FUNDAMENTAL CONCEPTS ON FOURIER ANALYSIS (WITH EXERCISES AND APPLICATIONS)

by

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Abstract

In this work we present the main concepts of Fourier Analysis (such as Fourier series, Fourier transforms, Parseval and Plancherel identities, correlation, and convolution) and illustrate them by means of examples and applications. Most of the concepts presented here can be found in the book *"A First Course in Fourier Analysis"* by David W.Kammler. Similarly, the examples correspond to over 15 problems posed in the same book which have been completely worked out in this report. As applications, we include Fourier's original approach to the heat flow using Fourier series and an application to filtering one-dimensional signals.

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Chapter 1

Introduction to Fourier Analysis

Fourier analysis, or frequency analysis, in the simplest sense, is the study of the effects of adding together sine and cosine functions. This type of analysis has become an essential tool in the study of a remarkably large number of engineering and scientific problems.

Daniel Bernoulli, while studying vibrations of a string in the 1750s, first suggested that a continuous function over the interval $(0, \pi)$ could be represented by an infinite series consisting only of sine functions. This suggestion was based on his physical intuition. Later, *J. B. Fourier* reopened the controversy while studying heat transfer. He argued, more formally, that a function continuous on an interval $(-\pi, \pi)$ could be represented as a linear combination of both sine and cosine functions of different frequencies which could later be combined to reconstruct the original function.[2]

The two main aspects of Fourier Analysis are the Fourier Series and Fourier Transforms.

The Fourier series of a periodic function (with period T) is defined, in rectangular form as

$$f(t) = \sum_{k=0}^{\infty} \left[A_k \cos\left(\frac{2\pi kt}{T}\right) + B_k \sin\left(\frac{2\pi kt}{T}\right) \right],$$

where the Fourier coefficients A_k and B_k are given by

$$A_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos\Big(\frac{2\pi kt}{T}\Big)\,dt, \qquad k = 1, 2, \ldots, \infty$$

$$A_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,dt,$$

$$B_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin\Big(\frac{2\pi kt}{T}\Big)\,dt, \qquad k = 1, 2, \ldots, \infty$$

$$B_0 = 0.$$

Given the function $f(t)$, we define the Fourier transform pair as

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-2\pi ist}\,dt,$$

$$f(t) = \int_{-\infty}^{\infty} F(s)e^{2\pi ist}\,ds.$$

The first equation is known as the *(direct) Fourier transform*, and the second equation is the *inverse Fourier transform*[2]

In this section, we will be dealing with functions defined on $\mathbb{R}$, $\mathbb{T}_p$, $\mathbb{Z}$, and $\mathbb{P}_N$. Besides this, we will focus on Gibbs Phenomenon. These applications stem from different useful properties of Fourier Transforms, some of them being Parseval Identities, Plancherel Identities, and Orthogonality relations while the remaining will be covered in the other sections.

1.1 Functions defined on $\mathbb{R}$

If f is any suitably regular complex-valued function defined on $\mathbb{R}$, then it can be analyzed using

$$F(s) = \int_{-\infty}^{\infty} f(x)e^{-2\pi isx}\,dx, \quad -\infty < s < \infty \tag{1.1}$$

This function F is also a complex-valued function defined on $\mathbb{R}$ from which we can synthesize the function f as

$$f(x) = \int_{-\infty}^{\infty} F(s)e^{2\pi isx}\,ds, \quad -\infty < x < \infty \tag{1.2}$$

The function F is said to be the Fourier Transform of f. (1.1) is the analysis equation and (1.2) is the synthesis equation for f.[1]

1.2 Example of functions defined on $\mathbb{R}$

In this section, we will illustrate how the Fourier representation (1.1)- (1.2) from section 1.1 is valid for the box function

$$f(x) := \begin{cases} 1 & \text{if } -\frac{1}{2} < x < \frac{1}{2}, \\ 1/2 & \text{if } x = \pm\frac{1}{2}, \\ 0 & \text{if } x < -\frac{1}{2} \text{ or } x > \frac{1}{2}. \end{cases}$$

(a) Evaluate the integral (1.1) in this particular case and thereby show that

$$F(s) := \begin{cases} 1 & \text{if } s = 0, \\ \frac{\sin(\pi s)}{\pi s} & \text{if } s \neq 0. \end{cases}$$

Solution: Using (1.1) we have,

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} f(x)e^{-2\pi isx}\,dx \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)e^{-2\pi isx}\,dx \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi isx}\,dx \\ &= \left[\frac{e^{-2\pi isx}}{-2\pi is}\right]_{-\frac{1}{2}}^{\frac{1}{2}} \\ &= \frac{e^{-\pi is}}{-2\pi is} + \frac{e^{\pi is}}{2\pi is} \\ &= \frac{\sin(\pi s)}{\pi s} \\ &= \begin{cases} 1 & \text{if } s = 0, \\ \frac{\sin(\pi s)}{\pi s} & \text{if } s \neq 0. \end{cases} \end{aligned}$$

(b) By using the fact that F is even, show that the synthesis equation (1.2) for f reduces to the identity

$$f(x) = \int_{0}^{\infty} 2\Big(\frac{\sin(\pi s)}{\pi s}\Big)\cos(2\pi sx)\,ds.$$

Solution: By (1.2) we have,

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} F(s)e^{2\pi isx}\,ds. \\ &= \int_{-\infty}^{\infty} F(s)[\cos(2\pi sx) + i\sin(2\pi sx)]\,ds. \end{aligned}$$

since $F(s)\sin(2\pi sx)$ is odd

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} F(s)\cos(2\pi sx)\, ds \\ &= 2\int_{0}^{\infty} F(s)\cos(2\pi sx)\, ds \\ &= 2\int_{0}^{\infty} \Big(\frac{sin(\pi s)}{\pi s}\Big)\cos(2\pi sx)\, ds. \end{aligned}$$

(c) Use integration by parts to verify that

$$\int_{0}^{\infty} e^{-ps}\cos(\pi qs)\, ds = \frac{p}{p^2+(\pi q)^2}, \quad p > 0.$$

Solution: Suppose,

$$I = \int_{0}^{\infty} e^{-ps}\cos(\pi qs)\, ds$$

Using integration by parts

$$\begin{aligned} I &= \Big[e^{-ps}\Big(\frac{\sin(\pi qs)}{\pi q}\Big)\Big]_0^{\infty} + \int_{0}^{\infty} pe^{-ps}\Big(\frac{\sin(\pi qs)}{\pi q}\Big)\, ds \\ &= \Big[e^{-ps}\Big(\frac{\sin(\pi qs)}{\pi q}\Big)\Big]_0^{\infty} - \Big[\Big(\frac{pe^{-ps}}{\pi^2 q^2}\Big)\cos(\pi qs)\Big]_0^{\infty} \\ &- \frac{p^2}{\pi^2 q^2}\int_{0}^{\infty} e^{-ps}\cos(\pi qs)\, ds \\ &= \frac{p}{\pi^2 q^2} - \frac{p^2}{\pi^2 q^2} I. \end{aligned}$$

Hence,

$$I + \frac{p^2}{\pi^2 q^2} I = \frac{p}{\pi^2 q^2}.$$

Therefore,

$$I = \int_{0}^{\infty} e^{-ps}\cos(\pi qs)\, ds = \frac{p}{p^2 + \pi^2 q^2}.$$

(d) Integrate the identity of (c) with respect to q from $q = 0$ to $q = a$ to obtain

$$\int_{0}^{\infty} e^{-ps}\Big(\frac{\sin(\pi as)}{\pi s}\Big)\, ds = \frac{1}{\pi}\arctan\Big(\frac{\pi a}{p}\Big).$$

Solution: From (c), we have

$$\int_0^\infty e^{-ps}\cos(\pi qs)\,ds = \frac{p}{p^2+\pi^2q^2}.$$

Then Integrating with respect to q from 0 to a,

$$\begin{aligned}
\int_{q=0}^{a}\int_{s=0}^{\infty} e^{-ps}\cos(\pi qs)\,ds\,dq &= \int_0^a \frac{p}{p^2+\pi^2q^2}\,dq \\
\int_0^\infty e^{-ps}\left[\frac{\sin(\pi qs)}{\pi s}\right]_0^a ds &= \frac{1}{\pi}\left[\arctan\left(\frac{\pi q}{p}\right)\right]_0^a \\
\int_0^\infty e^{-ps}\left(\frac{\sin(\pi as)}{\pi s}\right) ds &= \frac{1}{\pi}\arctan\left(\frac{\pi a}{p}\right).
\end{aligned}$$

(e) Let $p \to 0+$ in the identity of (d), and thereby show that

$$\int_0^\infty \frac{\sin(\pi as)}{\pi s}\,ds = \begin{cases} -\frac{1}{2} & \text{if } a<0 \\ 0 & \text{if } a=0 \\ \frac{1}{2} & \text{if } a>0. \end{cases}$$

Solution: From (d), we have

$$\int_0^\infty e^{-ps}\left(\frac{\sin(\pi as)}{\pi s}\right) ds = \frac{1}{\pi}\arctan\left(\frac{\pi a}{p}\right),$$

Thus,

$$\begin{aligned}
\lim_{p\to 0^+}\int_0^\infty e^{-ps}\left(\frac{\sin(\pi as)}{\pi s}\right) ds &= \lim_{p\to 0^+}\frac{1}{\pi}\arctan\left(\frac{\pi a}{p}\right) \\
\int_0^\infty \left(\lim_{p\to 0^+} e^{-ps}\right)\left(\frac{\sin(\pi as)}{\pi s}\right) ds &= \frac{1}{\pi}\lim_{p\to 0^+}\arctan\left(\frac{\pi a}{p}\right) \\
\int_0^\infty \frac{\sin \pi as}{\pi s} ds &= \frac{1}{\pi}\begin{cases} -\frac{\pi}{2} & \text{if } a<0 \\ 0 & \text{if } a=0 \\ \frac{\pi}{2} & \text{if } a>0 \end{cases} \\
&= \begin{cases} -\frac{1}{2} & \text{if } a<0 \\ 0 & \text{if } a=0 \\ \frac{1}{2} & \text{if } a>0. \end{cases}
\end{aligned}$$

(f) Use a trigonometric identity to write the integral from the synthesis equation of (b) in the form

$$\int_0^\infty \frac{2\sin(\pi s)\cos(2\pi sx)}{\pi s}\,ds = \int_0^\infty \frac{\sin[\pi(1+2x)s]}{\pi s}\,ds + \int_0^\infty \frac{\sin[\pi(1-2x)s]}{\pi s}\,ds.$$

Solution: We have,

$$\begin{aligned}
2\sin(\pi s)\cos(2\pi s x) &= \sin(\pi s + 2\pi s x) + \sin(\pi s - 2\pi s x) \\
\int_0^\infty \frac{2\sin(\pi s)\cos(2\pi s x)}{\pi s}\,ds &= \int_0^\infty \frac{\sin(\pi s + 2\pi s x) + \sin(\pi s - 2\pi s x)}{\pi s}\,ds \\
&= \int_0^\infty \frac{\sin(\pi s + 2\pi s x)}{\pi s}\,ds + \int_0^\infty \frac{\sin(\pi s - 2\pi s x)}{\pi s}\,ds \\
&= \int_0^\infty \frac{\sin[\pi(1+2x)s]}{\pi s}\,ds + \int_0^\infty \frac{\sin[\pi(1-2x)s]}{\pi s}\,ds.
\end{aligned}$$

(g) Finally, use the result of (e) (with $a = 1 \pm 2x$) to evaluate the integrals of (f) and thereby verify the synthesis identity from (b).

Solution: From (e) we have

$$\int_0^\infty \frac{\sin(\pi a s)}{\pi s}\,ds = \begin{cases} -\frac{1}{2} & \text{if } a < 0 \\ 0 & \text{if } a = 0 \\ \frac{1}{2} & \text{if } a > 0. \end{cases}$$

But from (f) we have,

$$\int_0^\infty \frac{2\sin(\pi s)\cos(2\pi s x)}{\pi s}\,ds = \int_0^\infty \frac{\sin[\pi(1+2x)s]}{\pi s}\,ds + \int_0^\infty \frac{\sin[\pi(1-2x)s]}{\pi s}\,ds$$

$$= \mathrm{I} + \mathrm{II}.$$

Then we have the following cases:

(i) When $1 + 2x < 0 \Rightarrow x < -\frac{1}{2}$

Then, I$=-\frac{1}{2}$ and II$=\frac{1}{2}$

And, when $1 - 2x < 0 \Rightarrow x > \frac{1}{2}$

We get, I$=\frac{1}{2}$ and II$=-\frac{1}{2}$

(ii) When $1 + 2x = 0 \Rightarrow x = -\frac{1}{2}$

Then, I=0 and II$=\frac{1}{2}$

And, when $1 - 2x = 0 \Rightarrow x = \frac{1}{2}$

We get, I$=\frac{1}{2}$ and II=0

(iii) When $1+2x>0 \Rightarrow -\frac{1}{2}<x$

Then, I=$\frac{1}{2}$ and II=$\frac{1}{2}$

And, when $1-2x>0 \Rightarrow x<\frac{1}{2}$

We get, I=$\frac{1}{2}$ and II=$\frac{1}{2}$

Combining these 3 cases, we have

$$\int_0^\infty \frac{2\sin(\pi s)\cos(2\pi s x)}{\pi s}\,ds = \int_0^\infty \frac{\sin[\pi(1+2x)s]}{\pi s}\,ds + \int_0^\infty \frac{\sin[\pi(1-2x)s]}{\pi s}\,ds$$

$$= \begin{cases} 1 & \text{if } -\frac{1}{2}<x<\frac{1}{2} \\ 1/2 & \text{if } x=\pm\frac{1}{2} \\ 0 & \text{if } x<-\frac{1}{2} \text{ or } x>\frac{1}{2}. \end{cases}$$

1.3 Functions defined on $\mathbb{T}_p$

Let $p>0$, a function f defined on $\mathbb{R}$ is *p-periodic* when,

$$f(x+p)=f(x), \quad -\infty<x<\infty.$$

This *p-periodic* function f is analyzed as,

$$F[k]=\frac{1}{p}\int_0^p f(x)e^{\frac{-2\pi ikx}{p}}dx, \quad k=0,\pm1,\pm1\ldots \tag{1.3}$$

F in this case is a complex-valued function on $\mathbb{Z}$. Using the function $F[k]$ we can construct f as,

$$f(x)=\sum_{k=-\infty}^{\infty} F[k]e^{\frac{2\pi ikx}{p}}, \quad -\infty<x<\infty \tag{1.4}$$

(1.3) is the analysis equation and (1.4) is the synthesis equation for *p-periodic* function f.[1]

1.4 Example of functions defined on $\mathbb{T}_p$

In this section, we will derive the cosine and sine transform pair for a regular real valued function.

Let g be a suitably regular real-valued function defined on the “right hand side” of $\mathbb{T}_p$, i.e. for $0\le x\le \frac{p}{2}$.

(a) Use (1.3)- (1.4) to observe the cosine transform pair

$$g(x) = G[0] + 2\sum_{k=1}^{\infty} G[k]\cos\left(\frac{2\pi kx}{p}\right), \quad 0 \le x \le \frac{p}{2}$$

and

$$G[k] = \frac{2}{p}\int_0^{\frac{p}{2}} g(x)\cos\left(\frac{2\pi kx}{p}\right)dx,$$

which shows how to synthesize g from cosine functions.

Solution: Using (1.4) we have

$$\begin{aligned} g(x) &= \sum_{k=-\infty}^{\infty} G[k]e^{\frac{2\pi ikx}{p}} \\ &= \sum_{k=-\infty}^{\infty} G[k]\cos\left(\frac{2\pi kx}{p}\right) \\ &= \sum_{k=-\infty}^{-1} G[-k]\cos\left(\frac{2\pi kx}{p}\right) + \sum_{k=1}^{\infty} G[k]\cos\left(\frac{2\pi kx}{p}\right) + G[0]\cos(0) \\ &= \sum_{k=1}^{\infty} G[k]\cos\left(\frac{2\pi kx}{p}\right) + \sum_{k=1}^{\infty} G[k]\cos\left(\frac{2\pi kx}{p}\right) + G[0] \\ &= G[0] + 2\sum_{k=1}^{\infty} G[k]\cos\left(\frac{2\pi kx}{p}\right), \quad 0 \le x \le \frac{p}{2}. \end{aligned}$$

Now, from (1.3) we have,

$$\begin{aligned} G[k] &= \frac{1}{p}\int_0^{p} g(x)e^{-\frac{2\pi ikx}{p}}\,dx, \quad k = 0, \pm 1, \pm 2, \ldots \\ &= \frac{1}{p}\int_0^{p} g(x)\cos\left(\frac{2\pi kx}{p}\right)dx \\ &= \frac{2}{p}\int_0^{\frac{p}{2}} g(x)\cos\left(\frac{2\pi kx}{p}\right)dx. \end{aligned}$$

(b) Derive the analogous sine transform pair

$$g(x) = 2\sum_{k=1}^{\infty} G[k]\sin\left(\frac{2\pi kx}{p}\right), \quad 0 < x < \frac{p}{2}$$

$$G[k] = \frac{2}{p}\int_0^{\frac{p}{2}} g(x)\sin\left(\frac{2\pi kx}{p}\right)dx.$$

Solution: From (1.4) we have

$$\begin{aligned}
g(x) &= \sum_{k=-\infty}^{\infty} G[k]e^{\frac{2\pi ikx}{p}} \\
&= \sum_{k=-\infty}^{\infty} G[k]\sin\Big(\frac{2\pi kx}{p}\Big) \\
&= \sum_{k=-\infty}^{-1} G[-k]\sin\Big(\frac{2\pi kx}{p}\Big) + \sum_{k=1}^{\infty} G[k]\sin\Big(\frac{2\pi kx}{p}\Big) + G[0]\sin(0) \\
&= \sum_{k=1}^{\infty} G[k]\sin\Big(\frac{2\pi kx}{p}\Big) + \sum_{k=1}^{\infty} G[k]\sin\Big(\frac{2\pi kx}{p}\Big) \\
&= 2\sum_{k=1}^{\infty} G[k]\sin\Big(\frac{2\pi kx}{p}\Big).
\end{aligned}$$

Now , from (1.3) we have

$$\begin{aligned}
G[k] &= \frac{1}{p}\int_0^p g(x)e^{-\frac{2\pi ikx}{p}}\,dx, \qquad\qquad k = 0, \pm1, \pm2, \ldots \\
&= \frac{1}{p}\int_0^p g(x)\sin\Big(\frac{2\pi kx}{p}\Big)\,dx \\
&= \frac{2}{p}\int_0^{\frac{p}{2}} g(x)\sin\Big(\frac{2\pi kx}{p}\Big)\,dx.
\end{aligned}$$

1.5 Functions defined on $\mathbb{Z}$

If f is any suitably regular function on $\mathbb{Z}$, then it can be analyzed using,

$$F(s) = \frac{1}{p}\sum_{n=-\infty}^{\infty} f[n]e^{\frac{-2\pi isn}{p}}. \tag{1.5}$$

This function F is a complex-valued function on $\mathbb{T}_p$ from which we can synthesize the function f as,

$$f[n] = \int_0^p F(s)e^{\frac{2\pi isn}{p}}\,ds. \tag{1.6}$$

(1.5) gives the analysis equation for function on $\mathbb{Z}$ and (1.6) is the synthesis equation of f.[1]

1.6 Functions defined on $\mathbb{P}_N$

If f is a complex-valued *N-periodic* function defined on $\mathbb{Z}$, then it can be analyzed using,

$$F[k] = \frac{1}{N}\sum_{0}^{N-1} f[n]e^{\frac{-2\pi ikn}{N}}, \quad k = 0, 1, \ldots, N-1 \tag{1.7}$$

F is the discrete Fourier Transform (DFT). Using this, we can construct the synthesis function f as,

$$f[n] = \sum_{k=0}^{N-1} F[k]e^{\frac{2\pi ikn}{N}}, \quad n = 0, \pm1, \pm1, \ldots \tag{1.8}$$

(1.7) gives an *N-periodic* discrete function on $\mathbb{Z}$ when k takes integer values, so, say F is a function on $\mathbb{P}_N$. (1.8) is the discrete synthesis equation of f.[1]

1.7 Notes

- F has real argument s when f is aperiodic (non-periodic) and the integer argument k when f is periodic.

  f has real argument x when F is aperiodic and the integer argument n when F is periodic.

- The argument of the exponentials that appear in the *synthesis-analysis* equations is the product of $\pm 2\pi i$, the argument s or k of F, the argument x or n of f, and the reciprocal $\frac{1}{p}$ or $\frac{1}{N}$ of the period if either f or F is periodic.

- Synthesis equation uses $+i$ exponential and all values of F to form f.

  Analysis equation uses $-i$ exponential and all values of f to form F.[1]

1.8 Parseval Identities

If f, g be functions on $\mathbb{R}$ with Fourier Transform F, G respectively, then

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx = \int_{-\infty}^{\infty} F(s)\overline{G(s)}\, ds. \tag{1.9}$$

If f, g be functions on $\mathbb{T}_p$ with Fourier Transform F, G respectively, then,

$$\int_0^p f(x)\overline{g(x)}\,dx = p\sum_{k=-\infty}^{\infty} F[k]\overline{G[k]}. \tag{1.10}$$

If f, g be functions on $\mathbb{Z}$ with Fourier Transform F, G respectively, then,

$$\sum_{n=-\infty}^{\infty} f[n]\overline{g[n]} = p\int_0^p F(s)\overline{G(s)}\,ds. \tag{1.11}$$

If f, g be functions on $\mathbb{P}_N$ with Fourier Transform F, G respectively, then,

$$\sum_{n=0}^{N-1} f[n]\overline{g[n]} = N\sum_{k=0}^{N-1} F[k]\overline{G[k]}. \tag{1.12}$$

These equations (1.9)- (1.12) are called Parseval's Identities (the bar on the function denotes the complex-conjugate of corresponding function).[1]

1.9 Example on Alternative forms of Parseval's Identity

In this section we will illustrate some concepts from section 1.8 by informally deriving alternative forms of the Parseval identities (1.9)- (1.12) using the corresponding synthesis-analysis equations and freely interchanging the limiting processes associated with integration and summation.

(a) Use (1.1)- (1.2) to show that

$$\int_{-\infty}^{\infty} f(x)g(x)\,dx = \int_{-\infty}^{\infty} F(s)G(-s)\,ds,$$

and,

$$\int_{-\infty}^{\infty} f(x)G(x)\,dx = \int_{-\infty}^{\infty} F(s)g(s)\,ds.$$

where F, G are Fourier Transform of the suitably regular functions f, g on $\mathbb{R}$.

