ON THE CONVERGENCE OF THE MILD RANDOM WALK ALGORITHM TO GENERATE RANDOM ONE-FACTORIZATIONS OF COMPLETE GRAPHS

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Abstract. The complete graph K_n, for n even, has a one-factorization (proper edge coloring) with $n-1$ colors. In the recent contribution [Dotan M., Linial N. (2017). ArXiv:1707.00477v2], the authors raised a conjecture on the convergence of the mild random walk on the Markov chain whose nodes are the colorings of K_n. The mild random walk consists in moving from a coloring C to a recoloring C' if and only if $\phi(C') \leq \phi(C)$, where ϕ is the potential function that takes its minimum at one-factorizations. We show the validity of such algorithm with several numerical experiments that demonstrate convergence in all cases (not just asymptotically) with polynomial cost. We prove several results on the mild random walk, we study deeply the properties of local minimum colorings, we give a detailed proof of the convergence of the algorithm for K_4 and K_6, and we raise new conjectures. We also present an alternative to the potential measure ϕ by consider the Shannon entropy, which has a strong parallelism with ϕ from the numerical standpoint.

Keywords: proper edge coloring; one-factorization; local minimum coloring; complete graph; mild random walk algorithm.

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1. Introduction

It is a standard result that the complete graph K_n with n vertices, for n even, has a one-factorization (proper edge coloring) with $n-1$ colors (in other words, its chromatic index is $\mathcal{X}'(K_n) = n-1$) [1, Th. 1]. Recall that a proper edge coloring is an edge coloring such that no two adjacent edges have the same color. Proper edge colorings may be identified with symmetric Latin squares. One-factorizations have been topics of considerable research in Graph Theory, see [3, 5, 6, 8, 9].

Given a coloring C of K_n with $n-1$ colors, we denote by $a_{\nu,\mu}(C)$ the total number of incident edges to the vertex ν of color μ. We define the potential of the vertex ν as $\phi(\nu) = \sum_\mu a_{\nu,\mu}(C)^2$. The potential of the coloring C is thus defined as $\phi(C) = \sum_\nu \phi(\nu) = \sum_\nu \sum_\mu a_{\nu,\mu}(C)^2$. One-factorizations are characterized in terms of the potential function: $n(n-1) \leq \phi(C) \leq n(n-1)^2$ for every C, where the lower bound is reached if and only if C is proper, and the upper bound is attained if and only if K_n is monochromatic [4, p. 4].

Another term that will be frequently used in this paper is recoloring: we say that C' is a recoloring of the coloring C if C and C' differ exactly by the color of one edge. The relation between the potential of C and a recoloring C' of it is the following: if the edge uv of C is painted with color μ_0, and after it is repainted with another color $\mu_1 \neq \mu_0$ to form C', then

$$\phi(C') = \phi(C) + 2(a_{u,\mu_1}(C) + a_{v,\mu_1}(C) + 2 - a_{u,\mu_0}(C) - a_{v,\mu_0}(C)). \tag{1.1}$$

This is readily seen by direct counting.

One may see colorings of K_n as states of a Markov chain $\mathcal{G}_n$, where one moves with uniform probability to recolorings having lower or equal potential. That is, the acceptance probability of moving from C to C' is $1/|N_\phi(C)|$, where $N_\phi(C)$ is the set of recolorings C' of C with $\phi(C') \leq \phi(C)$. This Markov chain may be referred to as mild random walk. We will denote by $\{X_n\}_{n=0}^\infty$ the sequence of states visited at each time step n.

In the recent preprint article [3] by Maya Dotan and Nati Linial (updated version of their preprint paper [4]), the following conjecture on the convergence of the mild random walk algorithm was raised:

Conjecture 1.1. *(*[3, Conjecture 1], [4, Conjecture 1]*) The mild random walk on $\mathcal{G}_n$ started from a uniformly random starting point reaches a one-factorization with probability $1 - o_n(1)$ in $\mathcal{O}(n^4)$ steps.*

This Conjecture 1.1 was analyzed and reformulated based on numerical evidence and analytical proofs in the recent Master's Thesis [2] written by one of the authors of this paper. In this paper, we have simplified and extended the main results from [2, Ch. 4].

The organization of the present paper is as follows. In Section 2, we describe and implement the mild random walk algorithm, and we perform numerical experiments to reformulate Conjecture 1.1. In Section 3, we give an equivalent condition for the convergence of the mild random walk algorithm based on local minimum colorings. Properties of local minimum colorings are studied deeply in this section. Section 4 derives a detailed proof of our forthcoming conjecture for K_4 and K_6, and provides some useful insights for K_n. Finally, Section 5 develops an alternative measure to the potential function ϕ, the Shannon entropy H of a coloring system. Numerical experiments establish the parallelism between the potential and entropy functions, and new conjectures and questions arise.

