \tau_1$ and with $X_{2,t}$ given by:

$$\begin{cases} dX_{2,t} = \mu_2(t, X_{2,t}; \theta_2)dt + \sigma_2(t, X_{2,t}; \theta_2)dW_t, \quad t \in [\tau_1, T] \\ X_{2,\tau_1} = M\ . \end{cases} \tag{2.4}$$

Consider next, the second regime switching moment defined by the first hitting time (after τ_1) of the lower threshold m,

$$\tau_2 = \inf\{t > \tau_1\ : X_{2,t} = m\} \wedge T\ . \tag{2.5}$$

Let us suppose again that $\tau_2 < T$ with non-zero probability, then it is obvious that by time τ_2 the process is at level m, and a new regime will start with the dynamics of the SDE (2.1). It is now possible to redefine the process $(X_t)_{t\geq 0}$ for $t \geq \tau_2$ by $X_t = X_{1,t}$ given by the solution of:

$$\begin{cases} dX_{1,t} = \mu_1(t, X_{1,t}; \theta_1)dt + \sigma_1(t, X_{1,t}; \theta_1)dW_t, \quad t \in [\tau_2, T] \\ X_{1,\tau_2} = m\ . \end{cases} \tag{2.6}$$

The existence and uniqueness of solutions for the SDE's (2.4) and (2.6) are given in the following proposition.

Proposition 2.1. *For $i = 1, 2$ and with $\nu_1 = m$, $\nu_2 = M$, $\tau(i) = \tau_2$ if $i = 1$ and $\tau(i) = \tau_1$ if $i = 2$, the processes defined by,*

$$\begin{cases} dX_{i,t} = \mu_i(t, X_{i,t}; \theta_i)dt + \sigma_i(t, X_{i,t}; \theta_i)dW_t, \quad t \in [\tau(i), T], \quad \theta_i \in \mathbf{\Theta}_i \\ X_{i,\tau(i)} = \nu_i\ , \end{cases} \tag{2.7}$$

exist and are uniquely determined if the usual conditions for the existence and uniqueness of solution are verified by the diffusion functions ($\mu_i(.)$ and $\sigma_i(.)$, $i = 1, 2$) of the next SDE's

$$\begin{cases} dX_{i,t} = \mu_i(t, X_{i,t}; \theta_i)dt + \sigma_i(t, X_{i,t}; \theta_i)dW_t, \quad t \in [t_0, T], \quad \theta_i \in \mathbf{\Theta}_i \\ X_{i,t_0} = Z_i\ , \end{cases} \tag{2.8}$$

with some initial conditions $Z_i \in L^2$, ensuring that an uniquely determined solution exists, for $i = 1, 2$ and any $t_0 \in [0, T]$.

Proof. We observe that, for $i = 1, 2$, equation (2.7) should be interpreted, for all $t \in [0, T]$, as

$$X_{i,t} = \nu_i + \int_0^t \mu_i(t, X_{i,t}; \theta_i)\mathbb{I}_{[\tau(i),T]}(u)du + \int_0^t \sigma_i(t, X_{i,t}; \theta_i)\mathbb{I}_{[\tau(i),T]}(u)dW_u\ . \tag{2.9}$$

Considering some standard theorem of existence and uniqueness (again, see [KS98, p. 289], [Øks03, p. 68] or [LS01, p. 134]), and since $\nu_1 = m$ and $\nu_2 = M$, it is clear that the initial random variable in (2.8) is in L^2. Moreover, as

$$|\mu_i(t, x; \theta_i)\mathbb{I}_{[\tau(i),T]}(t)| \leq |\mu_i(t, x; \theta_i)|$$

and

$$|\sigma_i(t, x; \theta_i)\mathbb{I}_{[\tau(i),T]}(t)| \leq |\mu_i(t, x; \theta_i)|\ ,$$

it is also clear that the conditions of integrability, of Lipschitz control and of sub-linear growth verified by the diffusion functions on (2.8) are still verified by the diffusion functions of the stochastic differential equation (2.7) and so the result is proved. □

It is possible to define $(\tau_n)_{n\geq 1}$ by induction, thus obtaining an increasing (in the strict sense) sequence of stopping times corresponding to the consecutive regime switching times. In [EM14] it was already proved that this stopping times are almost sure isolated when the same diffusion is considered in both regimes. When the diffusion differs from regime to regime a similar proof still holds.

Finally, we have the following theorem.

Theorem 2.2. *Consider the set $\mathbf{\Theta} = \mathbf{\Theta}_1 \times \mathbf{\Theta}_2$ for the two regime parameters and for $i = 1, 2$ let $\mu_i(t, x; \theta_i)$ and $\sigma_i(t, x; \theta_i)$ be real valued functions defined on $[0, T] \times \mathbb{R} \times \mathbf{\Theta}_i$ in such a way that for any random variable $Z_i \in L^2$, $\theta_i \in \mathbf{\Theta}_i$ and $t_0 \in [0, T]$, the usual conditions of integrability, Lipschitz control and sub-linear growth are verified by the coefficients of the stochastic differential equation*

$$\begin{cases} dX_{i,t} = \mu_i(t, X_{i,t}; \theta_i)dt + \sigma_i(t, X_{i,t}; \theta_i)dW_t, & t \in [t_0, T], \quad \theta_i \in \mathbf{\Theta}_i \\ X_{i,t_0} = Z_i \ , \end{cases}$$

ensuring the existence and the uniqueness of the solution. For $i \geq 0$, define