Solution: We have,

$$\int_{-\infty}^{\infty} f(x)g(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx\int_{-\infty}^{\infty} G(s)e^{2\pi isx}\,ds$$

Letting $s = -s'$,

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)g(x)\,dx &= -\int_{-\infty}^{\infty} f(x)\,dx \int_{\infty}^{-\infty} G(-s')e^{-2\pi i s' x}\,ds' \\ &= -\int_{\infty}^{-\infty} G(-s')\,ds' \int_{-\infty}^{\infty} f(x)e^{-2\pi i s' x}\,dx \\ &= -\int_{\infty}^{-\infty} G(-s')\,ds' F(s') \\ &= \int_{-\infty}^{\infty} F(s')G(-s')\,ds'\end{aligned}$$

Letting $s' = s$,

$$\int_{-\infty}^{\infty} f(x)g(x)\,dx = \int_{-\infty}^{\infty} F(s)G(-s)\,ds.$$

Also,

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)G(x)\,dx &= \int_{-\infty}^{\infty} f(x)\,dx \int_{-\infty}^{\infty} g(s)e^{-2\pi i s x}\,ds \\ &= \int_{-\infty}^{\infty} g(s)\,ds \int_{-\infty}^{\infty} f(x)e^{-2\pi i s x}\,dx \\ &= \int_{-\infty}^{\infty} g(s)\,ds F(s) \\ &= \int_{-\infty}^{\infty} F(s)g(s)\,ds.\end{aligned}$$

(b) Use (1.7)- (1.8) to show that

$$\sum_{n=0}^{N-1} f[n]g[n] = N\sum_{k=0}^{N-1} F[k]G[-k],$$

and,

$$\sum_{n=0}^{N-1} f[n]G[n] = \sum_{k=0}^{N-1} F[k]g[k],$$

where F, G are Fourier Transform of the functions f, g on $\mathbb{P}_N$.

Solution: We have,

$$\sum_{n=0}^{N-1} f[n]g[n] = \sum_{n=0}^{N-1} f[n] \sum_{k=0}^{N-1} G[k]e^{\frac{2\pi i k n}{N}}$$

Letting $k = -k'$,

$$\begin{aligned}
\sum_{n=0}^{N-1} f[n]g[n] &= \sum_{n=0}^{N-1} f[n] \sum_{k'=0}^{-(N-1)} G[-k']e^{\frac{-2\pi i k' n}{N}}, n = 0, \pm 1, \pm 2, \dots, \pm(N-1) \\
&= -\sum_{n=0}^{N-1} f[n] \sum_{k'=0}^{(N-1)} G[-k']e^{\frac{-2\pi i k' n}{N}} \\
&= \sum_{n=0}^{-(N-1)} f[n] \sum_{k'=0}^{(N-1)} G[-k']e^{\frac{-2\pi i k' n}{N}} \\
&= \sum_{k'=0}^{N-1} G[-k'] \sum_{n=0}^{-(N-1)} f[n]e^{\frac{-2\pi i k' n}{N}}, k' = 0, -1, -2, \dots, -(N-1) \\
&= \sum_{k'=0}^{N-1} G[-k']NF[k']
\end{aligned}$$

Letting $k' = k$,

$$\sum_{n=0}^{N-1} f[n]g[n] = N \sum_{k=0}^{N-1} F[k]G[-k].$$

Also,

$$\begin{aligned}
\sum_{n=0}^{N-1} f[n]G[n] &= \sum_{n=0}^{N-1} f[n] \frac{1}{N} \sum_{k=0}^{N-1} g[k]e^{-\frac{2\pi i k n}{N}} \\
&= \sum_{k=0}^{N-1} g[k] \frac{1}{N} \sum_{n=0}^{N-1} g[k]e^{-\frac{2\pi i k n}{N}} \\
&= \sum_{k=0}^{N-1} g[k]F[k] \\
&= \sum_{k=0}^{N-1} F[k]g[k].
\end{aligned}$$

(c) Use (1.3)- (1.4) to show that

$$\int_0^p f(x)g(x)\,dx = p \sum_{k=-\infty}^{\infty} F[k]G[-k],$$

where F, G are Fourier Transform of the suitably regular functions f, g on $\mathbb{T}_p$.

Solution: We have,

$$\int_0^p f(x)g(x)\,dx = \int_0^p f(x)\,dx \sum_{k=-\infty}^{\infty} G[k]e^{\frac{2\pi ikx}{p}}$$

Letting $k = -k'$.

$$\begin{aligned}\int_0^p f(x)g(x)\,dx &= \int_0^p f(x)\,dx \sum_{k'=\infty}^{-\infty} G[-k']e^{\frac{-2\pi ik'x}{p}} \\ &= \sum_{k'=\infty}^{-\infty} G[-k']p.\frac{1}{p}\int_0^p f(x)e^{\frac{-2\pi ik'x}{p}}\,dx \\ &= \sum_{k'=\infty}^{-\infty} G[-k']pF[k']\end{aligned}$$

Letting $k' = k$,

$$\int_0^p f(x)g(x)\,dx = p\sum_{k=-\infty}^{\infty} F[k]G[-k].$$

(d) Use (1.5)- (1.6) to show that

$$\sum_{n=-\infty}^{\infty} f[n]g[n] = p\int_0^p F(s)G(-s)\,ds,$$

where F, G are Fourier Transform of the suitably regular functions f, g on $\mathbb{Z}$.

Solution: We have,

$$\sum_{n=-\infty}^{\infty} f[n]g[n] = \sum_{n=-\infty}^{\infty} f[n]\int_0^p G(s)e^{\frac{2\pi isn}{p}}\,ds$$

Letting $s = -s'$,

$$\begin{aligned}\sum_{n=-\infty}^{\infty} f[n]g[n] &= \sum_{n=-\infty}^{\infty} f[n] - \int_0^{-p} G(-s')e^{\frac{-2\pi is'n}{p}}\,ds' \\ &= -\int_0^{-p} G(-s')\,ds' \sum_{n=-\infty}^{\infty} f[n]e^{\frac{-2\pi is'n}{p}} \\ &= -\int_0^{-p} G(-s')\,ds'F(s')\end{aligned}$$

Letting $s' = s$,

$$\sum_{n=-\infty}^{\infty} f[n]g[n] = \int_0^p G(-s)F(s)\,ds.$$

(e) Use (1.3)- (1.4) and (1.5)- (1.6) to show that

$$\int_0^p f(x)G(x)\,dx = \sum_{k=-\infty}^{\infty} F[k]g[k],$$

where F, G are Fourier Transform of the suitably regular functions f, g on $\mathbb{T}_p$, $\mathbb{Z}$ respectively.

Solution: We have,

$$\begin{aligned}\int_0^p f(x)G(x)\,dx &= \int_0^p f(x)\,dx \frac{1}{p} \sum_{k=-\infty}^{\infty} g[k] e^{\frac{-2\pi i x k}{p}} \\ &= \sum_{k=-\infty}^{\infty} g[k] \frac{1}{p} \int_0^p f(x) e^{\frac{-2\pi i x k}{p}}\,dx \\ &= \sum_{k=-\infty}^{\infty} g[k]F[k] \\ &= \sum_{k=-\infty}^{\infty} F[k]g[k].\end{aligned}$$

1.10 Plancherel Identities

Setting $f = g$ in equations (1.9)-(1.12) gives,

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds, \tag{1.13}$$

$$\int_0^p |f(x)|^2\,dx = p \sum_{k=-\infty}^{\infty} |F[k]|^2, \tag{1.14}$$

$$\sum_{n=-\infty}^{\infty} |f[n]|^2 = p \int_0^p |F(s)|^2\,ds, \tag{1.15}$$

$$\sum_{n=0}^{N-1} |f[n]|^2 = N \sum_{k=0}^{N-1} |F[k]|^2. \tag{1.16}$$

These above equations (1.13)- (1.16) are called Plancherel's Identities.[1]

1.11 Orthogonality

p-periodic complex exponentials on $\mathbb{R}$ are orthogonal, i.e.,

$$\int_0^p \exp\left(\frac{2\pi ikx}{p}\right)\exp\left(\frac{-2\pi ilx}{p}\right)dx = \begin{cases} p & \text{if } k = l, \\ 0 & \text{otherwise.} \end{cases} \tag{1.17}$$

where $k, l = 0, \pm 1, \pm 2, \ldots$

Corresponding discrete orthogonal relations are given as:

$$\sum_{n=0}^{N-1} \exp\left(\frac{2\pi ikn}{N}\right)\exp\left(\frac{-2\pi iln}{N}\right) = \begin{cases} N & \text{if } k = l, l \pm N, l \pm 2N, \ldots \\ 0 & \text{otherwise.} \end{cases} \tag{1.18}$$

where $k, l = 0, \pm 1, \pm 2, \ldots$ [1]

1.12 Example to prove orthogonality relations

In this section, we will derive the real version of the orthogonality relation of (1.17).

Let k, l be nonnegative integers. Use suitable trigonometric identities to show that

$$\int_0^p \cos\left(\frac{2\pi kx}{p}\right)\cos\left(\frac{2\pi lx}{p}\right)dx = \begin{cases} p & \text{if } k = l = 0 \\ \frac{p}{2} & \text{if } k = l \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$\int_0^p \cos\left(\frac{2\pi kx}{p}\right)\sin\left(\frac{2\pi lx}{p}\right)dx = 0,$$

and,

$$\int_0^p \sin\left(\frac{2\pi kx}{p}\right)\sin\left(\frac{2\pi lx}{p}\right)dx = \begin{cases} \frac{p}{2} & \text{if } k = l \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Solution: We have,

$$
\begin{aligned}
\int_0^p \cos\Big(\frac{2\pi kx}{p}\Big)\cos\Big(\frac{2\pi lx}{p}\Big)\,dx &= \frac{1}{2}\int_0^p 2\cos\Big(\frac{2\pi kx}{p}\Big)\cos\Big(\frac{2\pi lx}{p}\Big)\,dx \\
&= \frac{1}{2}\int_0^p \cos\Big[\frac{2\pi}{p}(k+l)x\Big] + \cos\Big[\frac{2\pi}{p}(k-l)x\Big]\,dx \\
&= \frac{1}{2}\left[\frac{\sin\frac{2\pi}{p}(k+l)x}{\frac{2\pi}{p}(k+l)} + \frac{\sin\frac{2\pi}{p}(k-l)x}{\frac{2\pi}{p}(k-l)}\right]_0^p \\
&= \frac{p}{4\pi}\left[\frac{\sin 2\pi(k+l)}{k+l} + \frac{\sin 2\pi(k-l)}{k-l}\right] \\
&= \begin{cases} p & \text{if } k=l=0 \\ \frac{p}{2} & \text{if } k=l\neq 0 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Also,

$$
\begin{aligned}
\int_0^p \cos\Big(\frac{2\pi kx}{p}\Big)\sin\Big(\frac{2\pi lx}{p}\Big)\,dx &= \frac{1}{2}\int_0^p 2\sin\Big(\frac{2\pi lx}{p}\Big)\cos\Big(\frac{2\pi kx}{p}\Big)\,dx \\
&= \frac{1}{2}\int_0^p \sin\Big[\frac{2\pi}{p}(l+k)x\Big] + \sin\Big[\frac{2\pi}{p}(l-k)x\Big]\,dx \\
&= \frac{1}{2}\left[\frac{-\cos\frac{2\pi}{p}(k+l)x}{\frac{2\pi}{p}(k+l)} - \frac{\cos\frac{2\pi}{p}(l-k)x}{\frac{2\pi}{p}(l-k)}\right]_0^p \\
&= \frac{1}{2}\left[\frac{\cos\frac{2\pi}{p}(k-l)x}{\frac{2\pi}{p}(k-l)} - \frac{\cos\frac{2\pi}{p}(k+l)x}{\frac{2\pi}{p}(k+l)}\right]_0^p \\
&= \frac{p}{4\pi}\left[\frac{\cos 2\pi(k-l)}{k-l} - \frac{\cos 2\pi(k+l)}{k+l} - \frac{1}{k-l} + \frac{1}{k+l}\right] \\
&= \frac{p}{4\pi}\left[\frac{2\sin^2\pi(k-l)}{k-l} - \frac{2\sin^2\pi(k+l)}{k+l}\right] \\
&= 0.
\end{aligned}
$$

And,

$$\begin{aligned}
\int_0^p \sin\Big(\frac{2\pi kx}{p}\Big)\sin\Big(\frac{2\pi lx}{p}\Big)\,dx &= \frac{1}{2}\int_0^p 2\sin\Big(\frac{2\pi kx}{p}\Big)\sin\Big(\frac{2\pi lx}{p}\Big)\,dx \\
&= \frac{1}{2}\int_0^p \cos\Big[\frac{2\pi}{p}(k-l)x\Big] - \cos\Big[\frac{2\pi}{p}(k+l)x\Big]\,dx \\
&= \frac{1}{2}\left[\frac{\sin\frac{2\pi}{p}(k-l)x}{\frac{2\pi}{p}(k-l)} - \frac{\sin\frac{2\pi}{p}(k+l)x}{\frac{2\pi}{p}(k+l)}\right]_0^p \\
&= \frac{p}{2}\left[\frac{\sin 2\pi(k-l)}{2\pi(k-l)} - \frac{\sin 2\pi(k+l)}{2\pi(k+l)}\right] \\
&= \begin{cases} \frac{p}{2} & \text{if } k = l \neq 0 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

1.13 Gibbs Phenomenon

The Gibbs phenomenon, named after the American physicist *J. Willard Gibbs*, is related to the study of the Fourier series. Consider a periodic function $f(x)$ with jump discontinuity. If this function is represented using a Fourier series, then the Fourier series is observed to have oscillations near the discontinuity. Adding more terms to the series causes the oscillations to move closer to the jump. Actually the overshoot ("a consequence of trying to approximate a discontinuous function with a partial sum of continuous functions") at the jump is larger than that of the function itself. Adding more terms does not cause the overshoot to disappear. Thus, in an interval containing a point of discontinuity, the partial sums approximation of $f(x)$ never approaches $f(x)$ uniformly. This phenomenon is referred to as the *Gibbs Phenomenon.*

It was in 1848 that *Henry Wilbraham* first noticed and analyzed Gibbs phenomenon, but it went unnoticed due to limited analysis. Later, *Albert Michelson* observed the phenomenon. He developed a device in 1898 that could compute and re-synthesize the Fourier series up to $n = \pm 79$. But the problem with this was that whenever the Fourier coefficients for a square wave were input to the machine, it would produced oscillations at points of discontinuities. So he thought the overshoot was a result of some problem with the machine. In 1899, *J. Willard Gibbs* observed that the oscillations were the result of synthesizing a discontinuous function with Fourier series. Later, *Maxime Bôcher* demonstrated the phenomenon in 1906 and named it the Gibbs phenomenon.[4,5]

1.13.1 Description.

Consider a rectangular function defined as,

$$s(x) := rect\left(\frac{x}{2\alpha}\right) = \begin{cases} 1 & \text{if } |x| < \alpha \\ 0 & \text{if } |x| > \alpha. \end{cases}$$

The wave represented by this function is given by figure 1.1.

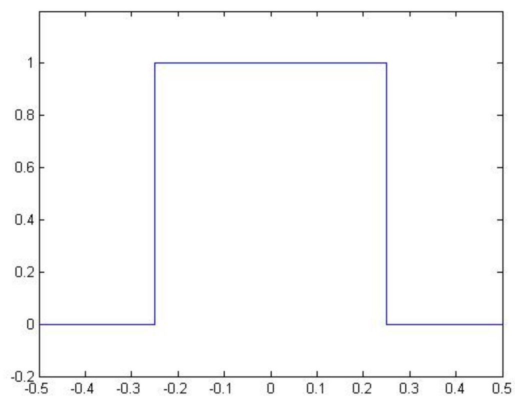

Figure 1.1: Rectangular wave s(x)

The Fourier series of this function is given as

$$S(x) = 2\alpha \sum_{k=-\infty}^{\infty} sinc(2\alpha k)e^{2\pi ikx},$$

where, the *sinc* function is given in chapter 3 by equation (3.2).

The K^{th} partial sums $S_K(x)$ is,

$$S_K(x) = 2\alpha \sum_{k=-K}^{K} sinc(2\alpha k)e^{2\pi ikx}.$$

It is clear from figures 1.2, 1.3, 1.4 and 1.5 that as the number of terms rises, the oscillations

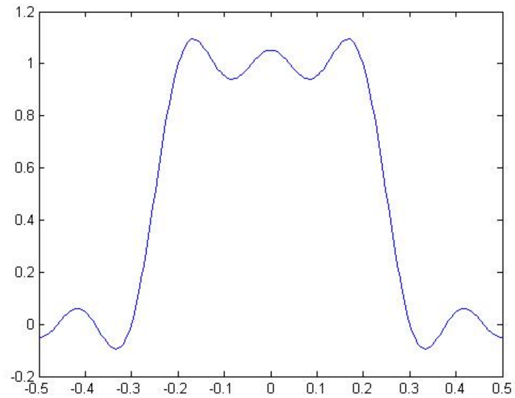

Figure 1.2: Gibbs Phenomenon for K=5

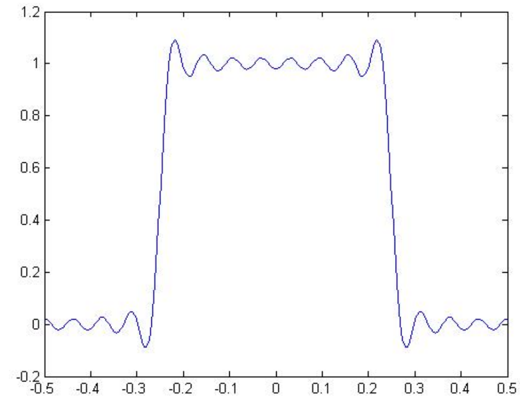

Figure 1.3: Gibbs Phenomenon for K=15

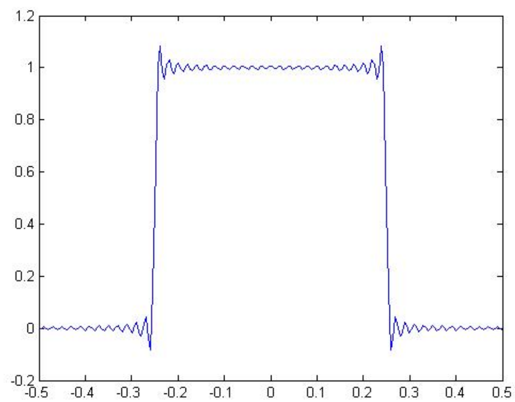

Figure 1.4: Gibbs Phenomenon for K=50

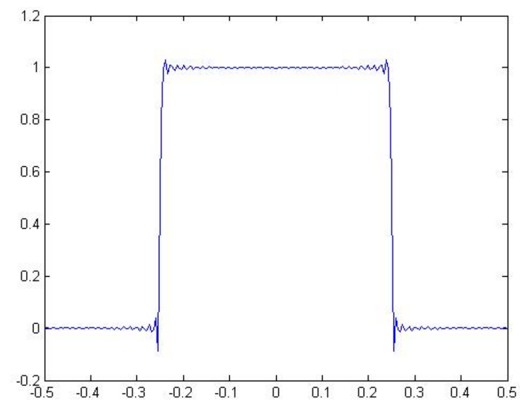

Figure 1.5: Gibbs Phenomenon for K=125

move closer and closer at the points of jump. It turns out that the Fourier series exceeds the height of the rectangular wave by

$$\frac{1}{2}\int_0^\pi \frac{\sin t}{t}\,dt - \frac{\pi}{4} = \frac{\pi}{2}(0.089490\dots),$$

or about 17.9 percent. Actually the jump in the partial series is approximately 18 percent more than that of the original function.

1.13.2 Gibbs phenomenon for w_0

Define $\xi := 2nx = \frac{x}{\frac{1}{2n}}$, so that ξ provides us with a measure of x in units of $\frac{1}{2n}$. We can then write

$$
\begin{aligned}
s_n(x) &= s_n(\frac{\xi}{2n}) \\
&= \sum_{k=1}^{n} \frac{\sin 2\pi k(\frac{\xi}{2n})}{\pi k} \\
&= \sum_{k=1}^{n} \frac{\sin \pi(\frac{k\xi}{n})}{\pi(\frac{k\xi}{n})} \frac{\xi}{n}.
\end{aligned}
$$

where $s_n(x)$ is the sequence of partial sums of Fourier series of the *1-periodic* saw-tooth function w_o given as

$$w_o := \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{2} - x & \text{if } 0 < x < 1. \end{cases} \tag{1.19}$$

that is continuously differentiable at all points of $\mathbb{T}_1$ except the origin. In terms of Fourier representation, we have

$$w_o = \sum_{k=1}^{\infty} \frac{\sin(2\pi kx)}{\pi k}. \tag{1.20}$$

and $s_n(x)$ is this sequence of partial sums of $w_o(x)$.

This $s_n(x)$ is a very good Riemann sum approximation

$$s_n(x) \simeq \zeta(\xi),$$

to the Gibbs Function,

$$\zeta(\xi) := \int_0^{\xi} \frac{\sin \pi u}{\pi u}\, du. \tag{1.21}$$

when n is large and $2nx$ is of modest size.[1]

1.13.3 Example of Gibbs Phenomenon on piece-wise smooth functions

To further illustrate the concepts of section 1.13, in this section we will study the Gibbs phenomenon associated with Fourier's representation of piecewise smooth functions on $\mathbb{R}$ with small regular tails.

