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IDENTITIES AND BASES IN PLACTIC, HYPOPLACTIC, SYLVESTER, AND RELATED MONOIDS

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Master in Mathematics and Applications

DOCTORATE IN MATHEMATICS

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To my friends and family.

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“Thálatta! Thálatta! — The Sea! The Sea!”
– The Ten Thousand,
in Xenophon’s *Anabasis*, bk IV, ch. 7, S 24, c. 370 BC

Abstract

The ubiquitous plactic monoid, also known as the monoid of Young tableaux, has deep connections to several areas of mathematics, in particular, to the theory of symmetric functions. An active research topic is the identities satisfied by the plactic monoids of finite rank. It is known that there is no “global" identity satisfied by the plactic monoid of every rank. In contrast, monoids related to the plactic monoid, such as the hypoplactic monoid (the monoid of quasi-ribbon tableaux), sylvester monoid (the monoid of binary search trees) and Baxter monoid (the monoid of pairs of twin binary search trees), satisfy global identities, and the shortest identities have been characterized.

In this thesis, we present new results on the identities satisfied by the hypoplactic, sylvester, #-sylvester and Baxter monoids. We show how to embed these monoids, of any rank strictly greater than 2, into a direct product of copies of the corresponding monoid of rank 2. This confirms that all monoids of the same family, of rank greater than or equal to 2, satisfy exactly the same identities. We then give a complete characterization of those identities, thus showing that the identity checking problems of these monoids are in the complexity class P, and prove that the varieties generated by these monoids have finite axiomatic rank, by giving a finite basis for them. We also give a subdirect representation of multihomogeneous monoids by finite subdirectly irreducible Rees factor monoids, thus showing that they are residually finite.

Keywords: Hypoplactic monoid, sylvester monoid, Baxter monoid, variety, identities, equational basis, axiomatic rank

Resumo

O ubíquo monóide plático, também conhecido como o monóide dos diagramas de Young, tem ligações profundas a várias áreas de Matemática, em particular à teoria das funções simétricas. Um tópico de pesquisa ativo é o das identidades satisfeitas pelos monóides pláticos de característica finita. Sabe-se que não existe nenhuma identidade "global" satisfeita pelos monóides pláticos de cada característica. Em contraste, sabe-se que monóides ligados ao monóide plático, como o monóide hipoplático (o monóide dos diagramas quasi-fita), o monóide silvestre (o monóide de árvores de busca binárias) e o monóide de Baxter (o monóide de pares de árvores de busca binária gémeas), satisfazem identidades globais, e as identidades mais curtas já foram caracterizadas.

Nesta tese, apresentamos novos resultados acerca das identidades satisfeitas pelos monóides hipopláticos, silvestres, silvestres-# e de Baxter. Mostramos como mergulhar estes monóides, de característica estritamente maior que 2, num produto direto de cópias do monóide correspondente de característica 2. Confirmamos assim que todos os monóides da mesma família, de característica maior ou igual a 2, satisfazem exatamente as mesmas identidades. A seguir, damos uma caracterização completa dessas identidades, mostrando assim que os problemas de verificação de identidades destes monóides estão na classe de complexidade P, e provamos que as variedades geradas por estes monóides têm característica axiomática finita, ao apresentar uma base finita para elas. Também damos uma representação subdireta de monóides multihomogéneos por monóides fatores de Rees finitos e subdiretamente irredutíveis, mostrando assim que são residualmente finitos.

Palavras-chave: Monóide hipoplático, monóide silvestre, monóide de Baxter, variedade, identidades, base equacional, característica axiomática

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Notation

$\phi(x)$: image of x under ϕ 5
$f \circ g$: f composed with g 5
$\mathbb{N}$: set of natural numbers 5
$\mathbb{N}_0$: set of natural numbers with zero 5
$X \backslash Y$: set difference of X and Y 5
$\phi\restriction_{X'}$: restriction of the map ϕ to X' 5
$\operatorname{im}\phi$: image of ϕ 5
Y^X : set of functions from X to Y 5
π_k : projection map to the k-th coordinate of a Cartesian product 5
1_M : identity of a monoid M 5
x^n : n-th power of x 6
$(S_i)_{i\in I}$: family of semigroups S_i 6
$\prod_{i\in I} S_i$: direct product of a family of semigroups $(S_i)_i$ 6
$J(x)$: principal ideal generated by x 6
$\langle X\rangle$: subsemigroup generated by X 6
$\triangle_X$: identity relation on a set X 7
$\leq$: partial order 7
$\bigwedge X$: meet of X 7
$\bigvee X$: join of X 7
$\ker\phi$: kernel of a homomorphism ϕ 7
S/ρ : quotient set of S by ρ 8
$[x]_\rho$: ρ-class of x 8
$\rho^\natural$: natural homomorphism 8
ρ_I : Rees congruence induced by I 8
S/I : Rees factor semigroup of S by I 8
$\rho^\#$: congruence generated by ρ 8
$\mathbf{Con}(S)$: set of congruences on S 8
X^+ : free semigroup on X 8
ε : empty word 8
X^* : free monoid on X 8
ι : inclusion embedding 9
$|u|$: length of a word u 9
$|u|_x$: number of occurrences of a letter x in a word u 9
$\operatorname{cont}(u)$: content of a word u 9
$\operatorname{supp}(u)$: support of a word u 9
$\langle X|\mathcal{R}\rangle$: semigroup or monoid presentation 9
$[w]_S$: congruence class of a word w representing an element of a monoid S defined by a presentation 9
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$\equiv_{\mathsf{plac}}$: plactic congruence 20

plac : infinite-rank plactic monoid 20

plac_n : plactic monoid of rank n 20

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$o_{x\leftarrow y}(u)$: number of occurrences of the symbol y before the first occurrence of the symbol x in a word u, when reading u from right-to-left 53

$o_{y\rightarrow x}(u)$: number of occurrences of the symbol y before the first occurrence of the symbol x in a word u, when reading u from left-to-right 53

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$\mathcal{B}_{\mathsf{baxt}}$: basis for the variety generated by the Baxter monoid 68

K_x : set of all elements y of a monoid M such that the element x is not in the principal ideal generated by y 72

1

Introduction

A semigroup identity is a formal equality that, when satisfied by a semigroup, holds for any evaluation of its variables in that semigroup (see Section 3.1). Many interesting semigroups satisfy identities: for instance, the ubiquitous bicyclic monoid satisfies Adjan's identity $xyyxxyxyyx \approx xyyxyxxyyx$, which is the shortest non-trivial identity satisfied by this monoid [Adj66].

The study of the identities satisfied by a semigroup S is of great importance, since, by Birkhoff's Theorem, the equational theory of S defines the variety generated by it. Besides the complete characterization of the identities satisfied by a semigroup, there are two other natural problems which arise when studying them: The first is the finite basis problem [Sap14; Vol01], which asks if the identities satisfied by S are consequences of those in some finite subset. There exist several powerful methods with which to approach the problem for finite semigroups, however, this is not the case for infinite semigroups. The second problem is the identity checking problem Check-Id(S) [KS95], a decision problem whose instance is an arbitrary identity $u \approx v$, and the answer to such an instance is 'YES' if S satisfies $u \approx v$, and 'NO' if it does not, and its computational complexity. It is well-known that, for any finite semigroup S, the problem Check-Id(S) is decidable, since there are only finitely many substitutions of the variables occurring in the identity by elements of S. Furthermore, Check-Id(S) belongs in the complexity class coNP. However, in the case of infinite semigroups, the brute-force approach used in the finite case does not work, and only recently there have been results on the computational complexity of identity checking for infinite semigroups, beyond undecidability and trivial or easy decidability in linear time [Che+20; DJK18; KV20].

The plactic monoid plac, whose elements can be identified with Young tableaux (see Section 4.1), has long been considered an important monoid, due to its numerous applications in different areas of mathematics, such as algebraic combinatorics [Lot02], representation theory [Ful97; Gre07], symmetric functions [Mac15; Sch77], Kostka-Foulkes polynomials [LS78] and crystal bases [BS17]. It was first studied by Schensted [Sch61] and Knuth [Knu70], and later studied in depth by Lascoux and Schützenberger [LS81]. By its definition via Schensted's insertion algorithm, the plactic monoid has decidable word problem.

The question of identities satisfied by the plactic monoid is actively studied [Izh19; KO15], since it is an infinite monoid with a powerful combinatorial structure. One of the initial motivations for the study of these identities was to obtain a more natural example of a finitely-generated polynomial-growth semigroup that does not satisfy non-trivial identities, than those given in [Shn93]. However, it is now known that each plactic monoid of finite rank satisfies non-trivial identities. For example, the plactic monoid of rank 2 also satisfies Adjan's identity $xyyxxyxyyx \approx xyyxyxxyyx$. The plactic monoid of rank 3 satisfies the identity $uvvuvu \approx uvuvvu$, where $u(x,y)$ and $v(x,y)$ are respectively the left and right side of Adjan's identity [KO15]. However, it does not satisfy Adjan's identity itself.

It is known that upper triangular tropical matrix semigroups satisfy non-trivial identities [Izh14; Okn15; Tay17]. Johnson and Kambites [JK21] gave a tropical representation of the plactic monoid of every finite rank, thus showing that they all satisfy non-trivial identities. Furthermore, they also show that every identity satisfied by the plactic monoid of finite rank n is also satisfied by the monoid of $n \times n$ upper triangular tropical matrices. This result, together with algorithms given in [DJK18] and [JT19], show that the identity checking problem for the plactic monoids of finite rank is in the complexity class **P**. On the other hand, Cain *et al* [Cai+17] showed that the plactic monoid of finite rank n does not satisfy any non-trivial identity of length less than or equal to n, thus showing that there is no single "global" identity satisfied by every plactic monoid of finite rank, and that the infinite-rank plactic monoid does not satisfy any non-trivial identity. Daviaud *et al* [DJK18] also show that the monoid of 2×2 upper triangular tropical matrices, the bicyclic monoid, and the plactic monoid of rank 2 satisfy exactly the same identities, expanding on the result obtained by Izhakian in [Izh19]. Since the bicyclic monoid is not finitely based [Shn89], none of these monoids are. The identities satisfied by the bicyclic monoid had previously been characterized by Pastijn [Pas06], in terms of the properties of associated polyhedral complexes.

In the context of combinatorial Hopf algebras, whose bases are indexed by combinatorial objects, the plactic monoid is used to construct the Hopf algebra of free symmetric functions **FSym** [DHT02; PR95], whose bases are indexed by standard Young tableaux. In this context, other monoids arise with similar combinatorial properties to those of the plactic monoid: the Hopf algebra **Sym** of non-commutative symmetric functions [Gel+95], whose bases are indexed by integer compositions, is obtained from the hypoplactic monoid hypo [KT97; Nov00], the monoid of quasi-ribbon tableaux (see Section 4.2); the Loday–Ronco Hopf algebra **PBT** [HNT05; LR98], whose bases are indexed by planar binary trees, is obtained from the sylvester monoid sylv [HNT05], the monoid of right strict binary search trees (see Section 4.3); the Baxter Hopf algebra **Baxter** [Gir12; Rea05], whose bases are indexed by Baxter permutations [Bax64], is obtained from the Baxter monoid baxt [Gir12], the monoid of pairs of twin binary search trees (see Section 4.4). Also arising in [Gir12] is the #-sylvester monoid, whose elements are identified with left strict binary search trees and whose properties can be derived from those of the sylvester monoid by parallel reasoning. These monoids are closely related to each other [Gir12, Proposition 3.7], as well as to the hypoplactic monoid [CM18a; Pri13]. These monoids satisfy identities, and the shortest identities have been characterized [CM18b]. Unlike in the case of the plactic monoid, these identities are satisfied regardless of rank. Related as well to the plactic monoid by its growth type [DK94], the Chinese monoid [Cas+01] embeds into a direct product of copies of the bicyclic monoid [JO11]. Furthermore, the Chinese and plactic monoids of rank 2 coincide, hence, they satisfy the same identities.

The main goal of this thesis is to present a systematic study of the identities satisfied by the hypoplactic, sylvester, #-sylvester and Baxter monoids. It also gives an alternate characterization of equality in words in the sylvester, #-sylvester and Baxter monoids, by introducing the concepts of right and left precedences, which serve the same purpose as inversions for the hypoplactic monoid. The author remarks that Theorems 7.1.5, 7.1.6 and 7.1.7 have been proven independently, in different ways, in [Cai+21, Theorems 6.9 and 6.13] and in [HZ21, Theorems 4.6, 4.7 and 4.10]. Furthermore, Theorems 5.2.13, 5.2.16 and 5.2.18 also arise as consequences of, respectively, [HZ21, Theorems 4.6, 4.7 and 4.10]. The thesis is structured as follows: Chapters 2, 3 and 4 provide the necessary background on, respectively, elementary semigroup theory and presentations; universal algebra and computational complexity; and monoids arising from insertion algorithms. In Chapter 5, we show how to embed the plactic-like monoids of rank strictly higher than 2 into direct products of copies of the corresponding monoid of rank 2, thus showing that they generate exactly the same variety and satisfy exactly the same identities. Chapter 6 gives the characterization of these identities, as well as results on the computational complexity of the identity checking problem for these monoids. In Chapter 7, we give finite bases for the varieties generated by these monoids, thus showing that they have finite axiomatic rank. We also give a subdirect representation of multihomogeneous monoids by finite subdirectly irreducible Rees factor monoids, thus showing that they are residually finite. Finally, Chapter 8 concludes the thesis with a discussion of open questions.

The majority of the results on the hypoplactic monoid are published in [CMR21a] by Cain, Malheiro, and the present author, while the results on the sylvester, #-sylvester and Baxter monoids are to appear in the submitted paper [CMR21b], by the same authors.

2

Elementary semigroup theory

This chapter gives elementary definitions and results on semigroup theory, and mostly follows Chapters 1 and 2 of [Cai16]. For further background on semigroups and monoids, see [How95]; for presentations, see [Hig92].

Throughout this thesis, maps are written *on the left*, e.g. $\phi(x)$, and composed *right to left*, e.g. $(f \circ g)(x) = f(g(x))$. The set of natural numbers (positive integers) is denoted by $\mathbb{N}$, and by $\mathbb{N}_0$ when including 0. For sets X, X', Y such that $X' \subseteq X$, and for a map $\phi : X \to Y$, we denote the set difference of X and Y by $X \backslash Y$, the restriction of the map ϕ to X' by $\phi\restriction_{X'}$ and the image of ϕ by $\operatorname{im}\phi$. We denote the set of functions from X to Y by Y^X. Let $(X_i)_{i \in I}$ be a family of sets. For each $k \in I$, the projection map $\pi_k : \prod_{i \in i} X_i \to X_k$ to the k-th coordinate of the Cartesian product $\prod_{i \in I} X_i$ is the map defined by

$$\pi_k(x) = x(k),$$

where $x(k) \in X_k$ denotes the k-th component of the tuple x.

2.1 Semigroups and monoids

Let S be a set. A binary operation on S is a map $\bullet : S \times S \to S$. This operation is associative if $x \bullet (y \bullet z) = (x \bullet y) \bullet z$, for any $x, y, z \in S$. A semigroup is an algebraic structure which consists of a non-empty set S equipped with an associative binary operation $\bullet$. We denote such a pair by $(S, \bullet)$, or simply S, if there is no need to distinguish the operation. In this case, we usually write xy instead of $x \bullet y$, for any $x, y \in S$, and we call the operation multiplication, and the element xy the product of x and y. Due to semigroup operations being associative, there is no ambiguity in writing $x_1 x_2 \cdots x_n$, where $x_1, x_2, \ldots, x_n \in S$ and $n \in \mathbb{N}$.

Let e be an element of S. If $ex = xe = x$, for all $x \in S$, then e is an identity element of S. A semigroup contains at most one identity element. A semigroup M with an identity element is called a monoid, and its identity is usually denoted by 1_M.

Let z be an element of S. If $zx = z$, for all $x \in S$, then z is called a left zero. If $xz = z$, for all $x \in S$, then z is called a right zero. If $zx = z = xz$, for all $x \in S$, then z is a zero. A semigroup contains at most one zero. If all elements of a semigroup S are left (respectively, right) zeros, then S is called a left zero semigroup (respectively, right zero semigroup).

For $x \in S$ and $n \in \mathbb{N}$, we define

$$x^n = \overbrace{xx\cdots x}^{n \text{ times}}.$$

In general, x^n is only defined for positive n, however, if S is a monoid, we define $x^0 = 1_S$. An element of the form x^n is called a power of x.

A semigroup S is commutative if $xy = yx$, for all $x, y \in S$.

Let $(S_i)_{i\in I}$ be a family of semigroups. The direct product of the family of semigroups $(S_i)_{i\in I}$ is the Cartesian product $\prod_{i\in I} S_i$ with componentwise multiplication: using tuple notation,

$$(\ldots, x_i, \ldots)(\ldots, y_i, \ldots) = (\ldots, x_i y_i, \ldots),$$

for $x_i, y_i \in S_i$, $i \in I$. Componentwise multiplication is associative: as such, the direct product is itself a semigroup. The direct product of a monoid is also a monoid.

Let T be a non-empty subset of a semigroup S. It is a subsemigroup if it is closed under multiplication. If it is also a monoid, then it is a submonoid of S. If it is closed under left and right multiplication by any element of S, then it is a two-sided ideal, or simply an ideal of S. Any ideal of S is a subsemigroup of S. For any $x \in S$, the principal ideal generated by x is the ideal

$$J(x) = \{x\} \cup \{xy : y \in X\} \cup \{yx : y \in X\} \cup \{yxz : y, z \in X\}.$$

Let $(T_i)_{i\in I}$ be a family of subsemigroups of S. If the intersection $\bigcap_{i\in I} T_i$ is non-empty, it is also a subsemigroup of S. Let X be a non-empty subset of S and let $(T_i)_{i\in I}$ be the family of all subsemigroups of S which contain X. Notice that $(T_i)_{i\in I}$ is non-empty, since it contains at least S. Furthermore, every subsemigroup in $(T_i)_{i\in I}$ contains X, hence $\bigcap_{i\in I} T_i$ is non-empty and, as such, a subsemigroup, the smallest which contains X. It is called the subsemigroup generated by X and denoted by $\langle X \rangle$.

On the other hand, if X is a subset of a semigroup S such that $\langle X \rangle = S$, then X is called a generating set of S, and we say X generates S. If S admits a finite generating set, we say S is finitely generated. If S is generated by a single element, we say S is a monogenic semigroup. The free monogenic semigroup $\langle x \rangle$ is the semigroup generated by a single element x, such that $x^m = x^n \implies m = n$, for any $m, n \in \mathbb{N}$.

Given a non-empty subset X of a monoid M, we define the submonoid generated by X in a similar way: it is the intersection of all submonoids of M which contain $X \cup \{1_M\}$. It is the smallest submonoid of M with identity 1_M which contains X. If $X \subseteq M$ is such that M is the submonoid generated by X, then X is called a monoid generating set for M and we say X generates M as a monoid. If M is generated by a single element, we say M is a monogenic monoid. The definition of free monogenic monoid is analogous to that of a free monogenic semigroup.

2.2 Binary relations, orders and lattices

A binary relation ρ on a set X is:

- reflexive if $x \,\rho\, x$ for all $x \in X$;
- symmetric if $x \,\rho\, y \implies y \,\rho\, x$ for all $x, y \in X$;
- anti-symmetric if $(x \,\rho\, y) \wedge (y \,\rho\, x) \implies x = y$, for all $x, y \in X$;
- transitive if $(x \,\rho\, y) \wedge (y \,\rho\, z) \implies x \,\rho\, z$ for all $x, y, z \in X$.

An equivalence relation ρ on a set X is a binary relation which is reflexive, symmetric and transitive, and it partitions the set X into equivalence classes, each made up of related elements. The set of equivalence classes is called the quotient set of S by ρ. The identity relation $\triangle_X = \left\{(x,x) : x \in X\right\}$ is an example of an equivalence relation.

A partial order on a set X is a binary relation which is reflexive, anti-symmetric and transitive, usually denoted by $\leq$. We write $x < y$ when $x \leq y$ and $x \neq y$, for any $x, y \in X$. A partially ordered set or poset is a set X equipped with a partial order $\leq$, denoted by $(X, \leq)$.

Let $(X, \leq)$ be a poset and let $Y \subseteq X$. An element $x \in X$ is a lower bound of Y if $x \leq y$, for all $y \in Y$. If $z \leq x$, for all $z \in X$ which are lower bounds of Y, then x is a greatest lower bound (alternatively, infimum or meet) of Y. The meet of Y, if it exists, is unique and denoted by $\bigwedge Y$, or, if $Y = \{a, b\}$, by $a \wedge b$. We define upper bound, least upper bound (or supremum or join), $\bigvee Y$ and $a \vee b$ in a similar fashion.

A poset $(X, \leq)$ is a lattice if the meet and join of any two elements of X exist. If the meet and join of any subset of X exist, then X is a complete lattice. A subset Y of X is a sublattice of X if it is closed under meets and joins.

2.3 Homomorphisms

Let S, T be semigroups. A map $\phi : S \to T$ is called a homomorphism if $\phi(xy) = \phi(x)\phi(y)$, for all $x, y \in S$. If a homomorphism is injective, it is called an embedding; if it is bijective, it is called an isomorphism. If there exists an isomorphism $\phi : S \to T$, then we say that S and T are isomorphic. Two semigroups which are isomorphic can be viewed as the same algebraic structure in different settings: the elements of each semigroup may differ by nature, but they interact with one another in the same way. If $\phi : S \to T$ is a surjective homomorphism, we say T is a homomorphic image of S.

The kernel of a homomorphism $\phi : S \to T$ is the binary relation

$$\ker \phi = \{(x, y) \in S \times S : \phi(x) = \phi(y)\}.$$

Notice that ϕ is an embedding if and only if its kernel is the identity relation.

A map $\phi : S \to T$ is called an anti-homomorphism if $\phi(xy) = \phi(y)\phi(x)$, for all $x, y \in S$. If it is bijective, then it is called an anti-isomorphism.

For monoids S, T and a map $\phi : S \to T$, monoid homomorphisms, embeddings, isomorphisms, anti-homomorphisms and anti-isomorphisms are defined in the same way, with the added condition that $\phi(1_S) = 1_T$.

2.4 Congruences and quotients

A binary relation ρ on a semigroup S is compatible if $x \,\rho\, y$ and $z \,\rho\, t$ imply $xz \,\rho\, yt$. A compatible equivalence relation is called a congruence.

Let ρ be a congruence on S. Let S/ρ denote the quotient set of S by ρ and, for any $x \in S$, let $[x]_\rho$ denote the ρ-class of x, that is, $[x]_\rho = \{y \in S : y \,\rho\, x\}$. The multiplication defined on S/ρ by

$$[x]_\rho [y]_\rho = [xy]_\rho,$$

for $x, y \in S$, is well-defined and associative, hence the quotient set of S by ρ is a semigroup, called the quotient or factor of S by ρ. The map $\rho^\natural : S \to S/\rho$, defined by $\rho^\natural(x) = [x]_\rho$, is a surjective homomorphism, called the natural homomorphism or natural map.

The kernel of a homomorphism $\phi : S \to T$ is a congruence.

Theorem 2.4.1 ([How76, Theorem 5.4]). *Let $\phi : S \to T$ be a homomorphism between semigroups and let ρ be a congruence on S such that $\rho \subseteq \ker\phi$. Then, there exists a homomorphism $\varphi : S/\rho \to T$ such that $\varphi \circ \rho^\natural = \phi$. Moreover, φ is injective if and only if $\rho = \ker\phi$.*

Let I be an ideal of S. Then $\rho_I = (I \times I) \cup \triangle_S$ is a congruence on S, called the Rees congruence induced by I. Its factor semigroup is called a Rees factor semigroup, and is denoted by S/I, and its elements are denoted by $[x]_I$. These are the ρ_I-classes, which comprise I itself, which is the zero element of S/I, and singleton sets $\{x\}$, for each $x \in S \setminus I$.

For any binary relation ρ on S, the smallest congruence on S which contains ρ is called the congruence generated by ρ, and is denoted by $\rho^\#$. It is easy to see that $\rho^\#$ is the intersection of all congruences which contain ρ. If $\rho = \{(x, y)\}$, for some $x, y \in S$ such that $x \neq y$, we say $\rho^\#$ is the principal congruence generated by (x, y).

The set of congruences on S, denoted by $\mathbf{Con}(S)$, admits $\subseteq$ as a partial order, and is a complete lattice.

Congruences on monoids and related definitions , as well as the analogue of Theorem 2.4.1 for monoids, arise in a natural way.

2.5 Alphabets, words and free semigroups

Let X be a non-empty set, referred to as an alphabet, whose elements are referred to as letters or symbols. The free semigroup over the alphabet X, denoted by X^+, is the set of all non-empty words over X, under the operation of word concatenation. If we include the empty word, denoted by ε, we obtain the free monoid over the alphabet X, denoted by X^*. A subset of X^* is called a language.

The free semigroup over the alphabet X has the following property, called the universal property: For any semigroup S and map $\phi : X \to S$, there is a unique homomorphism $\phi^+ : X^+ \to S$ extending ϕ, that is, $\phi^+ \circ \iota = \phi$, where $\iota : X \to X^+$ is the inclusion embedding. Equivalently, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] \arrow[dr, "\phi"'] & X^+ \arrow[d, "\phi^+"] \\ & S \end{tikzcd}$$

The free monoid has an analogous universal property, concerning monoids and monoid homomorphisms.

Notice that any free monogenic semigroups and monoids are, respectively, free semigroups and free monoids over alphabets with a single letter.

Let $u \in X^*$ be such that $u = u_1 \cdots u_k$, where $u_i \in X$. The length of u is the number of symbols which occur in it, namely k, and is denoted by $|u|$. For any $x \in X$, the number of occurrences of x in u is denoted by $|u|_x$. For any $1 \leq i \leq j \leq k$, the word $u_1 \cdots u_j$ is a prefix of u, the word $u_i \cdots u_k$ is a suffix of u and the word $u_i \cdots u_j$ is a factor of u. For any $i_1, \ldots, i_m \in \{1, \ldots, k\}$ such that $i_1 < \cdots < i_m$, the word $u_{i_1} \cdots u_{i_m}$ is a subsequence of u. The empty word can also be considered a prefix, suffix, factor or subsequence of any word.

The content of u, denoted by $\operatorname{cont}(u)$, is the infinite tuple of non-negative integers, indexed by X, whose x-th element is $|u|_x$. The support of u, denoted by $\operatorname{supp}(u)$, is the subset of symbols $x \in X$ such that $|u|_x \geq 1$. Notice that if two words share the same content, then they also share the same support.

2.6 Presentations

A semigroup presentation is a pair $\langle X | \mathcal{R} \rangle$, where X is an alphabet and $\mathcal{R}$ is a binary relation on X^+. The symbols of the alphabet X are called generators, while the elements of $\mathcal{R}$, which are pairs of words, are called defining relations. The presentation $\langle X | \mathcal{R} \rangle$ defines the semigroup $A^+/\mathcal{R}^\#$, up to isomorphism, and a semigroup S isomorphic to $A^+/\mathcal{R}^\#$ is presented by $\langle X | \mathcal{R} \rangle$.

Let S be a semigroup presented by $\langle X | \mathcal{R} \rangle$. Since S is isomorphic to $A^+/\mathcal{R}^\#$, there is a bijective correspondence between elements of S and $\mathcal{R}^\#$-classes. Then, we say a word $w \in X^*$ represents or is a representative of an element of S if its $\mathcal{R}^\#$-class corresponds to that element of S. We denote the $\mathcal{R}^\#$-class of w by $[w]_S$, to simplify the notation.

A presentation $\langle X | \mathcal{R} \rangle$ is finite if both X and $\mathcal{R}$ are finite. A semigroup is finitely presented if it admits a finite presentation. Finite presentability is independent of the generating set. A semigroup can be finitely generated but not finitely presented.

A presentation $\langle X | \mathcal{R} \rangle$ is homogeneous (respectively, multihomogeneous) if for every $(u, v) \in \mathcal{R}$, we have $|u| = |v|$ (respectively, $\operatorname{cont}(u) = \operatorname{cont}(v)$). In other words, in a homogeneous presentation, the defining relations preserve length, and, in a multihomogeneous presentation, they preserve content. Notice that any multihomogeneous presentation is also homogeneous. A semigroup is homogeneous (respectively, multihomogeneous) if it is presented by a homogeneous (respectively, multihomogeneous) presentation. We define the length of an element of a homogeneous semigroup as the length of its representatives, and we define the content and support of an element of a multihomogeneous semigroup as the content and support of its representatives.

Monoid presentations are defined in a similar fashion. The defining relations are pairs of words over X^*, and the monoid presentation $\langle X \mid \mathcal{R} \rangle$ presents the monoid $A^*/\mathcal{R}^\#$, up to (monoid) isomorphism. The well-known bicyclic monoid is defined by the presentation $\langle a, b \mid (ab, \varepsilon) \rangle$. Every element of the bicyclic monoid is uniquely represented by a word of the form $b^i a^j$, for some $i, j \in \mathbb{N} \cup \{0\}$.

3

Universal algebra and computational complexity

This chapter gives the necessary background on universal algebra and computational complexity. For a more in-depth look at universal algebra, see [Ber12; BS81; Mal73; MMT18]; and for computation and complexity theory, see [Pap94; Sip13].

3.1 Varieties and identities

This section mostly follows Chapter 8 of [Cai16] and Chapter 2 of [BS81].

An algebra $\mathbf{A}$ is a non-empty set A, called the universe of $\mathbf{A}$, equipped with a family of finitary functions $\{f_i : i \in I\}$. A function on a set A is a map $f_i : A^{\theta(f_i)} \mapsto A$, where θ is a map from the set of functions to $\mathbb{N} \cup \{0\}$. For each function f_i, the number $\theta(f_i)$ is called the arity of f_i. A function of arity 0 is a constant, a function of arity 1 is a unary function, and a function of arity 2 is a binary function. For example, in semigroups, the multiplication operation is a binary function, while in monoids, the identity element can be viewed as a constant.

