LINEAR AND MULTILINEAR SPHERICAL MAXIMAL FUNCTIONS

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CONTENTS

ABSTRACT

In dimensions $n \geq 2$ we obtain $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ boundedness for the multilinear spherical maximal function in the largest possible open set of indices and we provide counterexamples that indicate the optimality of our results. Moreover, we obtain weak type and Lorentz space estimates as well as counterexamples in the endpoint cases.

We also study a family of maximal operators that provides a continuous link connecting the Hardy-Littlewood maximal function to the spherical maximal function. Our theorems are proved in the multilinear setting but may contain new results even in the linear case. For this family of operators we obtain bounds between Lebesgue spaces in the optimal range of exponents.

Moreover, we provide multidimensional versions of the Kakeya, Nikodym, and Besicovitch constructions associated with a fixed rectifiable set. These yield counterexamples indicating that maximal operators given by translations of spherical averages are unbounded on all $L^p(\mathbb{R}^n)$ for $p < \infty$.

For lower-dimensional sets of translations, we obtain L^p boundedness for the associated maximally translated spherical averages and for the uncentered spherical maximal functions for a certain range of p that depends on the upper Minkowski dimension of the set of translations. This implies that the Nikodym sets associated with spheres have full Hausdorff dimension.

Chapter 1

Introduction

One of the most central objects in analysis is the Hardy-Littlewood maximal function

$$Mf(x) := \sup_{t>0} \frac{1}{v_n} \int_{\mathbb{B}^n} |f(x-ty)|dy \tag{1.1}$$

where $\mathbb{B}^n = B^n(0,1)$ is the unit ball in $\mathbb{R}^n$ and $v_n = |\mathbb{B}^n|$ is the volume of $\mathbb{B}^n$. It is well known that the Hardy-Littlewood maximal function is weak-type-(1,1) bounded, meaning that

$$\big|\{x \in \mathbb{R}^n \ : \ Mf(x) \ > \ \lambda\}\big| \leq \frac{C_n}{\lambda} \int_{\mathbb{R}^n} |f(y)|\, dy,$$

for some dimensional constant C_n independent of λ.

Since M is also trivially bounded on L^∞, we obtain that M maps $L^p(\mathbb{R}^n)$ to itself, which means that

$$\|Mf\|_{L^p} \lesssim \|f\|_{L^p}, \tag{1.2}$$

for all $p > 1$. Here, as well as in the following, we write $A \lesssim B$ if there is a constant C such that $A \leq CB$ and we write $A \lesssim_\alpha B$ to indicate that C depends on the parameter α.

An important consequence of the weak-type (1,1) bound of M is the Lebesgue differentiation theorem, which states that for any locally integrable function f on

$\mathbb{R}^n$ we have

$$\lim_{r\to 0}\frac{1}{v_n r^n}\int_{B^n(x,r)} f(y)\,dy = f(x)$$

for almost all $x\in\mathbb{R}^n$, where $B^n(x,r)$ is the n-dimensional ball centered at x with radius r.

The main objects in this work are spherical averages, where we are averaging over spheres centered at x, instead of taking averages over balls. Let $\mathbb{S}^{n-1}$ denote the unit sphere in $\mathbb{R}^n$, let $d\sigma_{n-1}$ denote the surface measure on $\mathbb{S}^{n-1}$ and $\omega_{n-1} = d\sigma(\mathbb{S}^{n-1})$. For any $t>0$, the spherical averages

$$\mathcal{A}_t f(x) := \frac{1}{\omega_{n-1}}\int_{\mathbb{S}^{n-1}} f(x-ty)d\sigma_{n-1}(y) \tag{1.3}$$

can be defined for functions $f\in\mathcal{S}(\mathbb{R}^n)$, the Schwartz class. For such a function f and a given point x, $A_t f(x)$ corresponds to the average of f on the sphere centered at x and radius $t>0$.

Spherical averages often appear as solutions to partial differential equations. For example, the spherical average $u(x,t) = \dfrac{1}{4\pi}\displaystyle\int_{\mathbb{S}^2} f(x-ty)d\sigma(y)$ is a solution to Darboux's equation

$$\begin{aligned}\sum_{i=1}^{3}\frac{\partial^2 u}{\partial x_i^2}(x,t) &= \frac{\partial^2 u}{\partial t^2}(x,t) + \frac{2}{t}\frac{\partial u}{\partial t}(x,t),\\ u(x,0) &= f(x),\\ \frac{\partial u}{\partial t}(x,0) &= 0.\end{aligned}$$

in $\mathbb{R}^3$.

The study of maximal spherical means was initiated by Stein [50], who obtained a bound for the linear spherical maximal function

$$Sf(x) := \sup_{t>0}\frac{1}{\omega_{n-1}}\int_{\mathbb{S}^{n-1}} |f(x-ty)|\,d\sigma_{n-1}(y) = \sup_{t>0} A_t|f|(x) \tag{1.4}$$

from $L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)$ when $n\geq 3$ and $p>\frac{n}{n-1}$ and showed that it is unbounded when $p \leq \frac{n}{n-1}$ and $n \geq 2$. The analogue of this result in dimension $n = 2$ was established by Bourgain in [7], who also obtained a restricted weak type estimate in [6] in the case $n \geq 3$. Later, Seeger, Tao, and Wright in [47] proved that the restricted weak type estimate does not hold in dimension $n = 2$. See Chapter 2 for the exact statements of these bounds.

A number of other authors have also studied the spherical maximal function; see for instance [9], [12], [39], and [46]. Extensions of the spherical maximal function to different settings have also been established by several authors; for instance see [8], [26], [14], and [35].

It is not obvious a priori whether Sf is well-defined for $f \in L^p$, since $\mathbb{S}^{n-1}$ is a set of measure zero in $\mathbb{R}^n$. However, since the Schwartz functions are dense in L^p for $1 \leq p < \infty$, the boundedness of S implies that we can define of Sf for $f \in L^p$ when $p > \frac{n}{n-1}$. See Chapter 3 below for an extension of the definition of Sf for $f \in L^1_{\text{loc}}$, under some necessary modification.

One corollary of the $L^p \to L^p$ boundedness result of the spherical maximal function is that the operator norm of the Hardy-Littlewood maximal function in 1.2 is bounded above by a constant independent of the dimension n. Indeed, noting that $Mf(x) \leq Sf(x)$ and since

$$\|S(f)\|_{L^p} \leq C_p \|f\|_{L^p}$$

for a constant C_p independent of the dimension, Stein and Strömberg showed in [52] that for $p > 1$,

$$\|M\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} \leq C_p$$

with C_p independent of the dimension n.

In the field of geometric measure theory, the boundedness of S has the interesting implication that there is no set of Lebesgue measure zero in $\mathbb{R}^n$ containing a sphere centered at every point $x \in \mathbb{R}^n$. This was proven independently for $n = 2$

by Marstrand in [36].

The bi(sub)linear analogue of Stein's spherical maximal function

$$S^2(f,g)(x) := \sup_{t>0} \frac{1}{\omega_{2n-1}} \int_{\mathbb{S}^{2n-1}} |f(x-ty)g(x-tz)| d\sigma_{2n-1}(y,z) \tag{1.5}$$

was first introduced in by Geba, Greenleaf, Iosevich, Palsson, and Sawyer [18] who obtained the first bounds for it but later improved bounds were provided by [3], [22], [29], and [33]. A multilinear (non-maximal) version of this operator when all input functions lie in the same space $L^p(\mathbb{R})$ was previously studied by Oberlin [43]. The authors in [3] provided an example that shows that the bilinear spherical maximal function is not bounded when $p \geq \frac{n}{2n-1}$. Jeong and Lee in [33] proved that the bilinear maximal function is pointwise bounded by the product of the linear spherical maximal function and the Hardy-Littlewood maximal function, which helped them establish boundedness in the optimal open set of exponents, along with some endpoint estimates. Certain analogous bounds have been obtained by Anderson and Palsson in [1] and [2] concerning the discrete multilinear spherical maximal function.

In Chapter 2 we extend the results of Jeong and Lee in the multilinear setting, obtaining $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_2}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ boundedness results for the operator

$$S^m(f_1,\dots,f_m)(x) := \sup_{t>0} \frac{1}{\omega_{mn-1}} \int_{\mathbb{S}^{mn-1}} \prod_{i=1}^{m} |f_i(x-ty_i)| d\sigma_{mn-1}(y_1,\dots,y_m), \tag{1.6}$$

for the optimal open set of exponents

$$p > \frac{n}{mn-1},$$

as well as some endpoint boundedness results. We also adapt the counterexample of Barrionuevo, Grafakos, He, Honzík, and Oliveira in [3] to show that our results are sharp. Moreover, we provide a counterexample that addresses a question raised by Jeong and Lee in [33] regarding the validity of a strong type $L^1 \times L^\infty \to L^1$

bound for the bilinear spherical maximal function.

The boundedness of the maximal operator S in [50] was obtain via the auxiliary family of operators

$$S_\alpha(f)(x) = \sup_{t>0} \frac{2}{\omega_{n-1}B(\frac{n}{2}, 1-\alpha)} \int_{\mathbb{B}^n} |f(x-ty)|(1-|y|^2)^{-\alpha} dy, \tag{1.7}$$

defined originally for Schwartz functions, where $0 \leq \alpha < 1$. Here B is the beta function defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$$

for $x, y > 0$.

For each $0 < \alpha < 1$, Stein obtained boundedness for the operator S_α from L^p to itself in the optimal range of exponents

$$p > \frac{n}{n-\alpha},$$

when $n \geq 3$. This was extended to the case $n = 2$ indirectly in [7] and more explicitly in [39]. In [31] the authors obtained boundedness results for maximal operators associated to a more general set of measures that includes the family S_α.

Far from being an artifact of the proof in [50], these averages arise naturally in the study of partial differential equations as well. Consider the Cauchy problem of the wave equation (in two dimensions)

$$\begin{aligned} \frac{\partial^2 u}{\partial x_1^2}(x,t) + \frac{\partial^2 u}{\partial x_2^2}(x,t) &= \frac{\partial^2 u}{\partial t^2}(x,t), \\ u(x,0) &= f(x), \\ \frac{\partial u}{\partial t}(x,0) &= g(x). \end{aligned}$$

Then

$$u(x,t) = c_0 S_{3/2,t} f(x) + c_1 t S_{1/2,t} g(x),$$

where $S_{\alpha,t}f(x) = \int_{\mathbb{B}^n} f(x-ty)\frac{dy}{(1-|y|^2)^\alpha}$. We note that while $S_{\alpha,t}$ has been initially defined for $0 \le \alpha \le 1$, we can extend the definition to the case $\alpha > 1$ by analytic continuation on α.

Moreover, family S_α provides a continuous link that connects M to S in the following explicit way: For any $f \in L^1_{loc}(\mathbb{R}^n)$ and any $x \in \mathbb{R}^n$ we have

$$\begin{aligned} &M(f)(x) \le S_\alpha(f)(x) \le S(f)(x), \\ &\lim_{\alpha\to 1^-} S_\alpha(f)(x) = S(f)(x), \\ &\lim_{\alpha\to 0^+} S_\alpha(f)(x) = M(f)(x). \end{aligned} \tag{1.8}$$

These assertions are contained in Theorem 3.1 and are proved in Chapter 3.

The multilinear Hardy-Littlewood maximal function is

$$M^m(f_1,\dots,f_m)(x) = \sup_{t>0}\frac{1}{v_{mn}}\int_{\mathbb{B}^{mn}}\prod_{i=1}^m |f_i(x-ty_i)|\,dy_1\cdots dy_m. \tag{1.9}$$

The uncentered version of this maximal operator first appeared in the work of Lerner, Ombrosi, Perez, Torres, Trujillo-Gonzalez [34] with the unit cube in place of the unit ball.

In analogy with the linear case, the boundedness of S^m implies that of M^m with constant independent of the dimension n. We have that

$$\|M^m(f_1,\dots,f_m)\|_{L^p(\mathbb{R}^n)} \le C(p_1,\dots,p_m)\prod_{i=1}^m \|f_i\|_{L^{p_i}}$$

with $C(p_1,\dots,p_m)$ independent of the dimension n.

In Chapter 3 we define the family of operators S^m_α, which is a multilinear version of S_α and that bridges the gap between S^m and M^m in manner similar to (1.8). We obtain $L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)$ in the optimal range of exponents

$$p > \frac{n}{mn-\alpha}.$$

This range interpolates between the bounds $p > \frac{1}{m}$ for M^m and $p > \frac{n}{mn-1}$ for S^m.

A corollary of the boundedness of S^m_α, which includes M^m and S^m (when $\alpha = 0$ and $\alpha = 1$ respectively) is the multilinear analogue of the Lebesgue differentiation theorem

$$\lim_{t \to 0} S^m_{\alpha,t}(f_1, \ldots, f_m)(x) = f_1(x) \cdots f_m(x).$$

An important counterpart to the operator M is the *uncentered* Hardy-Littlewood maximal function

$$M^{\mathrm{unc}} f(x) := \sup_{B^n(y,r) \ni x} \frac{1}{|B^n(y,r)|} \int_{B^n(y,r)} |f(z)| dz, \tag{1.10}$$

where the supremum is taken over all balls $B^n(y,r)$ in $\mathbb{R}^n$ that contain x.

Given the connections between the Hardy-Littlewood and the spherical maximal functions we have described above, it is natural to ask whether an uncentered version of S can be defined.

For a given direction of translation $u \in \mathbb{R}^n$ and a given radius $t > 0$, define the average

$$\mathcal{A}_{u,t} f(x) := \int_{\mathbb{S}^{n-1}} f(x + tu - ty) d\sigma(y).$$

For a compact subset T of $\mathbb{R}^n$, which will serve as a set of translations, we consider the maximal spherical translations

$$\mathcal{M}_T f(x) := \sup_{u \in T} \mathcal{A}_{u,1} |f|(x), \tag{1.11}$$

where we are considering averages over the unit spheres $x + u + \mathbb{S}^{n-1}$ with u varying in T. We also define the uncentered spherical maximal function

$$S_T f(x) := \sup_{\substack{t>0 \\ u \in T}} \mathcal{A}_{u,t} |f|(x), \tag{1.12}$$

which includes dilations.

The case that most resembles the uncentered Hardy-Littlewood maximal function is when $T = \overline{B^n}(0,1)$, the closed unit ball in $\mathbb{R}^n$. Then the operator S_T corresponds the averages over all spheres with x in their interior (in the Jordan-Brouwer sense) and $\mathcal{M}_T$ to all such unit spheres. However, by considering the characteristic function of an ε-neighborhood of the unit sphere, we can readily see that both S_T and $\mathcal{M}_T$ are unbounded on $L^p(\mathbb{R}^n)$ for any $p < \infty$ if $T = \overline{B^n}(0,1)$.

When $T = \mathbb{S}^{n-1}$, we instead are considering all spheres that pass through x. We will show that the operators M_T and S_T are unbounded as part of Corollary 4.1 when $T = \mathbb{S}^{n-1}$.

However, we can obtain boundedness results for $\mathcal{M}_T$ when the *upper Minkowski dimension* of T (see (5.7) below for a definition) is strictly less than $n-1$ and for S_T when it is strictly less than $n-2$. We denote by $\overline{\dim}_B T$ the upper Minkowski dimension (also known as the *upper box dimension*) of T. These results will be the subject of Chapter 5.

The counterexample for $\mathcal{M}_{\mathbb{S}^{n-1}}$ mentioned above follows from the construction of a Nikodym set in $\mathbb{R}^n$ associated with spheres, instead of hyperplanes. A classical Nikodym set is a set $A \subset \mathbb{R}^n$ of measure zero which contains a punctured hyperplane through every point: for every $y \in A$, there is a hyperplane $V_y \subset \mathbb{R}^n$ such that $y \in V_y$ and $V_y \setminus \{y\} \subset A$. Nikodym [42] proved the existence of such sets for $n = 2$, and Falconer [15] extended the result for all $n \geq 2$.

Chang and Csörnyei in [10] constructed Nikodym, Besicovitch, and Kakeya Sets associated with circles on $\mathbb{R}^2$. We prove the existence of the analogue of Nikodym sets for unit spheres instead of hyperplanes in Chapter 4. This is achieved by generalizing the Venetian blind construction from [10] to higher dimensions, which also implies the existence of Kakeya and Besicovitch sets for spheres.

The existence of Nikodym sets associated to spheres also yields a counterexample for an operator considered by Palsson and Sovine in [44], that is related to the triangle operator of Greenleaf and Iosevitch [27].

It has been conjectured that the classical Kakeya and Nikodym sets have full Hausdorff dimension. This is a very famous problem that remains open when $n \geq 3$. A similar question can be asked about their counterparts that are associated to spheres.

We will show in Chapter 5 that any Nikodym set for spheres has full Hausdorff dimension for any $n \geq 2$, by obtaining an L^2 bound for the Nikodym maximal function for spheres.

Chapter 2

The Multilinear Spherical Maximal Function

In this chapter we prove the $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ boundedness of the operator S^m and we provide counterexamples to show the optimality of the results obtained.

2.1 L^p, weak L^p, and Lorentz spaces

Our endpoint estimates involve weak Lebesgue as well as Lorentz spaces. The familiar L^p norm (or quasi-norm if $p < 1$) of a measurable function f is defined by

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}^n} |f(x)|^p \right)^{\frac{1}{p}} dx$$

for $0 < p < \infty$ and

$$\|f\|_{L^\infty} = \text{ess.sup}\, |f| = \inf \{\alpha \,:\, |\{x \,:\, |f(x)| > \alpha\}| > 0\} \,.$$

On the other hand, for $0 < p < \infty$, the weak-L^p (denoted $L^{p,\infty}$) norm of f is defined as

$$\|f\|_{L^{p,\infty}} = \inf \left\{ C > 0 \,:\, \big| \{x \,:\, |f(x)| > \alpha\} \big| \leq \frac{C^p}{\alpha^p} \quad \text{for all } \alpha > 0 \right\}.$$

The Lorentz $L^{p,q}$ norm of f is defined as

$$\|f\|_{L^{p,q}} = \left(\int_0^\infty \left(t^{\frac{1}{p}} f^*(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

for $0 < p, q < \infty$ where $f^*(t)$ is the decreasing rearrangement of f, defined on $[0, \infty)$ by

$$f^*(t) = \inf \left\{ \alpha > 0 \,:\, \big| \{x \,:\, |f(x)| > \alpha\} \big| \leq t \right\}.$$

The L^p, weak L^p and Lorentz spaces are the sets of all measurable functions with finite corresponding norm, under the equivalence relation identifying functions that are equal almost everywhere. The space $L^{\infty,\infty}$ is defined to be L^∞.

We refer the reader to [19, Chapter 1] for a detailed introduction to all of these spaces.

2.2 The boundedness region of S^m

Let $n \geq 2$, $1 \leq p_1, \ldots, p_m \leq \infty$, and $\sum_{j=1}^{m} \frac{1}{p_j} = \frac{1}{p}$. In this section we find the largest possible open set of exponents for which

$$\|S^m(f_1, \ldots, f_m)\|_{L^p} \lesssim \prod_{j=1}^{m} \|f_j\|_{L^{p_j}} \tag{2.1}$$

holds. At the endpoints of this open region we prove weak type estimates

$$\|S^m(f_1, \ldots, f_m)\|_{L^{p,\infty}} \lesssim \prod_{j=1}^{m} \|f_j\|_{L^{p_j}} \tag{2.2}$$

or Lorentz space estimates (when $n \geq 3$) of the form

$$\|S^m(f_1, \dots, f_m)\|_{L^{p,\infty}} \lesssim \Big(\prod_{j \neq k} \|f_j\|_{L^{p_j}}\Big) \|f_k\|_{L^{p_k,1}}, \quad k = 1, \dots, m. \tag{2.3}$$

We visualize the region of boundedness as a convex polytope with $2^m + m - 1$ vertices contained in the cube $[0,1]^m$ with coordinates $(\frac{1}{p_1}, \dots, \frac{1}{p_m})$ (see Figure 2.1). The closure of this region, which we denote by $\mathcal{R}$, is obtained as the intersection of the cube $[0,1]^m$ with the half space corresponding to the area of boundednsess:

$$\mathcal{R} := \left\{ \frac{1}{p_1}, \dots, \frac{1}{p_m} \in [0,1]^m \, : \, \sum_{j=1}^m \frac{1}{p_j} \leq \frac{mn-1}{n} \right\}.$$

Strong type boundedness at a point $(\frac{1}{p_1}, \dots, \frac{1}{p_m})$ means that (2.1) is satisfied; similarly for weak type and Lorentz space bounds. To better describe this region, we define

$$v_j = (1, \dots, 1, \frac{n-1}{n}, 1, \dots, 1) \quad \text{for } \; j = 1, \dots, m$$

and

$$V = \operatorname{conv}\{v_1, \dots, v_m\},$$

the closed convex hull of the v_j's. We also denote by $\partial\mathcal{R}$ the boundary of a region $\mathcal{R}$ in $\mathbb{R}^m$.

Theorem 2.1. *Let $n \geq 2$, $1 \leq p_1, \dots, p_m \leq \infty$ and $\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{p}$. Then the multilinear spherical maximal function S^m in (1.6) satisfies the following estimates:*

Case I: If $1 < p_j < \infty$ for all $j \in \{1, \dots, m\}$, then S^m is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ if and only if $p > \frac{n}{mn-1}$.

Case II: When $\frac{1}{p_j} \in \{0, 1\}$ for some j and $p > \frac{n}{mn-1}$ we have:

(a) At the vertex $(0, \dots, 0)$ the strong type estimate (2.1) holds.

(b) At the $2^m - 2$ vertices of $[0,1]^m$ except $(0, \dots, 0)$ and $(1, \dots, 1)$ the weak type estimate (2.2) holds.

