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AN APPLICATION OF CUBICAL COHOMOLOGY TO ADINKRAS AND SUPERSYMMETRY REPRESENTATIONS

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Abstract. An Adinkra is a class of graphs with certain signs marking its vertices and edges, which encodes off-shell representations of the super Poincaré algebra. The markings on the vertices and edges of an Adinkra are cochains for cubical cohomology. This article explores the cubical cohomology of Adinkras, treating these markings analogously to characteristic classes on smooth manifolds.

1. Introduction

Supersymmetry posits that the physical fields in nature form a representation of an algebra called the N-extended super Poincaré algebra, which is defined given two positive integers N and D (the number D represents the dimension of space-time). [26]

In recent years it has become apparent that even for $D = 1$, the representation theory is surprisingly intricate. [1, 2, 11, 12, 13, 17, 21, 22, 23, 24] In 2004, M. Faux and S. J. Gates introduced a graphical diagram called an *Adinkra* to describe some commonly studied representations of the super Poincaré algebra in $D = 1$ dimensions. [11] These diagrams are directed graphs, where the vertices and edges have certain markings and colorings. An Adinkra is similar in spirit to the concept of a Cayley graph or a Schreier graph in combinatorial group theory (see, for instance, Sec. 1.6 of [20]). These Adinkra diagrams have broadened our knowledge of the representation theory of the super Poincaré algebra in one dimension. [4, 5, 6, 7, 9, 18, 25]

In this paper, we will show how a certain combinatorial cohomology theory on the Adinkra, called cubical cohomology, can be used to determine which kinds of markings are possible.

1.1. Marked Graphs and Adinkras.

To define an Adinkra, it is first necessary to describe the kinds of markings to be put on vertices and edges.

A directed graph consists of V, a finite set of vertices, and a set of directed edges $E \subset V \times V$. If e is an edge, and $e = (v, w)$, then we say e points from v to w, and e is incident to both v and w.

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A *bipartition* is a partition of V into $V_0 \sqcup V_1$, so that each edge of the graph is incident with one element of V_0 and one element of V_1. The elements of V_0 are called *bosons* and the elements of V_1 are called *fermions*. We draw the elements of V_0 as empty circles, and the elements of V_1 as filled circles. A graph together with a bipartition is called a *bipartite graph*.

Fix a positive integer N. We choose N colors, one for each of the numbers in $\{1, \ldots, N\}$. An *edge coloring* is a mapping $\chi : E \to \{1, \ldots, N\}$. We draw each edge $e \in E$ using the color corresponding to $\chi(e)$.

A *dashing* is a mapping $\mu : E \to \{0, 1\}$. The edge $e \in E$ is drawn with a solid line if $\mu(e) = 0$ and drawn with a dashed line if $\mu(e) = 1$.[1]

In this paper, we use the term *marked graph* to mean a directed graph, together with a bipartition, an edge coloring, and a dashing. An example of a marked graph is Example 1.1.

Given a marked graph and a subset $J \subset \{1, \ldots, N\}$, we can take the subgraph given by the same vertex set, but for the edge set take $E' = \chi^{-1}(J)$; that is, the edges whose colors are in J. The bipartition of the vertices will be the same and the colorings and dashing of the edges will be restricted to E'. Connected components of this marked subgraph will be called J-color-faces of the marked graph. If k is an integer with $0 \leq k \leq N$, a k-face is a J-color-face where $\#J = k$ (the cardinality of a set J is denoted by $\#J$).

A *circuit* is a connected marked graph where each vertex is incident to exactly two edges. A *quadrilateral* is a circuit with 4 vertices.

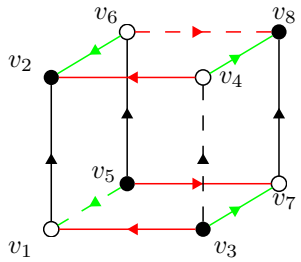
An *Adinkra* is a marked bipartite graph with the following properties: [11]

(1) (Color-regular) For every vertex v and color $j \in \{1, \ldots, N\}$, there is a unique edge incident with v of color j.
(2) (Square) Every 2-face is a quadrilateral.
(3) (Non-escheric) Suppose f is a 2-face. In traversing the boundary of f in either direction, the same number of edges are oriented along the direction of the path as are oriented against the path.
(4) (Odd) Every 2-face has an odd number of dashed edges.[2]

Example 1.1. *Consider the following Adinkra for $N = 3$:*

[1]This is equivalent to the notion of a *signed graph*, defined by Frank Harary, [14] where $+1$ and -1 are used instead of 0 and 1. The difference is that we will use additive notation instead of multiplicative notation for the group of order 2. The relationship of dashed edges with the supersymmetry representation actually suggests the multiplicative notation, and it may be that future work will motivate a switch to this notation, but since the purpose of this paper is to establish the relationship to cohomology, the additive notation will be used.

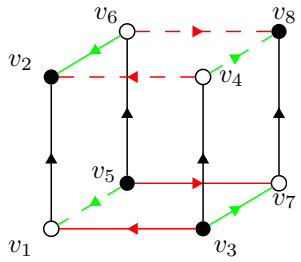
[2]In the language of Harary, [14] every 2-face is negative.



More generally, for all N, the 1-skeleton of an N-dimensional cube is the underlying graph of some Adinkra. In fact, all connected Adinkras are obtained by quotients of these cubes. [9]

Given an Adinkra, and a vertex v in the Adinkra, a *vertex switch* produces another Adinkra identical to the first, except that every edge incident with v changes dashedness; i.e., if it was dashed, it becomes solid, and vice versa. A vertex switch does not change the oddness of the dashing since every 2-face either does not contain the vertex being switched (in which case nothing happens to the edges in it) or it does, in which case two edges are switched. Therefore, the result is an Adinkra.

Example 1.2. *If we do a vertex switch on v_4 in the previous example, we get the following Adinkra. Note that each 2-face still has an odd number of dashed edges.*



1.2. **Purpose of this paper.** The purpose of this paper is to characterize the odd dashings on an Adinkra, up to vertex switches. The machinery to understand this is cubical cohomology, which is a combinatorial counterpart to simplicial cohomology, but using cubes instead of simplices.

This paper fits into a program to classify all Adinkras, as follows. Since the disjoint union of Adinkras is an Adinkra, it suffices to describe connected Adinkras. If we ignore the direction of the edges and the dashings, so that we are considering only the underlying undirected graph with the edge colorings, then these are obtained by taking the quotient of a cube by a binary doubly-even code. [9] The ways of assigning the directions of the edges were explored in the "Hanging Gardens" theorem. [4] This paper solves the problem of identifying the ways in which the dashings may be chosen, up to vertex switches.

1.3. **Review of Codes.** In this paper all codes will be binary linear block codes of length N. That is, they are linear subspaces of ${\mathbb{Z}_2}^N$ as vector spaces over $\mathbb{Z}_2$. We review a few basic concepts in the theory of codes to establish notation and

conventions. The interested reader can find a thorough introduction to the theory of codes in [15].

If $C \subset {\mathbb{Z}_2}^N$ is a code, then elements of C are called *codewords*. If a codeword $w = (w_1, \ldots, w_N) \in {\mathbb{Z}_2}^N$, the *weight* of w, denoted $\mathrm{wt}(w)$, is the number of j with $w_j = 1$. An *even* code is one where all codewords have even weights. A *doubly even* code is one where all codewords have weights divisible by 4.

A basis for C is called a *generating set*. A generating set is not unique, but all generating sets have the same cardinality: the *dimension* of the code, denoted as k, which is the dimension of C as a vector space. If a code is a binary linear block code of length N of dimension k, it is called an $[N, k]$ code.