$$\widehat{i} := \frac{1 - (-1)^i}{2} + 1 \ .$$

Then, there exists an increasing sequence of stopping times $(\tau_i)_{i\geq 0}$, (almost surely isolated), with $\tau_0 \equiv 0$, $\tau_0 < \tau_i < \tau_{i+1} \leq T$ for $i \geq 1$, and such that the stochastic integral equation

$$\begin{aligned} X_t = &\left(\sum_{i=0}^{+\infty} X_{\widehat{i},\tau_i} \mathbb{I}_{[\tau_i, \tau_{i+1}[}(t) \right) + \int_0^t \left(\sum_{i=0}^{+\infty} \mu_{\widehat{i}} \left(t, X_{\widehat{i},u}; \theta_{\widehat{i}} \right) \mathbb{I}_{[\tau_i, \tau_{i+1}[}(u) \right) du + \\ &+ \int_0^t \left(\sum_{i=0}^{+\infty} \sigma_{\widehat{i}} \left(t, X_{\widehat{i},u}; \theta_{\widehat{i}} \right) \mathbb{I}_{[\tau_i, \tau_{i+1}[}(u) \right) dW_u \ , \end{aligned} \tag{2.10}$$

defined with $X_0 = x_{1,0} < M$ for $t \in [0, T]$, has an unique almost surely continuous solution. This solution is a double diffusion process $(X_t)_{t\in[0,T]}$, with regime switching and represented by:

$$X_t = \sum_{i=0}^{+\infty} X_{\widehat{i},t} \mathbb{I}_{[\tau_i, \tau_{i+1}[}(t) \ , \tag{2.11}$$

with the excursion process $(X_{\widehat{i},t})_{t\in[\tau_i,\tau_{i+1}[}$ given by the unique solution of the stochastic differential equation

$$\begin{cases} dX_{\widehat{i},t} = \mu_{\widehat{i}}(t, X_{\widehat{i},t}; \theta_{\widehat{i}})dt + \sigma_{\widehat{i}}(t, X_{\widehat{i},t}; \theta_{\widehat{i}})dW_t, & \tau_i \leq t < \tau_{i+1} \\ X_{\widehat{i},\tau_i} = x_{1,0}\mathbb{I}_{\{i=0\}} + m\mathbb{I}_{\{\widehat{i}=1, i\neq 0\}} + M\mathbb{I}_{\{\widehat{i}=2\}} \ . \end{cases} \tag{2.12}$$

Remark 2.3. If $x_0 > m$ and we suppose that the starting regime is regime 2, similar results can be proved.

Remark 2.4. Given $\mathcal{T} = (\tau_n)_{n\geq 0}$, for $i \geq 0$ by construction $(X_{\widehat{i},t})_{t\in[\tau_i,\tau_{i+1}[}$ the excursion processes, are independent.

2.2. Double diffusion process with one threshold and delay. For the second double diffusion process, we start with one threshold m and a delay d, but in order to overcome the Brownian level sets problem described in [EM14], P. (2014), we consider a small but fixed $\varepsilon > 0$ to define auxiliary thresholds $m-\varepsilon$, $m+\varepsilon$ allowing us to define a threshold band from $m-\varepsilon$ to $m+\varepsilon$. In this context and in a similar way to what was done in the previous subsection, it is possible to define a new process $(X_t)_{t\geq 0}$ on the interval $[0,T]$, by joining the excursions of the diffusions $X_{1,t}$ and $X_{2,t}$ defined by the succession of the hitting moments of the (upper and lower) thresholds $m+\varepsilon$, $m-\varepsilon$ and after some delay d. Again and without loss of generality, let us assume that the process is initiated in the first regime, that is, $(X_t)_{t\geq 0} \equiv (X_{1,t})_{t\geq 0}$ as in equation (2.1), and we suppose that $x_{1,0} < m$. Let us consider that we have a first hitting time of the upper (auxiliary) threshold $m+\varepsilon$ given by:

$$\tau_1 = \inf\{t > 0 \ : X_{1,t} = m+\varepsilon\} \wedge T \ . \tag{2.13}$$

Suppose that we have $\tau_1 + d < T$ (at least, with non-zero probability) and that the process $(X_t)_{t\geq 0}$ is given for $t \in [0, \tau_1 + d]$ by $X_t = X_{1,t}$ and for $t \geq \tau_1 + d$, by the solution of the SDE (2.2) (with a random initial condition depending on a stopping time), that is, $X_t = X_{2,t}$ for $t \geq \tau_1 + d$ and with $X_{2,t}$ given by:

$$\begin{cases} dX_{2,t} = \mu_2(t, X_{2,t}; \theta_2)dt + \sigma_2(t, X_{2,t}; \theta_2)dW_t, \quad t \in [\tau_1 + d, T] \\ X_{2,\tau_1+d} = X_{1,\tau_1+d} \ . \end{cases} \tag{2.14}$$

Next, consider the second regime switching time, defined by the first hitting time (after τ_1) of the lower (auxiliary) threshold $m-\varepsilon$,

$$\tau_2 = \inf\{t > \tau_1 \ : X_{2,t} = m-\varepsilon\} \wedge T \ . \tag{2.15}$$

Let us suppose that $\tau_2 + d < T$ (again with non-zero probability), then by time τ_2 we can find the process at level $m-\varepsilon$ and a regime change happens at time $\tau_2 + d$. In a similar way to what have been done previously, the process $(X_t)_{t\geq 0}$ for $t \geq \tau_2 + d$ may be redefine by $X_t = X_{1,t}$ given by the solution of:

$$\begin{cases} dX_{1,t} = \mu_1(t, X_{1,t}; \theta_1)dt + \sigma_1(t, X_{1,t}; \theta_1)dW_t, \quad t \in [\tau_2 + d, T] \\ X_{1,\tau_2+d} = X_{2,\tau_2+d} \ . \end{cases} \tag{2.16}$$