2. Generation of one-factorizations via the mild random walk algorithm

We present the computational mild random walk algorithm. This probabilistic algorithm is defined as, from a coloring C, one moves to a recoloring C' with probability $1/|N_\phi(C)|$ if $C' \in N_\phi(C)$. That is, one needs to know $N_\phi(C)$ explicitly. In practice, this implementation is computationally expensive (we have to compute the potential of all recolorings). We suggest an alternative, which makes the Markov chain to move in the same way, but more efficiently in terms of computations. The alternative algorithm consists in:

i) Choose randomly a recoloring C' of C;
ii) If $C' \in N_\phi(C)$, move to it. Otherwise, do step i) again.

(if after doing step ii) we do step i) again, the probability of choosing a recoloring C' in step i) is independent of the choices done before).

In order to analyze the validity of Conjecture 1.1, we implement the mild random walk algorithm in the software R [7]. Colorings are seen as matrices where the color of the edge ij is represented by a number in the matrix entry (i, j), $i \neq j$. The numerical experiments will allow us to reformulate Conjecture 1.1, see the forthcoming Conjecture 2.1.

```
arbitrary_coloring <- function(n) {
  # Goal: with n-1 colors (1,2,...,n-1), to color uniformly the edges of Kn.
  # Input: n = size of Kn, we understand n even.
  # Output: matrix C such that C(i,j) is the color of the edge ij, i distinct j.
  #         The coloring is arbitrary and C(i,i) is taken as n (just notation).

  C <- diag(n,n,n) # initialize matrix and put n at the diagonal
  for(column in 1:(n-1)) # fill the lower part of C with colors
    C[(column+1):n,column] <- sample(1:(n-1),size=n-column,replace=TRUE,
                                     prob=rep(1/(n-1),n-1))
  C[upper.tri(C)] <- t(C)[upper.tri(C)] # make C symmetric
  return(C)
}
```

```
potential_function <- function(C) {
  # Goal: to compute the potential function of the coloring given by the matrix C.
  # Input: coloring matrix C.
  # Output: potential function.

  n <- dim(C)[1]
  phi <- 0
  for(vertex in 1:n)
    for(color in 1:(n-1))
      phi <- phi + sum(C[,vertex]==color)^2
```

```
  return(phi)
}
```

```
find_proper_coloring <- function(n) {
  # Goal: to find a proper coloring of Kn.
  # Input: n = size of Kn. n must be EVEN.
  # Output: a list having a coloring C, a boolean that is TRUE if C is a proper
  #         coloring and FALSE otherwise, and the number of steps to find C.

  C <- arbitrary_coloring(n)  # start with an arbitrary coloring
  phiC <- potential_function(C) # potential of C
  steps <- 0
  proper <- FALSE
  while(proper == FALSE & steps < n^10) {
    edge <- sample(1:n, size=2, replace=FALSE, prob=rep(1/n, n)) # choose edge
    color <- sample((1:(n-1))[-C[edge[1], edge[2]]], size=1, replace=FALSE,
                    prob=rep(1/(n-2), n-2)) # choose a color
    C1 <- C # recolor
    C1[edge[1], edge[2]] <- C1[edge[2], edge[1]] <- color # C1 is a recoloring
    phiC1 <- potential_function(C1)
    if(phiC1 <= phiC) { # step ii)
      C <- C1 # if the potential of C1 is smaller or equal, we move to C1
      phiC <- phiC1
      steps <- steps + 1 # count step whenever we move
    }
    if(phiC == n*(n-1)) # case when we finish the procedure
      proper <- TRUE
  }
  return(list(C, proper, steps))
}
```

In Figure 1, we analyze the convergence rate of the mild random walk algorithm. For each one of the even numbers n between 10 and 22, we have executed the above algorithm 30 times. In all of them, convergence is achieved in a finite number of steps (look at the plotted circles). A regression line in logarithmic scale allows estimating whether the algorithm has polynomial cost. Our results show that, indeed, the number of steps needed by the algorithm grows polynomially in n with exponent ≈ 2.51.

We notice several differences with respect to [3]. According to our numerical experiments, we conjecture that the algorithm always converges, and not just asymptotically. It would converge asymptotically if we moved only when the potential is strictly smaller (strict random walk), but this is not our setting. On the other hand, we conjecture that the polynomial time is smaller than $\mathcal{O}(n^4)$, in fact $\mathcal{O}(n^p)$, where $2 < p < 3$ and $p \approx 2.51$, see the regression line in Figure 1.