A generating set is often arranged as rows of a matrix, called the *generating matrix*. Given two generating matrices for C, there is a sequence of row operations taking one generating matrix to another.

2. Cubical cohomology

This paper uses a kind of combinatorial cohomology, similar to simplicial cohomology, but using cubes instead of simplices. This kind of homology/cohomology theory was introduced by Kan. [16] This description is simplified here because we do not use degeneracy maps, and because we are using $\mathbb{Z}_2$ coefficients and hence, we ignore orientation issues.

Definition 2.1. *A cubical complex is a sequence of sets $X_0, X_1, X_2, \ldots$, with maps*

$$\partial_m^\alpha : X_k \to X_{k-1}$$

for integers $k \geq 1$, $1 \leq m \leq k$, and $0 \leq \alpha \leq 1$, so that for $e \in X_k$, and $m < n \leq k$, there is a (multi-)set equality

$$\left\{\partial_m^\alpha \partial_n^\beta(e) \,|\, \alpha, \beta \in \{0,1\}\right\} = \left\{\partial_{n-1}^\beta \partial_m^\alpha(e) \,|\, \alpha, \beta \in \{0,1\}\right\}. \tag{1}$$

In the applications for this paper, we expect $\partial_m^0\partial_n^0(e)$, $\partial_m^0\partial_n^1(e)$, $\partial_m^1\partial_n^0(e)$, and $\partial_m^1\partial_n^1(e)$ to be all different, but if they are not, the multiplicity of each element is assumed to be preserved in the above multiset equality (1).

Example 2.1. *The N-dimensional cube (also called an N-cube) is the space $I^N = [0,1]^N$. It is a CW complex with 3^N different cells, each of the form $\prod_{i=1}^N Y_i$, where Y_i is either $\{0\}$, $\{1\}$, or $[0,1]$. The dimension of the cell is the number of factors Y_i that are equal to $[0,1]$. The set X_k is then the set of all cells of I^N of dimension k. Given a k-cell f in this CW complex, we can define $\partial_m^\alpha f$, where $m \in \{1, \ldots, k\}$ and $\alpha \in \{0,1\}$, to be the $k-1$-cell obtained by replacing the mth occurence of $[0,1]$ in the product with $\{\alpha\}$. Then (1) follows, and I^N defines a cubical complex.*

Let X be a cubical complex. For each integer k define the group of k-chains $C_k(X)$ to be the free $\mathbb{Z}_2$-vector space spanned by X_k. If $\sigma \in X_k$, we define

$$\partial\sigma = \sum_{n=1}^{k} \sum_{\alpha=0}^{1} \partial_n^\alpha(\sigma).$$

This operator is extended linearly to all of $C_k(X)$ for each k (for $\sigma \in C_0(X)$ we define $\partial\sigma = 0$). It is straightforward to show that $\partial\partial\sigma = 0$ for all σ.

We define the cubical homology of X, $H_k(X)$, to be the homology of this chain complex. Likewise, we define the cochain groups $C^k(X) = \mathrm{Hom}(C_k(X), \mathbb{Z}_2)$, define $d : C^k(X) \to C^{k+1}(X)$ so that $d\alpha(\sigma) = \alpha(\partial\sigma)$, and define the cubical cohomology of X, $H^k(X)$, to be the cohomology of this cochain complex.

In fact, these cubical homology groups and cohomology groups are isomorphic to the ordinary singular homology and cohomology groups of their geometric realizations, as can be proven using a spectral sequence using the filtration given by the skeleta of the cubical sets, as is commonly done with CW complexes.

Given a finite cubical complex X, we define the following special cocycles $\tilde{\omega}_k \in C^k(X)$: if f is any k-cell, let $\tilde{\omega}_k(f) = 1$. The fact that $d\tilde{\omega}_k = 0$ follows from the fact that every $k+1$ cell has an even number of k cells in its boundary. Let $w_k = [\tilde{\omega}_k] \in H^k(X)$ be its cohomology class.

Given a graph, we can construct a cubical complex X where X_0 is the set of vertices and X_1 is the set of edges. The boundary operators ∂_1^α send an edge e to the two vertices incident to e. Which is $\partial_1^0(e)$ and which is $\partial_1^1(e)$ is an arbitrary choice which must be determined beforehand, but will be otherwise unimportant in what follows.

If this graph is marked with an edge coloring that is color-regular and square (as in Section 1.1, we can add the 2-faces as elements of X_2. We choose an ordering of the colors, and given a $\{i,j\}$-face f, with $i < j$, we define $\partial_1^\alpha(f)$ to be the two edges of color i, and $\partial_2^\alpha(f)$ to be the two edges of color j in f. Again, which is $\alpha = 0$ and which is $\alpha = 1$ can be chosen arbitrarily.

More generally, given an Adinkra A, there is a cubical complex where X_k is the set of k-faces. If J is a set of colors, and f is a J-color-face, and if j is the ith color in J, then by looking at subgraphs obtained by deleting the edges of color j, we end up with either one or two $J - \{j\}$-color-faces. We define $\partial_i^0(f)$ and $\partial_i^1(f)$ to be these two $J-\{j\}$-color-faces if they are different, and to be equal to the one $J - \{j\}$-color-face if there is only one. Again, which is $\partial_i^0(f)$ and which is $\partial_i^1(f)$ is an arbitrary choice.

3. Dashed edges

If X is a cubical complex, a dashing on X can be recorded by a 1-cochain $\mu \in C^1(X)$ that assigns 1 to every dashed edge and 0 to every solid edge. A vertex switch on a set of vertices can be recorded as a 0-cochain that assigns 1 to a vertex if it is in the set and 0 otherwise.

Proposition 3.1. *The dashing μ is odd if and only if $d\mu = \tilde{\omega}_2$. If $\overline{\mu}$ is a dashing that is obtained from doing a vertex switch T to μ, then $\overline{\mu} = \mu + dT$.*

Proof. For μ to be odd means for every 2-cell f, $d\mu(f) = \mu(\partial f) = 1$.

A vertex switch T to μ is obtained by changing the dashing on an edge if and only if exactly one vertex incident to it is in the set. The 1-cocycle dT is 1 on every edge that has one vertex incident to an element of the set. □

Theorem 3.2. *Let X be a cubical complex. There exists an odd dashing on X if and only if $w_2 = 0$. The set of such is an affine space modeled on the set of 1-cocycles on X. Modulo vertex switchings, this is an affine space modeled $H^1(X)$.*

Proof. Since $d\tilde{\omega}_2 = 0$, the existence of a μ as given in Proposition 3.1 is equivalent to whether or not $w_2 = 0$ in $H^2(X)$.

If μ_0 is an odd dashing, then by Proposition 3.1, μ is an odd dashing if and only if $d\mu = d\mu_0 = \tilde{\omega}_2$, which is equivalent to saying that μ is μ_0 plus a 1-cocycle.