(a) Show that the glitch function defined in Exercise 1.40 of Chapter 1, 'A First Course in Fourier Analysis' by David.W.Kammler has the Fourier representation

$$z(x) := \frac{1}{2}\begin{cases} -e^{x} & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ e^{-x} & \text{if } x > 0 \end{cases} = \int_{s=0}^{\infty} \frac{4\pi s \sin(2\pi s x)}{1+4\pi^2 s^2}\, ds.$$

Solution: We have,

$$Z(s) = \frac{-2\pi i s}{1+4\pi^2 s^2}.$$

Then,

$$\begin{aligned} z(x) &= \int_{-\infty}^{\infty} Z(s)e^{2\pi i s x}\, ds, \quad -\infty < x < \infty \\ &= \int_{-\infty}^{\infty} \frac{-2\pi i s}{1+4\pi^2 s^2} e^{2\pi i s x}\, ds \\ &= \int_{-\infty}^{\infty} \frac{-2\pi i s}{1+4\pi^2 s^2} (\cos 2\pi s x + i \sin 2\pi s x)\, ds \\ &= \int_{-\infty}^{\infty} \frac{-2\pi i s}{1+4\pi^2 s^2} \cos(2\pi s x)\, ds + i\int_{-\infty}^{\infty} \frac{-2\pi i s}{1+4\pi^2 s^2} \sin(2\pi s x)\, ds \\ &= 0 + 2\int_{0}^{\infty} \frac{2\pi s}{1+4\pi^2 s^2} \sin(2\pi s x)\, ds \\ &= \int_{0}^{\infty} \frac{4\pi s}{1+4\pi^2 s^2} \sin(2\pi s x)\, ds. \end{aligned}$$

(b) Let

$$z_L(x) := \int_0^{L} \frac{4\pi s \sin(2\pi s x)}{1+4\pi^2 s^2}\, ds,$$

be the approximation to z that uses only the complex exponentials having frequencies in the band $-L \le s \le L$. Show that

$$z_L(x) = z(x) + \zeta(2Lx) - \frac{1}{2}sgn(x) + \mathbf{R}_L(x),$$

where ζ is the Gibbs function (1.21) and

$$\mathbf{R}_L(x) := \int_L^{\infty} \frac{\sin(2\pi s x)}{\pi s(1+4\pi^2 s^2)}\, ds.$$

Solution: We have,

$$z_L(x) = z(x) + \zeta(2Lx) - \frac{1}{2}sgn(x) + \mathbf{R}_L(x)$$

i.e.,

$$z_L(x) - z(x) - \zeta 2Lx) + \frac{1}{2}sgn(x) - \mathbf{R}_L(x) = 0$$

Then,

$$\begin{aligned}
z_L(x) - z(x) - \zeta(2Lx) + \frac{1}{2}sgn(x) - \mathbf{R}_L(x) &= \int_0^L \frac{4\pi s}{1+4\pi^2 s^2}\sin(2\pi s x)\,ds \\
&- \int_0^\infty \frac{4\pi s}{1+4\pi^2 s^2}\sin(2\pi s x)\,ds \\
&- \int_0^{2Lx} \frac{\sin \pi s}{\pi s}\,ds \\
&+ \frac{1}{2}sgn(x) - \int_L^\infty \frac{\sin(2\pi s x)}{\pi s(1+4\pi^2 s^2)}\,ds \\
&= -\int_L^\infty \frac{4\pi s}{1+4\pi^2 s^2}\sin(2\pi s x)\,ds \\
&- \int_L^\infty \frac{\sin(2\pi s x)}{\pi s(1+4\pi^2 s^2)}\,ds \\
&- \int_0^{2Lx} \frac{\sin \pi s}{\pi s}\,ds + \frac{1}{2}sgn(x) \\
&= -\int_L^\infty \frac{\sin 2\pi s x}{\pi s}\,ds - \int_0^{2Lx} \frac{\sin \pi s}{\pi s}\,ds \\
&+ \frac{1}{2}sgn(x),
\end{aligned}$$

Letting $x = \frac{1}{2}$,

$$\begin{aligned}
z_L(x) - z(x) - \zeta(2Lx) + \frac{1}{2}sgn(x) - \mathbf{R}_L(x) &= -\int_L^\infty \frac{\sin \pi s}{\pi s}\,ds - \int_0^L \frac{\sin \pi s}{\pi s}\,ds \\
&+ \frac{1}{2}sgn(\frac{1}{2}) \\
&= -\int_0^\infty \frac{\sin \pi s}{\pi s}\,ds + \frac{1}{2}sgn(\frac{1}{2}) \\
&= -\frac{1}{2}sgn(\frac{1}{2} + \frac{1}{2}sgn(\frac{1}{2}) \\
&= 0.
\end{aligned}$$

$$\text{Therefore,} z_L(x) - z(x) - \zeta(2Lx) + \frac{1}{2}sgn(x) - \mathbf{R}_L(x) = 0.$$

(c) Let f be a piecewise smooth function with small regular tails, and let F be the Fourier transform. Describe the appearance of the approximation

$$f_L(x) := \int_{-L}^{L} F(s)e^{2\pi isx}\,dx,$$

to the function in a neighborhood of some point where f has a jump discontinuity.

Solution: We have,

$$\begin{aligned}
f_L(x) &:= \int_{-L}^{L} F(s)e^{2\pi isx}\,dx \\
&= \int_{-L}^{L} \Big(\lim_{L\to\infty+} \int_{-L}^{L} f(x)e^{-2\pi isx}\,ds \Big) e^{2\pi isx}\,dx \\
&= \lim_{L\to\infty+} \int_{-L}^{L}\int_{-L}^{L} f(x)\,ds\,dx \\
&= \lim_{L\to\infty+} \int_{-L}^{L} f(x)\,dx \int_{-L}^{L} ds \\
&= \lim_{L\to\infty+} \int_{-L}^{L} 2Lf(x)\,dx.
\end{aligned}$$

Chapter 2

Convolution of Functions

In mathematics and in particular, functional analysis, convolution is a law of composition that combines two functions f and g to yield a third.[2]

Convolution is a mathematical tool with applications including statistics, computer vision, image and signal processing, and differential equations.

The concept of convolution is inherent in almost every field of the physical sciences and engineering. For instance, in mechanics, it is known as the superposition or Duchamel integral. In system theory, it plays a crucial role as the impulse response integral, and in optics as the point spread or smearing function.[3]

2.1 Definition of Convolution and Correlation

The convolution of two functions f and g, denoted $f*g$, is mathematically defined as follows:

$$(f * g)(t) = \int_{\mathbb{R}} f(x)g(t-x)\,dx. \tag{2.1}$$

Rather than a simple point to point multiplication, the convolution product is carried out as per the operation given in equation (2.1). Consider two functions x and h. From equation (2.1), we see that the convolution of x, h is given as the integral of the product of two functions $x(\tau)$ and $h(t-\tau)$. Figure 2.1 shows the function $x(t)$ and figure 2.2 shows the function $h(t)$. In figure 2.4, we first reflect $h(\tau)$ about origin to obtain $h(-\tau)$ and then shift it to the right by an amount $t = 2.5$s to obtain $h(t-\tau)$. Figure 2.3 shows the function

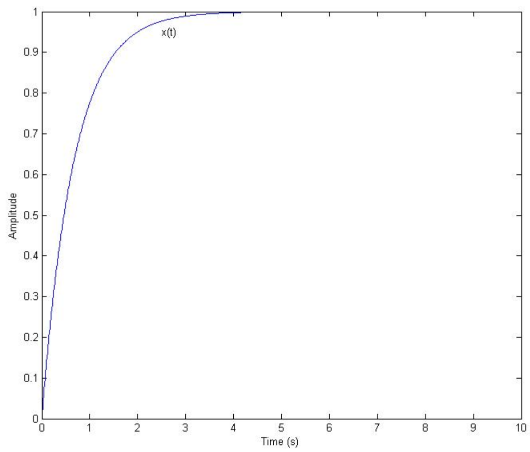

Figure 2.1: Signal x(t)

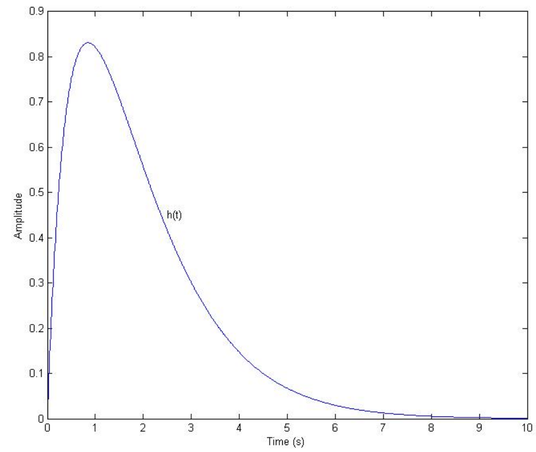

Figure 2.2: Signal h(t)

x along with the function h and figure 2.5 shows x with the reflected and shifted function $h(t-\tau)$. The product y of these two functions $x(\tau)$ and $h(t-\tau)$ at $t = 2.5$s is shown as the curve in the Figure 2.6. The convolution of x and h at t is the area under this product curve.

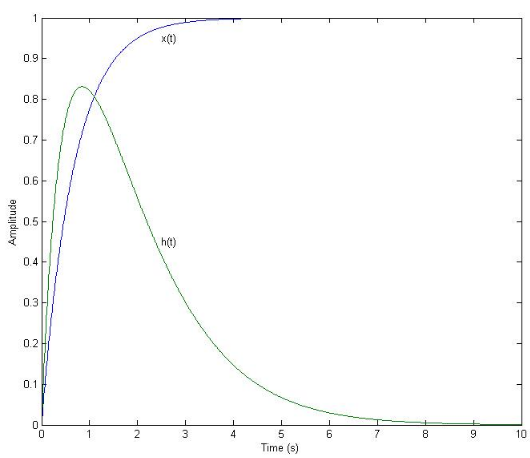

Figure 2.3: Signals x(t) and h(t)

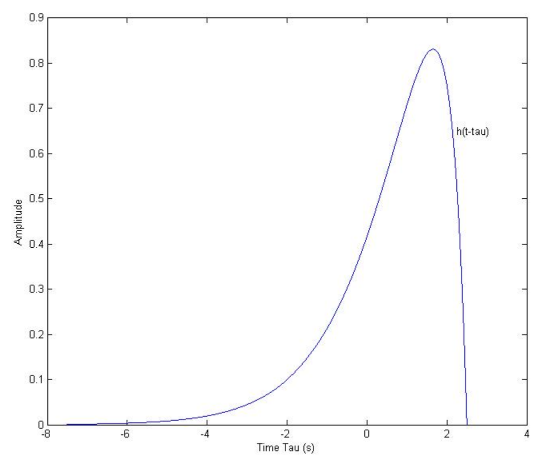

Figure 2.4: Shifted signal h(t-τ) at t=2.5s

The cross-correlation of two functions f and g, denoted as $(f \star g)(t)$, is mathematically

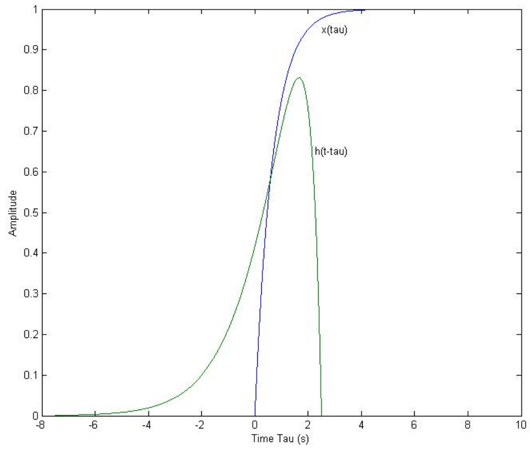

Figure 2.5: Signals x(τ) and h(t-τ)

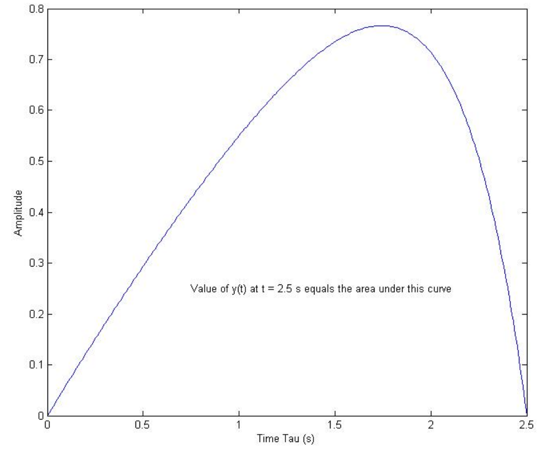

Figure 2.6: Product of signals x(τ) and h(t-τ) at t=2.5s

defined as,

$$(f \star g)(t) = \int_{\mathbb{R}} f(x)\overline{g(t+x)}\, dx. \tag{2.2}$$

where $\overline{g(t)}$ denotes the complex-conjugate of the function $g(t)$. When both the functions are real, then $\overline{g(t)} = g(t)$ and the cross-correlation becomes

$$(f \star g)(t) = \int_{\mathbb{R}} f(x)g(t+x)\, dx. \tag{2.3}$$

When we form the cross-correlation of two real functions, we simply displace the second function by an amount of t. However, if the functions are complex, then we take the complex-conjugate of the second function then displace it by t. When a function $f(t)$ is cross-correlated with itself, the result is known as the *auto-correlation product* and the resulting function is known as the *auto-correlation function*.[2]

2.1.1 Convolution of Functions on $\mathbb{R}$,$\mathbb{T}_p$,$\mathbb{Z}$ and $\mathbb{P}_N$

If f, g are two suitably regular functions, then the convolution product $(f * g)$ is,

$$(f * g)(x) = \int_{-\infty}^{\infty} f(u)g(x-u)\, du \quad \text{for } f, g, f*g \text{ functions on } \mathbb{R}. \tag{2.4}$$

$$(f * g)(x) = \int_{0}^{p} f(u)g(x-u)\, du \quad \text{for } f, g, f*g \text{ functions on } \mathbb{T}_p. \tag{2.5}$$

$$(f * g)[n] = \sum_{m=-\infty}^{\infty} f[m]g[n-m] \quad \text{for } f,g,f*g \text{ functions on } \mathbb{Z}. \tag{2.6}$$

$$(f * g)[n] = \sum_{m=o}^{N-1} f[m]g[n-m] \quad \text{for } f,g,f*g \text{ functions on } \mathbb{P}_N. \tag{2.7}$$ [1]

2.1.2 Example of computing convolution

Let $f(x) := e^{-\pi x^2}, \quad -\infty < x < \infty$.

(a) Verify that $f(u)f(x-u) = e^{-2\pi(u-\frac{x}{2})^2} e^{-\pi\frac{x^2}{2}}$.

Solution: We have,

$$\begin{aligned}
f(u)f(x-u) &= e^{-\pi u^2} e^{-\pi(x-u)^2} \\
&= e^{-\pi u^2} e^{-\pi(x^2-2xu+u^2)} \\
&= e^{-\pi u^2 - \pi x^2 + \pi 2xu - \pi u^2} \\
&= e^{-2\pi u^2 - \pi x^2 + \pi 2xu} \\
&= e^{-2\pi u^2 + \pi 2xu - \frac{\pi x^2}{2} - \frac{\pi x^2}{2}} \\
&= e^{-2\pi u^2 + \pi 2xu - \frac{\pi x^2}{2}} e^{-\frac{\pi x^2}{2}} \\
&= e^{-2\pi(u^2 - xu + \frac{x^2}{4})} e^{-\frac{\pi x^2}{2}} \\
&= e^{-2\pi(u^2 - 2u\frac{x}{2} + \frac{x^2}{4})} e^{-\frac{\pi x^2}{2}} \\
&= e^{-2\pi(u-\frac{x}{2})^2} e^{-\frac{\pi x^2}{2}}.
\end{aligned}$$

(b) Using (a), show that $(f*f)(x) = Ie^{-\pi\frac{x^2}{2}}$ where $I := \int_{-\infty}^{\infty} e^{-2\pi y^2}\, dy$.

Solution: From equation (2.4), we have,

$$\begin{aligned}
(f*f)(x) &= \int_{-\infty}^{\infty} f(u)f(x-u)\, du \\
&= \int_{-\infty}^{\infty} e^{-2\pi(u-\frac{x}{2})^2} e^{-\frac{\pi x^2}{2}}\, du
\end{aligned}$$

Let $u - \frac{x}{2} = y$ such that $du = dx$. Then

$$\begin{aligned}(f * f)(x) &= \int_{-\infty}^{\infty} e^{-2\pi y^2} e^{-\frac{\pi x^2}{2}}\, dy \\ &= e^{-\frac{\pi x^2}{2}} \int_{-\infty}^{\infty} e^{-2\pi y^2}\, dy \\ &= I e^{-\frac{\pi x^2}{2}}.\end{aligned}$$

2.1.3 Correlation of Functions on $\mathbb{R}$,$\mathbb{T}_p$,$\mathbb{Z}$ and $\mathbb{P}_N$

If f, g are two suitably regular functions, then the correlation product $(f \star g)$ is,

$$(f \star g)(x) = \int_{-\infty}^{\infty} \overline{f(u)} g(u+x)\, du \quad \text{for } f, g, f \star g \text{ functions on } \mathbb{R}. \tag{2.8}$$

$$(f \star g)(x) = \int_{0}^{p} \overline{f(u)} g(u+x)\, du \quad \text{for } f, g, f \star g \text{ functions on } \mathbb{T}_p. \tag{2.9}$$

$$(f \star g)[n] = \sum_{m=-\infty}^{\infty} \overline{f[m]} g[m+n] \quad \text{for } f, g, f \star g \text{ functions on } \mathbb{Z}. \tag{2.10}$$

$$(f \star g)[n] = \sum_{m=o}^{N-1} \overline{f[m]} g[m+n] \quad \text{for } f, g, f \star g \text{ functions on } \mathbb{P}_N.\ ^{1} \tag{2.11}$$

2.2 Mathematical Properties of Convolution

2.2.1 The Fourier Transform of $f * g$

If f, g be suitably regular functions on $\mathbb{R}$ and $q := f * g$, then using equation (1.1),

$$\begin{aligned}
Q(s) &= \int_{-\infty}^{\infty} q(x)e^{-2\pi isx}\,dx \\
&= \int_{-\infty}^{\infty} (f * g)(x)e^{-2\pi isx}\,dx \\
&= \int_{x=-\infty}^{\infty} \int_{u=-\infty}^{\infty} f(u)g(x-u)e^{-2\pi isx}\,du\,dx \\
&= \int_{u=-\infty}^{\infty} \int_{x=-\infty}^{\infty} f(u)g(x-u)e^{-2\pi isx}\,dx\,du \\
&= \int_{u=-\infty}^{\infty} f(u)e^{-2\pi isu} \int_{x=-\infty}^{\infty} g(x-u)e^{-2\pi is(x-u)}\,dx\,du \\
&= \int_{-\infty}^{\infty} f(u)e^{-2\pi isu}G(s)\,du \\
&= F(s)G(s).
\end{aligned}$$

Thus, we find that

$$\begin{aligned}
q(x) = (f * g)(x) \text{ on } \mathbb{R} \quad &\text{has FT} \quad Q(s) = F(s)G(s) \text{ on } \mathbb{R}, \\
q(x) = (f * g)(x) \text{ on } \mathbb{T}_p \quad &\text{has FT} \quad Q[k] = pF[k]G[k] \text{ on } \mathbb{Z}, \\
q[n] = (f * g)[n] \text{ on } \mathbb{Z} \quad &\text{has FT} \quad Q(s) = pF(s)G(s) \text{ on } \mathbb{T}_p, \\
q[n] = (f * g)[n] \text{ on } \mathbb{P}_N \quad &\text{has FT} \quad Q[k] = NF[k]G[k] \text{ on } \mathbb{P}_N.
\end{aligned}$$ [1]

2.2.2 Algebraic Structure

Consider suitably regular functions f, g and h. Let α and β be scalars. Some of the other familiar properties satisfied by the convolution product (2.4)- (2.7) is:

(a) Homogenous:

$$(\alpha f) * g = \alpha(f * g) = f * (\alpha g). \tag{2.12}$$

(b) Distributive:

$$f * (g + h) = (f * g) + (f * h), \tag{2.13}$$

$$(f + g) * h = (f * g) + (f * h). \tag{2.14}$$

(c) Linearity:

$$(\sum_{m=1}^{M} \alpha_m f_m) * (\sum_{n=1}^{N} \beta_n g_n) = \sum_{m=1}^{M} \sum_{n=1}^{N} \alpha_m \beta_n (f_m * g_n), \tag{2.15}$$

where f_1, ..., f_M, g_1, ..., g_N are suitably regular functions and α_1, ..., α_M, β_1, ..., β_N are scalars.

(d) Commutative:

$$f * g = g * f. \tag{2.16}$$

In order to show that f,g are commutative when they are suitably regular functions on $\mathbb{P}_N$, we need to prove the following two claims:

Let g be N-periodic on $\mathbb{Z}$ i.e.,

$$g[n] = g[n + pN] \quad \text{for all } n, p\epsilon\mathbb{Z}, \tag{2.17}$$

Claim1: Given $k\epsilon\mathbb{Z}$, we have

$$\sum_{m=0}^{N-1} g[m] = \sum_{m=k}^{k+N-1} g[m]. \tag{2.18}$$

In particular, since,

$$\sum_{m=0}^{N-1} g[m-k] = \sum_{m=k}^{k+N-1} g[m] \quad \text{(by change of variables)},$$

we obtain,

$$\sum_{m=0}^{N-1} g[m] = \sum_{m=0}^{N-1} g[m-k] \quad \text{for all k}\epsilon\mathbb{Z}. \tag{2.19}$$

Proof of Claim1: Given $k\epsilon\mathbb{Z}$, there exist $q\epsilon\mathbb{Z}$ and $r\epsilon\mathbb{Z}$ with $0 \leq r < N$ such that,

$$k = qN + r \qquad \text{(division algorithm)}$$

If $r = 0$ then equation (2.19) follows from equation (2.17) since $r = 0$ implies $k = qN$.

Suppose $r \geq 1$. Then,

$$\sum_{m=k}^{k+N-1} g[m] = \sum_{m=qN+r}^{qN+r+N-1} g[m] = \sum_{m=qN+r}^{qN-N-1} g[m] + \sum_{m=qN+N}^{qN+r+N-1} g[m].$$

But then,

$$\sum_{m=qN+r}^{qN-N-1} g[m] = \sum_{m'=r}^{N-1} g[m' + qN] = \sum_{m'=r}^{N-1} g[m'] \quad (\text{letting } m' = m - qN).$$

And,

$$\sum_{m=qN+N}^{qN+N+r-1} g[m] = \sum_{m'=0}^{r-1} g[m'+(q+1)N] = \sum_{m'=0}^{r-1} g[m'] \quad (\text{letting } m' = m - qN - N).$$

Then,

$$\begin{aligned}\sum_{m=k}^{k+N-1} g[m] &= \sum_{m'=r}^{N-1} g[m'] + \sum_{m'=0}^{r-1} g[m'] \\ &= \sum_{m'=0}^{N-1} g[m'] \\ &= \sum_{m=0}^{N-1} g[m].\end{aligned}$$

Hence equation (2.18) is obtained as desired.

Claim2: We have,

$$\sum_{m=0}^{N-1} g[m] = \sum_{m=0}^{N-1} g[-m], \tag{2.20}$$

for all N-periodic function g on $\mathbb{Z}$.