A type $\mathcal{T}$ of an algebra is a set of function symbols $\{\{_i : i \in I\}$, such that the arity of each function symbol is determined by a map $\vartheta : \{\{_i : i \in I\} \to \mathbb{N} \cup \{0\}$. We say an algebra $\mathbf{A}$ is of type $\mathcal{T}$ if there is a bijective correspondence between the functions of $\mathbf{A}$ and the function symbols of $\mathcal{T}$. Types are usually written as a list of pairs in the map ϑ, viewed as a set. For example, semigroups have type $\{(\bullet, 2)\}$, while monoids have type $\{(\bullet, 2), (1_M, 0)\}$. Notice that monoids can also be viewed as semigroups, hence they can be of type $\{(\bullet, 2)\}$ as well. While functions and function symbols are formally distinct, we use the same symbol to refer to operations with the same arity in different structures, for example, the different multiplications in different semigroups.

For the remainder of this section, let us fix a type $\mathcal{T} = \left\{(\{_i, \vartheta(\{_i)) : i \in I\right\}$. Let $\mathbf{A}$ be an algebra of type $\mathcal{T}$. An algebra $\mathbf{B}$ of type $\mathcal{T}$ is a subalgebra of $\mathbf{A}$ if its universe B is a non-empty subset of the universe A of $\mathbf{A}$, and its functions are the restriction of the functions of **A** to the universe B: for each $i \in I$ and $a_1, \dots, a_{\theta(f_i)} \in A$, we have

$$a_1, \dots, a_{\theta(f_i)} \in B \implies f_i\left(a_1, \dots, a_{\theta(f_i)}\right) \in B.$$

In particular, any constant of **A** must be in **B**. Let $X \subseteq A$. The subalgebra generated by X is the subalgebra of **A** whose universe is the intersection of all universes B of subalgebras **B** of **A** such that $X \subseteq B$.

Let **A** and **B** be algebras of type $\mathcal{T}$. Then a map $\phi : \mathbf{A} \to \mathbf{B}$ is a homomorphism if, for each $i \in I$ and $a_1, \dots, a_{\theta(f_i)} \in A$, we have

$$\phi\Big(f_i\left(a_1, \dots, a_{\theta(f_i)}\right)\Big) = f_i\Big(\phi(a_1), \dots, \phi\left(a_{\theta(f_i)}\right)\Big).$$

Notice that, on the left side of the equality, f_i is a function of **A**, while on the right side, it is a function of **B**. An injective homomorphism is an embedding, and a bijective homomorphism is an isomorphism. If $\phi : \mathbf{A} \to \mathbf{B}$ is a surjective homomorphism, we say **B** is a homomorphic image of **A**; if ϕ is an isomorphism, we say **B** and **A** are isomorphic.

Let **A** be an algebra of type $\mathcal{T}$. An equivalence relation ρ on A is a congruence on **A** if it satisfies the compatibility property, that is, for each $i \in I$ and $a_1, \dots, a_{\theta(f_i)}, b_1, \dots, b_{\theta(f_i)} \in A$, we have

$$\Big((a_1 \; \rho \; b_1) \wedge \cdots \wedge \left(a_{\theta(f_i)} \; \rho \; b_{\theta(f_i)}\right)\Big) \implies \Big(f_i\left(a_1, \dots, a_{\theta(f_i)}\right) \; \rho \; f_i\left(b_1, \dots, b_{\theta(f_i)}\right)\Big).$$

The compatibility property allows us to introduce an algebraic structure on the quotient set A/ρ, inherited from the algebra **A**. The quotient algebra of **A** by ρ, denoted by $\mathbf{A}/\rho$, is the algebra of type $\mathcal{T}$ whose universe is A/ρ and, for each $i \in I$ and $a_1, \dots, a_{\theta(f_i)} \in A$, we have

$$f_i\left([a_1]_\rho, \dots, [a_{\theta(f_i)}]_\rho\right) = \left[f_i(a_1, \dots, a_{\theta(f_i)})\right]_\rho.$$

The set of congruences on **A**, denoted by $\mathbf{Con}(\mathbf{A})$, admits $\subseteq$ as a partial order, and is a complete lattice. The natural homomorphism or natural map $\rho^\natural : \mathbf{A} \to \mathbf{A}/\rho$, defined by $\rho^\natural(x) = [x]_\rho$, is a surjective homomorphism.

For any binary relation ρ on **A**, the smallest congruence on **A** which contains ρ is called the congruence generated by ρ, and is denoted by $\rho^\#$. If $\rho = \{(x, y)\}$, for some $x, y \in A$ such that $x \neq y$, we say $\rho^\#$ is the principal congruence generated by (x, y).

Let $(\mathbf{A}_j)_{j \in J}$ be a family of algebras of type $\mathcal{T}$. The direct product of the algebras in $(\mathbf{A}_j)_{j \in J}$ is the algebra $\prod_{j \in J} \mathbf{A}_j$ of type $\mathcal{T}$, whose universe is the Cartesian product $\prod_{j \in J} A_j$, with the functions performed componentwise: for each $k \in J$ and $a_1, \dots, a_{\theta(f_i)} \in \prod_{j \in J} \mathbf{A}_j$, we have

$$\pi_k\left(f_i(a_1, \dots, a_{\theta(f_i)})\right) = f_i\left(\pi_k(a_1), \dots, \pi_k(a_{\theta(f_i)})\right).$$

Once again, notice that f_i is a function of $\prod_{j \in J} \mathbf{A}_j$ on the left side of the equality, while on the right side, it is a function of $\mathbf{A}_k$. The projection maps induce surjective homomorphisms $\pi_k : \prod_{j \in J} \mathbf{A}_j \to \mathbf{A}_k$.

Let X be a non-empty set. Consider the smallest set $F_{\mathcal{T}}X$ of words over the alphabet $X \cup \{f_i : i \in I\} \cup \{\,(\,\} \cup \{\,)\,\} \cup \{\,,\,\}$ such that

- $X \cup \{f_i : i \in I, \theta(f_i) = 0\} \subseteq F_{\mathcal{T}}X$;
- $w_1, \ldots, w_{\theta(f_i)} \in F_{\mathcal{T}}X \implies f_i\left(w_1, \ldots, w_{\theta(f_i)}\right) \in F_{\mathcal{T}}X,\ \forall i \in I.$

This set is the universe of an algebra $\mathbf{F}_{\mathcal{T}}X$ of type $\mathcal{T}$, called the absolutely free or term algebra of type $\mathcal{T}$. Its elements are usually called terms over X.

The absolutely free algebra of type $\mathcal{T}$ has the universal property: For any algebra $\mathbf{A}$ of type $\mathcal{T}$ and map $\phi : X \to \mathbf{A}$, there is a unique homomorphism $\hat{\phi} : \mathbf{F}_{\mathcal{T}}X \to \mathbf{A}$ extending ϕ, that is, $\hat{\phi} \circ \iota = \phi$, where $\iota : X \to \mathbf{F}_{\mathcal{T}}X$ is the inclusion embedding. Equivalently, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] \arrow[dr, "\phi"'] & \mathbf{F}_{\mathcal{T}}X \arrow[d, "\hat{\phi}"] \\ & \mathbf{A} \end{tikzcd}$$

For any term $w \in F_{\mathcal{T}}X$, a subterm of w is defined in the following way: w is itself a subterm of w, and if $f_i\left(w_1, \ldots, w_{\theta(f_i)}\right)$ is a subterm of w, then each w_j is a subterm of w. The content and support of a term w are defined in the same way as the content and support of a word given in Section 2.6, but here we only consider the symbols of X which occur in w, and disregard any symbol from $\{f_i : i \in I\} \cup \{\,(\,\} \cup \{\,)\,\} \cup \{\,,\,\}$.

A non-empty class $\mathcal{V}$ of algebras of type $\mathcal{T}$ is a variety of algebras of type $\mathcal{T}$ if it is closed under the taking of homomorphic images, subalgebras and direct products. Let $\mathcal{K}$ be a non-empty class of algebras of type $\mathcal{T}$. The intersection of all varieties of algebras of type $\mathcal{T}$ which contain $\mathcal{K}$ is itself a variety, called the variety generated by $\mathcal{K}$, denoted by $\mathcal{V}(\mathcal{K})$. This variety is the closure of $\mathcal{K}$ under taking subalgebras, homomorphic images and direct products.

Let $\mathcal{V}$ be a variety of algebras of type $\mathcal{T}$ and let X be a non-empty set. Let $\mathbf{A} \in \mathcal{V}$ and let $\phi : X \to \mathbf{A}$. Recall that there is a unique extension of ϕ to a homomorphism $\hat{\phi} : \mathbf{F}_{\mathcal{T}}X \to \mathbf{A}$. Since $\operatorname{im}\hat{\phi}$ (with the corresponding functions) is a subalgebra of $\mathbf{A}$, then $\operatorname{im}\hat{\phi} \in \mathcal{V}$. Let

$$\rho = \bigcap \{\ker\hat{\phi} : \phi \in \mathbf{A}^X, \mathbf{A} \in \mathcal{V}\}.$$

Then ρ is a congruence contained in $\ker\hat{\phi}$, for any $\phi \in \mathbf{A}^X$. For each $\mathbf{A} \in \mathcal{V}$ and $\phi \in \mathbf{A}^X$, there exists a unique homomorphism $\bar{\phi} : (\mathbf{F}_{\mathcal{T}}X)/\rho \to \mathbf{A}$ such that $\bar{\phi} \circ \rho^\natural = \hat{\phi}$. Hence, $\bar{\phi}$ is the unique homomorphism such that $\phi = \hat{\phi} \circ \iota = \bar{\phi} \circ \rho^\natural \circ \iota$, and the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] \arrow[dr, "\phi"'] & \mathbf{F}_{\mathcal{T}}X \arrow[r, two heads, "\rho^\natural"] \arrow[d, "\hat{\phi}"] & (\mathbf{F}_{\mathcal{T}}X)/\rho \arrow[dl, "\bar{\phi}"] \\ & \mathbf{A} & \end{tikzcd}$$

The algebra $(\mathbf{F}_{\mathcal{T}}X)/\rho$ is a subdirect product of the family $\left\{(\mathbf{F}_{\mathcal{T}}X)/\ker\hat{\phi} : \phi \in \mathbf{A}^X, \mathbf{A} \in \mathcal{V}\right\}$ of algebras of type $\mathcal{T}$, hence it is a member of $\mathcal{V}$ (see Section 3.2 for the definition of subdirect product). It is called the $\mathbf{V}$-free algebra over X, denoted by $\mathbf{F}_{\mathcal{V}}X$, and it satisfies the universal property as well. The definition of a $\mathcal{V}$-free algebra coincides with that of an absolutely free algebra when $\mathcal{V}$ is the variety of all algebras of type $\mathcal{T}$.

Let $\mathcal{S}$ be the variety of all semigroups. Then the definitions of absolutely free algebra and $\mathcal{V}$-free algebra coincide with that of a free semigroup.

An identity, over an alphabet of variables X, is a formal equality $u \approx v$, where u and v are terms in $F_{\mathcal{T}}X$, and is non-trivial if $u \neq v$. We say a variable x occurs in $u \approx v$ if $x \in \operatorname{supp}(u)$ or $x \in \operatorname{supp}(v)$. The rank of an identity is the number of distinct variables which occur in it. The size of an identity is the sum of the lengths of both sides of the identity. We say that the identity $u \approx v$ holds in an algebra $\mathbf{A}$ (or that $\mathbf{A}$ satisfies the identity $u \approx v$) if for every homomorphism $\psi : \mathbf{F}_{\mathcal{T}}X \to \mathbf{A}$ (also referred to as an evaluation), the equality $\psi(u) = \psi(v)$ holds in $\mathbf{A}$. In other words, $\mathbf{A}$ satisfies the identity $u \approx v$ if equality in $\mathbf{A}$ holds under every substitution of the variables of u and v by elements of $\mathbf{A}$. We say that $\mathbf{A}$ satisfies the identity $u \approx v$ up to equivalence if $\mathbf{A}$ satisfies all identities obtained by renaming variables or swapping both sides of the identity $u \approx v$.

An identity $u \approx v$ is balanced if $\operatorname{cont}(u) = \operatorname{cont}(v)$. We define the length, content and support of a balanced identity as the length, content and support of both sides of the identity, respectively. Any identity satisfied by a monoid that contains a free monogenic submonoid, such as the plactic and related monoids (see Chapter 4), must be balanced.

An identity $u \approx v$ is obtained by

- symmetry from an identity $p \approx q$ if $u = q$ and $v = p$;
- transitivity from identities $p \approx q$ and $p' \approx q'$ if $q = p'$, $u = p$ and $v = q'$;
- replacement from an identity $p \approx q$ if p is a subterm of u and v is obtained by replacing the occurrence of p in u by q;
- substitution from an identity $p \approx q$ if u and v are obtained from p and q, respectively, by replacing each occurrence of a variable by some term in $F_{\mathcal{T}}X$.

Let Σ be a non-empty set of identities, over an alphabet X. An identity $u \approx v$ is said to be a consequence of Σ if there exists a sequence of identities $u_1 \approx v_1, \ldots, u_k \approx v_k$, for some $k \in \mathbb{N}$, such that the last identity in the sequence is $u \approx v$, each $u_j \approx v_j$ belongs to Σ, or is a trivial identity, or is obtained from the previous identities in the sequence by reflexivity, transitivity, replacement or substitution. This sequence is called a formal deduction.

In the case of monoids, this is equivalent to having $k \in \mathbb{N}$, words $w_1, \ldots, w_k \in X^*$, and substitutions $\sigma_1, \ldots, \sigma_{k-1}$ of variables by elements of X^* (that is, endomorphisms of X^*), such that $u = w_1$, $v = w_k$ and, for $1 \leq i < k$, ,

$$\begin{aligned} w_i &= r_i \sigma_i(p_i) s_i, \\ w_{i+1} &= r_i \sigma_i(q_i) s_i, \end{aligned}$$

for some $p_i, q_i, r_i, s_i \in X^*$ such that $p_i \approx q_i$ or $q_i \approx p_i$ is in Σ.

A class $\mathcal{K}$ of algebras of type $\mathcal{T}$ is an equational class if there exists a set of identities Σ such that $\mathcal{K}$ is the class of all algebras of type $\mathcal{T}$ that satisfy all identities in Σ. Dually, a set of identities Σ is an equational theory if there exists a class $\mathcal{K}$ of algebras of type $\mathcal{T}$ such that Σ is the set of identities satisfied by all algebras in $\mathcal{K}$. When this holds, Σ is called the equational class of $\mathcal{K}$. Equational theories are closed under taking consequences.

The notions of equational class and variety coincide, as seen in the following theorem, known as Birkhoff's $\mathbb{HSP}$ Theorem:

Theorem 3.1.1 ([Bir35]). *$\mathcal{K}$ is an equational class if and only if $\mathcal{K}$ is a variety.*

An important corollary of this theorem is that, for any algebra $\mathbf{A}$, the identities satisfied by $\mathbf{A}$ must also be satisfied by all other algebras in $\mathcal{V}(\mathbf{A})$.

A set of identities $\mathcal{B}$ is an equational basis (or simply basis) of a variety $\mathcal{V}$ if the equational theory of $\mathcal{V}$ consists of all consequences of $\mathcal{B}$. A variety $\mathcal{V}$ is finitely based if it admits a finite basis. The axiomatic rank of $\mathcal{V}$ is the least natural number $r_a(\mathcal{V})$ such that $\mathcal{V}$ admits a basis $\mathcal{B}$, where the rank of each identity in $\mathcal{B}$ does not exceed $r_a(\mathcal{V})$. If no such natural number exists, we say that $\mathcal{V}$ has infinite axiomatic rank. Notice that if $\mathcal{V}$ is finitely based, then it has finite axiomatic rank.

A variety $\mathcal{V}$ is always generated by its $\mathcal{V}$-free algebra $\mathbf{F}_{\mathcal{V}}X$ over an infinite alphabet. However, $\mathcal{V}$ may also be generated by a $\mathcal{V}$-free algebra $\mathbf{F}_{\mathcal{V}}X_n$ over a finite alphabet X_n with n symbols, for some $n \in \mathbb{N}$. In such a case, the least natural number $r_b(\mathcal{V})$ such that $\mathcal{V}$ is generated by $\mathbf{F}_{\mathcal{V}}(X_{r_b(\mathcal{V})})$ is called the basis rank of $\mathcal{V}$. If $\mathcal{V}$ is generated by an algebra with a finite number of generators, the minimal such number coincides with the basis rank of $\mathcal{V}$.

In the context of semigroups and monoids, an equational theory Σ is left 1-hereditary if, for every identity $u \approx v$ of Σ and any variable $x \in \operatorname{supp}(u \approx v)$, the identity $u' \approx v'$ is in Σ, where u' (respectively, v') is the longest prefix of u (respectively, v) where x does not occur (see [Mas96; Vol01]). We define right 1-hereditary equational theories in a dual way. The equational theory of the variety generated by the bicyclic monoid, which coincides with that of the variety generated by the plactic monoid of rank 2 (see [JK21]), is both left and right 1-hereditary (see [Pas06; Shn89]).

3.2 Subdirectly irreducible and residually finite algebras

The definition of a subdirectly irreducible algebra is taken from [BS81, Chapter II, Section 8], while that of a residually finite algebra is taken from [Mal73, Chapter I, Section 2, Subsection 2.5] and [MMT18, Chapter 4, Section 4.5].

An algebra $\mathbf{A}$ is a subdirect product of an indexed family $(\mathbf{A}_i)_{i\in I}$ of algebras if $\mathbf{A}$ is a subalgebra of $\prod_{i\in I}\mathbf{A}_i$ such that $\pi_i(\mathbf{A}) = \mathbf{A}_i$, for all $i \in I$. An embedding $\phi : \mathbf{A} \longrightarrow \prod_{i\in I}\mathbf{A}_i$ is subdirect if $\phi(\mathbf{A})$ is a subdirect product of $(\mathbf{A}_i)_{i\in I}$.

An algebra $\mathbf{A}$ is subdirectly irreducible if for every subdirect embedding

$$\phi : \mathbf{A} \longrightarrow \prod_{i\in I}\mathbf{A}_i$$

there exists $i \in I$ such that $\pi_i \circ \phi : \mathbf{A} \longrightarrow \mathbf{A}_i$ is an isomorphism.

Another characterization of subdirectly irreducible algebras is more useful in practice:

Theorem 3.2.1 ([BS81, Theorem 8.4]). *An algebra* $\mathbf{A}$ *is subdirectly irreducible if and only if it is trivial or if there exists a minimum congruence in* $\mathbf{Con}(\mathbf{A})\backslash\{\triangle_{\mathbf{A}}\}$*, the congruence lattice of* $\mathbf{A}$*, excluding the identity relation. In the latter case the minimum congruence is a principal congruence.*

Subdirectly irreducible algebras are considered the building blocks of a variety of algebras $\mathcal{V}$, since every algebra of $\mathcal{V}$ is isomorphic to a subdirect product of subdirectly irreducible algebras of $\mathcal{V}$ [Bir44].

An algebra $\mathbf{A}$ is residually finite (or finitely approximable) if it is embeddable in a direct product of a family of finite algebras. Alternatively, an algebra $\mathbf{A}$ is residually finite if, for any $x, y \in \mathbf{A}$, there exists a finite algebra $\mathbf{B}$ and a homomorphism $\phi : \mathbf{A} \to \mathbf{B}$ such that $\phi(x) \neq \phi(y)$. A residually finite infinite algebra cannot be subdirectly irreducible (see [MMT18, Chapter 4, Section 4.5, Corollary 5]).

Finitely generated homogeneous semigroups are residually finite: Let S be a finitely generated homogeneous semigroup and let $l \in \mathbb{N}$. Notice that the set S_l of all elements of S with length strictly greater than l is an ideal of S. Furthermore, $S\backslash S_l$ is finite, since S is finitely generated and there are only finitely many words of length less than or equal to l. As such, for any two elements $x, y \in S$, we take the maximum l of the lengths of x and y and consider the natural homomorphism of S into S_l, which separates x and y. The same result can be proven for monoids. This is a folklore observation, thus it is difficult to find the formal statement and proof in the literature. As such, we further explore the subject in Section 7.2.

If a semigroup is both residually finite and finitely presented, then the word problem is solvable for such a semigroup [Eva78].

3.3 Computational complexity

In this section, we give a brief overview of basic concepts of algorithms, computational models and complexity theory. We do not go into detail on most of these subjects, and refer to [Pap94; Sip13] for further reading.

Intuitively speaking, an algorithm is just a collection of detailed instructions for solving a problem. Although no definitive formal definition exists, an algorithm is usually taken to be a set of operations which can be simulated by a Turing machine. This is due to the fact that all attempts to formalize the notion of computable functions have resulted in computational models equivalent in computing power to the Turing machine. This assumption is called the Church–Turing thesis, and we will assume it as true for the purposes of Chapter 6.

A Turing machine is a computational model with an infinite tape, divided into countably many cells, and a tape head which can read and write one symbol in each cell, and move around on the tape. It takes as input a string of symbols. In the initial configuration of the machine, the input string is written on the tape, which is blank everywhere else.

During its computations, the machine may store information on the tape, by writing over its cells, and it can read the information stored on the cells. The machine will continue its computations until it reaches an accepting or rejecting state. If it does not enter such a state, the machine will continue its computations, and will never halt.

Formally, a (deterministic) Turing machine is a 7-tuple $T = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$, where Q, Σ, Γ are all finite sets and

1. Q is the set of states,
2. Σ is the input alphabet not containing the blank symbol $\sqcup$,
3. Γ is the tape alphabet, where $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$,
4. $\delta : Q \times \Gamma \to Q \times \Gamma \times \{\leftarrow, \rightarrow\}$ is the transition function,
5. $q_0 \in Q$ is the initial state,
6. $q_{\text{accept}} \in Q$ is the accepting state;
7. $q_{\text{reject}} \in Q$ is the rejecting state, where $q_{\text{accept}} \neq q_{\text{reject}}$.

When a Turing machine is in a state q and the head is over a tape cell reading a symbol a, the machine will make its computation according to the image of (q, a) by δ, that is, if $\delta(q, a) = (r, b, D)$, the machine will write the symbol b over the symbol a, go to state r, and move its tape head to the direction D, where D can be either $\leftarrow$ (left) or $\rightarrow$ (right). Notice that the tape head only moves after writing over the cell. We also assume that the tape head does not move to the left of the leftmost cell of the tape which does not hold a blank symbol.

The configuration of a Turing machine is a triple (q, w, u), where q is the current state of the machine, w is the string of symbols in cells to the left of the tape head, including the cell where the tape head currently is, and u is the string of symbols in cells to the right of the tape head. We say a configuration (q, w, u) yields a configuration (q', w', u') if the Turing machine moves from the first configuration to the second in a single computation. In its start configuration, the tape head is at the leftmost cell of the tape, reading the leftmost symbol of the input, which is not a blank symbol.

We say that a Turing machine accepts an input w if there exists a finite sequence of configurations $C_1, \ldots, C_k$, where C_1 is the start configuration on input w, C_k is an accepting configuration (that is, the state of the configuration is q_{accept}), and each configuration in the sequence yields the consecutive one. The definition of a Turing machine rejecting an input is similarly defined. The collection of strings accepted by a Turing machine T is called the language recognized by T. A language is Turing-recognizable if some Turing machine recognizes it, and is Turing-decidable (or simply decidable) if some Turing machine recognizes it and halts on all inputs.

There exist several variants of the Turing machine, such as multitape or non-deterministic Turing Machines, and many other models of general purpose computations, such as Random Access Machines, which have unrestricted access to unlimited memory. One of the major reasons to assume the Church–Turing thesis is that all of these models of computation are equivalent in computing power, as long as they satisfy "reasonable" requirements, such as the ability to perform only a finite amount of work in a single step. As such, we do not give the description of the Turing machine corresponding to an algorithm in full detail, and only give a high-level description of an algorithm.

Let T be a deterministic Turing machine that halts on all inputs. The time complexity of T is the function $t : \mathbb{N} \to \mathbb{R}^+$, where $t(n)$ is the maximum number of computations that T uses on any input of length n. Time complexity is always computed as a function of the length of the string representing the input, and using a worst-case analysis.

Let f, g be functions of $\mathbb{N}$ into $\mathbb{R}^+$, the set of all positive real numbers. We say that $g(n)$ is an asymptotic upper bound for $f(n)$, and write $f(n) = O(g(n))$, if there exist positive integers c and n_0 such that, for every integer $n \geq n_0$,

$$f(n) \leq c \cdot g(n).$$

Let $t : \mathbb{N} \to \mathbb{R}^+$ be a function. The time complexity class $\mathsf{TIME}(t(n))$ is the collection of all languages that are decidable by a Turing machine with time complexity $O(t(n))$.

A variant of the Church–Turing thesis exists in the context of complexity theory, which states that all "reasonable" deterministic computational models are polynomially equivalent to a deterministic single-tape Turing machine, in the sense that any of these computational models can simulate another with only a polynomial increase in time complexity. We will also assume it as true for the purposes of Chapter 6.

The class P is the class of languages that are decidable in polynomial time on a deterministic single-tape Turing machine. That is,

$$\mathsf{P} = \bigcup_{k \in \mathbb{N}} \mathsf{TIME}(n^k).$$

This class plays a central role in complexity theory, since it is invariant for all computation models that are polynomially equivalent to a deterministic single-tape Turing machine.

For a given algebra $\mathbf{A}$, its identity checking problem $\textsc{Check-Id}(\mathbf{A})$ is the following combinatorial decision problem: the instance is an arbitrary identity $u \approx v$; the answer to such an instance is 'YES', if $\mathbf{A}$ satisfies the identity, and 'NO' otherwise. Notice that the algebra itself is fixed, as such, it is only the identity $u \approx v$ that serves as the input. Therefore, the time complexity of $\textsc{Check-Id}(\mathbf{A})$ should be measured only in terms of the size of the identity.

4

Monoids from insertion algorithms

This chapter gives the necessary background on the plactic and plactic-like monoids and their related combinatorial objects. Each section is roughly structured in the following way: we introduce the combinatorial object related to the monoid, we show how to construct this object using an insertion algorithm and we define a congruence on words using the objects obtained from each word, from which the monoid arises. Each element of these monoids can be uniquely identified with its corresponding combinatorial object, hence why we call them insertion monoids. We also give presentations for each monoid, and give alternative characterizations of equality of words in the plactic-like monoids, of which the characterizations for the sylvester, #-sylvester and Baxter monoids are new. Finally, we give an overview of the known results on the identities satisfied by these monoids.

4.1 The plactic monoid

This section gives a brief overview of the plactic monoid and its related combinatorial object and insertion algorithm, as well as results from [DJK18; JK21; JT19] on the identities satisfied by them. For more information, see [Lot02, Chapter 5].

Let $A = \{1 < 2 < 3 < \cdots\}$ denote the set of positive integers, viewed as an infinite ordered alphabet, and let $A_n = \{1 < \cdots < n\}$ denote the set of the first n positive integers, viewed as a finite ordered alphabet.

A Young tableau is a (finite) grid of cells, with top-left-aligned rows, filled with symbols from A, such that the entries in each row are weakly increasing from left to right, and the entries in each column are strictly increasing from top to bottom. An example of a Young tableau is

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 3 & 3 & 4 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 5 & 6 \\ \cline{1-2}\end{array} .$$

The following algorithm allows us to insert a symbol from A into an existing Young tableau, in order to obtain a new tableau:

Algorithm 1: *Schensted's algorithm.*

Input: A Young tableau T and a symbol $a \in A$.

Output: A Young tableau $T \leftarrow a$.

if *a is greater than or equal to any entry in the topmost row of T*, **then**
    add a as an entry at the rightmost end of T and output the resulting tableau;
else
    let z be the leftmost entry in the top row of T that is strictly greater than a. Replace z by a in the topmost row and recursively insert z into the tableau formed by the rows of T below the topmost. (Note that the recursion may end with an insertion into an 'empty row' below the existing rows of T).
return the resulting tableau.

Let $u \in A^*$. Using the insertion algorithm above, we can compute a unique Young tableau $\mathrm{P}_{\mathrm{plac}}^{\rightarrow}(u)$ from the word u: we start with the empty tableau and insert the symbols of u, one-by-one from left-to-right – see Example 4.1.1.

Example 4.1.1. Computing $\mathrm{P}_{\mathrm{plac}}^{\rightarrow}(2531613443)$:

$$\begin{array}{|c|}\hline 2\\\hline\end{array} \overset{5}{\leftarrow} \begin{array}{|c|c|}\hline 2&5\\\hline\end{array} \overset{3}{\leftarrow} \begin{array}{|c|c|}\hline 2&3\\\hline 5\\\cline{1-1}\end{array} \overset{1}{\leftarrow} \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1} 5\\\cline{1-1}\end{array} \overset{6}{\leftarrow} \begin{array}{|c|c|c|}\hline 1&3&6\\\hline 2\\\cline{1-1} 5\\\cline{1-1}\end{array} \overset{1}{\leftarrow} \begin{array}{|c|c|c|}\hline 1&1&6\\\hline 2&3\\\cline{1-2} 5\\\cline{1-1}\end{array} \overset{3}{\leftarrow}$$

$$\overset{3}{\leftarrow} \begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&3&6\\\hline 5\\\cline{1-1}\end{array} \overset{4}{\leftarrow} \begin{array}{|c|c|c|c|}\hline 1&1&3&4\\\hline 2&3&6\\\cline{1-3} 5\\\cline{1-1}\end{array} \overset{4}{\leftarrow} \begin{array}{|c|c|c|c|c|}\hline 1&1&3&4&4\\\hline 2&3&6\\\cline{1-3} 5\\\cline{1-1}\end{array} \overset{3}{\leftarrow} \begin{array}{|c|c|c|c|c|}\hline 1&1&3&3&4\\\hline 2&3&4\\\cline{1-3} 5&6\\\cline{1-2}\end{array}$$

We define the relation $\equiv_{\mathrm{plac}}$ on A^* as follows: For $u, v \in A^*$,

$$u \equiv_{\mathrm{plac}} v \iff \mathrm{P}_{\mathrm{plac}}^{\rightarrow}(u) = \mathrm{P}_{\mathrm{plac}}^{\rightarrow}(v).$$

This relation is a congruence on A^*, called the plactic congruence. The factor monoid $A^*/{\equiv_{\mathrm{plac}}}$ is the infinite-rank plactic monoid, denoted by plac. The congruence $\equiv_{\mathrm{plac}}$ naturally restricts to a congruence on A_n^*, and the factor monoid $A_n^*/{\equiv_{\mathrm{plac}}}$ is the plactic monoid of rank n, denoted by plac_n.