Conjecture 2.1. *Consider K_n, with n even, and all possible colorings of it with $n-1$ colors. Consider the algorithm in which we start from an arbitrary coloring, and we move with uniform probability from a coloring C to a recoloring C' of it if*

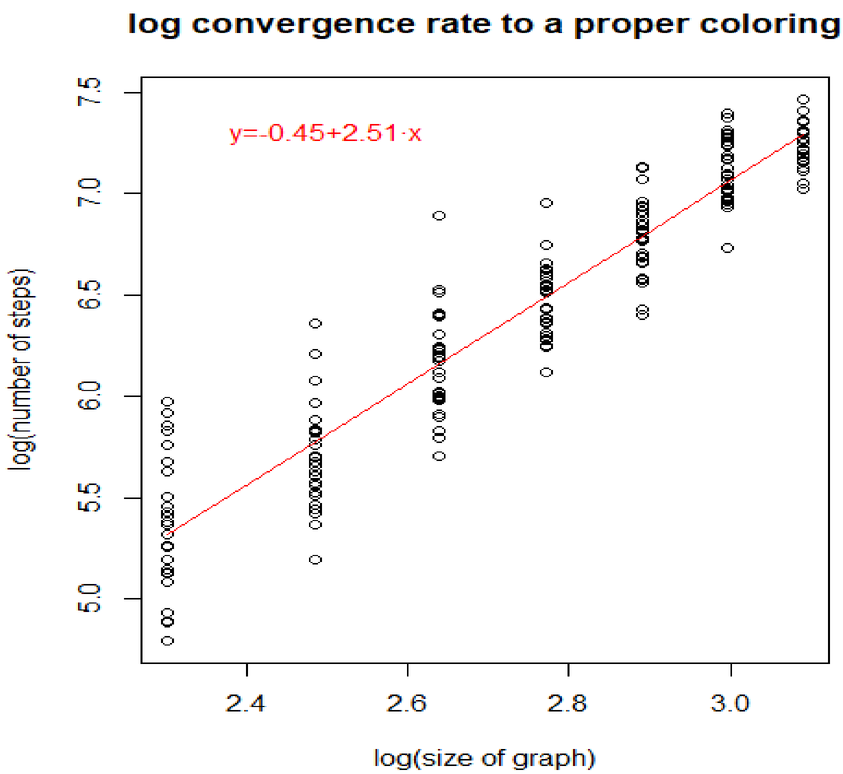

FIGURE 1. Steps required in the Markov chain to arrive at a proper coloring of K_n (circles). Regression line to study the cost in polynomial time.

and only if $\phi(C') \leq \phi(C)$ (mild random walk). Then the algorithm almost surely converges to a one-factorization of K_n for all n and, moreover, the number of steps required is $\mathcal{O}(n^p)$, where $2 < p < 3$.

3. Convergence of the mild random walk algorithm and local minimum colorings

Since proper colorings are absorbing states in $\mathcal{G}_n$ by the forthcoming Theorem 3.13 (denote the set of absorbing states by $\mathcal{B}_n$), the transition matrix P has the following structure:

$$P = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix}, \tag{3.1}$$

where I is the identity matrix with size $|\mathcal{B}_n| \times |\mathcal{B}_n|$, 0 is a matrix of zeros, R is the matrix of transition probabilities from a non-proper to a proper coloring, and Q is the matrix of transition probabilities between non-proper colorings.

As it is well-known, from Q we can compute the expected number of steps until the chain gets absorbed by a proper coloring. Indeed, the number of visits to a non-proper state j is $\sum_{n=0}^{\infty} \mathbb{1}_{\{X_n=j\}}$. Therefore, the expected number of visits to a non-proper state j having started at a non-proper state i is

$$\mathbb{E}\left[\sum_{n=0}^{\infty} \mathbb{1}_{\{X_n=j\}} \middle| X_0 = i\right] = \sum_{n=0}^{\infty} \mathbb{P}(X_n = j | X_0 = i) = \sum_{n=0}^{\infty} Q^n(i,j) = \left(\sum_{n=0}^{\infty} Q^n\right)(i,j).$$

Let $S = \sum_{j \notin \mathcal{B}_n} \sum_{n=0}^{\infty} \mathbb{1}_{\{X_n=j\}}$ be the random variable in $[0,\infty]$ that counts, at each experiment ω of the mild random walk algorithm, the number of steps until getting absorbed by a proper coloring. Then

$$\mathbb{E}[S|X_0 = i] = \sum_{j \notin \mathcal{B}_n} \left(\sum_{n=0}^{\infty} Q^n\right)(i,j) \in [0,\infty].$$

If we prove that $(\sum_{n=0}^{\infty} Q^n)(i,j) < \infty$, for all $i, j \notin \mathcal{B}_n$, then $\mathbb{E}[S] = \sum_{i \notin \mathcal{B}_n} \mathbb{E}[S|X_0 = i]\mathbb{P}(X_0 = i) < \infty$, which implies $S < \infty$ $\mathbb{P}$-almost surely. That is, from any starting coloring we reach with probability 1 a one-factorization in a finite number of steps.