Let $1 \leq k < m$. At each open k-dimensional face of $\partial[0,1]^m \cap \mathcal{R}$, described as *the set of all points* $(\frac{1}{p_1}, \dots, \frac{1}{p_m})$ *on the boundary of* $[0,1]^m \cap \mathcal{R}$ *with exactly* $m-k$ *fixed coordinates in* $\{0,1\}$*, we have:*

(c) *If all* $m-k$ *fixed coordinates are* 0*, then the strong type estimate* (2.1) *holds for all* $n \geq 2$*.*

(d) *If at least one fixed coordinate equals* 1*, then the strong type estimate* (2.1) *holds when* $n \geq 3$*.*

Case III: When $p = \frac{n}{mn-1}$ *(critical exponent), then we have when* $n \geq 3$*:*

(e) *On the boundary of* V *we have the Lorentz space estimate* (2.3)*.*

(f) *On the interior of* V *we have the weak type estimate* (2.2)*. More generally, we have*

$$\|S^m(f_1, \dots, f_m)\|_{L^{\frac{n}{mn-1},\infty}} \lesssim \prod_{j=1}^{m} \|f_j\|_{L^{p_j,s_j}}, \tag{2.4}$$

for all $s_1, \dots, s_m > 0$ *such that*

$$\sum_{j=1}^{m} \frac{1}{s_j} = \frac{mn-1}{n}.$$

Remarks. 1. *Using a well-known theorem of Stein and Strömberg [52], we can see that in the case of the largest open set, and the endpoint estimates (a) and (c) above, the implicit constant can be taken to be independent of the dimension* n*.*

2. *As was noted in [33], the method used in the proof of Theorem 2.1 also yields bounds for the stronger multi(sub)linear operator*

$$\mathscr{M}(f_1, \dots, f_m)(x) := \sup_{t_1,\dots,t_m>0} \left| \int_{\mathbb{S}^{mn-1}} \prod_{j=1}^{m} f_j(x - t_j y^j) d\sigma_{mn-1}(y^1, \dots, y^m) \right| \tag{2.5}$$

in the same ranges of p_j*'s as* S^m*. Also, trivially, the counterexamples provided for the unboundedness of* S^m *also work for* $\mathscr{M}$*.*

3. *When* $n = 1$ *and* $m \geq 3$*, estimates in the case* $L^p(\mathbb{R}) \times L^\infty(\mathbb{R}) \times \dots \times L^\infty(\mathbb{R}) \to L^p(\mathbb{R})$ *for* $p > \frac{m}{m-1}$ *follow from the classical theorem of Rubio de Francia in [45]. However the optimal results in the case* $n = 1$ *remain open.*

As an example we graph the area of boundedness for the trilinear spherical maximal function.

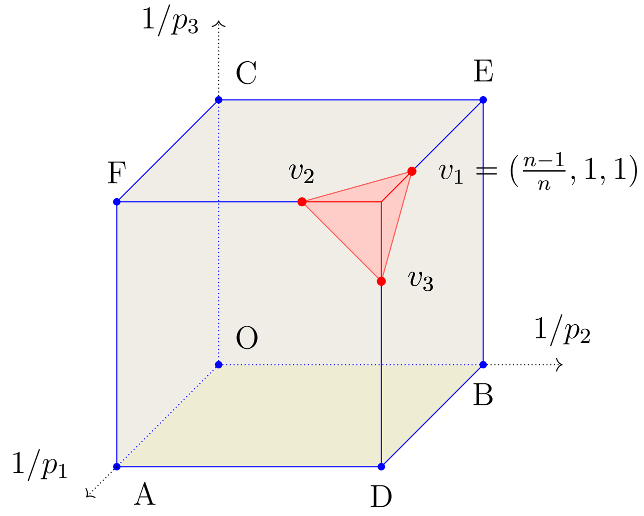

Figure 2.1: $L^{p_1} \times L^{p_2} \times L^{p_3} \to L^p$ boundedness of the trilinear spherical maximal operator $(n \geq 2)$.

The counterexamples claimed in Theorem 2.1 are contained in Proposition 2.1.

Proposition 2.1. *Let* $1 \leq p_1, \ldots, p_m \leq \infty$ *and* $\sum_{j=1}^{m} \frac{1}{p_j} = \frac{1}{p}$. *The multilinear spherical maximal function* S^m *is unbounded from* $L^{p_1} \times \cdots \times L^{p_m}$ *to* L^p *when*

$$p \leq \frac{n}{mn-1} \quad \text{and } n \geq 2.$$

The following proposition contributes a negative answer to a question posed by Jeong and Lee in [33] regarding a strong type $L^1(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ bound for the bilinear spherical maximal function.

Proposition 2.2. *Let* $p_j \in \{1, \infty\}$ *for all* $j = 1, \ldots, m$. *Then the strong type estimate*

$$\|S^m(f_1, \ldots, f_m)\|_{L^p} \lesssim \prod_{j=1}^{m} \|f_j\|_{L^{p_j}}$$

holds if and only if $p_j = \infty$ *for all* $j = 1, \ldots, m$.

The optimality of the weak type and Lorentz space estimates of Theorem 2.1 not covered in Proposition 2.2 remains open.

2.3 Proof of Theorem 2.1

The proof of our main theorem is based on the following slicing lemma, which was first used in [33] in the bilinear case. This result can be found for instance in [19, Appendix D.2], [3], or [29] and it is a special case of the co-area formula [16, Theorem 3.2.22], but we include a proof in Section 2.5 for the sake of completeness.

Lemma 2.1. *Let $1 \leq k < m$ and $n \geq 2$. For a function $F(y^1, \dots, y^m)$ defined in $\mathbb{R}^{mn}$ with $y^j \in \mathbb{R}^n$, $j = 1, \dots, m$, we have*

$$\begin{aligned} &\int_{\mathbb{S}^{mn-1}} F(y^1, \dots, y^m)\, d\sigma_{mn-1}(y^1, \dots, y^m) \qquad (2.6) \\ &= \int_{B^{kn}} \int_{r_{Y_k}\mathbb{S}^{(m-k)n-1}} F(y^1, \dots, y^m)\, d\sigma^{r_{Y_k}}_{(m-k)n-1}(y^{k+1}, \dots, y^m)\, \frac{dy^1 \cdots dy^k}{\sqrt{1 - \sum_{j=1}^k |y^j|^2}}, \end{aligned}$$

where $B^{kn} = B^{kn}(0,1)$ is the unit ball in $\mathbb{R}^{kn}$, $r_{Y_k} = \sqrt{1 - \sum_{j=1}^k |y^j|^2}$ and $d\sigma^{r_{Y_k}}_{(m-k)n-1}$ is the normalized surface measure on $r_{Y_k}\mathbb{S}^{(m-k)n-1}$.

Lemma 2.1 enables us to pointwise control the multilinear spherical maximal function in terms of a product of $m-1$ Hardy-Littlewood maximal operators and one spherical maximal operator.

Proof of Theorem 2.1. To avoid technicalities arising from interpolating sublinear operators, we consider the following linerization of the maximal operator. For a measurable function $\tau : \mathbb{R}^n \to [0, \infty)$ we define

$$T(f_1, \dots, f_m)(x) := \int_{\mathbb{S}^{mn-1}} \prod_{j=1}^m f_j(x - \tau(x) y^j) d\sigma_{mn-1}(y^1, \dots, y^m).$$

Since the boundedness of T on some spaces implies the boundedness for S^m on the same spaces, it is enough to show

$$\|T(f_1, \dots, f_m)\|_{L^p} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j}},$$

for some C independent of τ.

Since

$$|T(f_1,\ldots,f_m)| \leq S^m(f_1,\ldots,f_m),$$

applying Lemma 2.1 with $k = m-1$ yields the following m pointwise estimates:

$$T(f_1,\ldots,f_m)(x) \lesssim \prod_{j\neq k} Mf_j(x) \cdot Sf_k(x), \quad k = 1,\ldots,m, \tag{2.7}$$

where Mf_j is the Hardy-Littlewood maximal function of f_j and Sf_k is the linear spherical maximal function of f_k. As was covered in the introduction, for M it is well known that

$$\|Mf\|_{L^p} \lesssim \|f\|_{L^p}$$

for $p > 1$ and

$$\|Mf\|_{L^{1,\infty}} \lesssim \|f\|_{L^1}.$$

Also, for $n \geq 2$,

$$\|Sf\|_{L^p} \lesssim \|f\|_{L^p}$$

if and only if $p > \frac{n}{n-1}$. Therefore, by Hölder's inequality we obtain the following m estimates

$$\begin{aligned} \|T(f_1,\ldots,f_m)\|_{L^p} &\leq \prod_{j\neq k} \|Mf_j\|_{L^{p_j}} \cdot \|Sf_k\|_{L^{p_k}}, \quad k = 1,\ldots,m \\ &\lesssim \prod_{j=1}^{m} \|f_j\|_{L^{p_j}}, \end{aligned}$$

when

$$p_k > \tfrac{n}{n-1},\ 1 < p_1,\ldots,p_m \leq \infty, \text{ and } \sum_{j=1}^{m} \tfrac{1}{p_j} = \tfrac{1}{p}.$$

Thus, applying (complex) interpolation between these estimates and the trivial $L^\infty \times \cdots \times L^\infty \to L^\infty$ bound, we obtain the boundedness in the largest possible open set of exponents, as well as the endpoint estimates (a), (b) and (c) in the statement of the theorem (see [25] and [20, Theorem 7.2.2] for the interpolation result we used for (c)).

For the estimates in $(d)-(f)$, we will use Bourgain's restricted weak type endpoint bound for the linear spherical maximal function, which only holds for $n \geq 3$ (for $n = 2$ Seeger, Tao, and Wright [47] showed that the restricted weak type inequality fails in the linear case). So for $n \geq 3$ we have the following m estimates:

$$\|T(f_1, \ldots, f_m)\|_{L^{\frac{n}{mn-1},\infty}} \leq \prod_{j \neq k} \|Mf_j\|_{L^{1,\infty}} \cdot \|Sf_k\|_{L^{\frac{n}{n-1},\infty}}$$
$$\lesssim \prod_{j \neq k} \|f_j\|_{L^1} \|f_k\|_{L^{\frac{n}{n-1},1}}.$$

Interpolating these estimates with the estimates in (c), we conclude (d).

Moreover, trivially

$$\|T(\chi_{F_1}, \ldots, \chi_{F_m})\|_{L^{\frac{n}{mn-1},\infty}} \lesssim \prod_{j=1}^{m} |F_j|^{\frac{1}{p_j}}$$

for any measurable sets $F_1, \ldots, F_m$ and for all

$$1 \leq p_1, \ldots, p_m \leq \frac{n}{n-1} \quad \text{such that} \quad \sum_{j=1}^{m} \frac{1}{p_j} = \frac{mn-1}{m}.$$

Since $L^{\frac{n}{mn-1},\infty}(\mathbb{R}^n)$ is $\frac{n}{mn-1}$ - convex, the estimates imply

$$\|T(f_1, \ldots, f_m)\|_{L^{\frac{n}{mn-1},\infty}} \lesssim \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\frac{n}{mn-1}}}.$$

Thus, using multilinear interpolation we conclude (e) and (f) (see [4, Lemma 2.1 and Proposition 2.2] for the multilinear interpolation result used here). $\square$

2.4 Counterexamples

The counterexamples are obtained by an adaptation of the examples in [3, Proposition 7]. These are based on the original counterexamples of Stein [50].

Proof of Proposition 2.1. We consider the functions

$$f_j(y) = |y|^{-\frac{n}{p_j}} \log\left(\frac{1}{|y|}\right)^{-\frac{m}{p_j}} \chi_{|y|\le \nu_j},$$

where $\nu_j = e^{-m/n}/100$ when $j \le m-1$ and $\nu_m = e^{-m/n}/2$. Then $f_j \in L^{p_j}(\mathbb{R}^n)$. Indeed

$$\begin{aligned}
\int_{\mathbb{R}^n} |f_j(y)|^{p_j} dy &= \int_{B^n(0,\nu_j)} |y|^{-n} \log\left(\frac{1}{|y|}\right)^{-m} dy \\
&= \int_0^{\nu_j} |r|^{-1} \log\left(\frac{1}{|r|}\right)^{-m} dy \\
&= \int_{1/\nu_j}^{\infty} u^{-1} \log(u)^{-m} du \\
&= \int_{\log(1/\nu_j)}^{\infty} v^{-m} dv < \infty, \qquad \text{since } m > 1.
\end{aligned}$$

For $s > 0$, let

$$S_s^m(f_1,\dots,f_m)(x) := \int_{\mathbb{S}^{mn-1}} \prod_{j=1}^m f_j(x - sy^j) d\sigma_{mn-1}(y^1,\dots,y^m).$$

Since the mapping

$$(y^1,\dots,y^m) \mapsto (Ay^1,\dots,Ay^m),$$

with $A \in SO_n$ is an isometry on $\mathbb{S}^{mn-1}$, we see that we estimate

$$S^m(f_1,\dots,f_m) \ge S^m_{\sqrt{m}R}(f_1,\dots,f_m)(Re_1),$$

for some R large, where $e_1 = (1,0,\dots,0) \in \mathbb{R}^n$.

We let

$$Y = (y^1,\dots,y^{m-1}) \in \mathbb{R}^{(m-1)n}, \quad y^m =: z = (z', z_n),$$

with

$$z' = (y_1^m,\dots,y_{n-1}^m) \in \mathbb{R}^{n-1} \quad \text{and } z_n = \sqrt{1 - \sum_{j=1}^{m-1} |y^j|^2 - |z'|^2}.$$

Moreover, we define

$$E = (e_1, \dots, e_1) \in \mathbb{R}^{(m-1)n}.$$

Applying Lemma 2.1 with $k = m-1$, we have that

$$S^m_{\sqrt{m}R}(f_1, \dots, f_m)(Re_1) =$$
$$= \int_{B^{(m-1)n}} \prod_{j=1}^{m-1} f_j(Re_1 - \sqrt{m}Ry^j) \int_{r_Y \mathbb{S}^{n-1}} f_m(Re_1 - \sqrt{m}Rz) d\sigma^{r_Y}_{n-1}(z) \frac{dY}{\sqrt{1-|Y|^2}}.$$

First we focus on the inner integral, namely

$$I = \int_D |Re_1 - \sqrt{m}Rz|^{-\frac{n}{pm}} \left(-\log |Re_1 - \sqrt{m}Rz|\right)^{-\frac{m}{pm}} d\sigma^{r_Y}_{n-1}(z),$$

where

$$D = r_Y \mathbb{S}^{n-1} \cap B^n\left(\frac{e_1}{\sqrt{m}}, \frac{\nu_m}{\sqrt{m}R}\right).$$

For

$$z_0 \in r_Y \mathbb{S}^{n-1} \cap \partial B^n\left(\frac{e_1}{\sqrt{m}}, \frac{\nu_m}{\sqrt{m}R}\right),$$

we let θ be the angle between the vectors z_0 and e_1, which is the largest one between $z \in D$ and e_1. Then θ is small since R is large and

$$|D| \sim (\sqrt{1-|Y|^2}\theta)^{n-1} \sim \theta^{n-1}.$$

Using the fact that for θ small,

$$\theta^2 \sim \sin^2\theta = 1 - \cos^2\theta \sim 1 - \cos\theta,$$

and the law of cosines

$$\frac{1}{4mR^2} = 1 - |Y|^2 + \frac{1}{m} - 2\sqrt{1-|Y|^2}\frac{1}{\sqrt{m}}\cos\theta,$$

we obtain that

$$\theta^2 \sim \frac{1}{4mR^2} - \left(\sqrt{1-|Y|^2} - \frac{1}{\sqrt{m}}\right)^2.$$

In turn, since

$$\sqrt{1-|Y|^2} > \frac{1}{2}$$

when $R > 2(m-1)$, we have that

$$\left|\sqrt{1-|Y|^2} - \frac{1}{\sqrt{m}}\right| \leq \frac{4(m-1)}{100\sqrt{m}R}$$

and thus we conclude that $\theta \geq C/R$. The same calculation also yields that

$$\left|\sqrt{1-|Y|^2} - \frac{1}{\sqrt{m}}\right| \lesssim \left|\frac{E}{\sqrt{m}} - Y\right|,$$

that will be used later in the proof.

Hence, we can bound I from below by

$$\int_0^\theta \int_{r_Y \sin\alpha \mathbb{S}^{n-2}} |Re_1 - \sqrt{m}Rz|^{-\frac{n}{pm}} \left(-\log|Re_1 - \sqrt{m}Rz|\right)^{-\frac{m}{pm}} d\sigma_{n-2}^{r_Y \sin\alpha} d\alpha,$$

where

$$|z| = r_{_Y} = \sqrt{1-|Y|^2} \approx \frac{1}{\sqrt{m}} \quad \text{and} \quad z = r_{_Y}\cos\alpha.$$

By symmetry, it suffices to consider the case $r_{_Y} < \frac{1}{\sqrt{m}}$. Let β be the angle such that

$$\left|\frac{e_1}{\sqrt{m}} - z\right| = 2\left|r_{_Y} - \frac{1}{\sqrt{m}}\right|.$$

Then

$$\beta \approx \left|r_{_Y} - \frac{1}{\sqrt{m}}\right|.$$

On the other hand, when $\alpha = 0$, we have

$$\left|\frac{e_1}{\sqrt{m}} - z\right| = \left|r_{_Y} - \frac{1}{\sqrt{m}}\right|.$$

Therefore, for all $\alpha \in [0, \beta]$ we have that

$$\left|z - \frac{e_1}{\sqrt{m}}\right| \approx \left|r_{_Y} - \frac{1}{\sqrt{m}}\right| \lesssim \left|Y - \frac{E}{\sqrt{m}},\right|$$

as was noted above. Thus, using Lemma 2.2, we obtain

$$\begin{aligned}
&I \geq \\
&\geq C\int_0^\theta \int_{r_Y \sin\alpha \mathbb{S}^{n-2}} \frac{|Re_1 - \sqrt{m}Rz|^{1-n}}{|Re_1 - \sqrt{m}Rz|^{\frac{n}{pm}-n+1} \left(-\log|Re_1 - \sqrt{m}Rz|\right)^{\frac{m}{pm}}} d\sigma_{n-2}^{r_Y \sin\alpha}(z) d\alpha \\
&\geq CR^{1-n}|RE - \sqrt{m}RY|^{-\frac{n}{pm}+n-1}\left(-\log|RE - \sqrt{m}RY|\right)^{-\frac{m}{pm}} |\sqrt{m}r_{_Y} - 1|^{1-n} \\
&\qquad\qquad\qquad\qquad \cdot \int_0^{C(1-\sqrt{m}r_{_Y})} \sin^{n-2}\alpha \, d\alpha \\
&\geq CR^{1-n}|RE - \sqrt{m}RY|^{-\frac{n}{pm}+n-1}\left(-\log|RE - \sqrt{m}RY|\right)^{-\frac{m}{pm}}.
\end{aligned}$$

Also, for any $0 \leq j \leq m-1$, we have the trivial bound

$$|Re_1 - \sqrt{m}Ry^j| \leq \left(\sum_{j=1}^{m-1} |Re_1 - \sqrt{m}Ry^j|^2\right)^{1/2} = |RE - \sqrt{m}RY|.$$

Thus using Lemma 2.2 again, we see that

$$\begin{aligned}
&M_{\sqrt{m}R}(f_1, \ldots, f_m)(Re_1) \\
&\geq CR^{1-n}\int_{B^{(m-1)n}(\frac{E}{\sqrt{m}}, \frac{1}{50\sqrt{m}R})} |RE - \sqrt{m}RY|^{-\frac{n}{p}+n-1}\left(-\log|RE - \sqrt{m}RY|\right)^{-\frac{m}{p}} dY \\
&\geq CR^{1-mn}\int_{B^{(m-1)n}(0, \frac{1}{50})} |w|^{-\frac{n}{p}+n-1}\left(-\log|w|\right)^{-\frac{m}{p}} dw \\
&\geq CR^{1-mn}\int_0^{\frac{1}{50}} r^{-\frac{n}{p}+mn-2}\left(-\log r\right)^{-\frac{m}{p}} dr \\
&= \begin{cases} CR^{1-mn} & \text{if } p = \frac{n}{mn-1} \\ \infty & \text{if } p < \frac{n}{mn-1}. \end{cases}
\end{aligned}$$

We therefore conclude that $S^m(f_1, \ldots, f_m)$ is not in L^p for any $p < \frac{n}{mn-1}$ and when

$p = \frac{n}{mn-1}$, and

$$S^m(f_1, \ldots, f_m)(x) \gtrsim |x|^{1-mn}$$

for all $|x|$ large enough and thus it is also not in $L^{\frac{n}{mn-1}}(\mathbb{R}^n)$ when $p = \frac{n}{mn-1}$. $\square$

Proof of Proposition 2.2. The $L^\infty \times \cdots \times L^\infty \to L^\infty$ holds trivially and the multilinear maximal function is unbounded from $L^1 \times \cdots \times L^1 \to L^{\frac{1}{m}}$ by Proposition 2.1. Let $1 \leq k < m$. By symmetry, it is enough to show that the strong type estimate (2.1) fails at the point $(1, \ldots, 1, 0, \ldots, 0)$. Let

$$f_1(y) = \cdots = f_k(y) = \chi_{B^n\left(0,\frac{1}{2}\right)}(|y|)$$

and

$$f_{k+1}(y) = \cdots = f_m = 1.$$

Then, similar to the proof of Proposition 2.1, we obtain the following pointwise bound

$$\begin{aligned} S^m(f_1, \ldots, f_m)(x) = S^m(f_1, \ldots, f_m)(Re_1) &\geq S^m_{\sqrt{m}R}(f_1, \ldots, f_m)(Re_1) \\ &= \int_{\mathbb{S}^{mn-1}} \prod_{j=1}^{k} \chi_{B^n\left(0,\frac{1}{2}\right)}(|Re_1 - \sqrt{m}Ry^j|) d\sigma_{mn-1}(y^1, \ldots, y^m) \\ &\geq \int_{\mathbb{S}^{mn-1}} \chi_{B^{kn}\left(0,\frac{1}{2}\right)}(|RE_k - \sqrt{m}RY_k|) d\sigma_{mn-1}(y^1, \ldots, y^m), \end{aligned}$$

where

$$E_k = (e_1, \ldots, e_1) \quad \text{and} \quad Y_k = (y^1, \ldots, y^k)$$

are vectors in $\mathbb{R}^{kn}$. Applying Lemma 2.1, we have that

$$
\begin{aligned}
&\int_{\mathbb{S}^{mn-1}} \chi_{B^{kn}(0,\frac{1}{2})}(|RE_k - \sqrt{m}RY_k|) d\sigma_{mn-1}(y^1, \dots, y^m) \\
&= \int_{\mathbb{B}^{kn}} \chi_{B^{kn}(0,\frac{1}{2})}(|RE_k - \sqrt{m}RY_k|) \\
&\qquad\qquad\qquad \int_{r_{Y_k}\mathbb{S}^{(m-k)n-1}} d\sigma^{r_Y}_{[(m-k)n-1]}(y^{k+1}, \dots, y^m) \frac{dY_k}{\sqrt{1-|Y_k|^2}} \\
&\gtrsim \int_{\mathbb{B}^{kn}} \chi_{B^{kn}(0,\frac{1}{2})}(|RE_k - \sqrt{m}RY_k|) dY_k,
\end{aligned}
$$

since

$$
1 - |Y_k|^2 \geq \frac{1}{2m}
$$

when R is large enough. Therefore we obtain

$$
\begin{aligned}
S^m(f_1, \dots, f_m)(x) =& S^m(f_1, \dots, f_m)(Re_1) \\
\geq& S^m_{\sqrt{m}R}(f_1, \dots, f_m)(Re_1) \\
\gtrsim& R^{-kn},
\end{aligned}
$$

and thus S^m does not map $L^1 \times \cdots \times L^1 \times L^\infty \times \cdots \times L^\infty \to L^{\frac{1}{k}}$. $\square$