It follows from the definition of $\equiv_{\mathrm{plac}}$ that each element $[u]_{\mathrm{plac}}$ of plac can be identified with the Young tableau $\mathrm{P}_{\mathrm{plac}}^{\rightarrow}(u)$.

Recall that the content of u describes the number of occurrences of each symbol of A in u. It is immediate from the definition of the plactic monoid that if $u \equiv_{\mathrm{plac}} v$, then u and v share the same content. Thus, we can define the content of an element of plac as the content of any word which represents it. Furthermore, since two words with the same content also share the same support, we can also define the support of an element of plac as the support of any word which represents it.

Notice that plac_n is (isomorphic to) a submonoid of plac, for each $n \in \mathbb{N}$, and, for $n, m \in \mathbb{N}$, if $n \leq m$, then plac_n is (isomorphic to) a submonoid of plac_m.

The plactic monoid can also be defined by the presentation $\langle A \,|\, \mathcal{R}_{\mathsf{plac}} \rangle$, where

$$\mathcal{R}_{\mathsf{plac}} = \{(acb, cab) : a \leq b < c\} \tag{4.1}$$
$$\cup \{(bac, bca) : a < b \leq c\}. \tag{4.2}$$

The defining relations are known as the plactic relations. A presentation for the plactic monoid of rank n, for some $n \in \mathbb{N}$, can be obtained by restricting generators and relations of the above presentation to generators in A_n. Notice that these relations are content-preserving, hence the plactic monoids are multihomogeneous. As such, the finite-rank plactic monoids are residually finite. Furthermore, since they are also infinite monoids, they are not subdirectly irreducible. The infinite-rank plactic monoid is also residually finite, but not subdirectly irreducible (see Section 7.2).

Recall that a non-trivial identity is an identity where the two sides of the formal equality are not the same word. The plactic monoid of rank 1 is monogenic and thus commutative, and so satisfies the non-trivial identity $xy \approx yx$. On the other hand, the plactic monoid of rank 2 satisfies exactly the same identities as the monoid of 2×2 upper triangular tropical matrices [Izh19, Corollary 7.19] and the bicyclic monoid [DJK18, Theorem 4.1], hence, it satisfies Adjan's identity

$$xyyxxyxyyx \approx xyyxyxxyyx,$$

the shortest non-trivial identity satisfied by the bicyclic monoid. Furthermore, the variety generated by plac_2 is not finitely based [Shn89]. The plactic monoid of rank 3 satisfies the identity

$$uvvuvu \approx uvuvvu,$$

where $u(x,y)$ and $v(x,y)$ are respectively the left and right side of Adjan's identity [KO15, Theorem 2.6], however, it does not satisfy Adjan's identity itself [KO15, p. 111–2]. Notice that this identity has sixty variables x or y on each side.

On a more general note, the identities satisfied by the plactic monoids of finite rank are dependent on the rank. The plactic monoid of finite rank n does not satisfy any non-trivial identity of length less than or equal to n [Cai+17, Proposition 3.1]. As such, the infinite-rank plactic monoid does not satisfy any non-trivial identity [Cai+17, Theorem 3.2]. Johnson and Kambites give a faithful representation of the plactic monoid of every finite rank as a monoid of upper triangular matrices over the tropical semiring [JK21, Theorem 2.8], thus showing that the plactic monoids of finite rank each satisfy non-trivial identities [JK21, Theorem 3.1]. This is due to the fact that upper triangular tropical matrix monoids satisfy non-trivial identities [Izh14; Okn15; Tay17]. On the other hand, every identity satisfied by the plactic monoid of finite rank n is also satisfied by the monoid of $n \times n$ upper triangular tropical matrices [JK21, Theorem 4.4]. Furthermore, the variety generated by plac_3 coincides with the variety generated by the monoid of 3×3 upper triangular tropical matrices [JK21, Corollary 4.5]. These results, together with algorithms, given in [DJK18] and [JT19], that check if identities are satisfied in upper triangular tropical matrix monoids in polynomial time, imply that the identity checking problem for the plactic monoids of finite rank is in the complexity class P.

4.2 The hypoplactic monoid

This section gives a brief overview of the hypoplactic monoid and its related combinatorial object and insertion algorithm, as well as results from [CM18b]. For more information, see [Nov00] and [CM17].

A quasi-ribbon tableau is a (finite) grid of cells, aligned so that the leftmost cell in each row is below the rightmost cell of the previous row, filled with symbols from A, such that the entries in each row are weakly increasing from left to right, and the entries in each column are strictly increasing from top to bottom. An example of a quasi-ribbon tableau is

$$\begin{array}{|c|c|c|cc} \hline 1 & 2 & 4 & & \\ \hline \multicolumn{2}{c|}{} & 5 & & \\ \cline{3-4} \multicolumn{2}{c|}{} & 6 & \multicolumn{1}{c|}{6} & \\ \cline{3-5} \multicolumn{3}{c|}{} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{8} \\ \cline{4-5} \end{array}\ .$$

Observe that the same symbol cannot appear in two different rows of a quasi-ribbon tableau.

The following algorithm allows us to insert a symbol from A into an existing quasi-ribbon tableau, in order to obtain a new quasi-ribbon tableau:

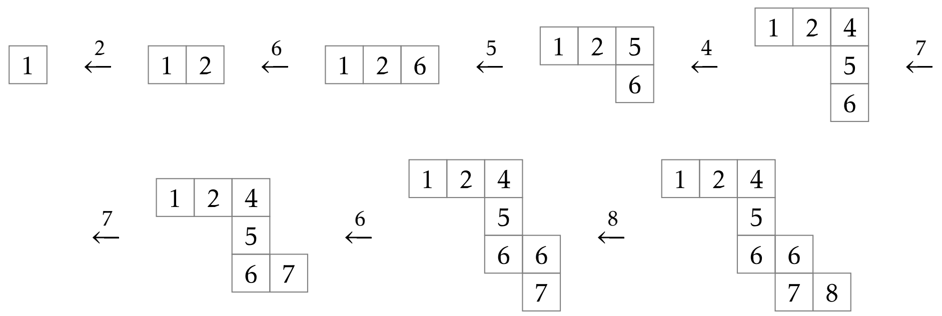
Algorithm 2: *Krob–Thibon algorithm.*

Input: A quasi-ribbon tableau T and a symbol $a \in A$.
Output: A quasi-ribbon tableau $T \leftarrow a$.

if *there is no entry in T that is less than or equal to a,* **then**
  output the tableau obtained by creating a new cell, labelled with a, and gluing T by its top-leftmost entry to the bottom of this new cell;
else
  let x be the bottom-rightmost entry of T that is less than or equal to a. Separate T in two parts, such that one part is from the top left down to and including x. Create a new cell, labelled with a, to the right of x and glue the remaining part of T (below and to the right of x) onto the bottom of the new entry a.
return the resulting tableau.

Let $u \in A^*$. Using the insertion algorithm above, we can compute a unique quasi-ribbon tableau $\mathrm{P}_{\overrightarrow{\mathsf{hypo}}}(u)$ from the word u: we start with the empty tableau and insert the symbols of u, one-by-one from left-to-right – see Example 4.2.1.

Example 4.2.1. Computing $\mathrm{P}_{\mathsf{hypo}}^{\rightarrow}(12654768)$:



We define the relation $\equiv_{\mathsf{hypo}}$ on A^* in a way analogous to the definition of the plactic congruence: For $u,v \in A^*$,

$$u \equiv_{\mathsf{hypo}} v \iff \mathrm{P}_{\mathsf{hypo}}^{\rightarrow}(u) = \mathrm{P}_{\mathsf{hypo}}^{\rightarrow}(v).$$

This relation is a congruence on A^*, called the hypoplactic congruence. The factor monoid $A^*/{\equiv_{\mathsf{hypo}}}$ is the infinite-rank hypoplactic monoid, denoted by hypo. The congruence $\equiv_{\mathsf{hypo}}$ naturally restricts to a congruence on A_n^*, and the factor monoid $A_n^*/{\equiv_{\mathsf{hypo}}}$ is the hypoplactic monoid of rank n, denoted by hypo_n.

It follows from the definition of $\equiv_{\mathsf{hypo}}$ that each element $[u]_{\mathsf{hypo}}$ of hypo can be identified with the quasi-ribbon tableau $\mathrm{P}_{\mathsf{hypo}}^{\rightarrow}(u)$. As with the case of the plactic monoid, we can define the content of an element of hypo as the content of any word which represents it, and the support of an element of hypo as the support of any word which represents it.

Notice that hypo_n is (isomorphic to) a submonoid of hypo, for each $n \in \mathbb{N}$, and, for $n,m \in \mathbb{N}$, if $n \leq m$, then hypo_n is (isomorphic to) a submonoid of hypo_m.

Let $u \in A_n^*$. Suppose $\operatorname{supp}(u) = \{a_1 < \cdots < a_k\}$, for some $k \in \mathbb{N}$. We say u has an a_{i+1}-a_i **inversion**, for $1 \leq i \leq k-1$, if it admits $a_{i+1}a_i$ as a subsequence. In other words, when reading u from left-to-right, there is at least an occurrence of a_{i+1} before the last occurrence of a_i. Notice that we only consider inversions of consecutive elements of the support of u.

Example 4.2.2. The word 31214 has 3-2 and 2-1 inversions, but no 4-3 inversion. On the other hand, 21341 has a 2-1 inversion, but neither a 4-3 nor a 3-2 inversion.

The following characterization of the hypoplactic monoid is a consequence of [Nov00, Subsection 4.2, Theorem 4.18 and Note 4.10]:

Proposition 4.2.3. *For $u,v \in A^*$, we have that $u \equiv_{\mathsf{hypo}} v$ if and only if u and v share exactly the same content and inversions.*

By the previous result, we can say that an element $[u]_{\mathsf{hypo}}$ of hypo has an a_{i+1}-a_i inversion if the word u itself, and hence any other word in $[u]_{\mathsf{hypo}}$, has an a_{i+1}-a_i inversion. This characterization will be extensively used throughout the rest of this paper.

The hypoplactic monoid can also be defined by the presentation $\left\langle A \middle| \mathcal{R}_{\mathsf{hypo}} \right\rangle$, where

$$
\begin{aligned}
\mathcal{R}_{\mathsf{hypo}} =& \Big\{(acb, cab) : a \leq b < c\Big\} && (4.3)\\
& \cup \Big\{(bac, bca) : a < b \leq c\Big\} && (4.4)\\
& \cup \Big\{(cadb, acbd) : a \leq b < c \leq d\Big\} && (4.5)\\
& \cup \Big\{(bdac, dbca) : a < b \leq c < d\Big\}. && (4.6)
\end{aligned}
$$

The first two relations are the plactic relations, while the last two are known as the hypoplactic relations. A presentation for the hypoplactic monoid of rank n, for some $n \in \mathbb{N}$, can be obtained by restricting generators and relations of the above presentation to generators in A_n. Notice that the hypoplactic monoids are multihomogeneous, hence the finite-rank hypoplactic monoids are residually finite, but not subdirectly irreducible. The infinite-rank hypoplactic monoid is also residually finite, but not subdirectly irreducible (see Section 7.2).

The following non-trivial identities are satisfied by hypo:

$$
\begin{aligned}
xyxy \approx xyyx \approx yxxy \approx yxyx;\\
xxyx \approx xyxx.
\end{aligned}
$$

Furthermore, up to equivalence (that is, up to renaming variables or swapping both sides of the identities), these are the shortest non-trivial identities satisfied by hypo [CM18b, Proposition 12]. Notice that these identities are satisfied by hypoplactic monoids of all ranks, in contrast to the plactic monoid case.

4.3 The sylvester and #-sylvester monoids

This section gives a brief overview of the sylvester and #-sylvester monoids and their related combinatorial objects and insertion algorithms, as well as results from [CM18b]. We introduce new characterizations of equality of words in these monoids, analogous to the one given in [Nov00] for the hypoplactic monoid, as well as some new notation. For more information on the sylvester monoid, see [HNT05] and [CM18a]; on binary search trees, see [Knu70] and [AU92]; on graph theory, see [Die17].

A graph is a pair $G = (V, E)$ of sets, where $E \subseteq V \times V$. The elements of V are called vertices and the elements of E are called edges. An edge $\{x, y\}$ is usually written as xy. Two vertices $x, y \in V$ are adjacent if xy is an edge of G. We say $G' = (V', E')$ is a subgraph of G if $V' \subseteq V$ and $E' \subseteq E$. If G' is a subgraph of G which contains all edges $xy \in E$ with $x, y \in V'$, then G' is called an induced subgraph of G.

A (simple) path is a non-empty graph $P = (V, E)$, where $V = \{x_1, x_2 \ldots, x_k\}$ with all vertices v_i distinct from each other and $E = \{x_1x_2, x_2x_3 \ldots, x_{k-1}x_k\}$. The length of the path is the number of its edges. A cycle is a graph $C = (V, E \cup \{x_kx_1\})$, where $P = (V, E)$ is a path. Paths and cycles are usually denoted by their sequence of vertices, for example,

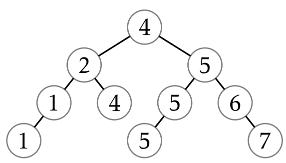
$P = x_1 x_2 \ldots x_k$. We say a path or a cycle is in a graph G if it is a subgraph of G. A graph is connected if it is non-empty and any two vertices are contained in a path in G.

A connected graph which does not contain any cycles is called a tree, and its vertices are usually called nodes. Any two nodes of a tree are linked by a unique path in the tree. A rooted tree T is a tree with a fixed node r, called the root. The length of the unique path from the root to a node x is called the depth of the node x. Let $P = x_1 x_2 \ldots x_k$ be the path starting at the root $r = x_1$ of the tree and ending in x_k. Then, x_i is a child of x_{i-1}, for $1 < i \leq k$, and a parent of x_{i+1}, for $1 \leq i < k$. Furthermore, x_i is a descendant of all nodes x_j, for all $1 \leq j < i$, and an ancestor of all nodes x_l, for all $i < l \leq k$. The lowest common ancestor of two nodes x and y is the node which is a common ancestor of x and y but whose child nodes are not. The nodes of a tree with no child nodes are called leaves. A subtree is a subgraph of a rooted tree induced by a node x and all its descendants. Notice that a subtree is a rooted tree itself, with x as its root.

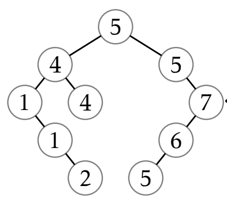
A rooted binary tree is a rooted tree where each node has at most two child nodes, ordered from left to right. A subtree of a rooted binary search tree is a left subtree (respectively, right subtree) of a node x if its root is a left child (respectively, right child) of x. The nodes of a left (respectively, right) subtree of a node x are left descendants (respectively, right descendants) of x.

A labelled tree is a tree where each node has a label or value associated with it. Throughout the rest of the thesis, we will assume that the labels are symbols from the infinite ordered alphabet $A = \{1 < 2 < 3 < \cdots\}$. For brevity, we will write 'the node a' instead of 'the node labelled with a'.

A right strict binary search tree is a labelled rooted binary tree where the label of each node is greater than or equal to the label of every node in its left subtree, and strictly less than the label of every node in its right subtree. A left strict binary search tree is a labelled rooted binary tree where the label of each node is strictly greater than the label of every node in its left subtree, and less than or equal to the label of every node in its right subtree. The following are examples of, respectively, right and left strict binary search trees:



(4.7)



. (4.8)

The (left-to-right) postfix or postorder traversal of a labelled rooted binary tree T is the sequence of nodes obtained by recursively performing the postorder traversal of the left subtree of the root of T, then recursively performing the postorder traversal of the right subtree of the root of T, and then adding the root of T to the sequence. The (left-to-right) postfix reading of a labelled rooted binary tree T is the word $\mathrm{Post}(T)$ obtained by listing the labels of the nodes in the order visited during the postorder traversal. For example, the postfix reading of the tree given in (4.7) is 1142557654.

The (left-to-right) prefix or preorder traversal of a labelled rooted binary tree T is the sequence of nodes obtained by first adding the root of T to the sequence, then recursively performing the preorder traversal of the left subtree of the root of T, and then recursively performing the preorder traversal of the right subtree of the root of T. The (left-to-right) prefix reading of a labelled rooted binary tree T is the word $\mathrm{Pre}(T)$ obtained by listing the labels of the nodes visited during the preorder traversal. For example, the prefix reading of the tree given in (4.8) is 5411245765.

The infix or inorder traversal of a labelled rooted binary tree T is the sequence of nodes obtained by recursively performing the inorder traversal of the left subtree of the root of T, then adding the root of T to the sequence, and then recursively performing the inorder traversal of the right subtree of the root of T. The following result is immediate from the definitions of right and left strict binary search trees:

Proposition 4.3.1 ([CM19, Proposition 6.5]). *For any right or left strict binary search tree T, if a node a is encountered before a node b in an inorder traversal, then $a \leq b$.*

In other words, the inorder traversal visits nodes in weakly increasing order, in right or left strict binary search trees.

Let T be a labelled rooted binary tree. Consider two nodes a and b of T, as well as their lowest common ancestor c. If the node a is in the left subtree of the node c or coincides with it, and the node b is in the right subtree of the node c or coincides with it, and the nodes a and b do not both coincide with the node c, then we say that the node a is to the left of the node b, and the node b is to the right of the node a. It is immediate to see that a node a is to the left of a node b if and only if the inorder traversal visits the node a before the node b, hence a is less than or equal to b. Furthermore, if we consider a subset of nodes of T, the definition of leftmost and rightmost nodes follows naturally.

Let T be a right or left strict binary search tree, and let $a \in A$ be a symbol which labels a node of T. We say that a node a is topmost if all other nodes a are its descendants.

Lemma 4.3.2 ([CM19, Lemma 6.6]). *If a node a occurs on a path descending from the root to a leaf, in a right or left strict binary search tree, then all other nodes a occur in that path as well. Thus, there is a unique topmost node a.*

The following is a generalization of the results obtained in [CM19, Subsection 6.4]:

Lemma 4.3.3. *Let T be a right (respectively, left) strict binary search tree, and let $a \in A$ be such that more than one node in T is labelled by a. Choosing a node a, one of the following holds:*

- *the node a is topmost;*

- *the node a is not a topmost node, and is the left (respectively, right) child of another node a;*
- *the node a is not a topmost node, and the right (respectively, left) child of a topmost node b, and a is the least symbol greater than (respectively, greatest symbol less than) the label of every topmost node in the subtree rooted at the node b.*

The following algorithm allows us to insert a symbol from A into an existing right strict binary search tree, as a leaf node in the unique place that maintains the property of being a right strict binary search tree:

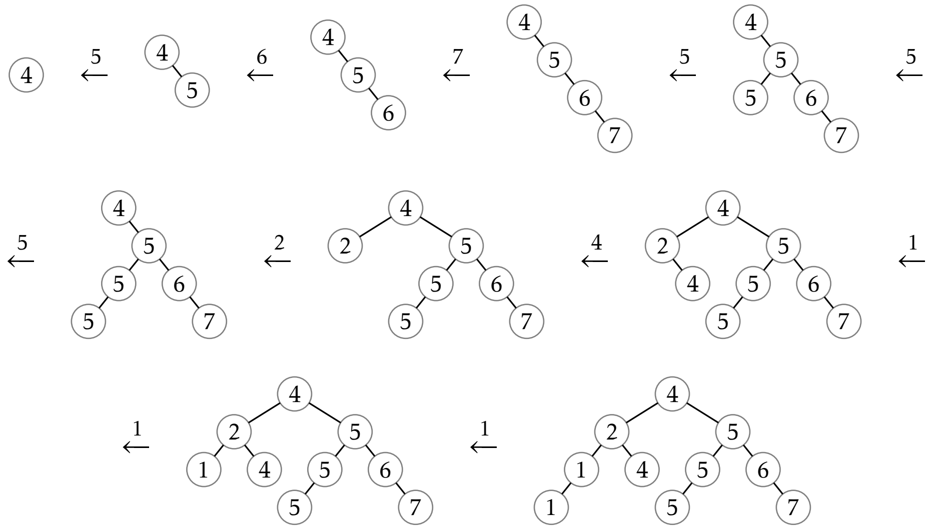
Algorithm 3: *Right strict leaf insertion.*

Input: A right strict binary search tree T and a symbol $a \in A$.
Output: A right strict binary search tree $T \leftarrow a$.

if T *is empty*, **then**
  create a node and label it a;
else
  examine the label x of the root node; if $a > x$, recursively insert a into the right subtree of the root node; otherwise recursively insert a into the left subtree of the root node;
return the resulting tree.

Let $u \in A^*$. Using the insertion algorithm above, we can compute a unique right strict binary search tree $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(u)$ from the word u: we start with the empty tree and insert the symbols of u, one-by-one from right-to-left – see Example 4.3.4.

Example 4.3.4. Computing $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(1142557654)$:



Notice that, for any right strict binary search tree T, we have that $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}((\mathrm{Post}(T))) = T$, that is, the right strict insertion algorithm, with the postfix reading of T as input, gives back T. As such, any right strict binary search tree can be obtained as an output of the right strict insertion algorithm.

We define the relation $\equiv_{\mathsf{sylv}}$ on A^* in a way analogous to the definition of the plactic and hypoplactic congruences: for $u,v \in A^*$,

$$u \equiv_{\mathsf{sylv}} v \iff \mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(u) = \mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(v).$$

This relation is a congruence on A^*, called the sylvester congruence. The factor monoid $A^*/{\equiv_{\mathsf{sylv}}}$ is the infinite-rank sylvester monoid, denoted by sylv. The congruence $\equiv_{\mathsf{sylv}}$ naturally restricts to a congruence on A_n^*, and the factor monoid $A_n^*/{\equiv_{\mathsf{sylv}}}$ is the sylvester monoid of rank n, denoted by sylv_n.

It follows from the definition of $\equiv_{\mathsf{sylv}}$ that each element $[u]_{\mathsf{sylv}}$ of sylv can be identified with the right strict binary search tree $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(u)$. As such, for each right strict binary search tree T, the set of words $u \in A^*$ such that $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(u) = T$ is called the sylvester class of T, and the postfix reading of T is called the canonical word of the sylvester class of T. As with the previous monoids, we can define the content of an element of sylv as the content of any word which represents it, and the support of an element of sylv as the support of any word which represents it. We define the content and support of a right strict binary search tree as the content and support of its corresponding sylvester class.

Notice that sylv_n is (isomorphic to) a submonoid of sylv, for each $n \in \mathbb{N}$, and, for $n,m \in \mathbb{N}$, if $n \leq m$, then sylv_n is (isomorphic to) a submonoid of sylv_m.

The sylvester monoid can also be defined by the presentation $\langle A \,|\, \mathcal{R}_{\mathsf{sylv}} \rangle$, where

$$\mathcal{R}_{\mathsf{sylv}} = \{(caub, acub) : a \leq b < c, u \in A^*\}. \tag{4.9}$$

These defining relations are known as the sylvester relations. A presentation for the sylvester monoid of rank n, for some $n \in \mathbb{N}$, can be obtained by restricting generators and relations of the above presentation to generators in A_n. Notice that the sylvester monoids are multihomogeneous, hence the finite-rank sylvester monoids are residually finite, but not subdirectly irreducible. The infinite-rank sylvester monoid is also residually finite, but not subdirectly irreducible (see Section 7.2).

The sylvester monoid satisfies the non-trivial identity $xyxy \approx yxxy$. Furthermore, up to equivalence, this is the shortest non-trivial identity satisfied by sylv [CM18b, Proposition 20].

We now give a new alternative characterization of the sylvester monoid, inspired by the characterization of the hypoplactic monoid using inversions (see [Nov00]). Let $u \in A^*$ and let $a,b \in \mathrm{supp}(u)$ be such that $a < b$. We say u has a b-a right precedence if, when reading u from right to left, b occurs before the first occurrence of a and, for any $c \in \mathrm{supp}(u)$ such that $a < c < b$, c does not occur before the first occurrence of a. The number of occurrences of b before the first occurrence of a is the index of the right precedence.

Notice that, by the definition of a right precedence, for any given $a \in \mathrm{supp}(u)$, there is at most one $b \in \mathrm{supp}(u)$ such that u has a b-a right precedence (of index k, for some $k \in \mathbb{N}$). On the other hand, u can have several right precedences of the form b-x, for a fixed b.

Example 4.3.5. The word 3123 has a 2-1 and a 3-2 right precedence, both of index 1, however, it does not have a 3-1 right precedence, since 2 occurs before the first occurrence of 1; the word 2313 has a 3-1 right precedence of index 1 and a 3-2 right precedence of index 2; and the word 3132 has a 2-1 right precedence of index 1, and does not have a 3-1 right precedence.

In order to prove that the sylvester monoid can be characterized using only the content and right precedences of words, we require some lemmata:

Lemma 4.3.6. *Let $u \in A^*$, and let $a, b \in \mathrm{supp}(u)$ be such that $a < b$ and b occurs at least k times in u, for some $k \in \mathbb{N}$. Then, u has a b-a right precedence of index k if and only if the topmost node a in $\mathrm{P}_{\mathsf{sylv}}^{\leftarrow}(u)$ has exactly k ancestor nodes labelled with b, and no ancestor nodes labelled with c, for any $c \in \mathrm{supp}\big(\mathrm{P}_{\mathsf{sylv}}^{\leftarrow}(u)\big)$ such that $a < c < b$.*

Proof. Suppose that u has a b-a right precedence of index k. It is clear that, as a consequence of the insertion algorithm 3, the topmost node a in $\mathrm{P}_{\mathsf{sylv}}^{\leftarrow}(u)$) corresponds to the rightmost occurrence of a in u. Thus, there are exactly k symbols b inserted before a, when computing $\mathrm{P}_{\mathsf{sylv}}^{\leftarrow}(u)$. Notice that, by Lemma 4.3.2, the corresponding nodes must be in a single path from the root to any leaf of $\mathrm{P}_{\mathsf{sylv}}^{\leftarrow}(u)$. Furthermore, since no symbol c occurs before the rightmost symbol a, when reading u from right-to-left, for any $a < c < b$, we have that the rightmost symbol a must be inserted as a left child of a node b, in particular, the node corresponding to the k-th inserted symbol b. This is due to the fact that, during the "searching" step of the insertion algorithm, the symbol a will satisfy exactly the same criteria as the last inserted symbol b, except when checking the k-th node b. Thus, the topmost node a in $\mathrm{P}_{\mathsf{sylv}}^{\leftarrow}(u)$ must have exactly k ancestor nodes labelled with b, and no ancestor nodes labelled with c.

Suppose now that the topmost node a in $\mathrm{P}_{\mathsf{sylv}}^{\leftarrow}(u)$ has exactly k ancestor nodes labelled with b, and no ancestor nodes labelled with c, for any $c \in \mathrm{supp}\big(\mathrm{P}_{\mathsf{sylv}}^{\leftarrow}(u)\big)$ such that $a < c < b$. It is clear, by the previous paragraph, that u cannot have a b-a right precedence of index different than k. Assume, in order to obtain a contradiction, that u does not have a b-a right precedence. Then, either no b occurs before the first occurrence of a, or there exists some $c \in \mathrm{supp}\big(\mathrm{P}_{\mathsf{sylv}}^{\leftarrow}(u)\big)$ such that $a < c < b$ and which occurs before the first occurrence of a.

In the first case, the symbol a is inserted before any symbol b, when computing $\mathrm{P}_{\mathsf{sylv}}^{\leftarrow}(u)$. Hence, the topmost node a in $\mathrm{P}_{\mathsf{sylv}}^{\leftarrow}(u)$ cannot have any ancestor nodes labelled with b. In the second case, the symbol c is inserted before any symbol a, when computing $\mathrm{P}_{\mathsf{sylv}}^{\leftarrow}(u)$. Assume, without loss of generality, that c is minimal. Therefore, by the same reasoning given before, the topmost node a in $\mathrm{P}_{\mathsf{sylv}}^{\leftarrow}(u)$ must be a left descendant of the topmost node

c. In both cases, we reach a contradiction, hence, u must have a b-a right precedence of index k. □

It is clear, from the previous lemma, that all words in the same sylvester class must share exactly the same right precedences. However, it is not immediate that two words which share the same content and right precedences will produce exactly the same output, when considered as inputs for the insertion algorithm 3.

Lemma 4.3.7. *Let T be a right strict binary search tree, and let $a \in \operatorname{supp}(T)$. The topmost node a is a left descendant of some node if and only if there exists $b \in \operatorname{supp}(T)$ such that $a < b$ and all words in the sylvester class of T have a b-a right precedence of index k, where k is the number of ancestor nodes of the topmost node a labelled with b.*

Proof. If the words in the sylvester class of T have a b-a right precedence, then the topmost node a must be a descendant of a node b, by Lemma 4.3.6. Since $a < b$, it must be a left descendant. On the other hand, if the topmost node a is a left descendant of some node, then the label of that node must be strictly greater than a. Let b be the lowest possible label of the nodes of which the topmost node a is a left descendant. The result follows from Lemma 4.3.6. □

The following is a generalization of [CM18b, Lemma 19], given without proof, since the original proof requires only a slight alteration in its last paragraph in order to hold for this generalization:

Lemma 4.3.8. *Let $u, v, w \in A^*$ be such that $\operatorname{cont}(u) = \operatorname{cont}(v)$ and $\operatorname{supp}(uv) \subseteq \operatorname{supp}(w)$. Then, $uw \equiv_{\mathsf{sylv}} vw$.*

Now, we are ready to prove our main result:

Proposition 4.3.9. *For $u, v \in A^*$, we have that $u \equiv_{\mathsf{sylv}} v$ if and only if u and v share exactly the same content and right precedences.*

Proof. We already know that if two words u and v are in the same sylvester class, then they share the same content and right precedences, by Lemma 4.3.6. Suppose now that u and v share the same content and right precedences.