The second part of Proposition 3.1 shows that modulo vertex switching is precisely the quotient by 1-coboundaries. □

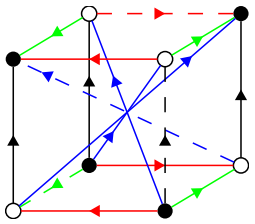
Remark: This is remarkably similar to the situation with spin structures on manifolds: the existence of a spin structure on an orientable manifold M is determined by the second Stiefel–Whitney class of the tangent bundle $w_2(TM) \in H^2(M; \mathbb{Z}_2)$. A necessary and sufficient condition for spin structures to exist is $w_2(TM) = 0$, and the difference of two such spin structures is characterized by an element in $H^1(M; \mathbb{Z}_2)$, so that the set of spin structures is in bijection with $H^1(M; \mathbb{Z}_2)$. As it turns out, there is a relationship, explained in [10], which uses the work of Cimasoni and Reshetikhin, where edge dashings on surface graphs correspond to spin structures. [3]

Example 3.3. *When X is an N-dimensional cube, then since the geometric realization of X is a CW complex, cubical cohomology coincides with ordinary singular cohomology. Since $H^2(X) = 0$, $w_2 = 0$, and an odd dashing exists. Since $H^1(X) = 0$, this odd dashing is unique up to vertex switching.*

Example 3.4. *We now consider a quotient of a 4-cube by the code*

$$d_4 = \{(0,0,0,0), (1,1,1,1)\}.$$

The graph is obtained by taking two copies of the 3-cube (colored black, red, and green), connected by blue lines representing the fourth color. When we identify antipodal vertices, we are left with only one 3-cube, and the blue lines connect antipodal vertices in the 3-cube. The result is the following:



As a CW complex, the 4-cube is a 4-ball, with its boundary 3-sphere as its 3-skeleton. The quotient acts on the 3-sphere by antipodal mapping, and sends the 4-cell to itself in a way that the boundary is two copies of the 3-sphere. Thus, the corresponding chain complex is that of the real projective space $\mathbb{RP}^4$. It is known that $H^1(\mathbb{RP}^4; \mathbb{Z}_2) \cong \mathbb{Z}_2$, so there are two odd dashings up to vertex sign flips. These correspond to the chiral[3] and twisted chiral multiplets. We will show this in Example 6.6, once we have the more general theory in place.

[3]More precisely, the dimensional reduction to $D = 1$ of the $D = 4$ chiral multiplet

4. The Cubical Cohomology of an Adinkra

4.1. **Computation of Cohomology Groups.** It was proved in [9] that every connected Adinkra is a quotient of the graph of I^N by a binary linear code C of length N. Specifically, the quotients are realized by compositions of reflection maps of the following type.

For every $1 \leq j \leq N$, define the reflection map $r_j : I^N \to I^N$ to be reflection in the plane $x_j = 1/2$, and for all $(t_1, \ldots, t_N) \in \{0,1\}^N$, define

$$r_{(t_1,\ldots,t_N)} = r_1^{t_1} \circ \cdots \circ r_N^{t_N}.$$

These maps preserve the cubical structure described above, and thus define maps ρ_j and $\rho_{(t_1,\ldots,t_N)}$ among the k-cells. This can be viewed as an action of ${\mathbb{Z}_2}^N$ on the set of k-cells of I^N.

If C is a code of length N, then by restriction we get an action of C on the set of k-cells of I^N. Quotienting by this action of C, we get a cubical complex. Its 1-skeleton is then a graph. The result in [9] is that every connected Adinkra has as its graph the quotient of the 1-skeleton of I^N by some doubly even code C.

In this section we compute some homology and cohomology groups of such a quotient of an N-cube by a doubly even binary linear code C.

Theorem 4.1. *Let A be a quotient of an N-cube by a doubly even binary linear $[N,k]$ block code C. Then we have the following.*

$$H_0(A) \cong H^0(A) \cong \mathbb{Z}_2$$
$$H_1(A) \cong H^1(A) \cong {\mathbb{Z}_2}^k$$

Proof. Since A is connected, $H_0(A) \cong \mathbb{Z}_2$. To compute $H_k(A)$ for $k \leq 2$, we can replace I^N with its 3-skeleton, denoted $I^{N(3)}$, and A with $I^{N(3)}/C$. Since the minimal weight of a non-zero codeword in C is greater than or equal to 4, the fixed points of C on I^N lie in cells of dimension 4 and higher. Thus, C acts freely on $I^{N(3)}$, and $I^{N(3)}$ is the universal covering space of $I^{N(3)}/C$. Then $H_1(A) \cong H_1(I^{N(3)}/C) \cong \pi_1(I^{N(3)}/C) \cong C \cong {\mathbb{Z}_2}^k$. The cohomology groups follow from the universal coefficient theorem. □

Remark: It is possible to prove that $H^2(A) \cong (\mathbb{Z}_2)^{k+\binom{k}{2}}$ through a more involved argument, replacing $I^{N(3)}/C$ with the classifying space $BC \cong (\mathbb{RP}^\infty)^k$, but that computation will not be used in this paper. This can be used to prove the existence of odd dashing if and only if C is doubly even, a fact already proved in Ref. [9].

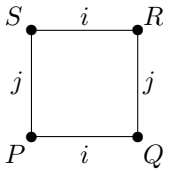
4.2. **Lifting paths and squares.** To represent homology classes in A it will be convenient to have a combinatorial version of path lifting.

Lemma 4.2. *Let A be a graph with edge coloring that is color-regular. Suppose v_0 is any vertex of A. Given a sequence of colors $j_1, \ldots, j_r$, there is a unique path starting at v_0 of length r so that the ith edge in the path has color j_i.*

Proof. This can be proved by induction on the length r of the sequence of colors, using the color-regularity property. □

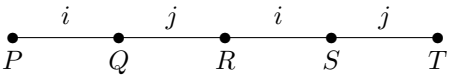
Likewise, it will be useful to lift squares, and so we prove the following lemma:

Lemma 4.3. *Let A be a graph with edge coloring that is color-regular, and for which the 2-faces are quadrilaterals. Suppose v_0 is any vertex of A. Given two different colors i and j, and the graph G with edge colors*



there is a unique mapping of G to A where P is sent to v_0 and edges are sent to edges of the same color. There is a unique $\{i,j\}$-color-face in A with this boundary.

Proof. We first view this square as a path with sequence of colors i, j, i, j:



By Lemma 4.2, there is a unique path in A that lifts this path where P goes to v_0. Since there are two colors, this path is part of an $\{i,j\}$-color-face σ. Since the 2-faces of A are quadrilaterals, the path must close, so that P and T go to the same vertex in A. Thus, the square G maps into A. The boundary of σ is that path. □

4.3. **Generators for homology and cohomology.** For $j \in \{1,\dots,N\}$, we define 1-cochain ϵ_j to be 1 on all edges of color j, and 0 on all other edges.

Proposition 4.4. *The 1-cochain ϵ_j is a 1-cocycle.*

Proof. Let σ be any 2-face. Then $d\epsilon_j(\sigma) = \epsilon_j(\partial\sigma) = \sum_{n=1}^{2}(\epsilon_j(\partial_n^1\sigma) + \epsilon_j(\partial_n^0\sigma))$. The edge $\partial_n^1\sigma$ is of color j if and only if $\partial_n^0\sigma$ is. So this sum is zero modulo 2. □

We will sometimes use the notation ϵ_j to denote the cohomology class represented by ϵ_j.

Theorem 4.1 says that $H_1(A)$ is isomorphic to the code C, but it will be useful to have a more constructive description of this isomorphism.

Definition 4.1. *Pick a vertex v_0 from the Adinkra A. Let $w = (w_1,\dots,w_N) \in C$ be a word of length N and weight r. Let $1 \le j_1 < j_2 < \cdots < j_r \le N$ be the integers so that $w_{j_i} = 1$.*

We use Lemma 4.2 to find a path ${}^{v_0}P_w$ in A starting from v_0 so that the ith edge is of color j_i for all i. We interpret ${}^{v_0}P_w$ as a sum in $C_1(A)$ of its edges. When v_0 is understood, we will write P_w.

Example 4.5. *Example 3.4 has as its code $\{(0,0,0,0),(1,1,1,1)\}$. Take the codeword $w = (1,1,1,1)$, so that $j_1 < j_2 < j_3 < j_4$ is $1 < 2 < 3 < 4$. Pick the bosonic vertex on the lower left of the diagram. The ordering of the j_i means we go up from this vertex along color 1 (black), then to the right along color 2 (red), then to the back face using color 3 (green), then across the diagonal along color 4 (blue). Note that this brings us back to the starting point. This path is $P_{(1,1,1,1)}$. The*

interested reader may verify that a similar path, starting at any other vertex, will also form a loop.