The existence and uniqueness of solutions for the SDE's (2.14) and (2.16) is proved in the same way as in Proposition 2.1 with the only difference regarding the initial condition, that is, the integrability of the initial random variable. That integrability condition is easy to prove since, for instance:

$$\mathbb{E}\left[|X_{1,\tau_1+d}|^2\right] \leq \mathbb{E}\left[\sup_{0\leq t\leq T} |X_{1,t}|^2\right] < +\infty \ . \tag{2.17}$$

The *double diffusion process with one threshold and delay* may be defined inductively in this way by gluing together solutions to standard stochastic differential equations defined between the stopping times, $(\tau_n + d)_{n\geq 0}$, as in equation (2.11) of theorem 2.2.

Remark 2.5. Notice that, formally the second process can be considered as the first one with delay but in practice and for ε small enough the threshold band will work as a unique threshold and is like if the change in regime is driven by the crossing of the threshold.

3. On the thresholds estimation

For a fixed $\omega \in \Omega$ and the corresponding trajectory of the double diffusion process, consider $C_n(\omega) = \{X_{t_1}(\omega), \ldots, X_{t_{p_n}}(\omega)\}$ a finite set of $p_n \in \mathbb{N}$ discrete observations at times $t_1, ..., t_{p_n}$ in $[0, T]$ and let us assume that for all $n \geq 1$ we have that $C_n(\omega) \subseteq C_{n+1}(\omega)$. We also assume that, for any $\omega \in \Omega$, we know the regime for each observation $X_{t_j}(\omega)$, that knowledge allowing us to define a random variable R_{t_j} such that $R_{t_j}(\omega) = 1$ if $X_{t_j}(\omega)$ is in regime 1 and $R_{t_j}(\omega) = 2$ if $X_{t_j}(\omega)$ is in regime 2.

3.1. Two thresholds. If we are in the context of the *double diffusion process with two thresholds*, then we can define a natural partition of the observations set, given by:

$$C_n^M(\omega) = \{X_{t_j}(\omega) : R_{t_j}(\omega) = 1,\ j = 1, \ldots, p_n\} \tag{3.1}$$

and

$$C_n^m(\omega) = \{X_{t_j}(\omega) : R_{t_j}(\omega) = 2,\ j = 1, \ldots, p_n\}\ . \tag{3.2}$$

Remark 3.1. An important remark is that:

- If $R_{t_j}(\omega) = 1$, meaning that $X_{t_j}(\omega)$ is in regime 1, then $X_{t_j}(\omega) \leq M$;
- If $R_{t_j}(\omega) = 2$, meaning that $X_{t_j}(\omega)$ is in regime 2, then $X_{t_j}(\omega) \geq m$.

The next result shows that, if we have an hypothesis of increasing number of observations, we can define strongly consistent estimators for the thresholds m and M.

Theorem 3.2. *Let $\widehat{M}_n(\omega) = \max C_n^M(\omega)$ and $\widehat{m}_n(\omega) = \min C_n^m(\omega)$. Suppose that the observations are distributed in a regular mesh in $[0, T]$, in the sense that,*

$$\lim_{n \to +\infty} \max_{1 \leq i \leq p_n - 1} |t_{i+1} - t_i| = 0\ . \tag{3.3}$$

Then, if there is at least one regime change from the first regime to the second regime, the sequence $(\widehat{M}_n)_{n \geq 1}$ of estimators of the upper threshold M, is strongly consistent and, if there is at least one regime change from the second regime to the first regime, the sequence $(\widehat{m}_n)_{n \geq 1}$ of estimators of the lower threshold m, is also consistent (strongly), that is:

$$\lim_{n \to +\infty} \widehat{M}_n = M \text{ a.s. and } \lim_{n \to +\infty} \widehat{m}_n = m \text{ a.s.} \tag{3.4}$$

Proof. We only write the proof of the consistency of $\widehat{M}_n(\omega)$, being similar to $\widehat{m}_n(\omega)$. For simplicity, write $\widehat{M}(\omega) := \lim_{n \to +\infty} \widehat{M}_n(\omega)$, as $C_n(\omega) \subseteq C_{n+1}(\omega)$ we have that $\widehat{M}_n \leq \widehat{M}_{n+1}$ and from Remark 3.1, we have that $\widehat{M}(\omega) = \lim_{n \to +\infty} \widehat{M}_n(\omega) \leq M$. If we suppose, that for some $\varepsilon > 0$ we have $\widehat{M}(\omega) < M - \varepsilon$ and we consider the first regime switching time (from regime 1 to regime 2), that is:

$$\tau(\omega) = \inf \{t > 0 : X_t(\omega) = M\}\ ,$$

from the uniform, almost surely, continuity of the trajectory $(X_t(\omega))_{t \in [0,T]}$, there exists $\eta > 0$ such that for $|r - s| < \eta$, with $r, s \in [0, T]$, we have $|X_r(\omega) - X_s(\omega)| < \varepsilon$. Next, it is enough to select some $n_0 \in \mathbb{N}$ for which is valid $\max_{1 \leq i \leq p_{n_0} - 1} |t_{i+1} - t_i| < \eta$ and to select $i_0 \in \{1, \ldots, p_{n_0}\}$ such that $\tau(\omega) \in [t_{i_0}, t_{i_0+1}]$. As a consequence, we have that

$$|\tau(\omega) - t_{i_0}| \leq |t_{i_0+1} - t_{i_0}| < \eta$$

and so

$$|X_{\tau(\omega)}(\omega) - X_{t_{i_0}}(\omega)| = |M - X_{t_{i_0}}(\omega)| < \varepsilon \ ,$$

thus implying that $|\widehat{M}_{n_0}(\omega) - M| < \varepsilon$ and furthermore $|\widehat{M}(\omega) - M| < \varepsilon$, contradicting the initial hypothesis of $\widehat{M}(\omega) < M - \varepsilon$, completing the proof. □

Remark 3.3. When the regimes are not known, we propose a practical procedure to estimate the thresholds and all the other remaining parameters for this kind of process with regime changes. The method is easily described in a few simple steps.