The following two propositions give a condition under which $(\sum_{n=0}^{\infty} Q^n)(i,j) < \infty$. They come from standard results from linear algebra and Markov chains, and we reference [2, Prop. 4.4, Prop. 4.5] for a detailed proof.

Proposition 3.1. *If* $\lim_{n\to\infty} Q^n = 0$*, then* $I - Q$ *is invertible and* $(I-Q)^{-1} = \sum_{n=0}^{\infty} Q^n$*. In particular,* $(\sum_{n=0}^{\infty} Q^n)(i,j) < \infty$*.*

Proposition 3.2. *Consider a finite Markov chain with transition probability matrix of the form (3.1). If every non-absorbing state has a positive probability of being absorbed at some step, then* $\lim_{n\to\infty} Q^n = 0$*.*

We say that C is a local minimum coloring if $\phi(C) \leq \phi(C')$ for every recoloring C' of C. As a consequence of these two propositions, the convergence of the mild random walk is equivalent to the following fact: for every local minimum C, there exists a finite sequence of steps in our Markov chain until we reach a coloring C_1 such that $\phi(C_1) < \phi(C)$. Hence, the study of local minimum colorings is seen to be crucial.

Hereafter, we show properties of local minimum colorings. Given a fixed coloring C and an edge uv, we will denote by $\mu(uv)$ the color of the edge uv. We will also write $c_C(\mu)$ for the number of times color μ appears in coloring C. Let $c_{uv,\mu}(C) := a_{u,\mu}(C) + a_{v,\mu}(C)$ be the total number of edges of color μ incident to uv.

Lemma 3.3. *([3, Lemma 1]) If C is a local minimum coloring of K_n, then the monochromatic connected subgraphs of K_n are isolated edges or 2-paths. That is, $a_{u,\mu(uv)}(C) + a_{v,\mu(uv)}(C) \leq 3$ for all edge uv.*

Theorem 3.4 (Characterization of local minimum colorings of K_n)**.** *Let C be a coloring of K_n. The following conditions are equivalent:*

- *C is a local minimum.*
- *The monochromatic connected subgraphs of K_n are isolated edges or 2-paths, and each edge of every monochromatic 2-path has every color incident at least once. That is, $a_{u,\mu(uv)}(C) + a_{v,\mu(uv)}(C) \leq 3$ for all edge uv, and if for some edge uv the equality holds, then $a_{u,\mu}(C) + a_{v,\mu}(C) \geq 1$ for each color μ.*

Proof. Suppose that C is a local minimum. By Lemma 3.3, $a_{u,\mu(uv)}(C)+a_{v,\mu(uv)}(C) \leq 3$, for every edge uv. Suppose an edge uv such that $a_{u,\mu(uv)}(C) + a_{v,\mu(uv)}(C) = 3$. Then, by expression (1.1), for all $\mu_1 \neq \mu(uv)$ we have $a_{u,\mu_1}(C) + a_{v,\mu_1}(C) + 2 \geq a_{u,\mu(uv)}(C) + a_{v,\mu(uv)}(C) = 3$, that is, $a_{u,\mu_1}(C) + a_{v,\mu_1}(C) \geq 3 - 2 = 1$.

On the other hand, let us assume the second point of the theorem. Suppose by contradiction that there exists C' recoloring of C such that $\phi(C') < \phi(C)$. Assume that C' consists in repainting an edge u_0v_0 with color $\mu_0(u_0v_0) \neq \mu(u_0v_0)$ (recall that $\mu(u_0v_0)$ is the color of uv in C). Since $\phi(C') < \phi(C)$, by expression (1.1)

$$a_{u_0,\mu(u_0v_0)}(C) + a_{v_0,\mu(u_0v_0)}(C) > a_{u_0,\mu_0(u_0v_0)}(C) + a_{v_0,\mu_0(u_0v_0)}(C) + 2.$$

By Lemma 3.3, $a_{u_0,\mu(u_0v_0)}(C)+a_{v_0,\mu(u_0v_0)}(C) \leq 3$. It cannot be stricly less than 3, otherwise $a_{u_0,\mu_0(u_0v_0)}(C)+a_{v_0,\mu_0(u_0v_0)}(C) < 0$, which is not possible. Thus, $a_{u_0,\mu(u_0v_0)}(C)+a_{v_0,\mu(u_0v_0)}(C) = 3$, and we obtain $3 > a_{u_0,\mu_0(u_0v_0)}(C) + a_{v_0,\mu_0(u_0v_0)}(C) + 2$. But by hypothesis, $a_{u_0,\mu_0(u_0v_0)}(C) + a_{v_0,\mu_0(u_0v_0)}(C) \geq 1$, which is a contradiction. □

Lemma 3.5. *([3, p. 3]) If C is a local minimum coloring of K_n, then the number of monochromatic 2-paths of C is $\psi(C) = \frac{\phi(C)-n(n-1)}{2}$.*