Proposition 4.6. *Fix a vertex v_0 in A. For every codeword $w \in C$, ${}^{v_0}P_w$ is a 1-cycle. The mapping $\Pi\colon C \to H_1(A)$ sending $w \in C$ to the homology class $[{}^{v_0}P_w] \in H_1(A)$ is a monomorphism, and if C is doubly even, then Π is an isomorphism.*

Proof. There is a vertex $x = (x_1, \ldots, x_N)$ in $\{0,1\}^N$ that goes to v_0 under the quotient of I^N by C. Use Lemma 4.2 to find a path ${}^{x}\tilde{P}_w$ in I^N starting at x so that the ith edge is of color j_i. Under the quotient by C, this path results in a similar path starting from v_0 so that the color of the ith edge is j_i for all i. By the uniqueness of such a path, this must be ${}^{v_0}P_w$. Since ${}^{x}\tilde{P}_w$ ends at $x + w$ and $w \in C$, it must be that ${}^{v_0}P_w$ ends at v_0. Therefore, ${}^{v_0}P_w$ is a cycle.

We now show that the map Π is a homomorphism. Suppose v and w are two words in C, with corresponding sequence of colors $i_1 < \ldots < i_q$ and $j_1 < \ldots < j_r$, respectively.

Consider the color sequence $i_1, \ldots, i_q, j_1, \ldots, j_r$. Write this sequence as $k_1^0, \ldots, k_s^0$. We obtain a path F^0 starting at v_0 following this sequence of colors. This path consists of a sequence of vertices $v_0^0, \ldots, v_s^0$ and edges $e_1^0, \ldots, e_s^0$ so that for each i, e_i^0 is incident with v_{i-1}^0 and v_i^0, and is of color k_i^0. As a homology class, $[F^0] = [P_v] + [P_w]$.

At each stage t, we perform a sequence of adjacent swaps, that is, for some index $1 \leq n < s$, we define

$$k_i^t = \begin{cases} k_n^{t-1}, & \text{if } i = n+1 \\ k_{n+1}^{t-1}, & \text{if } i = n \\ k_i^{t-1}, & \text{otherwise.} \end{cases}$$

The choice of n at each stage t is given by the bubble sort algorithm to obtain for some t a sequence k_i^t that is non-decreasing. We can ensure that for no t do we swap identical colors.

For each t, we produce a path F^t starting at v_0 and following the colors in k_i^t. More specifically, F^t consists of a sequence of vertices $v_0^t = v_0, \ldots, v_s^t$ and edges $e_1^t, \ldots, e_s^t$ with e_i^t incident to v_{i-1}^t and v_i^t and color k_i^t, for all i. Consider the swap done at stage t, and let n be the index such that $k_n^t = k_{n+1}^{t-1}$. If we consider the vertex v_{n-1} in F^{t-1}, and let S be the square in A starting at v_{n-1} with colors k_n^t and k_{n+1}^t (see Prop. 4.3), then S is a face so that ∂S is the union of the edges $e_n^{t-1}, e_{n+1}^{t-1}, e_n^t, e_{n+1}^t$. Thus, $F^t = F^{t-1} + S$. In this way, by induction, $[F^t] = [F^0]$ for all t.

Let t be the stage at which the k_i^t are non-decreasing. Then Π^t consists of a path, possibly with backtracks when two adjacent colors are identical. Note that runs of identical colors may be at most 2 edges long. When there is a backtrack corresponding to a run of 2 identical colors, then as an element of $C_1(A)$, these two edges cancel, and the corresponding pair of colors may be deleted. The result of removing such pairs shows that Π^t is the path starting at v_0 with colors corresponding to the non-zero entries in the codeword $v + w$. Since C is a finite group, this suffices to prove that Π is a homomorphism.

To establish that Π has trivial kernel, let $w = (w_1, \ldots, w_N)$ be any non-zero codeword, and let j be any index so that $w_j = 1$. Then $\epsilon_j([P_w]) = \sum_{i=1}^{r} \epsilon_j(e_i) = 1$, and therefore, $[P_w] \neq 0$. Thus, Π is a monomorphism.

If C is doubly even, then Theorem 4.1 applies and C and $H_1(A)$ are vector spaces of the same finite dimension over $\mathbb{Z}_2$. Thus, Π being a monomorphism implies it is an isomorphism. □

5. Classification of odd dashings modulo vertex switches

Suppose we have an Adinkra A that is a quotient of an N-cube by a doubly even code C. We know from Theorem 3.2 that the set of odd dashings modulo vertex switches is an affine space modeled on $H^1(A)$, which, by Theorem 4.1, is isomorphic to $(\mathbb{Z}_2)^k$. We will now describe a particular isomorphism from the set of odd dashings modulo vertex switches to $(\mathbb{Z}_2)^k$. The point is to describe a complete invariant $(s_1, \ldots, s_k) \in (\mathbb{Z}_2)^k$ for the set of odd dashings modulo vertex switches which can be used in computations.

Definition 5.1. *Suppose $c_1, \ldots, c_k$ is a generating set for the code C. Pick a vertex v_0 of A. As in Section 4.3, for each i, let ${}^{v_0}P_{c_i}$ be the path starting from v_0 and corresponding to the colors indicated by c_i. For each $\mu \in C^1(A)$ associated with an odd dashing, define ${}^{v_0}s(\mu) = (s_1, \ldots, s_k) \in (\mathbb{Z}_2)^k$ with s_i equal to the number, modulo 2, of dashed edges of μ along ${}^{v_0}P_{c_i}$. As before, when v_0 is understood, we can write $s(\mu)$.*[4]

Example 5.1. *In Example 4.5, we followed a path that involved zero dashed edges. So s in this case is 0. The interested reader can try this with other starting points, and should note in this case that s is 1 if the starting vertex is a fermion, and 0 if the starting vertex is a boson.*

Theorem 5.2. *The map s from the set of odd dashings modulo vertex switches to $(\mathbb{Z}_2)^k$ is a bijection.*

Proof. First, note that s is unchanged under vertex switches (since P_c is a cycle and dT is a coboundary), so the domain of s can be taken to be odd dashings modulo vertex sign switches. Also, s is affine-linear. That is, if we let μ_0 be an odd dashing, then $s(\mu) - s(\mu_0)$ is a linear function of $\mu - \mu_0$. The map s is then a bijection if and only if the corresponding linear function is one as well. Since the dimensions of the domain and range are equal, the only thing we need to check is that this linear map is an injection.

Suppose $s(\mu) - s(\mu_0) = 0$; in other words, $\mu(P_{c_i}) - \mu_0(P_{c_i}) = 0$ for all i. By linearity, $\mu(P_c) - \mu_0(P_c) = 0$ for all $c \in C$. Therefore, $\mu - \mu_0 = 0$ in cohomology, and $\mu - \mu_0 = dT$ for some vertex switch T. □

Remark. Even though the paths $[P_{c_i}]$ generate $H_1(A)$, the application of μ to such a 1-cycle is not well-defined on the homology class; that is, it may differ when a path is modified to something homologous. The reason is that μ is not a 1-cocycle: $d\mu$ is $\tilde{\omega}_2$, not 0.

[4]More invariantly, but less convenient computationally, ${}^{v_0}s(\mu)$ could be viewed as a linear functional on the code C.

Along these lines, note that the actual sequence $s_1, \ldots, s_k$ depends on v_0. It also depends on the ordering of the colors when lifting the sequence of colors to P_{c_i}, which we chose by specifying $j_1 < \cdots < j_r$. But if these choices are fixed for a given Adinkra, then this theorem says that $s(\mu)$ is a complete invariant for dashings modulo vertex switches. We will consider the effect of changing v_0 in Section 6.