- Chose some values $\widetilde{m}$ and $\widetilde{M}$ for the two thresholds and for this choice split the observations in a set of observation in first regime 1 and other of observations in the second regime.
- Use traditional estimators for the diffusion parameters, in order to get an estimator $\widehat{\theta}_1$ for the parameter value θ_1 (using the observations assigned to the first regime, by the previous step procedure) and use the same approach to get $\widehat{\theta}_2$ (using the observations in the second regime) as an estimator of θ_2.
- Minimize some kind of contrast function, for instance, of the conditional least squares type,

$$\mathrm{CLS}_{\widetilde{m},\widetilde{M}} := \sum_i \left(X_i - \mathbb{E}_{\widehat{\theta}_1,\widehat{\theta}_2,\widetilde{m},\widetilde{M}}\left[X_i \mid X_1, \dots X_{i-1}\right] \right)^2 \ , \tag{3.5}$$

  over $\widetilde{m}$ and $\widetilde{M}$.

3.2. One threshold and delay. If we are in the context of the *double diffusion process with one threshold and delay* and supposing that the delay d parameter is known, we are able to build an estimator for the threshold m. Again, if we know the sequence $R_{t_1}, R_{t_2}, ..., R_{t_{p_n}}$ and we know the delay d, we can divide the observations $X_{t_1}, X_{t_2}, ..., X_{t_{p_n}}$ in two sets using that: $R_{t_j} = 1 \Rightarrow X_{t_j - d} \leq m + \varepsilon$ and $R_{t_j} = 2 \Rightarrow X_{t_j - d} \geq m - \varepsilon$. In practice, we only consider the random variables R_{t_j} such that $t_j \geq d$, in order keep the time $t_j - d$ in the interval $[0, T]$. From the uniform continuity of the process trajectory $(X_t(\omega))_{t\in[0,T]}$ and the fact that the hitting times are (almost sure) isolated, there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$, we have:

$$R_{t_j} = 1 \Rightarrow \exists t_{j_1} \in \{t_1, ..., t_{p_n}\}, \exists k_1 \in \mathbb{N}_0 : t_j - d \in [t_{j_1}, t_{j_1+1}[\wedge t_{j_1} \in]\tau_{2k_1}, \tau_{2k_1+1}] \tag{3.6}$$

and we can define:

$$\widetilde{X}_j^-(\omega) := X_{t_{j_1}}(\omega) \tag{3.7}$$

and the set

$$C_n^-(\omega) = \{\widetilde{X}_j^-(\omega) : R_{t_j}(\omega) = 1, t_j \geq d\} \tag{3.8}$$

as the set of the observations that are equal or smaller than $m + \varepsilon$. In a similar way,

$$R_{t_j} = 2 \Rightarrow \exists t_{j_2} \in \{t_1, ..., t_{p_n}\}, \exists k_2 \in \mathbb{N}_0 : t_j - d \in [t_{j_2}, t_{j_2+1}[\wedge t_{j_2} \in]\tau_{2k_2+1}, \tau_{2k_2+2}] \tag{3.9}$$

and we can define:

$$\widetilde{X}_j^+(\omega) := X_{t_{j_2}}(\omega) \tag{3.10}$$

and the set

$$C_n^+(\omega) = \{\widetilde{X}_j^+(\omega) : R_{t_j}(\omega) = 2, t_j \geq d\} \tag{3.11}$$

where the observations are equal or larger than $m-\varepsilon$. We have the next result on consistency.

Theorem 3.4. *Consider $\widehat{m}_n^-(\omega) = \max C_n^-(\omega)$ and $\widehat{m}_n^+(\omega) = \min C_n^+(\omega)$ and suppose that:*

$$\lim_{n\to+\infty} \max_{1\leq i\leq p_n-1} |t_{i+1}-t_i| = 0 \ . \tag{3.12}$$

Then, if there is at least one change from the first to the second regime,

$$\lim_{n\to+\infty} \widehat{m}_n^- = m+\varepsilon \ a.s. \tag{3.13}$$

and if there is at least one change from the second to the first regime,

$$\lim_{n\to+\infty} \widehat{m}_n^+ = m-\varepsilon \ a.s. \tag{3.14}$$

and then

$$\widehat{m}_n = \frac{\widehat{m}_n^- + \widehat{m}_n^+}{2} \tag{3.15}$$

is an (almost sure) consistent estimator for the threshold m.

The proof of the results in equations (3.13) and (3.14), follows the same ideas as the ones presented in the proof of Theorem 3.2 and then the conclusion in equation (3.15) is straightforward.

Remark 3.5. For the *double diffusion process with one threshold and delay* and when the regimes are not known, we can adapt in a straightforward way the procedure presented in Remark 3.3 in order to get a practical procedure to estimate the threshold, the delay and all the other parameters of the model.