If the roots of $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(u)$ and $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(v)$ are labelled differently, the words in the sylvester class of the tree whose root label is greater have a right precedence involving the root label of the other tree, while the words in the sylvester class of the tree whose root label is lesser do not. This contradicts our hypothesis, hence, we will assume that the two trees $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(u)$ and $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(v)$ have the same root. Let $T_{u,v}$ be the maximum induced tree of $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(u)$ and $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(v)$, that is, $T_{u,v}$ is an induced subgraph of both $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(u)$ and $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(v)$ that is a tree and has as many vertices as possible. Notice that $T_{u,v}$ is a right strict binary search tree and, by our previous assumption, has at least one node, which corresponds to the shared root of $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(u)$ and $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(v)$. This will also be the root node of $T_{u,v}$.

Let $t \in A^*$ be such that $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(t) = T_{u,v}$. Then, there exist u', v' such that $u't \equiv_{\mathsf{sylv}} u$ and $v't \equiv_{\mathsf{sylv}} v$. By the observation made after Lemma 4.3.6 and our hypothesis, we have that $u't$, u, $v't$ and v all have the same content and right precedences. Furthermore, u' and v' have the same content.

If $\operatorname{supp}(u') = \operatorname{supp}(v') \subseteq \operatorname{supp}(t)$, then, by Lemma 4.3.8, we have that $u \equiv_{\mathsf{sylv}} v$. Assume, in order to obtain a contradiction, that there is some symbol $a \in \operatorname{supp}(u)$ which does not occur in t. Then, the topmost node a in either $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(u)$ or $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(v)$ is not a node of $T_{u,v}$. We can assume, without loss of generality, that the topmost node a is not a descendant of any node labelled by another symbol in $\operatorname{supp}(u)$ which does not occur in t.

Suppose that u and v do not have a b-a right precedence, for any $b \in \operatorname{supp}(u)$. Then, by Lemma 4.3.7, the topmost node a is not a left descendant of any node in $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(u)$ or in $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(v)$. As such, it must be a right descendant of the rightmost node of $T_{u,v}$. Since we assumed that the topmost node a is not a descendant of any node labelled by another symbol in $\operatorname{supp}(u)$ which does not occur in t, then the topmost node a is the right child of the rightmost node of $T_{u,v}$ in $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(u)$.

Assume, in order to obtain a contradiction, that the topmost node a is not the right child of the rightmost node of $T_{u,v}$ in $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(v)$. Then, there exists a node c in $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(v)$ which is the right child of the rightmost node of $T_{u,v}$ and of which the topmost node a is a right descendant. This node must be topmost and, since it is not a left descendant of any node in $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(v)$, then v does not have a b-c right precedence, for any $b \in \operatorname{supp}(u)$, by Lemma 4.3.7. But u and v share the same right precedences, as such, by Lemma 4.3.7 once again, the topmost node c is not a left descendant of any node in $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(u)$. Then, it must be a right descendant of the topmost node a in $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(u)$. Since this is a contradiction, we can assume that the topmost node a is also a right child of the rightmost node of $T_{u,v}$ in $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(v)$. But this contradicts the maximality of $T_{u,v}$.

Suppose now that u and v have a b-a right precedence of index k. By Lemma 4.3.6, the topmost node a has exactly k ancestor nodes labelled with b, and no ancestor nodes labelled with c, for any $c \in \operatorname{supp}(u)$ such that $a < c < b$, in both $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(u)$ and $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(v)$. Consider the unique path P_u from the topmost node b to the topmost node a in $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(u)$. The nodes in P_u, excluding the node a, must be either labelled with b or with a symbol strictly less than a, since all nodes in P_u except for the topmost node b are in the left subtree of a node b. Furthermore, for each label strictly less than a, there is a single node with that label in P_u, since the topmost node a must be in the right subtree of such nodes. As such, by Lemma 4.3.3, those nodes are topmost. Even more so, they must occur in P_u in increasing order of the label. As such, none of those nodes have an ancestor node with label greater than their own label, but less than b. The same can be said about the unique path P_v from the topmost node b to the topmost node a in $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(v)$. Notice that, since u and v have the same right precedences, they share, in particular, b-d right precedences, where d is the label of a topmost node in P_u or P_v with label strictly less than a.

On the other hand, by our assumption that the topmost node a is not a descendant of any node labelled by another symbol in $\operatorname{supp}(u)$ which does not occur in t, we have that the labels of all nodes in P_u, except for a, must occur in t. As such, the topmost nodes with label strictly less than a in P_u, as well as the topmost node b, must be nodes of $T_{u,v}$. This implies that all nodes in P_u, except for possibly a sequence of nodes b at the end of the path and node a itself, are in $T_{u,v}$. Suppose there is a node d in P_v, with label strictly less than a, that is not in $T_{u,v}$. This node d must be an ancestor of the topmost node a in $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(v)$, but in $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(u)$, it must be a left descendant of the topmost node a, since its label must be strictly greater than all other labels in P_u, except for a and b, and as such, when computing $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(u)$, the symbol d is inserted in the left subtree of the topmost node a. This implies that v has a b-d right precedence, while u has an a-d right precedence. This contradicts our hypothesis, hence all nodes in P_v with label strictly less than a are in $T_{u,v}$. But since u and v share the same right precedences, this implies, by the observations made in the previous paragraph, that P_u and P_v are equal paths. As such, the topmost node a is a child of the same node of $T_{u,v}$ in both $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(u)$ and $\mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(v)$. But this contradicts the maximality of $T_{u,v}$.

Therefore, all symbols in $\mathrm{supp}(u)$ occur in t, hence $u \equiv_{\mathsf{sylv}} v$. □

In light of the previous result, we say that an element $[u]_{\mathsf{sylv}}$ of sylv has a b-a right precedence (of index k) if the word u itself, and hence any other word in $[u]_{\mathsf{sylv}}$, has a b-a right precedence (of index k).

The sylvester and hypoplactic monoids are closely related (see [CM18a; Pri13]). In fact, the hypoplactic monoid is a homomorphic image of the sylvester monoid. Notice that, for any right strict binary search tree with support $\{a_1 < \cdots < a_k\}$, all words in its sylvester class have an a_{i+1}-a_i inversion if and only if a node a_{i+1} appears in a right subtree of a node a_i. Hence, the map defined by $[u]_{\mathsf{sylv}} \mapsto [u]_{\mathsf{hypo}}$, for $u \in A^*$, is a surjective homomorphism from sylv onto hypo. Therefore, hypo is in the variety generated by sylv, and thus must satisfy all identities satisfied by sylv, by Birkhoff's Theorem. By restricting the map to sylv_n, we obtain a surjective homomorphism from sylv_n onto hypo_n.

There is an analogue of the sylvester monoid for left strict binary search trees, arising from their natural insertion algorithm, which is dual to Algorithm 3:

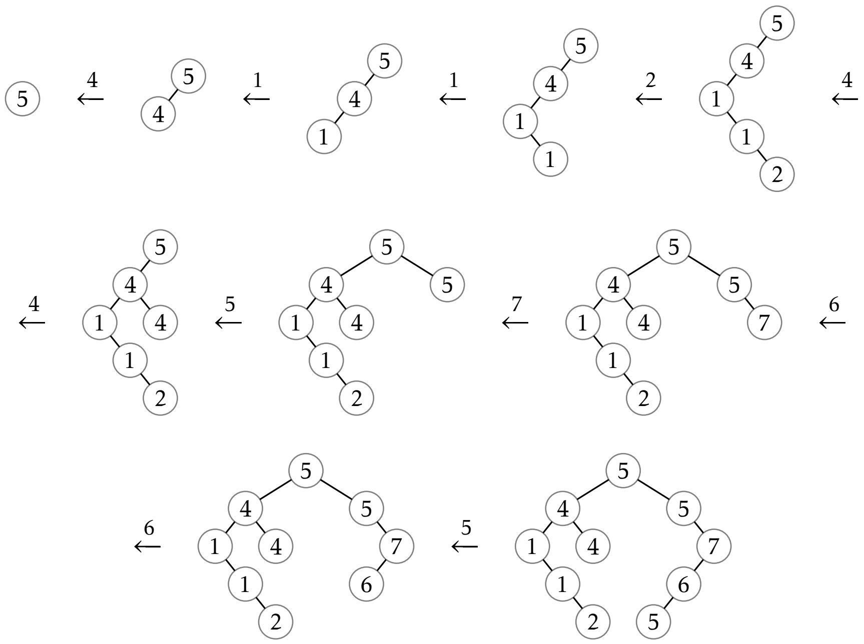
Algorithm 4: *Left strict leaf insertion.*

Input: A left strict binary search tree T and a symbol $a \in A$.
Output: A left strict binary search tree $T \leftarrow a$.

```
if T is empty, then
    create a node and label it a;
else
    examine the label x of the root node; if a < x, recursively insert a into the left
      subtree of the root node; otherwise recursively insert a into the right subtree
      of the root node;
return the resulting tree.
```

Let $u \in A^*$. Using the insertion algorithm above, we can compute a unique left strict binary search tree $\mathrm{P}_{\mathsf{sylv}^\#}^{\rightarrow}(u)$ from the word u: we start with the empty tree and insert the symbols of u, one-by-one from left-to-right – see Example 4.3.10.

Example 4.3.10. Computing $\mathrm{P}_{\mathsf{sylv}^\#}^{\rightarrow}(5411245765)$:



Notice that, for any left strict binary search tree T, we have that $\mathrm{P}_{\mathsf{sylv}^\#}^{\rightarrow}(\mathrm{Pre}(T)) = T$, that is, the left strict insertion algorithm, with the prefix reading of T as input, gives back T. As such, any left strict binary search tree can be seen as an output of the left strict insertion algorithm.

We define the #-sylvester congruence $\equiv_{\mathsf{sylv}^\#}$, the infinite-rank #-sylvester monoid $\mathsf{sylv}^\#$, the #-sylvester monoid of rank n $\mathsf{sylv}_n^\#$, and the content and support of a #-sylvester class in a similar fashion as before.

The #-sylvester monoid can also be defined by the presentation $\left\langle A \,\middle|\, \mathcal{R}_{\mathsf{sylv}^\#} \right\rangle$, where

$$\mathcal{R}_{\mathsf{sylv}^\#} = \{(buac, buca) : a < b \leq c, u \in A^*\}. \tag{4.10}$$

These defining relations are known as the #-sylvester relations. Notice that the #-sylvester monoids are multihomogeneous, hence the finite-rank #-sylvester monoids are residually finite, but not subdirectly irreducible. The infinite-rank #-sylvester monoid is also residually finite, but not subdirectly irreducible (see Section 7.2).

The #-sylvester monoid satisfies the non-trivial identity $yxyx \approx yxxy$. Furthermore, up to equivalence, this is the shortest non-trivial identity satisfied by $\mathsf{sylv}^\#$ [CM18b, Proposition 24].

The sylvester and #-sylvester monoids of finite rank n are anti-isomorphic: an anti-isomorphism from sylv_n to $\mathsf{sylv}_n^\#$ arises by taking a right strict binary search tree, reflecting it about a vertical axis, and renumbering the label i of each node to $n-i+1$, thus obtaining a left strict binary search tree. Similarly, we can define an anti-isomorphism from

$\mathsf{sylv}_n^{\#}$ to sylv_n. These anti-isomorphisms allow us to easily deduce results for finite-rank #-sylvester monoids from results for the sylvester monoids. However, notice that these anti-isomorphisms do not arise in the infinite rank case, as there is no way to renumber the labels of the nodes. In fact, we have the following:

Proposition 4.3.11. *There is no anti-isomorphism from* sylv *to* $\mathsf{sylv}^{\#}$.

Proof. Suppose, in order to obtain a contradiction, that there exists such an anti-isomorphism $\sigma : \mathsf{sylv} \to \mathsf{sylv}^{\#}$. Notice that, for any element of the form $[a]_{\mathsf{sylv}}$, where $a \in A$, we have that $\left|\sigma\left([a]_{\mathsf{sylv}}\right)\right| = 1$, since $\left|\sigma\left([a]_{\mathsf{sylv}}\right)\right| > 1$ implies that

$$\left|[a]_{\mathsf{sylv}}\right| = \left|\sigma^{-1}\left(\sigma\left([a]_{\mathsf{sylv}}\right)\right)\right| > 1,$$

along with the fact that the identity of sylv must be mapped to the identity of $\mathsf{sylv}^{\#}$. Thus, σ maps the sylv-classes of generators into $\mathsf{sylv}^{\#}$-classes of generators.

Thus, we have that $\sigma\left([1]_{\mathsf{sylv}}\right) = [x]_{\mathsf{sylv}^{\#}}$, for some $x \in A$. Since we are considering the infinite-rank case, there must exist $b, y \in A$ such that $y > x$ and $\sigma\left([b]_{\mathsf{sylv}}\right) = [y]_{\mathsf{sylv}^{\#}}$. Notice that, since σ maps 1 to x, then $b > 1$. As such, we have that $b11 \equiv_{\mathsf{sylv}} 1b1$, but on the other hand,

$$\sigma\left([b11]_{\mathsf{sylv}}\right) = [xxy]_{\mathsf{sylv}^{\#}} \neq [xyx]_{\mathsf{sylv}^{\#}} = \sigma\left([1b1]_{\mathsf{sylv}}\right),$$

which contradicts the hypothesis that σ is an anti-isomorphism. $\square$

We now give an alternative characterization of the #-sylvester monoid, parallel to the one given for the sylvester monoid. Let $u \in A^*$ and let $a, b \in \operatorname{supp}(u)$ be such that $a < b$. We say u has an *a*-*b* left precedence of index k if, when reading u from left to right, a occurs before the first occurrence of b and, for any $c \in \operatorname{supp}(u)$ such that $a < c < b$, the symbol c does not occur before the first occurrence of b. The number of occurrences of a before the first occurrence of b is the index of the left precedence.

Notice that, by the definition of a left precedence, for any given $b \in \operatorname{supp}(u)$, there is at most one $a \in \operatorname{supp}(u)$ such that u has an *a*-*b* left precedence (of index k, for some $k \in \mathbb{N}$). On the other hand, u can have several left precedences of the form *a*-*x*, for a fixed a.

Example 4.3.12. The word 1231 has a 1-2 and a 2-3 left precedence, both of index 1, while 1312 has a 1-2 left precedence of index 2 and a 1-3 left precedence of index 1, and 3121 has a 1-2 left precedence of index 1.

The following proposition mirrors Proposition 4.3.9:

Proposition 4.3.13. *For* $u, v \in A^*$, *we have that* $u \equiv_{\mathsf{sylv}^{\#}} v$ *if and only if* u *and* v *share exactly the same content and left precedences.*

In light of the previous result, we say that an element $[u]_{\mathsf{sylv}^{\#}}$ of $\mathsf{sylv}^{\#}$ has an *a*-*b* left precedence (of index k) if the word u itself, and hence any other word in $[u]_{\mathsf{sylv}^{\#}}$, has an *a*-*b* left precedence (of index k).

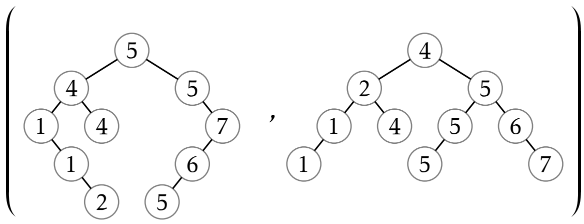
The hypoplactic monoid is also a homomorphic image of the #-sylvester monoid. Notice that, for any left strict binary search tree with support $\{a_1 < \cdots < a_k\}$, all words in its #-sylvester class have an a_{i+1}-a_i inversion if and only if a node a_i appears in a left subtree of a node a_{i+1}. Hence, the map defined by $[u]_{\mathsf{sylv}^\#} \mapsto [u]_{\mathsf{hypo}}$, for $u \in A^*$, is a surjective homomorphism from $\mathsf{sylv}^\#$ onto hypo. Hence, hypo is in the variety generated by $\mathsf{sylv}^\#$, and thus must satisfy all identities satisfied by $\mathsf{sylv}^\#$, by Birkhoff's Theorem. By restricting the map to $\mathsf{sylv}^\#_n$, we obtain a surjective homomorphism from $\mathsf{sylv}^\#_n$ onto hypo_n.

4.4 The Baxter monoid

This section gives a brief overview of the Baxter monoid and its related combinatorial object and insertion algorithm, as well as results from [CM18b]. Due to its connection with the sylvester and #-sylvester monoids, we also give an alternative characterization of this monoid, derived from the characterization of sylv and $\mathsf{sylv}^\#$ given in the previous section. For more information, see [Gir12] and [CM18a].

The canopy of a rooted binary tree T is the word over $\{0,1\}$ obtained by doing an inorder traversal of T, outputting 1 when an empty left subtree is encountered and 0 when an empty right subtree is encountered, then omitting the first and last symbols of the resulting word (which correspond, respectively, to the empty left subtree of the leftmost node and the empty right subtree of the rightmost node).

A pair of twin binary search trees consists of a left strict binary search tree T_L and a right strict binary search tree T_R , such that T_L and T_R have the same content, and the canopies of T_L and T_R are complementary, in the sense that the i-th symbol of the canopy of T_L is 0 (respectively 1) if and only if the i-th symbol of the canopy of T_R is 1 (respectively 0). The following is an example of a pair of twin binary search trees:



Let $u \in A^*$. Due to [Gir12, Proposition 4.5], for each $u \in A^*$, the pair of binary search trees

$$\left(\mathrm{P}^{\rightarrow}_{\mathsf{sylv}^\#}(u), \mathrm{P}^{\leftarrow}_{\mathsf{sylv}}(u)\right)$$

is a pair of *twin* binary search trees. As such, by defining $\mathrm{P}_{\mathsf{baxt}}(u)$ as this pair, we can use Algorithms 3 and 4 to compute a unique pair of twin binary search trees $\mathrm{P}_{\mathsf{baxt}}(u)$ from the word u.

We define the Baxter congruence $\equiv_{\mathsf{baxt}}$, the infinite-rank Baxter monoid baxt, the Baxter monoid of rank n baxt_n, and the content and support of a Baxter class in a similar fashion as before.

The Baxter monoid can also be defined by the presentation $\langle A | \mathcal{R}_{\mathsf{baxt}} \rangle$, where

$$\mathcal{R}_{\mathsf{baxt}} = \{(cudavb, cuadvb) : a \le b < c \le d, u, v \in A^*\} \tag{4.11}$$
$$\cup \{(budavc, buadvc) : a < b \le c < d, u, v \in A^*\}. \tag{4.12}$$

These defining relations are known as the Baxter relations. Notice that the Baxter monoids are multihomogeneous, hence the finite-rank Baxter monoids are residually finite, but not subdirectly irreducible. The infinite-rank Baxter monoid is also residually finite, but not subdirectly irreducible (see Section 7.2).

The Baxter monoid satisfies the non-trivial identities

$$yxxyxy \approx yxyxxy \quad \text{and} \quad xyxyxy \approx xyyxxy.$$

Furthermore, up to equivalence, these are the shortest non-trivial identities satisfied by baxt [CM18b, Proposition 26].

The Baxter, sylvester and #-sylvester monoids are closely related, due to [Gir12, Proposition 3.7]: For $u, v \in A^*$, we have that $u \equiv_{\mathsf{baxt}} v$ if and only if $u \equiv_{\mathsf{sylv}} v$ and $u \equiv_{\mathsf{sylv}^\#} v$. As a consequence of this, and Propositions 4.3.9 and 4.3.13, we have the following:

Corollary 4.4.1. *For $u, v \in A^*$, $u \equiv_{\mathsf{baxt}} v$ if and only if u and v share exactly the same content and left and right precedences.*

In light of the previous result, we say that an element $[u]_{\mathsf{baxt}}$ of baxt has a b-a right (respectively, left) precedence of index k if the word u itself, and hence any other word in $[u]_{\mathsf{baxt}}$, has a b-a right (respectively, left) precedence of index k.

It is also easy to see that the maps defined by

$$[u]_{\mathsf{baxt}} \mapsto [u]_{\mathsf{sylv}} \quad \text{and} \quad [u]_{\mathsf{baxt}} \mapsto [u]_{\mathsf{sylv}^\#},$$

for $u \in A^*$, are surjective homomorphisms from baxt onto, respectively, sylv and $\mathsf{sylv}^\#$. Therefore, both sylv and $\mathsf{sylv}^\#$, as well as hypo, are monoids in the variety generated by baxt, and thus must satisfy all identities satisfied by baxt, by Birkhoff's Theorem. By restricting the maps to baxt_n, we obtain surjective homomorphism from baxt_n onto sylv_n and $\mathsf{sylv}_n^\#$.

On the other hand, the map defined by $[u]_{\mathsf{baxt}} \mapsto \big([u]_{\mathsf{sylv}^\#}, [u]_{\mathsf{sylv}}\big)$, for $u \in A^*$, is an embedding of baxt into $\mathsf{sylv}^\# \times \mathsf{sylv}$, due to [Gir12, Proposition 3.7]. Therefore, baxt is a monoid in the variety generated by $\{\mathsf{sylv}^\#, \mathsf{sylv}\}$, and thus must satisfy all identities which are satisfied by both sylv and $\mathsf{sylv}^\#$, by Birkhoff's Theorem. By restricting the map to baxt_n, we obtain an embedding of baxt_n into $\mathsf{sylv}_n^\# \times \mathsf{sylv}_n$.

5

Embedding results

In this chapter, we prove that the hypoplactic, sylvester, #-sylvester and Baxter monoids, of rank greater than or equal to 2, satisfy identities regardless of rank. We do this by constructing embeddings of these monoids of any rank greater than 2 into *direct products* of copies of the corresponding monoid of rank 2, as it is not possible to embed one into another. Thus, monoids of the same family generate the same variety and, by Birkhoff's Theorem, satisfy exactly the same identities. We also show that the basis rank of the varieties generated by these monoids is 2.

The results in Subsections 5.1.1 and 5.2.1 have appeared in [CMR21a], while the results in Subsections 5.1.2 and 5.2.2 are to appear in the submitted paper [CMR21b].

5.1 Non-existence of embedding into a monoid of lesser rank

We first show that it is not possible to embed a plactic-like monoid of finite rank greater than 2 into the corresponding monoid of lesser rank. This stands in contrast with the case of the free monoid, where all countable-rank free monoids can be embedded into the free monoid of rank 2.

5.1.1 Hypoplactic monoid case

It is not possible to embed a hypoplactic monoid of finite rank greater than 2 into a hypoplactic monoid of lesser rank:

Proposition 5.1.1. *For all $n > m \geq 1$, there is no embedding of hypo_n into hypo_m.*

Proof. First of all, notice that hypo_1 is isomorphic to the free monogenic monoid, which is commutative, and hypo_n is non-commutative, for any $n \geq 2$. Thus, there is no embedding of hypo_n into hypo_1.

If there exists an embedding of hypo_n into hypo_m, for some $n > m \geq 2$, then, since hypo_m is a submonoid of hypo_{n-1}, there must also exist an embedding of hypo_n into hypo_{n-1}. As such, we just need to prove that this second embedding cannot exist.

Suppose, in order to obtain a contradiction, that there exists $n \geq 3$ such that we have an embedding $\phi : \mathsf{hypo}_n \to \mathsf{hypo}_{n-1}$. Without loss of generality, suppose n is the smallest positive integer in such conditions.

Observe that $\operatorname{supp}\Big(\phi\left([1\cdots(n-1)]_{\mathsf{hypo}_n}\right)\Big) = A_{n-1}$, that is, the image of the product of all generators of hypo_n, except for n, has all the possible symbols of A_{n-1}. Indeed, if we had

$$\operatorname{supp}\Big(\phi\left([1\cdots(n-1)]_{\mathsf{hypo}_n}\right)\Big) \subsetneq A_{n-1},$$

we would be able to construct an embedding from the submonoid of hypo_n isomorphic to hypo_{n-1}, generated by all generators of hypo_n except for n, into a submonoid of hypo_{n-1} isomorphic to hypo_{n-2}. This contradicts the minimality of n.

Hence, $\phi\left([1\cdots(n-1)]^2_{\mathsf{hypo}_n}\right)$ has all the possible inversions of symbols of A_{n-1}. If we multiply this element by any other element of hypo_{n-1}, either on the left or the right, we obtain the same result, by Proposition 4.2.3. Thus, by left and right multiplying by $\phi([n]_{\mathsf{hypo}_n})$, we have that

$$\phi\left([n]_{\mathsf{hypo}_n} \cdot [1\cdots(n-1)]^2_{\mathsf{hypo}_n}\right) = \phi\left([1\cdots(n-1)]^2_{\mathsf{hypo}_n} \cdot [n]_{\mathsf{hypo}_n}\right).$$

On the other hand, we have that

$$[n]_{\mathsf{hypo}_n} \cdot [1\cdots(n-1)]^2_{\mathsf{hypo}_n} \neq [1\cdots(n-1)]^2_{\mathsf{hypo}_n} \cdot [n]_{\mathsf{hypo}_n},$$

because the left-hand side has a n-$n-1$ inversion and the right-hand side does not. This contradicts our hypothesis that ϕ is injective. Hence, for all $n \geq 2$, there is no embedding of hypo_n into hypo_{n-1}. As such, there is no embedding of hypo_n into hypo_m, for $n > m \geq 2$. □

Corollary 5.1.2. *There is no embedding of* hypo *into* hypo_n*, for any* $n \in \mathbb{N}$.

Proof. If such an embedding existed, for some $n \in \mathbb{N}$, then, by restricting the embedding to the first $n+1$ generators of hypo, we would obtain an embedding of hypo_{n+1} into hypo_n, which contradicts the previous proposition. □

5.1.2 Binary search tree monoids case

It is also not possible to embed a sylvester monoid of finite rank greater than 2 into a sylvester monoid of lesser rank:

Proposition 5.1.3. *For all* $n > m \geq 1$*, there is no embedding of* sylv_n *into* sylv_m.

The overall structure of the proof parallels that of Proposition 5.1.1, but we use the new characterization of right precedences in sylv and reach a contradiction using different words.

Proof. First of all, notice that sylv_1 is isomorphic to the free monogenic monoid, which is commutative, and sylv_n is non-commutative, for any $n \geq 2$. Thus, there is no embedding of sylv_n into sylv_1.

If there exists an embedding of sylv_n into sylv_m, for some $n > m \geq 2$, then, since sylv_m is a submonoid of sylv_{n-1}, there must also exist an embedding of sylv_n into sylv_{n-1}. As such, we just need to prove that this second embedding cannot exist.

Suppose, in order to obtain a contradiction, that there exists $n \geq 3$ such that we have an embedding $\phi : \mathsf{sylv}_n \to \mathsf{sylv}_{n-1}$. Without loss of generality, suppose n is the smallest positive integer in such conditions.

Observe that $\operatorname{supp}\Big(\phi\left([2 \cdots n]_{\mathsf{sylv}_n}\right)\Big) = \mathcal{A}_{n-1}$, that is, the image of the product of all generators of sylv_n, except for 1, has all the symbols of $\mathcal{A}_{n-1}$. Indeed, if we had

$$\operatorname{supp}\Big(\phi\left([2 \cdots n]_{\mathsf{sylv}_n}\right)\Big) \subsetneqq \mathcal{A}_{n-1},$$

we would be able to construct an embedding from the submonoid of sylv_n isomorphic to sylv_{n-1}, generated by all generators of sylv_n except for 1, into a submonoid of sylv_{n-1} isomorphic to sylv_{n-2}. This contradicts the minimality of n.

Hence, since all symbols of $\mathcal{A}_{n-1}$ already occur in $\phi\left([2 \cdots n]_{\mathsf{sylv}_n}\right)$, if we multiply this element by any other element of sylv_{n-1} to the left, we obtain an element with the same right precedences as $\phi\left([2 \cdots n]_{\mathsf{sylv}_n}\right)$. Thus, by Proposition 4.3.9, since $\phi([12]_{\mathsf{sylv}})$ and $\phi([21]_{\mathsf{sylv}_n})$ have the same content, we have that

$$\phi\left([12]_{\mathsf{sylv}_n} \cdot [2 \cdots n]_{\mathsf{sylv}_n}\right) = \phi\left([21]_{\mathsf{sylv}_n} \cdot [2 \cdots n]_{\mathsf{sylv}_n}\right).$$

On the other hand, we have that

$$[12]_{\mathsf{sylv}_n} \cdot [2 \cdots n]_{\mathsf{sylv}_n} \neq [21]_{\mathsf{sylv}_n} \cdot [2 \cdots n]_{\mathsf{sylv}_n},$$

since the left-hand side has a 2-1 right precedence of index 2, and the right-hand side has a 2-1 right precedence of index 1.

This contradicts our hypothesis that ϕ is injective. Hence, for all $n \geq 2$, there is no embedding of sylv_n into sylv_{n-1}. As such, there is no embedding of sylv_n into sylv_m, for $n > m \geq 2$. □

Corollary 5.1.4. *There is no embedding of* sylv *into* sylv_n*, for any* $n \in \mathbb{N}$.

Proof. If such an embedding existed, for some $n \in \mathbb{N}$, then, by restricting the embedding to the first $n+1$ generators of sylv, we would obtain an embedding of sylv_{n+1} into sylv_n, which contradicts the previous proposition. □

If there existed an embedding of a #-sylvester monoid of finite rank greater than 2 into a #-sylvester monoid of lesser rank, then we would be able to compose it with the anti-isomorphisms given in Section 4.3, thus obtaining an embedding for the sylvester case. Therefore, no such embedding exists, as well as no embedding of the infinite-rank #-sylvester monoid into a #-sylvester monoid of finite rank. We can also prove the corresponding result for the Baxter monoid:

Proposition 5.1.5. *For all $n > m \geq 1$, there is no embedding of* baxt_n *into* baxt_m.