Proof of Claim2:

$$\begin{aligned}\sum_{m=0}^{N-1} g[-m] &= \sum_{m=0}^{N-1} g[-m + N - 1] \quad (\text{by } \textit{equation } (2.19) \text{ with } k = N - 1) \\ &= \sum_{m'=0}^{N-1} g[m'] \quad (\text{with } m' = N - 1 - m).\end{aligned}$$

(e) Associativity:

$$f * (g * h) = (f * g) * h. \tag{2.21}$$

(f) Identities:

$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n = \pm 1, \pm 2, \ldots \end{cases} \qquad \text{and} \qquad \delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n = 1, 2, \ldots, N-1. \end{cases} \tag{2.22}$$

These functions serve as identities for the convolution product of functions on $\mathbb{Z}$, $\mathbb{P}_N$, i.e.,

$$\delta * f = f * \delta = f.$$

but we don't have any functions on $\mathbb{R}$ or $\mathbb{T}_p$ that serves as identity.

2.2.3 Showing that $f * g$ is Commutative

In this section we will show that the convolution product is commutative, i.e., $f_1 * f_2 = f_2 * f_1$ when

(a) f_1, f_2 are suitably regular functions on $\mathbb{R}$.

Solution: We have,

$$(f_1 * f_2)(x) = \int_{-\infty}^{\infty} f_1(u) f_2(x-u)\, du$$

Let $u^{'} = x - u$ such that $du^{'} = -du$. Then,

$$\begin{aligned} (f_1 * f_2)(x) &= -\int_{\infty}^{-\infty} f_1(x-u') f_2(u')\, du' \\ &= \int_{-\infty}^{\infty} f_2(u') f_1(x-u')\, du' \\ &= (f_2 * f_1)(x). \end{aligned}$$

(b) f_1, f_2 are suitably regular functions on $\mathbb{T}_p$.

Solution: We have,

$$(f_1 * f_2)(x) = \int_{0}^{p} f_1(u) f_2(x-u)\, du$$

Let $u' = x - u$ such that $du' = -du$. Then,

$$\begin{aligned}(f_1 * f_2)(x) &= -\int_x^{x-p} f_1(x-u')f_2(u')\,du' \\ &= -\int_0^{-p} f_1(x-u')f_2(u')\,du' \\ &= \int_0^{p} f_2(u')f_1(x-u')\,du' \\ &= (f_2 * f_1)(x).\end{aligned}$$

(c) f_1, f_2 are suitably regular functions on $\mathbb{Z}$.

Solution: We have,

$$(f_1 * f_2)[n] = \sum_{m=-\infty}^{\infty} f_1[m]f_2[n-m]$$

Let $m' = n - m$ such that $m = n - m'$. Then,

$$\begin{aligned}(f_1 * f_2)[n] &= \sum_{m'=\infty}^{-\infty} f_1[n-m']f_2[m'] \\ &= \sum_{m'=-\infty}^{\infty} f_2[m']f_1[n-m'] \\ &= (f_2 * f_1)[n].\end{aligned}$$

(d) f_1, f_2 are suitably regular functions on $\mathbb{P}_N$.

Solution: We have,

$$\begin{aligned}(f_1 * f_2)[n] &= \sum_{m=0}^{N-1} f_1[m]f_2[n-m] \\ &= \sum_{m=0}^{N-1} f_1[-m]f_2[m-n] \qquad \text{(by (4))} \\ &= \sum_{m=0}^{N-1} f_1[-m+n]f_2[m] \qquad \text{(by (3) with } k=-n\text{)} \\ &= (f_2 * f_1)[n].\end{aligned}$$

2.2.4 Showing that $f * g$ is Associative

In this section we will show that the convolution product is *associative*,

i.e., $f_1 * (f_2 * f_3) = (f_1 * f_2) * f_3$ when

(a) f_1, f_2, f_3 are suitably regular functions on $\mathbb{R}$.

Solution: We have,

$$\begin{aligned}
(f_1 * f_2) * f_3(x) &= \big[\int_{-\infty}^{\infty} f_1(u) f_2(x-u)\, du\big] * f_3(x) \\
&= \int_{-\infty}^{\infty} \big[\int_{-\infty}^{\infty} f_1(u) f_2(v-u)\, du\big] f_3(x-v)\, dv \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(u) f_2(v-u) f_3(x-v)\, dv\, du
\end{aligned}$$

Let $y = v - u$ such that $dy = dv$. Then,

$$\begin{aligned}
(f_1 * f_2) * f_3(x) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(u) f_2(y) f_3(x-y-u)\, dy\, du \\
&= \int_{-\infty}^{\infty} f_1(u) \big[\int_{-\infty}^{\infty} f_2(y) f_3(x-u-y)\, dy\big]\, du \\
&= \int_{-\infty}^{\infty} f_1(u) [f_2 * f_3](x-u)\, du \\
&= f_1 * (f_2 * f_3)(x).
\end{aligned}$$

(b) f_1, f_2, f_3 are suitably regular functions on $\mathbb{T}_p$.

Solution: We have,

$$\begin{aligned}
(f_1 * f_2) * f_3(x) &= \big[\int_0^p f_1(u) f_2(x-u)\, du\big] * f_3(x) \\
&= \int_0^p \big[\int_0^p f_1(u) f_2(v-u)\, du\big] f_3(x-v)\, dv \\
&= \int_0^p \int_0^p f_1(u) f_2(v-u) f_3(x-v)\, dv\, du
\end{aligned}$$

Let $y = v - u$ such that $dy = dv$. Then,

$$\begin{aligned}(f_1 * f_2) * f_3(x) &= \int_0^p \int_0^p f_1(u) f_2(y) f_3(x - y - u)\, dy\, du \\ &= \int_0^p f_1(u) \Big[\int_0^p f_2(y) f_3(x - u - y)\, dy\Big]\, du \\ &= \int_0^p f_1(u) [f_2 * f_3](x - u)\, du \\ &= f_1 * (f_2 * f_3)(x).\end{aligned}$$

(c) f_1, f_2, f_3 are suitably regular functions on $\mathbb{Z}$.

Solution: We have,

$$\begin{aligned}(f_1 * f_2) * f_3[n] &= \Big[\sum_{k=-\infty}^{\infty} f_1[k] f_2[n-k]\Big] * f_3[n] \\ &= \sum_{q=-\infty}^{\infty} \Big[\sum_{k=-\infty}^{\infty} f_1[k] f_2[q-k]\Big] f_3[n-q] \\ &= \sum_{k=-\infty}^{\infty} f_1[k] \sum_{q=-\infty}^{\infty} f_2[q-k] f_3[n-q]\end{aligned}$$

Let $m = n - q$. Then,

$$\begin{aligned}(f_1 * f_2) * f_3[n] &= \sum_{k=-\infty}^{\infty} f_1[k] \sum_{m=-\infty}^{\infty} f_2[n-m-k] f_3[m] \\ &= \sum_{k=-\infty}^{\infty} f_1[k] \sum_{m=-\infty}^{\infty} f_2[(n-k)-m] f_3[m] \\ &= \sum_{k=-\infty}^{\infty} f_1[k] (f_2 * f_3)[n-k] \\ &= f_1 * (f_2 * f_3)[n].\end{aligned}$$

(d) f_1, f_2, f_3 are suitably regular functions on $\mathbb{P}_N$.

Solution: We have,

$$
\begin{aligned}
(f_1 * f_2) * f_3[n] &= \Big[\sum_{k=0}^{N-1} f_1[k] f_2[n-k]\Big] * f_3[n] \\
&= \sum_{q=0}^{N-1}\sum_{k=0}^{N-1} f_1[k] f_2[q-k] f_3[n-q] \\
&= \sum_{k=0}^{N-1} f_1[k] \sum_{q=0}^{N-1} f_2[q-k] f_3[n-q]
\end{aligned}
$$

Let $m = q - k$. Then,

$$
\begin{aligned}
(f_1 * f_2) * f_3[n] &= \sum_{k=0}^{N-1} f_1[k] \sum_{m=-k}^{-k+(N-1)} f_2[m] f_3[n-m-k] \\
&= \sum_{k=0}^{N-1} f_1[k] \sum_{m=0}^{(N-1)} f_2[m] f_3[(n-k)-m] \\
&= \sum_{k=0}^{N-1} f_1[k] (f_2 * f_3)[n-k] \\
&= f_1 * (f_2 * f_3)[n].
\end{aligned}
$$

2.2.5 Translation Invariance

Consider two suitably regular functions f, g on $\mathbb{R}$ and let $-\infty < a < \infty$. Then function $g(x+a)$ is the translation of $g(x)$ by a, and

$$
\begin{aligned}
f(x) * g(x+a) &= \int_{-\infty}^{\infty} f(u) g((x-u)+a)\, du \\
&= \int_{-\infty}^{\infty} f(u) g((x+a)-u)\, du \\
&= (f * g)(x+a).
\end{aligned}
$$

Hence, the convolution product is translation invariant. It is also the case for suitably regular functions on $\mathbb{T}_p$, $\mathbb{Z}$ or $\mathbb{P}_{\mathbb{N}}$.[1]

2.2.6 Differentiation of $f * g$

If f, g be suitably regular functions on $\mathbb{R}$, we can write

$$\begin{aligned}(f * g)'(x) &= \frac{d}{dx}\int_{-\infty}^{\infty} f(u)g(x-u)\,du \\ &= \int_{-\infty}^{\infty} f(u)g'(x-u)\,du \\ &= (f * g')(x).\end{aligned}$$

Since, $f * g = g * f$, it follows

$$\begin{aligned}(f * g)' &= f' * g = f * g' \\ (f * g)'' &= f'' * g = f' * g' = f * g'' \\ &\vdots \\ (f * g)^{(n)} &= f^{(m)} * g^{(n-m)}, \quad m = 0, 1, \ldots, n.\end{aligned}$$

when f has (m) derivatives and g has $(n-m)$ derivatives for some $m = 0, 1, 2, \ldots, n$.[1]

2.2.7 Example on Differentiation

In this section, we will show how to use the differentiation rule to find convolution products of functions.

Let $f_n(x) := x^n e^{-\pi x^2}, n = 0, 1, 2$. Consider the convolution product $f_0 * f_0$ from Exercise 2.6 of Chapter 2, 'A First Course in Fourier Analysis' by David.W.Kammler.

(a) Use the differentiation rule to find $f_0 * f_1$.

Solution: Using the hint given, we have

$$\begin{aligned}
f_0 * f_1 &= f_0 * \left(-\frac{1}{2\pi}\right) f_0' \\
&= \left(-\frac{1}{2\pi}\right) f_0 * f_0' \\
&= \left(-\frac{1}{2\pi}\right) (f_0 * f_0)' \\
&= \left(-\frac{1}{2\pi}\right) \left(\frac{1}{\sqrt{2}} e^{-\frac{\pi x^2}{2}}\right)' \\
&= \frac{1}{2\sqrt{2}} (x e^{-\frac{\pi x^2}{2}}) \\
&= \frac{x}{2} \left(\frac{1}{\sqrt{2}} e^{-\frac{\pi x^2}{2}}\right) \\
&= \frac{x}{2} (f_0 * f_0).
\end{aligned}$$

(b) Use the differentiation rule to find $f_1 * f_1$.

Solution: We have,

$$\begin{aligned}
f_1 * f_1 &= \left(-\frac{1}{2\pi}\right) f_0' * \left(-\frac{1}{2\pi}\right) f_0' \\
&= \left(\frac{1}{4\pi^2}\right) f_0' * f_0' \\
&= \frac{1}{4\pi^2} (f_0 * f_0)^2 \\
&= \frac{1}{4\pi^2} \left(-\frac{\pi x}{\sqrt{2}} e^{-\frac{\pi x^2}{2}}\right)' \\
&= -\frac{1}{4\sqrt{2}\pi} (x e^{-\frac{\pi x^2}{2}})' \\
&= -\frac{1}{4\sqrt{2}\pi} (-\pi x^2 e^{-\frac{\pi x^2}{2}} + e^{-\frac{\pi x^2}{2}}) \\
&= \frac{1}{4\sqrt{2}\pi} (\pi x^2 - 1) e^{-\frac{\pi x^2}{2}}.
\end{aligned}$$

(c) Use the differentiation rule to find $f_0 * f_2$.

Solution: We have,

$$\begin{aligned}
f_0 * f_2 &= f_0 * \frac{1}{2\pi}(f_0 - f_1') \\
&= \frac{1}{2\pi}\left[f_0 * (f_0 - f_1')\right] \\
&= \frac{1}{2\pi}[f_0 * f_0 - f_0 * f_1'] \\
&= \frac{1}{2\pi}[f_0 * f_0] - \frac{1}{2\pi}[f_0 * f_1'] \\
&= \frac{1}{2\pi}[f_0 * f_0]' - \frac{1}{2\pi}[f_0 * f_1]' \\
&= \frac{1}{2\sqrt{2}\pi}[xe^{-\frac{\pi x^2}{2}}] + \frac{1}{4\sqrt{2}\pi}[(\pi x^2 - 1)e^{-\frac{\pi x^2}{2}}] \\
&= \frac{1}{2\sqrt{2}\pi}[xe^{-\frac{\pi x^2}{2}}] + \frac{1}{4\sqrt{2}}[x^2 e^{-\frac{\pi x^2}{2}}] - \frac{1}{4\sqrt{2}\pi}[e^{-\frac{\pi x^2}{2}}].
\end{aligned}$$

Chapter 3

FT Calculus for Functions on $\mathbb{R}$

3.1 Definition of functions

3.1.1 The Box Function

The *box function* is defined as,

$$\prod(x) := \begin{cases} 1 & \text{if } -\frac{1}{2} < x < \frac{1}{2} \\ 0 & \text{if } x < -\frac{1}{2} \text{ or } x > \frac{1}{2}. \end{cases} \tag{3.1}$$

and the *cardinal sinc* (sinc function) is defined as,

$$sinc(s) := \frac{\sin(\pi s)}{\pi s}, \quad s \neq 0, \tag{3.2}$$

Since, $f := \prod$ is even, we get,

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} \prod(x) e^{-2\pi i s x}\, dx \\ &= \int_{-\infty}^{\infty} \prod(x) \cos(2\pi s x)\, dx \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(2\pi s x)\, dx \\ &= \left[\frac{\sin(2\pi s x)}{2\pi s}\right]_{-\frac{1}{2}}^{\frac{1}{2}} \\ &= \frac{\sin \pi s}{\pi s} \\ &= sinc(s). \end{aligned}$$

Thus, the box function $f(x) = \prod(x)$ has FT $F(s) = sinc(s)$.[1]

3.1.2 The Heaviside Step Function

The *Heaviside step* function is defined as,

$$h(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0. \end{cases} \tag{3.3}$$

This h is used to create functions that vanish on a half line.[1]

3.1.3 Example on Heaviside Step Function

Let $u(x) := e^{-\alpha x}h(x)$ where $a > 0$ and $h(x)$ is the Heaviside step, and let $u_1 := u, u_2 := u * u, u_3 := u * u * u, \ldots$.

(a) Find the Fourier Transform of u_1 and then use the convolution rule to deduce that u_{n+1} has the Fourier Transform $U_{n+1}(s) = (\alpha + 2\pi is)^{-n-1}$.

Solution: We have,

$u(x) := e^{-\alpha x}h(x)$ and $h(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$

Then, $u_1(x) = u(x) = \begin{cases} e^{-\alpha x} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$

Now,

$$\begin{aligned} U_1(s) &= \int_{-\infty}^{\infty} e^{-\alpha x}e^{-2\pi isx}\,dx \\ &= \int_{0}^{\infty} e^{-\alpha x}e^{-2\pi isx}\,dx \\ &= \int_{0}^{\infty} e^{-(\alpha+2\pi is)x}\,dx \\ &= \left[-\frac{e^{-(\alpha+2\pi is)x}}{\alpha + 2\pi is}\right]_0^{\infty} \\ &= \frac{1}{\alpha + 2\pi is}. \end{aligned}$$

Then, $u_{n+1}(x) = u * u * u * \ldots * u(x)$.

And,

$$\begin{aligned}
U_{n+1}(s) &= U(s).U(s).\dots.U(s) \\
&= \big[U(s)\big]^{n+1} \\
&= \frac{1}{(\alpha+2\pi is)^{n+1}} \\
&= (\alpha+2\pi is)^{-n-1}.
\end{aligned}$$

(b) Use the power scaling rule and the fact that

$$U_{n+1}(s) = \frac{(-2\pi i)^{-n}}{n!}\frac{d^n}{ds^n}(\alpha+2\pi is)^{-1},$$

to deduce that $u^{n+1}(x) = x^n e^{-\alpha x \frac{h(x)}{n!}}, \qquad n = 0, 1, \dots.$

Solution: We have,

$$U_{n+1}(s) = \frac{(-2\pi i)^{-n}}{n!}\frac{d^n}{ds^n}(\alpha+2\pi is)^{-1},$$

Also,

$$g(x) := xf(x) \text{ has the FT } G(s) = (-2\pi i)^{-1}F'(s),$$

But,

$$F'(s) = \frac{d}{ds}\int_{-\infty}^{\infty} f(x)e^{-2\pi isx}\,dx.$$

From (a),

$$(\alpha+2\pi is)^{-1} = \int_0^{\infty} u(x)e^{-2\pi isx}\,dx,$$

Thus,

$$U^n(s) = \frac{d^n}{ds^n}(\alpha+2\pi is)^{-1} = \frac{d^n}{ds^n}\int_0^{\infty} u(x)e^{-2\pi isx}\,dx = \frac{d^n}{ds^n}\int_{-\infty}^{\infty} u(x)e^{-2\pi isx}\,dx.$$

Hence,

$$U_{n+1}(s) = \frac{(-2\pi i)^{-n}}{n!}U^n(s).$$

This implies

$$u_{n+1}(x) = \frac{x^n u(x)}{n!} = \frac{x^n e^{-\alpha x}h(x)}{n!}, \quad n = 0, 1, \dots$$

3.1.4 The Truncated Decaying Exponential

The *truncated decaying exponential* function is defined as,

$$f(x) := e^{-x}h(x), \tag{3.4}$$

where h is the Heaviside function given by (3.1.2).

Then, its Fourier Transform is,

$$\begin{aligned}
F(s) &= \int_{-\infty}^{\infty} e^{-x}e^{-2\pi isx}\,dx \\
&= \lim_{L\to+\infty}\int_0^L \frac{d}{dx}\left[\frac{-e^{-(1+2\pi is)x}}{1+2\pi is}\right]dx \\
&= \lim_{L\to+\infty}\frac{1-e^{-(1+2\pi is)L}}{1+2\pi is} \\
&= \frac{1}{1+2\pi is} \\
&= \frac{1}{1+4\pi^2s^2} - \frac{2\pi is}{1+4\pi^2s^2}.
\end{aligned}$$

So, the truncated decaying exponential $f(x) := e^{-x}h(x)$ has FT $F(s) = \frac{1}{1+2\pi is}$.[1]

3.1.5 The Unit Gaussian

The *Unit Gaussian* function is defined as,

$$f(x) := e^{-\pi x^2}. \tag{3.5}$$

Since f is even,

$$F(s) = \int_{-\infty}^{\infty} e^{-\pi x^2}e^{-2\pi isx}\,dx = \int_{-\infty}^{\infty} e^{-\pi x^2}\cos(-2\pi sx)\,dx,$$

Since the integrand and its derivative with respect to s rapidly approaches 0 as $x \to \pm\infty$, we can write

$$F'(s) = \int_{-\infty}^{\infty} e^{-\pi x^2}\frac{\partial}{\partial s}\big(\cos(2\pi sx)\big)\,dx,$$

Hence,

$$\begin{aligned} F'(s) + 2\pi s F(s) &= \int_{-\infty}^{\infty} e^{-\pi x^2} (-2\pi x) \sin(2\pi s x) + (2\pi s) \cos(2\pi s x)\, dx \\ &= \int_{-\infty}^{\infty} \frac{d}{dx} \left[e^{-\pi x^2} \sin(2\pi s x) \right] dx \\ &= 0. \end{aligned}$$

It follows that,

$$\frac{d}{ds}\left[e^{\pi s^2} F(s)\right] = e^{\pi s^2}\left[F'(s) + 2\pi s F(s)\right] = 0.$$

So that,

$$e^{\pi s^2} F(s) = F(0), \quad -\infty < s < \infty$$

Then,

$$F(0) = \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx.$$

Also,

$$\begin{aligned} F(0)^2 &= \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx \int_{-\infty}^{\infty} e^{-\pi y^2}\, dy \\ &= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} e^{-\pi(x^2+y^2)}\, dy\, dx \\ &= \int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} e^{-\pi r^2} r\, d\theta\, dr \\ &= 1. \end{aligned}$$

And so, $F(0) = 1$. Hence, $F(s) = e^{-\pi s^2}$.

So, unit gaussian $f(x) = e^{-\pi x^2}$ has FT $F(s) = e^{-\pi s^2}$.[1]

3.2 Rules for Finding FT

3.2.1 Linearity

If c, c_1, c_2, ... be complex scalars and $g(x) := cf(x)$. Then,

$$G(s) = \int_{-\infty}^{\infty} cf(x) e^{-2\pi i s x}\, dx = c \int_{-\infty}^{\infty} f(x) e^{-2\pi i s x}\, dx = cF(s).$$

Thus,

$$g(x) := cf(x) \text{ has FT } G(s) = cF(s). \tag{3.6}$$

Again, if $g(x) := f_1(x) + f_2(x)$, then

$$\begin{aligned} G(s) &= \int_{-\infty}^{\infty} \Big[f_1(x) + f_2(x) \Big] e^{-2\pi i s x} \, dx \\ &= \int_{-\infty}^{\infty} f_1(x) e^{-2\pi i s x} \, dx + \int_{-\infty}^{\infty} f_2(x) e^{-2\pi i s x} \, dx \\ &= F_1(s) + F_2(s). \end{aligned}$$

Thus,

$$g(x) := f_1(x) + f_2(x) \text{ has FT } G(s) = F_1(s) + F_2(s). \tag{3.7}$$

From equation (3.6) and equation (3.7), we have

$g(x) := c_1 f_1(x) + \ldots + c_m f_m(x)$ has FT $G(s) = c_1 F_1(s) + \ldots + c_m F_m(s)$.[1]

3.2.2 Reflection and Conjugation

The reflection rule is defined as,

$$g(x) := f(-x) \text{ has the FT } G(s) = F(-s.)$$

We verify this by writing,

$$G(s) = \int_{-\infty}^{\infty} f(-x) e^{-2\pi i s x} \, dx = \int_{-\infty}^{\infty} f(u) e^{-2\pi i (-s) u} \, du = F(-s).$$

The conjugation rule is defined as,

$$g(x) := \overline{f(x)} \text{ has the FT } G(s) = \overline{F(-s)}.$$

We verify this by writing,

$$G(s) = \int_{-\infty}^{\infty} \overline{f(x)} e^{-2\pi i s x} \, dx = \int_{-\infty}^{\infty} \overline{f(x) e^{-2\pi i (-s) x}} \, dx = \overline{F(-s)}.$$ [1]

3.2.3 Example using Parseval's and Conjugation

Let $a > 0$, $b > 0$. Here, we use Parseval's identity along with conjugation rule defined in above sections to

(a) show that:

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{ab(a+b)}.$$

Solution: Let,

$f(x) = \frac{1}{(x^2+a^2)}$ has FT $F(s) = \frac{\pi}{a}e^{-2\pi as}$,

and, $\overline{g(x)} = \frac{1}{(x^2+b^2)}$ has FT $\overline{G(s)} = \frac{\pi}{b}e^{-2\pi bs}$.