Theorem 3.6. *Let C be a local minimum coloring of K_n. Then each edge of every monochromatic 2-path has at least one color incident only once.*

Proof. Let uv be an edge of a monochromatic 2-path: $a_{u,\mu(uv)}(C) + a_{v,\mu(uv)}(C) = 3$. Suppose by contradiction that $a_{u,\mu}(C) + a_{v,\mu}(C) \geq 2$ for all $\mu \neq \mu(uv)$. The number of edges incident to uv with color distinct from $\mu(uv)$ is $2n-5$. But according to the inequality $a_{u,\mu}(C) + a_{v,\mu}(C) \geq 2$, there should be $2(n-2) = 2n-4$ edges incident to uv with color distinct from $\mu(uv)$. This is a contradiction. □

Theorem 3.7. *Let C be a local minimum coloring of K_n. If it has a unique α-colored 2-path, then there is a γ-colored 2-path such that α is not incident to its center.*

Proof. Let us suppose by contradiction that every vertex has an incident edge of color α. Thus, C has an α-colored 2-path, plus isolated edges of color α. This gives an odd number n of vertices, contradicting the assumption of n being even. Thus, there must exist a vertex w with no incident edge of color α. Since the degree of w is $n-1$ and there are $n-2$ remaining colors, an incident color must be repeated, say γ. Then there is a γ-colored 2-path such that α is not incident to its center w. $\square$

Theorem 3.8. *Let C be a local minimum coloring of K_n. Then*

$$c_C(\mu) \leq \left\lfloor \frac{2n}{3} \right\rfloor,$$

for each color μ.

Proof. Suppose by contradiction that $c_C(\mu) > \lfloor 2n/3 \rfloor$, for certain color μ. We distinguish cases according to the value $n \bmod 3$:

- Case $n \equiv 0 \bmod 3$. In this case, $\lfloor 2n/3 \rfloor = 2n/3$, therefore, our assumption becomes $c_C(\mu) > 2n/3$. At least $2n/3+1$ edges are painted μ. These edges can go alone or forming 2-paths, by Theorem 3.4. The way these $2n/3+1$ cover less vertices is when there are $n/3$ 2-paths and an isolated edge, which takes up $3 \cdot n/3 + 2 = n+2$ vertices. This is a contradiction, as K_n has n vertices.
- Case $n \equiv 1 \bmod 3$. In this case, $\lfloor 2n/3 \rfloor = (2n-2)/3$ (since $2n-2 \equiv 0 \bmod 3$). Our assumption on μ becomes $c_C(\mu) > (2n-2)/3$. At least $(2n-2)/3+1$ edges are painted μ. These edges can go alone or forming 2-paths, by Theorem 3.4. The manner these $(2n-2)/3+1$ edges cover less vertices is with $(n-1)/3$ 2-paths and a single edge, which take up $3 \cdot (n-1)/3 + 2 = n+1$ vertices, and this gives a contradiction.
- Case $n \equiv 2 \bmod 3$. In this case, $\lfloor 2n/3 \rfloor = (2n-1)/3$ (notice that $2n - 1 \equiv 0 \bmod 3$). Our surmise on μ becomes $c_C(\mu) > (2n-1)/3$. At least $(2n-1)/3+1$ edges are painted μ. These edges can go alone or forming 2-paths, by Theorem 3.4. The way these $(2n-1)/3+1 = (2n+2)/3$ edges occupy less vertices is with $(n+1)/3$ 2-paths, which cover $3 \cdot (n+1)/3 = n+1$ vertices, and this gives once again a contradiction.

This finishes the proof. Notice that this upper-bound is tight, because for $n=4$ and $n=6$ it is reached. $\square$

Conjecture 3.9. *Let C be a local minimum coloring of K_n. Then*

$$c_C(\mu) \geq \left\lceil \frac{n}{3} \right\rceil,$$

for each color μ.

Theorem 3.10. *Let C be a local minimum coloring of K_n. Then $c_C(\mu) \geq 2$ if $n=4$ and $n=6$. Thus, Conjecture 3.9 holds for $n=4$ and $n=6$.*

Proof. For K_4, the result is clear, since if there were a color repeated only once, then there would be a monochromatic connected subgraph with 3 edges, which is not possible by Theorem 3.4.