5.1. **Edge sign flips.** To enumerate all Adinkras, it will be useful to find all odd dashings modulo vertex switches for a given Adinkra. This involves inverting the map s, meaning given a k-tuple $(s_1, \ldots, s_k)$, we wish to find an odd dashing μ. Actually, we assume that some other odd dashing μ_0 is already given, with some $s(\mu_0) = (\sigma_1, \ldots, \sigma_k)$. Then we describe what to do to μ_0 to change $s(\mu_0)$ to any other k-tuple. This will be achieved with what are called edge sign flips, described next.

If we are given an odd dashing, and j is a color, then an *edge sign flip of color* j takes the dashing and reverses the dashing of all edges of color j. An example of this was reversing the dashing on the blue edges in Example 3.4.

Proposition 5.3. *Let A be an Adinkra with odd dashing described by $\mu_1 \in C^1(A)$. Let j be a color. The result of an edge sign flip of color j is a dashing described by $\mu_2 = \mu_1 + \epsilon_j$. This resulting dashing is odd. If G is the generating matrix for the code, then $s(\mu_2)$ is $s(\mu_1)$ plus the jth column of G, taken modulo 2.*

Proof. Recall that ϵ_j takes the value 1 on every edge of color j, and 0 on all other edges. Therefore, adding it modulo 2 to μ_1 reverses every edge of color j. We recall that $d\epsilon_j = 0$, so μ_2 is odd if μ_1 was. For each i we see that $s_i(\mu_2) - s_i(\mu_1) = \epsilon_j(P_{c_i}) = (c_i)_j = G_{ij}$. □

More generally, we can consider a series of edge sign flips: for every word $x = (x_1, \ldots, x_N) \in (\mathbb{Z}_2)^N$ we perform an edge sign flip of every color j where $x_j = 1$.

Theorem 5.4. *Let A be an Adinkra that is an N-cube quotiented by a code C, and suppose A has odd dashing μ_0. Let $x \in (\mathbb{Z}_2)^N$ be a word of length N. The result of performing a series of edge sign flips using the colors j where $x_j = 1$ on μ_0 results in another odd dashing μ_x.*

If we write x and $s(\mu)$ as column vectors, and G is the generating matrix for the code, then

$$s(\mu_x) \equiv s(\mu_0) + Gx \pmod 2. \tag{2}$$

Proof. The result of the series of edge sign flips using x on μ_0 is

$$\mu_x \equiv \mu_0 + \sum_{j=1}^{N} x_j \epsilon_j.$$

If $w = (w_1, \ldots, w_N) \in C$ is a codeword, we take the above equation and evaluate it on the 1-chain P_w, and the result is

$$s_a(\mu_x) \equiv s_a(\mu_0) + \sum_{j=1}^{N} x_j w_j \pmod 2$$

for all a. In particular for the ith generating word of G,

$$w_j = G_{i,j},$$

and then (2) follows from writing this statement in matrix form. □

We are now able to find an odd dashing for each k-tuple $(s_1, \ldots, s_k)$, using the following:

Algorithm 1. *Input:*

- *a generating matrix G for a code C*
- *a vertex v_0 of the Adinkra $A = I^N/C$*
- *an odd dashing μ_0 on A*
- *a k-tuple $(s_1, \ldots, s_k) \in (\mathbb{Z}_2)^N$*

Output: an odd dashing μ on A with ${}^{v_0}s(\mu) = (s_1, \ldots, s_k)$

Procedure:

Use Gauss–Jordan elimination on G. Then G is in reduced row-echelon form, with leading 1s in columns $j_1 < \cdots < j_k$. Compute ${}^{v_0}s(\mu_0)$. Let

$$(\sigma_1, \ldots, \sigma_k) = (s_1, \ldots, s_k) - s(\mu_0) \pmod 2.$$

Perform an edge sign flip on μ_0 for color j_i for every i with $\sigma_i = 1$. Let μ be the resulting dashing after all these edge flips are accomplished.

By Theorem 5.4,

$${}^{v_0}s(\mu) = {}^{v_0}s(\mu_0) + (\sigma_1, \ldots, \sigma_k),$$

which is $(s_1, \ldots, s_k)$.

Edge sign flips are powerful enough to generate all of the different odd dashings up to vertex switches, as we see in the following theorem.

Corollary 5.5. *Let A be a connected Adinkra, and let μ_1 and μ_2 be two odd dashings on A. There is a sequence of edge sign flips and vertex switches that turns μ_1 into μ_2.*

Proof. Every connected Adinkra is a quotient of an N-cube by a code C. Let G be a generating matrix for C. Pick a vertex v_0 in the Adinkra. Apply the above Algorithm to G, v, μ_1, and ${}^{v_0}s(\mu_2) - {}^{v_0}s(\mu_1)$, to find an odd dashing μ_3 with $s(\mu_3) = s(\mu_2)$. By Theorem 5.2, μ_2 can be obtained from μ_3 by a series of vertex switches. □

6. Changing the starting vertex v_0

We now have a complete invariant, ${}^{v_0}s(\mu)$, of the odd dashing. But it depends on the starting vertex v_0. Let us consider the effect of changing v_0 on the k-tuple ${}^{v_0}s(\mu)$. As before, we assume the Adinkra A is connected, and therefore, it is a quotient of I^N by a code C.

Let v_1 be another vertex in A. There is a path P connecting v_0 to v_1. If we follow the sequence of colors in P, we form a word x, where $x_j = 1$ if and only if color j appears in the path P.

Theorem 6.1. *Suppose A is an Adinkra, with odd dashing given by μ. Let v_0, v_1, and x be as above. If we write x and $s(\mu)$ as column vectors, and G is the generating matrix for the code, then*

$$ {}^{v_1}s(\mu) \equiv {}^{v_0}s(\mu) + Gx \pmod 2.$$

Proof. As in the proof of Theorem 5.4, we actually prove that if $w = (w_1, \ldots, w_n)$ is a codeword in C, then

$$\mu\left({}^{v_1}P_w\right) \equiv \mu\left({}^{v_0}P_w\right) + \sum_{j=1}^{N} x_j w_j \pmod 2. \tag{3}$$

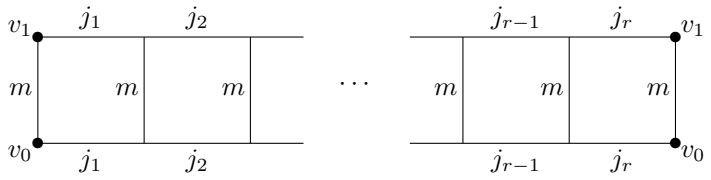
We take for w each generator of C, that is, each row of the matrix G. Gathering this into matrix notation, we then would have the desired result.

As before, let $j_1 < \cdots < j_r$ be the integers so that $w_{j_i} = 1$.

To prove (3), first consider the situation where the x is all zeros except for a single 1 at position m. That is, $x_m = 1$ and $x_i = 0$ for all $i \neq m$.

Case 1: $w_m = 0$

If $w_m = 0$, then consider the following diagram.