2.5 Lemmas 2.1 and 2.2

Proof of Lemma 2.1. For $y^j = (y^j_1, \dots, y^j_n) \in \mathbb{R}^n$, $j = 1, \dots, m$, we set

$$
Y = (y^1, \dots, y^k) \in \mathbb{R}^{kn}, \quad (y^{k+1}, \dots, y^m) =: Z = (Z', z_n),
$$

with

$$
Z' = (y^{k+1}, \dots, y^{m-1}, y^m_1, \dots, y^m_{n-1}) \in \mathbb{R}^{(m-k)n-1}
$$

and

$$
z_n = \sqrt{1 - \sum_{j=1}^{k} |y^j|^2 - |Z'|^2}.
$$

For the sake of clarity in notation, we write

$$r_{_Y} = \sqrt{1 - \sum_{j=1}^{k} |y^j|^2}$$

instead of $r_{_{Y_k}}$. Setting

$$Z/r_{_Y} = W = (W', w_n),$$

we express the right hand side of (2.6) as

$$\begin{aligned}
&\int_{\mathbb{B}^{kn}} \int_{r_Y \mathbb{S}^{(m-k)n-1}} F(Y,Z) d\sigma^{r_Y}_{[(m-k)n-1]}(Z) \frac{dY}{\sqrt{1-|Y|^2}} \\
&= \int_{\mathbb{B}^{kn}} r_{_Y}{}^{(m-k)n-1} \int_{\mathbb{S}^{(m-k)n-1}} F(Y, r_{_Y} W) d\sigma_{[(m-k)n-1]}(W) \frac{dY}{\sqrt{1-|Y|^2}} \\
&= \int_{\mathbb{B}^{kn}} r_{_Y}{}^{(m-k)n-1} \\
&\qquad \cdot \int_{\mathbb{B}^{(m-k)n-1}} [F(Y, r_{_Y} W', r_{_Y} w_n) + F(Y, r_{_Y} W', -r_{_Y} w_n)] \frac{dW'}{\sqrt{1-|W'|^2}} \frac{dY}{\sqrt{1-|Y|^2}} \\
&= \int_{\mathbb{B}^{kn}} \int_{r_Y \mathbb{B}^{(m-k)n-1}} [F(Y, Z', z_n) + F(Y, Z', -z_n)] \frac{dZ'}{\sqrt{1-|W'|^2}} \frac{dY}{\sqrt{1-|Y|^2}} \\
&= \int_{\mathbb{B}^{kn}} \int_{r_Y \mathbb{B}^{(m-k)n-1}} [F(Y, Z', z_n) + F(Y, Z', -z_n)] \frac{dZ' dY}{\sqrt{1-|Y|^2-|Z'|^2}},
\end{aligned}$$

as one can verify that

$$\sqrt{1-|W'|^2}\sqrt{1-|Y|^2} = \sqrt{1-|Y|^2-|Z'|^2}.$$

Using that $\mathbb{B}^{mn-1}$ is equal to the disjoint union

$$\mathbb{B}^{mn-1} = \bigcup \Big\{ \big(Y, r_{_Y} v\big) \;:\; v \in \mathbb{B}^{(m-k)n-1} \Big\},$$

we see that the last integral is equal to

$$\int_{\mathbb{B}^{mn-1}} [F(Y, Z', z_n) + F(Y, Z', -z_n)] \frac{dY dZ'}{\sqrt{1-|Y|^2-|Z'|^2}},$$

which, in turn, is equal to

$$\int_{\mathbb{S}^{mn-1}} F(Y,Z)d\sigma_{mn-1}(Y,Z)$$

using [19, Appendix D.5]. $\square$

Lemma 2.2. *Let $r_1, r_2 > 0$, $t, s < e^{-\frac{r_2}{r_1}}$ and $t \leq Cs$ for some $C \geq 1$. Then there exists an absolute constant C' (depending only on C, r_1, r_2) such that*

$$s^{-r_1}\left(\log\frac{1}{s}\right)^{-r_2} \leq C't^{-r_1}\left(\log\frac{1}{t}\right)^{-r_2}. \tag{2.8}$$

Proof. For $x > 0$ we define

$$F(x) = x^{r_1}\left(\log x\right)^{-r_2}.$$

Differentiating F, we see that F is increasing when $x > e^{\frac{r_2}{r_1}}$ and so for $s < e^{-\frac{r_2}{r_1}}$,

$$\begin{aligned} F\left(\frac{1}{s}\right) =& s^{-r_1}\left(\log\frac{1}{s}\right)^{-r_2} \\ \leq& C^{r_1}(Cs)^{-r_1}\left(\log\frac{1}{Cs}\right)^{-r_2} \\ =& C^{r_1}F\left(\frac{1}{Cs}\right) \\ \leq& C'F\left(\frac{1}{t}\right) \\ =& C't^{-r_1}\left(\log\frac{1}{t}\right)^{-r_2}. \end{aligned}$$

$\square$

Chapter 3

Families between the Hardy-Littlewood and Spherical maximal functions

3.1 The families S_α and S_α^m

In this chapter, we denote by $d\sigma_{\kappa-1}$ the surface measure on unit sphere $\mathbb{S}^{\kappa-1}$ in $\mathbb{R}^\kappa$, v_κ the measure of the unit ball in $\mathbb{R}^\kappa$ and $\omega_{\kappa-1} = d\sigma_{\kappa-1}(\mathbb{S}^{\kappa-1})$ is the total measure of $\mathbb{S}^{\kappa-1}$. Recall that $\kappa v_\kappa = \omega_{\kappa-1}$ for any integer $\kappa \geq 2$. We also use the notation $\mathbb{B}^\kappa$ for the unit ball in $\mathbb{R}^\kappa$ and $R\,\mathbb{B}^\kappa$ for the ball of radius $R > 0$ centered at the origin in $\mathbb{R}^\kappa$. The space of Schwartz functions on $\mathbb{R}^\kappa$ is denoted by $\mathcal{S}(\mathbb{R}^\kappa)$.

We will study multilinear versions of S, S_α and of M. We introduce the family of operators

$$S_\alpha^m(f_1,\ldots,f_m)(x) = \frac{2/\omega_{mn-1}}{B(\frac{mn}{2},1-\alpha)} \sup_{t>0} \int_{\mathbb{B}^{mn}} \prod_{i=1}^m |f_i(x-ty_i)| \frac{dy}{(1-|y|^2)^\alpha}, \qquad (3.1)$$

defined initially for functions $f_i \in \mathcal{S}(\mathbb{R}^n)$ and $0 \leq \alpha < 1$. This is a multilinear extension of the operator $S_\alpha = S_\alpha^1$ introduced in (1.7).

We recall the definition of the multilinear spherical maximal operator

$$S^m(f_1,\dots,f_m)(x) = \sup_{t>0} \frac{1}{\omega_{mn-1}} \int_{\mathbb{S}^{mn-1}} \prod_{i=1}^m |f_i(x-t\theta_i)|\, d\sigma_{mn-1}(\theta_1,\dots,\theta_m), \quad (3.2)$$

given also for functions $f_i \in \mathcal{S}(\mathbb{R}^n)$. When $m=1$, S^m reduces to S in (1.4).

We would like to extend the definitions of the operators in (3.1) and (3.2) to functions in f_i in $L^1_{loc}(\mathbb{R}^n)$, which is the space of all locally integrable functions, meaning that $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ if and only if $f \in L^1(K)$ for every compact $K \subset \mathbb{R}^n$.

Fix f_i in $L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$; then

$$t \mapsto F(t) = t^{mn-1} \int_{\mathbb{S}^{mn-1}} \prod_{i=1}^m |f_i(x-t\theta_i)| d\sigma_{mn-1}(\theta_1,\dots,\theta_m) \quad (3.3)$$

is integrable over any interval $[0,L]$, which implies that the integrals in (3.2) are finite for almost all $t>0$. Likewise, if F is as in (3.3) and $t \in (0,L)$, then

$$\begin{aligned} \int_{\mathbb{B}^{mn}} \prod_{i=1}^m |f_i(x-ty_i)| \frac{dy}{(1-|y|^2)^\alpha} &= \int_0^1 \frac{F(tr)}{(1-r^2)^\alpha} \frac{dr}{t^{mn-1}} \\ &\le \frac{1}{t^{mn-\alpha}} \int_0^L \frac{F(s)\, ds}{(t-s)^\alpha}, \end{aligned} \quad (3.4)$$

and the last integral is the convolution (evaluated at t) of the L^1 functions $F\chi_{[0,L]}$ and $s^{-\alpha}\chi_{(0,L]}$ on the real line, hence it is finite a.e. on $(0,L)$. We conclude that the integral in (3.1) is finite for almost all $t>0$ for $f_i \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

Now, one cannot properly define the supremum of a family $\{A_t\}_{t>0}$ ($A_t \ge 0$) which satisfies $A_t < \infty$ for almost all $t>0$. But it is possible to define the essential supremum of $\{A_t\}_{t>0}$, which is practically the supremum restricted over the subset of $(0,\infty)$ on which $A_t < \infty$. So to extend the definitions of the operators in (3.2) and (3.1) to functions $f_i \in L^1_{loc}(\mathbb{R}^n)$ for any $x \in \mathbb{R}^n$ by replacing the supremum in these expressions by the essential supremum ess.sup. However, this adjustment is not needed when

$$f_i \in L^{p_i}(\mathbb{R}^n) \quad \text{with} \quad \sum_{i=1}^{n} \frac{1}{p_i} = \frac{1}{p} < \frac{mn-\alpha}{n},$$

since, in that case, the corresponding averages vary continuously in t. See Corollary 3.1 below. Based on this discussion we provide the following definition.

Definition 3.1. *Let $t > 0$, $f_i \in L^1_{loc}(\mathbb{R}^n)$ for $1 \le i \le m$, and $x \in \mathbb{R}^n$. We define*

$$S^m_{\alpha,t}(f_1,\ldots,f_m)(x) = \frac{2/\omega_{mn-1}}{B(mn/2, 1-\alpha)} \int_{\mathbb{B}^{mn}} \prod_{i=1}^{m} f_i(x - ty_i) \frac{dy}{(1-|y|^2)^\alpha} \tag{3.5}$$

and

$$S^m_\alpha(f_1,\ldots,f_m)(x) = \operatorname*{ess.sup}_{t>0} S^m_{\alpha,t}(|f_1|,\ldots,|f_m|)(x) \tag{3.6}$$

for $0 \le \alpha < 1$. We also define

$$S^m_{1,t}(f_1,\ldots,f_m)(x) = \frac{1}{\omega_{mn-1}} \int_{\mathbb{S}^{mn-1}} \prod_{i=1}^{m} f_i(x - t\theta_i)\, d\sigma_{mn-1}(\theta_1,\ldots,\theta_m) \tag{3.7}$$

and

$$S^m(f_1,\ldots,f_m)(x) = \operatorname*{ess.sup}_{t>0} S^m_{1,t}(|f_1|,\ldots,|f_m|)(x). \tag{3.8}$$

We prove the following results:

Theorem 3.1. *Let $0 < \alpha < 1$. Given $f_i \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ we have*

$$M^m(f_1,\ldots,f_m)(x) \le S^m_\alpha(f_1,\ldots,f_m)(x) \le S^m(f_1,\ldots,f_m)(x) \tag{3.9}$$

$$\lim_{\alpha\to 1^-} S^m_\alpha(f_1,\ldots,f_m)(x) = S^m(f_1,\ldots,f_m)(x) \tag{3.10}$$

$$\lim_{\alpha\to 0^+} S^m_\alpha(f_1,\ldots,f_m)(x) = M^m(f_1,\ldots,f_m)(x). \tag{3.11}$$

These statements are valid even when some of the preceding expressions equal ∞.

As M^m is pointwise controlled by the product of the Hardy-Littlewood operators acting on each function, this operator is bounded from $L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)$

to $L^p(\mathbb{R}^n)$ in the full range of exponents

$$1 < p_1, \ldots, p_m \leq \infty \quad \text{and} \ \ 1/p_1 + \cdots + 1/p_m = 1/p.$$

Boundedness for S^m holds in the smaller region

$$n/(mn-1) < p \leq \infty$$

as shown in Chapter 2. So it is expected that S^m_α are bounded in some intermediate regions. This is the content of the following result.

Theorem 3.2. *Let $n \geq 2$, $0 \leq \alpha < 1$, and $1 < p_i \leq \infty$. Define p by $\sum_{i=1}^m \frac{1}{p_i} = \frac{1}{p}$. Then there is a constant $C = C(m, \alpha, p_1, \ldots, p_m)$ such that*

$$\|S^m_\alpha(f_1, \ldots, f_m)\|_{L^p(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)} \tag{3.12}$$

for all $f_i \in L^{p_i}(\mathbb{R}^n)$ if and only if

$$\frac{n}{mn-\alpha} < p \leq \infty.$$

Moreover, if (3.12) holds, then the constant C can be chosen to be independent of the dimension (as indicated by the parameters on which it is claimed to depend).

Remark 3.1. *As a consequence, we obtain dimensionless $L^{p_1} \times \cdots \times L^{p_m} \to L^p$ bounds for the multilinear Hardy-Littlewood maximal function M^m. Precisely, we have*

$$\|M^m(f_1, \ldots, f_m)\|_{L^p} \leq C(p_1, \ldots, p_m) \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)},$$

where

$$\frac{1}{m} < p \leq \infty, \quad \sum_{i=1}^n \frac{1}{p_i} = \frac{1}{p}$$

and $C(p_1, \ldots, p_m)$ does not depend of n.

This extents the result of Stein and Strömberg [52] to the multilinear setting.

In Figure 3.1 we graph the range of boundedness for the bilinear operator S^2_α.

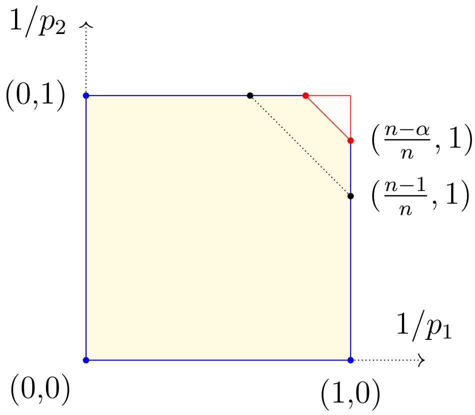

Figure 3.1: Range of $L^{p_1} \times L^{p_2} \to L^p$ boundedness of S^2_α when $0 \le \alpha \le 1$ and $n \ge 2$. The bilinear spherical maximal function is bounded below the black dotted line, while the bilinear Hardy-Littlewood maximal function is bounded on the entire square.

The estimates in (3.12) imply that when $f_i \in L^{p_i}(\mathbb{R}^n)$ with

$$\sum_{i=1}^{n} \frac{1}{p_i} = \frac{1}{p} < \frac{mn-\alpha}{n},$$

then for almost all $x \in \mathbb{R}^n$, the averages

$$S^m_{\alpha,t}(f_1, \dots, f_m)(x)$$

are finite uniformly in $t > 0$.

Corollary 3.1. *Let $0 \le \alpha \le 1$ and suppose that for all $1 \le i \le m$, $f_i \in L^{p_i}(\mathbb{R}^n)$ where $1 < p_i \le \infty$ satisfy $\sum_{i=1}^m \frac{1}{p_i} = \frac{1}{p} < \frac{mn-\alpha}{n}$. Then for almost every x in $\mathbb{R}^n$, the function*

$$t \mapsto S^m_{\alpha,t}(f_1, \dots, f_m)(x)$$

is well defined and continuous in $t \in (0, \infty)$. Therefore in Definition 3.1, for almost all $x \in \mathbb{R}^n$, we can replace the essential supremum by a supremum in both (3.6) and (3.8).

Corollary 3.2. *Let $0 \le \alpha \le 1$ and suppose that for all $1 \le i \le m$, $f_i \in L^{p_i}_{loc}(\mathbb{R}^n)$ where $1 < p_i \le \infty$ satisfy $\sum_{i=1}^m \frac{1}{p_i} = \frac{1}{p} < \frac{mn-\alpha}{n}$. Then for almost every $x \in \mathbb{R}^n$,*

$$\lim_{t\to 0} S^m_{\alpha,t}(f_1,\dots,f_m)(x) = f_1(x)\cdots f_m(x). \tag{3.13}$$

Parts of Theorem 3.1 may be new even when $m = 1$. Theorem 3.2 is only new when $m \ge 2$ as the case $m = 1$ was considered in [50]. The proofs of these theorems can be suitably adapted to the measures

$$\frac{q}{B(mn/q, 1-\alpha)} \frac{d\vec{y}}{(1-|\vec{y}\,|^q)^\alpha}$$

for any $q > 0$ in lieu of

$$\frac{2}{B(\frac{mn}{2}, 1-\alpha)} \frac{d\vec{y}}{(1-|\vec{y}\,|^2)^\alpha}$$

in (3.1). To simplify the notation in our proofs, we adopt the following conventions:

$$\begin{aligned}
&\vec{y} = (y_1,\dots,y_m) \in (\mathbb{R}^n)^m && [\vec{f}\,] = (f_1,\dots,f_m)\\
&d\vec{y} = dy_1\cdots dy_m && (f_1\otimes\cdots\otimes f_m)(\vec{y}\,) = f_1(y_1)\cdots f_m(y_m)\\
&\vec{\theta} = (\theta_1,\dots,\theta_m) \in \mathbb{S}^{mn-1} && \otimes\vec{f} = f_1\otimes\cdots\otimes f_m\\
&\overline{x} = \underbrace{(x,\dots,x)}_{m \text{ times}} \in (\mathbb{R}^n)^m && |\otimes\vec{f}\,| = |f_1|\otimes\cdots\otimes|f_m|
\end{aligned}$$

Two main ideas are used in the proof of Theorem 3.1; integration by parts and the fundamental theorem of calculus, both with respect to the radial coordinate. Theorem 3.2 is based on a slicing formula that allows us to control S^m_α by the product of the Hardy-Littlewood maximal operators acting on $m-1$ input functions and of S_α (defined in (1.7)) acting on the remaining function. This gives estimates near the vertices of the region on which boundedness is claimed, while the remaining bounds are obtained by multilinear interpolation.

3.2 The proof of Theorem 3.1

Before we discuss the proof of Theorem 3.1 we note that when $\alpha = 0$, equality holds in the first inequality in (3.9), since

$$\frac{\omega_{mn-1}}{2} B\left(\frac{mn}{2}, 1\right) = \frac{\omega_{mn-1}}{2} \frac{2}{mn} = v_{mn}.$$

That is,

$$M^m[\vec{f}](x) = S_0^m[\vec{f}](x).$$

In fact (3.9) is valid even when $\alpha = 0$, since

$$\begin{aligned}
M^m[\vec{f}](x) &= S_0^m[\vec{f}](x) \\
&= \sup_{t>0} \frac{1}{v_{mn}} \int_0^1 \int_{\mathbb{S}^{mn-1}} |\otimes \vec{f}|(\overline{x} - tr\vec{\theta}) d\sigma_{mn-1}(\vec{\theta}) r^{mn-1} dr \\
&\leq \frac{1}{v_{mn}} \int_0^1 \sup_{t'>0} \int_{\mathbb{S}^{mn-1}} |\otimes \vec{f}|(\overline{x} - t'\vec{\theta}) d\sigma_{mn-1}(\vec{\theta}) r^{mn-1} dr \\
&= mn\, S^m[\vec{f}](x) \int_0^1 r^{mn-1} dr \\
&= S^m[\vec{f}](x).
\end{aligned}$$

Proof of Theorem 3.1. First we show that for any $0 < \alpha < 1$ we have

$$S_\alpha^m[\vec{f}](x) \leq S^m[\vec{f}](x)$$

for any $x \in \mathbb{R}^n$. Indeed, we have

$$\begin{aligned}
&\frac{1}{\omega_{mn-1}} \frac{2}{B(\frac{mn}{2}, 1-\alpha)} \operatorname{ess.sup}_{t>0} \int_{\mathbb{B}^{mn}} |\otimes \vec{f}|(\overline{x} - t\vec{y})(1 - |\vec{y}|^2)^{-\alpha} d\vec{y} \\
\leq& \frac{1}{\omega_{mn-1}} \frac{2}{B(\frac{mn}{2}, 1-\alpha)} \int_0^1 \frac{r^{mn-1}}{(1-r^2)^\alpha} \operatorname{ess.sup}_{t>0} \int_{\mathbb{S}^{mn-1}} |\otimes \vec{f}|(\overline{x} - rt\vec{\theta}) d\sigma_{mn-1}(\vec{\theta}) dr \\
\leq& \frac{1}{\omega_{mn-1}} \frac{2}{B(\frac{mn}{2}, 1-\alpha)} \left(\int_0^1 \frac{r^{mn-1}}{(1-r^2)^\alpha} dr \right) \operatorname{ess.sup}_{t'>0} \int_{\mathbb{S}^{mn-1}} |\otimes \vec{f}|(\overline{x} - t'\vec{\theta}) d\sigma(\vec{\theta}) \\
=& S^m[\vec{f}](x),
\end{aligned}$$

since

$$\int_0^1 \frac{r^{mn-1}}{(1-r^2)^\alpha} dr = \frac{1}{2} B\left(\frac{mn}{2}, 1-\alpha\right).$$

This concludes the proof of the second inequality in (3.9).