Proof. The proof follows the same reasoning as given in the proof of Proposition 5.1.3: Instead of considering the elements

$$[12]_{\mathsf{sylv}_n} \cdot [2\cdots n]_{\mathsf{sylv}_n} \quad \text{and} \quad [21]_{\mathsf{sylv}_n} \cdot [2\cdots n]_{\mathsf{sylv}_n},$$

we consider, respectively, the elements

$$[2\cdots n]_{\mathsf{baxt}_n} \cdot [12]_{\mathsf{baxt}_n} \cdot [2\cdots n]_{\mathsf{baxt}_n} \quad \text{and} \quad [2\cdots n]_{\mathsf{baxt}_n} \cdot [21]_{\mathsf{baxt}_n} \cdot [2\cdots n]_{\mathsf{baxt}_n},$$

which are different in baxt_n but whose images under an embedding would have the same content and left and right precedences. □

Corollary 5.1.6. *There is no embedding of* baxt *into* baxt_n*, for any $n \in \mathbb{N}$.*

5.2 Embedding into a direct product of copies of monoids of rank 2

5.2.1 Embeddings of the hypoplactic monoids

In this subsection, we prove that the hypoplactic monoids of rank greater than or equal to 2 satisfy exactly the same identities and that the basis rank of the variety generated by hypo is 2.

For any $i, j \in A$, with $i < j$, define a map from A to hypo_2 in the following way: For any $a \in A$,

$$a \longmapsto \begin{cases} [1]_{\mathsf{hypo}_2} & \text{if } a = i; \\ [2]_{\mathsf{hypo}_2} & \text{if } a = j; \\ [21]_{\mathsf{hypo}_2} & \text{if } i < a < j; \\ [\varepsilon]_{\mathsf{hypo}_2} & \text{otherwise}; \end{cases}$$

and extend it to a homomorphism $f^{ij}_{\mathsf{hypo}} : A^* \longrightarrow \mathsf{hypo}_2$, in the usual way.

Notice that, for any $w \in A^*$, its image under f^{ij}_{hypo} is the hypoplactic class of the word obtained from w by replacing any occurrence of i by 1; any occurrence of j by 2; any occurrence of an a, with $i < a < j$, by 21; and erasing the occurrences of any other element.

Lemma 5.2.1. f^{ij}_{hypo} *factors to give a homomorphism* $\varphi^{ij}_{\mathsf{hypo}} : \mathsf{hypo} \longrightarrow \mathsf{hypo}_2$.

Proof. Since hypo is given by the presentation $\langle A \mid \mathcal{R}_{\mathsf{hypo}} \rangle$, we just need to verify that both sides of the plactic and hypoplactic relations have the same image under f^{ij}_{hypo}. Let $a, b, c, d \in A$. Assume, without loss of generality, that f^{ij}_{hypo} does not map any symbol to $[\varepsilon]_{\mathsf{hypo}_2}$.

If $a \leq b < c \leq d$, then either f^{ij}_{hypo} maps at least one symbol to $[21]_{\mathsf{hypo}_2}$, or f^{ij}_{hypo} maps a and b to $[1]_{\mathsf{hypo}_2}$, and c and d to $[2]_{\mathsf{hypo}_2}$. Thus, by Proposition 4.2.3,

$$f^{ij}_{\mathsf{hypo}}(acb) = f^{ij}_{\mathsf{hypo}}(cab) \quad \text{and} \quad f^{ij}_{\mathsf{hypo}}(acbd) = f^{ij}_{\mathsf{hypo}}(cadb),$$

since both classes in each side of the equalities have the same content and 2-1 inversions.

If $a < b \leq c$, then either f^{ij}_{hypo} maps at least one symbol to $[21]_{\mathsf{hypo}_2}$, or f^{ij}_{hypo} maps a to $[1]_{\mathsf{hypo}_2}$, and b and c to $[2]_{\mathsf{hypo}_2}$. Thus,

$$f^{ij}_{\mathsf{hypo}}(bac) = f^{ij}_{\mathsf{hypo}}(bca).$$

since both classes in each side of the equality have the same content and 2-1 inversions.

If $a < b \leq c < d$, then f^{ij}_{hypo} maps b and c to $[21]_{\mathsf{hypo}_2}$. Thus, by the same reasoning,

$$f^{ij}_{\mathsf{hypo}}(bdac) = f^{ij}_{\mathsf{hypo}}(dbca).$$

Hence, $\mathcal{R}_{\mathsf{hypo}} \subseteq \ker f^{ij}_{\mathsf{hypo}}$. Therefore, the congruence generated by $\mathcal{R}_{\mathsf{hypo}}$, which is the hypoplactic congruence $\equiv_{\mathsf{hypo}}$, is contained in $\ker f^{ij}_{\mathsf{hypo}}$. As such, by Theorem 2.4.1, there exists a homomorphism $\varphi^{ij}_{\mathsf{hypo}} : \mathsf{hypo} \longrightarrow \mathsf{hypo}_2$ such that $\varphi^{ij}_{\mathsf{hypo}} \circ \equiv^{\natural}_{\mathsf{hypo}} = f^{ij}_{\mathsf{hypo}}$. $\square$

Let $w \in A_n^*$, for some $n \geq 3$. Assume $\operatorname{supp}(w) = \{a_1 < \cdots < a_k\}$, for some $k \in \mathbb{N}$. Observe that, ranging $1 \leq i < k$, we can get the number of occurrences of a_i and a_{i+1} in w from the images of $[w]_{\mathsf{hypo}}$ under the maps $\varphi^{a_i a_{i+1}}_{\mathsf{hypo}}$. Furthermore, we can check if w has an a_{i+1}-a_i inversion: Since no element $a \in A$ such that $a_i < a < a_{i+1}$ occurs in w, each occurrence of 1 in (a word in) $\varphi^{a_i a_{i+1}}_{\mathsf{hypo}}([w]_{\mathsf{hypo}})$ corresponds to an occurrence of a_i in w and, similarly, each occurrence of 2 in $\varphi^{a_i a_{i+1}}_{\mathsf{hypo}}([w]_{\mathsf{hypo}})$ corresponds to an occurrence of a_{i+1} in w. Thus, $\varphi^{a_i a_{i+1}}_{\mathsf{hypo}}([w]_{\mathsf{hypo}})$ has a 2-1 inversion if and only if w has an a_{i+1}-a_i inversion. Hence, we get the following lemma:

Lemma 5.2.2. *Let $u, v \in A_n^*$. Then, $u \equiv_{\mathsf{hypo}} v$ if and only if $\varphi^{ij}_{\mathsf{hypo}}([u]_{\mathsf{hypo}}) = \varphi^{ij}_{\mathsf{hypo}}([v]_{\mathsf{hypo}})$, for all $1 \leq i < j \leq n$.*

Proof. The direct implication is trivial, due to the fact that $\varphi^{ij}_{\mathsf{hypo}}$ is well-defined as a map, for all $1 \leq i < j \leq n$. The proof of the converse follows from the previous observations, as well as Proposition 4.2.3. $\square$

Example 5.2.3. Checking the inversions for the words given in Example 4.2.2:

$$[31214]_{\mathsf{hypo}_4} \underset{\varphi^{12}_{\mathsf{hypo}}}{\longmapsto} [121]_{\mathsf{hypo}_2} \qquad [21341]_{\mathsf{hypo}_4} \underset{\varphi^{12}_{\mathsf{hypo}}}{\longmapsto} [211]_{\mathsf{hypo}_2}$$

$$[31214]_{\mathsf{hypo}_4} \underset{\varphi^{13}_{\mathsf{hypo}}}{\longmapsto} [21211]_{\mathsf{hypo}_2} \qquad [21341]_{\mathsf{hypo}_4} \underset{\varphi^{13}_{\mathsf{hypo}}}{\longmapsto} [21121]_{\mathsf{hypo}_2}$$

$$[31214]_{\mathsf{hypo}_4} \underset{\varphi^{14}_{\mathsf{hypo}}}{\longmapsto} [2112112]_{\mathsf{hypo}_2} \qquad [21341]_{\mathsf{hypo}_4} \underset{\varphi^{14}_{\mathsf{hypo}}}{\longmapsto} [2112121]_{\mathsf{hypo}_2}$$

$$[31214]_{\mathsf{hypo}_4} \underset{\varphi^{23}_{\mathsf{hypo}}}{\longmapsto} [21]_{\mathsf{hypo}_2} \qquad [21341]_{\mathsf{hypo}_4} \underset{\varphi^{23}_{\mathsf{hypo}}}{\longmapsto} [12]_{\mathsf{hypo}_2}$$

$$[31214]_{\mathsf{hypo}_4} \underset{\varphi^{24}_{\mathsf{hypo}}}{\longmapsto} [2112]_{\mathsf{hypo}_2} \qquad [21341]_{\mathsf{hypo}_4} \underset{\varphi^{24}_{\mathsf{hypo}}}{\longmapsto} [1212]_{\mathsf{hypo}_2}$$

$$[31214]_{\mathsf{hypo}_4} \underset{\varphi^{34}_{\mathsf{hypo}}}{\longmapsto} [12]_{\mathsf{hypo}_2} \qquad [21341]_{\mathsf{hypo}_4} \underset{\varphi^{34}_{\mathsf{hypo}}}{\longmapsto} [12]_{\mathsf{hypo}_2}$$

For each $n \in \mathbb{N}$, with $n \geq 3$, consider the index set $I_n = \{(i,j) : 1 \leq i < j \leq n\}$, and let $I := \bigcup_{n\in\mathbb{N}} I_n$. Now, consider the map

$$\phi_{\mathsf{hypo}_n} : \mathsf{hypo}_n \longrightarrow \prod_{I_n} \mathsf{hypo}_2,$$

whose (i,j)-th component is given by $\varphi^{ij}_{\mathsf{hypo}}([w]_{\mathsf{hypo}})$, for $w \in A_n^*$ and $(i,j) \in I_n$.

Proposition 5.2.4. *The map ϕ_{hypo_n} is an embedding.*

Proof. It is clear that ϕ_{hypo_n} is a homomorphism. It follows from the definition of ϕ_{hypo_n} and Lemma 5.2.2 that, for any $u,v \in A_n^*$, we have

$$u \equiv_{\mathsf{hypo}_n} v \iff \phi_{\mathsf{hypo}_n}([u]_{\mathsf{hypo}}) = \phi_{\mathsf{hypo}_n}([v]_{\mathsf{hypo}}),$$

hence ϕ_{hypo_n} is an embedding. □

Thus, for each $n \in \mathbb{N}$, we can embed hypo_n into a direct product of $\binom{n}{2}$ copies of hypo_2. Similarly, we can embed hypo into a direct product of infinitely many copies of hypo_2. Consider the map

$$\phi_{\mathsf{hypo}} : \mathsf{hypo} \longrightarrow \prod_{I} \mathsf{hypo}_2,$$

whose (i,j)-th component is given by $\varphi^{ij}_{\mathsf{hypo}}([w]_{\mathsf{hypo}})$, for $w \in A^*$ and $(i,j) \in I$.

Proposition 5.2.5. *The map ϕ_{hypo} is an embedding.*

Proof. It is clear that ϕ_{hypo} is a homomorphism. Notice that, for any word $w \in A^*$, there must exist $n \in \mathbb{N}$ such that $w \in A_n^*$. Furthermore, for $(i,j) \in I_n$, we have that the (i,j)-th component of $\phi_{\mathsf{hypo}}([w]_{\mathsf{hypo}})$ is equal to the (i,j)-th component of $\phi_{\mathsf{hypo}_n}([w]_{\mathsf{hypo}})$. Thus, for $u,v \in A^*$, if $\phi_{\mathsf{hypo}}([u]_{\mathsf{hypo}}) = \phi_{\mathsf{hypo}}([v]_{\mathsf{hypo}})$, then $\phi_{\mathsf{hypo}_n}([u]_{\mathsf{hypo}}) = \phi_{\mathsf{hypo}_n}([v]_{\mathsf{hypo}})$, for some $n \in \mathbb{N}$ such that $u,v \in A_n^*$.

It follows from Lemma 5.2.2 that, for any $u,v \in A^*$, we have

$$u \equiv_{\mathsf{hypo}} v \iff \phi_{\mathsf{hypo}}([u]_{\mathsf{hypo}}) = \phi_{\mathsf{hypo}}([v]_{\mathsf{hypo}}),$$

hence ϕ_{hypo} is an embedding. □

As such, all hypoplactic monoids of rank strictly greater than 2 are in the variety generated by hypo_2. Since hypo_2 is a submonoid of hypo and hypo_n, for any $n \geq 3$, they all generate the same variety, which we will denote by $\mathcal{V}_{\mathsf{hypo}}$. Thus, by Birkhoff's Theorem, we have the following result:

Theorem 5.2.6. *For any $n \geq 2$, hypo and hypo_n satisfy exactly the same identities.*

Another consequence of $\mathcal{V}_{\mathsf{hypo}}$ being generated by hypo_2 is the following:

Proposition 5.2.7. *The basis rank of $\mathcal{V}_{\mathsf{hypo}}$ is 2.*

Proof. Since $\mathcal{V}_{\mathsf{hypo}}$ is generated by hypo_2, and hypo_2 is defined by a presentation where the alphabet has two generators, then $r_b\left(\mathcal{V}_{\mathsf{hypo}}\right)$ is less than or equal to 2.

On the other hand, notice that any monoid generated by a single element is commutative. Since hypo is not commutative, $\mathcal{V}_{\mathsf{hypo}}$ cannot be generated by any single monoid which is itself generated by a single element. As such, $r_b\left(\mathcal{V}_{\mathsf{hypo}}\right)$ is strictly greater than 1.

Hence, the basis rank of $\mathcal{V}_{\mathsf{hypo}}$ is 2. □

5.2.2 Embeddings of the sylvester, #-sylvester and Baxter monoids

In this subsection, we prove that the sylvester monoids of ranks greater than or equal to 2 satisfy exactly the same identities and that the basis rank of the variety generated by sylv is 2. By parallel reasoning, we prove the same results for the #-sylvester monoids of ranks greater than or equal to 2. Furthermore, as a consequence of [Gir12, Proposition 3.7], we also obtain the same results for the Baxter monoids of ranks greater than or equal to 2.

For any $i, j \in A$, with $i < j$, define a map from A to sylv_2 in the following way: For any $a \in A$,

$$a \longmapsto \begin{cases} [1]_{\mathsf{sylv}_2} & \text{if } a = i; \\ [2]_{\mathsf{sylv}_2} & \text{if } a = j; \\ [21]_{\mathsf{sylv}_2} & \text{if } i < a < j; \\ [\varepsilon]_{\mathsf{sylv}_2} & \text{otherwise;} \end{cases}$$

and extend it to a homomorphism $f^{ij}_{\mathsf{sylv}} : A^* \longrightarrow \mathsf{sylv}_2$, in the usual way. This homomorphism is analogous to the one given in Subsection 5.2.1. The proofs of the following lemmata and propositions make use of the new characterization using right precedences for the sylvester monoid (see Proposition 4.3.9).

Notice that, for any $w \in A^*$, its image under f^{ij}_{sylv} is the sylvester class of the word obtained from w by replacing any occurrence of i by 1; any occurrence of j by 2; any occurrence of an a, with $i < a < j$, by 21; and erasing the occurrences of any other element.

Lemma 5.2.8. *f^{ij}_{sylv} factors to give a homomorphism $\varphi^{ij}_{\mathsf{sylv}} : \mathsf{sylv} \longrightarrow \mathsf{sylv}_2$.*

Proof. Since sylv is given by the presentation $\left\langle A \middle| \mathcal{R}_{\mathsf{sylv}} \right\rangle$, we just need to verify that both sides of the sylvester relations have the same image under f^{ij}_{sylv}.

Let $a, b, c \in A$ and $u \in A^*$ be such that $a \leq b < c$. If f^{ij}_{sylv} maps either a or c to $[\varepsilon]_{\mathsf{sylv}_2}$, then the images of $caub$ and $acub$ under f^{ij}_{sylv} coincide. Assume, without loss of generality, that f^{ij}_{sylv} does not map any symbol to $[\varepsilon]_{\mathsf{sylv}_2}$. Then, f^{ij}_{sylv} maps a to $[1]_{\mathsf{sylv}_2}$, b to either $[21]_{\mathsf{sylv}_2}$ or $[1]_{\mathsf{sylv}_2}$, and c to $[2]_{\mathsf{sylv}_2}$. Notice that b is mapped to an element with no right precedences. As such, we have that

$$f^{ij}_{\mathsf{sylv}}(caub) = f^{ij}_{\mathsf{sylv}}(acub),$$

since both sides of the equality have no right precedences. Hence, $\mathcal{R}_{\mathsf{sylv}} \subseteq \ker f^{ij}_{\mathsf{sylv}}$ and, by Theorem 2.4.1 once again, there exists a homomorphism $\varphi^{ij}_{\mathsf{sylv}} : \mathsf{sylv} \to \mathsf{sylv}_2$ such that $\varphi^{ij}_{\mathsf{sylv}} \circ \equiv^{\natural}_{\mathsf{sylv}} = f^{ij}_{\mathsf{sylv}}$. ☐

Let $w \in A_n^*$, for some $n \geq 3$. Assume $\mathrm{supp}(w) = \{a_1 < \cdots < a_k\}$, for some $k \in \mathbb{N}$. Observe that, ranging $1 \leq i < k$, we can get the number of occurrences of a_i and a_{i+1} in w from the images of $[w]_{\mathsf{sylv}}$ under the maps $\varphi^{a_i a_{i+1}}_{\mathsf{sylv}}$, since, when reading any word in $\varphi^{a_i a_{i+1}}_{\mathsf{sylv}}([w]_{\mathsf{sylv}})$, every occurrence of 1 corresponds exactly to an occurrence of a_i in w, and every occurrence of 2 corresponds exactly to an occurrence of a_{i+1}.

Recall that there is at most one index j, with $1 \leq i < j \leq k$, such that w has an a_j-a_i right precedence. Ranging $1 \leq i < j \leq k$, we can check if w has an a_j-a_i right precedence: If w does not have any b-a_i right precedence, for $b < a_j$, then no b occurs before a_i, when reading w from right-to-left. As such, when reading any word in $\varphi^{a_i a_j}_{\mathsf{sylv}}([w]_{\mathsf{sylv}})$ from right-to-left, the first occurrence of 1 corresponds to the first occurrence of a_i, when reading w from right-to-left, and all occurrences of 2 before the first occurrence of 1 correspond to all occurrences of a_j before the first occurrence of a_i. Thus, $\varphi^{a_i a_j}_{\mathsf{sylv}}([w]_{\mathsf{sylv}})$ has a 2-1 right precedence if and only if w has an a_j-a_i right precedence, and the indexes must coincide. Hence, we get the following lemma:

Lemma 5.2.9. *Let $u, v \in A_n^*$. Then, $u \equiv_{\mathsf{sylv}} v$ if and only if $\varphi^{ij}_{\mathsf{sylv}}([u]_{\mathsf{sylv}}) = \varphi^{ij}_{\mathsf{sylv}}([v]_{\mathsf{sylv}})$, for all $1 \leq i < j \leq n$.*

Proof. The direct implication is trivial, due to the fact that $\varphi^{ij}_{\mathsf{sylv}}$ is well-defined as a map, for all $1 \leq i < j \leq n$. The proof of the converse follows from the previous observations, as well as Proposition 4.3.9. ☐

Example 5.2.10. Checking the right precedences for the words given in Example 4.3.5

$$[3123]_{\mathsf{sylv}_3} \underset{\varphi^{12}_{\mathsf{sylv}}}{\longmapsto} [12]_{\mathsf{sylv}_2} \qquad [2313]_{\mathsf{sylv}_3} \underset{\varphi^{12}_{\mathsf{sylv}}}{\longmapsto} [21]_{\mathsf{sylv}_2}$$
$$[3123]_{\mathsf{sylv}_3} \underset{\varphi^{13}_{\mathsf{sylv}}}{\longmapsto} [21212]_{\mathsf{sylv}_2} \qquad [2313]_{\mathsf{sylv}_3} \underset{\varphi^{13}_{\mathsf{sylv}}}{\longmapsto} [21212]_{\mathsf{sylv}_2}$$
$$[3123]_{\mathsf{sylv}_3} \underset{\varphi^{23}_{\mathsf{sylv}}}{\longmapsto} [212]_{\mathsf{sylv}_2} \qquad [2313]_{\mathsf{sylv}_3} \underset{\varphi^{23}_{\mathsf{sylv}}}{\longmapsto} [122]_{\mathsf{sylv}_2}$$

$$[3132]_{\mathsf{sylv}_3} \underset{\varphi^{12}_{\mathsf{sylv}}}{\longmapsto} [12]_{\mathsf{sylv}_2}$$
$$[3132]_{\mathsf{sylv}_3} \underset{\varphi^{13}_{\mathsf{sylv}}}{\longmapsto} [21221]_{\mathsf{sylv}_2}$$
$$[3132]_{\mathsf{sylv}_3} \underset{\varphi^{23}_{\mathsf{sylv}}}{\longmapsto} [221]_{\mathsf{sylv}_2}$$

For each $n \in \mathbb{N}$, with $n \geq 3$, let I_n and I be the index sets defined in Subsection 5.2.1. Now, consider the map

$$\phi_{\mathsf{sylv}_n} : \mathsf{sylv}_n \longrightarrow \prod_{I_n} \mathsf{sylv}_2,$$

whose (i,j)-th component is given by $\varphi^{ij}_{\mathsf{sylv}}([w]_{\mathsf{sylv}})$, for $w \in A_n^*$ and $(i,j) \in I_n$.

Proposition 5.2.11. *The map ϕ_{sylv_n} is an embedding.*

Proof. It is clear that ϕ_{sylv_n} is a homomorphism. It follows from the definition of ϕ_{sylv_n} and Lemma 5.2.9 that we have

$$u \equiv_{\mathsf{sylv}_n} v \iff \phi_{\mathsf{sylv}_n}([u]_{\mathsf{sylv}}) = \phi_{\mathsf{sylv}_n}([v]_{\mathsf{sylv}}),$$

for any $u, v \in A_n^*$. Hence ϕ_{sylv_n} is an embedding. □

Thus, for each $n \in \mathbb{N}$, we can embed sylv_n into a direct product of $\binom{n}{2}$ copies of sylv_2. Similarly, we can embed sylv into a direct product of infinitely many copies of sylv_2. Consider the map

$$\phi_{\mathsf{sylv}} : \mathsf{sylv} \longrightarrow \prod_I \mathsf{sylv}_2,$$

whose (i, j)-th component is given by $\varphi^{ij}_{\mathsf{sylv}}([w]_{\mathsf{sylv}})$, for $w \in A^*$ and $(i, j) \in I$.

Proposition 5.2.12. *The map ϕ_{sylv} is an embedding.*

Proof. It is clear that ϕ_{sylv} is a homomorphism. Notice that, for any word $w \in A^*$, there must exist $n \in \mathbb{N}$ such that $w \in A_n^*$. Furthermore, for $(i, j) \in I_n$, we have that the (i, j)-th component of $\phi_{\mathsf{sylv}}([w]_{\mathsf{sylv}})$ is equal to the (i, j)-th component of $\phi_{\mathsf{sylv}_n}([w]_{\mathsf{sylv}})$. Thus, for $u, v \in A^*$, if $\phi_{\mathsf{sylv}}([u]_{\mathsf{sylv}}) = \phi_{\mathsf{sylv}}([v]_{\mathsf{sylv}})$, then $\phi_{\mathsf{sylv}_n}([u]_{\mathsf{sylv}}) = \phi_{\mathsf{sylv}_n}([v]_{\mathsf{sylv}})$, for some $n \in \mathbb{N}$ such that $u, v \in A_n^*$.

It follows from Lemma 5.2.9 that, for any $u, v \in A^*$, we have

$$u \equiv_{\mathsf{sylv}} v \iff \phi_{\mathsf{sylv}}([u]_{\mathsf{sylv}}) = \phi_{\mathsf{sylv}}([v]_{\mathsf{sylv}}),$$

hence ϕ_{sylv} is an embedding. □

As such, all sylvester monoids of rank higher than 2 are in the variety generated by sylv_2. Since sylv_2 is a submonoid of sylv and sylv_n, for any $n \geq 3$, they all generate the same variety, which we will denote by $\mathcal{V}_{\mathsf{sylv}}$. Thus, by Birkhoff's Theorem, we have the following result:

Theorem 5.2.13. *For any $n \geq 2$, sylv and sylv_n satisfy exactly the same identities.*

Another consequence of $\mathcal{V}_{\mathsf{sylv}}$ being generated by sylv_2 is the following:

Corollary 5.2.14. *The basis rank of $\mathcal{V}_{\mathsf{sylv}}$ is 2.*

Proof. The reasoning is identical to that given for the proof of Proposition 5.2.7. □

By parallel reasoning, we can also prove that the #-sylvester monoids of rank greater than or equal to 2 embed into a direct product of copies of $\mathsf{sylv}_2^{\#}$. For any $i, j \in A$, with $i < j$, define the homomorphism $f^{ij}_{\mathsf{sylv}^{\#}} : A^* \longrightarrow \mathsf{sylv}_2^{\#}$ in an identical fashion to f^{ij}_{hypo}, mapping a word to a #-sylvester class instead of a hypoplactic class.

As with the case of f^{ij}_{hypo} (see Lemma 5.2.1), we have that $f^{ij}_{\mathsf{sylv}^\#}$ factors to give a homomorphism $\varphi^{ij}_{\mathsf{sylv}^\#} : \mathsf{sylv}^\# \longrightarrow \mathsf{sylv}^\#_2$. Furthermore, for a word $w \in A_n^*$, with $n \geq 3$, we can deduce the number of occurrences of each symbol in w, and its left precedences, by looking at the images of $[w]_{\mathsf{sylv}^\#}$ under $\varphi^{ij}_{\mathsf{sylv}^\#}$, ranging $1 \leq i < j \leq n$. Thus, for $u, v \in A_n^*$, we have that $u \equiv_{\mathsf{sylv}^\#_n} v$ if and only if $\varphi^{ij}_{\mathsf{sylv}^\#}([u]_{\mathsf{sylv}^\#}) = \varphi^{ij}_{\mathsf{sylv}^\#}([v]_{\mathsf{sylv}^\#})$, for all $1 \leq i < j \leq n$.

Example 5.2.15. Checking the left precedences for the words given in Example 4.3.12

$$[1231]_{\mathsf{sylv}^\#_3} \underset{\varphi^{12}_{\mathsf{sylv}^\#}}{\longmapsto} [121]_{\mathsf{sylv}^\#_2} \qquad [1312]_{\mathsf{sylv}^\#_3} \underset{\varphi^{12}_{\mathsf{sylv}^\#}}{\longmapsto} [112]_{\mathsf{sylv}^\#_2}$$
$$[1231]_{\mathsf{sylv}^\#_3} \underset{\varphi^{13}_{\mathsf{sylv}^\#}}{\longmapsto} [12121]_{\mathsf{sylv}^\#_2} \qquad [1312]_{\mathsf{sylv}^\#_3} \underset{\varphi^{13}_{\mathsf{sylv}^\#}}{\longmapsto} [12121]_{\mathsf{sylv}^\#_2}$$
$$[1231]_{\mathsf{sylv}^\#_3} \underset{\varphi^{23}_{\mathsf{sylv}^\#}}{\longmapsto} [12]_{\mathsf{sylv}^\#_2} \qquad [1312]_{\mathsf{sylv}^\#_3} \underset{\varphi^{23}_{\mathsf{sylv}^\#}}{\longmapsto} [21]_{\mathsf{sylv}^\#_2}$$

$$[3121]_{\mathsf{sylv}^\#_3} \underset{\varphi^{12}_{\mathsf{sylv}^\#}}{\longmapsto} [121]_{\mathsf{sylv}^\#_2}$$
$$[3121]_{\mathsf{sylv}^\#_3} \underset{\varphi^{13}_{\mathsf{sylv}^\#}}{\longmapsto} [21211]_{\mathsf{sylv}^\#_2}$$
$$[3121]_{\mathsf{sylv}^\#_3} \underset{\varphi^{23}_{\mathsf{sylv}^\#}}{\longmapsto} [21]_{\mathsf{sylv}^\#_2}$$

For each $n \in \mathbb{N}$, we can embed $\mathsf{sylv}^\#_n$ into a direct product of $\binom{n}{2}$ copies of $\mathsf{sylv}^\#_2$, using the embedding

$$\phi_{\mathsf{sylv}^\#_n} : \mathsf{sylv}^\#_n \longrightarrow \prod_{I_n} \mathsf{sylv}^\#_2,$$

whose (i,j)-th component is given by $\varphi^{ij}_{\mathsf{sylv}^\#}([w]_{\mathsf{sylv}^\#})$, for $w \in A_n^*$ and $(i,j) \in I_n$. Similarly, we can embed $\mathsf{sylv}^\#$ into a direct product of infinitely many copies of $\mathsf{sylv}^\#_2$, using the embedding

$$\phi_{\mathsf{sylv}^\#} : \mathsf{sylv}^\# \longrightarrow \prod_{I} \mathsf{sylv}^\#_2,$$

whose (i,j)-th component is given by $\varphi^{ij}_{\mathsf{sylv}^\#}([w]_{\mathsf{sylv}^\#})$, for $w \in A^*$ and $(i,j) \in I$.

As such, all #-sylvester monoids of rank higher than 2 are in the variety generated by $\mathsf{sylv}^\#_2$. Since $\mathsf{sylv}^\#_2$ is a submonoid of $\mathsf{sylv}^\#$ and $\mathsf{sylv}^\#_n$, for any $n \geq 3$, they all generate the same variety, which we will denote by $\mathbf{V}_{\mathsf{sylv}^\#}$. Thus, by Birkhoff's Theorem, we have the following result:

Theorem 5.2.16. *For any $n \geq 2$, $\mathsf{sylv}^\#$ and $\mathsf{sylv}^\#_n$ satisfy exactly the same identities.*

We also obtain the following result:

Corollary 5.2.17. *The basis rank of $\mathbf{V}_{\mathsf{sylv}^\#}$ is 2.*

Notice that the embeddings of the #-sylvester monoids of finite rank greater than or equal to 2 into a direct product of copies of $\mathsf{sylv}^\#_2$ can also be obtained using the anti-isomorphisms between sylvester and #-sylvester monoids of finite rank, and the previously obtained embeddings. However, we cannot use this argument for the infinite rank case, due to Proposition 4.3.11. On the other hand, since anti-isomorphisms exist in the finite case, we can conclude that, due to Theorems 5.2.13 and 5.2.16, any monoid anti-isomorphic to sylv (respectively, $\mathsf{sylv}^{\#}$) is in the variety generated by $\mathsf{sylv}^{\#}$ (respectively, sylv).