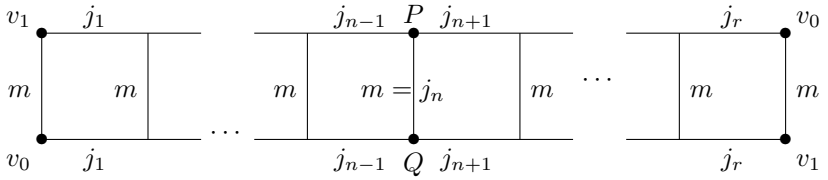
Use Lemma 4.3 iteratively to map this figure into A with the bottom left vertex going to v_0, and the edges going to edges colored with the labels shown. The leftmost edge connects v_0 to v_1. Then the bottom edge of this diagram goes to the path ${}^{v_0}P_w$, and the top edge goes to the path ${}^{v_1}P_w$. Both of these paths are circuits, and so the left edge of this diagram matches the right edge. Let F be the sum of the faces that this diagram gets sent to in A. The boundary of F is ${}^{v_0}P_w + {}^{v_1}P_w$. Therefore we have (mod 2):

$$\begin{aligned} \mu({}^{v_1}P_w) &\equiv \mu({}^{v_0}P_w + \partial F) \\ &\equiv \mu({}^{v_0}P_w) + \mu(\partial F) \\ &\equiv \mu({}^{v_0}P_w) + d\mu(F) \\ &\equiv \mu({}^{v_0}P_w) + \tilde{\omega}_2(F). \end{aligned}$$

But $\tilde{\omega}_2(F)$ is the number of 2-faces in F, which is r, the weight of the codeword w, which is even. Therefore $\mu({}^{v_1}P_w) \equiv \mu({}^{v_0}P_w) \pmod 2$.

Case 2: $w_m = 1$

In this case, $j_n = m$ for some n, and we have the following diagram.



Again, use Lemma 4.3 iteratively to map this figure into A with the bottom left vertex going to v_0, and the edges going to edges colored with the labels shown. The leftmost edge connects v_0 to v_1.

The path ${}^{v_0}P_w$ is in the image of the path that goes from the lower left corner along the lower edge until the point Q, then going up to P, then going across the top edge to the point in the upper right. Similarly, the path ${}^{v_1}P_w$ is in the image of the path that goes along the top from the upper left to P, then down to Q, then along the bottom edge to the lower right. Again, let F be the sum of the faces that this diagram gets sent to in A. As before,

$$\mu({}^{v_1}P_w) \equiv \mu({}^{v_0}P_w) + \tilde{\omega}_2(F) \pmod 2,$$

but $\tilde{\omega}_2(F)$, the number of 2-faces in F, is now $r-1$, which is odd. Therefore $\mu({}^{v_1}P_w) \equiv \mu({}^{v_0}P_w) + 1$.

General case: We return to the situation where we consider words x with $\mathrm{wt}(x) > 1$. We iterate the above procedure for every j so that $x_j = 1$. Note that ρ_x is the composition of the ρ_j for which $x_j = 1$. The result follows. □

6.1. **Node choice symmetry.** We have seen how the k-tuple ${}^{v_0}s(\mu) = (s_1, \ldots, s_k)$ captures the properties of μ that are invariant under vertex switches. But it depends on the choice of a vertex v_0, as we have just seen. These vertices can sometimes be easily distinguished: for instance, some are bosons and others are fermions; some have many arrows pointing away from it, others have few; and so on. But in other cases, there is no property that can distinguish them, and these are node choice symmetries.

Definition 6.1. *A node choice symmetry of an Adinkra A is a permutation of the vertices and edges in A that preserves*

1. *the incidence relation between edges and vertices,*
2. *the bipartition (V_0, V_1) of the vertices,*
3. *the colors of the edges, and*
4. *the orientation of the arrows on the edges.*

That is, it preserves all the features of the Adinkra except for the dashedness.

Let us assume A is connected. By the color-regular property of Adinkras, criteria 1 and 3 in the definition guarantees that the permutation is determined uniquely by where it sends a given vertex v_0 (to, say, v_1). The path from v_0 to v_1 can be described as a word $x \in (\mathbb{Z}_2)^N$. The permutation given by ρ_x satisfies criterion 1 and 3, and by uniqueness, must be the original permutation. Therefore, every node choice symmetry is given by ρ_x for some x.

Criterion 2 is equivalent to $\mathrm{wt}(x)$ being even, and criterion 4 can be described in terms of height assignments by saying that $\rho_x(v)$ and v have the same height (equivalently, the same engineering dimension) for every vertex v. This motivates the definition of the following code:

Definition 6.2. *The node choice code $\mathcal{H}$ of a connected Adinkra is the set of words $x \in (\mathbb{Z}_2)^N$ so that for all vertices v in the Adinkra, $\rho_x(v)$ has the same engineering dimension as v.*

Note that $\mathcal{H}$ is a group under addition modulo 2, and is therefore a linear code. Suppose A is a quotient of I^N by the doubly even code C. Then C is a subgroup of $\mathcal{H}$. The map $x \mapsto \rho_x$ describes an action of $\mathcal{H}$ on the Adinkra via node choice symmetries, and C is the subgroup that acts trivially on the Adinkra. All node choice symmetries are obtained in this way.[5] Criterion 2 implies $\mathcal{H}$ is an even code. There is a generating matrix H whose rows are the generating words for $\mathcal{H}$.

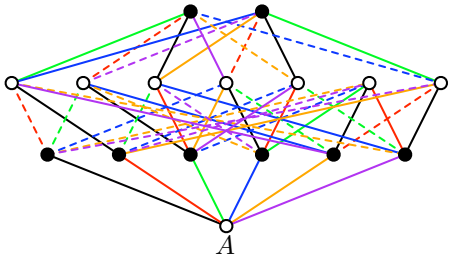
The point here is that by considering the invariant k-tuple ${}^{v_0}s(\mu)$, one might be under the impression that there are 2^k odd dashings on $A = I^N/C$, where k is the dimension of the code C. But the node choice symmetries act on the vertices, and therefore on the possible k-tuples by changing v_0. If we identify the various possible k-tuples, the result will give us invariants that identify the odd dashings on an Adinkra modulo vertex switches.

Theorem 6.2. *Let A be a quotient of I^N by a code C with generating matrix G. Let H be the generating matrix for the node choice code. Let μ be an odd dashing on A, and let v_0, v_1 be vertices of A. There exists a node choice symmetry from v_0 to v_1 if and only if there exists a column vector y so that*

$$GH^T y = {}^{v_1}s(\mu) - {}^{v_0}s(\mu).$$

Proof. If ρ_x is a node choice symmetry, then since the columns of H^T generate the node choice group, x can be written as $H^T y$ for some column vector y. The result then follows from Theorem 6.1. □

Example 6.3. *Consider the $N = 6$ example given below. This is an Adinkra but the arrows have been removed for the sake of clarity. Instead, the vertices have been placed at heights so that every arrow is assumed to point upward along its edge. The fact that this can be done is explained in* [4].



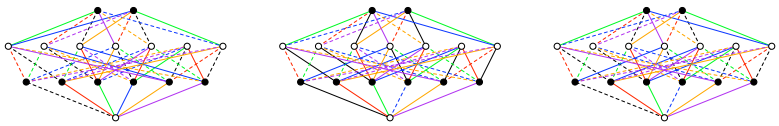
j	jth color
1	black
2	orange
3	red
4	purple
5	blue
6	green

[5] In [8] we defined the node choice group to be $\mathcal{H}$, and a node choice symmetry to be an element of $\mathcal{H}$, whereas by our definition, a node choice symmetry is given by the actual permutation ρ_x, which is determined by an element in $\mathcal{H}/C$. This distinction will not be relevant for our purposes.