4. Practical application

For the practical application and to illustrate the estimation procedure we consider that the underlying processes are Brownian motion with drift (BMD) and geometric Brownian motion (GBM). When the underlying process in regime 1 is the Brownian motion with drift, the stochastic differential equation (2.1) reduces to:

$$dX_{1,t} = \mu_1 dt + \sigma_1 dW_t \ , \tag{4.1}$$

with (strong) solution in $[t, t+\Delta]$,

$$X_{1,t+\Delta} = X_{1,t} + \mu\Delta + \sigma_1(W_{t+\Delta} - W_t), \tag{4.2}$$

and when the underlying process (for regime 2) is the geometric Brownian motion, equation (2.2) is:

$$dX_{2,t} = \mu_2 X_{2,t} dt + \sigma_2 X_{2,t} dW_t \tag{4.3}$$

with solution in $[t, t+\Delta]$,

$$X_{2,t+\Delta} = X_{2,t} e^{\left(\mu_2 - \frac{1}{2}\sigma_2^2\right)\Delta + \sigma_2(W_{t+\Delta} - W_t)}. \tag{4.4}$$

In this situation, instead of using the conditional least squares function (3.5) we can also build a conditional log-likelihood function, noting that from equation (4.2) the conditional distribution of $X_{1,t+\Delta}$ given $X_{1,t}$ is normal and from equation (4.4) the conditional distribution of $X_{1,t+\Delta}$ given $X_{1,t}$ is log-normal. For simplicity, let us suppose that the observations $X_0, X_1, ..., X_n$ are collected at equally spaced times $t_0, t_1, ..., t_n$, with $\Delta = t_{i+1} - t_i, i = 0, 1, ..., n-1$.

4.1. Double diffusion process with two thresholds. The construction of the regimes and the estimation of the parameters for the *double diffusion process with two thresholds*, is implemented in the following way:

(1) For fixed values for the lower and upper thresholds, respectively $\widetilde{m}$ and $\widetilde{M}$ (selected from a double grid of values with a small size interval), the observations set is divided into two new sets, corresponding to the first and second regimes, in the following way: starts with the first observation X_0, that is supposed to belong to regime 1, that is, $\widetilde{R}_0 = 1$. Next, the second observation X_1 is considered and if $X_1 \leq \widetilde{M}$ then X_1 is also classified in the first regime ($\widetilde{R}_1 = 1$) and the procedure continues by classifying observations in regime 1 until we find the first observation larger than $\widetilde{M}$, say X_k, then this observation is the first one that should be considered in the regime 2, that is, $\widetilde{R}_k = 2$ and the procedure continues by classifying observations in regime 2 until we get the first next observation that is smaller than $\widetilde{m}$, that observation is now considered belonging to regime 1 and the procedure restarts on regime 1 and is repeated until the end of the observations is reached.

(2) After that, the conditional estimators for the diffusion parameters, namely μ_i and σ_i for $i = 1, 2$ are computed, using the observations in the corresponding regime. The conditional maximum likelihood estimators are, for regime 1:

$$\widehat{\mu}_1 = \frac{1}{n_1\Delta}\sum_{i=0}^{n-1}(X_{i+1} - X_i)\mathbb{I}_{\{\widetilde{R}_i=1\}} \tag{4.5}$$

and

$$\widehat{\sigma}_1^2 = \frac{1}{n_1\Delta}\sum_{i=0}^{n-1}(X_{i+1} - X_i - \widehat{\mu}_1\Delta)^2\mathbb{I}_{\{\widetilde{R}_i=1\}} \tag{4.6}$$

and for regime 2,

$$\widehat{\mu}_2 = \frac{1}{n_2\Delta}\sum_{i=0}^{n-1}\ln\left(\frac{X_{i+1}}{X_i}\right)\mathbb{I}_{\{\widetilde{R}_i=j\}} + \frac{\widehat{\sigma}_2^2}{2} \tag{4.7}$$

with

$$\widehat{\sigma}_2^2 = \frac{1}{n_2\Delta}\sum_{i=0}^{n-1}\left(\ln\left(\frac{X_{i+1}}{X_i}\right) - \frac{1}{n_2\Delta}\sum_{k=0}^{n-1}\ln\left(\frac{X_{k+1}}{X_k}\right)\mathbb{I}_{\{\widetilde{R}_k=2\}}\right)^2\mathbb{I}_{\{\widetilde{R}_i=2\}}, \tag{4.8}$$

where $n_j = \sum_{i=0}^{n-1}\mathbb{I}_{\{\widehat{R}_i=j\}}$, $j = 1, 2$ is the number of observations in regime j.

(3) In the end, both threshold estimators (lower and upper) are chosen as the ones that maximize the conditional log-likelihood function, $Cl_n(\widetilde{m}, \widetilde{M})$:

$$Cl_n(\widetilde{m}, \widetilde{M}) = \sum_{i=0}^{n-1}\left\{\left[-\frac{1}{2}\ln(2\pi\widehat{\sigma}_1^2\Delta) - \frac{(x_{i+1} - x_i - \widehat{\mu}_1\Delta)^2}{2\widehat{\sigma}_1^2\Delta}\right]\mathbb{I}_{\{\widetilde{R}_i=1\}}\right.$$
$$\left.+\left[-\frac{\ln(2\pi\widehat{\sigma}_2^2\Delta)}{2} - \ln(x_{i+1}) - \frac{1}{2\widehat{\sigma}_2^2\Delta}\left(\ln\left(\frac{x_{i+1}}{x_i}\right) - \left(\widehat{\mu}_2 - \frac{\widehat{\sigma}_2^2}{2}\right)\Delta\right)^2\right]\mathbb{I}_{\{\widetilde{R}_i=2\}}\right\} \tag{4.9}$$

that is,

$$\left(\widehat{m}, \widehat{M}\right) = \operatorname{argmax}_{(\widetilde{m}, \widetilde{M})} Cl_n(\widetilde{m}, \widetilde{M}) \, . \tag{4.10}$$