Next we prove the first inequality in (3.9). That is, for a fixed $x \in \mathbb{R}^n$ and $0 < \alpha < 1$, we show that

$$M^m[\vec{f}](x) \leq S_\alpha^m[\vec{f}](x).$$

If for some $x \in \mathbb{R}^n$ we had $M^m[\vec{f}](x) = \infty$, we would also have that $S_\alpha^m[\vec{f}](x) = \infty$ as

$$(1 - |\vec{y}|^2)^{-\alpha} \geq 1$$

when $|\vec{y}| < 1$. So we may assume that $M^m[\vec{f}](x) < \infty$ in the calculation below. For fixed $t > 0$ we define

$$H_t(r) = \int_0^r s^{mn-1}\left(\int_{\mathbb{S}^{mn-1}} |\otimes \vec{f}|(\overline{x} - ts\vec{\theta})\, d\sigma(\vec{\theta})\right) ds = \int_{|\vec{y}| \leq r} |\otimes \vec{f}|(\overline{x} - t\vec{y}) d\vec{y},$$

for $r > 0$. As each f_j is locally integrable, the integral on the right converges absolutely, and thus the expressions in the parentheses are finite for almost all $s > 0$ and moreover, the s-integral converges absolutely. Thus $H_t(r)$ is the integral from 0 to r of an L^1 function. Then, the Lebesgue differentiation theorem gives

$$\frac{d}{dr} H_t(r) = H_t'(r) = r^{mn-1} \int_{\mathbb{S}^{mn-1}} |\otimes \vec{f}|(\overline{x} - tr\vec{\theta}) d\sigma(\vec{\theta}) \qquad \text{for almost all } r > 0.$$

Moreover, for any $r > 0$ we have

$$\begin{aligned} \operatorname*{ess.sup}_{t>0} \frac{1}{v_{mn} r^{mn}} H_t(r) &= \operatorname*{ess.sup}_{t>0} \frac{1}{v_{mn}} H_{rt}(1) \\ &= \operatorname*{ess.sup}_{t'>0} \frac{1}{v_{mn}} H_{t'}(1) \\ &= M^m[\vec{f}](x) < \infty, \end{aligned}$$

where in the last equality we replaced the essential supremum by the supremum, using the continuity of the function

$$t \mapsto M_t^m(f_1, \ldots, f_m)(x) = \frac{1}{v_{mn}} \int_{\mathbb{B}^{mn}} \prod_{i=1}^{m} |f_i(x - ty_i)| dy_1 \cdots dy_m,$$

for any $f_i \in L^1_{loc}(\mathbb{R}^n)$, which can be obtained by an application of the Lebesgue dominated convergence theorem. Let

$$c_{mn,\alpha} = \frac{2}{\omega_{mn-1} B(\frac{mn}{2}, 1 - \alpha)}.$$

For any $0 < b < 1$ we write

$$\begin{aligned}
&S_\alpha^m[\vec{f}](x) \\
&\quad \geq \operatorname*{ess.sup}_{t>0} c_{mn,\alpha} \int_0^b H_t'(r) \frac{1}{(1-r^2)^\alpha} dr \\
&\quad = \operatorname*{ess.sup}_{t>0} c_{mn,\alpha} \left[H_t(b) \frac{1}{(1-b^2)^\alpha} - \int_0^b H_t(r) \frac{-2\alpha r}{(1-r^2)^{\alpha+1}} dr \right] \\
&\quad \geq \operatorname*{ess.sup}_{t>0} c_{mn,\alpha} \left[H_t(b) \frac{1}{(1-b^2)^\alpha} - \int_0^b M^m[\vec{f}](x) \frac{-2\alpha r}{(1-r^2)^{\alpha+1}} v_{mn} r^{mn} dr \right] \\
&\quad = c_{mn,\alpha} \left[M^m[\vec{f}](x) \frac{v_{mn} b^{mn}}{(1-b^2)^\alpha} - \int_0^b M^m[\vec{f}](x) \frac{-2\alpha r}{(1-r^2)^{\alpha+1}} v_{mn} r^{mn} dr \right] \\
&\quad = c_{mn,\alpha} M^m[\vec{f}](x) v_{mn} \left[\frac{b^{mn}}{(1-b^2)^\alpha} - \int_0^b \frac{-2\alpha r}{(1-r^2)^{\alpha+1}} r^{mn} dr \right] \\
&\quad = c_{mn,\alpha} M^m[\vec{f}](x) v_{mn} \left[mn \int_0^b (1-r^2)^{-\alpha} r^{mn-1} dr \right],
\end{aligned}$$

where all the previous steps make use of the assumption that $M^m[\vec{f}](x) < \infty$. Letting $b \to 1^-$ we obtain the first inequality in (3.9). So we established both inequalities in (3.9) for $f_i \in L^1_{loc}(\mathbb{R}^n)$.

Our next goal is to show that

$$\varliminf_{\alpha \to 1^-} S_\alpha^m[\vec{f}](x) \geq S^m[\vec{f}](x), \tag{3.14}$$

where $\varliminf$ denotes the limit inferior. Let us fix f_j in $L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. We define

$$G_{\vec{f}}(t) = \int_{\mathbb{S}^{mn-1}} |\otimes \vec{f}|(\overline{x} - t\vec{\theta}) d\sigma_{mn-1}(\vec{\theta}).$$

We observed earlier that for any $L < \infty$ we have

$$\int_0^L t^{mn-1} G_{\vec{f}}(t)\, dt \le \prod_{i=1}^m \int_{(|x|+1)\mathbb{B}^n} |f_i(y_i)|\, dy_i < \infty$$

thus $G_{\vec{f}}(t) < \infty$ for almost all $t > 0$. So let us fix a $t > 0$ for which $G_{\vec{f}}(t) < \infty$. For this t we will show that

$$\lim_{\alpha \to 1^-} \int_0^1 G_{\vec{f}}(rt) \frac{2r^{mn-1}(1-r^2)^{-\alpha}}{B(\frac{mn}{2}, 1-\alpha)} dr = G_{\vec{f}}(t). \tag{3.15}$$

Once (3.15) is shown, we deduce

$$\varliminf_{\alpha \to 1^-} \sup_{t'>0} \int_0^1 G_{\vec{f}}(rt') \frac{2r^{mn-1}(1-r^2)^{-\alpha}}{B(\frac{mn}{2}, 1-\alpha)} dr \ge G_{\vec{f}}(t)$$

and taking the supremum on the right over all $t > 0$ for which $G_{\vec{f}}(t) < \infty$, yields (3.14). Notice that the supremum over these t's is the essential supremum which appears in the definition of this operator.

To prove (3.15), it will suffice to show that

$$\lim_{\alpha \to 1^-} \int_0^1 \left| G_{\vec{f}}(rt) - G_{\vec{f}}(t) \right| \frac{2r^{mn-1}(1-r^2)^{-\alpha}}{B(\frac{mn}{2}, 1-\alpha)} dr = 0. \tag{3.16}$$

For smooth functions with compact support φ_i we have

$$\lim_{\alpha \to 1^-} \int_0^1 \left| G_{\vec{\varphi}}(rt) - G_{\vec{\varphi}}(t) \right| \frac{2r^{mn-1}(1-r^2)^{-\alpha}}{B(\frac{mn}{2}, 1-\alpha)} dr = 0 \tag{3.17}$$

as

$$\begin{aligned} \left| \prod_{i=1}^m \left| \varphi_i(x - rt\theta_i) \right| - \prod_{i=1}^m \left| \varphi_i(x - t\theta_i) \right| \right| \le & \left| \prod_{i=1}^m \varphi_i(x - rt\theta_i) - \prod_{i=1}^m \varphi_i(x - t\theta_i) \right| \\ \le & C\, t(1-r) \end{aligned}$$

and this factor cancels the singularity of $(1-r^2)^{-\alpha}$ while

$$\lim_{\alpha\to 1^-} B(\frac{mn}{2},1-\alpha)=+\infty.$$

Let us suppose that $0<\epsilon<1$ is given. For our given $f_i \in L^1_{loc}(\mathbb{R}^n)$, fixed $t>0$, and $x\in\mathbb{R}^n$, we pick φ_i smooth functions with compact support such that

$$\|f_i-\varphi_i\|_{L^1((\frac{1}{t}+1)(|x|+1)\mathbb{B}^n)}\le \epsilon.$$

This implies that

$$\big\|f_1\otimes\cdots\otimes f_m-\varphi_1\otimes\cdots\otimes\varphi_m\big\|_{L^1((\frac{1}{t}+1)(|x|+1)\mathbb{B}^{mn})}\le C'\epsilon, \tag{3.18}$$

where

$$C'=\sum_{i=1}^m \prod_{\substack{1\le j\le m\\ j\ne i}}\Big(\|f_j\|_{L^1((\frac{1}{t}+1)(|x|+1)\mathbb{B}^n)}+1\Big), \tag{3.19}$$

using the identity (valid for complex numbers a_i, b_i)

$$a_1a_2\cdots a_m-b_1b_2\cdots b_m=\sum_{i=1}^m b_1\cdots b_{i-1}(a_i-b_i)a_{i+1}\cdots a_m. \tag{3.20}$$

In view of (3.17), the proof of (3.16) will be a consequence of the estimate:

$$\int_0^1 \big|Q(\vec{f},\vec{\varphi},tr,t)\big|\frac{2r^{mn-1}(1-r^2)^{-\alpha}}{B(\frac{mn}{2},1-\alpha)}dr\le C''\epsilon \tag{3.21}$$

where

$$Q(\vec{f},\vec{\varphi},tr,t)=\big(G_{\vec{f}}(rt)-G_{\vec{f}}(t)\big)-\big(G_{\vec{\varphi}}(rt)-G_{\vec{\varphi}}(t)\big).$$

Notice that this function is integrable in r over $[0,1]$. Thus the fundamental theorem of calculus applies, in the sense that

$$r^KQ(\vec{f},\vec{\varphi},tr,t)=\frac{d}{dr}\int_0^r s^KQ(\vec{f},\vec{\varphi},ts,t)ds$$

for almost all r in $[0,1]$. (K here is a fixed positive power.)

For any $0 < b < 1$ we have

$$\begin{aligned}
&\int_0^b \left|Q(\vec{f},\vec{\varphi},tr,t)\right| \frac{2r^{mn-1}(1-r^2)^{-\alpha}}{B(\frac{mn}{2},1-\alpha)}dr \\
=&\int_0^b \frac{d}{dr}\int_0^r s^{mn-1}\left|Q(\vec{f},\vec{\varphi},ts,t)\right|ds \ \frac{2(1-r^2)^{-\alpha}}{B(\frac{mn}{2},1-\alpha)}dr \\
=&\left(\int_0^b s^{mn-1}\left|Q(\vec{f},\vec{\varphi},ts,t)\right|ds\right)\frac{2(1-b^2)^{-\alpha}}{B(\frac{mn}{2},1-\alpha)} \\
&\qquad\qquad -\int_0^b\left(\int_0^r s^{mn-1}\left|Q(\vec{f},\vec{\varphi},ts,t)\right|ds\right)\frac{2(-2\alpha r)(1-r^2)^{-\alpha-1}}{B(\frac{mn}{2},1-\alpha)}dr \\
\le&\left(\int_0^1 s^{mn-1}\left|Q(\vec{f},\vec{\varphi},ts,t)\right|ds\right)\left[\frac{2(1-b^2)^{-\alpha}}{B(\frac{mn}{2},1-\alpha)}-\int_0^b\frac{2(-2\alpha r)(1-r^2)^{-\alpha-1}}{B(\frac{mn}{2},1-\alpha)}dr\right] \\
=&\left(\int_0^1 s^{mn-1}\left|Q(\vec{f},\vec{\varphi},ts,t)\right|ds\right)\left[\int_0^b\frac{2(1-r^2)^{-\alpha}}{B(\frac{mn}{2},1-\alpha)}dr\right] \\
\le&\left(\int_0^1 s^{mn-1}\left|Q(\vec{f},\vec{\varphi},ts,t)\right|ds\right)\left[\int_0^b\frac{2(1-r)^{-\alpha}}{B(\frac{mn}{2},1-\alpha)}dr\right] \\
\le&\int_0^1 s^{mn-1}\left|Q(\vec{f},\vec{\varphi},ts,t)\right|ds\, 2^{mn-1},
\end{aligned}$$

as $(1-\alpha)B(mn,1-\alpha)$ is bounded from below by some constant $C'(mn)$.

It remains to show that the integral

$$\int_0^1 s^{mn-1}\left|Q(\vec{f},\vec{\varphi},ts,t)\right|ds$$

is bounded by a constant multiple of ϵ. But this integral is controlled by

$$\int_{\mathbb{B}^{mn}}\left|\overrightarrow{|f|}(\overline{x}-t\vec{y})-\overrightarrow{|\varphi|}(\overline{x}-t\vec{y})\right|d\vec{y}+\int_{\mathbb{B}^{mn}}\left|\overrightarrow{|f|}(\overline{x}-\vec{y})-\overrightarrow{|\varphi|}(\overline{x}-\vec{y})\right|d\vec{y}$$

which is bounded by

$$\int_{\mathbb{B}^{mn}}\left|\left|\prod_{i=1}^m f_i(x-ty_i)\right|-\left|\prod_{i=1}^m \varphi_i(x-ty_i)\right|\right|d\vec{y}+\int_{\mathbb{B}^{mn}}\left|\left|\prod_{i=1}^m f_i(x-y_i)\right|-\left|\prod_{i=1}^m \varphi_i(x-y_i)\right|\right|d\vec{y}$$

which, in turn, is bounded by

$$\int_{\mathbb{B}^{mn}}\left|\prod_{i=1}^m f_i(x-ty_i)-\prod_{i=1}^m\varphi_i(x-ty_i)\right|d\vec{y}+\int_{\mathbb{B}^{mn}}\left|\prod_{i=1}^m f_i(x-y_i)-\prod_{i=1}^m\varphi_i(x-y_i)\right|d\vec{y}.$$

Then using (3.20) we obtain that the preceding expression is bounded by $2C'\epsilon$, where C' is as in (3.19). This proves (3.21), which as observed earlier, implies (3.14).

Finally, we prove (3.11). To do this, in view of (3.9), we fix x in $\mathbb{R}^n$ and f_i in $L^1_{loc}(\mathbb{R}^n)$. It will suffice to show that

$$\overline{\lim_{\alpha\to 0^+}}\, S_\alpha^m[\vec{f}\,](x)\le M^m[\vec{f}\,](x). \tag{3.22}$$

For $t>0$ we set

$$K_t(\alpha)=\frac{2}{\omega_{mn-1}B(\frac{mn}{2},1-\alpha)}\int_{\mathbb{B}^{mn}}|\vec{f}\,|(\overline{x}-t\vec{y}\,)(1-|\vec{y}\,|^2)^{-\alpha}d\vec{y}\,.$$

Since we are taking the limit as $\alpha\to 0^+$ we may consider $\alpha<1/2$. By the triangle inequality, for $0<\alpha<1/2$ we have

$$K_t(\alpha)\le K_t(0)+\big|K_t(\alpha)-K_t(0)\big|\le K_t(0)+\alpha\sup_{0\le\beta\le 1/2}|K_t'(\beta)|, \tag{3.23}$$

where we denoted by $K_t'(\beta)$ the derivative of K_t with respect to β. Let us temporarily assume that f_i are bounded functions. Fix $0\le\beta\le 1/2$. We write

$$\begin{aligned}
&|K_t'(\beta)|\\
&=\frac{2/\omega_{mn-1}}{B\left(\frac{mn}{2},1-\beta\right)^2}\Bigg|B\left(\frac{mn}{2},1-\beta\right)\int_{\mathbb{B}^{mn}}|\otimes\vec{f}\,|(\overline{x}-t\vec{y})\Big(\ln\frac{1}{1-|\vec{y}\,|^2}\Big)\frac{d\vec{y}}{(1-|\vec{y}\,|^2)^{\beta}}\\
&\qquad\qquad-\left(\frac{d}{d\beta}B\left(\frac{mn}{2},1-\beta\right)\right)\int_{\mathbb{B}^{mn}}|\otimes\vec{f}\,|(\overline{x}-t\vec{y})(1-|\vec{y}\,|^2)^{-\beta}d\vec{y}\Bigg|\\
&\le\frac{2\prod_{i=1}^m\|f_i\|_{L^\infty}}{B\left(\frac{mn}{2},1-\beta\right)}\int_0^1(1-r^2)^{-\beta}r^{mn-1}\Big(\ln\frac{1}{1-r^2}\Big)\,dr\\
&\qquad+\frac{2\prod_{i=1}^m\|f_i\|_{L^\infty}}{B\left(\frac{mn}{2},1-\beta\right)^2}\left(\int_0^1 s^{-\beta}\Big(\ln\frac{1}{s}\Big)(1-s)^{\frac{mn}{2}-1}\,ds\right)\frac{1}{2}B\left(\frac{mn}{2},1-\beta\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{2\prod_{i=1}^m \|f_i\|_{L^\infty}}{B(\frac{mn}{2},1)}\bigg[\int_0^1 (1-r^2)^{-\frac{1}{2}} r^{mn-1}\Big(\ln\frac{1}{1-r^2}\Big)\,dr \\
&\qquad\qquad\qquad\qquad + \frac{1}{2}\int_0^1 s^{-\frac{1}{2}}\Big(\ln\frac{1}{s}\Big)(1-s)^{\frac{mn}{2}-1}\,ds\bigg] \\
&= C_{\vec{f},mn},
\end{aligned}
$$

where we used that $0 \leq \beta \leq 1/2$ and that

$$
\begin{aligned}
\left|\frac{d}{d\beta}B\left(\frac{mn}{2},1-\beta\right)\right| &= \int_0^1 s^{-\beta}\Big(\ln\frac{1}{s}\Big)(1-s)^{\frac{mn}{2}-1}ds \\
&\leq \int_0^1 s^{-\frac{1}{2}}\Big(\ln\frac{1}{s}\Big)(1-s)^{\frac{mn}{2}-1}ds.
\end{aligned}
$$

Taking the essential supremum in (3.23) with respect to $t>0$, we conclude for $\alpha < 1/2$ that

$$
S_\alpha^m[\vec{f}](x) \leq M^m[\vec{f}](x) + \alpha\, C_{\vec{f},mn}.
$$

Therefore for every $x \in \mathbb{R}^n$ we obtain

$$
\varlimsup_{\alpha\to 0^+} S_\alpha^m[\vec{f}](x) \leq M^m[\vec{f}](x),
$$

under the assumption that f_i are bounded functions. We now remove this assumption on the f_i. Given f_i in $L^1_{loc}(\mathbb{R}^n)$, define

$$
f_i^k = f_i\chi_{|f_i|\leq k},
$$

for $k = 1,2,3,\ldots$. Then

$$
|f_i^1| \leq |f_i^2| \leq |f_i^3| \leq \cdots \leq |f_i|,
$$

amd

$$
|f_i^k| \uparrow |f_i| \quad \text{as} \;\; k\to\infty,
$$

and the functions f_i^k are bounded.

Let $[\vec{f^k}\,] = (f_1^k, \ldots, f_m^k)$. For each $k = 1, 2, 3, \ldots$ and each $t > 0$ the monotonicity of S_α^m in each variable and the preceding argument for bounded functions give

$$\begin{aligned}\overline{\lim_{\alpha\to 0^+}}\, S_\alpha^m[\vec{f}\,](x) \geq & \overline{\lim_{\alpha\to 0^+}}\, S_\alpha^m[\vec{f^k}\,](x) \\ \geq & \frac{2\omega_{mn-1}^{-1}}{B(\frac{mn}{2}, 1-\alpha)} \int_{\mathbb{B}^{mn}} \prod_{i=1}^{m} |f_i^k(x - ty_i)| \frac{dy}{(1-|y|^2)^\alpha}.\end{aligned}$$

Ignoring the middle term and letting $k \to \infty$ we obtain

$$\overline{\lim_{\alpha\to 0^+}}\, S_\alpha^m[\vec{f}\,](x) \geq \frac{2\omega_{mn-1}^{-1}}{B(\frac{mn}{2}, 1-\alpha)} \int_{\mathbb{B}^{mn}} \prod_{i=1}^{m} |f_i(x - ty_i)| \frac{dy}{(1-|y|^2)^\alpha}$$

via the Lebesgue monotone convergence theorem. Taking the essential supremum over all $t > 0$ yields inequality (3.22), and thus concludes the proof of (3.11). $\square$

3.3 The derivation of Theorem 3.2

Proof of Theorem 3.2. For any $0 \leq \alpha < 1$, we prove that the estimate

$$S_\alpha^m(f_1, \ldots, f_m)(x) \leq S_\alpha(f_k)(x) \prod_{i\neq k} M(f_i)(x), \tag{3.24}$$

is valid for all $f_i \in L^1_{loc}(\mathbb{R}^n)$ and all $x \in \mathbb{R}^n$, where S_α is defined in (1.7) and M is the Hardy-Littlewood maximal operator on $\mathbb{R}^n$. For any fixed $t > 0$, we set

$$S_{\alpha,t}^m(f_1, \ldots, f_m)(x) = c_{mn,\alpha} \int_{\mathbb{B}^{mn}} \prod_{i=1}^{m} |f_i(x - ty^i)|(1-|y|^2)^{-\alpha} dy$$

where

$$c_{mn,\alpha} = \frac{2}{\omega_{mn-1} B(mn/2, 1-\alpha)}.$$

For $y^i \in \mathbb{R}^n$ we set

$$y = (y^1, \dots, y^m) \qquad \text{and} \qquad \hat{y}^k = (y^1, \dots, y^{k-1}, y^{k+1}, \dots, y^m).$$

Then for a fixed $k \in \{1, 2, \dots, m\}$ we have

$$\begin{aligned}
&c_{mn,\alpha}^{-1}\, S_{\alpha,t}^m(f_1, \dots, f_m)(x) \\
&= \int_{\mathbb{B}^{mn}} \prod_{i=1}^m |f_i(x - ty^i)|(1 - |y|^2)^{-\alpha} dy \\
&= \int_{\mathbb{B}^{(m-1)n}} \int_{\sqrt{1-|\hat{y}^k|^2}\mathbb{B}^n} \prod_{i=1}^m |f_i(x - ty^i)|\, (1 - |\hat{y}^k|^2)^{-\alpha} \left(1 - \left|\frac{y^k}{\sqrt{1-|\hat{y}^k|^2}}\right|^2\right)^{-\alpha} dy^k d\hat{y}^k \\
&= \int_{\mathbb{B}^{(m-1)n}} \prod_{i\neq k} |f_i(x - ty^i)| \int_{\mathbb{B}^n} |f_k(x - t\sqrt{1 - |\hat{y}^k|^2} u^k)| \frac{(1-|\hat{y}^k|^2)^{\frac{n}{2}-\alpha}}{(1-|u^k|^2)^\alpha} du^k d\hat{y}^k \\
&\le \int_{\mathbb{B}^{(m-1)n}} \prod_{i\neq k} |f_i(x - ty^i)|\, \operatorname*{ess.sup}_{t>0} \int_{\mathbb{B}^n} |f_k(x - tu^k)|(1-|u^k|^2)^{-\alpha} du^k \frac{d\hat{y}^k}{(1-|\hat{y}^k|^2)^{\alpha-\frac{n}{2}}} \\
&\le c_{n,\alpha}^{-1}\, S_\alpha(f_k)(x) \cdot \sup_{t>0} \int_{\mathbb{B}^{(m-1)n}} \prod_{i\neq k} |f_i(x - ty^i)| \frac{d\hat{y}^k}{(1 - |\hat{y}^k|^2)^{\alpha-\frac{n}{2}}},
\end{aligned}$$

with $c_{n,\alpha} = 2/(\omega_{n-1} B(n/2, 1 - \alpha))$.