For K_6, let us assume that there is a color μ repeated once at an edge u_1u_2. Then there are four vertices, say v_1, v_2, v_3, v_4, with no incident edge of color μ. Since the degree of each v_i is 5 and there are four possible incident remaining colors, each v_i must be the center of a monochromatic 2-path. For each one of those monochromatic 2-paths, no edge can go from a v_i to a v_j, otherwise such an edge would not have μ incident, which is not possible by Theorem 3.4. Thus, the four monochromatic 2-paths with centers v_1, v_2, v_3, v_4, respectively, must have their endpoints at u_1 and u_2, so they must be made up with distinct colors, say $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, respectively. Moreover, for each v_i, its three remaining incident edges (the ones that do not belong to the monochromatic 2-path of center v_i) must be painted with the three remaining colors distinct from μ and α_i. Fix a color α_1. Then v_1 has α_1 in its monochromatic 2-path, and the other v_j, $j > 1$, have α_1 repeated once. Therefore α_1 is repeated five times, but this is not possible by Theorem 3.8. □

Theorem 3.11. *Let C be a local minimum coloring. Then, for every edge uv and every color μ',*

$$c_{uv,\mu(uv)}(C) - c_{uv,\mu'}(C) \leq 2.$$

Equality holds if and only if $\phi(C') = \phi(C)$, where C' is the recoloring of C constructed by changing the color of the edge uv to μ'.

Proof. By expression (1.1), $\phi(C') = \phi(C) + 2(c_{uv,\mu'}(C) + 2 - c_{uv,\mu}(C))$. As C is a local minimum, $\phi(C) \leq \phi(C') = \phi(C) + 2(c_{uv,\mu'}(C) + 2 - c_{uv,\mu}(C))$. Hence, $c_{uv,\mu}(C) - c_{uv,\mu'}(C) \leq 2$, as wanted. Equality holds if and only if $\phi(C) = \phi(C')$. □

Corollary 3.12. *The moves permitted to maintain the potential of a local minimum coloring are:*

- *if uv is part of a monochromatic 2-path, you can change the color of uv by the color of an edge incident once in uv;*
- *if uv is an independent edge, you can change the color of uv by a non-incident color to uv.*

Proof. Let C be a local minimum coloring. Let uv be any edge of C. By Theorem 3.4, either uv is part of a monochromatic 2-path or it is independent of the other edges with its same color. If uv is part of an α-colored 2-path, then $c_{uv,\alpha}(C) = 3$. By Theorem 3.11, for each color μ', $c_{uv,\mu'} \geq 3 - 2 = 1$. Thus, if C' is a recoloring of C consisting in changing the color α of uv by β, then $\phi(C') = \phi(C)$ if and only if $c_{uv,\beta} = 1$, or in words, there is a single β-colored edge incident to uv. On the other hand, if uv is an independent α-colored edge, then the unique way of having $\phi(C') = \phi(C)$ is by changing α by a color not incident to uv. □

Theorem 3.13 (Characterization of one-factorizations of K_n)**.** *Let C be a coloring of K_n. The following conditions are equivalent:*

- *C is a one-factorization.*
- *$\phi(C') > \phi(C)$ for every recoloring C' of C.*

Proof. Suppose that C is a one-factorization. If we recolor an edge of C to obtain C', then we obtain a monochromatic 2-path, therefore C' cannot be a one-factorization. Thus, $\phi(C') > n(n-1) = \phi(C)$.

On the other hand, assume that $\phi(C') > \phi(C)$ for every recoloring C' of C. Then C is a local minimum. Suppose by contradiction that C is not proper. Then there exists a monochromatic 2-path. Let uv be an edge of this monochromatic 2-path of color $\mu(uv)$. By Theorem 3.6, there exists a color $\bar{\mu}$ incident only once to uv. By Corollary 3.12, we can change the color of uv by $\bar{\mu}$ so that the new recoloring C' possesses the same potential as C, which is a contradiction. □

4. Proof of the convergence of the mild random walk algorithm for K_4 and K_6

In this section, we prove Conjecture 2.1 for K_4 and K_6 by using counting methods.

Theorem 4.1. *The mild random walk algorithm converges for K_4.*

Proof. Let C be a non-proper local minimum coloring of K_4 made up with three colors. We need to prove that there exists a finite sequence of steps in $\mathcal{G}_4$ until we reach a coloring C_1 such that $\phi(C_1) < \phi(C)$. Since C is not proper, there is a monochromatic 2-path, say of color α, by Theorem 3.4. Moreover, there cannot be more edges of color α. By Theorem 3.7, there is another monochromatic 2-path, say of color γ, such that its center does not have an α-colored incident edge. There is an edge uv from the α-colored 2-path that has a γ-colored incident edge only once. By Corollary 3.12, we can change the color of uv to γ maintaining the potential. The new recoloring has a 3-path of color γ, so it is not a local minimum, and its potential can be decreased at a next step. □

Theorem 4.2. *The mild random walk algorithm converges for K_6.*

Proof. Let C be a non-proper local minimum coloring of K_6 painted with five colors. We have to prove that there exists a finite sequence of steps in $\mathcal{G}_6$ until we can decrease its potential. By Theorem 3.8 and Theorem 3.10, each color in C is repeated between two and four times, therefore the possible repetitions of colors are 22344, 23334 and 33333.