4.2. Double diffusion process with one threshold and delay. The estimation procedure for the *double diffusion process with one threshold and delay* being similar to the previous one is implemented in a very similar way:

(1) Consider $\widetilde{d}$ and $\widetilde{m}$ as fixed values for the delay and the threshold, we split again the observations in two sets, $\widetilde{R}_1(\widetilde{d}, \widetilde{m})$ and $\widetilde{R}_2(\widetilde{d}, \widetilde{m})$ corresponding to the first regime and the second regime, respectively. Naturally, we limit our search procedure for d taking values in the set with the time distances between any two observations, that is, taking only the values $p\Delta, p = 1, ..., n-1$ with $\Delta = t_{i+1} - t_i, i = 1, ..., n-1$ and having some upper bond $dmax$. The regime classification is done by starting with the initial observation X_0 and if $X_0 \leq m$ we consider X_{0+d} in the first regime ($\widetilde{R}_{0+d} = 1$) or else we consider X_{0+d} in the second regime ($\widetilde{R}_{0+d} = 2$). Next, we move to X_1 and repeat the classification procedure for the observation X_{1+d}, continuing until the end of the observations.

(2) Next and as before, we compute the conditional estimators for all the diffusion parameters by using the observations in each regime.

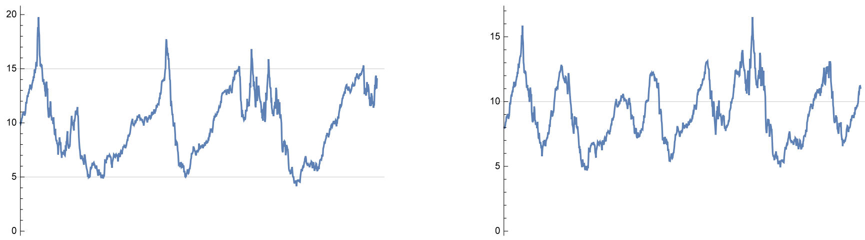
(3) Finally, the delay and threshold estimates are chosen as the values that maximize the $Cl_n(\widetilde{d}, \widetilde{m})$, that is,

$$\begin{aligned} Cl_n(\widetilde{d}, \widetilde{m}) = \sum_{i \geq dmax}^{n-1} \Bigg\{ & \left[-\frac{1}{2} \ln(2\pi\widehat{\sigma}_1^2 \Delta) - \frac{(x_{i+1} - x_i - \widehat{\mu}_1 \Delta)^2}{2\widehat{\sigma}_1^2 \Delta} \right] \mathbb{I}_{\{\widetilde{R}_i = 1\}} \\ + & \left[-\frac{\ln(2\pi\widehat{\sigma}_2^2 \Delta)}{2} - \ln(x_{i+1}) - \frac{1}{2\widehat{\sigma}_2^2 \Delta} \left(\ln\left(\frac{x_{i+1}}{x_i}\right) - \left(\widehat{\mu}_2 - \frac{\widehat{\sigma}_2^2}{2}\right)\Delta \right)^2 \right] \mathbb{I}_{\{\widetilde{R}_i = 2\}} \Bigg\} \end{aligned} \tag{4.11}$$

Remark 4.1. Notice that, the first $dmax$ observations are discarded since we do not categorize those observation in any regime, we do this to have the same number of terms in the sum in order to compare the $Cl_n(\widetilde{d}, \widetilde{m})$ function for different values of $\widetilde{d}$.

4.3. Simulated data. Next, we implement the estimation procedure for both double diffusion models, different combinations on the parameters and for different dimension of samples in order to observe the asymptotic behavior of the estimators. To give some insight on the kind of trajectories we are studying, we present in Figure 1 two simulated trajectories, one for the *double diffusion process with two thresholds* and another for the *double diffusion process with one threshold and delay*. In both trajectories, plotted in Figure 1, the first regime is Brownian motion with drift and the second is geometric Brownian motion. The results are based in the simulation of trajectories of the process, using the transition densities of the Brownian motion with drift and the geometric Brownian motion, applying a discretization step of $\Delta = 0.01$ and the regime classification is done using a double grid for m and M with mesh 0.1.

In Tables 1 and 2 is presented the estimation procedure results for the *double diffusion process with two thresholds*, for samples with dimension $n = 500$ and $n = 1000$, respectively, and for different combinations of volatility, that is, for



(A) Process with two thresholds, where $\mu_1 = 5$, $\sigma_1 = 2$, $\mu_2 = -1$, $\sigma_2 = 0.5$, $m = 5$ and $M = 15$.

(B) Process with one threshold and delay, where $\mu_1 = 5$, $\sigma_1 = 2$, $\mu_2 = -1$, $\sigma_2 = 0.5$, $m = 10$ and $d = 30$.

FIGURE 1. Simulated trajectories for the double diffusion process

different values of σ_1 and σ_2. We think that by keeping all the other parameters fixed and by changing the volatility, that is, by increasing or decreasing the random noise in the simulated trajectories, is the best way to get some conclusions regarding the procedure effectiveness.