Once again, we can also prove that the Baxter monoids of rank greater than or equal to 2 embed into a direct product of copies of baxt_2. For any $i, j \in A$, with $i < j$, define the homomorphism $f^{ij}_{\mathsf{baxt}} : A^* \longrightarrow \mathsf{baxt}_2$ in an identical fashion to f^{ij}_{hypo}, mapping a word to a Baxter class instead of a hypoplactic class.

As before with the case of f^{ij}_{hypo}, we have that f^{ij}_{baxt} factors to give a homomorphism $\varphi^{ij}_{\mathsf{baxt}} : \mathsf{baxt} \longrightarrow \mathsf{baxt}_2$. Furthermore, for a word $w \in A_n^*$, with $n \geq 3$, we can deduce the number of occurrences of each symbol in w, and its left and right precedences, by looking at the images of $[w]_{\mathsf{baxt}}$ under $\varphi^{ij}_{\mathsf{baxt}}$, ranging $1 \leq i < j \leq n$. Thus, for $u, v \in A_n^*$, we have that $u \equiv_{\mathsf{baxt}_n} v$ if and only if $\varphi^{ij}_{\mathsf{baxt}}([u]_{\mathsf{baxt}}) = \varphi^{ij}_{\mathsf{baxt}}([v]_{\mathsf{baxt}})$, for all $1 \leq i < j \leq n$.

For each $n \in \mathbb{N}$, we can embed baxt_n into a direct product of $\binom{n}{2}$ copies of baxt_2, using the embedding

$$\phi_{\mathsf{baxt}_n} : \mathsf{baxt}_n \longrightarrow \prod_{I_n} \mathsf{baxt}_2,$$

whose (i,j)-th component is given by $\varphi^{ij}_{\mathsf{baxt}}([w]_{\mathsf{baxt}})$, for $w \in A_n^*$ and $(i,j) \in I_n$. Similarly, we can embed baxt into a direct product of infinitely many copies of baxt_2, using the embedding

$$\phi_{\mathsf{baxt}} : \mathsf{baxt} \longrightarrow \prod_{I} \mathsf{baxt}_2,$$

whose (i,j)-th component is given by $\varphi^{ij}_{\mathsf{baxt}}([w]_{\mathsf{baxt}})$, for $w \in A^*$ and $(i,j) \in I$.

As such, all Baxter monoids of rank higher than 2 are in the variety generated by baxt_2. Since baxt_2 is a submonoid of baxt and baxt_n, for any $n \geq 3$, they all generate the same variety, which we will denote by $\mathcal{V}_{\mathsf{baxt}}$. Thus, by Birkhoff's Theorem, we have the following result:

Theorem 5.2.18. *For any $n \geq 2$, baxt and baxt_n satisfy exactly the same identities.*

Another consequence is the following:

Corollary 5.2.19 (Cor. 5.2.20, p.47). *The basis rank of $\mathcal{V}_{\mathsf{baxt}}$ is 2.*

We also have that all Baxter monoids of rank greater than or equal to 2 embed into a direct product of copies of $\mathsf{sylv}^{\#}_2$ and sylv_2. Since all Baxter monoids of rank greater than or equal to 2 embed into the direct product of the #-sylvester and sylvester monoids of the same rank (see Section 4.4), the following diagrams commute:

$$\begin{tikzcd}
\mathsf{baxt}_n \arrow[r, hook] \arrow[d, hook] & \mathsf{sylv}^{\#}_n \times \mathsf{sylv}_n \arrow[d, hook] \\
\prod\limits_{I_n} \mathsf{baxt}_2 \arrow[r, hook] & \left(\prod\limits_{I_n} \mathsf{sylv}^{\#}_2\right) \times \left(\prod\limits_{I_n} \mathsf{sylv}_2\right)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{baxt} \arrow[r, hook] \arrow[d, hook] & \mathsf{sylv}^{\#} \times \mathsf{sylv} \arrow[d, hook] \\
\prod\limits_{I} \mathsf{baxt}_2 \arrow[r, hook] & \left(\prod\limits_{I} \mathsf{sylv}^{\#}_2\right) \times \left(\prod\limits_{I} \mathsf{sylv}_2\right)
\end{tikzcd}$$

As such, baxt is a monoid in the variety generated by $\{\mathsf{sylv}^{\#}_2, \mathsf{sylv}_2\}$. This implies, by Birkhoff's Theorem, that baxt must satisfy all identities which are satisfied by both $\mathsf{sylv}^{\#}_2$ and sylv_2. Furthermore, by the observations made in Section 4.4, we have the following corollary:

Corollary 5.2.20. *The identities satisfied by the Baxter monoids of rank greater than or equal to 2 are exactly those identities which are simultaneously satisfied by the #-sylvester and sylvester monoids of rank greater than or equal to 2.*

6

Identities satisfied by the plactic-like monoids

In this chapter, we obtain a complete characterization of the identities satisfied, respectively, by the hypoplactic, sylvester, #-sylvester and Baxter monoids. These identities and those satisfied by their respective monoids of rank 2 are exactly the same. As such, we shall use the monoids of rank 2 to obtain a characterization of those identities.

Most of the results in Section 6.1 have appeared in [CMR21a], except for Corollary 6.1.5, while the results in Section 6.2 are to appear in the submitted paper [CMR21b].

6.1 Characterization of the identities satisfied by the hypoplactic monoid

Observe that, for each element of hypo_2, there is at most one other distinct element of hypo_2 which has the same content as it. Indeed, an element of hypo_2 with support $\{1,2\}$ either has a 2-1 inversion or not.

Theorem 6.1.1. *The identities $u \approx v$ satisfied by* hypo *are exactly the balanced identities such that u admits xy as a subsequence if and only if v does too, for any variables $x, y \in \operatorname{supp}(u \approx v)$.*

Proof. We first prove by contradiction that an identity satisfied by hypo_2 must satisfy the stated conditions. Suppose $u \approx v$ is an identity satisfied by hypo_2. Since hypo contains the free monogenic submonoid, we know that any identity satisfied by hypo must be a balanced identity. Thus, we assume $u \approx v$ is a balanced identity.

Assume, in order to obtain a contradiction, that there exist variables $x, y \in \operatorname{supp}(u \approx v)$, such that u admits xy as a subsequence, but v does not. Observe that, since both x and y occur in v, then v must admit yx as a subsequence.

Then, taking the evaluation ψ of X in hypo_2 such that $\psi(x) = [2]_{\mathsf{hypo}_2}$, $\psi(y) = [1]_{\mathsf{hypo}_2}$ and $\psi(z) = [\varepsilon]_{\mathsf{hypo}_2}$, for all other variables $z \in X$, we have

$$\psi(u) = [2^{|u|_x} 1^{|u|_y}]_{\mathsf{hypo}_2} \quad \text{and} \quad \psi(v) = [1^{|v|_y} 2^{|v|_x}]_{\mathsf{hypo}_2},$$

where $|u|_x = |v|_x \geq 1$ and $|u|_y = |v|_y \geq 1$. Notice that $\psi(u)$ has a 2-1 inversion, but $\psi(v)$ does not. Therefore, by Proposition 4.2.3, we have that $\psi(u) \neq \psi(v)$, which contradicts our hypothesis that $u \approx v$ is an identity satisfied by hypo_2.

We now prove by contradiction that an identity which satisfies the previously mentioned conditions must also be satisfied by hypo_2. Suppose that $u \approx v$ is a balanced identity, such that u admits xy as a subsequence if and only if v does too, for any variables $x, y \in \operatorname{supp}(u \approx v)$. Suppose, in order to obtain a contradiction, that there is some evaluation ψ of X in hypo_2 such that $\psi(u) \neq \psi(v)$.

Notice that, since $u \approx v$ is a balanced identity, then $\psi(u)$ and $\psi(v)$ have the same content. Since $u \approx v$ is non-trivial and $\psi(u) \neq \psi(v)$, then $\operatorname{supp}(\psi(u)) = \operatorname{supp}(\psi(v)) = \{1,2\}$ and, by Proposition 4.2.3, $\psi(u)$ has a 2-1 inversion but $\psi(v)$ does not, or the converse.

We assume, without loss of generality, that $\psi(u)$ has a 2-1 inversion and $\psi(v)$ does not. Note that $\psi(v)$, as a congruence class of words, has only one word, of the form $1^a 2^b$, for some $a, b \geq 1$. Then, v must be of the form $v = v_1 v_2$ or $v = v_1 z v_2$, with $z \in X$, where for each variable x which occurs in v_1, $\psi(x)$ has support $\{1\}$; for each variable y which occurs in v_2, $\psi(y)$ has support $\{2\}$; and $\psi(z)$ has support $\{1,2\}$.

Notice that z is a variable that occurs in neither v_1 nor v_2, and that no variable occurs simultaneously in v_1 and v_2. Also notice that, for any variables x occurring in v_1 and y occurring in v_2, v admits xz, xy and zy as subsequences (if there exists a variable z in the previously mentioned conditions), but not zx, yx nor yz. Thus, by our hypothesis, u must be of the form $u = u_1 u_2$ or $u = u_1 z u_2$, where $\operatorname{cont}(u_1) = \operatorname{cont}(v_1)$ and $\operatorname{cont}(u_2) = \operatorname{cont}(v_2)$. Hence, $\psi(u_1) = \psi(v_1)$ and $\psi(u_2) = \psi(v_2)$, by the observations in the previous paragraph.

Therefore, we either have

$$\psi(u) = \psi(u_1)\psi(u_2) = \psi(v_1)\psi(v_2) = \psi(v)$$

or

$$\psi(u) = \psi(u_1)\psi(z)\psi(u_2) = \psi(v_1)\psi(z)\psi(v_2) = \psi(v).$$

Thus, we have reached a contradiction, hence, there is no evaluation ψ of X in hypo_2 such that $\psi(u) \neq \psi(v)$. Thus, $u \approx v$ is an identity satisfied by hypo_2.

Since all identities satisfied by hypo must also be satisfied by hypo_2, we obtain the stated result. □

Recall that two identities are equivalent if one can be obtained from the other by renaming variables or swapping both sides of the identities. With this characterization, we recover as a corollary the following result:

Corollary 6.1.2 ([CM18b, Proposition 12]). *The following non-trivial identities are satisfied by* hypo*:*

$$xyxy \approx xyyx \approx yxxy \approx yxyx;$$
$$xxyx \approx xyxx.$$

Furthermore, up to equivalence, these are the shortest non-trivial identities satisfied by hypo.

We also easily obtain some important non-trivial identities satisfied by hypo:

Example 6.1.3. The following non-trivial identities are satisfied by hypo:

$$xyzxty \approx yxzxty; \tag{L}$$

$$xzxytx \approx xzyxtx; \tag{M}$$

$$xzytxy \approx xzytyx. \tag{R}$$

These identities form a basis for the variety generated by the hypoplactic monoid, as stated in Theorem 7.1.1.

It is well known that the set of all balanced identities is the equational theory of the variety $\mathbf{C}$ of all commutative monoids, which is generated by the free monogenic monoid. On the other hand, the set J_2 of all identities $u \approx v$ where u and v share exactly the same subsequences of length at most 2 is the equational theory of the *pseudovariety* $\mathfrak{J}_2$, which, due to Eilenberg's correspondence (see [Eil76; Pin86]), corresponds to the class of all piecewise testable languages of height 2 (see [Sim72]). This pseudovariety is generated by $\mathcal{C}_3$, the 5-element monoid of all order preserving and extensive transformations of the chain $1 < 2 < 3$ (see [Vol04]). Thus, the equational theory of $\mathcal{V}(\mathcal{C}_3)$, the *variety* generated by $\mathcal{C}_3$, is J_2. It is easy to see that the equational theory of $\mathcal{V}_{\mathsf{hypo}}$ is the intersection of the set of all balanced identities and the set J_2. As such, we have the following corollary of Theorem 6.1.1, suggested by the anonymous referee of [CMR21a]:

Corollary 6.1.4. *$\mathcal{V}_{\mathsf{hypo}}$ is the varietal join $\mathbf{C} \vee \mathcal{V}(\mathcal{C}_3)$, and is generated by the free monogenic monoid and the monoid $\mathcal{C}_3$.*

Since there exist polynomial-time algorithms that check if two words over an alphabet X, with length up to some $k \in \mathbb{N}$, have the same content and share exactly the same subsequences of length at most 2 (see, for example, [Bar+20, Theorem 8]), we have the following corollary:

Corollary 6.1.5. *The decision problem* Check-Id(hypo) *belongs to the complexity class* P.

Let $u\restriction_{x,y}$ be the word obtained from u by eliminating every occurrence of a symbol other than x or y. An alternative characterization of the identities satisfied by hypo is the following:

Corollary 6.1.6. *The identities $u \approx v$ satisfied by hypo are exactly the balanced identities such that, for any variables $x, y \in \operatorname{supp}(u \approx v)$, the identity $u\restriction_{x,y} \approx v\restriction_{x,y}$ is satisfied by hypo.*

Proof. Let ψ be an evaluation of $\{x, y\}$ in hypo_2. Extend it to an evaluation of all variables in $u \approx v$ by mapping all other variables to the empty word. Then

$$\psi(u\restriction_{x,y}) = \psi(u) = \psi(v) = \psi(v\restriction_{x,y}),$$

since $u \approx v$ is satisfied by hypo. Hence $u\restriction_{x,y} \approx v\restriction_{x,y}$ is also satisfied by hypo.

Suppose then that the identity $u \approx v$ is balanced and satisfies the following property: For any variables $x, y \in \operatorname{supp}(u \approx v)$, the identity $u\restriction_{x,y} \approx v\restriction_{x,y}$ is satisfied by hypo.

By Theorem 6.1.1, to prove that $u \approx v$ is satisfied by hypo, we only need to show that u admits xy (or yx) as a subsequence if and only if v does too, for any variables $x, y \in \operatorname{supp}(u \approx v)$.

Let $x, y \in \operatorname{supp}(u \approx v)$. By our hypothesis, we have that $u\restriction_{x,y} \approx v\restriction_{x,y}$ is an identity satisfied by hypo. By Theorem 6.1.1, we know that $u\restriction_{x,y}$ admits xy (or yx) as a subsequence if and only if $v\restriction_{x,y}$ does too.

Notice that $u\restriction_{x,y}$ is the unique subsequence of u with the same number of occurrences of x and y as u. Therefore, we can conclude that $u\restriction_{x,y}$ admits xy (or yx) as a subsequence if and only if u does too. The same can be said about $v\restriction_{x,y}$ and v.

Therefore, u admits xy (or yx) as a subsequence if and only if v does too. Hence, by Theorem 6.1.1, we conclude that $u \approx v$ is an identity satisfied by hypo. □

It is very easy to verify if a balanced identity, over a two-symbol alphabet, is satisfied by hypo, by the following complete characterization:

Corollary 6.1.7. *The non-trivial identities, over the two-symbol alphabet $\{x, y\}$, satisfied by* hypo *are balanced identities such that neither side of the identity is of the form $x^a y^b$ or $y^b x^a$, for some $a, b \in \mathbb{N}$.*

Proof. Suppose first that $u \approx v$ is a non-trivial identity, over the two-symbol alphabet $\{x, y\}$, satisfied by hypo. By Theorem 6.1.1, this identity is balanced. Suppose, without loss of generality, that $u = x^a y^b$, for some $a, b \in \mathbb{N}$. Then, since u does not admit yx as a subsequence, v cannot as well. Since $u \approx v$ is a balanced identity, we conclude that $v = x^a y^b = u$, which contradicts our hypothesis that $u \approx v$ is non-trivial.

Since this argument can be applied to all other possible cases, we conclude that neither side of the identity is of the form $x^a y^b$ or $y^b x^a$.

Conversely, let $u \approx v$ be a non-trivial, balanced identity, over the two-symbol alphabet $\{x, y\}$, such that neither side of the identity is of the form $x^a y^b$ or $y^b x^a$, for some $a, b \in \mathbb{N}$. Then, since $u \approx v$ is a non-trivial identity, both x and y must occur at least once in both u and v.

Observe that the only words over $\{x, y\}$, where both x and y occur, that do not admit yx as a subsequence, are words of the form $x^a y^b$. Similarly, the only words over $\{x, y\}$, where both x and y occur, that do not admit xy as a subsequence, are words of the form $y^b x^a$. Thus, both u and v admit xy and yx as subsequences. Hence, by Theorem 6.1.1, $u \approx v$ is satisfied by hypo. □

The following corollary will be important in Subsection 7.1.1:

Corollary 6.1.8. *The shortest non-trivial identity, with n variables, satisfied by* hypo*, is of length $n + 2$.*

Proof. It is immediate, by the previous theorem, that for variables $x, a_1 \dots a_{n-1}$,

$$xa_1 \dots a_{n-1}xx \approx xxa_1 \dots a_{n-1}x$$

is an identity satisfied by hypo.

On the other hand, assume, in order to obtain a contradiction, that there exists a non-trivial identity $u \approx v$, with n variables, satisfied by hypo, of length strictly less than $n+2$, such that $\mathrm{supp}(u \approx v) = \{a_1, \dots, a_n\}$. By Theorem 6.1.1, if u admits a subsequence xy, for variables $x, y \in \mathrm{supp}(u \approx v)$, then v must also admit such a subsequence. Thus, we easily conclude that the identity cannot be of length n, otherwise it would be trivial. As such, there must be some variable z which occurs at least twice in u and v. Assume, without loss of generality, that $z = a_n$.

Then $u \approx v$ must be of length $n+1$. Hence, a_n occurs exactly twice and all other variables $a_1, \dots, a_{n-1}$ occur only once, in u and v. Assume, without loss of generality, that u admits the subsequence $a_1 \cdots a_{n-1}$. Then, u is of the form

$$w_1\, a_n\, w_2\, a_n\, w_3,$$

where w_1, w_2, w_3 are (possibly empty) words over $\mathrm{supp}(u \approx v)$ such that $w_1 w_2 w_3 = a_1 \cdots a_{n-1}$. Notice that, for $i \in \{1, \dots, n-1\}$, if a_i occurs in w_1, then u admits $a_i a_n$ as a subsequence, but not $a_n a_i$; if a_i occurs in w_2, then u admits both $a_i a_n$ and $a_n a_i$ as subsequences; a_i occurs in w_3, then u admits $a_n a_i$ as a subsequence, but not $a_i a_n$. Once again, by Theorem 6.1.1, v must admit exactly these subsequences. Hence, we can conclude that $u = v$.

This contradicts the hypothesis that $u \approx v$ is a non-trivial identity satisfied by hypo. As such, there is no non-trivial identity, with n variables, satisfied by hypo, of length $n+1$. ☐

6.2 Characterization of the identities satisfied by the sylvester, #-sylvester and Baxter monoids

For a word u over an alphabet of variables X, and for variables $x, y \in \mathrm{supp}(u)$, we denote the number of occurrences of y before the first occurrence of x in u, when reading u from right-to-left (respectively, from left-to-right), by $o_{x \leftarrow y}(u)$ (respectively, $o_{y \rightarrow x}(u)$).

Theorem 6.2.1. *The identities $u \approx v$ satisfied by sylv are exactly the balanced identities such that $o_{x \leftarrow y}(u) = o_{x \leftarrow y}(v)$, for any variables $x, y \in \mathrm{supp}(u \approx v)$.*

Proof. We first prove by contradiction that an identity satisfied by sylv_2 must satisfy the stated conditions. Suppose $u \approx v$ is an identity satisfied by sylv_2. Since sylv_2 contains the free monogenic submonoid, we know that any identity satisfied by sylv_2 must be a balanced identity. Thus, we assume $u \approx v$ is a balanced identity.

Suppose, in order to obtain a contradiction, that there exist variables $x, y \in \mathrm{supp}(u \approx v)$, such that $o_{x \leftarrow y}(u) \neq o_{x \leftarrow y}(v)$. Then, if we consider the words $u\restriction_{x,y}$ and $v\restriction_{x,y}$, obtained from

u and v, respectively, by eliminating every occurrence of a variable other than x or y, we have that $u\restriction_{x,y}$ admits the suffix $xy^{o_{x\leftarrow y}(u)}$ and $v\restriction_{x,y}$ admits the suffix $xy^{o_{x\leftarrow y}(v)}$.

Taking the evaluation ψ of X in sylv_2 such that $\psi(x) = [1]_{\mathsf{sylv}_2}$, $\psi(y) = [2]_{\mathsf{sylv}_2}$ and $\psi(z) = [\varepsilon]_{\mathsf{sylv}_2}$, for all other variables $z \in X$, we have

$$\psi(u) = \psi\left(u\restriction_{x,y}\right) = [u']_{\mathsf{sylv}_2} \cdot [12^{o_{x\leftarrow y}(u)}]_{\mathsf{sylv}_2} \text{ and } \psi(v) = \psi\left(v\restriction_{x,y}\right) = [v']_{\mathsf{sylv}_2} \cdot [12^{o_{x\leftarrow y}(v)}]_{\mathsf{sylv}_2},$$

for some words $u', v' \in A_2^*$. Since $o_{x\leftarrow y}(u) \neq o_{x\leftarrow y}(v)$, we have that $\psi(u)$ and $\psi(v)$ cannot share a 2-1 right precedence of the same index. Thus, by Proposition 4.3.9, we have that $\psi(u) \neq \psi(v)$, which contradicts our hypothesis that $u \approx v$ is an identity.

We now prove by contradiction that an identity which satisfies the previously mentioned conditions must also be satisfied by sylv_2. Suppose that $u \approx v$ is a balanced identity, such that $o_{x\leftarrow y}(u) = o_{x\leftarrow y}(v)$, for any variables $x, y \in \operatorname{supp}(u \approx v)$. Suppose, in order to obtain a contradiction, that there is some evaluation ψ of X in sylv_2 such that $\psi(u) \neq \psi(v)$.

Notice that, since $u \approx v$ is a balanced identity, then $\psi(u)$ and $\psi(v)$ have the same content. As such, since $u \approx v$ is non-trivial, we have that $\operatorname{supp}(\psi(u)) = \operatorname{supp}(\psi(v)) = \{1,2\}$, and, by Proposition 4.3.9, either $\psi(u)$ and $\psi(v)$ have 2-1 right precedences of different indexes, or one of them has a 2-1 right precedence and the other does not. Assume, without loss of generality, that words in $\psi(u)$ admit a suffix of the form 12^a, and words in $\psi(v)$ admit a suffix of the form 12^b, for some $a, b \in \mathbb{N}_0$ such that $a > b$. Notice that this assumption covers both the case where $\psi(v)$ has a right precedence and the case where it does not. Furthermore, the assumption implies that $\psi(u)$ has a 2-1 right precedence of index a.

Observe that u must be of the form $u = u_1 z u_2$, with $z \in X$ and $u_2 \in X^*$, such that $\psi(u_2)$ has support $\{2\}$ and $\psi(z)$ has either support $\{1,2\}$ or support $\{1\}$. As such, $\psi(zu_2)$ has a 2-1 right precedence of index a, the same as $\psi(u)$. Furthermore, notice that z cannot occur in u_2. Thus, by our hypothesis, we have that $o_{z\leftarrow x}(u) = o_{z\leftarrow x}(v)$, for any variable $x \in \operatorname{supp}(u_2)$. Therefore, v must also be of the form $v = v_1 z v_2$, where v_2 has the same content as u_2. But this implies that $\psi(v_2)$ has support $\{2\}$, hence $\psi(zv_2)$ also has a 2-1 right precedence of index a. Since $\psi(zv_2)$ must also have the same right precedence as $\psi(v)$, we have reached a contradiction.

Thus, there is no evaluation ψ of X in sylv_2 such that $\psi(u) \neq \psi(v)$. Therefore, $u \approx v$ is an identity satisfied by sylv_2.

Since all identities satisfied by sylv must also be satisfied by sylv_2, we obtain the stated result. □

By parallel reasoning, we can obtain the characterization of the identities satisfied by the #-sylvester monoid:

Theorem 6.2.2. *The identities $u \approx v$ satisfied by $\mathsf{sylv}^\#$ are exactly the balanced identities such that, for any variables $x, y \in \operatorname{supp}(u \approx v)$, $o_{x\rightarrow y}(u) = o_{x\rightarrow y}(v)$.*

The characterization of the identities satisfied by the Baxter monoid is an immediate consequence of Theorems 6.2.1 and 6.2.2, and Corollary 5.2.20:

Theorem 6.2.3. *The identities $u \approx v$ satisfied by* baxt *are exactly the balanced identities such that, for any variables $x, y \in \operatorname{supp}(u \approx v)$, $o_{x\to y}(u) = o_{x\to y}(v)$ and $o_{x\leftarrow y}(u) = o_{x\leftarrow y}(v)$.*

With these characterizations, we recover the following corollaries:

Corollary 6.2.4 ([CM18b, Proposition 20])**.** *The sylvester monoid satisfies the non-trivial identity $xyxy \approx yxxy$. Furthermore, up to equivalence, this is the shortest non-trivial identity satisfied by* sylv.

Corollary 6.2.5 ([CM18b, Proposition 24])**.** *The #-sylvester monoid satisfies the non-trivial identity $yxyx \approx yxxy$. Furthermore, up to equivalence, this is the shortest non-trivial identity satisfied by* $\mathsf{sylv}^{\#}$.

Corollary 6.2.6 ([CM18b, Proposition 26])**.** *The Baxter monoid satisfies the non-trivial identities $yxxyxy \approx yxyxxy$ and $xyxyxy \approx xyyxxy$. Furthermore, up to equivalence, these are the shortest non-trivial identities satisfied by* baxt.

The following corollaries are useful alternative characterizations of the identities satisfied by sylv, $\mathsf{sylv}^{\#}$ and baxt. They imply that, when reading both sides of an identity satisfied by the sylvester, #-sylvester or Baxter monoid, the first occurrence of a variable is read at the same time in both words:

Corollary 6.2.7. *The identities $u \approx v$ satisfied by* sylv *(respectively,* $\mathsf{sylv}^{\#}$*) are balanced identities such that, for any $x \in \operatorname{supp}(u \approx v)$, the longest suffix (respectively, prefix) of u where x does not occur has the same content as the longest suffix (respectively, prefix) of v where x does not occur.*

Proof. We give the proof for the sylv case. The reasoning for the $\mathsf{sylv}^{\#}$ case is parallel.

Let $u = u_1 x u_2$ and $v = v_1 x v_2$, where u_2 and v_2 are words where x does not occur. Notice that, due to Theorem 6.2.1, for any variable y which occurs in u_2 or v_2, we have that $o_{x\leftarrow y}(u) = o_{x\leftarrow y}(v)$. The result follows immediately. □

Corollary 6.2.8. *The identities $u \approx v$ satisfied by* baxt *are balanced identities such that, for any $x \in \operatorname{supp}(u \approx v)$, the longest prefix of u where x does not occur has the same content as the longest prefix of v where x does not occur, and the longest suffix of u where x does not occur has the same content as the longest suffix of v where x does not occur.*

Proof. The result follows from the previous corollary. □

These alternate characterizations allow us to obtain algorithms that check if identities are satisfied by the sylvester, #-sylvester and Baxter monoids in polynomial time. For brevity's sake, we only show the algorithm for the sylvester case:

Proposition 6.2.9. *Algorithm 5 is sound and complete, and has time complexity $\mathcal{O}(k^2 \log(k))$, where k is the length of the word u, for input $u \approx v$.*

Algorithm 5: *Identity checking algorithm for the sylvester monoid.*

Input: An identity $u \approx v$.
Output: True if sylv satisfies $u \approx v$, False otherwise.

```
1  if |u| ≠ |v| then return False;
2  k ← |u|;
3  C[1,...,k], D[1,...,k] ← [0,...,0];
4  s⃖ ← ∅;
5  for 0 ≤ i ≤ k − 1 do
6      if u_{k−i} = v_{k−i} then
7          if u_{k−i} ∉ supp(s⃖) then
8              if C ≠ D then
9                  return False;
10             else
11                 append u_{k−i} to s⃖;
12                 j ← |s⃖|;
13                 C[j] ← C[j] + 1; D[j] ← D[j] + 1;
14         else
15             j ← index of u_{k−i} in s⃖;
16             C[j] ← C[j] + 1; D[j] ← D[j] + 1;
17     else
18         if u_{k−i}, v_{k−i} ∈ supp(s⃖) then
19             j ← index of u_{k−i} in s⃖;
20             l ← index of v_{k−i} in s⃖;
21             C[j] ← C[j] + 1; D[l] ← D[l] + 1;
22         else
23             return False;
24 if C ≠ D then return False;
25 return True
```

Proof. Algorithm 5 first checks if u and v have the same length, in line 1. If they do not, then $u \approx v$ is not a balanced identity, and as such, is not satisfied by sylv. This is done in $2k+1$ time, in the worst-case scenario where the length of v is greater than the length of u.

The algorithm scans u and v, from right-to-left, in the **for** loop in line 5. The arrays C and D stand for, respectively, the content vectors of the suffixes of u and v read so far, while the word $\overleftarrow{s}$ stands for the support of these suffixes. Notice that, since u is of length k, then at most k variables occur in u. Hence, C and D have length k.

In each iteration of the loop, the algorithm checks if the symbol which is being read in u is the same as the one being read in v. If they are the same, and do not occur in $\overleftarrow{s}$, this means that this is the first occurrence of a variable x. The algorithm checks if the arrays C and D are equal. If they are not, this implies that the longest suffix of u where x

does not occur does not have the same content as the longest suffix of v where x does not occur. Hence, by Corollary 6.2.7, $u \approx v$ is not satisfied by sylv. If C and D are equal, then the algorithm registers the new variable in $\overleftarrow{s}$ and updates the content vectors C and D. On the other hand, if x occurs already in $\overleftarrow{s}$, the algorithm simply updates C and D.

If the symbols which are being read in u and v are different, then the algorithm checks if they both occur in $\overleftarrow{s}$. If that does not happen, then that means at least one of them is the first occurrence of a variable in one of the words, but not in the other. Hence, by Corollary 6.2.7, $u \approx v$ is not satisfied by sylv. Otherwise, if they both occur in $\overleftarrow{s}$, the algorithm simply updates C and D.

After the final iteration of the loop, the algorithm checks if C and D are equal, to verify if the content of u is the same as the content of v. This is done because two words might have the same support, but their content might differ.