Next, we use the following fact concerning multilinear approximate identities: Suppose that

$$\phi : \mathbb{R}^{\kappa n} \to \mathbb{C}$$

has an integrable radially decreasing majorant Φ, and let $\phi_t(\vec{y}\,) = t^{-\kappa n}\phi(\vec{y}/t)$. If $*$ denotes convolution on $\mathbb{R}^{\kappa n}$, then the estimate

$$\sup_{t>0} |(\otimes \vec{f}\,) * \phi_t(\overline{x})| \le \|\Phi\|_{L^1(\mathbb{R}^{\kappa n})} M^m[\vec{f}\,](x) \tag{3.25}$$

is valid for all locally integrable functions f_j on $\mathbb{R}^n$, $j = 1, \dots, \kappa$. This follows by applying [19, Corollary 2.12] to the function

$$(x_1, \dots, x_\kappa) \mapsto \otimes \vec{f}(x_1, \dots, x_\kappa)$$

on $\mathbb{R}^{\kappa n}$ and using that the κn-dimensional Hardy-Littlewood maximal function of $\otimes \vec{f}$ at the point $(x, \ldots, x) \in (\mathbb{R}^n)^\kappa$ equals $M^m[\vec{f}](x)$.

Returning to the previous calculation, for $\hat{y}^k \in \mathbb{R}^{(m-1)n}$ we consider the function $\phi(\hat{y}^k) = (1 - |\hat{y}^k|^2)_+^{\frac{n}{2}-\alpha}$. Using that $n \geq 2$ (hence $n/2 - \alpha \geq 0$), we calculate that

$$\|\phi\|_{L^1(\mathbb{R}^{(m-1)n})} = \frac{\omega_{(m-1)n-1}}{2} B\left(\frac{(m-1)n}{2}, \frac{n}{2} + 1 - \alpha\right).$$

Using (3.25) for $\kappa = m - 1$, we can see that

$$\sup_{t>0} \int_{\mathbb{B}^{(m-1)n}} \prod_{i \neq k} |f_i(x - ty^i)| \frac{d\hat{y}^k}{(1 - |\hat{y}^k|^2)^{\alpha - \frac{n}{2}}} \leq \|\phi\|_{L^1(\mathbb{R}^{\kappa n})} M^{m-1}[\hat{f}^k](x),$$

where $[\hat{f}^k] = (f_1, \ldots, f_{k-1}, f_{k+1}, f_m)$. Using the well known fact that

$$\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

and the identity

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)},$$

one can verify that

$$c_{mn,\alpha} \cdot c_{n,\alpha}^{-1} \cdot \|\phi\|_{L^1} = 1.$$

Thus we conclude that

$$S_{\alpha,t}^m(f_1, \ldots, f_m)(x) \leq S_\alpha(f_k)(x) M^{m-1}[\hat{f}^k](x).$$

Taking the essential supremum of $S_{\alpha,t}^m(f_1, \ldots, f_m)(x)$ over $t > 0$ yields

$$S_\alpha^m(f_1, \ldots, f_m)(x) \leq S_\alpha(f_k)(x) M^{m-1}[\hat{f}^k](x). \tag{3.26}$$

Since (3.26) holds for $\alpha = 0$, we have that

$$M_\alpha^m[\vec{f}] \leq M(f_1)(x) M^{m-1}[\hat{f}^1](x).$$

Therefore, consecutive applications of (3.26) conclude the proof of (3.24).

We now turn to the boundedness of S_α when $m = 1$. It was shown in [50] that S_α is bounded on L^p for $\frac{n}{n-\alpha} < p \le \infty$ when $n \ge 3$. We remark that this result also holds when $n = 2$. We now provide a sketch of a proof valid in all dimensions $n \ge 2$. To do this, for $f \in \mathcal{S}(\mathbb{R}^n)$, we express $S_\alpha f$ as a maximal multiplier operator

$$S_\alpha f(x) = \frac{2\pi^\alpha}{\omega_{n-1}} \frac{\Gamma(1-\alpha)}{B(\frac{n}{2}, 1-\alpha)} \sup_{t>0} \left| \int_{\mathbb{R}^n} \widehat{f}(\xi) \frac{J_{\frac{n}{2}-\alpha}(2\pi t|\xi|)}{|t\xi|^{\frac{n}{2}-\alpha}} e^{2\pi i x\cdot\xi} d\xi \right|$$

using the identity in [19, Appendix B.5]. To derive this we use the Bochner-Riesz multiplier $(1-|x|^2)^{-\alpha}$ with a negative exponent, viewed as a kernel. Then the Fourier transform expression for $(1-|x|^2)^z$ when Re $z > 0$ is also valid for Re $z > -1$ by analytic continuation. Notice that in this range of z, the kernel remains locally integrable. Using properties of Bessel functions, the multiplier

$$m_\alpha(\xi) = \frac{J_{\frac{n}{2}-\alpha}(2\pi|\xi|)}{|\xi|^{\frac{n}{2}-\alpha}}$$

is a smooth function which satisfies for all multi-indices γ

$$|\partial_\xi^\gamma m_\alpha(\xi)| \le \frac{C_{n,\gamma}}{|\xi|^{\frac{n+1}{2}-\alpha}}$$

and the exponent

$$a = \frac{n+1}{2} - \alpha$$

is strictly bigger than $\frac{1}{2}$ (since $n \ge 2$ and $\alpha < 1$). Then the hypotheses of [45, Theorem B] apply and we obtain that S_α is bounded on $L^p(\mathbb{R}^n)$ (when restricted to Schwartz functions) for

$$p > \frac{2n}{n+2a-1} = \frac{n}{n-\alpha}.$$

(In [45, Theorem B] there is an upper restriction on p, but as S_α is bounded on L^∞ this does not apply here.) Then S_α extends to general $f \in L^p(\mathbb{R}^n)$ for $p > \frac{n}{n-\alpha}$

by density, and this extension coincides with that given in Definition 3.1.

We now use (3.24) to obtain that

$$\|S_\alpha^m(f_1,\dots,f_m)\|_{L^p(\mathbb{R}^n)} \le C\prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)} \tag{3.27}$$

for all $f_i \in L^{p_i}$, when $1 < p_i \le \infty$ for $i \neq k$ and $\frac{n}{n-\alpha} < p_k \le \infty$. Here the constant $C = C(m,\alpha,p_1,\dots,p_m)$ doesn't depend on the dimension n, since

$$S_\alpha(f)(x) \le S_1(f)(x)$$

and

$$\|S_1(f)(x)\|_{L^p(\mathbb{R}^n)} \le c\,\|f\|_{L^p(\mathbb{R}^n)}$$

for a constant c independent of n (see [52]).

To describe geometrically the points $(1/p_1,\dots,1/p_m)$ for which we claim boundedness for S_α^m, consider the cube $Q = [0,1]^m$ and let V be the set of all of its vertices except for the vertex $(1,1,\dots,1)$. Then $|V| = 2^m - 1$. We consider the intersection of Q with the half-space H of $\mathbb{R}^m$ described by

$$H = \big\{(t_1,\dots,t_m):\ t_1+\cdots+t_m \le \tfrac{mn-\alpha}{n}\big\}.$$

Then $Q\cap H$ has $2^m - 1 + m$ vertices, namely the set V union the m points

$$(1,\dots,1,\tfrac{mn-\alpha}{n},1,\dots,1),$$

where $\frac{mn-\alpha}{n}$ ranges over the m slots. We claim that S_α^m satisfies strong L^p bounds in the interior of $Q\cap H$. To see this, we interpolate between estimates at the vertices of $Q\cap H$.

Precisely, the interpolation works as follows: Let W be the vertices of $Q\cap H$ that do not belong to V and let W' be a finite union of open balls centered at the points of W intersected with $Q\cap H$. We interpolate between points $P =$

$(1/p_1, \dots, 1/p_m)$ in $V \cup W'$. If $P \in V$, then we have an estimate $L^{p_1} \times \cdots \times L^{p_m}$ to L^p for S_α^m, as at least one coordinate $1/p_k$ is 0 (i.e., $p_k = \infty$), and we apply (3.27) for this k. Now if P lies in W', then there is a $k \in \{1, \dots, m\}$ such that $p_k > \frac{n}{n-\alpha}$ and p_i are near 1 for all $i \neq k$. Using estimate (3.27) again for this choice of k, we obtain that S_α^m maps $L^{p_1} \times \cdots \times L^{p_m}$ to L^p at this point P. Applying the m-linear version of the Marcinkiewicz interpolation theorem [25], we deduce the boundedness of S_α^m in the interior of $Q \cap H$. Similar reasoning provides weak type bounds on all the faces of $Q \cap H$, except possibly on the H face, on which we don't know if there are any bounds at all.

Finally we show the optimality of the range $p > \frac{n}{mn-\alpha}$. We consider the action of S_α^m on characteristic functions; specifically, let

$$f_1 = \cdots = f_m = \chi_{\mathbb{B}^n}.$$

Since the characteristic functions belong in all L^p spaces, in the definition of $S_\alpha^m[\vec{f}\,]$ we can replace the essential supremum by the supremum (see Corollary 3.1). Therefore for $|x|$ sufficiently large it is enough to pick $t = \sqrt{m}\,|x|$ in order to write the estimate

$$\begin{aligned} c_{mn,\alpha}^{-1}\, S_\alpha^m(f_1, \dots, f_m)(x) &\geq \int_{\mathbb{B}^{mn}} \prod_{i=1}^m |f_i(x - \sqrt{m}\,|x|\,y_i)| \cdot (1 - |\vec{y}\,|^2)^{-\alpha} d\vec{y} \\ &\geq \int_{|\vec{y} - \frac{\overline{x}}{\sqrt{m}\,|x|}| \leq \frac{1}{\sqrt{m}\,|x|}} (1 - |\vec{y}\,|^2)^{-\alpha} d\vec{y} \\ &\geq 2^{-\alpha} \int_{|\vec{y} - \frac{\overline{x}}{\sqrt{m}\,|x|}| \leq \frac{1}{\sqrt{m}\,|x|}} (1 - |\vec{y}\,|)^{-\alpha} d\vec{y} \end{aligned} \tag{3.28}$$

since

$$\left|\vec{y} - \frac{\overline{x}}{\sqrt{m}\,|x|}\right| \leq \frac{1}{\sqrt{m}\,|x|} \implies \left|x - \sqrt{m}\,|x|\,y_j\right| \leq 1 \qquad \text{for all } j = 1, \dots, m.$$

The point

$$\overline{\theta_x} = \frac{\overline{x}}{\sqrt{m}|x|}$$

lies on the sphere $\mathbb{S}^{mn-1}$. A simple geometric argument gives that the integral in (3.28) expressed in polar coordinates $\vec{y} = r\vec{\theta}$ is at least

$$\int_{1-\frac{c}{|x|}}^{1} (1-r)^{-\alpha} r^{mn-1} \int_{|\vec{\theta}-\overline{\theta_x}|\le \frac{c}{|x|}} d\sigma_{mn-1}(\vec{\theta})\, \frac{dr}{2^\alpha} \ge \frac{2^{-\alpha}}{|x|^{1-\alpha}} \frac{C(m,n)}{|x|^{mn-1}} = \frac{2^{-\alpha} C(m,n)}{|x|^{mn-\alpha}}$$

for some small constants c, C (depending on n and m). We conclude the proof by noting that the function

$$|x|^{-mn+\alpha} \chi_{|x|\ge 100}$$

does not lie in $L^p(\mathbb{R}^n)$ for $p \le \frac{n}{mn-\alpha}$. $\square$

We proved Theorem 3.2 working directly with L^{p_i} functions. Alternatively, we could have worked with a dense family of L^{p_i} and then extend to L^{p_i} by density. There is no ambiguity in this extension, in view of the following proposition.

Proposition 3.1. *Let $0 < p_1, \dots, p_m, p \le \infty$. Suppose that T is a subadditive operator in each variable*[1] *that satisfies the estimate*

$$\|T(f_1, \dots, f_m)\|_{L^p} \le K \|f_1\|_{L^{p_1}} \cdots \|f_m\|_{L^{p_m}} \tag{3.29}$$

for all functions f_j in a dense subspace of L^{p_j}. Then T admits a unique bounded subadditive extension from $L^{p_1} \times \cdots \times L^{p_m}$ to L^p with the same bound.

Proof. For any $j \in \{1, \dots, m\}$, given $f_j \in L^{p_j}$ pick sequences a_j^k, b_j^l, $k, l = 1, 2, 3, \dots$ in the given dense subspace of L^{p_j} which converge to f_j in L^{p_j}. Using the idea proving (3.20) and the subadditivity of T in each variable we obtain:

$$\left|T(a_1^k, a_2^k, \dots, a_m^k) - T(b_1^l, b_2^l, \dots, b_m^l)\right| \le$$

$$\sum_{i=1}^{m} \Big[\left|T(b_1^l, \dots, b_{i-1}^l, a_i^k - b_i^l, a_{i+1}^k, \dots, a_m^k)\right| + \left|T(a_1^k, \dots, a_{i-1}^k, b_i^k - a_i^l, b_{i+1}^l, \dots, b_m^l)\right| \Big].$$

[1]this means $|T(\dots, f+g, \dots)| \le |T(\dots, f, \dots)| + T(\dots, g, \dots)|$ for all f, g

Applying the L^p (quasi norm) and hypothesis (3.29) we deduce

$$\begin{aligned}
&\big\|T(a_1^k, a_2^k, \dots, a_m^k) - T(b_1^l, b_2^l, \dots, b_m^l)\big\|_{L^p} \\
&\qquad \le C_p K \sum_{i=1}^{m} \|a_i^k - b_i^l\|_{L^{p_i}} \prod_{j\neq i} \big[\|a_j^k\|_{L^{p_j}} + \|b_j^l\|_{L^{p_j}}\big].
\end{aligned} \tag{3.30}$$

Taking $b_j^l = a_j^l$ in (3.30) we conclude that the sequence

$$\big\{T(a_1^k, a_2^k, \dots, a_m^k)\big\}_{k=1}^{\infty}$$

is Cauchy in L^p and thus it has a limit

$$\overline{T}(f_1, \dots, f_m).$$

This limit does not depend on the choice of the sequences a_j^k converging to f_j, as we can choose $l = k$ in (3.30) and let $k \to \infty$. Thus T has a unique extension $\overline{T}$. This extension is also bounded with the same bound and is subbadditive by density. $\square$

3.4 The proofs of Corollaries 3.1 and 3.2

Next we discuss the proof of Corollary 3.1. The case $m = 1$ of this result is contained in [51, Chapter XI Section 3.5].

Proof. It suffices to prove the assertion for almost all x in a ball $N\,\mathbb{B}^n$, as $\mathbb{R}^n$ is a countable union of $N\,\mathbb{B}^n$ over $N = 1, 2, \dots$. Let us fix such a ball $N\,\mathbb{B}^n$. It will suffice to prove the continuity of

$$t \mapsto S_{\alpha,t}(f_1, \dots, f_m)(x)$$

on $(0, R)$ for every $R > 0$. Fix such an $R > 0$ as well. Then we may replace each

f_i by

$$g_i = f_i \chi_{(N+R)\mathbb{B}^n}$$

as

$$S_{\alpha,t}(f_1, \dots, f_m)(x) = S_{\alpha,t}(g_1, \dots, g_m)(x)$$

when $x \in N\mathbb{B}^n$ and $0 < t < R$. As g_i have compact support and lie in L^{p_i}, they also lie in L^{q_i}, where $q_i < p_i$ are chosen so that

$$\frac{1}{q} = \sum_{i=1}^{m} \frac{1}{q_i} < \frac{mn - \alpha}{n}.$$

The purpose of introducing $q_i < p_i$ was to replace all infinite indices p_i by finite ones, as there is no good dense subspace of L^∞.

We pick sequences φ_j^k of smooth compactly supported functions with $\varphi_j^k \to g_j$ in $L^{q_j}(\mathbb{R}^n)$ (since $q_j < \infty$) and consider the sequence

$$\operatorname*{ess.sup}_{t>0} S_{\alpha,t}^m(g_1 - \varphi_1^k, \dots, g_m - \varphi_m^k), \qquad m = 1, 2, 3, \dots.$$

By (3.12) if $\alpha < 1$ (or by Chapter 2 if $\alpha = 1$) this sequence converges to zero in $L^q(\mathbb{R}^n)$, thus there is a subsequence that converges to zero a.e. This implies that there is a subset E of $\mathbb{R}^n$ of measure zero such that for all $x \in \mathbb{R}^n \setminus E$ we have

$$\lim_{k\to\infty} \left\| S_{\alpha,t}^m[\vec{g}](x) - S_{\alpha,t}^m[\vec{\varphi^k}](x) \right\|_{L^\infty((0,\infty),dt)} = 0,$$

i.e.,

$$S_{\alpha,t}^m[\vec{\varphi^k}](x) \to S_{\alpha,t}^m[\vec{f}](x)$$

uniformly in $t > 0$. Since $S_{\alpha,t}^m[\vec{\varphi^k}](x)$ is continuous in t, we conclude that $S_{\alpha,t}^m[\vec{g}](x)$ is also continuous in t, for almost every $x \in \mathbb{R}^n$. $\square$

To prove Corollary 3.2, we will need a proposition analogous to [19, Theorem 2.1.14]. Let (X, μ) and (Y, ν) be σ finite measure spaces and let $0 < p_j \le \infty$, $j = 1, \dots, m$, and $0 < q < \infty$. Let D_j be a dense subspace of $L^{p_j}(X, \mu)$. Suppose that for all $t > 0$, T_t is an m-linear operator defined on $L^{p_1}(X,\mu)\times\cdots\times L^{p_m}(X,\mu)$ with values in the space of measurable functions defined a.e. on Y. Assume that for all $f_j \in L^{p_j}$, the function

$$y \mapsto T_*(f_1,\dots,f_m)(y) = \sup_{t>0} |T_t(f_1,\dots,f_m)(y)|$$

is ν-measurable on Y.

Proposition 3.2. *Let $0 < p_i \le \infty$, $1 \le i \le m$, $0 < q < \infty$ and T_t and T_* as in the previous discussion. Suppose that there is a constant B such that*

$$\|T_*(f_1,\dots,f_m)\|_{L^{q,\infty}} \le B \prod_{j=1}^{m} \|f_j\|_{L^{p_j}} \tag{3.31}$$

for all $f_j \in L^{p_j}(X,\mu)$. Also suppose that for all $\varphi_j \in D_j$

$$\lim_{t\to 0} T_t(\varphi_1,\dots,\varphi_m) = T(\varphi_1,\dots,\varphi_m) \tag{3.32}$$

exists and is finite ν-a.e. Then for all functions $f_j \in L^{p_j}(X,\mu)$ the limit in (3.32) exists and is finite ν-a.e., and defines an m-linear operator which uniquely extends T defined on $D_1\times\cdots\times D_m$ and which is bounded from $L^{p_1}\times\cdots\times L^{p_m}$ to $L^{q,\infty}(X)$.

Proof. Given $f_j \in L^{p_j}$ we define the oscillation of $\vec{f}$ for $y \in Y$ by setting

$$O_{\vec{f}}(y) = \limsup_{\varepsilon\to 0}\limsup_{\theta\to 0} \left|T_\varepsilon[\vec{f}](y) - T_\theta[\vec{f}](y)\right|.$$

We will show that for all $f_j \in L^{p_j}$ and all $\delta > 0$,

$$\nu\big(\{y \in Y \,:\, O_{\vec{f}}(y) > \delta\}\big) = 0. \tag{3.33}$$

Once (3.33) is established, we obtain that $O_{\vec{f}}(y) = 0$ for ν-almost all y, which implies that $T_t[\vec{f}](y)$ is Cauchy for ν-almost all y, and it therefore converges ν-a.e. to some $T[\vec{f}](y)$ as $t \to 0$. The operator T defined this way on $L^{p_1}(X)\times\cdots\times$

$L^{p_m}(X)$ is linear and extends T given in (3.32) defined on $D_1 \times \cdots \times D_m$.