TABLE 1. Estimates for 100 replicates of 500 observations from the double diffusion process with two thresholds (BMD vs GBM) with $m = 5$, $M = 15$, $\mu_1 = 5$, $\mu_2 = -1$ and different values for σ

	$\widehat{m}$	$\widehat{M}$	$\widehat{\mu}_1$	$\widehat{\mu}_2$	$\widehat{\sigma}_1$	$\widehat{\sigma}_2$	
$\sigma_1 = 0.5$	5.010	15.001	4.950	−1.165	0.497	0.495	mean
$\sigma_2 = 0.5$	0.030	0.010	0.278	0.463	0.021	0.027	sd
$\sigma_1 = 1.0$	5.052	14.994	4.879	−1.445	0.993	0.989	mean
$\sigma_2 = 1.0$	0.081	0.024	0.532	0.935	0.040	0.070	sd
$\sigma_1 = 1.0$	5.037	14.998	4.921	−1.148	0.993	0.496	mean
$\sigma_2 = 0.5$	0.068	0.014	0.558	0.423	0.042	0.028	sd
$\sigma_1 = 0.5$	5.012	15.000	4.942	−1.587	0.496	0.981	mean
$\sigma_2 = 1.0$	0.033	0.000	0.275	1.055	0.020	0.062	sd

TABLE 2. Estimates for 100 replicates of 1000 observations from the double diffusion process with two thresholds (BMD vs GBM) with $m = 5$, $M = 15$, $\mu_1 = 5$, $\mu_2 = -1$ and different values for σ

	$\widehat{m}$	$\widehat{M}$	$\widehat{\mu}_1$	$\widehat{\mu}_2$	$\widehat{\sigma}_1$	$\widehat{\sigma}_2$	
$\sigma_1 = 0.5$	5.019	15.000	4.989	−1.071	0.497	0.499	mean
$\sigma_2 = 0.5$	0.039	0.000	0.202	0.269	0.014	0.019	sd
$\sigma_1 = 1.0$	5.043	15.000	4.957	−1.166	0.994	0.995	mean
$\sigma_2 = 1.0$	0.070	0.000	0.408	0.621	0.026	0.043	sd
$\sigma_1 = 1.0$	5.046	14.999	4.958	−1.064	0.993	0.499	mean
$\sigma_2 = 0.5$	0.054	0.001	0.402	0.265	0.028	0.018	sd
$\sigma_1 = 0.5$	5.021	15.000	4.988	−1.288	0.498	0.990	mean
$\sigma_2 = 1.0$	0.041	0.000	0.186	0.710	0.012	0.048	sd

Next, in Tables 3 and 4 is presented the estimation procedure results for the *double diffusion process with one threshold and delay*, in the same conditions as before but with only one threshold at $m = 10$ and delay $d = 30$. For the simulation of trajectories of the process we continue using the transition densities of the underlying diffusions with a discretization step of $\Delta = 0.01$, and the regime classification is done using a grid for m with mesh 0.1 and with $d \in \{1, ..., 100\}$. As

TABLE 3. Estimates for 100 replicates of 500 observations from the double diffusion process with one threshold and delay (BMD vs GBM) with $m = 10$, $d = 30$, $\mu_1 = 5$, $\mu_2 = -1$ and different values for σ

	$\widehat{m}$	$\widehat{d}$	$\widehat{\mu}_1$	$\widehat{\mu}_2$	$\widehat{\sigma}_1$	$\widehat{\sigma}_2$	
$\sigma_1 = 0.5$	10.000	30.000	4.921	-1.226	0.498	0.497	mean
$\sigma_2 = 0.5$	0.000	0.000	0.277	0.439	0.021	0.029	sd
$\sigma_1 = 1.0$	10.001	29.990	4.928	-1.865	0.994	0.996	mean
$\sigma_2 = 1.0$	0.010	0.100	0.584	0.796	0.035	0.066	sd
$\sigma_1 = 1.0$	10.003	30.000	4.957	-1.307	0.993	0.499	mean
$\sigma_2 = 0.5$	0.017	0.000	0.542	0.416	0.041	0.028	sd
$\sigma_1 = 0.5$	10.000	30.000	4.908	-1.637	0.500	0.988	mean
$\sigma_2 = 1.0$	0.000	0.000	0.276	0.772	0.018	0.063	sd

TABLE 4. Estimates for 100 replicates of 1000 observations from the double diffusion process with one threshold and delay (BMD vs GBM) with $m = 10$, $d = 30$, $\mu_1 = 5$, $\mu_2 = -1$ and different values for σ

	$\widehat{m}$	$\widehat{d}$	$\widehat{\mu}_1$	$\widehat{\mu}_2$	$\widehat{\sigma}_1$	$\widehat{\sigma}_2$	
$\sigma_1 = 0.5$	10.000	30.000	4.979	-1.153	0.499	0.499	mean
$\sigma_2 = 0.5$	0.000	0.000	0.209	0.277	0.014	0.018	sd
$\sigma_1 = 1.0$	10.000	30.000	4.951	-1.644	0.999	0.997	mean
$\sigma_2 = 1.0$	0.000	0.000	0.410	0.600	0.025	0.045	sd
$\sigma_1 = 1.0$	10.000	30.000	4.970	-1.190	0.998	0.500	mean
$\sigma_2 = 0.5$	0.000	0.000	0.430	0.295	0.027	0.019	sd
$\sigma_1 = 0.5$	10.000	30.000	4.980	-1.586	0.499	0.999	mean
$\sigma_2 = 1.0$	0.000	0.000	0.201	0.572	0.014	0.044	sd

we can observe, the results obtained suggest that the estimation procedure gives good results, getting good approximations for the true values of the parameters. As expected, for higher values of volatility (higher σ) the standard deviation for the estimates are larger but the standard deviation of the estimates decreases with the increasing of k, that is, when the number of observations increases, suggesting the consistency of the estimators.