We claim that, from the local minimum colorings with structure 22344, we can reach a new coloring in $\mathcal{G}_6$ with structure 23334, and from 23334 another coloring in

$\mathcal{G}_6$ with repetitions 33333. Thus, it will suffice to deal with a local minimum coloring C having repetitions 33333.

Indeed, let C be a local minimum coloring with structure 22344 or 23334. The common feature of both cases is that there is a color repeated four times, say α, and another color repeated twice, say β. If we can repaint an α-colored edge with color β maintaining the potential, the structure 22344 will have reached 23334, and the structure 23334 will have become 33333, as wanted. Thus, we need to prove that an α-colored edge can be repainted with color β maintaining the potential. Since α is repeated four times in K_6, it appears in the form of two monochromatic 2-paths. There is an edge uv among the four edges of the two α-colored 2-paths that has a β-colored incident edge only once. By Corollary 3.12, we can repaint uv with β maintaining the potential, and we are done.

Thereby, it is enough to deal with a local minimum coloring C having repetitions 33333. If C is not a one-factorization, then it must have a monochromatic 2-path, say of color α. Since α is repeated three times, there is another independent edge of color α. Label the vertices of K_6 as $1, 2, 3, 4, 5, 6$. We may assume without loss of generality that 123 is the α-colored 2-path and that 56 is the α-colored independent edge. By Theorem 3.7, there is another monochromatic 2-path, say of color β, such that its center does not have an α-colored incident edge. Thus, its center must be vertex 4. We derive that the possibilities for the β-colored 2-path are: 342, 341, 346, 345, 241, 246, 245, 146, 145 and 645. Having placed the β-colored 2-path, there are some available positions for the independent edge of color β, by taking into account Theorem 3.4. See Table 1.

β-colored 2-path	342	341	346	345	241	246	245	146	145	645
Independent edge of color β	16, 15	26, 25	15, 25	16, 26	36, 35	15, 13, 35	16, 13, 36	25, 35	36, 26	13

TABLE 1. Possible positions of the β-colored edges in K_6.

For example, we plot in Figure 2 the two cases from the first column of Table 1.

Using Corollary 3.12, it is easy to see that from each coloring from Table 1 one can arrive at a non-local minimum coloring in $\mathcal{G}_6$. For example, from C_1 from Figure 2, since 23 is the only α-colored edge incident to 34, we can repaint 34 with α to obtain an α-colored 3-path, which yields a non-local minimum coloring. This completes the proof. □

Some of the ideas utilized for K_4 and K_6 could be applied for K_n, although our counting method cannot be carried out for a general n. An example of a solvable

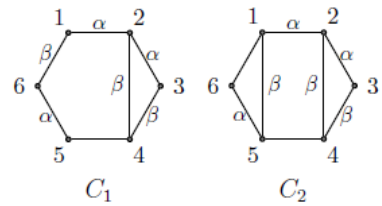

FIGURE 2. Colorings C_1 and C_2 corresponding to the first column of Table 1.

coloring in K_n is the following. Let C be a non-proper local minimum coloring of K_n. As C is non-proper, it has a monochromatic 2-path, say of color α. By Theorem 3.7, there is a color β such that $c_{uv,\beta}(C) = 1$. If β is part of a monochromatic 2-path, then, by changing the color of uv by β, we obtain a recoloring C' such that $\phi(C') = \phi(C)$ and with a 3-path of color β. By Theorem 3.4, C' is not a local minimum, so its potential can be decreased in a next step, and we are done with this case. It would remain the difficult case in which the unique β-colored incident edge to uv is not part of a monochromatic 2-path.

5. Entropy

Consider K_n and $n-1$ colors. Let $\Omega = \{(\nu,\mu) : \nu \text{ vertex}, \mu \text{ color}\}$ be a sample space, with σ-algebra $\mathcal{F} = 2^\Omega$ and probability measure $\mathbb{P}(\nu,\mu) = \frac{a_{\nu,\mu}(C)}{n(n-1)} = p_{\nu,\mu}$. For each coloring C of K_n, we have $n(n-1)$ events of the form $\{(\nu,\mu)\}$, each one of probability $p_{\nu,\mu}$. The Shannon entropy H of the system C is the expected value of the amount of information: $H(C) = -\sum_{\nu,\mu} p_{\nu,\mu} \log p_{\nu,\mu}$ (here, $0 \log 0 = 0$). This function measures the chaos-uncertainty-surprise in the system C. In our case, when K_n is monochromatic, there is no uncertainty, so the entropy should be minimum; otherwise, one-factorizations present the most uncertainty, so the entropy is maximum. It is easy to prove that $\log n \le H(C) \le \log(n(n-1))$, being the lower bound attained if C is monochromatic and the upper bound reached if C is proper.