The code is d_6, generated by 111100 *and* 001111. *The generating matrix is thus*

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

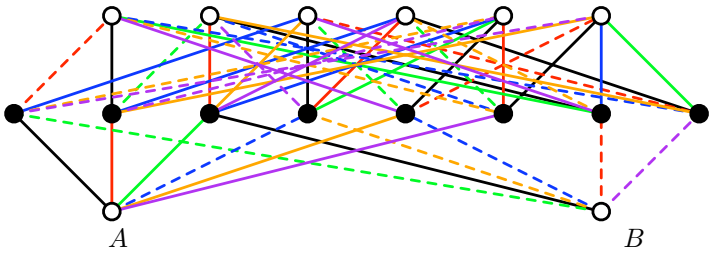
Since d_6 is two-dimensional, there should be $2^2 = 4$ different dashings. The matrix G is not in reduced row-echelon form, but since we have $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as colums, we can see that the other 3 dashings can be obtained by edge sign flips with color 1, color 5, and with both, as illustrated below.



There are no node choice symmetries, since any such symmetry must send the bottom node to itself. Because a node choice symmetry is determined by where it sends any one vertex, the only node choice symmetry is the identity.

Take v_0 to be the lowest vertex, labelled A in the diagram. We trace a path with colors 1, 2, 3, then 4 (in that order) from v_0 and count the number of dashed edges, modulo 2. This is s_1. We do the same with the colors 3, 4, 5, then 6 to get s_2.

Example 6.4. *Now consider a different hanging of the same $N = 6$ Adinkra:*



In this case, G is the same, but there is now a node choice symmetry that sends A to B. This can be described as a path from A to B following color 1 (black), then color 6 (green). Hence, the matrix H is

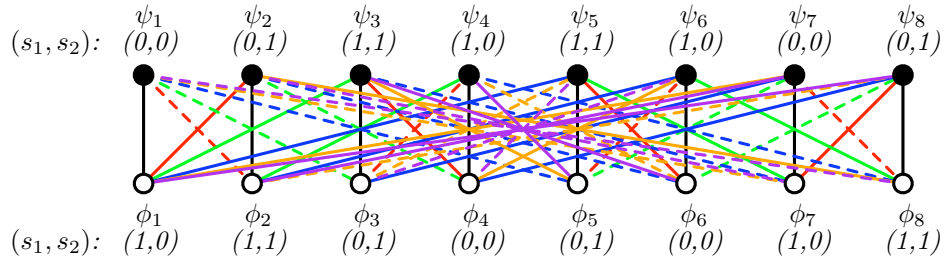
$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The first two rows are simply the rows in G, to acknowledge that C is a subgroup of $\mathcal{H}$; they will be irrelevant anyway. We compute:

$$GH^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore the effect of a node choice symmetry on (s_1, s_2) is to either do nothing, or to toggle both s_1 and s_2. We can thus use $s_1 + s_2$ as a complete invariant of dashings, modulo vertex switches and node choice symmetries.

Example 6.5. *We now consider the same $N = 6$ Adinkra, but hung as a valise; that is, all bosons on one level, and all fermions on the other. In this case, the arrows go from bosons to fermions, which in* [11] *was termed a* base Adinkra.[6]



Given any two bosons, there is a node choice symmetry that takes one to the other. Thus, the node choice group is the set of even weight words. This is generated by words of weight 2 where the 1s are adjacent.

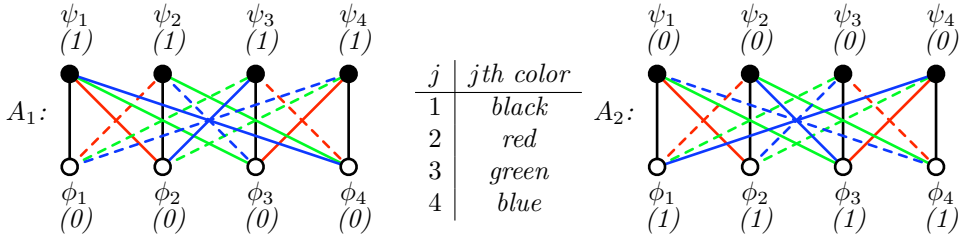
$$H = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Then we have

$$GH^T = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

and note that the columns span the whole space $(\mathbb{Z}_2)^2$. In particular, a node symmetry given by $y = (0,0,0,1,0)$, corresponding to $x = H^T y = (0,0,0,1,1,0)$, involving ρ_4 followed by ρ_5, will change the value of s_1 without changing s_2. Likewise, $y = (0,1,0,0,0)$, corresponding to $x = H^T y = (0,1,1,0,0,0)$, involving ρ_2 followed by ρ_3, will change the value of s_2 without changing s_1. And doing both changes both s_1 and s_2. Therefore, the valise Adinkra for d_6 has only one dashing up to node choice symmetries.

Example 6.6. *In the case $N = 4$ with the code d_4 generated by* 1111*, consider a valise Adinkra. There are two dashings, shown below.*



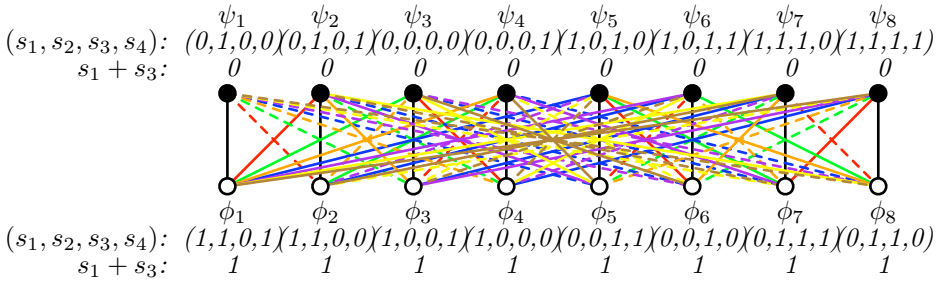
j	jth color
1	black
2	red
3	green
4	blue

[6]A similar example could be obtained if the arrows go from fermions to bosons; this is the Klein flip of a base Adinkra. A base Adinkra or its Klein flip is called a Valise Adinkra.

The numbers in the parentheses are the values of $s_1(\mu)$ at each vertex. For the Adinkra A_1 on the left, every boson has $s_1 = 0$ and every fermion has $s_1 = 1$. We do an edge sign flip on any edge (in this case, blue) and obtain the Adinkra A_2 on the right, where every boson has $s_1 = 0$ and every fermion has $s_1 = 0$. Since every node choice symmetry preserves the bipartition of the vertices, no node choice symmetry can swap the two, and so A_1 and A_2 are really different. Note that GH^T is $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$, which confirms this result. If we do a Klein flip on A_1 (swap V_0 and V_1 in the bipartition), then do vertex lifts so that the bosons are still on the bottom of the diagram, the result will be (up to vertex switches) A_2.

The distinction between the two dashings is the same as that between the chiral and the twisted chiral superfield, except that the chiral and twisted chiral superfields are not Valise Adinkras.

Example 6.7. *Consider the valise Adinkra for $N = 8$ with $C = e_8$:*



Color key:

j	j*th color*
1	*black*
2	*red*
3	*green*
4	*orange*
5	*blue*
6	*purple*
7	*yellow*
8	*brown*

Here the generating matrix for e_8 is

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

The node choice code has generating matrix

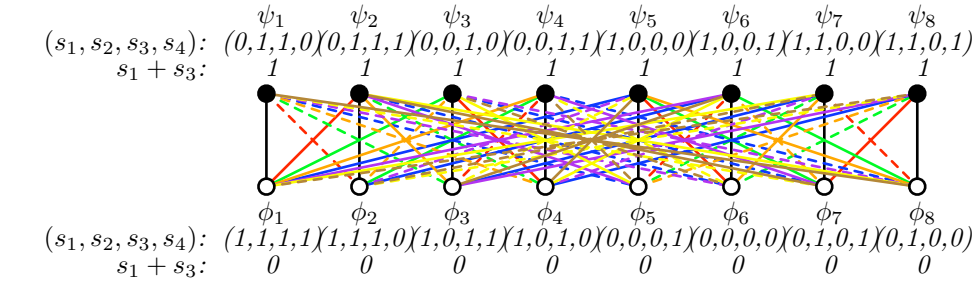
$$H = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

and we compute

$$GH^T = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Note that the first and third rows are the same, but other than that, the rows are linearly independent. Thus, node choice symmetries can produce $2^3 = 8$ different $s(\mu)$ sequences. These correspond to the 8 bosons: note in the figure that the $s(\mu)$ sequences are listed at each vertex. As long as the symmetry sends bosons to bosons, we cannot alter $s_1 + s_3$, which is truly invariant. But other non-trivial linear combinations of the s_a can always be changed. Hence, there are two distinct dashings on an $N = 8$ valise Adinkra that are a quotient of I^8 by the code e_8.