It is clear that the algorithm is sound and complete, since it always detects when a new variable is read, if it is read at the same time in both u and v, and if the content of the suffixes is the same, as well as if the content of u and v is the same.

Taking into consideration that operations of addition and comparing numbers are logarithmic time in a Turing machine model, and that accessing coordinates of vectors is a linear-time operation, we have that comparing the content vectors C and D takes at most $\mathcal{O}(k\log(k))$ time, and updating them takes $\mathcal{O}(k\log(k))$ time as well. On the other hand, checking if a variable occurs in $\overleftarrow{s}$ takes $\mathcal{O}(k)$ time. As such, each iteration of the **for** loop has time complexity $\mathcal{O}(k\log(k))$. Since there are k iterations of the loop, and no other part of the algorithm takes as much time as the loop, we can conclude that Algorithm 5 has time complexity $\mathcal{O}\left(k^2\log(k)\right)$. □

Corollary 6.2.10. *The decision problem* CHECK-ID(sylv) *belongs to the complexity class* P.

By parallel reasoning, we can also construct an algorithm that checks if identities hold in $\mathsf{sylv}^{\#}$, with time complexity $\mathcal{O}(k^2\log(k))$. From that algorithm and Algorithm 5, we can construct another algorithm for the Baxter case, also with polynomial time complexity. As such, we also have the following corollary:

Corollary 6.2.11. *The decision problems* CHECK-ID($\mathsf{sylv}^{\#}$) *and* CHECK-ID(baxt) *belong to the complexity class* P.

We can easily obtain some important non-trivial identities satisfied by these monoids:

Example 6.2.12. Recall that the following non-trivial identities, given in Example 6.1.3, are satisfied by the hypoplactic monoid:

$$xyzxty \approx yxzxty; \tag{L}$$

$$xzxytx \approx xzyxtx; \tag{M}$$

$$xzytxy \approx xzytyx. \tag{R}$$

The sylvester monoid satisfies (L), but satisfies neither (M) nor (R). On the other hand, the #-sylvester monoid satisfies (R), but satisfies neither (L) nor (M).

The Baxter monoid satisfies the following non-trivial identities:

$$xzyt\ xy\ rxsy \approx xzyt\ yx\ rxsy; \tag{O}$$

$$xzyt\ xy\ rysx \approx xzyt\ yx\ rysx. \tag{E}$$

These identities form bases for the varieties generated by the sylvester monoid, as stated in Theorem 7.1.5, the #-sylvester monoid, as stated in Theorem 7.1.6, and the Baxter monoid, as stated in Theorem 7.1.7.

The following corollaries will be important in Subsection 7.1.2:

Corollary 6.2.13. *The shortest non-trivial identities, with n variables, satisfied by* sylv *or by* $\mathsf{sylv}^{\#}$*, are of length $n+2$.*

Proof. Since any identity satisfied by sylv must also be satisfied by hypo, and since the shortest non-trivial identity, with n variables, satisfied by hypo, is of length $n+2$ (see 6.1.8), then a non-trivial identity, with n variables, satisfied by sylv, must be of length at least $n+2$.

On the other hand, by Theorem 6.2.1, it is immediate that

$$xya_1 \dots a_{n-2}yx \approx yxa_1 \dots a_{n-2}yx$$

is an identity satisfied by sylv, for variables $x, y, a_1, \dots, a_{n-2}$.

The reasoning for identities satisfied by $\mathsf{sylv}^{\#}$ is parallel to the one given previously. □

Corollary 6.2.14. *The shortest non-trivial identity, with n variables, satisfied by* baxt*, is of length $n+4$.*

Proof. It is immediate, by Theorem 6.2.3, that for variables $x, y, a_1 \dots a_{n-2}$,

$$xy\ xy\ a_1 \dots a_{n-2}yx \approx xy\ yx\ a_1 \dots a_{n-2}yx$$

is an identity satisfied by baxt.

On the other hand, let $u \approx v$ be a non-trivial identity, with n variables, satisfied by baxt, such that $u = wxu'$ and $v = wyv'$, for some words w, u', v' over the alphabet of variables X. Notice that, since $u \approx v$ is satisfied by baxt, it must be balanced, hence $\mathrm{cont}(xu') = \mathrm{cont}(yv')$. Therefore, x must occur in v' and y must occur in u'.

Observe that y must occur in w, otherwise, we would have $o_{x \to y}(u) > o_{x \to y}(v)$. Similarly, x must occur in w. On the other hand, by Corollary 6.2.8, x must occur in u' and y must occur in v', since $|u'| = |v'|$. Therefore, x and y both occur at least three times each in u and v. Since $u \approx v$ is an identity with n variables, it must be of length at least $n+4$. □

Finally, we can also clarify the relation between the Baxter monoids and the plactic monoids:

Corollary 6.2.15. *The variety generated by* baxt *is strictly contained in the variety generated by* plac_2.

Proof. Let $u \approx v$ be an identity satisfied by plac_2. Thus, it must be a balanced identity. Let $x, y \in \operatorname{supp}(u \approx v)$. Suppose, in order to obtain a contradiction, that $o_{x \to y}(u) > o_{x \to y}(v)$. Let

$$u = u_1\, y\, u_2 \quad \text{and} \quad v = v_1\, y\, v_2,$$

where u_1 (respectively, v_1) is the longest prefix of u (respectively, v) where y does not occur. Since the equational theory of the variety generated by plac_2 is left 1-hereditary (see Section 3.1), then $u_1 \approx v_1$ must be satisfied by plac_2. Hence, it must be a balanced identity. But $|u_1|_x = o_{x \to y}(u) > o_{x \to y}(v) = |v_1|_x$. We have reached a contradiction, hence, $o_{x \to y}(u) \not> o_{x \to y}(v)$.

By this reasoning, we prove that $o_{x \to y}(u) = o_{x \to y}(v)$ and $o_{x \leftarrow y}(u) = o_{x \leftarrow y}(v)$. Hence, by Theorem 6.2.3, $u \approx v$ must be satisfied by baxt.

On the other hand, it is well-known that the shortest non-trivial identity satisfied by the bicyclic monoid is Adjan's identity $xyyxxyxyyx \approx xyyxyxxyyx$ (see [Adj66]). Since plac_2 satisfies exactly the same identities as the bicyclic monoid (see [DJK18, Theorem 4.1]), plac_2 does not satisfy any non-trivial identity of length less than 10. But baxt satisfies an identity of length 6, as seen in Corollary 6.2.6. Thus, not all identities satisfied by baxt are satisfied by plac_2.

Therefore, as a consequence of Birkhoff's Theorem, the variety generated by baxt is strictly contained in the variety generated by plac_2. □

7

Varieties generated by the plactic-like monoids

In this chapter, we obtain finite bases for $\mathcal{V}_{\mathsf{hypo}}$, $\mathcal{V}_{\mathsf{sylv}}$, $\mathcal{V}_{\mathsf{sylv}^\#}$ and $\mathcal{V}_{\mathsf{baxt}}$, using the identities given in Examples 6.1.3 and 6.2.12, and also their axiomatic ranks. We then show how to subdirectly represent finitely generated multihomogeneous monoids by subdirectly irreducible monoids.

The results in Subsection 7.1.1 have appeared in [CMR21a], while the results in Subsection 7.1.2 are to appear in the submitted paper [CMR21b]. The results in Section 7.2 are new, to the best of the author's knowledge.

7.1 Finite bases and axiomatic rank

7.1.1 The axiomatic rank of the variety generated by the hypoplactic monoid

The aim of this subsection is to prove that not only $\mathcal{V}_{\mathsf{hypo}}$ has finite axiomatic rank, but that it is also finitely based. We give a basis for $\mathcal{V}_{\mathsf{hypo}}$ with three identities, all of them over a four-symbol alphabet, each of length 6. This basis is minimal, in the sense that no identity in this basis is a consequence of the others, and also that each identity is of minimal length, for identities satisfied by hypo over a four-symbol alphabet. Furthermore, we also prove that there exists no basis for $\mathcal{V}_{\mathsf{hypo}}$ with only identities over an alphabet with at most three variables, thus showing that the axiomatic rank of $\mathcal{V}_{\mathsf{hypo}}$ is 4.

Theorem 7.1.1. $\mathcal{V}_{\mathsf{hypo}}$ *admits a finite basis* $\mathcal{B}_{\mathsf{hypo}}$*, consisting of the following identities:*

$$xyzxty \approx yxzxty; \tag{L}$$

$$xzxytx \approx xzyxtx; \tag{M}$$

$$xzytxy \approx xzytyx. \tag{R}$$

Proof. Let $\mathcal{B}_{\mathsf{hypo}}$ be the set comprising the three identities (L), (M) and (R). Notice that these identities are the ones given in Example 6.1.3.

The proof will be by induction, in the following sense: We order identities by the length of the common prefix of both sides of the identity. The induction will be on the length of the suffix, that is, the length of the identity minus the length of the common prefix.

The base case for the induction, for identities of length n (with $n \geq 4$), is the identities of the form

$$wxy \approx wyx,$$

where w is a word of length $n-2$ and x, y are variables. Observe that, since any identity $u \approx v$ satisfied by hypo is a balanced identity, there are no non-trivial identities, of length n, with a common prefix of length greater than $n-2$, satisfied by hypo. That is, the suffix after the common prefix must have length at least 2. Furthermore, since wxy admits a subsequence xy and wyx admits a subsequence yx, then x and y must both occur in w. Thus, w is of the form

$$w_1xw_2yw_3 \quad \text{or} \quad w_1yw_2xw_3,$$

for some words w_1, w_2, w_3. Therefore, by replacing z with w_2, and t by w_3, and, if necessary, renaming x and y, we can immediately deduce this identity from the identity (R) of $\mathcal{B}_{\mathsf{hypo}}$. Notice that, when $n = 4$, the base case corresponds to the identities given in Corollary 6.1.2.

The idea of the proof of the induction step is that, for any identity $u \approx v$, we can apply identities of $\mathcal{B}_{\mathsf{hypo}}$, finitely many times, to deduce a new identity $u \approx u^*$ from $u \approx v$, such that u^* is "closer" to v than u, in the sense that u^* and v have a common prefix which is strictly longer than the common prefix of u and v. Notice that $u^* \approx v$ is a consequence of $\mathcal{B}_{\mathsf{hypo}}$, by the induction hypothesis. As such, we can conclude that $u \approx v$ is a consequence of $\mathcal{B}_{\mathsf{hypo}}$.

The technical part of the proof allows us to show that there is always a way to shuffle some variables of u in such a way that we obtain u^*. We show that these variables must occur several times in u, thus allowing us to apply the identities of $\mathcal{B}_{\mathsf{hypo}}$ to shuffle u and obtain u^*.

Let $u \approx v$ be a non-trivial identity satisfied by hypo. Then, it must be a balanced identity, by Theorem 6.1.1. Since $u \approx v$ is a non-trivial identity, we must have $u = wxu'$ and $v = wyv'$, for some words w, u', v' over $\operatorname{supp}(u \approx v)$ and variables $x, y \in \operatorname{supp}(u \approx v)$ such that $x \neq y$. Notice that u' and v' cannot be the empty word, otherwise, we would have $u = wx$ and $v = wy$, which contradicts the fact that $\operatorname{cont}(u) = \operatorname{cont}(v)$.

On the other hand, since $\operatorname{cont}(xu') = \operatorname{cont}(yv')$, we have that y occurs in u'. Thus, to distinguish the leftmost y in u', we have that

$$xu' = u_1ayu_2,$$

for some variable a and words u_1 and u_2, such that y does not occur in u_1 and $a \neq y$. Once again, since $\operatorname{cont}(xu') = \operatorname{cont}(yv')$, we have that a occurs in v'. Thus, to distinguish the leftmost a in v', we have that

$$v' = v_1av_2,$$

for some words v_1 and v_2, such that a does not occur in v_1. To sum up, we have that

$$u = wu_1ayu_2 \quad \text{and} \quad v = wyv_1av_2,$$

where y does not occur in u_1 and a does not occur in v_1.

Notice that u admits ay as a subsequence, hence, v must also do so. Thus, either a occurs in w, or y occurs in v_2, since a does not occur in v_1. But if y occurs in v_2, then it must also occur in u_2, since $\mathrm{cont}(xu') = \mathrm{cont}(yv')$ and y does not occur in u_1.

On the other hand, notice that v admits ya as a subsequence, hence, u must also do so. Thus, either y occurs in w, or a occurs in u_2, since y does not occur in u_1.

As such, we have four possible cases to look at:

Case 1. Both variables y and a occur in w. Then, we can deduce the word wu_1yau_2 from u, by applying the identity (R), renaming x to a.

Case 2. Both variables y and a occur in u_2. Then, we can deduce the word wu_1yau_2 from u, by applying the identity (L), renaming x to a.

Case 3. Variable y occurs in both w and u_2. Then, we can deduce the word wu_1yau_2 from u, by applying the identity (M), renaming x to y and y to a.

Case 4. Variable a occurs in both w and u_2. Then, we can deduce the word wu_1yau_2 from u, by applying the identity (M), renaming x to a.

Observe that we can repeatedly apply this reasoning until we obtain a word of the form

$$u^* = wyu'',$$

for some word u'', since the only restriction imposed on the variable a was that $a \neq y$. Thus, we have proven that, for any non-trivial identity $u \approx v$ satisfied by hypo, we can obtain a new word u^* from u such that the common prefix of u^* and v is strictly longer than the common prefix of u and v, by applying identities of $\mathcal{B}_{\mathsf{hypo}}$ finitely many times.

By induction, we conclude that $u \approx v$ is a consequence of $\mathcal{B}_{\mathsf{hypo}}$, thus proving that $\mathcal{B}_{\mathsf{hypo}}$ is a basis for $\mathcal{V}_{\mathsf{hypo}}$. □

Since $\mathcal{V}_{\mathsf{hypo}}$ admits a finite basis, it has finite axiomatic rank. In order to determine the axiomatic rank of $\mathcal{V}_{\mathsf{hypo}}$, we first check if all identities in $\mathcal{B}_{\mathsf{hypo}}$ are necessary in order to obtain a basis for $\mathcal{V}_{\mathsf{hypo}}$. It is easy to see that right zero semigroups satisfy identities (L) and (M), but not the identity (R), and left zero semigroups satisfy (R) and (M), but not (L). Thus, (L) and (R) are not consequences of the other identities in $\mathcal{B}_{\mathsf{hypo}}$.

On the other hand, consider the monoid $R_1(2) = \left\langle e, g \,\middle|\, ge = e = eg^2, g^3 = g, e^2 = e^3 = e^2g \right\rangle$, with multiplication table:

	1	g	e	eg	0
1	1	g	e	eg	0
g	g	1	e	eg	0
e	e	eg	0	0	0
eg	eg	e	0	0	0
0	0	0	0	0	0

This monoid is a noncryptic monoid (that is, Green's relation $\mathcal{H}$ is not a congruence on it), that belongs to a family of monoids which generate all minimal noncryptic monoid varieties (see [PV06]). It consists of an ideal extension of a null semigroup $K = \{e, eg\}$ by the 2-element cyclic group with a zero adjoined $C_2^0 = \{0, 1, g\}$. Notice that $g^2 = 1$ and $e^2 = 0$.

The monoid $R_1(2)$ satisfies (L) and (R): If we substitute all variables with elements of C_2^0, we get equality by commutativity. If, after substituting, we have more than one element of K on each side, both sides equal 0. So we are left with the case of substituting a variable that appears once (either z or t) with an element of K and everything else with elements of C_2^0. But since z and t are in the same position on both sides of the identities, and whichever elements of C_2^0 are substituted for x and y commute, the result is the same. However, $R_1(2)$ does not satisfy (M): Taking the evaluation ψ such that $\psi(y) = e$ and $\psi(x) = \psi(z) = \psi(t) = g$, we get

$$\psi(xzxytx) = g^3 eg^2 = e \neq eg = g^2 eg^3 = \psi(xzyxtx).$$

Since no identity of $\mathcal{B}_{\mathsf{hypo}}$ is a consequence of the other identities also in $\mathcal{B}_{\mathsf{hypo}}$, we conclude that $\mathcal{B}_{\mathsf{hypo}}$ is minimal, in the sense that it does not contain any proper subset which is also a basis for $\mathbf{\mathcal{V}}_{\mathsf{hypo}}$.

We now show that the identities (L) and (R) are required to be in any basis for $\mathbf{\mathcal{V}}_{\mathsf{hypo}}$ which contains only identities over an alphabet with four variables:

Proposition 7.1.2. *Neither of the identities* (L) *or* (R) *is a consequence of the set of non-trivial identities, satisfied by* hypo, *over an alphabet with four variables, excluding itself (but not the other) and equivalent identities.*

Proof. We prove the result for the identity (L). Parallel reasoning shows the analogous result for (R).

Let $X = \{x, y, z, t\}$ be an alphabet with four variables and let $\mathcal{S}$ be the set of all non-trivial identities, satisfied by hypo, over X, excluding (L) and equivalent identities. Suppose, in order to obtain a contradiction, that (L) is a consequence of $\mathcal{S}$. As such, there must exist a non-trivial identity $u \approx v$ in $\mathcal{S}$, and a substitution σ, such that

$$xyzxty = w_1 \sigma(u) w_2,$$

where w_1, w_2 are words over X, and $\sigma(u) \neq \sigma(v)$. Notice that $u \approx v$ must be balanced, and that there must be at least two variables in $\operatorname{supp}(u \approx v)$, otherwise, $u \approx v$ would be a trivial identity.

Observe that if the substitution σ maps some variables in $\operatorname{supp}(u \approx v)$ to the empty word, then $\sigma(u) = \sigma(u')$ and $\sigma(v) = \sigma(v')$, where u' and v' are words obtained by eliminating every occurrence of such variables in u and v, respectively. Hence, we have that

$$xyzxty = w_1 \sigma(u') w_2.$$

Notice that $u' \approx v'$ is an identity satisfied by hypo, which cannot be trivial, otherwise we would have $\sigma(u) = \sigma(u') = \sigma(v') = \sigma(v)$. Thus, $u' \approx v'$ is also in $\mathcal{S}$. On the other hand, notice that σ does not map any variable occurring in u' and v' to the empty word. As such, any case where σ maps any variable to the empty word guarantees the existence of another case where it does not map any variable to the empty word.

Therefore, we can assume, without loss of generality, that σ does not map any variable to the empty word. Due to this, and since (L) is an identity where x and y occur two times, and t and z each occur one time, we have that each variable in $\mathrm{supp}(u \approx v)$ can occur at most two times, and only two variables can occur more than one time. Furthermore, by Corollary 6.1.8, which gives us a lower bound for the length of the identities, we have that $u \approx v$ is of length at least 4. Thus, up to renaming of variables, x and y occur exactly twice in $u \approx v$, and t and z can occur at most one time.

Suppose now, in order to obtain a contradiction, that $w_1 \neq \varepsilon$. Then, since $u \approx v$ is of length at least 4, we must have w_1 of length at most 2, that is, w_1 is either x or xy. Therefore, x can occur only once in $\sigma(u)$. But x and y occur twice in u, and σ does not map any variable to the empty word, hence, there must be at least two variables which occur twice in $\sigma(u)$. However, only x and y occur twice in $xyzxty$. We have reached a contradiction, hence, $w_1 = \varepsilon$. Using a similar argument, we can also conclude that $w_2 = \varepsilon$. Therefore, we have that

$$xyzxty = \sigma(u).$$

As such, we can immediately conclude that only up to three variables occur in $u \approx v$: If $u \approx v$ were to be a four-variable identity, then it would be of length 6, and σ would be simply renaming the variables, thus implying that $u \approx v$ was equivalent to (L), which contradicts our hypothesis.

Suppose that $u \approx v$ is a two-variable identity. Hence, it is of length 4 and x and y both occur twice in it. Since $xyzxty = \sigma(u)$, then either $\sigma(x)$ or $\sigma(y)$ must be a single variable, and the other must be of length 2. Since no variable occurs more than twice in $\sigma(u)$, this implies that three variables occur twice in $\sigma(u)$. But x and y are the only variables which occur twice in $xyzxty$. We have reached a contradiction, hence, $u \approx v$ is not a two-variable identity.

Then, $u \approx v$ must be a three-variable identity. Hence, it is of length 5, with x and y occurring twice and z occurring once in it. Notice that $\sigma(x)$ and $\sigma(y)$ must be single variables, otherwise, the length of $\sigma(u)$ would be greater than 6. These variables cannot be z or t, since they occur only once in $xyzxty$. Therefore, $\sigma(z)$ must be a factor of $xyzxty$ of length 2. But neither x nor y can occur in $\sigma(z)$, hence, this factor cannot exist, and subsequently, σ cannot exist.

As such, we can conclude that (L) is not a consequence of the set of non-trivial identities, satisfied by hypo, over an alphabet with four variables, excluding (L) itself and equivalent identities. □

Therefore, we can conclude that $\mathcal{V}_{\mathsf{hypo}}$ does not admit any basis with only identities over an alphabet with two or three variables. In other words, we have that:

Corollary 7.1.3. *The axiomatic rank of* $\mathcal{V}_{\mathsf{hypo}}$ *is* 4.

Another consequence of the previous proposition is the following:

Corollary 7.1.4. *Any basis for* $\mathcal{V}_{\mathsf{hypo}}$ *with only identities over an alphabet with four variables must contain the identities* (L) *and* (R), *or equivalent identities.*

Furthermore, since (M) is not a consequence of (L) and (R), any basis for $\mathcal{V}_{\mathsf{hypo}}$ with only identities over an alphabet with four variables must contain at least three identities, one of which must be either (M), an equivalent identity, or an identity of which (M) is a consequence.

7.1.2 The axiomatic rank of the varieties generated by the sylvester, #-sylvester and Baxter monoids

Now, we prove that that the varieties generated by the sylvester, #-sylvester and Baxter monoids are finitely based and, therefore, have finite axiomatic rank. We give bases for $\mathcal{V}_{\mathsf{sylv}}$ and $\mathcal{V}_{\mathsf{sylv}^{\#}}$ with one identity each, of length 6, over a four-symbol alphabet. Trivially, these bases are minimal with regards to the number of identities in the basis; they are also minimal with regards to the number of variables occurring in these identities, and the length of these identities. We also give a basis for $\mathcal{V}_{\mathsf{baxt}}$, with two identities, of length 10, over a six-symbol alphabet. This basis is also minimal with regards to the number of identities in the basis, the number of variables occurring in these identities, and the length of these identities.

Furthermore, we also prove that there exist no bases for $\mathcal{V}_{\mathsf{sylv}}$ or $\mathcal{V}_{\mathsf{sylv}^{\#}}$ with only identities over an alphabet with at most three variables, thus showing that the axiomatic rank of $\mathcal{V}_{\mathsf{sylv}}$ and $\mathcal{V}_{\mathsf{sylv}^{\#}}$ is 4. We also prove that there exists no basis for $\mathcal{V}_{\mathsf{baxt}}$ with only identities over an alphabet with at most five variables, thus showing that the axiomatic rank of $\mathcal{V}_{\mathsf{baxt}}$ is 6.

Theorem 7.1.5. $\mathcal{V}_{\mathsf{sylv}}$ *admits a finite basis* $\mathcal{B}_{\mathsf{sylv}}$, *consisting of the following identity:*

$$xyzxty \approx yxzxty. \tag{L}$$

Proof. The proof follows the same overall strategy as the proof of Theorem 7.1.1. Let $\mathcal{B}_{\mathsf{sylv}}$ be the set of identities which contains only the identity (L). Notice that this identity is given in Example 6.2.12.

The proof will be by induction, in the following sense: We order identities by the length of the common suffix of both sides of the identity. The induction will be on the length of the prefix, that is, the length of the identity minus the length of the common suffix.

The base case for the induction, for identities of length n (with $n \geq 4$), is the identities of the form

$$xyw \approx yxw,$$

where w is a word of length $n-2$ and x, y are variables. Observe that, since any identity $u \approx v$ satisfied by sylv is a balanced identity, there are no non-trivial identities, of length n, with a common suffix of length greater than $n-2$, satisfied by sylv. That is, the prefix before the common suffix must have length at least 2. Furthermore, x and y must both occur in w, otherwise, we would have $o_{x \leftarrow y}(xyw) > o_{x \leftarrow y}(yxw)$ or $o_{y \leftarrow x}(xyw) < o_{y \leftarrow x}(yxw)$. Thus, w is of the form

$$w_1 x w_2 y w_3 \quad \text{or} \quad w_1 y w_2 x w_3,$$

for some words w_1, w_2, w_3. Therefore, by replacing z with w_1, and t by w_2, and, if necessary, renaming x and y, we can immediately deduce this identity from the identity (L). Notice that, when $n = 4$, the base case corresponds to the identity given in Corollary 6.2.4.

Let $u \approx v$ be a non-trivial identity satisfied by sylv. Then, it must be a balanced identity, by Theorem 6.2.1. Since $u \approx v$ is a non-trivial identity, we must have $u = u'xw$ and $v = v'yw$, for some words w, u', v' over $\operatorname{supp}(u \approx v)$ and variables $x, y \in \operatorname{supp}(u \approx v)$ such that $x \neq y$. Notice that u' and v' cannot be the empty word, otherwise, we would have $u = xw$ and $v = yw$, which contradicts the fact that $\operatorname{cont}(u) = \operatorname{cont}(v)$.

On the other hand, since $\operatorname{cont}(u'x) = \operatorname{cont}(v'y)$, we have that y occurs in u'. Thus, to distinguish the rightmost y in u', we have that

$$u'x = u_1 y a u_2,$$

for some variable a and words u_1 and u_2, such that y does not occur in u_2 and $a \neq y$. Once again, since $\operatorname{cont}(u'x) = \operatorname{cont}(v'y)$, we have that a occurs in v'. Thus, to distinguish the rightmost a in v', we have that

$$v' = v_1 a v_2,$$

for some words v_1 and v_2, such that a does not occur in v_2. To sum up, we have that

$$u = u_1 y a u_2 w \quad \text{and} \quad v = v_1 a v_2 y w,$$

where y does not occur in u_2 and a does not occur in v_2.

By Theorem 6.2.1, y must occur in w, otherwise, we would have $o_{y \leftarrow x}(u) > o_{y \leftarrow x}(v)$. Thus, a must also occur in w, otherwise, we would have $o_{a \leftarrow y}(u) < o_{a \leftarrow y}(v)$. As such, we can deduce the word $u_1 a y u_2 w$, by applying the identity (L) to u, renaming x to a and replacing z and t by the appropriate words.

Observe that we can repeatedly apply this reasoning until we obtain a word of the form

$$u^* = u'' y w,$$

for some word u'', since the only restriction imposed on the variable a was that $a \neq y$. Thus, we have proven that, for any non-trivial identity $u \approx v$ satisfied by sylv, we can obtain a new word u^* from u such that the common suffix of u^* and v is strictly longer than the common suffix of u and v, by applying the identity (L) finitely many times.

By induction, we conclude that $u \approx v$ is a consequence of $\mathcal{B}_{\mathsf{sylv}}$, thus proving that $\mathcal{B}_{\mathsf{sylv}}$ is a basis for $\mathcal{V}_{\mathsf{sylv}}$. □

By the same reasoning, we can also prove the following result:

Theorem 7.1.6. *$\mathcal{V}_{\mathsf{sylv}^\#}$ admits a finite basis $\mathcal{B}_{\mathsf{sylv}^\#}$, consisting of the following identity:*

$$xzytxy \approx xzytyx. \tag{R}$$

Proof. The proof follows the same reasoning as the proof of Theorem 7.1.5, the main difference being that, within a set of identities of the same length, they are ordered on the length of the common prefix of both sides of the identity, and the induction is on the length of the suffix. The induction step resorts to Theorem 6.2.2. □

We also use the same reasoning to prove the following theorem:

Theorem 7.1.7. *$\mathcal{V}_{\mathsf{baxt}}$ admits a finite basis $\mathcal{B}_{\mathsf{baxt}}$, consisting of the following identities:*

$$xzyt\,xy\,rxsy \approx xzyt\,yx\,rxsy; \tag{O}$$

$$xzyt\,xy\,rysx \approx xzyt\,yx\,rysx. \tag{E}$$

Proof. The proof follows the same overall strategy as the proof of Theorem 7.1.5. As such, we only give the reasoning for the base case and the induction step.

The base case for the induction on the length of the prefix before the common suffix, for identities of length n (with $n \geq 6$), is the identities of the form

$$xyxyw \approx xyyxw,$$

where w is a word of length $n-4$ and x, y are variables. Notice that both sides of the identity must have a prefix of the form xy, due to Corollary 6.2.8. By the same reason, observe that, since any identity $u \approx v$ satisfied by baxt is a balanced identity, there are no non-trivial identities, of length n, with a common suffix of length greater than $n-4$, satisfied by baxt. Furthermore, x and y must both occur in w, otherwise, we would have $o_{x \leftarrow y}(xyw) > o_{x \leftarrow y}(yxw)$. Thus, w is of the form

$$w_1 x w_2 y w_3 \quad \text{or} \quad w_1 y w_2 x w_3,$$

for some words w_1, w_2, w_3. Therefore, by replacing z and t with the empty word, r with w_1, and s by w_2, and, if necessary, renaming x and y, we can immediately deduce this identity from the identity (O) or the identity (E), depending on the form of w. Notice that, when $n = 6$, the base case corresponds to the identities given in Corollary 6.2.6.

Let $u \approx v$ be a non-trivial identity satisfied by baxt. Then, it must be a balanced identity, by Theorem 6.2.3. Since $u \approx v$ is a non-trivial identity, we must have $u = u'xw$

and $v = v'yw$, for some words w, u', v' over $\mathrm{supp}(u \approx v)$ and variables $x, y \in \mathrm{supp}(u \approx v)$ such that $x \neq y$. By Corollary 6.2.8, we have that x must occur at least once in u' and w and at least twice in v', and y must occur at least twice in u' and at least once in v' and w. Thus, to distinguish the leftmost y in u', we have that

$$u'x = u_1 y a u_2$$

for some variable a and words u_1 and u_2, such that y does not occur in u_2 and $a \neq y$. Notice that y must occur in u_1. Since $\mathrm{cont}(u'x) = \mathrm{cont}(v'y)$, we have that a occurs in v'. Thus, to distinguish the rightmost a in v', we have that

$$v' = v_1 a v_2,$$

for some words v_1 and v_2, such that a does not occur in v_2. To sum up, we have that

$$u = u_1 y a u_2 w \quad \text{and} \quad v = v_1 a v_2 y w,$$

where y occurs in u_1 but not in u_2 and a does not occur in v_2.