The entropy is an alternative measure to the potential. We can consider an alternative mild random walk algorithm, in which we move from a coloring C to a recoloring C' if $H(C) \le H(C')$. We implement the mild random walk algorithm based on the entropy in the software R.

```
entropy <- function(C) {
  # Goal: to compute the entropy of the coloring given by the matrix C.
  #       It belongs to the interval [-log(1/n),-log(1/(n*(n-1)))].
  # Input: coloring matrix C.
  # Output: entropy of C.

  n <- dim(C)[1]
```

```
  H <- 0
  for(vertex in 1:n) {
    for(color in 1:(n-1)) {
      p <- sum(C[,vertex]==color)/(n*(n-1))
      H <- H + ifelse(p == 0, 0, -p*log(p))
    }
  }
  return(H)
}
```

```
find_proper_coloring_entropy <- function(n) {
  # Goal: to find a proper coloring of Kn via the entropy measure.
  # Input: n = size of Kn. n must be EVEN.
  # Output: a list having a coloring C, a boolean that is TRUE if C is a proper
  #          coloring and FALSE otherwise, and the number of steps to find C.

  C <- arbitrary_coloring(n) # start with an arbitrary coloring C
  entropyC <- entropy(C) # entropy of coloring C
  steps <- 0
  proper <- FALSE
  while(proper == FALSE & steps < n^10) {
    edge <- sample(1:n,size=2,replace=FALSE,prob=rep(1/n,n)) # choose arbitrary edge
    color <- sample((1:(n-1))[-C[edge[1],edge[2]]],size=1,replace=FALSE,
                    prob=rep(1/(n-2),n-2)) # choose color
    C1 <- C # recolor
    C1[edge[1],edge[2]] <- C1[edge[2],edge[1]] <- color # C1 is a recoloring of C
    entropyC1 <- entropy(C1)
    if(round(entropyC,4) <= round(entropyC1,4)) {
      C <- C1 # if the entropy of C1 is greater or equal, we move to C1
      entropyC <- entropyC1
      steps <- steps + 1 # count step whenever we move
    }
    if(round(entropyC,4) == round(log(n*(n-1)),4)) # case when we finish
      proper <- TRUE
  }
  return(list(C,proper,steps))
}
```

Figure 3 depicts the convergence rate of the mild random walk algorithm based on the entropy measure. For each one of the even numbers n between 10 and 22, we have executed the presented code 30 times. In all cases, convergence is achieved in a finite number of steps (look at the plotted circles). The regression line shows that the number of steps needed by the algorithm increases polynomially in n with exponent ≈ 2.59.

The numerical results are similar to those obtained with the potential function. This makes us raise the following conjecture, analogous to the potential function setting.

Conjecture 5.1. *Consider K_n, with n even, and all possible colorings of it with $n-1$ colors. Consider the algorithm in which we start from an arbitrary coloring, and we move with uniform probability from a coloring C to a recoloring C' of it if*

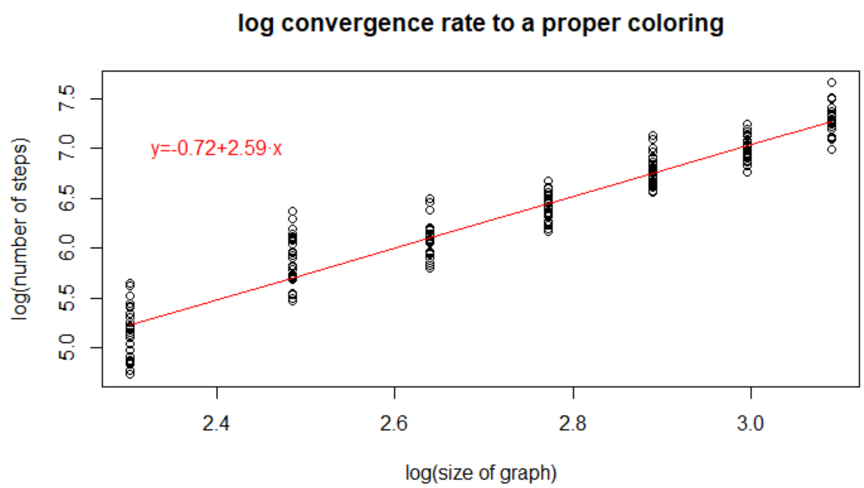

FIGURE 3. Steps required in the Markov chain to arrive at a proper coloring of K_n via the entropy algorithm. Regression line to study the cost in polynomial time.

and only if $H(C) \leq H(C')$ *(mild random walk). Then the algorithm almost surely converges to a one-factorization of* K_n *for all* n *and, moreover, the number of steps required is* $\mathcal{O}(n^p)$*, where* $2 < p < 3$.

Conflict of Interest Statement

The authors declare that there is no conflict of interests regarding the publication of this article.

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