Then, using Parseval's identity,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} &= \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx \\
&= \int_{-\infty}^{\infty} F(s)\overline{G(s)}\,ds \\
&= \int_{-\infty}^{\infty} \frac{\pi}{a}e^{-2\pi as}\frac{\pi}{b}e^{-2\pi bs}\,ds \\
&= \frac{\pi^2}{ab}\int_{-\infty}^{\infty} e^{-2\pi(a+b)s}\,ds \\
&= \frac{2\pi^2}{ab}\int_{0}^{\infty} e^{-2\pi(a+b)s}\,ds \\
&= -\frac{2\pi^2}{ab(2\pi(a+b))}\left[e^{-2\pi(a+b)s}\right]_0^{\infty} \\
&= -\frac{\pi}{ab(a+b)}\left[e^{-\infty} - e^{0}\right] \\
&= \frac{\pi}{ab(a+b)}.
\end{aligned}$$

(b) show that:

$$\int_{-\infty}^{\infty} \frac{\sin(\pi ax)}{x(x^2+b^2)}\,dx = \frac{\pi}{b^2}(1-e^{-\pi ab}).$$

Solution: Let,

$f(x) = \frac{\sin(\pi a x)}{x}$ has FT $F(s) = \pi \prod(as)$,

and, $\overline{g(x)} = \frac{1}{(x^2+b^2)}$ has FT $\overline{G(s)} = \frac{\pi}{b} e^{-2\pi b s}$.

Then, using Parseval's identity,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\sin(\pi a x)}{x(x^2+b^2)}\, dx &= \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx \\
&= \int_{-\infty}^{\infty} F(s)\overline{G(s)}\, ds \\
&= \int_{-\infty}^{\infty} \pi \prod(as) \frac{\pi}{b} e^{-2\pi b s}\, ds \\
&= \frac{\pi^2}{b} \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-2\pi b s}\, ds \\
&= \frac{2\pi^2}{b} \int_{0}^{\frac{a}{2}} e^{-2\pi b s}\, ds \\
&= \frac{2\pi^2}{b(-2\pi b)} \Big[e^{-2\pi b s} \Big]_0^{\frac{a}{2}} \\
&= -\frac{\pi}{b^2} \Big[e^{-\pi a b} - 1 \Big] \\
&= \frac{\pi}{b^2} \Big[1 - e^{-\pi a b} \Big].
\end{aligned}$$

3.2.4 Translation and Modulation

The translation rule (or shift rule) is defined as,

$$g(x) := f(x - x_0) \text{ has the FT } G(s) = e^{-2\pi i s x_0} F(s),$$

where x_0 is a real parameter. We verify this by writing,

$$G(s) = \int_{-\infty}^{\infty} f(x - x_0) e^{-2\pi i s x}\, dx = e^{-2\pi i s x_0} \int_{-\infty}^{\infty} f(u) e^{-2\pi i s u}\, du = e^{-2\pi i s x_0} F(s).$$

The modulation rule (or transform shift rule) is defined as,

$$g(x) := e^{2\pi i s_0 x} f(x) \text{ has the FT } G(s) = F(s - s_0),$$

where s_0 is a real parameter. We verify this by writing,

$$G(s) = \int_{-\infty}^{\infty} e^{2\pi i s_0 x} f(x) e^{-2\pi i s x}\, dx = \int_{-\infty}^{\infty} f(x) e^{-2\pi i (s - s_0) x}\, dx = F(s - s_0).$$ [1]

3.2.5 Dilation

The dilation rule (or similarity rule) is defined as,

$$g(x) := f(ax) \text{ has the FT } G(s) = \frac{1}{|a|} F(\frac{s}{a}),$$

where $a \neq 0$ is a real parameter. We verify this by writing,

$$\begin{aligned} G(s) &= \int_{-\infty}^{\infty} f(ax) e^{-2\pi i s x}\, dx \\ &= \begin{cases} \frac{1}{a} \int_{-\infty}^{\infty} f(u) e^{-2\pi i (\frac{s}{a}) u}\, du & \text{if } a > 0 \\ \frac{1}{a} \int_{\infty}^{-\infty} f(u) e^{-2\pi i (\frac{s}{a}) u}\, du & \text{if } a < 0 \end{cases} \\ &= \frac{1}{|a|} F(\frac{s}{a}). \end{aligned}$$ [1]

3.2.6 Inversion

The inversion rule is defined as,

$$g(x) := F(x) \text{ has the FT } G(s) = f(-s).$$

We verify this by writing,

$$G(s) = \int_{-\infty}^{\infty} F(x) e^{-2\pi i s x}\, dx = \int_{-\infty}^{\infty} F(x) e^{2\pi i (-s) x}\, dx = f(-s).$$ [1]

3.2.7 Derivative and Power Scaling

The derivative rule is defined as,

$$g(x) := f'(x) \text{ has the FT } G(s) = 2\pi i s F(s),$$

where f is a suitably regular function on $\mathbb{R}$. We verify this by writing,

$$\begin{aligned} G(s) &= \int_{-\infty}^{\infty} f'(x)e^{-2\pi isx}\,dx \\ &= \Big[e^{-2\pi isx}f(x)\Big]_{-\infty}^{\infty} + 2\pi is\int_{-\infty}^{\infty} f(x)e^{-2\pi isx}\,dx \\ &= 2\pi isF(s). \end{aligned}$$

The power scaling rule is defined as,

$$g(x) := xf(x) \text{ has the FT } G(s) = (-2\pi i)^{-1}F'(s),$$

where f is a suitably regular function on $\mathbb{R}$. We verify this by writing,

$$\begin{aligned} G(s) &= \int_{-\infty}^{\infty} xf(x)e^{-2\pi isx}\,dx \\ &= \left(\frac{1}{-2\pi i}\right)\frac{d}{ds}\int_{-\infty}^{\infty} f(x)e^{-2\pi isx}\,dx \\ &= (-2\pi i)^{-1}F'(s). \end{aligned}$$ [1]

3.2.8 Examples on how to find function f using above rules

In this section, we illustrate the method to find the inverse Fourier Transforms when Fourier Transform $F(s)$ is given as

(a) $\int_{-\infty}^{\infty} F(-s)e^{2\pi isx}\,ds$;

Solution: Letting $q = -s$ such that $dq = -ds$, we have

$$\begin{aligned} \int_{-\infty}^{\infty} F(-s)e^{2\pi isx}\,ds &= -\int_{\infty}^{-\infty} F(q)e^{-2\pi iqx}\,dq \\ &= \int_{-\infty}^{\infty} F(q)e^{2\pi iq(-x)}\,dq \\ &= f(-x). \end{aligned}$$

(b) $\int_{-\infty}^{\infty} \overline{F(-s)}e^{2\pi isx}\,ds$;

Solution: Letting $q = -s$ such that $dq = -ds$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \overline{F(-s)} e^{2\pi isx}\, ds &= -\int_{\infty}^{-\infty} \overline{F(q)} e^{-2\pi iqx}\, dq \\ &= \int_{-\infty}^{\infty} \overline{F(q)} e^{-2\pi iqx}\, dq \\ &= \overline{\int_{-\infty}^{\infty} F(q) e^{-2\pi iqx}\, dq} \\ &= \overline{f(x)}.\end{aligned}$$

(c) $\int_{-\infty}^{\infty} F(s-5) e^{2\pi isx}\, ds$;

Solution: Letting $q = s - 5$ such that $dq = ds$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} F(s-5) e^{2\pi isx}\, ds &= \int_{-\infty}^{\infty} F(q) e^{2\pi i(q+5)x}\, dq \\ &= e^{10\pi ix} \int_{-\infty}^{\infty} F(q) e^{2\pi iqx}\, dq \\ &= e^{10\pi ix} f(x).\end{aligned}$$

(d) $\int_{-\infty}^{\infty} F(2s) e^{2\pi isx}\, ds$;

Solution: Letting $q = 2s$ such that $dq = 2ds$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} F(2s) e^{2\pi isx}\, ds &= \frac{1}{2} \int_{-\infty}^{\infty} F(q) e^{2\pi ip\frac{x}{2}}\, dq \\ &= \frac{1}{2} f\left(\frac{x}{2}\right).\end{aligned}$$

(e) $\int_{-\infty}^{\infty} s^2 F(s) e^{2\pi isx}\, ds$;

Solution:

$$\begin{aligned}\int_{-\infty}^{\infty} s^2 F(s) e^{2\pi isx}\, ds &= \frac{1}{(2\pi i)^2} \int_{-\infty}^{\infty} F(s) \frac{d^2}{dx^2} e^{2\pi isx}\, ds \\ &= \frac{1}{(2\pi i)^2} \frac{d^2}{dx^2} \int_{-\infty}^{\infty} F(s) e^{2\pi isx}\, ds \\ &= \frac{1}{(2\pi i)^2} f''(x).\end{aligned}$$

(f) $\int_{-\infty}^{\infty} \cos(2\pi s)F(s)e^{2\pi isx}\,ds$;

Solution:

$$\begin{aligned}\int_{-\infty}^{\infty} \cos(2\pi s)F(s)e^{2\pi isx}\,ds &= \frac{1}{2}\int_{-\infty}^{\infty}(e^{2\pi is}+e^{-2\pi is})F(s)e^{2\pi isx}\,ds \\ &= \frac{1}{2}\Big[\int_{-\infty}^{\infty} F(s)e^{2\pi i(x+1)s}\,ds + \int_{-\infty}^{\infty} F(s)e^{2\pi i(x-1)s}\,ds\Big] \\ &= \frac{1}{2}\Big[f(x+1)+f(x-1)\Big].\end{aligned}$$

(g) $\int_{-\infty}^{\infty} F'''(s)e^{2\pi isx}\,ds$;

Solution:

$$\begin{aligned}\int_{-\infty}^{\infty} F'''(s)e^{2\pi isx}\,ds &= \frac{1}{-2\pi ix}\int_{-\infty}^{\infty} F''(s)e^{2\pi isx}\,ds \\ &= \frac{1}{(-2\pi ix)^2}\int_{-\infty}^{\infty} F'(s)e^{2\pi isx}\,ds \\ &= \frac{1}{(-2\pi ix)^3}\int_{-\infty}^{\infty} F(s)e^{2\pi isx}\,ds \\ &= (-2\pi i)^{-3}x^{-3}f(x).\end{aligned}$$

(h) $\int_{-\infty}^{\infty} sF(2s)e^{-2\pi isx}\,ds$;

Solution:

$$\begin{aligned}\int_{-\infty}^{\infty} sF(2s)e^{-2\pi isx}\,ds &= \frac{1}{-2\pi i}\int_{-\infty}^{\infty} F(2s)\frac{d}{dx}e^{-2\pi isx}\,ds \\ &= \frac{1}{-2\pi i}\frac{d}{dx}\int_{-\infty}^{\infty} F(2s)e^{-2\pi isx}\,ds\end{aligned}$$

Letting $q = 2s$ such that $dq = 2ds$,

$$\begin{aligned}\int_{-\infty}^{\infty} sF(2s)e^{-2\pi isx}\,ds &= \frac{1}{-2\pi i}\frac{d}{dx}\int_{-\infty}^{\infty} F(q)e^{-2\pi i(\frac{q}{2})x}\frac{1}{2}\,dq \\ &= \frac{1}{(-4\pi i)}\frac{d}{dx}\int_{-\infty}^{\infty} F(q)e^{2\pi ip(-\frac{x}{2})}\,dq \\ &= \frac{1}{(-4\pi i)}f'(-\frac{x}{2}).\end{aligned}$$

(i) $\int_{-\infty}^{\infty} \frac{1}{2}\Big[F(s) + \overline{F(s)}\Big] e^{2\pi isx}\, ds$;

Solution:

$$\int_{-\infty}^{\infty} \frac{1}{2}\Big[F(s) + \overline{F(s)}\Big] e^{2\pi isx}\, ds = \frac{1}{2}\Big[\int_{-\infty}^{\infty} F(s)e^{2\pi isx}\, ds + \int_{-\infty}^{\infty} \overline{F(s)}e^{2\pi isx}\, ds\Big]$$
$$= \frac{1}{2}\left[f(x) + \overline{\int_{-\infty}^{\infty} F(s)e^{2\pi is(-x)}\, ds}\right]$$
$$= \frac{1}{2}\big[f(x) + \overline{f(-x)}\big].$$

3.2.9 Example to find Cross-correlation and Plancherel's Identity

The cross-correlation product $f_1 \star f_2$ of the suitably regular functions f_1, f_2 is defined by (2.8). In this example, we illustrate the way to derive the Fourier Transforms for the cross-correlation, auto-correlation and Plancherel's Identity for these function.

(a) Let us derive the cross-correlation rule:

$g(x) := (f_1 \star f_2)(x)$ has the FT $G(s) = \overline{F_1(s)}F_2(s)$.

Solution: We have,

$g(x) := (f_1 \star f_2)(x)$.

Then,

$$G(s) = \int_{-\infty}^{\infty} g(x)e^{-2\pi isx}\, dx$$
$$= \int_{-\infty}^{\infty} (f_1 \star f_2)(x)e^{-2\pi isx}\, dx$$
$$= \int_{x=-\infty}^{\infty} \int_{u=-\infty}^{\infty} \overline{f_1(u)} f_2(u+x)e^{-2\pi isx}\, dx\, du$$
$$= \int_{x=-\infty}^{\infty} \left[\overline{\int_{u=-\infty}^{\infty} f_1(u)e^{-2\pi isu}\, du}\right] f_2(u+x)e^{-2\pi i(u+x)s}\, dx$$
$$= \int_{-\infty}^{\infty} \overline{F_1(s)} f_2(u+x)e^{-2\pi i(u+x)s}\, dx$$
$$= \overline{F_1(s)}F_2(s).$$

(b) Specializing (a) we obtain the autocorrelation rule:

$g(x) := (f \star f)(x)$ has the FT $G(s) = |F(s)|^2$.

Solution: Using $f_1 = f_2 = f$ in (a), we get,

$g(x) := (f \star f)(x)$ has FT $G(s) = \overline{F(s)}F(s) = |F(s)|^2$.

(c) Then, using (b) obtain Plancherel's identity

$$\int_{-\infty}^{\infty} |f(u)|^2 \, du = \int_{-\infty}^{\infty} |F(s)|^2 \, ds.$$

Solution:

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(u)|^2 \, du &= \int_{-\infty}^{\infty} f(u)\overline{f(u)} \, du \\
&= \int_{-\infty}^{\infty} f(u) \left[\overline{\int_{-\infty}^{\infty} F(s) e^{2\pi i s u} \, ds} \right] du \\
&= \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} \overline{F(s)} e^{-2\pi i s u} \, ds \right] du \\
&= \int_{-\infty}^{\infty} \overline{F(s)} \left[\int_{-\infty}^{\infty} f(u) e^{-2\pi i s u} \, du \right] ds \\
&= \int_{-\infty}^{\infty} \overline{F(s)} F(s) \, ds \\
&= \int_{-\infty}^{\infty} |F(s)|^2 \, ds.
\end{aligned}$$

3.2.10 Convolution and Multiplication

The convolution rule is defined as,

$$g(x) := (f_1 * f_2)(x) \text{ has the FT } G(s) = F_1(s)F_2(s),$$

where $f_1,\ f_2$ is a suitably regular function on $\mathbb{R}$. We verify this by writing,

$$\begin{aligned}
G(s) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(u)f_2(x-u)e^{-2\pi isx}\,du\,dx \\
&= \int_{-\infty}^{\infty} f_1(u)\int_{-\infty}^{\infty} f_2(x-u)e^{-2\pi isx}\,dx\,du \\
&= \int_{-\infty}^{\infty} f_1(u)e^{-2\pi isu}\int_{-\infty}^{\infty} f_2(x-u)e^{-2\pi is(x-u)}\,dx\,du \\
&= \int_{-\infty}^{\infty} f_1(u)e^{-2\pi isu}F_2(s)\,du \\
&= F_1(s)F_2(s).
\end{aligned}$$

The multiplication rule is defined as,

$$g(x) := f_1(x)f_2(x) \text{ has the FT } G(s) = (F_1 * F_2)(s),$$

where $f_1,\ f_2$ is a suitably regular function on $\mathbb{R}$. We verify this by writing,

$$\begin{aligned}
G(s) &= \int_{-\infty}^{\infty} f_1(x)f_2(x)e^{-2\pi isx}\,dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(u)e^{2\pi iux}f_2(x)e^{-2\pi isx}\,du\,dx \\
&= \int_{-\infty}^{\infty} F_1(u)\int_{-\infty}^{\infty} f_2(x)e^{-2\pi ix(s-u)}\,dx\,du \\
&= \int_{-\infty}^{\infty} F_1(u)F_2(s-u)\,du \\
&= (F_1 * F_2)(s).
\end{aligned}$$ [1]

3.2.11 Example using Convolution and Multiplication Rule

Let f, g be piecewise smooth functions with small regular tails, and let F, G be the corresponding Fourier transforms. In this exercise we will illustrate that the multiplication rule can be used with fg, fG, Fg, FG, and the convolution rule can be used with $f * g$, $F * G$ to

(a) show the following

$$\int_{-\infty}^{\infty}(f * g)(x)e^{-2\pi isx}\,dx = F(s)G(s).$$

Solution: We have,

$$\begin{aligned}
\int_{-\infty}^{\infty}(f*g)(x)e^{-2\pi isx}\,dx &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)g(x-u)\,du\right]e^{-2\pi isx}\,dx \\
&= \int_{-\infty}^{\infty} f(u)e^{-2\pi isu}\left[\int_{-\infty}^{\infty} g(x-u)e^{-2\pi i(x-u)s}\,dx\right]du \\
&= \int_{-\infty}^{\infty} f(u)e^{-2\pi isu}G(s)\,du \\
&= F(s)G(s).
\end{aligned}$$

(b) show the following

$$\int_{-\infty}^{\infty} F(s)G(s)e^{2\pi isx}\,ds = (f*g)(x).$$

Note. Together (a)-(b) establish the convolution rule for $f*g$ and the multiplication rule for FG.

Solution: We have,

$$\begin{aligned}
\int_{-\infty}^{\infty} F(s)G(s)e^{2\pi isx}\,ds &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)e^{-2\pi isu}\,du\right]G(s)e^{-2\pi isx}\,ds \\
&= \int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} G(s)e^{2\pi i(x-u)s}\,ds\right]du \\
&= \int_{-\infty}^{\infty} f(u)g(x-u)\,du \\
&= (f*g)(x).
\end{aligned}$$

(c) show the following

$$\int_{-\infty}^{\infty} f(x)g(x)e^{-2\pi isx}\,dx = (F*G)(s).$$

Solution: We have,

$$\int_{-\infty}^{\infty} f(x)g(x)e^{-2\pi isx}\,dx = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} F(u)e^{2\pi iux}\,du\right]g(x)e^{-2\pi isx}\,dx$$
$$= \int_{-\infty}^{\infty} F(u)\left[\int_{-\infty}^{\infty} g(x)e^{-2\pi i(s-u)x}\,dx\right]du$$
$$= \int_{-\infty}^{\infty} F(u)G(s-u)\,du$$
$$= (F*G)(s).$$

(d) show the following

$$\int_{-\infty}^{\infty} (F*G)(s)e^{2\pi isx}\,ds = f(x)g(x).$$

Note. Together (c)-(d) establish the convolution rule for $F*G$ and the multiplication rule for fg.

Solution: We have,

$$\int_{-\infty}^{\infty} (F*G)(s)e^{2\pi isx}\,ds = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} F(u)G(s-u)\,du\right]e^{2\pi isx}\,ds$$
$$= \int_{-\infty}^{\infty} F(u)e^{2\pi iux}\left[\int_{-\infty}^{\infty} G(s-u)e^{2\pi i(s-u)x}\,ds\right]du$$
$$= \int_{-\infty}^{\infty} F(u)e^{2\pi iux}g(x)\,du$$
$$= f(x)g(x).$$

(e) show the following

$$\int_{-\infty}^{\infty} f(x)G(-x)e^{-2\pi isx}\,dx = (F*g)(s).$$

Solution: We have,

$$\int_{-\infty}^{\infty} f(x).G(-x)e^{-2\pi isx}\,dx = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} F(u)e^{2\pi iux}\,du\right]G(-x)e^{-2\pi isx}\,dx$$
$$= \int_{-\infty}^{\infty} F(u)\left[\int_{-\infty}^{\infty} G(-x)e^{2\pi i(s-u)(-x)}\,dx\right]du$$

Let $v = -p$ such that $dv = -dx$. Then,

$$= \int_{-\infty}^{\infty} F(u)\left[\int_{-\infty}^{\infty} G(v)e^{2\pi i(s-u)(v)}\,dv\right]du$$
$$= \int_{-\infty}^{\infty} F(u)g(s-u)\,du$$
$$= (F * g)(s).$$

Chapter 4

FT Calculus for Functions on $\mathbb{T}_p$, $\mathbb{Z}$ and $\mathbb{P}_N$

4.1 Fourier Series

4.1.1 Introduction

Suppose f is a *p-periodic* function. Then, as we saw in Chapter 1, the Fourier Series is given as,

$$f(x) = \sum_{k=-\infty}^{\infty} F[k] e^{\frac{2\pi i k x}{p}}, \quad -\infty < x < \infty \tag{4.1}$$

and F in this case is a complex-valued function on $\mathbb{Z}$. The function $F[k]$ can be constructed by evaluating the integral from the analysis equation,

$$F[k] = \frac{1}{p} \int_0^{\infty} f(x) e^{\frac{-2\pi i k x}{p}} dx, \quad k = 0, \pm 1, \pm 1 \ldots \tag{4.2}$$

Recall from Chapter 1 that the synthesis equation (1.4) for f on $\mathbb{T}_p$ can be written as the analysis equation,

$$\frac{f(-s)}{p} = \frac{1}{p} \sum_{n=-\infty}^{\infty} F[n] e^{\frac{-2\pi s n}{p}},$$

for F on $\mathbb{Z}$.