To approximate $O_{\vec{f}}(y)$ we use density. Given $0 < \eta < 1$, we find $\varphi_j \in D_j$ such that $\|f_j - \varphi_j\|_{L^{p_j}} < \eta$, $j = 1, \ldots, m$. Without a loss of generality, we also assume that $\|\varphi_i\|_{L^{p_i}} \leq 2\|f_i\|_{L^{p_i}}$. Since

$$T_t[\vec{\varphi}] \to T[\vec{\varphi}] \quad \nu - \text{a.e.},$$

it follows that $O_{\vec{\varphi}} = 0$ ν-a.e. Using (3.20), we write

$$T_t[\vec{f}] - T_t[\vec{\varphi}] = \sum_{i=1}^{m} T_t(\varphi_1, \ldots, \varphi_{i-1}, f_i - \varphi_i, f_{i+1}, \ldots, f_m)$$

and from this we obtain

$$O_{\vec{f}} \leq O_{\vec{\varphi}} + \sum_{i=1}^{m} O_{(\varphi_1, \ldots, \varphi_{i-1}, f_i - \varphi_i, f_{i+1}, \ldots, f_m)} \qquad \nu\text{-a.e.}$$

Now, for any $\delta > 0$ we have

$$\begin{aligned}
&\nu\big(\{y \in Y \,:\, O_{\vec{f}}(y) > \delta\}\big) \\
&\leq \nu\Big(\Big\{y \in Y \,:\, \sum_{i=1}^{m} O_{(\varphi_1, \ldots, \varphi_{i-1}, f_i - \varphi_i, f_{i+1}, \ldots, f_m)} > \delta\Big\}\Big) \\
&\leq \nu\Big(\Big\{y \in Y \,:\, \sum_{i=1}^{m} 2T_*(\varphi_1, \ldots, \varphi_{i-1}, f_i - \varphi_i, f_{i+1}, \ldots, f_m) > \delta\Big\}\Big) \\
&\leq \sum_{i=1}^{m} \nu\Big(\Big\{y \in Y \,:\, 2T_*(\varphi_1, \ldots, \varphi_{i-1}, f_i - \varphi_i, f_{i+1}, \ldots, f_m) > \frac{\delta}{m}\Big\}\Big) \\
&\leq \sum_{i=1}^{m} \Big[\Big(2B\frac{m}{\delta}\Big)\|\varphi_1\|_{L^{p_1}} \cdots \|\varphi_{i-1}\|_{L^{p_{i-1}}} \|f_i - \varphi_i\|_{L^{p_i}} \|f_{i+1}\|_{L^{p_{i+1}}} \cdots \|f_m\|_{L^{p_m}}\Big]^q \\
&\leq \Big(2^m B\frac{m}{\delta}\Big)^q \eta^q \sum_{i=1}^{m} \big(\prod_{j \neq i} \|f_j\|_{L^{p_j}}^q\big).
\end{aligned}$$

Letting $\eta \to 0$, we deduce (3.33). We conclude that $T_t[\vec{f}]$ is a Cauchy sequence and hence it converges ν-a.e. to some $T[\vec{f}]$ which satisfies the claimed assertions. $\square$

We now prove Corollary 3.2

Proof. It suffices to prove the assertion for almost all x in a ball $N\mathbb{B}^n$, as $\mathbb{R}^n$ is a countable union of balls. Let us fix a ball $N\mathbb{B}^n$. Then we replace the given f_i in $L^{p_i}_{loc}$ by $g_i = f_i\chi_{(N+1)\mathbb{B}^n}$ since

$$S_{\alpha,t}(f_1,\dots,f_m)(x) = S_{\alpha,t}(g_1,\dots,g_m)(x)$$

when $x \in N\mathbb{B}^n$ and $0 < t < 1$. As g_i have compact support and lie in L^{p_i}, they also lie in L^{q_i}, where $q_i < p_i$ are chosen so that

$$\frac{1}{q} = \sum_{i=1}^{m} \frac{1}{q_i} < \frac{mn-\alpha}{n}.$$

As $q_i < \infty$, the space of smooth functions with compact support is a dense subspace of L^{q_i}. Now (3.13) is easily shown to hold for smooth functions with compact support f_i, when $0 \le \alpha \le 1$, thus (3.32) holds with $T_t = S_{\alpha,t}$. Moreover (3.31) holds by Theorem 3.2 if $\alpha < 1$ or by Chapter 2 if $\alpha = 1$. By Proposition 3.2, for $t < 1$, we obtain that for almost all $x \in N\mathbb{B}^n$ we have

$$\lim_{t\to 0} S^m_{\alpha,t}(f_1,\dots,f_m)(x) = \lim_{t\to 0} S^m_{\alpha,t}(g_1,\dots,g_m)(x) = \prod_{j=1}^{m} g_j(x) = \prod_{j=1}^{m} f_j(x),$$

thus (3.13) holds for all g_i in L^{q_i}, in particular for our given f_i in $L^{p_i}_{loc}$. □

Chapter 4

Nikodym, Besicovitch, and Kakeya Sets Associated with Spheres

In [10] the authors constructed Nikodym, Besicovitch, and Kakeya Sets associated with circles on $\mathbb{R}^2$. In this chapter we extend this construction to the higher dimensional setting, proving the existence of Nikodym, Besicovitch, and Kakeya sets associated with spheres. These sets provide counterexamples for maximal operators given by translations of spherical averages are unbounded on all $L^p(\mathbb{R}^n)$ for $p < \infty$.

4.1 Introduction and statements of results

In this chapter and in the next, we will denote with $d\sigma$ the normalized surface measure, since the normalizing factor $\frac{1}{\omega_{n-1}}$ will not play any important role. For a given vector $u \in \mathbb{R}^n$ and a given radius $t > 0$, we define the average

$$\mathcal{A}_{u,t}f(x) := \int_{\mathbb{S}^{n-1}} f(x + tu - ty)d\sigma(y).$$

The vector u corresponds to a direction of translation. The (centered) spherical maximal function which originated in Stein [50] corresponds to the case $u = 0$:

$$Sf(x) := \sup_{t>0} \int_{\mathbb{S}^{n-1}} |f(x - ty)|\, d\sigma(y) = \sup_{t>0} \mathcal{A}_{0,t}|f|(x). \tag{4.1}$$

For a fixed $u \in \mathbb{S}^{n-1}$, let

$$M_u f(x) := \sup_{t>0} \mathcal{A}_{u,t}|f|(x) \tag{4.2}$$

be the maximal average of f over all spheres $x + t(u + \mathbb{S}^{n-1})$ with $t > 0$ varying. It was shown in [S91] that for $n = 2$, M_u is bounded from $L^p(\mathbb{R}^2)$ to itself for $p > 2$, uniformly in u.

In the following we will be allowing u to vary as well. For a compact subset T of $\mathbb{R}^n$, which will serve as a set of translations, we consider the maximal spherical translations

$$\mathcal{M}_T f(x) := \sup_{u\in T} \mathcal{A}_{u,1}|f|(x), \tag{4.3}$$

where we are considering averages over the unit spheres $x+u+\mathbb{S}^{n-1}$ with u varying in T. We also consider the uncentered spherical maximal function

$$S_T f(x) := \sup_{\substack{t>0 \\ u\in T}} \mathcal{A}_{u,t}|f|(x), \tag{4.4}$$

which includes dilations. These operators are initially defined for $f \in \mathcal{S}(\mathbb{R}^n)$, the Schwartz class. When $T = \{0\}$ we recover the spherical maximal function defined in (5.1). Clearly

$$\mathcal{M}_T f(x) \leq S_T f(x)$$

for any $x \in \mathbb{R}^n$.

When $T = \overline{B^n}(0,1)$, the closed unit ball in $\mathbb{R}^n$, the operator S_T corresponds the averages over all spheres with x in their interior (in the Jordan-Brouwer sense) and $\mathcal{M}_T$ to all such unit spheres. By considering the characteristic function of an ε-neighborhood of the unit sphere, we can readily see that both S_T and $\mathcal{M}_T$ are unbounded on $L^p(\mathbb{R}^n)$ for any $p < \infty$ if $T = \overline{B^n}(0, 1)$.

When $T = \mathbb{S}^{n-1}$, we instead are considering all spheres that pass through x. We will show that the operators M_T and S_T are unbounded as part of Corollary 4.1 when $T = \mathbb{S}^{n-1}$. However, we obtain boundedness results for $\mathcal{M}_T$ when the *upper Minkowski dimension* of T (see (5.7) below for a definition) is strictly less than $n - 1$ and for S_T when it is strictly less than $n - 2$. We denote by $\overline{\dim}_B T$ the upper Minkowski dimension (also known as the *upper box dimension*) of T. These results will be the subject of Chapter 5.

The counterexample for $\mathcal{M}_{\mathbb{S}^{n-1}}$ mentioned above follows from the construction of a Nikodym set in $\mathbb{R}^n$ associated with spheres, instead of hyperplanes. A classical Nikodym set is a set $A \subset \mathbb{R}^n$ of measure zero which contains a punctured hyperplane through every point: for every $y \in A$, there is a hyperplane $V_y \subset \mathbb{R}^n$ such that $y \in V_y$ and $V_y \setminus \{y\} \subset A$. Nikodym [42] proved the existence of such sets for $n = 2$, and Falconer [15] extended the result for all $n \geq 2$.

We prove the existence of the analogue of Nikodym sets for unit spheres instead of hyperplanes. Our result is a higher-dimensional analogue of [10, Theorem 6.9], specialized to spheres.

Theorem 4.1 (Nikodym set for spheres). *There exists a set $A \subset \mathbb{R}^n$ such that:*

1. *A has Lebesgue measure zero.*
2. *For all $y \in \mathbb{R}^n$, there is a point $p_y \in \mathbb{R}^n$ and an $(n-1)$-plane V_y containing 0 such that $y \in p_y + \mathbb{S}^{n-1}$ and $p_y + (\mathbb{S}^{n-1} \setminus V_y) \subset A$.*

Also, the mappings $y \mapsto p_y$ and $y \mapsto V_y$ are Borel.

To prove Theorem 4.1, we consider two closely related problems: the Kakeya needle problem and the existence of Besicovitch sets. These problems were originally studied in the case of the line segment in the plane by Besicovitch in [5]. See [38] for classical and recent results in this area. By adapting a construction of Cunningham [13], Héra and Laczkovich showed in [28] that a sufficiently short circular arc can be moved (via rigid motions) to any position in the plane leaving a trace of arbitrarily small area; this can be considered the analogue of the Kakeya needle problem for circular arcs.

The results in [28] were extended by Csörnyei and the first author, who showed that if one removes a neighborhood of two diametrically opposite points from a circle, the resulting set can be moved to any other position in arbitrarily small area [10, Corollary 1.3]. In fact, they studied a Kakeya needle problem variant for all rectifiable sets, not just circles. See [10, Theorem 1.2].

In Section 4.2, we prove the following, which provides a higher-dimensional generalization of the result in [10] about circles mentioned in the previous paragraph.

Theorem 4.2 (Kakeya needle problem for spheres)**.** *Let $\epsilon > 0$ be arbitrary. Then between the origin and any prescribed point in $\mathbb{R}^n$, there exists a polygonal path $P = \bigcup_{i=1}^m L_i$ with each L_i a line segment, and for each i there exists an $(n-1)$-plane V_i containing 0, such that*

$$\left| \bigcup_i \bigcup_{p \in L_i} (p + \{x \in \mathbb{S}^{n-1} : \operatorname{dist}(x, V_i) > \epsilon\}) \right| < \epsilon. \tag{4.5}$$

By considering the limit $\epsilon \to 0$ in the appropriate sense, Theorem 4.2, we obtain Theorem 4.3 (see [10, Section 6]).

Theorem 4.3 (Besicovitch set for spheres)**.** *For every path P_0 in $\mathbb{R}^n$ and for any neighborhood of P_0, there is a path P in this neighborhood with the same endpoints as P_0 and there is a $(n-1)$-plane V_p containing 0 such that*

$$\left| \bigcup_{p \in P} (p + (\mathbb{S}^{n-1} \setminus V_p)) \right| = 0. \tag{4.6}$$

Also, the mapping $p \mapsto V_p$ is Borel.

One can see that Theorem 4.3 implies Theorem 4.1; the idea is to take countably many translates of the set in (4.6). (see [10, Section 6] for details.)

We use the set A of Theorem 4.1 to construct counterexamples for two operators that arise in harmonic analysis. Kakeya sets were first used to construct counterexamples in harmonic analysis in the celebrated work of C. Fefferman [17] on the ball multiplier.

We provide counterexamples for two maximally translated spherical averages. The first operator, $\mathcal{M}_{\mathbb{S}^{n-1}}$, has already been defined in (5.3). The characteristic function of the set A in Theorem 4.1 serves as a counterexample to the boundedness of $\mathcal{M}_{\mathbb{S}^{n-1}}$. Since

$$\mathcal{M}_{\mathbb{S}^{n-1}}f(x) \leq S_{\mathbb{S}^{n-1}}f(x),$$

we will conclude that $S_{\mathbb{S}^{n-1}}$ is unbounded as well. We have the following corollary:

Corollary 4.1. *For any $n \geq 2$, the maximal operator $\mathcal{M}_{\mathbb{S}^{n-1}}$ is not bounded from $L^p(\mathbb{R}^n)$ to itself for any $1 \leq p < \infty$.*

Proof. Let A be the set of measure zero in Theorem 4.1. Since A is a Lebesgue measurable set, the regularity of the Lebesgue measure implies that for any $\varepsilon > 0$ there exists an open set $G_\varepsilon \supset A$ such that $|G_\varepsilon| < \varepsilon$. Then at each x we can choose the sphere $\Sigma_x = p_x + \mathbb{S}^{n-1}$ with $x \in \Sigma_x$ contained in A up to a set of $(n-1)$-dimensional measure zero. Since $A \subset G_\varepsilon$, for any $x \in \mathbb{R}^n$ we have

$$\mathcal{M}_{\mathbb{S}^{n-1}}\chi_{G_\varepsilon}(x) \geq \int_{\Sigma_x} \chi_{G_\varepsilon}(y)d\sigma(y) = 1.$$

Thus for each $p \in [1, \infty)$ there does not exist a constant c_p such that

$$\|\mathcal{M}_{\mathbb{S}^{n-1}}f\|_{L^p} \leq c_p\|f\|_{L^p}$$

for all $f \in L^p(\mathbb{R}^n)$, since $\mathcal{M}_{\mathbb{S}^{n-1}}\chi_{G_\varepsilon}(x) = 1$ for all $x \in \mathbb{R}^n$ while $\|\chi_{G_\varepsilon}\|_{L^p} < \varepsilon^{1/p}$. □

The second operator, S_*, was introduced by Palsson and Sovine in [44], to estimate the triangle operator

$$T(f,g)(x) = \int_{\Omega} f(x-u)g(y-v)d\mu_{\Omega}(u,v),$$

which is the averaging operator corresponding to the surface measure μ_Ω on the submanifold Ω of $\mathbb{R}^{2n}$

$$\Omega = \left\{(u,v) \in \mathbb{R}^{2n} \,:\, |u| \,=\, |v| \,=\, |u-v| \,=\, 1\right\}.$$

The Triangle Operator was first introduced for $n = 1$ by Greenleaf and Iosevich in [27], where $L^{p_1} \times L^{p_2} \to L^p$ estimates for T were used to show that if the Hausdorff dimension of a compact set $E \subset \mathbb{R}^2$ is greater than $\frac{7}{4}$, then the set of three-point configurations determined by E has positive three-dimensional measure. It was the extended in [44] to higher dimensions and, more recently, a trilinear version of T, the pyramid averaging operator, was introduced by Neuman in [41] in order to study the four-point configuration problem.

In [44] it was noted that a geometric majorization of T, similar to the one utilized by Jeong and Lee in [33] for the bilinear spherical maximal function, yields the natural pointwise estimate

$$T(f,g)(x) \leq Sf(x) \cdot \sup_{u\in\mathbb{S}^{n-1}} |S_u f(x)|,$$

for every $x \in \mathbb{R}^n$, where $Sf(x)$ is the spherical maximal function of f defined in (5.1) and for $u \in \mathbb{S}^{n-1}$,

$$S_u g(x) := \int_{H_u} g(x-y)d\sigma_{H_u}(y),$$

is the averaging operator over the $(n-2)$-sphere

$$H_u := \{v \in \mathbb{S}^{n-1} \,:\, |u-v| = 1\} = \mathbb{S}^{n-1} \cap \{v \,:\, u \cdot v = 1/2\}$$

and σ_{H_u} is the surface measure on H_u.

In [44] the authors posed the question of the $L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ boundedness of

$$S_* g(x) := \sup_{u \in \mathbb{S}^{n-1}} |S_u g(x)|.$$

Such a bound would imply an $L^{p_1} \times L^{p_2} \to L^p$ bound for T. However, no such bound is possible and the counterexample follows from the set A of Theorem 4.1 once again.

Corollary 4.2. *For any $n \geq 2$, the operator S_* is not bounded on $L^p(\mathbb{R}^n)$ for any $1 \leq p < \infty$.*

Corollary 4.2 is proved in Section 4.3.

4.2 The Kakeya needle problem

The proof of Theorem 4.2 is a generalization of the proof of [10, Theorem 1.2] to higher dimensions. At first we focus on spheres, but in Subsection 4.2.2 we will extend the construction to general rectifiable sets.

4.2.1 The Venetian Blind Construction

Let $\mathbb{P}^{n-1}$ be the quotient of $\mathbb{S}^{n-1}$ where we identify antipodal points together. We will refer to an element of $\mathbb{P}^{n-1}$ as a *direction* of $\mathbb{R}^n$. Note that if $\theta_1, \theta_2 \in \mathbb{P}^{n-1}$, then the dot product $\theta_1 \cdot \theta_2$ is well-defined up to a sign. For $\theta \in \mathbb{P}^{n-1}$, let $\theta^\perp \subset \mathbb{R}^n$ denote the orthogonal complement.

We begin with the following basic estimate, which provides a higher dimensional analogue of [10, Lemma 3.3], specialized to spheres. We denote the k Hausdorff measure by $\mathcal{H}^k$. We refer the reader to [37] for the definitions of the Hausdorff measure and dimension.

Lemma 4.1 (Basic estimate)**.** *Let $\epsilon > 0$. Let $L \subset \mathbb{R}^n$ be a line segment in the* *direction $\theta \in \mathbb{P}^{n-1}$. Suppose $R \subset \{x \in \mathbb{S}^{n-1} : |x \cdot \theta| \leq \epsilon\}$. Then*

$$|L + R| \leq \epsilon \mathcal{H}^1(L)\mathcal{H}^{n-1}(R).$$

Proof. Without loss of generality, assume $\theta = e_1 = (1, 0, \dots, 0)$. Let

$$P : \mathbb{R}^n \to \mathbb{R}$$

be the orthogonal projection onto $e_1^{\perp}$. Let $R = R^+ \cup R^-$, where

$$R^+ = \{(x_1, \dots, x_n) \in R : x_1 \geq 0\}.$$

Let $f : \mathbb{R}^{n-1} \to \mathbb{R}$ be given by

$$f(y) := \sqrt{1 - |y|^2}.$$

Using the hypothesis on R, an elementary computation gives

$$\begin{aligned} \mathcal{H}^{n-1}(R^+) =& \int_{P(R^+)} \sqrt{1 + |\nabla f(y)|^2}\, dy \\ \geq& \frac{1}{\epsilon}\mathcal{H}^{n-1}(P(R^+)). \end{aligned}$$

A similar inequality holds for R^-. By Fubini's theorem, we have

$$|L + R| \leq \mathcal{H}^1(L)\left(\mathcal{H}^{n-1}(P(R^+)) + \mathcal{H}^{n-1}(P(R^-))\right) \leq \epsilon \mathcal{H}^1(L)\mathcal{H}^{n-1}(R).$$

□

In the following lemma, we will think of $\mathbb{R}^2$ as the subspace of $\mathbb{R}^n$ spanned by $e_1 = (1, 0, \dots, 0)$ and $e_2 = (0, 1, 0, \dots, 0)$. The set $\mathbb{P}^1 \subset \mathbb{P}^{n-1}$ will denote the set of directions of $\mathbb{R}^2$. We also identify the direction $(\cos\alpha, \sin\alpha) \in \mathbb{P}^1$ with $\alpha \in \mathbb{R}/\pi\mathbb{Z}$.

Lemma 4.2 (The iterated Venetian blind construction)**.** *Let $\epsilon > 0$. Let $L \subset$*

$\mathbb{R}^2 \subset \mathbb{R}^n$ be a line segment in direction e_1. Then there exists a polygonal path $P \subset \mathbb{R}^2 \subset \mathbb{R}^n$ with the same endpoints as L such that we can decompose $P = G \cup B$, where the "good" part G and the "bad" part B satisfy the following:

1. *G is a finite union of parallel line segments of some direction $\theta_G \in (0, \frac{\pi}{2}) \subset \mathbb{P}^1$.*

2. *B is a finite union of parallel line segments of some direction $\theta_B \in (-\frac{\pi}{2}, 0) \subset \mathbb{P}^1$.*

3. *We have $\theta_G > \frac{\pi}{2} - \epsilon$, and $\theta_G - \theta_B < \frac{\pi}{2}$.*

4. *There exists a constant $c \in (0,1)$ depending only on θ_B and θ_G such that $\mathcal{H}^1(B) \leq c\mathcal{H}^1(L)$.*

5. *For any compact set*

$$R \subset \mathbb{S}^{n-1} \cap \bigcup_{\theta \in [0, \theta_G]} \theta^\perp, \tag{4.7}$$

we have

$$|G + R| \lesssim \epsilon \mathcal{H}^1(L) \mathcal{H}^{n-1}(R). \tag{4.8}$$

The implied constant is absolute. (Recall that we think of $[0, \theta_G]$ as a subset of $\mathbb{P}^1$, which itself is a subset of $\mathbb{P}^{n-1}$.)

6. *For any compact set*

$$R \subset \mathbb{S}^{n-1} \cap \bigcup_{\theta \in I_B} \theta^\perp. \tag{4.9}$$

Then

$$|B + R| \lesssim \epsilon \mathcal{H}^1(L) \mathcal{H}^{n-1}(R) \tag{4.10}$$

Remark 4.1. *The set G is "good" because the interval of directions associated with it, $[0, \theta_G]$, is large, i.e., close to $\pi/2$. On the other hand, the interval of directions associated with the "bad" set B is small. However, the redeeming property of B is (4), which means that when we iterate this iterated Venetian blind construction, the total size of the "bad" set will decrease to zero very quickly.*

The iterated Venetian blind construction used to prove Lemma 4.2 has essentially already appeared in [10, Section 4]. We provide a sketch below.

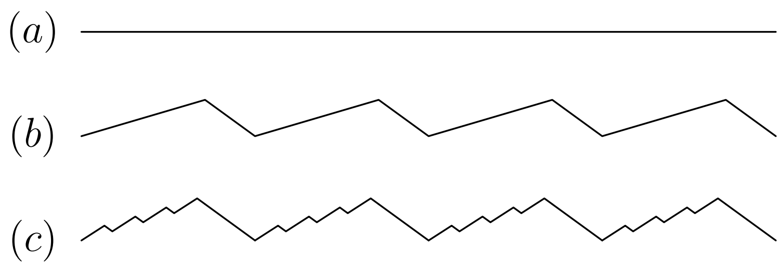

Figure 4.1: Two steps in the Venetian blind construction, starting with (a) as the initial segment.

Proof. The fundamental procedure in our construction is taking a line segment L and replacing it with a "basic zigzag" with the same endpoints. A *basic zigzag* is a polygonal path which is made up of N congruent and equally spaced segments in some direction interlaced with N congruent segments in some other direction. (See Figure 4.1(b) for an example.)

Fix $\epsilon > 0$ and $L \subset \mathbb{R}^2 \subset \mathbb{R}^n$ a line segment in direction e_1. Let

$$0 < \beta \leq \gamma < \frac{\pi}{4}$$

be angles depending on ϵ, to be determined later.