The other dashing can be obtained by edge flipping any color, for instance Q_8 (brown):



Example 6.8. *Suppose more generally we have any connected Valise Adinkra $A = I^N/C$. The node choice code is the set E of even weight words in $(\mathbb{Z}_2)^N$. The matrix H is then*

$$H = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}.$$

Let the ith row of H be denoted h_i.

Consider the generating matrix G for the code C. Let z be the length N word $(1, \ldots, 1)$.

The dimension D of the span of the columns of GH^T can be determined by doing Gauss–Jordan elimination. This dimension is k minus the number of rows

of 0*s of the reduced row-echelon form of* GH^T*. Thus,* $D = k$ *if and only if no non-trivial linear combination of the rows of* GH^T *is zero. Such a linear combination can be done on* G*, resulting in a non-trivial linear combination of the generating words, i.e., a codeword* $w \in C$*, with* $wH^T = 0$ *(here* w *is written as a row matrix). But unless* $w = 0$ *or* $w = z$*, there will be a row of* H*,* h_i*, so that*

$$\sum_{j=1}^{N} w_j h_{i,j} \equiv 1 \pmod 2.$$

To see this, simply locate a column of w *where* w *changes from* 0 *to* 1 *or vice versa. So the only non-trivial codeword* w *with* $wH^T = 0$ *is* z*.*

This demonstrates that if $z \notin C$*, then* $D = k$*, and therefore any* $s(\mu)$ *can be turned into* 0 *by choosing* v_0 *appropriately. In other words, if* $z \notin C$*, then there is only one odd dashing up to node choice symmetries and vertex switches.*

If $z \in C$*, then since there is only one non-trivial* w *with* $\sum_{j=1}^{N} w_j h_{i,j} \equiv 1 \pmod 2$*, there is only one row of* 0*s in the reduced row-echelon form of* GH^T*. Thus,* $D = k-1$*. Choose the generating set for* C *so that* z *is one of the generating words, say, the first one. Then* $s_1(\mu)$ *is not altered by any node choice symmetry (since all elements of* $\mathcal{H}$ *are even, they do nothing to* z*), but the other* s_a *for* $a > 1$ *can be turned into* 0 *by choosing* v_0 *appropriately. In other words, if* $z \in C$*, then there are two odd dashings up to node choice symmetries and vertex switches.*

Since C *is doubly even, a necessary condition for* $z \in C$ *is that* N *is a multiple of* 4*. And when* N *is a multiple of* 4*, if* C *is maximal, then* $z \in C$*, since otherwise, the span of* C *and* z *forms a larger doubly even code.*

This is in accordance to the following observations. First, valise representations of the N*-extended super Poincaré algebra are in one-to-one correspondence with supermodules of the Clifford algebra* $\mathrm{Cl}(0, N)$*, which in turn are in one-to-one correspondence with representations of* $\mathrm{Cl}(0, N+1)$*.* [8] *Second, when* N *is a multiple of* 4*, there are exactly two isomorphism classes of such irreducible representations, and otherwise, there is just one such.* [19]

6.2. **Dashing invariants.** The results of the previous section indicate that the sequence $(s_1, \ldots, s_k)$ does not in general work as an invariant for a dashing, because of node choice symmetries. But the methods of the previous section also suggest a solution: find linear combinations of the s_i that are invariant under every node choice symmetry.

Algorithm 2. *Input: code generator matrix* G *and Node choice code generator matrix* H

Output: A set of linear combinations of the s_i *that is invariant under node choice symmetries*

Procedure:

Create an augmented matrix, consisting of GH^T *on the left, and a* $k \times k$ *identity matrix on the right.*

Perform Gauss–Jordan elimination on this augmented matrix.

For each row of the resulting matrix where the left side has only zeros, we will create an invariant. On such a row, look on the right side of the matrix. For each column i *on the right with a 1, include a summand* s_i*.*

Example 6.9. *Take the $N = 6$ Example 6.4 above. Here we have*

$$GH^T = \left[\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 1 \end{array}\right],$$

which we augment with an identity matrix, as demanded in the algorithm:

$$\left[\begin{array}{ccc|cc} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right].$$

Doing Gauss–Jordan elimination gives

$$\left[\begin{array}{ccc|cc} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right],$$

and there is one row which has 0 *on the left of the line: the second row. The right side of this is*

$$1 \quad 1,$$

which indicates that $s_1 + s_2$ is an invariant of both node symmetries and vertex switches.

7. Other applications of cubical cohomology to Adinkras

This paper has been about odd dashings. But cubical cohomology has other applications to the mathematics of Adinkras.

7.1. **Bipartitions.** The existence and classification of bipartitions is analogous to that of odd dashings, but where the role of w_2 is replaced by w_1. Just as a dashing gives rise to a 1-cochain, a partition of the vertices is equivalent to a 0-cochain μ, where bosons are assigned 0, and fermions are assigned 1. This is a bipartition if and only if every edge is incident with one boson and one fermion, i.e., if for every edge e, $\mu(\partial e) = 1$. This is equivalent to saying that $d\mu = \tilde{\omega}_1$. Since $d\tilde{\omega}_1 = 0$, this is possible if and only if $w_1 = 0$ in $H^1(A)$.

This choice of μ is unique up to an element of $H^0(A)$. If A is connected, then there are only two choices, up to swapping all bosons and fermions (called a Klein flip in [11]). If A is not connected, then a Klein flip could be done independently in each connected component.

7.2. **Arrows.** If we define cohomology of an Adinkra using $\mathbb{Z}$ coefficients instead of $\mathbb{Z}_2$ coefficients, then an orientation (that is, drawing an arrow on every edge) on a cubical graph gives rise to a one-cochain μ in $C^1(A;\mathbb{Z})$ in the following manner: for every oriented edge e going from v_p to v_q, we define $\mu(e)$ to be 1 if the arrow points from v_p to v_q, and -1 otherwise. Then the orientation is non-escheric if and only if $d\mu = 0$. Since $H^1(A;\mathbb{Z}) = 0$, a non-escheric orientation μ can be described as df for some 0-cochain f. Such a cochain is an assignment of an integer to each vertex, which if we imagine is a placement of the vertices at different "heights" on the page. Since μ takes its values in ± 1, it must be that edges connect vertices of adjacent height. This height assignment is unique up to adding an integer additive constant for each connected component of the Adinkra.

This height function provides what in particle physics is called engineering dimension or mass dimension. Quantities in physics come with units of time, distance, mass, or combinations of these. If we choose units so that the speed of light

$c = 1$ and Planck's constant $\hbar = 1$, then units can be written as a power of the unit of mass. This power is the engineering dimension. Assuming the transformation rules do not involve constants that have dimension, the function that takes each field and returns twice the engineering dimension is a height assignment.

A non-cohomological proof of these facts, and an application of these ideas to classifying all non-escheric orientations, is given in [4].

The escheric central charge example in [11] admits a similar 1-form with values in $\mathbb{Z}_4$.

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