Hence, every Fourier Series (1.4) tells us that,

$$f(x) \text{ has FT } F[k], \tag{4.3}$$

$$F[k] \text{ has FT } \frac{f(-s)}{p}. \text{[1]} \tag{4.4}$$

4.1.2 Direct Integration

Suppose f, g are suitably regular functions. Then, using integration by parts formula,

$$\begin{aligned}\int_a^b f(x)q(x)\,dx &= f(x)q^{(-1)}(x) - \int_a^b f'(x)q^{(-1)}(x)\,dx \\ &= \Big[f(x)q^{(-1)}(x) - f'(x)q^{(-2)}(x)\Big]_a^b + \int_a^b f''(x)q^{(-2)}(x)\,dx,\end{aligned}$$

and so on..

Here, $q^{(-1)}$, $q^{(-2)}$, ... are successive antiderivatives of $q(x)$.

When f is a polynomial, the integrated terms will eventually disappear from the above equation such that,

$$\int_a^b f(x)q(x)\,dx = f(x)q^{(-1)}(x) + \ldots + (-1)^{(n-1)} f^{(n-1)}(x)q^{(-n)}(x)\Big|_a^b, \quad f^{(n)} \equiv 0 \tag{4.5}$$

(4.5) is known as Kronecker's Rule.[1]

4.1.3 Elementary Rules

Linearity

If c_1, c_2, ..., c_m be scalars. Then,

$$g(x) := c_1 f_1(x) + \ldots + c_m f_m(x) \quad \text{has FT } G[k] = c_1 F_1[k] + \ldots + c_m F_m[k]. \tag{4.6}$$

Reflection and Conjugation

The reflection rule is defined as,

$$g(x) := f(-x) \text{ has the FT } G[k] = F[-k]. \tag{4.7}$$

The conjugation rule is defined as,

$$g(x) := \overline{f(x)} \text{ has the FT } G[k] = \overline{F[-k]}. \tag{4.8}$$

Translation and Modulation

The translation rule (or shift rule) is defined as,

$$g(x) := f(x - x_0) \text{ has the FT } G[k] = e^{-2\pi ik\frac{x_0}{p}}F[k], \quad -\infty < x_0 < \infty \tag{4.9}$$

where x_0 is a real parameter.

The modulation rule (or transform shift rule) is defined as,

$$g(x) := e^{2\pi ik_0\frac{x}{p}}f(x) \text{ has the FT } G[k] = F[k - k_0], \quad k_0 = 0, \pm 1, \dots \tag{4.10}$$

where s_0 is a real parameter.

Convolution and Multiplication

The convolution rule is defined as,

$$g(x) := (f_1 * f_2)(x) \text{ has the FT } G[k] = pF_1[k]F_2[k]. \tag{4.11}$$

The multiplication rule is defined as,

$$g(x) := f_1(x)f_2(x) \text{ has the FT } G[k] = (F_1 * F_2)[k]. \tag{4.12}$$

Derivative

The derivative rule,

$$g(x) := f'(x) \text{ has the FT } G[k] = \big(\frac{2\pi ik}{p}\big)F[k]. \tag{4.13}$$

can be used when f is continuous and f' is piecewise smooth.

4.1.4 Example to compute Fourier Series and Fourier transforms using the above properties

In this section, we will use the above properties to first find the Fourier Series for the given functions and then compute their Fourier Transforms.

Let b, r be 2π-*periodic* function on $\mathbb{R}$ with

$$b(x) := \begin{cases} 1 & \text{for } 0 < x < \pi \\ 0 & \text{for } \pi < x < 2\pi, \end{cases} \qquad r(x) := \begin{cases} x & \text{for } 0 < x < \pi \\ 0 & \text{for } \pi < x < 2\pi. \end{cases}$$

Verify that b, r have the Fourier Series

$$b(x) = \frac{1}{2} + \sum_{k\neq 0} \frac{i\big[(-1)^k - 1\big]}{2\pi k} e^{ikx}, \qquad r(x) = \frac{\pi}{4} + \sum_{k\neq 0} \Big[\frac{i(-1)^k}{2k} + \frac{(-1)^k - 1}{2\pi k^2}\Big] e^{ikx},$$

and then use the Fourier transforms calculus to find the Fourier series for

(a) $t(x) := (b * b)(x) = r(x) + r(-x)$; **(b)** $f(x) := b(x) - b(-x) = 2b(x) - 1 = t'(x)$;

(c) $g(x) := r(x) + r(x - \pi)$; **(d)** $j(x) := r(\pi - x) - r(\pi + x) = 2\pi w_0(\frac{x}{2\pi})$;

(e) $d(x) := b(x - \frac{\pi}{4}) - b(x + \frac{\pi}{4})$; **(f)** $p_n(x) := b(x)\sin(nx), n = 1, 2, \ldots$.

Solution: We have,

$$\begin{aligned} B[0] &= \frac{1}{2\pi}\int_0^{2\pi} b(x) e^{-2\pi i(0)\frac{x}{2\pi}}\, dx \\ &= \frac{1}{2\pi}\int_0^{\pi} dx \\ &= \frac{1}{2}, \end{aligned}$$

And,

$$\begin{aligned} B[k] &= \frac{1}{2\pi}\int_0^{2\pi} b(x) e^{-2\pi ik\frac{x}{2\pi}}\, dx \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx}\, dx \\ &= \frac{1}{2\pi}\left[\frac{e^{-ikx}}{-ik}\right]_0^{\pi} \\ &= \frac{1}{-2\pi ik}\Big[\cos kx - \sin kx\Big]_0^{\pi} \\ &= \frac{i}{2\pi k}\big[\cos k\pi - 1\big] \\ &= \frac{i}{2\pi k}\big[(-1)^k - 1\big]. \end{aligned}$$

Thus,

$$b(x) = \frac{1}{2} + \sum_{k\neq 0} \frac{i\big[(-1)^k - 1\big]}{2\pi k} e^{ikx}.$$

Also,

$$\begin{aligned}
R[0] &= \frac{1}{2\pi}\int_0^{2\pi} r(x)e^{-2\pi i(0)\frac{x}{2\pi}}\,dx \\
&= \frac{1}{2\pi}\int_0^{\pi} x\,dx \\
&= \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{\pi} \\
&= \frac{\pi}{4},
\end{aligned}$$

And,

$$\begin{aligned}
R[k] &= \frac{1}{2\pi}\int_0^{2\pi} r(x)e^{-2\pi ik\frac{x}{2\pi}}\,dx \\
&= \frac{1}{2\pi}\int_0^{\pi} xe^{-ikx}\,dx \\
&= \frac{1}{2\pi}\left[x\int_0^{\pi} e^{-ikx}\,dx - \int_0^{\pi}\int_0^{\pi} e^{-ikx}\,dx\,dx\right] \\
&= \frac{1}{2\pi}\left[\frac{xe^{-ikx}}{-ik} + \frac{1}{ikx}\int_0^{\pi} e^{-ikx}\,dx\right] \\
&= \frac{1}{2\pi}\left[\frac{ixe^{-ikx}}{k} + \frac{e^{-ikx}}{k^2}\right]_0^{\pi} \\
&= \left[\frac{i\pi\cos(\pi k)}{2\pi k} + \frac{\cos(\pi k)-\cos(0)}{2\pi k^2}\right] \\
&= \left[\frac{i(-1)^k}{2k} + \frac{(-1)^k-1}{2\pi k^2}\right].
\end{aligned}$$

Thus,

$$r(x) = \frac{\pi}{4} + \sum_{k\neq 0}\Big[\frac{i(-1)^k}{2k} + \frac{(-1)^k-1}{2\pi k^2}\Big]e^{ikx}.$$

Then,

(a) $t(x) := (b*b)(x) = r(x) + r(-x)$.

Since,

$$r(x) = \frac{\pi}{4} + \sum_{k\neq 0}\Big[\frac{i(-1)^k}{2k} + \frac{(-1)^k-1}{2\pi k^2}\Big]e^{ikx},$$

and

$$r(-x) = \frac{\pi}{4} + \sum_{k\neq 0}\Big[\frac{i(-1)^k}{2k} + \frac{(-1)^k-1}{2\pi k^2}\Big]e^{-ikx}.$$

Then,

$$t(x) = \frac{\pi}{2} + \sum_{k\neq 0}\Big[\frac{i(-1)^k}{2k} + \frac{(-1)^k - 1}{2\pi k^2}\Big]\Big(\frac{e^{ikx} + e^{-ikx}}{2k}\Big)(2k)$$
$$= \frac{\pi}{2} + \sum_{k\neq 0}\Big[i(-1)^k + \frac{(-1)^k - 1}{\pi k}\Big]\cos(kx).$$

(b) $f(x) := b(x) - b(-x) = 2b(x) - 1 = t'(x)$

Since,

$$b(x) = \frac{1}{2} + \sum_{k\neq 0}\frac{i\big[(-1)^k - 1\big]}{2\pi k}e^{ikx},$$

Then,

$$f(x) = 2\left[\frac{1}{2} + \sum_{k\neq 0}\frac{i\big[(-1)^k - 1\big]}{2\pi k}e^{ikx}\right] - 1$$
$$= \sum_{k\neq 0}\frac{i\big[(-1)^k - 1\big]}{\pi k}e^{ikx}.$$

(c) $g(x) := r(x) + r(x - \pi)$

Since,

$$r(x) = \frac{\pi}{4} + \sum_{k\neq 0}\Big[\frac{i(-1)^k}{2k} + \frac{(-1)^k - 1}{2\pi k^2}\Big]e^{ikx},$$

and

$$r(x-\pi) = \frac{\pi}{4} + \sum_{k\neq 0}\Big[\frac{i(-1)^k}{2k} + \frac{(-1)^k - 1}{2\pi k^2}\Big]e^{ik(x-\pi)}.$$

Then,

$$\begin{aligned}
g(x) &= \frac{\pi}{2} + \sum_{k\neq 0}\left[\frac{i(-1)^k}{2k} + \frac{(-1)^k-1}{2\pi k^2}\right]\big(1+(-1)^k\big)e^{ikx} \\
&= \frac{\pi}{2} + \sum_{k\neq 0}\left[\frac{i(-1)^k}{2k}\big(1+(-1)^k\big)e^{ikx} + \frac{(-1)^k-1}{2\pi k^2}\big(1+(-1)^k\big)e^{ikx}\right] \\
&= \frac{\pi}{2} + \sum_{k\neq 0}\left[\frac{i(-1)^k}{2k}\big(1+(-1)^k\big)e^{ikx} + \frac{(-1)^{2k}-1^2}{2\pi k^2}e^{ikx}\right] \\
&= \frac{\pi}{2} + \sum_{k\neq 0}\frac{i(-1)^k + i(-1)^{2k}}{2k}e^{ikx} \\
&= \frac{\pi}{2} + \sum_{k\neq 0}\frac{i\big(1+(-1)^k\big)}{2k}e^{ikx}.
\end{aligned}$$

(d) $j(x) := r(\pi - x) - r(\pi + x) = 2\pi w_0(\frac{x}{2\pi})$

Since,

$$r(\pi - x) = \frac{\pi}{4} + \sum_{k\neq 0}\left[\frac{i(-1)^k}{2k} + \frac{(-1)^k-1}{2\pi k^2}\right]e^{ik(\pi-x)},$$

and

$$r(\pi + x) = \frac{\pi}{4} + \sum_{k\neq 0}\left[\frac{i(-1)^k}{2k} + \frac{(-1)^k-1}{2\pi k^2}\right]e^{ik(\pi+x)}.$$

Then,

$$\begin{aligned}
j(x) &= \sum_{k\neq 0}\left[\frac{i(-1)^k}{2k} + \frac{(-1)^k-1}{2\pi k^2}\right]\Big(-e^{ik\pi}(e^{ikx} - e^{-ikx})\Big) \\
&= \sum_{k\neq 0}\left[\frac{i(-1)^k}{2k} + \frac{(-1)^k-1}{2\pi k^2}\right]\Big(-2ik(-1)^k\sin(kx)\Big) \\
&= \sum_{k\neq 0}\left[-1 + \frac{-1+i(-1)^k}{\pi k}\right]\sin(kx) \\
&= \sum_{k\neq 0}\left[\frac{i(-1)^k-1}{\pi k} - 1\right]\sin(kx).
\end{aligned}$$

(e) $d(x) := b\big(x - \frac{\pi}{4}\big) - b\big(x + \frac{\pi}{4}\big)$

Since,

$$b\Big(x - \frac{\pi}{4}\Big) = \frac{1}{2} + \sum_{k\neq 0}\frac{i\big[(-1)^k - 1\big]}{2\pi k}e^{ik(x-\frac{\pi}{4})},$$

and,

$$b\big(x+\frac{\pi}{4}\big) = \frac{1}{2} + \sum_{k\neq 0} \frac{i\big[(-1)^k - 1\big]}{2\pi k} e^{ik(x+\frac{\pi}{4})}.$$

Then,

$$\begin{aligned} d(x) &= \sum_{k\neq 0} \frac{i\big[(-1)^k - 1\big]}{2\pi k} \Big[e^{ik(x-\frac{\pi}{4})} - e^{ik(x+\frac{\pi}{4})}\Big] \\ &= \sum_{k\neq 0} \frac{i\big[(-1)^k - 1\big]}{2\pi k} (-e^{ikx}) \left[\frac{e^{i\frac{\pi}{4}} - e^{-i\frac{\pi}{4}}}{2i}\right] 2\pi \\ &= \sum_{k\neq 0} \frac{i\big[(-1)^k - 1\big]}{2\pi k} \Big(-2ie^{ikx}\sin\big(\frac{\pi}{4}\big)\Big) \\ &= \sum_{k\neq 0} \frac{i\big[(-1)^k - 1\big]}{2\pi k} (-\sqrt{2}ie^{ikx}) \\ &= \sqrt{2}\sum_{k\neq 0} \frac{-i^2\big[(-1)^k - 1\big]}{2\pi k} (e^{ikx}) \\ &= \sum_{k\neq 0} \frac{\big[(-1)^k - 1\big]}{\sqrt{2}\pi k} (e^{ikx}). \end{aligned}$$

(f) $p_n(x) := b(x)\sin(nx)$, $n = 1, 2, \ldots$

Since,

$$b(x) = \frac{1}{2} + \sum_{k\neq 0} \frac{i\big[(-1)^k - 1\big]}{2\pi k} e^{ikx},$$

Then,

$$\begin{aligned} p_n(x) &= \left[\frac{1}{2} + \sum_{k\neq 0} \frac{i\big[(-1)^k - 1\big]}{2\pi k} e^{ikx}\right] \sin(nx) \\ &= \frac{\sin(nx)}{2} + \left[\sum_{k\neq 0} \frac{i\big[(-1)^k - 1\big]}{2\pi k} e^{ikx}\right] \sin(nx). \end{aligned}$$

4.1.5 Poisson's Relation

Let f be a piecewise smooth function on $\mathbb{R}$ that has small regular tails. Using *Poisson relation*, we have,

$$g(x) := \sum_{m=-\infty}^{\infty} f(x - mp) \ (f \text{ on } \mathbb{R} \text{ and } g \text{ on } \mathbb{T}_p) \text{ has the FS } \sum \frac{1}{p} F\Big(\frac{k}{p}\Big) e^{2\pi i k \frac{x}{p}}. \quad (4.14)$$ [1]

4.1.6 Bernoulli Function

Suppose,

$$w_0(x) = \begin{cases} 0 & \text{if } x = 0, 1, \\ \frac{1}{2} - x & \text{if } 0 < x < 1. \end{cases} \tag{4.15}$$

Then, taking antiderivatives, we have polynomials,

$$w_1(x) = -\frac{x^2}{2} + \frac{x}{2} - \frac{1}{12}, \qquad 0 \le x \le 1 \tag{4.16}$$

$$w_2(x) = -\frac{x^3}{6} + \frac{x^2}{4} - \frac{x}{12}, \qquad 0 \le x \le 1 \tag{4.17}$$

$$\vdots$$

with,

$$w_n'(x) = w_{n-1}(x), \quad n = 1, 2, \ldots (\text{and } x \neq 0, 1 \text{ when } n = 1), \tag{4.18}$$

$$\int_0^1 w_n(x)\, dx = 0, \qquad n = 0, 1, \ldots \tag{4.19}$$

But we have

$$w_0(x) = \frac{1}{2} - x, \quad 0 < x < 1 \text{ has FS } \sum_{k \neq 0} \frac{e^{2\pi i k x}}{2\pi i k}. \tag{4.20}$$

Using (1.11) with (1.9)- (1.10), and the derivative rule, we obtain the 1-periodic *Bernoulli* functions,

$$w_n(x) = \sum_{k \neq 0} \frac{e^{2\pi i k x}}{(2\pi i k)^{n+1}}, \quad n = 0, 1, \ldots, \quad -\infty < x < \infty \tag{4.21}$$

These functions have been constructed so that,[1]

$$w_n^{(m)}(0+) - w_n^{(m)}(0-) = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m = 0, 1, \ldots, n-1, n+1, \ldots \end{cases} \tag{4.22}$$

4.1.7 Laurent Series

A *Laurent series*

$$C(z) = \sum_{k=-\infty}^{\infty} c_k z^k, \quad z \epsilon \mathbb{C}, \quad a \le |z| \le b \tag{4.23}$$

is a complex power series that may contain terms with z^{-1}, z^{-2}, ... as well as terms 1, z, z^2,....

If the Laurent series (4.23) converges within some non-degenerate closed annulus that contains the unit circle then, we can produce FS by setting $z = e^{2\pi i\frac{x}{p}}$, i.e.,

$$f(x) := C(e^{2\pi i\frac{x}{p}}) \text{ has the FS } \sum_{k=-\infty}^{\infty} c_k e^{2\pi i\frac{x}{p}}. \quad (4.24)$$

[1]

4.1.8 Dirichlet Kernel

The *Dirichlet kernel*, named after *Johann Peter Gustav Lejeune Dirichlet* is the collection of functions

$$D_n(x) = \frac{\sin{(n+\frac{1}{2})x}}{\sin(\frac{x}{2})} \quad \text{which have FS } \sum_{k=-n}^{n} e^{ikx}. \quad (4.25)$$

The convolution of $D_n(x)$ with any function f of period 2π is the n^{th}-degree Fourier series approximation to f, i.e.,

$$(D_n * f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)D_n(x-y)\,dy = \sum_{k=-n}^{n}\hat{f}(k)e^{ikx}, \quad (4.26)$$

where

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx. \quad (4.27)$$

is the k^{th} Fourier coefficient of f by Chapter 1.

4.1.9 Dilation and Grouping rule

When f is a *p-periodic* function on $\mathbb{R}$, and $m = 1, 2, \ldots$, the dilate $f(mx)$ is *p-periodic* as well as $\frac{p}{m}$*-periodic.*

The Dilation rule (or similarity rule) is defined as,

$$g(x) := f(mx) \text{ has the FT } G[k] = \begin{cases} F\left[\frac{k}{m}\right] & \text{if } k = 0, \pm m, \pm 2m, \ldots \\ 0 & \text{otherwise.} \end{cases} \quad (4.28)$$

We verify this by writing,

$$g(x) = \sum_{k=-\infty}^{\infty} F[k]e^{2\pi ikm\frac{x}{p}} = \sum_{m|k} F\left[\frac{k}{m}\right]e^{2\pi ik\frac{x}{p}}.$$

Let f be a *p-periodic* function and $m = 1, 2, \ldots$. We sum the $\frac{p}{m}$ translates of f to produce a $\frac{p}{m}$-*periodic* function,

$$f_m(x) := f(x) + f\big(x - \frac{p}{m}\big) + \ldots + f\Big(x - \frac{(m-1)p}{m}\Big),$$

that has *p-periodic* dilate $f_m\big(\frac{x}{m}\big)$.

The Grouping rule,

$$g(x) := \sum_{l=0}^{m-1} f\big(\frac{x}{m} - \frac{lp}{m}\big) \text{ has FT } G[k] = mF[mk]. \tag{4.29}$$

is verified as,

$$\begin{aligned} g(x) &= \sum_{l=0}^{m-1} \sum_{k=-\infty}^{\infty} F[k] e^{2\pi i k \frac{(\frac{x}{m} - \frac{lp}{m})}{p}} \\ &= \sum_{k=-\infty}^{\infty} F[k] e^{2\pi i k \frac{x}{mp}} \sum_{l=0}^{m-1} e^{-2\pi i k \frac{l}{m}} \\ &= \sum_{k=-\infty}^{\infty} F[k] e^{2\pi i k \frac{x}{mp}} \begin{cases} m & \text{if } k = 0, \pm m, \pm 2m, \ldots \\ 0 & \text{otherwise} \end{cases} \\ &= \sum_{k=-\infty}^{\infty} mF[mk] e^{2\pi i k \frac{x}{p}}. \end{aligned}$$ [1]

4.1.10 Example on how to compute Fourier Transform of given function f

Let f be a suitably regular function on $\mathbb{T}_p$, $-\infty < x_0 < \infty$, and $m = 1, 2, 3, \ldots$.

What can you infer about the Fourier transform F of these given functions if you know that:

(a) $\overline{f(x)} = f(x)$?

Solution: Integrating on both sides from 0 to p with respect to x and then dividing by p, we have,

$$\frac{1}{p} \int_0^p \overline{f(x)} e^{-2\pi i k \frac{x}{p}}\, dx = \frac{1}{p} \int_0^p f(x) e^{-2\pi i k \frac{x}{p}}\, dx$$
$$\frac{1}{p} \int_0^p \overline{f(x) e^{-2\pi i (-k) \frac{x}{p}}}\, dx = \frac{1}{p} \int_0^p f(x) e^{-2\pi i k \frac{x}{p}}\, dx$$
$$\overline{F[-k]} = F[k].$$

(b) $\overline{f(x)} = f(-x)$?

Solution: Integrating on both sides from 0 to p with respect to x and then dividing by p, we have,

$$\frac{1}{p}\int_0^p \overline{f(x)}e^{-2\pi ik\frac{x}{p}}\,dx = \frac{1}{p}\int_0^p f(-x)e^{-2\pi ik\frac{x}{p}}\,dx$$
$$\frac{1}{p}\int_0^p \overline{f(x)e^{-2\pi i(-k)\frac{x}{p}}\,dx} = \frac{1}{p}\int_0^p f(-x)e^{-2\pi i(-k)\frac{-x}{p}}\,dx$$
$$\overline{F[-k]} = F[-k].$$

(c) $f(x+\frac{p}{m}) = f(x)$?