4.4. Real data. We also discuss our estimation procedure results using real data. In [ME16] we used the Akaike information criterion (AIC, see [Aka73]) and Bayesian information criterion (BIC, see [Sch78]) to show that for a significant number of companies chosen from the Nasdaq index, the double diffusion model with one threshold, delay and the GBM in both regimes as the underlying process is preferable to the single regime GBM for data adjustment. For some particular companies

we also compared the predictive values of the two models. In this section, we consider daily stock prices from the Broadcom Company (BRCM) and from the year of 2013, we then project the expected prices for the year of 2014, just like in [ME16], but this time considering the possibility of different underlying processes in each regime.
To compare the performance of the different models we compute the cumulative absolute errors (AE) and the cumulative quadratic errors (QE) between the expected and the observed prices in 2014. In Figure 2 we plot the BRCM company's 2014 daily prices with the expected prices of the processes with regimes and without regimes. The expected prices are obtained recursively as the expected value of X_{i+1} given X_i, that is, $\mathbb{E}[X_{i+1}|X_i = x_i] = x_i e^{\hat{\mu}\Delta}, i = 1, ..., 252$, where $\hat{\mu}$ is $\hat{\mu}_{1,n_1}$ and $\hat{\mu}_{2,n_2}$ in the double diffusion models or $\hat{\mu}_n$ in the single diffusion models.

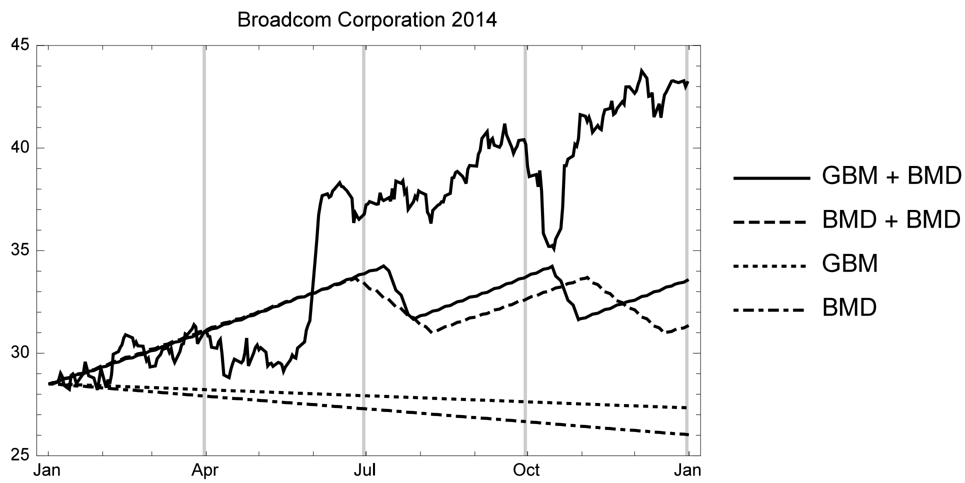


FIGURE 2. 2014 BRCM data with the expected evolution for the prices with and without regimes

Notice that, as can be observed in Tables 5 and 6 for all criteria the double diffusion model is always preferable to the single diffusion model, and from the double diffusion model the one with better adjustment do the BRCM data is the double diffusion model with GBM in the first regime and BMD in the second regime.

TABLE 5. Estimates for the for BRCM company data considering the single diffusion process (model with a single regime)

	μ	σ	AIC	BIC	AE	QE
GBM	-0.042	0.283	365.88	372.72	1894.20	21406.10
BMD	-2.487	8.421	366.70	373.54	2057.32	24964.70

5. Conclusions

In this paper we studied auto-induced regime switching double diffusion models with thresholds — and in the case of one threshold, with a delay — to the case of

TABLE 6. Estimates for the for BRCM company data considering the double diffusion process with one threshold and delay

	m	d	μ_1	σ_1	μ_2	σ_2	AIC	BIC	AE	QE
GBM vs GBM	33.8	8	0.353	0.228	−1.838	0.499	325.98	346.29	1060.90	7285.96
BMD vs BMD	32.8	19	10.830	5.637	−22.134	11.509	320.97	341.27	1144.26	8397.87
BMD vs GBM	32.8	19	10.830	5.637	−0.674	0.377	332.31	352.61	1143.69	8399.13
GBM vs BMD	33.8	8	0.353	0.228	−59.586	14.445	315.28	335.58	1057.37	7218.15

different functional forms for the drifts and volatilities and we gave conditions to ensure the existence and uniqueness of this kind of processes as solutions of stochastic differential equations. We also proposed estimators for the model thresholds and proved their consistency when the regimes are known. Whenever the regimes are unknown an alternative estimation procedure is proposed and a simulation study is presented to ascertain the quality of the procedure. Finally, real data is used with the implemented procedure to show that the double diffusion model can be useful for price data fitting prediction and pricing.

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Department of of Mathematics, FCT NOVA, New University of Lisbon, Campus de Caparica, 2829-516 Monte de Caparica, Portugal
Email address: pjpm@fct.unl.pt

Don State Technical University, Gagarin Square 1, Rostov-on-Don 344000, Russian Federation
Email address: krasnad@yandex.ru

Department of of Mathematics, FCT NOVA, New University of Lisbon, Campus de Caparica, 2829-516 Monte de Caparica, Portugal
Email address: mle@fct.unl.pt