Suppose, in order to obtain a contradiction, that a does not occur in u_1. This implies that $|u'|_y = o_{y \to a}(u)$. But $|u'|_y = |v'|_y + 1$, hence

$$o_{y \to a}(v) \leq |v'|_y < |u'|_y = o_{y \to a}(u).$$

Thus, by Theorem 6.2.3, we obtain a contradiction. As such, a must occur in u_1. By the same theorem, a must occur in w as well, otherwise, we would have $o_{a \leftarrow y}(u) < o_{a \leftarrow y}(v)$. Therefore, y and a both occur at least once in u_1 and w. As such, we can deduce the word $u_1 a y u_2 w$, by applying the identity (O) or the identity (E) to u, depending on where y and a occur in u_1 and w, renaming x to a and replacing the remaining variables by the appropriate words. $\square$

The identities in $\mathcal{B}_{\mathsf{baxt}}$ also form a basis for either of the varieties generated by, respectively, the monoids $2\mathfrak{Cob}_1^\circ$ and $\overline{2\mathfrak{Cob}_1^\circ}$, the endomorphism monoids (or local monoids) at object 1 of the categories $2\mathfrak{Cob}^\circ$ and $\overline{2\mathfrak{Cob}^\circ}$ (see [AV20, Theorem 5.7]). These are intermediary categories between, respectively, the category $2\mathfrak{Cob}$ of 2-cobordisms and the (undeformed) partition category $\mathfrak{P}$, and the regular category $\overline{2\mathfrak{Cob}}$ (a regular version of $2\mathfrak{Cob}$) and $\mathfrak{P}$, which appear in mathematical physics and representation theory. Furthermore, the varieties generated by any other endomorphism monoids of $2\mathfrak{Cob}^\circ$ and $\overline{2\mathfrak{Cob}^\circ}$, as well as the varieties generated by any endomorphism monoids of $2\mathfrak{Cob}$ and $\overline{2\mathfrak{Cob}}$, are not finitely based (see [AV20, Theorem 5.6]).

Since they admit finite bases, the varieties $\mathcal{V}_{\mathsf{sylv}}$, $\mathcal{V}_{\mathsf{sylv}^\#}$ and $\mathcal{V}_{\mathsf{baxt}}$ have finite axiomatic rank. By Proposition 7.1.2, we know that the identities (L) and (R) are not consequences of the set of non-trivial identities, satisfied by hypo, over an alphabet with four variables, excluding themselves and equivalent identities. Since the identities satisfied by sylv and $\mathsf{sylv}^\#$ must also be satisfied by hypo, we can conclude the following:

Corollary 7.1.8. *The identity* (L) *is not a consequence of the set of non-trivial identities, satisfied by* sylv, *over an alphabet with four variables, excluding* (L) *itself and equivalent identities. Furthermore, any basis for* $\mathcal{V}_{\mathsf{sylv}}$ *with only identities over an alphabet with four variables must contain the identity* (L), *or an equivalent identity.*

Corollary 7.1.9. *The identity* (R) *is not a consequence of the set of non-trivial identities, satisfied by* $\mathsf{sylv}^{\#}$, *over an alphabet with four variables, excluding* (R) *itself and equivalent identities. Furthermore, any basis for* $\mathcal{V}_{\mathsf{sylv}^{\#}}$ *with only identities over an alphabet with four variables must contain the identity* (R), *or an equivalent identity.*

Hence, $\mathcal{V}_{\mathsf{sylv}}$ and $\mathcal{V}_{\mathsf{sylv}^{\#}}$ do not admit any bases with only identities over an alphabet with two or three variables. In other words, we have that:

Corollary 7.1.10. *The axiomatic rank of* $\mathcal{V}_{\mathsf{sylv}}$ *and* $\mathcal{V}_{\mathsf{sylv}^{\#}}$ *is* 4.

We now show that the identities (O) and (E) must be in any basis for $\mathcal{V}_{\mathsf{baxt}}$ which contains only identities over an alphabet with six variables:

Proposition 7.1.11. *Neither of the identities* (O) *or* (E) *is a consequence of the set of non-trivial identities, satisfied by* baxt, *over an alphabet with six variables, excluding itself (but not the other) and equivalent identities.*

Proof. We prove the result for the identity (O). Parallel reasoning shows the analogous result for (E).

Let $X = \{x, y, z, t, r, s\}$ be an alphabet with six variables and let $\mathcal{S}$ be the set of all non-trivial identities, satisfied by baxt, over X, excluding (O) and equivalent identities. Suppose, in order to obtain a contradiction, that (O) is a consequence of $\mathcal{S}$. As such, there must exist a non-trivial identity $u \approx v$ in $\mathcal{S}$, and a substitution σ, such that

$$xzyt\ xy\ rxsy = w_1 \sigma(u) w_2,$$

where w_1, w_2 are words over X, and $\sigma(u) \neq \sigma(v)$. Notice that $u \approx v$ must be balanced, and that there must be at least two variables in $\operatorname{supp}(u \approx v)$, otherwise, $u \approx v$ would be a trivial identity.

By the same reasoning as in the proof of 7.1.2, we can assume, without loss of generality, that σ does not map any variable to the empty word. Due to this, and since only x and y occur three times in $xzyt\ xy\ rxsy$, and all other variables each occur one time, we have that each variable in $\operatorname{supp}(u \approx v)$ can occur at most three times, and only two variables can occur more than one time. Furthermore, by Corollary 6.2.14, which gives us a lower bound for the length of the identities, we have that $u \approx v$ is of length at least 6. Notice that it is exactly of length 6 if only two variables occur in it. Thus, up to renaming of variables, x and y occur exactly three times in $u \approx v$, and t, z, r and s can occur at most one time.

Suppose now, in order to obtain a contradiction, that $w_1 \neq \varepsilon$. Then, since $u \approx v$ is of length at least 6, we must have w_1 of length at most 4, that is, w_1 is either x, xz, xzy or

$xzyt$. Therefore, x can occur only twice in $\sigma(u)$. But x and y occur three times in u, and σ does not map any variable to the empty word, hence, there must be at least two variables which occur three times in $\sigma(u)$. However, only x and y occur three times in $xzyt\ xy\ rxsy$. We have reached a contradiction, hence, $w_1 = \varepsilon$. Using a similar argument, we can also conclude that $w_2 = \varepsilon$. Therefore, we have that

$$xzyt\ xy\ rxsy = \sigma(u).$$

As such, we can immediately conclude that only up to five variables occur in $u \approx v$: If $u \approx v$ were to be a six-variable identity, then it would be of length 10, and σ would be simply renaming the variables, thus implying that $u \approx v$ was equivalent to (O), which contradicts our hypothesis.

Notice that, regardless of the number of variables occurring in $u \approx v$, we have that both $\sigma(x)$ and $\sigma(y)$ are a single variable, otherwise, more than two variables would have to occur three times in $xzyt\ xy\ rxsy$, or one variable would have to occur six times. Furthermore, $\sigma(x)$ and $\sigma(y)$ can only be x or y, since these are the only variables occurring three times in $xzyt\ xy\ rxsy$. Hence, if $u \approx v$ is an identity where up to five variables occur, then $u \approx v$ cannot be a two-variable identity, and, furthermore, there is at least one variable z occurring in $u \approx v$ such that $\sigma(z)$ is of length at least 2, and neither x nor y can occur in $\sigma(z)$. This is impossible, since x or y occur in every factor of $xzyt\ xy\ rxsy$ of length 2.

As such, we can conclude that (O) is not a consequence of the set of non-trivial identities, satisfied by baxt, over an alphabet with six variables, excluding (O) itself and equivalent identities. $\square$

Therefore, we can conclude that $\mathcal{V}_{\mathsf{baxt}}$ does not admit any basis with only identities over an alphabet with up to five variables. In other words, we have that:

Corollary 7.1.12. *The axiomatic rank of $\mathcal{V}_{\mathsf{baxt}}$ is 6. Furthermore, any basis for $\mathcal{V}_{\mathsf{baxt}}$ with only identities over an alphabet with six variables must contain the identities* (O) *and* (E)*, or equivalent identities.*

7.2 Subdirect representations of multihomogeneous monoids

As seen in Section 3.2, finitely generated homogeneous monoids are residually finite. We now show that multihomogeneous monoids are residually finite, by classifying whether a Rees factor monoid of a multihomogeneous monoid is subdirectly irreducible or not, then constructing a subdirect representation by finite subdirectly irreducible Rees factor monoids. We also show that this representation can be used to prove that finitely generated homogeneous monoids are residually finite, however, we also show that it is not possible to subdirectly represent these monoids by subdirectly irreducible Rees factor monoids alone. We are unsure if these results are new, mainly due to the difficulty of finding results on multihomogeneous monoids, compared to the more general case of homogeneous monoids, and the different terminology used in the literature.

Let M be a multihomogeneous monoid, and let $\langle X|\mathcal{R}\rangle$ be a monoid presentation for M. Recall that $J(x)$ denotes the principal ideal generated by $x \in M$. Let

$$K_x = \{y \in M : x \notin J(y)\}.$$

It is easy to see that K_x is an ideal: Let $y \in K_x$ and $z \in M$. Assume, in order to obtain a contradiction, that $yz \notin K_x$, that is, $x \in J(yz)$. Then, there exist $p,q \in M$ such that $x = pyzq$, which implies that $x \in J(y)$. Similarly, $zy \notin K_x$ implies that $x \in J(y)$. Hence, $yz, zy \in K_x$, for any $y \in K_x$ and $z \in M$.

Notice that all elements $y \in M\backslash K_x$ are such that $x \in J(y)$. On the other hand, since M is multihomogeneous, the content of the product of two elements of M is greater than the content of any one of those elements, when considering the componentwise order. As such, if $x \in J(y)$, then the content of x is greater than the content of y. Therefore, since there are finitely many elements of M whose content is less than or equal to the content of x, the Rees factor monoid M/K_x is finite.

Proposition 7.2.1. *Let I be an ideal of a multihomogeneous monoid M. Then, M/I is subdirectly irreducible if and only if $I = K_x$, for some $x \in M$.*

Proof. First of all, notice that, for $x \in M$, all elements of M whose length is greater than or equal to the length of x are in K_x, except for x itself.

For any $y,z \in M\backslash K_x$ such that $y \neq z$ and the length of y is less than or equal to the length of z, there exist $p,q \in M\backslash K_x$ such that $pyq = x$ and $pzq \in K_x$: Since $y \notin K_x$, there exist $p,q \in M\backslash K_x$ such that $pyq = x$. On the other hand, since $y \neq z$ and M is multihomogeneous, then $\operatorname{cont}(y) \neq \operatorname{cont}(z)$, hence $\operatorname{cont}(x) = \operatorname{cont}(pyq) \neq \operatorname{cont}(pzq)$. Since the length of y is less than or equal to the length of z, then the length of pzq is greater than or equal to the length of x, hence $pzq \neq x$ and, as such, $pzq \in K_x$.

As such, we have that any principal congruence in M/K_x contains the principal congruence generated by $([x]_{K_x}, K_x)$. Thus, $\mathbf{Con}(M/K_x)\backslash\left\{\triangle_{M/K_x}\right\}$ has a minimum element, which implies that M/K_x is subdirectly irreducible.

On the other hand, let I be an ideal of M such that M/I is subdirectly irreducible. Suppose, in order to obtain a contradiction, that there is no $x \in M\backslash I$ such that $J(x)\backslash I = \{x\}$. As such, there must exist a sequence $(x_k)_{k\in\mathbb{N}}$ of elements in $M\backslash I$ such that $x_k \neq x_{k+1}$ and $x_{k+1} \in J(x_k)\backslash I$. Notice that the elements in this sequence are pairwise distinct, since M is multihomogeneous, therefore any element in $J(x_k)$ other than x_k itself must be of length strictly greater than x_k. Assume, without loss of generality, that $x_1 \neq 1_M$. As such, the length of x_k is greater than or equal to k, for each $k \in \mathbb{N}$. Thus, if we take the principal congruence ρ_k of M/I generated by the pair $([x_k]_I, I)$, for each $k \in \mathbb{N}$, we can see that $\rho_{k+1} \subsetneqq \rho_k$. As such, we have an infinite chain of congruences on M/I. Notice that, for any $x,y \in M\backslash I$ such that $x \neq y$ and l is the maximum of the lengths of x and y, the pair $([x]_I, [y]_I)$ is not in ρ_{l+1}, since neither x nor y are in $J(x_{l+1})$. Therefore, the intersection of all congruences ρ_k is the identity relation, which implies that $\mathbf{Con}(M/I)\backslash\{\triangle_{M/I}\}$ does not have a minimum congruence, and consequently M/I is not subdirectly irreducible. This contradicts our hypothesis.

Therefore, there exists at least an element $x \in M \backslash I$ such that $J(x) \backslash I = \{x\}$. Let z be such an element. Suppose, in order to obtain a contradiction, that there exists y in $M \backslash I$ such that $z \notin J(y)$. Then, $(J(y) \cap J(z)) \backslash I = \emptyset$, hence, the intersection of the principal congruence generated by the pair $([y]_I, I)$ and the principal congruence generated by the pair $([z]_I, I)$ is the identity relation $\triangle_{M/I}$. As such, $\mathbf{Con}(M/I) \backslash \{\triangle_{M/I}\}$ does not have a minimum congruence, and consequently M/I is not subdirectly irreducible. This contradicts our hypothesis.

Therefore, there is only be one element $x \in M \backslash I$ such that $J(x) \backslash I = \{x\}$, and all other elements in $y \in M \backslash I$ are such that $x \in J(y)$. Therefore, $K_x \subseteq I$. On the other hand, if there were some $y \in I$ such that $y \notin K_x$, then $x \in J(y) \subseteq I$, which contradicts our hypothesis. Hence, $I = K_x$. ☐

Define $\delta : M \longrightarrow \prod_{x \in M} M/K_x$ in the following way: For $x, y \in M$,

$$\pi_x(\delta(y)) = [y]_{K_x}.$$

Proposition 7.2.2. *The map δ is a subdirect embedding of M into the indexed family $(M/K_x)_{x \in M}$ of subdirectly irreducible monoids.*

Proof. It is clear that, by its definition, δ is a morphism. Let $p, q \in M$ be such that $p \neq q$. Then $p, q \in M \backslash K_{pq}$. Hence, we have that

$$\pi_{pq}(\delta(p)) = [p]_{K_{pq}} \neq [q]_{K_{pq}} = \pi_{pq}(\delta(q)),$$

therefore, $\delta(p) \neq \delta(q)$. Thus, δ is an embedding. Furthermore, it is easy to see that $\delta(M)$ is a subdirect product of the indexed family $(M/K_x)_{x \in M}$ of subdirectly irreducible monoids, since $\pi_x \circ \delta$ is the natural surjective map from M to M/K_x, for each $x \in M$. ☐

We have managed to obtain a subdirect representation of a multihomogeneous monoid by its finite subdirectly irreducible Rees factor monoids. As such, we have the following corollary:

Corollary 7.2.3. *Multihomogeneous monoids are residually finite.*

In particular, we have that the plactic-like monoids of infinite rank are residually finite. It is also easy to see that all multihomogeneous monoids are infinite, hence, we have the following

Corollary 7.2.4. *No multihomogeneous monoid is subdirectly irreducible.*

In the more general case of homogeneous monoids, the map δ is not necessarily an embedding into a direct product of finite monoids: For example, consider the monoid M presented by $\langle \mathbb{N} | \mathcal{R} \rangle$, where $\mathcal{R}$ is such that $abc \ \mathcal{R}\ 111$, for all $a, b, c \in \mathbb{N}$. Then, for all elements x of length greater than or equal to 3, the Rees factor monoid M/K_x is not finite, since all elements $[a]_M$ are in $M\backslash K_x$, for each $a \in \mathbb{N}$.

On the other hand, if we discard all infinite monoids from the definition of δ, we no longer have an embedding. To clarify this, let M' be the subset of elements x of M such that M/K_x is finite and define $\delta' : M \longrightarrow \prod_{x \in M'} M/K_x$ in the following way: For $x \in M'$ and $y \in M$,

$$\pi_x(\delta'(y)) = [y]_{K_x}.$$

Notice that the elements of M' are of the form $[a]_M$ or $[ab]_M$, where $a, b \in \mathbb{N}$ are not necessarily distinct. As such, all elements y of length greater than or equal to 3 are in K_x, for each $x \in M'$. Thus, we have that, for all $x \in M'$,

$$\pi_x(\delta'([123]_M)) = \Big[[123]_M\Big]_{K_x} = K_x = \Big[[1234]_M\Big]_{K_x} = \pi_x(\delta'([1234]_M)),$$

hence $\delta'([123]_M) = \delta'([1234]_M)$. As such, δ' is not an embedding.

In the case of *finitely generated* homogeneous monoids, the δ is still an embedding into a direct product of finite monoids, since having a finite number of generators implies that there are finitely many words of length less than or equal to the length of an element $x \in M$, hence $M\backslash K_x$ is finite. This gives us an alternative proof to show that finitely generated homogeneous monoids are residually finite. However, Proposition 7.2.1 no longer holds, since the Rees factor monoids may no longer be subdirectly irreducible: For example, let $X = \{a, b\}$ and let $\mathcal{R}$ be such that

$$aab \;\mathcal{R}\; aba \;\mathcal{R}\; abb \;\mathcal{R}\; baa \;\mathcal{R}\; bab \;\mathcal{R}\; bba.$$

Let M be the monoid presented by $\langle X | \mathcal{R} \rangle$. It is clear that M is a finitely generated homogeneous monoid. Furthermore, it is easy to see that, for any word $w \in X^*$ such that $|w| \geq 3$ and $\mathrm{supp}(w) = \{a, b\}$, then $w \;\mathcal{R}\; ba^{|w|-1}$. In other words, for words of length $k \geq 3$, there are only three possible elements of M which they can represent: a^k and b^k are, respectively, the sole representatives of $[a^k]_M$ and $[b^k]_M$, and all other words are representatives of $[ba^{k-1}]_M$.

As such, for $w \in X^*$ such that $|w| \geq 3$ and $\mathrm{supp}(w) = \{a, b\}$, we have that, for any $p, q \in X^*$,

$$[p]_M \cdot [ab]_M \cdot [q]_M = [w]_M \iff [p]_M \cdot [ba]_M \cdot [q]_M = [w]_M.$$

Hence, in $M/K_{[w]_M}$, the intersection of the principal congruence generated by the pair

$$\left(\Big[[ab]_M\Big]_{K_{[w]_M}}, \Big[[ba]_M\Big]_{K_{[w]_M}}\right)$$

and the principal congruence generated by the pair

$$\left(\Big[[w]_M\Big]_{K_{[w]_M}}, K_{[w]_M}\right)$$

is the identity relation $\triangle_{M/K_{[w]_M}}$. Therefore $\mathbf{Con}(M/K_{[w]_M})\backslash\left\{\triangle_{M/K_{[w]_M}}\right\}$ does not have a minimum congruence, consequently $M/K_{[w]_M}$ is not subdirectly irreducible.

This shows that Proposition 7.2.2 does not hold for the case of finitely generated homogeneous monoids, that is, the embedding δ no longer gives a subdirect representation of a finitely generated homogeneous monoid by subdirectly irreducible Rees factor monoids.

On the other hand, if we discard all non-subdirectly irreducible monoids from the definition of δ, we no longer have an embedding. To clarify this, let M'' be the subset of elements x of M such that M/K_x is subdirectly irreducible. Define $\delta'' : M \longrightarrow \prod_{x \in M''} M/K_x$ in the following way: For $x \in M''$ and $y \in M$,

$$\pi_x(\delta''(y)) = [y]_{K_x}.$$

For any word $w \in X^*$ such that $|w| \geq 3$ and $\mathrm{supp}(w) = \{a, b\}$ and any $k \in \mathbb{N}$, since $M/K_{[w]_M}$ is not subdirectly irreducible, and $[w]_M \in K_{a^k} \cap K_{b^k}$, we have that $[w]_M \in K_x$, for all $x \in M''$. Therefore, for any $x \in M''$, we have that

$$\pi_x(\delta''([aab]_M)) = \Big[[aab]_M\Big]_{K_x} = K_x = \Big[[aabb]_M\Big]_{K_x} = \pi_x(\delta''([aabb]_M)),$$

hence $\delta''([aab]_M) = \delta''([aabb]_M)$. As such, δ'' is not an embedding.

However, the proof of Proposition 7.2.1 still allows us to show that any subdirectly irreducible Rees factor monoid of M is of the form K_x, for some $x \in M$. As such, we can conclude the following:

Corollary 7.2.5. *Finitely generated homogeneous monoids are not subdirectly represented by subdirectly irreducible Rees factor monoids.*

In conclusion, we have that multihomogeneous monoids are residually finite and subdirectly represented by finite subdirectly irreducible Rees factor monoids; finitely generated homogeneous monoids are residually finite, but not subdirectly represented by subdirectly irreducible Rees factor monoids; homogeneous monoids are neither necessarily residually finite, nor subdirectly represented by subdirectly irreducible Rees factor monoids.

8

Open problems

In this thesis, we have proved that the hypoplactic (respectively, sylvester, #-sylvester and Baxter) monoids of rank greater than or equal to 2 satisfy exactly the same identities. We have obtained a characterization of these identities and shown that the identity checking problems for these monoids are in the complexity class P. Furthermore, we have obtained finite bases for the varieties generated by each of these monoids. However, there exist other plactic-like monoids, which are yet to be studied in such depth.

Recall that plactic-like monoids are quotients of the free monoid over an ordered alphabet, whose elements can be uniquely identified with combinatorial objects. Examples of other plactic-like monoids, besides those studied in this thesis, are the taiga monoid [Pri13], the monoid of binary search trees with multiplicities; the stalactic monoid [HNT08; Pri13], the monoid of stalactic tableaux; the left and right patience sorting monoids [CMS19; Rey07], the monoids of patience sorting tableaux; and the stylic monoid [AR21], the monoid of N-tableaux. With the exception of the stylic monoid, the identities satisfied by these monoids have been first studied in [CM18b] and, in the case of the left and right patience sorting monoids, in [CMS19]. It has been proven independently in [Cai+21, Corollary 5.10] and in [HZ21, Theorem 4.2] that the taiga and stalactic monoids of rank greater than or equal to 2 generate the same variety, which admits a finite basis consisting of the identity $xyx \approx yxx$. However, there are still questions to ask about these monoids:

Open Problems 1. Do all plactic-like monoids (except the plactic monoid itself) of rank higher than 2 embed into a direct product of copies of the corresponding monoid of rank 2? Can we obtain a complete characterization of the identities satisfied by these monoids? Do the identity checking problems for these monoids belong to the complexity class P? Do the varieties generated by these monoids admit finite bases?

Since the plactic-like monoids arise from the study of combinatorial Hopf algebras, whose bases are indexed by combinatorial objects, it is natural to ask if we can deduce results on these Hopf algebras from what we obtained on the plactic-like monoids. In particular, we ask the following:

Open Problems 2. Can we deduce identities satisfied by combinatorial Hopf algebras from the identities satisfied by their corresponding plactic-like monoids?

In this thesis, the plactic monoid of rank n was defined as a quotient of the free monoid over the ordered alphabet $A_n = \{1 < \cdots < n\}$, by a presentation and by Young tableaux and the Schensted algorithm. Another possible way for the plactic monoid of rank n to arise is from the crystal basis for the q-analogue of the special linear Lie algebra $\mathfrak{sl}_n$, that is, the type A_{n+1} simple Lie algebra, which links the plactic monoid to Kashiwara's theory of crystal graphs [KN94]. The plactic congruence corresponds to isomorphisms between connected components of the crystal graphs. As such, the classical plactic monoid can be defined in terms of crystals of type A_n, and thus is said to be of type A_n.

Similarly, generalizations of the classical plactic monoid arise from the crystals of representations of other quantum algebras, namely symplectic Lie algebras $\mathfrak{sp}_n$ (the type C_n simple Lie algebra), special orthogonal Lie algebras of odd and even rank $\mathfrak{so}_{2n+1}$ and $\mathfrak{so}_{2n}$ (the type B_n and D_n simple Lie algebras), and the exceptional simple Lie algebra G_2 [KS04; Lec02; Lec03; Lec07]. The plactic monoids of types B_n, C_n, D_n and G_2 can also be defined by presentations or by Young tableaux and the insertion algorithm, as detailed in [Lec07] and [CGM19].

On the other hand, the hypoplactic monoid [CM17] and the sylvester and Baxter monoids [CM18a] have been defined using a purely combinatorial notion of quasi-crystals. However, the corresponding monoids of types B_n, C_n, D_n and G_2 are yet to be defined. As such, we ask the following:

Open Problems 3. Is it possible to define all plactic-like monoids in terms of crystals or similar structures? Can generalized versions of these monoids, of types B_n, C_n, D_n and G_2, also be defined? If such is possible, then can we obtain characterizations of the identities satisfied by these monoids, and other related results? And are these results connected to the previously obtained results on the monoids of type A_n?

Structures are a generalization of algebras, defined as sets endowed with some relations (not necessarily functions). Automatic presentations for structures were first introduced by Khoussainov and Nerode [KN95] to extend finite model theory to infinite structures, in such a way that preserves the decidability of important decision problems. Informally, an automatic presentation for a structure consists of a regular language of abstract representatives for the elements of the structure such that the relations of the structure are all recognizable by synchronous finite automata. A structure that admits an automatic presentation is said to be FA-presentable. Many characterization and classification results have been obtained for FA-presentable structures of various kinds (see [Rub08]).

This concept was first applied to semigroups in [Cai+09], where a complete classification of finitely generated FA-presentable cancellative semigroups was given, along with a complete list of FA-presentable one-relation semigroups. In [Cai+10], the interaction of automatic presentations and several semigroup constructions was studied, and classifications for FA-presentable finitely generated Clifford semigroups, completely simple semigroups, and completely 0-simple semigroups were also given. In [CRT12], unary FA-presentable semigroups were studied, and unary FA-presentable completely simple semigroups were classified. Stuart Margolis asked [A. Cain, personal communication, 2008] if all monoids admitting automatic presentations satisfy non-trivial identities. All known examples satisfy identities, and the question is still open. In particular, Brough, Cain and Malheiro proved that finite rank plactic and hypoplactic monoids admit automatic presentations [A. Cain, personal communication]. As such, we ask the following:

Open Problems 4. Do all plactic-like monoids admit automatic presentations? Do all monoids admitting automatic presentations satisfy non-trivial identities?

As mentioned in Section 4.1, the plactic monoid of finite rank n does not satisfy any non-trivial identity of length less than or equal to n [Cai+17, Proposition 3.1]. This is the only known result on a lower bound for the minimum length of the identities satisfied by finite-rank plactic monoids. Recall that plac_n must satisfy at least one non-trivial identity [JK21, Theorem 3.1]. Let l be the length of said identity. Then, by [Cai+17, Proposition 3.1], all plactic monoids of rank strictly greater than l do not satisfy that identity, hence they do not generate the same variety as plac_n. However, it is still unknown if plac_m and plac_n generate the same variety, for any $n < m \leq l$. It is conceivable that there exists a hierarchy of plactic monoids of finite ranks, where each level consists of monoids which satisfy exactly the same identities.

The case of upper triangular tropical matrix monoids is similar, however, there is a method to construct identities satisfied by these monoids, given in [Izh14] (on which there is a mistake which voids the validity of the method, but which is corrected in an erratum) and in [Tay17, Section 3.2]. Unfortunately, the length of these identities is not explicitly stated, only an upper bound is given in [Izh14, Section 5]. However, due to the mistakes in the paper, this upper bound is for the length of identities which are not necessarily satisfied by upper triangular tropical matrix monoids, as shown in [Tay17, Section 3.2, Counterexample 3.2.1].

Although the length of the identities satisfied by the monoid of $n \times n$ upper triangular tropical matrices, given in [Tay17, Section 3.2], is not explicitly stated, it is easy to obtain, by noticing that the shortest $(n-1)$-power words of C and $[n-1]$ used in their construction are in fact non-cyclic de Bruijn sequences of order $n-1$ on a size-2 alphabet [Bru46]. As such, these words have length $2^{n-1}+n-2$, and the identities have length $2^{n+1}+4n-4$, which gives us an upper bound for the minimum length of the identities satisfied by upper triangular tropical matrix monoids. However, this upper bound is not tight, since there exist shorter identities satisfied by these monoids [Tay17, Proposition 3.2.30].

The plactic monoid of rank n must satisfy all identities satisfied by the monoid of $d \times d$ upper triangular tropical matrices [JK21, Corollary 3.3], where $d = \lfloor n^2/4 \rfloor + 1$, and all identities satisfied by plac_n must be satisfied by the monoid of $n \times n$ upper triangular tropical matrices [JK21, Theorem 4.4]. As such, the minimum length l of the identities satisfied by plac_n is such that

$$n \leq l \leq 2^{\lfloor n^2/4 \rfloor + 2} + n^2.$$

Recall that the variety generated by plac_2 coincides with the variety generated by the monoid of 2×2 upper triangular tropical matrices [Izh19, Corollary 7.19], and the variety generated by plac_3 coincides with the variety generated by the monoid of 3×3 upper triangular tropical matrices [JK21, Corollary 4.5]. However, it is still unknown if, for higher ranks, the varieties generated by plactic monoids and upper triangular tropical matrix monoids coincide.

Open Problems 5. Can these upper and lower bounds be improved, both in the case of the finite-rank plactic monoids and the case of upper triangular tropical matrix monoids? Do plactic monoids or upper triangular tropical matrix monoids of different finite ranks generate the same variety, and if so, which ones? Does the variety generated by a finite-rank plactic monoid always coincide with a variety generated by an upper triangular tropical matrix monoid?

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2021 Duarte Ribeiro IDENTITIES AND BASES IN PLACTIC, HYPOPLACTIC, SYLVESTER, AND RELATED MONOIDS