Solution: Integrating on both sides from 0 to p with respect to x and then dividing by p, we have,

$$\frac{1}{p}\int_0^p f(x+\frac{p}{m})e^{-2\pi ik\frac{x}{p}}\,dx = \frac{1}{p}\int_0^p f(x)e^{-2\pi ik\frac{x}{p}}\,dx$$
$$\frac{1}{p}\int_0^p f(x+\frac{p}{m})e^{-2\pi ik\frac{(x+\frac{p}{m})}{p}}e^{2\pi i\frac{k}{m}}\,dx = \frac{1}{p}\int_0^p f(x)e^{-2\pi ik\frac{x}{p}}\,dx$$
$$e^{2\pi i\frac{k}{m}}F[k] = F[k].$$

(d) $f(x+\frac{p}{2}) = -f(x)$?

Solution: Integrating on both sides from 0 to p with respect to x and then dividing by p, we have,

$$\frac{1}{p}\int_0^p f(x+\frac{p}{2})e^{-2\pi ik\frac{x}{p}}\,dx = -\frac{1}{p}\int_0^p f(x)e^{-2\pi ik\frac{x}{p}}\,dx$$
$$\frac{1}{p}\int_0^p f(x+\frac{p}{2})e^{-2\pi ik\frac{(x+\frac{p}{2})}{p}}e^{2\pi i\frac{k}{2}}\,dx = -\frac{1}{p}\int_0^p f(x)e^{-2\pi ik\frac{x}{p}}\,dx$$
$$e^{\pi ik}F[k] = -F[k].$$

(e) $f(x_0+x) = f(x_0-x)$?

Solution: Integrating on both sides from 0 to p with respect to x and then dividing by p, we have,

$$\frac{1}{p}\int_0^p f(x+x_0)e^{-2\pi ik\frac{x}{p}}\,dx = \frac{1}{p}\int_0^p f(x_0-x)e^{-2\pi ik\frac{x}{p}}\,dx$$
$$\frac{1}{p}\int_0^p f(x+x_0)e^{-2\pi ik\frac{(x+x_0)}{p}}e^{2\pi ik\frac{x_0}{p}}\,dx = \frac{1}{p}\int_0^p f(x_0-x)e^{-2\pi i(-k)\frac{x_0-x}{p}}e^{-2\pi ik\frac{x_0}{p}}\,dx$$
$$e^{2\pi ik\frac{x_0}{p}}F[k] = e^{-2\pi ik\frac{x_0}{p}}F[-k]$$
$$F[k] = e^{-4\pi ik\frac{x_0}{p}}F[-k].$$

(f) $f(x_0+x) = -f(x_0-x)$?

Solution: Integrating on both sides from 0 to p with respect to x and then dividing by p, we have,

$$\frac{1}{p}\int_0^p f(x+x_0)e^{-2\pi ik\frac{x}{p}}\,dx = -\frac{1}{p}\int_0^p f(x_0-x)e^{-2\pi ik\frac{x}{p}}\,dx$$
$$\frac{1}{p}\int_0^p f(x+x_0)e^{-2\pi ik\frac{(x+x_0)}{p}}e^{2\pi ik\frac{x_0}{p}}\,dx = -\frac{1}{p}\int_0^p f(x_0-x)e^{-2\pi i(-k)\frac{x_0-x}{p}}e^{-2\pi ik\frac{x_0}{p}}\,dx$$
$$e^{2\pi ik\frac{x_0}{p}}F[k] = -e^{-2\pi ik\frac{x_0}{p}}F[-k]$$
$$F[k] = -e^{-4\pi ik\frac{x_0}{p}}F[-k].$$

(g) $\int_0^p f(x)\,dx = 1$?

Solution: We know by equation (1.3) that,

$$F[k] = \frac{1}{p}\int_0^p f(x)e^{-2\pi ik\frac{x}{p}}\,dx$$

Taking $k=0$, we get

$$F[0] = \frac{1}{p}\int_0^p f(x)\,dx$$

But then,

$$\int_0^p f(x)\,dx = 1$$

So,

$$F[0] = \frac{1}{p}.$$

(h) $\int_0^p |f(x)|^2\,dx = 1$?

Solution: We have,

$$\frac{1}{p}\int_0^p f(x)\overline{f(x)}\,dx = 1$$
$$\frac{1}{p}\int_0^p f(x)\overline{\sum_{k=-\infty}^{\infty} F[k]e^{2\pi ik\frac{x}{p}}}\,dx = 1$$
$$\frac{1}{p}\int_0^p f(x)\sum_{k=-\infty}^{\infty} \overline{F[k]}e^{-2\pi ik\frac{x}{p}}\,dx = 1$$
$$p\sum_{k=-\infty}^{\infty} \overline{F[k]}\frac{1}{p}\int_0^p f(x)e^{-2\pi ik\frac{x}{p}}\,dx = 1$$
$$p\sum_{k=-\infty}^{\infty} F[k]\overline{F[k]} = 1$$
$$\sum_{k=-\infty}^{\infty} |F[k]|^2 = \frac{1}{p}.$$

(i) If you know that,

$$f(x) = \frac{1}{p}\int_0^p f(u)\Big[\frac{\sin\big[(2m+1)\pi\frac{(x-u)}{p}\big]}{\sin\big[\pi\frac{(x-u)}{p}\big]}\Big]\,du?$$

Solution: We have,

$$\begin{aligned} f(x) &= \frac{1}{p}\int_0^p f(u)\Big[\frac{\sin\big[(2m+1)\pi\frac{(x-u)}{p}\big]}{\sin\big[\pi\frac{(x-u)}{p}\big]}\Big]\,du \\ &= \frac{1}{p}\int_0^p f(u)D_n(x-u)\,du \\ &\text{where } D_n(x-u) \text{ is Dirichlet's Kernel (4.25)} \\ &= \frac{1}{p}(f * D_n)(x) \end{aligned}$$

Integrating on both sides from 0 to p with respect to x and then dividing by p, we have,

$$\frac{1}{p}\int_0^p f(x)e^{-2\pi ik\frac{x}{p}}\,dx = \frac{1}{p}\int_0^p (f * D_n)(x)e^{-2\pi ik\frac{x}{p}}\,dx.$$

Then, using Subsection 2.2.1, we get,

$$F[k] = pF[k]D_n[k] \quad \text{(where } D_n[k] \text{ is the FT of } D_n(x)\text{)}.$$

4.2 Direct Fourier Transform

4.2.1 Direct Summation

Some of the commonly used DFT can be found by evaluating the finite sum from the analysis equation,

$$F[k] = \frac{1}{N}\sum_{n=0}^{N-1} f[n]e^{\frac{-2\pi ikn}{N}}, \quad k = 0, 1, \ldots, N-1 \tag{4.30}$$

When $m = 1, 2, \ldots$ evenly divide N, we define the discrete comb,

$$c_m[n] := \begin{cases} 1 & \text{if } n = 0, \pm m, \pm 2m, \ldots \text{on } \mathbb{P}_N \\ 0 & \text{otherwise.} \end{cases} \tag{4.31}$$

We verify that,

$$f[n] := c_m[n] \text{ has the FT } F[k] = \frac{1}{m}C_{\frac{N}{m}}[k], \tag{4.32}$$

by,

$$\begin{aligned} G[k] &= \frac{1}{N}\sum_{n=0}^{N-1} C_m[n]e^{-2\pi ik\frac{n}{N}} \\ &= \frac{1}{N}\sum_{n,=0}^{m'-1} e^{-2\pi ik\frac{mn'}{mm'}}, \quad \text{where } m' = \frac{N}{m} \\ &= \begin{cases} \frac{m'}{N} & \text{if } k = 0, \pm m', \pm 2m', \ldots \\ 0 & \text{otherwise} \end{cases} \\ &= \frac{1}{m}C'_m[k]. \end{aligned}$$ [1]

4.2.2 Basic Rules

Linearity

If c_1, c_2, …, c_m be scalars. Then,

$$g[n] := c_1f_1[n] + \ldots + c_mf_m[n] \quad \text{has FT } G[k] = c_1F_1[k] + \ldots + c_mF_m[k]. \tag{4.33}$$

Reflection and Conjugation

The reflection rule is defined as,

$$g[n] := f[-n] \text{ has the FT } G[k] = F[-k]. \tag{4.34}$$

The conjugation rule is defined as,

$$g[n] := \overline{f[n]} \text{ has the FT } G[k] = \overline{F[-k]}. \tag{4.35}$$

Translation and Modulation

The translation rule (or shift rule) is defined as,

$$g[n] := f[n - n_0] \text{ has the FT } G[k] = e^{-2\pi ik\frac{n_0}{N}}F[k], \quad n_0 = 0, \pm 1, \ldots \tag{4.36}$$

The modulation rule (or transform shift rule) is defined as,

$$g[n] := e^{2\pi ik_0\frac{n}{N}}f[n] \text{ has the FT } G[k] = F[k - k_0], \quad k_0 = 0, \pm 1, \ldots \tag{4.37}$$

Convolution and Multiplication

The convolution rule is defined as,

$$g[n] := (f_1 * f_2)[n] \text{ has the FT } G[k] = NF_1[k]F_2[k]. \tag{4.38}$$

The multiplication rule is defined as,

$$g[n] := f_1[n]f_2[n] \text{ has the FT } G[k] = (F_1 * F_2)[k]. \tag{4.39}$$

Inversion

The inversion rule is defined as,

$$g[n] := F[k] \text{ has the FT } G[k] = \frac{1}{N}f[-k]. \tag{4.40}$$

4.2.3 Rules that link functions on $\mathbb{P}_N$ with functions on $\mathbb{P}_{\frac{N}{m}}$,$\mathbb{P}_{mN}$

Zero packing rule

The zero packing rule (or up-sampling rule)

$$g[n] := \begin{cases} f[\frac{n}{m}] & \text{if } n = 0, \pm m, \pm 2m, \ldots \\ 0 & \text{otherwise} \end{cases} \quad \text{(with } f \text{ on } \mathbb{P}_{\frac{N}{m}},\ g \text{ on } \mathbb{P}_N \text{) has FT } G[k] = \frac{1}{m}F[k]. \tag{4.41}$$

is verified by,

$$\begin{aligned}G[k] &:= \frac{1}{N}\sum_{n=0}^{N-1} g[n]e^{-2\pi ik\frac{n}{N}}\\ &= \frac{1}{N}\sum_{n'=0}^{\frac{N}{m}-1} f[n']e^{-2\pi ikm\frac{n'}{N}}\\ &= \frac{1}{m}F[k].\end{aligned}$$ [1]

Repeat rule

The repeat rule

$$g[n] := f[n] \quad (\text{with } f \text{ on } \mathbb{P}_{\frac{N}{m}},\, g \text{ on } \mathbb{P}_N\ ) \text{ has FT } G[k] = \begin{cases} F[\frac{k}{m}] & \text{if } k = 0, \pm m, \pm 2m, \dots \\ 0 & \text{otherwise.}\end{cases} \tag{4.42}$$

is verified by,

$$\begin{aligned}\sum_{k=0}^{N-1} G[k]e^{2\pi ik\frac{n}{N}} &= \sum_{k'=0}^{\frac{N}{m}-1} F[k']e^{2\pi ik'm\frac{n}{N}}\\ &= f[n] := g[n].\end{aligned}$$ [1]

Summation rule

The summation rule

$$g[n] := \sum_{l=0}^{m-1} f[n-lN] \quad (\text{with } f \text{ on } \mathbb{P}_{mN},\, g \text{ on } \mathbb{P}_N\ ) \text{ has FT } G[k] = mF[mk]. \tag{4.43}$$

is verified by,

$$\begin{aligned}G[k] &:= \frac{1}{N}\sum_{n=0}^{N-1}\Big(\sum_{l=0}^{M-1} f[n-lN]\Big)e^{-2\pi ik\frac{n}{N}}\\ &= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{l=0}^{M-1} f[n-lN]e^{-2\pi ik\frac{(n-lN)}{N}}\\ &= \frac{1}{N}\sum_{n'=0}^{mN-1} f[n']e^{-2\pi ik\frac{n'}{mN}}\\ &= mF[mk].\end{aligned}$$ [1]

Decimation rule

The decimation rule (or down-sampling rule or sampling rule)

$$g[n] := f[mn] \quad \text{(with } f \text{ on } \mathbb{P}_{mN},\ g \text{ on } \mathbb{P}_N \text{) has FT } G[k] = \sum_{l=0}^{m-1} F[k-lN]. \tag{4.44}$$

is verified by,

$$\begin{aligned}
\sum_{k=0}^{N-1} G[k])e^{2\pi ik\frac{n}{N}} &:= \sum_{k=0}^{N-1}\Big(\sum_{l=0}^{m-1} F[k-lN]\Big)e^{2\pi ik\frac{n}{N}} \\
&= \sum_{k=0}^{N-1}\sum_{l=0}^{m-1} F[k-lN]e^{2\pi i(k-lN)\frac{mn}{mN}} \\
&= \sum_{k'=0}^{mN-1} F[k']e^{2\pi ik'\frac{mn}{mN}} \\
&= f[mn] = g[n].
\end{aligned}$$ [1]

4.2.4 Example on how to compute DFT using zero packing and translation rule

Let (A, B, C, D), (E, F, G, H) be the discrete Fourier transforms of (a, b, c, d), (e, f, g, h), respectively. In this section, we make use of the zero packing and translation rule to find the discrete Fourier transform of

(a) (a,0,b,0,c,0,d,0);

Solution: Let $f = (a, b, c, d)$ and $g = (e, f, g, h)$. Then,

Let $h_1 = (a, 0, b, 0, c, 0, d, 0)$

Using the zero packing rule, define,

$$h_1[n] := \begin{cases} f[\frac{n}{2}] & \text{if } n = 0, \pm m, \pm 2m, \dots \\ 0 & \text{otherwise.} \end{cases}$$

(with f on $\mathbb{P}_4$, h_1 on $\mathbb{P}_8$) has FT

$$H_1[k] = \frac{1}{2}F[k] \qquad k = 0, 1, 2, \dots, 7$$

Then,

$$H_1[k] = \frac{1}{2}(A, B, C, D, A, B, C, D).$$

(b) (0,e,0,f,0,g,0,h);

Solution: Let $h_2 = (0, e, 0, f, 0, g, 0, h)$

Using the zero packing rule, define,

$$h_2[n] := \begin{cases} g[\frac{n}{2}] & \text{if } n = 0, \pm m, \pm 2m, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

(with g on $\mathbb{P}_4$, h_2 on $\mathbb{P}_8$) has FT

$$H_2[k] = \frac{1}{2}G[k], \qquad k = 0, 1, 2, \ldots, 7$$

Then,

$$H_2[k] = \frac{1}{2}(E, F, G, H, E, F, G, H).$$

(c) (a,e,b,f,c,g,d,h).

Solution: Let $h_3 = (a, e, b, f, c, g, d, h)$

Using the zero packing and translation rule, define,

$$h_3[n] := \begin{cases} f[\frac{n}{2}] + g[\frac{n}{2}] & \text{if } n = 0, \pm m, \pm 2m, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

(with $f + g$ on $\mathbb{P}_4$, h_3 on $\mathbb{P}_8$) has FT

$$H_3[k] = \frac{1}{2}(F[k] + G[k]), \qquad k = 0, 1, 2, \ldots, 7$$

Then,

$$H_3[k] = \frac{1}{2}(A + E, B + F, C + G, D + H, A + E, B + F, C + G, D + H).$$

4.2.5 Dilation

First consider the the case where dilation parameter $m = 1, 2, \ldots, N-1$ and $N = 2, 3, \ldots$ are relatively prime , i.e, $gcd(m, N) = 1$.

The P-dilation rule

$$g[n] := f[mn] \quad (gcd(m, N) = 1\ ) \text{ has FT } G[k] = F[m'k] \quad \text{where } mm' \equiv 1(mod \equiv N). \tag{4.45}$$

is verified by,

$$\begin{aligned} G[k] &= \frac{1}{N}\sum_{n=0}^{N-1} f[mn]e^{-2\pi ik\frac{n}{N}} \\ &= \frac{1}{N}\sum_{n=0}^{N-1} f[mn]e^{-2\pi i(km')\frac{mn}{N}} \\ &= \frac{1}{N}\sum_{n'=0}^{mN-1} f[n']e^{-2\pi i(km')\frac{n'}{N}} \\ &= F[m'k]. \end{aligned}$$

Next consider the the case where dilation parameter $m = 1, 2, \ldots, N$ is a divisor of N. Then, the D-dilation rule

$$g[n] := f[mn] \quad \text{(with m—N) has FT } G[k] = \begin{cases} \sum_{l=0}^{m-1} F[\frac{k}{m} - l\frac{N}{m}] & \text{if } m|k \\ 0 & \text{otherwise.} \end{cases} \tag{4.46}$$

is verified by,

$$\begin{aligned} \sum_{k=0}^{N-1} G[k]e^{2\pi ik\frac{n}{N}} &:= \sum_{k'=0}^{\frac{N}{m}-1}\Big(\sum_{l=0}^{m-1} F[k' - l\frac{N}{m}]\Big)e^{2\pi ik'm\frac{n}{N}} \\ &= \sum_{k'=0}^{\frac{N}{m}-1}\sum_{l=0}^{m-1} F[k' - l\frac{N}{m}]e^{2\pi i(k'-l\frac{N}{m})\frac{mn}{N}} \\ &= \sum_{k=0}^{N-1} F[k]e^{2\pi ik\frac{mn}{N}} \\ &= f[mn] = g[n]. \end{aligned}$$ [1]

4.2.6 Poisson Relations

Suppose first that f is absolutely summable function on $\mathbb{Z}$ with the FT F on $\mathbb{T}_p$.

Then, we can use the summation rule

$$g[n] := \sum_{m=-\infty}^{\infty} f[n - mN] \quad (\text{with } f \text{ in } \mathbb{Z},\, g \text{ in } \mathbb{P}_N\ ) \text{ has FT } G[k] = \frac{P}{N} f\Big(k\frac{P}{N}\Big). \tag{4.47}$$

by writing,

$$\begin{aligned}
G[k] &= \frac{1}{N} \sum_{n=0}^{N-1} g[n] e^{-2\pi i k \frac{n}{N}} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=-\infty}^{\infty} f[n - mN] e^{-2\pi i k \frac{n}{N}} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=-\infty}^{\infty} f[n'] e^{-2\pi i k \frac{n'}{N}} \\
&= \frac{P}{N} F\Big[k\frac{P}{N}\Big].
\end{aligned}$$

Let $f \epsilon \mathbb{T}_p$ that has absolutely summable Fourier coefficients $F[k]$. Then, the sampling rule is,

$$g[n] := f\Big[n\frac{P}{N}\Big] \quad (\text{with } f \text{ in } \mathbb{T}_p,\, g \text{ in } \mathbb{P}_N\ ) \text{ has FT } G[k] = \sum_{m=-\infty}^{\infty} F[k - mN]. \tag{4.48}$$

Suppose now that f is a smooth function on $\mathbb{R}$ with small regular tails. The function f and its FT are then linked by Poisson sum formula, and we can use sample-sum rule,

$$g[n] := \sum_{m=-\infty}^{\infty} f\Big[n\frac{n - mN}{\sqrt{N}}\Big] \quad (\text{with } f \text{ in } \mathbb{R},\, g \text{ in } \mathbb{P}_N\ ) \text{ has FT}$$

$$G[k] = \frac{1}{|a|\sqrt{N}} \sum_{m=-\infty}^{\infty} F\Big[\frac{1}{a} - \frac{k - mN}{\sqrt{N}}\Big]. \tag{4.49}$$

Here, a is a nonzero real dilation parameter.[1]

Chapter 5

Applications of Fourier Analysis

5.1 Heat Flow

Consider an idealized long thin rod, with the assumption that heat energy neither enters nor leaves the rod and that it is neither created nor destroyed.

Let $T(x,t)$ be the heat at time t at the point x in the rod. The heat equation in one dimension is derived from two physical laws, Fourier's Law and the Conservation of energy.

Fourier's law states that the heat T is transported in the direction opposite to the temperature gradient of u and is proportional to it, i.e.,

$$T(x,t) = -\kappa \nabla u, \tag{5.1}$$

where κ is the proportionality constant also known as the *thermal conductivity*.

Also, the heat T is related to the mass m and temperature u via the following formula,

$$T(x,t) = \lambda m u(x,t). \tag{5.2}$$

where λ is known as the specific heat.

Let us now consider an infinitesimal piece from the rod with length $[x, x+\Delta x]$. Then, if the rod has a cross section A, then this piece has volume $A\Delta x$. Further assuming that the density of the material is ρ, then the infinitesimal mass of our infinitesimal volume element is simply given by

$$\Delta m = \rho A \Delta x, \tag{5.3}$$

Therefore, based on (5.1), the equivalent heat for our volume element is described by,

$$T(x,t) = \lambda m u = \lambda \rho S \Delta x u. \tag{5.4}$$

Also, we know that rate of change of heat is given by the difference in rate of heat flowing in and rate of heat flowing out.

Mathematically written as,

$$\frac{\partial T}{\partial t} = T_{in}(x,t)S - T_{out}(x+\Delta x, t)S. \tag{5.5}$$

Rate of change of heat can also be found by differentiating (5.4),

$$\frac{\partial T}{\partial t} = \lambda \rho A \Delta x \frac{\partial u}{\partial t}. \tag{5.6}$$

Substituting (5.6) in (5.5),

$$\lambda \rho A \Delta x \frac{\partial u}{\partial t} = A[T(x,t) - T(x+\Delta x, t)].$$

Dividing by Δx and A,

$$\lambda \rho \frac{\partial u}{\partial t} = -\left[\frac{T_{out}(x+\Delta x, t) - T_{in}(x,t)}{\Delta x}\right].$$

If $\Delta x \to 0$, then,

$$\lambda \rho \frac{\partial u}{\partial t} = -\frac{\partial T}{\partial x}.$$

Finally using Fourier law of Heat condition (5.1) in one dimension,

$$T = -\kappa \frac{\partial u}{\partial x},$$

we have,

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2},$$

where $a^2 = \frac{\kappa}{\lambda \rho}$ is the *thermal diffusivity* constant.[6,7]

Fourier used the equations (1.3)- (1.4) and (1.1)- (1.2) to solve problems involving the flow of heat in solids. He showed that the temperature $u(x,t)$ at times $t \geq 0$ and coordinate x along a thin insulated rod of uniform cross section is a solution of the diffusion equation

$$u_t(x,t) = a^2 u_{xx}(x,t) \quad [1] \tag{5.7}$$

Let f be a generalized function on $\mathbb{R}$. If the solution of the equation (5.7) exists that has the initial temperature,

$$u(x,0) = f(x) \tag{5.8}$$

then, we can Fourier transform (5.7) and (5.8) to obtain,

$$U_t(s,t) = a^2(2\pi is)^2 U(s,t), \qquad U(s,0) = F(s).$$

Using the fact that the initial value problem

$$y'(t) + \alpha y(t) = 0, \qquad y(0) = A$$

has the solution

$$y(t) = Ae^{-\alpha t}, \qquad t \geq 0,$$

we then write,

$$U(s,t) := e^{-4\pi^2 a^2 s^2 t} F(s), \qquad t \geq 0. \tag{5.9}$$

The Gaussian factor

$$K(s,t) := e^{-4\pi^2 a^2 s^2 t}, \qquad t \geq 0$$

is the Fourier transform of the *diffusion kernel*

$$k(x,t) := \begin{cases} \delta(x) & \text{if } t = 0 \\ \frac{1}{\sqrt{4\pi a^2 t}} e^{-\frac{x^2}{4a^2 t}} & \text{if } t > 0 \end{cases}$$

Then,

$$U(s,t) = K(s,t)F(s),$$

is the Fourier Transform of

$$u(x,t) = k(x,t) * f(x)$$ [1]

Fourier observed that the function

$$e^{2\pi isx} e^{-4\pi^2 a^2 s^2 t},$$

satisfies the equation (5.7) for every choice of the real parameter s.