Fix $N_1 \in \mathbb{N}$ (to be determined later), and apply the basic zigzag construction to L to get a zigzag

$$Z_1 \subset R^2$$

with N_1 segments in directions γ and N_1 segments in direction $-\beta$. We write

$$Z_1 = G_1 \cup B_1,$$

where G_1 is the union of line segments in direction γ and B_1 is the union of line segments in direction $-\beta$

Next fix $N_2 \in \mathbb{N}$, and repeat the zigzag procedure on each segment of G_1, but now with directions 2γ and $-\beta$. The end result is a polygonal path

$$Z_2 = G_2 \cup B_2 \cup B_1.$$

See Figure 4.1(c) for an example. Here G_2 is the union of the line segments in direction 2γ, and B_2 is the union of line segments in direction $-\beta$ created in this step.

At the end of the jth step, we have a polygonal path

$$Z_j = G_j \cup \bigcup_{i=1}^{j} B_i.$$

All the segments in G_j have direction $m\gamma$. (Note that there are $N_1 \cdots N_j$ congruent segments in G_j.) All the segments in

$$\bigcup_{i=1}^{j} B_i$$

have direction $-\beta$.

We stop the iteration at $j = k$, where k satisfies

$$k\gamma \in [\pi/2 - 2\beta - \gamma, \pi/2 - 2\beta).$$

We set $G = G_k$, $B = \bigcup_{i=1}^{k} B_i$, $\theta_G = k\gamma$, and $\theta_B = -\beta$. Then (1) and (2) are satisfied. By choosing γ and β small enough and by the definition of k, we can satisfy (3).

Next, we show that (4) follows from some elementary Euclidean geometry. Let L_G and L_B be line segments in directions θ_G and θ_B respectively, such that $L \cup L_G \cup L_B$ is a triangle. Then $\mathcal{H}^1(L_B) = \mathcal{H}^1(B)$, and (4) follows from

$$k\gamma > \pi - (k\gamma + \beta)$$

and the law of sines.

Finally, we check the volume estimates (5) and (6). For each $i = 1, \ldots, k$, let

$$E_i = \mathbb{S}^{n-1} \cap \bigcup_{\theta \in [(i-1)\gamma, i\gamma]} \theta^{\perp}. \tag{4.11}$$

Let R be a compact set such that

$$R \subset \bigcup_{i=1}^{k} E_i.$$

This is precisely the condition (4.7). By Lemma 4.1,

$$|G_i + (R \cap E_i)| \leq \gamma \mathcal{H}^1(G_i)\mathcal{H}^{n-1}(R \cap E_i) \leq \gamma \mathcal{H}^1(L)\mathcal{H}^{n-1}(R \cap E_i).$$

Furthermore, by choosing each N_i sufficiently large (i.e., making the basic zigzags very fine), G_j is contained in a small neighborhood of G_i whenever $j \geq i$. This, along with compactness, implies that

$$|G_j + (R \cap E_i)| \lesssim \gamma \mathcal{H}^1(L)\mathcal{H}^{n-1}(R \cap E_i)$$

for all $j \geq i$. By taking $j = k$ and by summing over all i, we get

$$|G + R| \lesssim \gamma \mathcal{H}^1(L)\mathcal{H}^{n-1}(R) \tag{4.12}$$

which implies (5) if we choose γ small enough depending on ϵ. To obtain (6), we use Lemma 4.1 again, and choose β small enough depending on ϵ. $\square$

Lemma 4.3 (Iterating the iterated Venetian blind construction)**.** *Let $\epsilon > 0$. Let $L \subset \mathbb{R}^2 \subset \mathbb{R}^n$ be a line segment in direction e_1. Then there exists a polygonal path*

$$P = \bigcup_{i=1}^{m} L_i \subset \mathbb{R}^2,$$

with each L_i a line segment, such that for each i there exists an interval $I_i \subset \mathbb{P}^1$ such that

1. *Each I_i has length at most ϵ.*

2. *We have*
$$|\bigcup_i (L_i + \mathbb{S}^{n-1} \cap \bigcup_{\theta \in \mathbb{P}^1 \setminus I_i} \theta^\perp)| < \epsilon. \tag{4.13}$$

The idea of the proof of Lemma 4.3 is to apply the iterated Venetian blind construction to the initial line segment L to get a polygonal path, then to apply the construction again to each segment of the polygonal path, and so on.

Each time we apply Lemma 4.2, we have a choice of direction of the iterated Venetian blind (clockwise or counter-clockwise). By making these choices carefully, we can reduce the size of the exceptional set of directions I_i for most segments in our polygonal path. By Lemma 4.2(4), the total length of the segments for which the exceptional set of directions remains large goes to zero. For the complete details, see [10, Section 4].

Observe that Lemma 4.3 immediately implies Theorem 4.2.

4.2.2 Translates of rectifiable sets

The theorems for spheres can easily be generalized to $(n-1)$-rectifiable sets in $\mathbb{R}^n$. The main modification is replacing Lemma 4.1 with Lemma 4.4, below. We do not need these generalizations, but they may be of independent interest, we state the theorems below.

Let $\mathbb{P}^{n-1}$ be the quotient of $\mathbb{S}^{n-1}$ where we identify antipodal points together. We will refer to an element of $\mathbb{P}^{n-1}$ as a *direction* of $\mathbb{R}^n$. Note that if $\theta_1, \theta_2 \in \mathbb{P}^{n-1}$, then the dot product $\theta_1 \cdot \theta_2$ is well-defined up to a sign. We consider $\mathbb{P}^{n-1}$ as a metric space, with the metric induced by the arc-length metric on $\mathbb{S}^{n-1}$. That is, the distance between θ_1 and θ_2 is $\arccos|\theta_1 \cdot \theta_2|$. For $\theta \in \mathbb{P}^{n-1}$, we define the ball

$$B(\theta, r) \subset \mathbb{P}^{n-1}$$

of radius r via this metric.

Let $E \subset \mathbb{R}^n$. We say that E is $(n-1)$-*rectifiable* if there exist countably many C^1 hypersurfaces $\Gamma_i \subset \mathbb{R}^n$ such that

$$\mathcal{H}^{n-1}(E \setminus \bigcup_i \Gamma_i) = 0. \tag{4.14}$$

For $x \in E$, we define

$$N_x \in \mathbb{P}^{n-1}$$

to be the direction such that for all Γ_i containing x, the direction N_x is normal to Γ_i at x. While N_x may not be defined at all points in E, it is a standard fact that N_x is defined for $\mathcal{H}^{n-1}$-almost every $x \in E$. If N_x is defined, we refer to it as the *direction normal to E at x.*

The following is the analogue of [10, Theorem 1.2].

Theorem 4.4 (Kakeya needle problem for translations)**.** *Let $E \subset \mathbb{R}^n$ be a $(n-1)$-rectifiable set of finite $\mathcal{H}^{n-1}$-measure. Let $\epsilon > 0$ be arbitrary. Then between the origin and any prescribed point in $\mathbb{R}^n$, there exists a polygonal path*

$$P = \bigcup_{i=1}^{m} L_i$$

with each L_i a line segment, and for each i there exists a direction $\theta_i \in \mathbb{P}^{n-1}$, such that

$$\left|\bigcup_i \bigcup_{p \in L_i} (p + E_i)\right| < \epsilon, \tag{4.15}$$

where

$$E_i = \{x \in E : |N_x \cdot \theta_i| > \epsilon\}. \tag{4.16}$$

The following is the analogue of [10, Theorem 6.2]. See also [10, Remark 6.3].

Theorem 4.5 (Besicovitch set for translations)**.** *Suppose $E \subset \mathbb{R}^n$ can be covered by finitely many C^1 hypersurfaces. Then for every path P_0 in $\mathbb{R}^n$, and for any neighborhood of P_0, there is a path P in this neighborhood with the same endpoints as P_0, and for every $p \in P$, there exists $\theta_p \in \mathbb{P}^{n-1}$ such that*

$$|\bigcup_{p \in P}(p + \{x \in E_0 : N_x \cdot \theta_p \neq 0\}| = 0. \tag{4.17}$$

Also, the mapping $p \mapsto \theta_p$ is Borel.

The following is the analogue of [10, Theorem 6.10].

Theorem 4.6 (Nikodym set for translations)**.** *Suppose $E \subset \mathbb{R}^n$ can be covered by finitely many C^1 hypersurfaces. Let Γ be a rectifiable hypersurface. Then there exists a set $A \subset \mathbb{R}^n$ which satisfies the following:*

1. *A has Lebesgue measure zero;*
2. *For all $y \in \mathbb{R}^n$, there is a $\theta_y \in \mathbb{P}^{n-1}$ and a $p_y \in \mathbb{R}^n$ such that $y \in p_y + \Gamma$, and $p_y + E_y \subset A$, where $E_y = \{x \in E : N_x \cdot \theta_y \neq 0\}$.*

Also, the mappings $y \mapsto p_y$ and $y \mapsto V_y$ are Borel.

The proof of Theorem 4.4 uses the same iterated Venetian blind construction as in Lemma 4.2. To obtain the volume estimates ((5) and (6) of Lemma 4.2), we use the following lemma in place of Lemma 4.1.

Lemma 4.4 (Basic estimate)**.** *Let $E \subset \mathbb{R}^n$ be a $(n-1)$ rectifiable set. Let $\delta > 0$ be sufficiently small, and let $\theta \in \mathbb{P}^{n-1}$ be an arbitrary direction. Let R be a subset* *of E such that*

$$|N_x \cdot \theta| \leq \delta \text{ for every } x \in R. \tag{4.18}$$

Then if we translate R by a vector v of direction θ, the total area covered is bounded above by a constant multiple of $\delta\mathcal{H}^{n-1}(R)|v|$.

Proof. We follow the proof of [10, Lemma 3.3]. We fix $\delta > 0$, a direction θ, and a vector v of direction θ. Let R be a subset of E such that

$$|N_x \cdot \theta| \leq \delta$$

for every $x \in R$.

For each $x \in R$ there is a C^1 hypersurface Γ_i from (4.14) containing x. We choose a partition

$$R = \bigcup_i R_i$$

such that

$$R_i \subset \Gamma_i$$

for each i. Because of (4.18), we can further partition each

$$R_i = \bigcup_j R_{i,j}$$

into countably many pieces such that for each $R_{i,j}$, there exists a direction $w_{i,j}$ orthogonal to θ such that $R_{i,j}$ is the graph of a Lipschitz function $f_{i,j}$ in the $(w_{i,j}^{\perp}, w_{i,j})$ coordinate system, with Lipschitz constant bounded by a constant multiple of δ.

Fix i and j. Without loss of generality, we can assume that $\theta = e_1$ and that $w_{i,j} = e_n$. When we translate $R_{i,j}$ by the vector v, for each fixed $t \in \mathbb{R}$ we obtain

$$\#\{x \in \mathbb{R}^{n-1} : \ f_{i,j}(x) = t, \ (x, f_{i,j}(x)) \in R_{i,j}\}$$

many (not necessary disjoint) horizontal line segments on the line $\mathbb{R}^{n-1} \times \{t\}$, each of length $|v|$. Therefore, by Fubini's theorem, the volume covered is at most

$$|v| \int \#\{x \in \mathbb{R} : \ f_{i,j}(x) = t, \ (x, f_{i,j}(x)) \in R_{i,j}\} \, dt \lesssim \delta \mathcal{H}^{n-1}(R_{i,j})|v|,$$

where the second inequality follows from the co-area formula (see [16, Theorem 3.2.22]) and the fact that f has Lipschitz constant $\lesssim \delta$. Summing over i and j, we obtain the lemma. $\square$

Remark 4.2. *If, e.g., the set E is a hyperplane, then Theorem 4.4, Theorem 4.5, and Theorem 4.6 do not say anything useful because every point on E has the same normal direction. If we use rotations in addition to translations, we can shrink the set we need to delete in these theorems. In two dimensions, this is explained in [10, Section 5], and the result generalizes to higher dimensions as well. However we will not these results, so we omit the details.*

4.3 The Operator S_*

In the proof of Corollary 4.2 extra care is required to make sure that the deleted set of the sphere in Theorem 4.1 does not intersect the manifold of integration H_u in a set of positive $\mathcal{H}^{n-2}$ measure.

Proof of Corollary 4.2. For $u \in \mathbb{S}^{n-1}$, the operator $S_u g(x)$ is the average of $g(x - \cdot)$ over the submanifold

$$H_u := \{v \in \mathbb{S}^{n-1} \ : \ |u - v| = 1\} = \mathbb{S}^{n-1} \cap \{v \ : \ u \cdot v = 1/2\},$$

which is a $(n-2)$-dimensional sphere of radius $\sqrt{3}/2$ centered at $\frac{1}{2}u$ lying inside the hyperplane containing $\frac{1}{2}u$ with normal direction u. It is the intersection of two unit spheres, one centered at the origin and the other at $u \in \mathbb{S}^{n-1}$.

Let A be the set of measure zero in Theorem 4.1. Then for each $x \in \mathbb{R}^n$ there exists a unit sphere

$$\Sigma_x = p_x + \mathbb{S}^{n-1}$$

for which we have $x \in \Sigma_x$ and

$$\mathcal{H}^{n-1}(\Sigma_x \setminus A) = 0.$$

In particular, there exists a $(n-1)$-manifold $V_x \ni 0$ such that

$$p_x + (\mathbb{S}^{n-1} \setminus V_x) \subset A.$$

Let

$$z_x = p_x - x$$

and note that $z_x \in \mathbb{S}^{n-1}$. For this z_x consider the manifold

$$H_{z_x} = (z_x + \mathbb{S}^{n-1}) \cap \mathbb{S}^{n-1}.$$

Since

$$\Sigma_x \cap A$$

is a $(n-2)$-sphere with radius 1, center p_x, and it is contained in the affine space

$$p_x + V_x,$$

we conclude that

$$(x + H_{z_x}) \cap A$$

has full $(n-2)$-dimensional measure.

To see this, fix $x \in \mathbb{R}^n$. We translate the space so that $x = 0$ and subsequently we rotate the space so that $z_x = e_1$. Then

$$H_{z_x} = \mathbb{S}^{n-1} \cap \{e_1 = 1/2\}$$

is a $(n-2)$-sphere with radius $\sqrt{3}/2$. Let W_x be intersection

$$V_x \cap \{e_1 = 1/2\}.$$

Then W_x is a $(n-2)$-dimensional affine space or the empty set. Identifying $\{e_1 = 1/2\}$ with $\mathbb{R}^{n-1}$ we see that

$$H_{z_x} \setminus A = H_{z_x} \setminus W_x$$

is the intersection of a sphere in $\mathbb{R}^{n-1}$ with an affine space in $\mathbb{R}^{n-1}$ (or with the empty set) and thus

$$H_{z_x} \cap A$$

has full measure.

Thus for every $x \in \mathbb{R}^n$ there exists a direction $z_x \in \mathbb{S}^{n-1}$ such that the $(n-2)$-dimensional manifold

$$x + H_{z_x}$$

is contained in A, except for a set of $(n-2)$-dimensional measure zero.

It follows that the operator S_* is not bounded for any $p < \infty$ using a counterexample similar to the one in Corollary 4.1. Since A is a Lebesgue measurable set, the regularity of the Lebesgue measure implies that for any $\varepsilon > 0$ there exists an open set $G_\varepsilon \supset A$ such that $|G_\varepsilon| < \varepsilon$. Then for any $x \in \mathbb{R}^2$

$$S_* \chi_{G_\varepsilon}(x) \geq \int_{H_{z_x}} \chi_{G_\varepsilon}(x-y) d\sigma(y) = 1.$$

Thus for each $p \in [1, \infty)$ there cannot exist a constant c_p such that

$$\|S_* g\|_{L^p} \leq c_p \|g\|_{L^p},$$

since for $g(x) = \chi_{G_\varepsilon}(x)$ we have $S_* \chi_{G_\varepsilon}(x) = 1$ for all $x \in \mathbb{R}^n$ while $\|\chi_{G_\varepsilon}\|_{L^p} < \varepsilon^{1/p}$.

□

Chapter 5

Uncentered Spherical Maximal Functions

Recall that in Chapter 4 we defined the average

$$\mathcal{A}_{u,t}f(x) := \int_{\mathbb{S}^{n-1}} f(x + tu - ty)d\sigma(y)$$

for a given direction of translation $u \in \mathbb{R}^n$ and a given radius $t > 0$. The (centered) spherical maximal function which originated in Stein [50] corresponds to the case $u = 0$:

$$Sf(x) := \sup_{t>0} \int_{\mathbb{S}^{n-1}} |f(x - ty)| \, d\sigma(y) = \sup_{t>0} \mathcal{A}_{0,t}|f|(x). \tag{5.1}$$

For a fixed $u \in \mathbb{S}^{n-1}$, let

$$M_u f(x) := \sup_{t>0} \mathcal{A}_{u,t}|f|(x) \tag{5.2}$$

be the maximal average of f over all spheres $x + t(u + \mathbb{S}^{n-1})$ with $t > 0$ varying. It was shown in [S91] that for $n = 2$, M_u is bounded from $L^p(\mathbb{R}^2)$ to itself for $p > 2$, uniformly in u.

For a compact subset T of $\mathbb{R}^n$, which will serve as a set of translations, we consider the maximal spherical translations

$$\mathcal{M}_T f(x) := \sup_{u \in T} \mathcal{A}_{u,1}|f|(x), \tag{5.3}$$

where we are considering averages over the unit spheres $x+u+\mathbb{S}^{n-1}$ with u varying in T. We also consider the uncentered spherical maximal function

$$S_T f(x) := \sup_{\substack{t>0 \\ u \in T}} \mathcal{A}_{u,t}|f|(x), \tag{5.4}$$

which includes dilations. These operators are initially defined for $f \in \mathcal{S}(\mathbb{R}^n)$, the Schwartz class. When $T = \{0\}$ we recover the spherical maximal function defined in (5.1). Clearly

$$\mathcal{M}_T f(x) \leq S_T f(x)$$

for any $x \in \mathbb{R}^n$.

When $T = \overline{B^n}(0,1)$, the closed unit ball in $\mathbb{R}^n$, the operator S_T corresponds the averages over all spheres with x in their interior (in the Jordan-Brouwer sense) and $\mathcal{M}_T$ to all such unit spheres. By considering the characteristic function of an ε-neighborhood of the unit sphere, we can readily see that both S_T and $\mathcal{M}_T$ are unbounded on $L^p(\mathbb{R}^n)$ for any $p < \infty$ if $T = \overline{B^n}(0,1)$.

When $T = \mathbb{S}^{n-1}$, we instead are considering all spheres that pass through x. We have showed that the operators M_T and S_T are unbounded as part of Corollary 4.1 when $T = \mathbb{S}^{n-1}$.

However, we can obtain boundedness results for $\mathcal{M}_T$ when the *upper Minkowski dimension* of T (see (5.7) below for a definition) is strictly less than $n-1$ and for S_T when it is strictly less than $n-2$. We denote by $\overline{\dim}_B T$ the upper Minkowski dimension (also known as the *upper box dimension*) of T. For the maximal spherical translations operator we have the following result:

Theorem 5.1. *Let $n \geq 2$ and let $T \subset \mathbb{R}^n$ be a compact set. If $\overline{\dim}_B T < n-1$,* *then $\mathcal{M}_T$ is bounded on $L^p(\mathbb{R}^n)$ for all*

$$p > 1 + (n - \overline{\dim}_B T)^{-1}.$$

Therefore, for $n \geq 2$, when $\overline{\dim}_B T < n - 1$ and $p > 1 + (n - \overline{\dim}_B T)^{-1}$, the operator $\mathcal{M}_T$, initially defined for Schwartz functions $f \in \mathcal{S}$, can be uniquely extended to $L^p(\mathbb{R}^n)$ by continuity. For the uncentered spherical maximal function we have:

Theorem 5.2. *For $n \geq 3$ let $T \subset \mathbb{R}^n$ be a compact set. If $\overline{\dim}_B T < n - 2$, then S_T is bounded on $L^p(\mathbb{R}^n)$ for all*

$$p > 1 + (n - \overline{\dim}_B T - 1)^{-1}.$$

Similarly, for $n \geq 3$, when $\overline{\dim}_B T < n - 2$ and $p > 1 + (n - \overline{\dim}_B T - 1)^{-1}$, S_T admits a unique extension to $L^p(\mathbb{R}^n)$. Moreover, the boundedness of S_T implies the almost everywhere convergence of the uncentered spherical means.

Corollary 5.1. *For $n \geq 3$ let $T \subset \mathbb{R}^n$ be a compact set. Suppose $\overline{\dim}_B T < n - 2$,*

$$p > 1 + (n - \overline{\dim}_B T - 1)^{-1},$$

and $f \in L^p_{loc}(\mathbb{R}^n)$. Then for almost every $x \in \mathbb{R}^n$,

$$\lim_{t \to 0} \int_{\mathbb{S}^{n-1}} f(x + tu - ty) d\sigma(y) = f(x) \qquad \textit{uniformly in } u \in T. \tag{5.5}$$

That is, for almost every $x \in \mathbb{R}^n$,

$$\lim_{t \to 0} \sup_{u \in T} \left| \int_{\mathbb{S}^{n-1}} f(x + tu - ty) d\sigma(y) - f(x) \right| = 0. \tag{5.6}$$

Proof of Corollary 5.1. Let T and p be as in the hypothesis of Corollary 5.1. Let $f \in L^p$. For any $\epsilon > 0$ we can write $f = g + h$ where h is smooth with compact support, and $\|g\|_{L^p} \le \epsilon$. Since h is continuous,

$$\lim_{t\to 0} \sup_{u\in T} |\mathcal{A}_{u,t} h - h| = 0$$

and thus

$$\begin{aligned}
\| \lim_{t\to 0} \sup_{u\in T} |\mathcal{A}_{u,t} f - f| \|_{L^p} \le & \| \lim_{t\to 0} \sup_{u\in T} |\mathcal{A}_{u,t} g - g| \|_{L^p} \\
\le & \| \sup_{t\to 0} \sup_{u\in T} \mathcal{A}_{u,t} |g| \|_{L^p} + \|g\|_{L^p} \\
\lesssim & \|g\|_{L^p} \\
\le & \epsilon,
\end{aligned}$$

where in the third inequality we use Theorem 5.2. Letting $\epsilon \to 0$, we obtain (5.6). □

Theorems 5.1 and 5.2 have the following geometric implications.