For the temperature in a rod, Fourier wrote

$$u(x,t) = \int_{-\infty}^{\infty} A(s) e^{2\pi i s x} e^{-4\pi^2 a^2 s^2 t}\, ds.$$

with the intention of choosing the amplitude function $A(s)$, $\quad -\infty < s < \infty$, to make his formula agree with the known initial temperature $u(x,0)$ at time $t = 0$, i.e., to make

$$u(x,0) = \int_{-\infty}^{\infty} A(s) e^{2\pi i s x}\, ds. \tag{5.10}$$

This identity is actually the synthesis equation (1.2) for the function $u(x,0)$ so we can use the corresponding analysis equation (1.1) to write,

$$A(s) = \int_{-\infty}^{\infty} u(x,0) e^{-2\pi i s x}\, dx, \qquad\qquad -\infty < s < \infty,$$

thereby expressing A in terms of the initial temperature. In this way Fourier solved the heat flow problem for a doubly infinite rod.[1]

For the temperature in a *ring* of circumference $p > 0$, Fourier used the *p-periodic* solutions

$$e^{2\pi i k x/p} e^{-4\pi^2 a^2 k/p^2 t}, \qquad\qquad k = 0, \pm 1, \pm 2, \ldots$$

of the diffusion equation (with $s = k/p$)to write

$$u(x,t) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x/p} e^{-4\pi^2 a^2 k/p^2 t},$$

with the intention of choosing the coefficients $c_k, k = 0, \pm 1, \pm 2, \ldots$ to make

$$u(x,0) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x/p}.$$

This identity is actually the synthesis equation (1.4) for the function $u(x,0)$ so we can use the corresponding analysis equation (1.3) to write,

$$c_k = \frac{1}{p}\int_0^p u(x,0) e^{-2\pi i k x/p}\, dx, \qquad\qquad k = 0, \pm 1, \pm 2, \ldots$$

in terms of the known initial temperature. In this way Fourier solved the heat flow problem for a ring.[1]

5.2 Example of Heat Flow

In order to illustrate the concepts from the above section 5.1, we will show how to find the temperature function for box function.

Fourier derived the formula

$$u(x,t) = \int_{-\infty}^{\infty} A(s) e^{2\pi i s x} e^{-4\pi^2 a^2 s^2 t}\, ds.$$

for the temperature at the point x, $-\infty < x < \infty$, at times $t \geq 0$ along an infinite one dimensional rod with thermal diffusivity a^2. Suppose that when $t < 0$, the rod is held at the uniform temperature $u = 0$. At time $t = 0$, that portion of the rod from $x = -\frac{1}{2}$ to $x = +\frac{1}{2}$ is instantaneously heated to the temperature $u = 100$ thereby producing the initial temperature

$$\int_{-\infty}^{\infty} A(s) e^{2\pi i s x}\, ds = u(x,t) := \begin{cases} 100 & \text{if } |x| < \frac{1}{2} \\ 0 & \text{if } |x| > \frac{1}{2}. \end{cases}$$

Use the analysis equation (1.1) together with the Fourier transform pair of Exercise 1.6 of Chapter 1, *'A First Course in Fourier Analysis'* by David.W.Kammler to find A and thereby produce a formula for $u(x,t)$.

Solution: Using analysis equation (1.1), we have,

$$\begin{aligned}
A(s) &= \int_{-\infty}^{\infty} u(x,0+) e^{-2\pi i s x}\, dx, -\infty < s < \infty \\
&= \int_{-\infty}^{-\frac{1}{2}} u(x,0+) e^{-2\pi i s x}\, dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} u(x,0+) e^{-2\pi i s x}\, dx + \int_{\frac{1}{2}}^{\infty} u(x,0+) e^{-2\pi i s x}\, dx \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} 100 e^{-2\pi i s x}\, dx \\
&= 100 \left[\frac{e^{-2\pi i s x}}{-2\pi i s} \right]_{-\frac{1}{2}}^{\frac{1}{2}} \\
&= \frac{100}{\pi s} \left[\frac{e^{\pi i s} - e^{-\pi i s}}{2i} \right] \\
&= \left(\frac{100}{\pi s} \right) \sin(\pi s) \\
&= \begin{cases} 100 & \text{if } s = 0 \\ 100 \frac{\sin(\pi s)}{\pi s} & \text{if } s \neq 0 \end{cases}
\end{aligned}$$

Then,

$$
\begin{aligned}
u(x,t) &= \int_{-\infty}^{\infty} A(s) e^{2\pi i s x} e^{-4\pi^2 a^2 s^2 t}\, ds \\
&= \int_{-\infty}^{\infty} 100\Big(\frac{\sin(\pi s)}{\pi s}\Big) e^{2\pi i s x} e^{-4\pi^2 a^2 s^2 t}\, ds \\
&= \int_{-\infty}^{\infty} 100\Big(\frac{\sin(\pi s)}{\pi s}\Big) [\cos(2\pi s x) + i \sin(2\pi s x)] e^{-4\pi^2 a^2 s^2 t}\, ds
\end{aligned}
$$

Using odd and even functions,

$$
\begin{aligned}
u(x,t) &= 2\int_{0}^{\infty} 100\Big(\frac{\sin(\pi s)}{\pi s}\Big) e^{-4\pi^2 a^2 s^2 t} \cos(2\pi s x)\, ds \\
&= 200\int_{0}^{\infty} \Big(\frac{\sin(\pi s)}{\pi s}\Big) e^{-4\pi^2 a^2 s^2 t} \cos(2\pi s x)\, ds.
\end{aligned}
$$

5.3 Convolution Factor as Filters

In Fourier Analysis, convolution is basically a means of filtering. Consider a signal x given by figure 5.1. Then, the Fourier Transform of x is given by X as shown in figure 5.2.

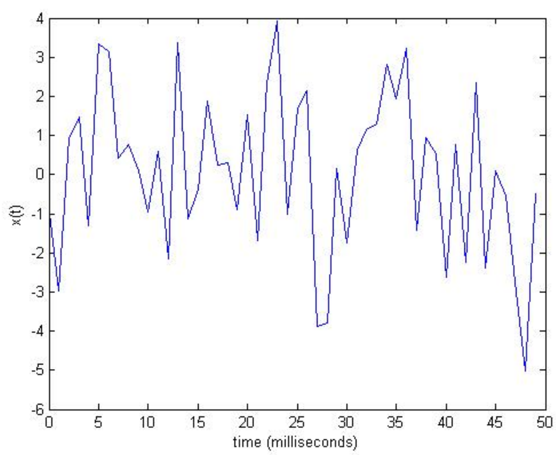

Figure 5.1: x(t)=Original Signal

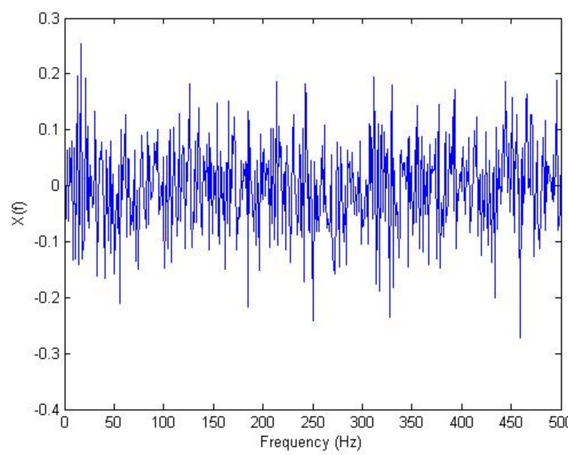

Figure 5.2: $|X(f)|$=Fourier Transform of the Signal x(t)

Suppose we want to keep the components of $x(t)$ that vibrate at frequencies between 75 and 125Hz. Let H be the Fourier transform of a signal h such that H is 1 for frequencies between 75 and 125Hz and zero at other frequencies. The signals h and H are shown in figure 5.3 and figure 5.4 respectively.

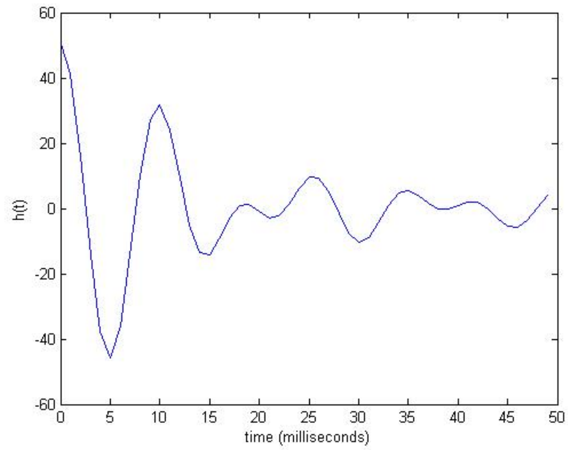

Figure 5.3: h(t)= Band Pass Filter

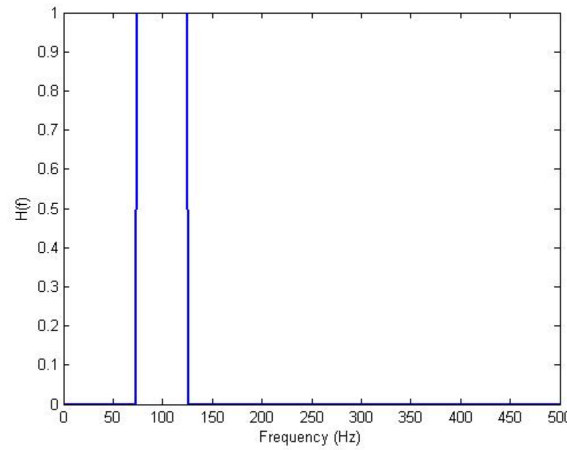

Figure 5.4: H(f)=Fourier Transform of the Band Pass Filter h(t)

The product $Y(f)$ of $|X(f)|H(f)$ is given by figure 5.6. Taking the inverse of this product gives the convolution of x and h i.e., $y = x * h$ as given by 5.5.

Here, y is the desired filtered version of x and h is called a Band-Pass Filter.

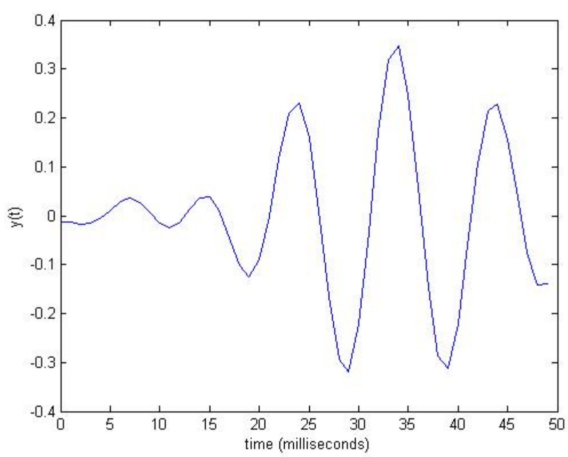

Figure 5.5: y(t)=Convoution of Signals x(t) and h(t)

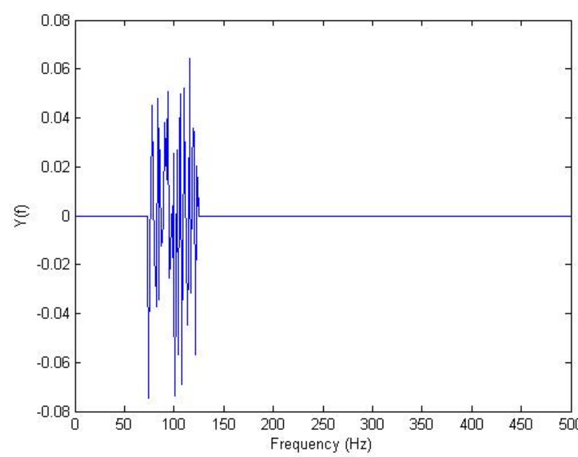

Figure 5.6: Y(f)=Product of $|X(f)|$ and H(f)

Now, consider G to be the Fourier transform of a function g, G being 1 only for frequencies between -50 and 50 Hz and zero at other frequencies. As in above case, if we take the product for X and G and then take the inverse of that product, the resulting figure again is a smooth component of x that "sounds" at frequencies between -50 and 50. g is called a Low-Pass Filter.

5.4 Convolution and Correlation

Consider two finite functions $f[n]$ and $g[n]$. By the Cauchy-Schwartz inequality, we have,

$$\left|\sum f[n]g[n]\right| \leq \left(\sum f[n]^2\right)^{\frac{1}{2}} \left(\sum g[n]^2\right)^{\frac{1}{2}}$$

with equality if and only if $g = f$ or $g = -f$. Suppose that,

$$\sum f[n]^2 = \sum g[n]^2 = 1 \tag{5.11}$$

Then,

$$-1 \leq \sum f[n]g[n] \leq 1.$$

If the equalities in (5.11) are not met, we normalize f and g as follows

$$\acute{f}[n] = \frac{f[n]}{(\sum f[n]^2)^{1/2}}$$

$$\sum \acute{f}[n]^2 = \sum \frac{f[n]^2}{(\sum f[n]^2)} = \frac{\sum f[n]^2}{\sum f[n]^2} = 1$$

Here $C(f, g) = \sum f[n]g[n]$ is the correlation between f and g.

If $C(f, g) = 1$, then $f = g$.

If $C(f, g) = -1$, then $f = -g$.

If $C(f, g) = 0$, then f is orthogonal to g.

Then,

$$(f * \tilde{g})(x) = \sum f[n]g[n - x] = C(f, g(\cdot - x)).$$

where we set $\tilde{g}[n] = g[-n]$ and x is the 'shifting' parameter.

Similarly, consider two suitably regular functions f, g be on $\mathbb{R}$. Then,

$$(f \star g)(x) = \int_{-\infty}^{\infty} \overline{f(u)} g(u + x)\, du$$

Using change of variable $u = -v$, we have,

$$(f \star g)(x) = \int_{-\infty}^{\infty} \overline{f(-v)} g(x-v)\, dv = (\tilde{f} * g)(x). \tag{5.12}$$

Here $\tilde{f}$ is the hermitian conjugate of f i.e.,

$$\tilde{f}(x) = \overline{f(-x)} \quad \text{for } f \text{ on } \mathbb{R} \text{ or } \mathbb{T}_p,$$
$$\tilde{f}[n] = \overline{f[-n]} \quad \text{for } f \text{ on } \mathbb{Z} \text{ or } \mathbb{P}_N.$$

This relation (5.12) works for any function defined on $\mathbb{R}$, $\mathbb{T}_p$, $\mathbb{Z}$ and $\mathbb{P}_N$.

Since, $\tilde{\tilde{f}} = f$, we have,

$$f * g = (\tilde{\tilde{f}} * g) = \tilde{f} \star g. \tag{5.13}$$

The relations given by equations (5.12) and (5.13) is taken in reference to Chapter 2 of 'A First Course in Fourier Analysis' by David.W.Kammler, pg 91.

5.5 Fourier Analytical Method for Noise Reduction

Suppose that a noisy time series/signal is given by Figure 5.7.

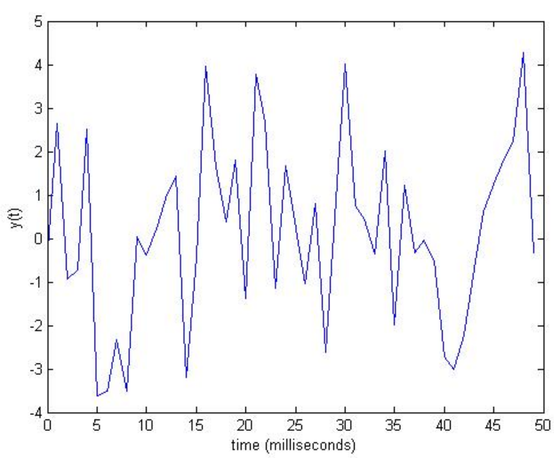

Figure 5.7: y(t)=Signal Corrupted with Zero-Mean Random Noise

Under the assumption that the noisy time series above is representing an oscillatory periodic phenomenon, we use Fourier Analysis to remove noise from it. Taking Fourier Transform, we obtain Figure 5.8.

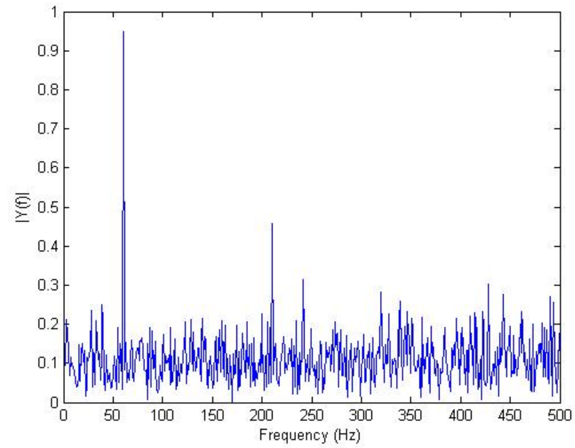

Figure 5.8: $|Y(t)|$=Single-Sided Amplitude Spectrum of y(t)

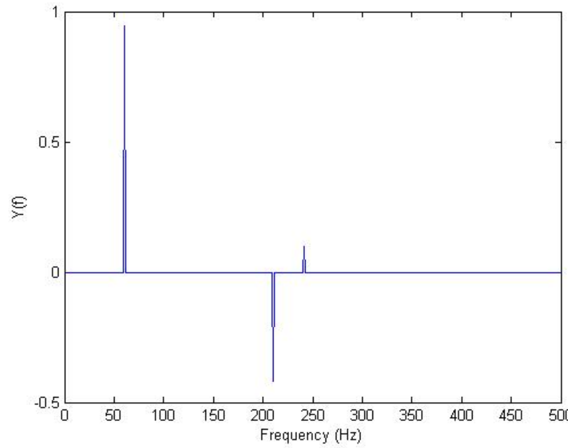

Figure 5.9: Y1(t)=Decimated Single-Sided Amplitude Spectrum of y(t)

We see two predominant frequencies, namely 60 and 210 Hz. There are also many contributions from other frequencies which encode the noise (in particular the higher frequencies).

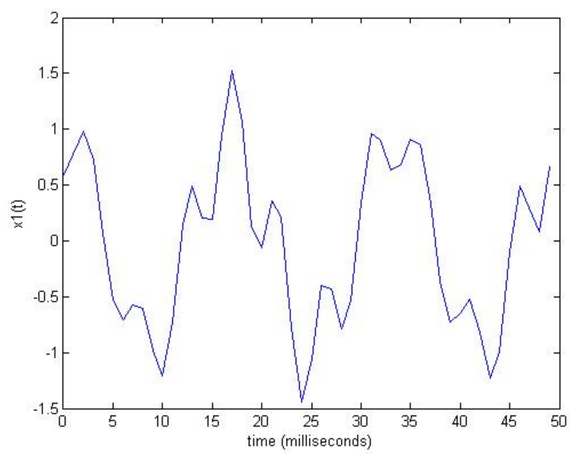

Figure 5.10: x1(t)=Signal reconstructed by using ifft of Y1

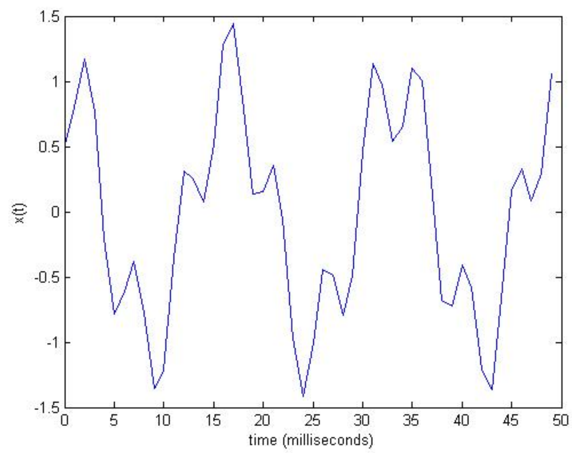

Figure 5.11: Original Signal x(t)=$\cos(2\pi 60t) - 0.5\cos(2\pi 210t)$

To get rid of most of the frequencies we use threshold or decimation method. We fix a T and say, every frequency with intensity less than T will be dropped (thresholding) or given a parameter N we can keep only the N frequencies with biggest intensities (decimation). For sake of illustration, consider the decimated signal, given by Figure 5.9.

Thus, only the predominant frequencies are retained. The good news is that this map of frequencies can now be taken back to the "time series side" by the means of the inverse Fourier Transform. This yields Figure 5.10.

This is a de-noised version of the original signal y(t). To verify how effective this method is, let us say that the signal y(t) was the function x(t)=$\cos(2\pi 60t) - 0.5\cos(2\pi 210t)$ plus some MATLAB generated random noise. The function x(t) looks as given by Figure 5.11.

This is very similar to the reconstructed signal x1 above. In fact, their correlation is illustrated by the following Figure 5.12.

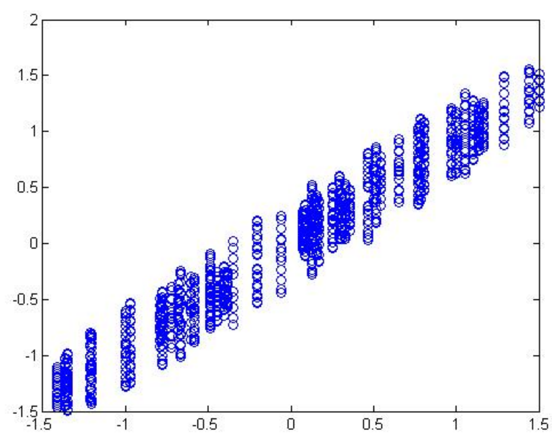

Figure 5.12: Correlation between the signals x(t) and x1(t)

Notice that when coming up with the de-noised signal x1(t), we knew nothing about the existence of the "underlying" function x(t).

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