Corollary 5.2. *Every Nikodym set for spheres has full Hausdorff dimension.*

Moreover, if we have a Nikodym set for spheres A as in Theorem 4.1, then the set $T := \{y - p_y : y \in \mathbb{R}^n\} \subset \mathbb{S}^{n-1}$ of "relative positions of the spheres" must be have full upper Minkowski dimension in $\mathbb{S}^{n-1}$. We prove this in Section 5.3.

Corollary 5.3. *Let $T \subset \mathbb{S}^{n-1}$ be a compact set with $\overline{\dim}_B T < n-1$. Suppose that a subset A of $\mathbb{R}^n$ has the property that for each $y \in \mathbb{R}^n$ there exists a point $p_y \in \mathbb{R}^n$ such that $y - p_y \in T$, and*

$$\mathcal{H}^{n-1}\left((p_y + \mathbb{S}^{n-1}) \cap A\right) > 0.$$

Then A has positive measure.

Proof. Let $T \subset \mathbb{S}^{n-1}$ and A be as above. If A had zero measure, then the operator $\mathcal{M}_T$ would be unbounded, however this contradicts 5.1, since $\overline{\dim}_B T < n-1$ and thus $\mathcal{M}_T$ is bounded.

Suppose that A has zero measure. We prove that $\mathcal{M}_T$ is unbounded with an argument similar to corollary 4.1. Since A is a Lebesgue measurable set, the regularity of the Lebesgue measure implies that for any $\varepsilon > 0$ there exists an open set $G_\varepsilon \supset A$ such that $|G_\varepsilon| < \varepsilon$. Then at each x we can choose the sphere $\Sigma_x = p_x + \mathbb{S}^{n-1}$ with $x \in \Sigma_x$ and $\Sigma_x \cap A$ has positive $(n-1)$-dimensional measure zero. Since $A \subset G_\varepsilon$, for any $x \in \mathbb{R}^n$ we have

$$\begin{aligned}\mathcal{M}_T \chi_{G_\varepsilon}(x) &\geq \int_{\Sigma_x} \chi_{G_\varepsilon}(y) d\sigma(y)\\ &\geq \mathcal{H}^{n-1}\left((p_x + \mathbb{S}^{n-1}) \cap A\right)\\ &\geq c_x\end{aligned}$$

for some $c_x > 0$ independent of ε. On the other hand,

$$\chi_{G_\varepsilon} \to 0 \quad \text{pointwise almost everywhere}$$

and thus $\mathcal{M}_T$ is unbounded. □

Corollary 5.4. *Let $T \subset \mathbb{S}^{n-1}$ be a compact set with $\overline{\dim}_B T < n-2$. Suppose that a subset A of $\mathbb{R}^n$ has the property that for each $y \in \mathbb{R}^n$ there exists a point $p_y \in \mathbb{R}^n$ such that*

$$\frac{y - p_y}{|y - p_y|} \in T,$$

and

$$\mathcal{H}^{n-1}\left((p_y + |y - p_y|\mathbb{S}^{n-1}) \cap A\right) > 0.$$

Then A has positive measure.

We omit the proof, since it is similar to the proof of Corollary 5.3.

The range of boundedness in the Theorems 5.1 and 5.2 might not be optimal, however, we will show that there are lower bounds for the range of p where $\mathcal{M}_T$ (and thus S_T) are bounded.

Theorem 5.3. *Let $n \geq 2$ and let $0 \leq s \leq 1$. If $\mathcal{M}_T$ is bounded on $L^p(\mathbb{R}^n)$ for all $\overline{\dim}_B T = s$, then*

$$p \geq 1 + \frac{s}{n-1}.$$

Theorem 5.4. *Let $n \geq 2$ and let $1 < s \leq n-1$. If $\mathcal{M}_T$ is bounded on $L^p(\mathbb{R}^n)$ for all $\overline{\dim}_B T = s$, then*

$$p \geq 1 + \frac{s - \lceil s \rceil + 1}{n - \lceil s \rceil + 1}.$$

5.1 The Decomposition

Recall that one definition of the upper Minkowski dimension is

$$\overline{\dim}_B T = \inf \left\{ s \geq 0 : \limsup_{r \to 0} \frac{|\{x \in \mathbb{R}^n : \operatorname{dist}(x, T) < r\}|}{r^{n-s}} < \infty \right\}. \tag{5.7}$$

(See, e.g., [37, Section 5.5].)

Since $\overline{\dim}_B T = \overline{\dim}_B \overline{T}$, we can assume that T is compact. In the following θ will denote an element of the fixed compact set T.

We begin with a dyadic decomposition. We fix a radial function $\rho_0 \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that $\rho_0(\xi) = 1$ for $|\xi| \leq 1$ and $\rho_0(\xi) = 0$ when $|\xi| \geq 2$. For $j \geq 1$ we let

$$\rho_j(\xi) = \rho_0(2^{-j}\xi) - \rho_0(2^{1-j}\xi)$$

and we observe that $\sum_{j=0}^\infty \rho_j \equiv 1$. We define $\sigma_j = (\rho_j \widehat{\sigma})^\vee$ and

$$\begin{aligned}
\mathcal{M}_T^j f(x) &:= \sup_{\theta \in T} |(f * \sigma_j)(x + \theta)| \\
S_T^j f(x) &:= \sup_{\substack{t>0 \\ \theta \in T}} \left| \int_{\mathbb{R}^n} f(x + t\theta - ty)\sigma_j(y) dy \right| = \sup_{\substack{t>0 \\ \theta \in T}} \left| (f * (\sigma_j)_t)(x + t\theta) \right|,
\end{aligned}$$

and note that $\mathcal{A}_{\theta,1} f(x) = \sum_{j=0}^\infty \mathcal{A}_\theta^j f(x)$. where for any function ϕ defined on

$\mathbb{R}^n$,

$$(\phi)_t(x) := t^{-n}\phi\left(\frac{x}{t}\right)$$

is the L^1-normalized dilation. Then

$$\mathcal{M}_T f(x) \le \sum_{j=0}^{\infty} \mathcal{M}_T^j |f|(x)$$

and

$$S_T f(x) \le \sum_{j=0}^{\infty} S_T^j |f|(x).$$

The following are our main estimates for $\mathcal{M}_T^j$ and S_T^j.

Lemma 5.1. *Let $T \subset \mathbb{R}^n$ be compact. Then for all $f \in L^2(\mathbb{R}^n)$ and all $s > \overline{\dim}_B T$,*

$$\|\mathcal{M}_T^j f\|_{L^2(\mathbb{R}^n)} \lesssim_{T,s} 2^{j\left(\frac{s}{2}-\frac{n-1}{2}\right)} \|f\|_{L^2(\mathbb{R}^n)} \tag{5.8}$$

$$\|S_T^j f\|_{L^2(\mathbb{R}^n)} \lesssim_{T,s} 2^{j\left(\frac{s+1}{2}-\frac{n-1}{2}\right)} \|f\|_{L^2(\mathbb{R}^n)}. \tag{5.9}$$

Lemma 5.2. *Let $T \subset \mathbb{R}^n$ be any compact set. For any $f \in L^1(\mathbb{R}^n)$ and any $\epsilon > 0$,*

$$\|\mathcal{M}_T^j f\|_{L^1(\mathbb{R}^n)} \lesssim_{T,\epsilon} 2^{j(1+\epsilon)} \|f\|_{L^1(\mathbb{R}^n)} \tag{5.10}$$

$$\|S_T^j f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim_{T,\epsilon} 2^{j(1+\epsilon)} \|f\|_{L^1(\mathbb{R}^n)}. \tag{5.11}$$

When $\overline{\dim}_B T < n-1$, interpolating between (5.8) and (5.10) we conclude that $\mathcal{M}_T$ is bounded on $L^p(\mathbb{R}^n)$ when

$$1 + 1/(n - \overline{\dim}_B T) < p \le 2.$$

Since $\mathcal{M}_T$ is trivially bounded on $L^\infty(\mathbb{R}^n)$, Theorem 5.1 follows. Similarly, interpolating between (5.9) and (5.11) yields Theorem 5.2. The rest of this section contains the proofs of Lemma 5.1 and Lemma 5.2.

5.2 The L^2 bound

Proof of Lemma 5.1. Let $T \subset \mathbb{R}^n$ be a compact set and fix $s > \overline{\dim}_B T$. In this proof, the implied constants can depend on T and s.

Define ϕ by $\widehat{\phi}(\xi) = \rho_0(2^{-j-2}\xi)$, so that $\widehat{\phi}(\xi) = 1$ for $|\xi| \leq 2^{j+2}$ and $\widehat{\phi}(\xi) = 0$ for $|\xi| \geq 2^{j+3}$. We observe that since ρ_0 is a Schwartz function,

$$\sup_{\theta \in T} |\phi(x+\theta)| \lesssim \frac{2^{jn}}{(1 + 2^j \operatorname{dist}(x,T))^{100n}} =: \Phi(x) \tag{5.12}$$

We begin with some estimates on Φ, which will be the key estimates needed in our proof. We claim the following:

$$\|\Phi\|_{L^1(\mathbb{R}^n)} \lesssim 2^{js} \tag{5.13}$$

$$\left\| \frac{\partial}{\partial t} (\Phi)_t(x) \right\|_{L^1(\mathbb{R}^n)} \lesssim t^{-1} 2^{j(s+1)}. \tag{5.14}$$

By $s > \overline{\dim}_B T$, (5.7), and the boundedness of T, we have

$$|\{x \in \mathbb{R}^n : \operatorname{dist}(x,T) < r\}| \lesssim r^{n-s} \max(r,1)^s$$

for all $r \geq 0$, so

$$\begin{aligned}
&\int_{\mathbb{R}^n} (1 + 2^j \operatorname{dist}(x,T))^{-100n}\, dx \\
&= \int_{\operatorname{dist}(x,T) \leq 2^{-j}} + \sum_{\ell=-j+1}^{\infty} \int_{2^{\ell-1} < \operatorname{dist}(x,T) \leq 2^\ell} \\
&\lesssim \sum_{\ell=-j}^{\infty} (2^{j+\ell})^{-100n} |\{x \in \mathbb{R}^n : \operatorname{dist}(x,T) < 2^\ell\}| \\
&\lesssim \sum_{\ell=-j}^{\infty} (2^{j+\ell})^{-100n} (2^\ell)^{n-s} \max(2^\ell, 1)^s \\
&\lesssim (2^{-j})^{n-s},
\end{aligned}$$

which proves (5.13). For (5.14), we compute

$$\frac{\partial}{\partial t}(\Phi)_t(x) = -t^{-1}(n(\Phi)_t(x) + (\widetilde{\Phi})_t(x)) \tag{5.15}$$

where

$$\widetilde{\Phi}(x) := x \cdot \nabla\Phi(x).$$

Since $\operatorname{dist}(x, T)$ is a 1-Lipschitz function, it follows that for almost every $x \in \mathbb{R}^n$, $\nabla \operatorname{dist}(x, T)$ exists and $|\nabla \operatorname{dist}(x, T)| \leq 1$. Thus Φ is Lipschitz, and for almost every x, $\widetilde{\Phi}(x)$ is defined and satisfies

$$|\widetilde{\Phi}(x)| \lesssim \frac{2^{j(n+1)}}{(1 + 2^j \operatorname{dist}(x, T))^{100n}},$$

Arguing as in the proof of (5.13), we have

$$\|\widetilde{\Phi}\|_1 \lesssim 2^{j(s+1)}. \tag{5.16}$$

By (5.13), (5.15), and (5.16), we obtain (5.14).

Now we can prove (5.8) and (5.9). We may assume that $f \geq 0$. First we focus on (5.8). By the fact that the Fourier transform of $(f * \sigma_j)^2$ is supported in $B(0, 2^{j+2})$ and (5.12),

$$\begin{aligned} \mathcal{M}_T^j f(x)^2 = \sup_{\theta \in T} |(f * \sigma_j)(x + \theta)|^2 &= \sup_{\theta \in T} |((f * \sigma_j)^2 * \phi)(x + \theta)| \\ &\lesssim (|f * \sigma_j|^2 * \Phi)(x). \end{aligned} \tag{5.17}$$

Therefore,

$$\begin{aligned} \|\mathcal{M}_T^j f\|_2^2 &\lesssim \left\||f * \sigma_j|^2 * \Phi\right\|_1^2 \\ &\leq \|f * \sigma_j\|_2^2 \, \|\Phi\|_1 \end{aligned}$$

By Plancherel and the familiar estimate $|\widehat{\sigma}(\xi)| \lesssim (1+|\xi|)^{-\frac{n-1}{2}}$ (see, e.g., [19, Appendix B.4]), we have

$$\begin{aligned}
\|f * \sigma_j\|_2 &= \|\widehat{f}\rho_j\widehat{\sigma}\|_2 \\
&\leq \|\widehat{f}\|_2\|\rho_j\widehat{\sigma}\|_\infty \\
&\lesssim 2^{-j(\frac{n-1}{2})}\|f\|_2,
\end{aligned}$$

which combined with (5.13) yields (5.8).

For (5.9), we proceed similarly. By the fact that the Fourier transform of $(f*(\sigma_j)_t)^2$ is supported in $B(0, 2^{j+2}/t)$ and (5.12),

$$\begin{aligned}
S_T^j f(x)^2 &= \sup_{\theta\in T, t>0} |(f*(\sigma_j)_t)(x+t\theta)|^2 \\
&= \sup_{\theta\in T, t>0} |((f*(\sigma_j)_t)^2 * \phi_t)(x+t\theta)| \\
&\lesssim \sup_{t>0}((f*(\sigma_j)_t)^2 * (\Phi)_t)(x).
\end{aligned}$$

Since σ_j is smooth and Φ is Lipschitz, the function

$$t \mapsto ((f*(\sigma_j)_t)^2 * (\Phi)_t)(x)$$

is absolutely continuous for almost all $x \in \mathbb{R}^n$ by Radamacher's theorem. By the fundamental theorem of calculus,

$$\int_1^s \frac{\partial}{\partial t}((f*(\sigma_j)_t)^2 * (\Phi)_t)(x)\,dt = (f*(\sigma_j)_s)^2 * (\Phi)_s)(x) - (|f*\sigma_j|^2 * \Phi)(x).$$

Thus for $s > 1$

$$\begin{aligned}
(f*(\sigma_j)_s)^2 * (\Phi)_s)(x) &\leq \left|\int_1^s \frac{\partial}{\partial t}((f*(\sigma_j)_t)^2 * (\Phi)_t)(x)\,dt\right| + \left|(f*\sigma_j)^2 * \Phi(x)\right| \\
&\leq \int_1^s \left|\frac{\partial}{\partial t}((f*(\sigma_j)_t)^2 * (\Phi)_t)(x)\right|\,dt + (|f*\sigma_j|^2 * \Phi)(x) \\
&\leq \int_0^\infty \left|\frac{\partial}{\partial t}((f*(\sigma_j)_t)^2 * (\Phi)_t)(x)\right|\,dt + (|f*\sigma_j|^2 * \Phi)(x).
\end{aligned}$$

Similarly, for $0 < s < 1$,

$$\begin{aligned}(f * (\sigma_j)_s)^2 * (\Phi)_s)(x) &\le \left|\int_1^s \frac{\partial}{\partial t}((f * (\sigma_j)_t)^2 * (\Phi)_t)(x)\, dt\right| + \left|(f * \sigma_j)^2 * \Phi(x)\right| \\ &\le \int_s^1 \left|\frac{\partial}{\partial t}((f * (\sigma_j)_t)^2 * (\Phi)_t)(x)\right| dt + (|f * \sigma_j|^2 * \Phi)(x) \\ &\le \int_0^\infty \left|\frac{\partial}{\partial t}((f * (\sigma_j)_t)^2 * (\Phi)_t)(x)\right| dt + (|f * \sigma_j|^2 * \Phi)(x).\end{aligned}$$

Since the right hand side is independent of s, we can take supremum over all $s > 0$ to conclude that

$$S_T^j f(x)^2 = (|f * \sigma_j|^2 * \Phi)(x) + \int_0^\infty \left|\frac{\partial}{\partial t}((f * (\sigma_j)_t)^2 * (\Phi)_t)(x)\right| dt.$$

By Leibniz's rule and integrating in x,

$$\begin{aligned}\|S_T^j f\|_2^2 \lesssim &\int_{\mathbb{R}^n} (|f * \sigma_j|^2 * \Phi)(x)\, dx \\ &+ \int_{\mathbb{R}^n}\int_0^\infty \left|(\frac{\partial}{\partial t}[(f * (\sigma_j)_t)^2] * (\Phi)_t)(x)\right| dt\, dx \\ &+ \int_{\mathbb{R}^n}\int_0^\infty \left|((f * (\sigma_j)_t)^2 * \frac{\partial}{\partial t}(\Phi)_t)(x)\right| dt\, dx \\ := &\, I + II + III.\end{aligned}$$

In the proof of (5.8) above, we showed

$$I \lesssim 2^{j(s-(n-1))}\|f\|_2^2.$$

To estimate II, we use Fubini, Young's inequality, and

$$\|(\Phi)_t\|_1 = \|\Phi\|_1 \lesssim 2^{js}.$$

Then we argue as in the proof of [19, Lemma 6.5.2]. This gives us

$$II \lesssim 2^{js} \int_{\mathbb{R}^n}\int_0^\infty \left|\frac{\partial}{\partial t}[(f * (\sigma_j)_t)^2(x)]\right| dt\, dx$$

$$\lesssim 2^{js}(2^{j(\frac{1}{2}-\frac{n-1}{2})}\|f\|_2)^2.$$

For III, we use (5.14), Fubini, Young, and Plancherel to get

$$\begin{aligned} III &\lesssim 2^{j(s+1)} \int_{\mathbb{R}^n} \int_0^\infty |(f * (\sigma_j)_t)(x)|^2 \, \frac{dt}{t} \, dx \\ &\lesssim 2^{j(s+1)} \int_{\mathbb{R}^n} \int_0^\infty \left| \widehat{f}(\xi) \widehat{\sigma}_j(t\xi) \right|^2 \, \frac{dt}{t} \, d\xi \\ &\lesssim 2^{j(s+1)-j(n-1)} \|f\|_2^2, \end{aligned}$$

which completes the proof. □

5.3 Proof Corollary 5.2

With the results in Chapter 3 in mind, we define

$$N_\alpha f(x) := \sup_{\theta \in \mathbb{S}^{n-1}} c_{n,\alpha} \int_{|y| \le 1} |f(x-y-\theta)|(1-|y|^2)^{-\alpha} dy,$$

for $0 < \alpha < 1$, where $c_{n,\alpha} = \frac{2}{\omega_{n-1} B(\frac{n}{2}, 1-\alpha)}$ and B is the Beta function.

The argument underlined in Section 5.2 can be applied to the operator N_α, showing that it is bounded on L^2. We will be use this to show that any Nikodym set associated to spheres has full dimension

Proof of Corollary 5.2. Let $d\sigma^\alpha = c_{n,\alpha}(1-|y|^2)^{-\alpha} dy$ and let ρ_j, ϕ, and Φ be as above. Then we define $\sigma_j^\alpha = \left(\rho_j \widehat{\sigma^\alpha}\right)^\vee$ and

$$N_\alpha^j f(x) = \sup_{\theta \in T} |(f * \sigma_j)(x+\theta)|$$

and we observe that

$$N_\alpha^j f(x)^2 \lesssim (|f * \sigma_j^\alpha|^2 * \Phi)(x).$$

As was shown in Chapter 3, using the identity in [19, Appendix B.5] and the properties of Bessel functions, the multiplier

$$\widehat{\sigma_j^\alpha}(\xi) = c'_{n,\alpha}\frac{J_{\frac{n}{2}-\alpha}(2\pi|\xi|)}{|\xi|^{\frac{n}{2}-\alpha}}\rho_j(\xi)$$

is a smooth function which satisfies for all multi-indices γ

$$|\partial_\xi^\gamma \widehat{\sigma_j^\alpha}(\xi)| \lesssim_{n,\alpha,\gamma} 2^{-j(\frac{n+1}{2}-\alpha)}.$$

We proceed as in the proof on Lemma 5.1 to obtain that

$$\|N_\alpha^j f\|_{L^2} \lesssim_{n,\alpha} 2^{-(1-\alpha)j}\|f\|_{L^2}$$

and thus N_α is bounded from L^2 to L^2 for any $0 < \alpha < 1$.

From this, we can immediately conclude that the Minkowski dimension of the Nikodym set for spheres is n. Indeed let A_δ be a δ-neighborhood of $A \cap [-10, 10]^n$, where A is a Nikodym set associated to spheres. Then for every $x \in [-5, 5]^n$ there exists a $\theta_x \in \mathbb{S}^{n-1}$ such that $x \in \theta_x + \mathbb{S}^{n-1}$ and

$$\theta_x + \mathbb{S}^{n-1} \subset A_\delta.$$

Thus for $f = \chi_{A_\delta}$,

$$\begin{aligned} N_\alpha \chi_{A_\delta}(x) &\gtrsim_{n,\alpha} \int_{|y|\leq 1} |\chi_{A_\delta}(x-y-\theta_x)|(1-|y|^2)^{-\alpha}dy \\ &= \int_{1-d\leq|y|\leq 1}(1-|y|^2)^{-\alpha}dy \\ &\gtrsim_{n,\alpha} \delta^{1-\alpha}. \end{aligned}$$

Thus, using the L^2 to L^2 bound for N_α, we get that

$$\delta^{1-\alpha} \lesssim_{n,\alpha} \|N_\alpha\chi_{A_\delta}\|_{L^2} \lesssim_{n,\alpha} \|\chi_{A_\delta}\|_{L^2} \lesssim_{n,\alpha} |A_\delta|^{1/2}.$$

Since this holds for all $0 < \delta < 1$, we conclude that $\underline{\dim}_B A \geq n - 2(1-\alpha)$ for all $\alpha < 1$ and thus $\underline{\dim}_B A = n$.

To address the question of the Hausdorff dimension of the Nikodym set for spheres, we define $\mathbb{S}_\delta = \{x \in \mathbb{R}^n : 1 - \delta \leq |x| \leq 1\}$ and define the Nikodym maximal function for spheres:

$$\widetilde{N}_\delta f(x) = \sup_{\theta \in \mathbb{S}^{n-1}} \frac{1}{|\mathbb{S}_\delta|} \int_{\mathbb{S}_\delta} |f(x - y - \theta)| dy$$

From the definition of $\mathbb{S}_\delta$ we conclude that

$$|\mathbb{S}_\delta| \sim \delta.$$

For $y \in S_\delta$, we have

$$(1 - |y|^2)^{-\alpha} \gtrsim_\alpha \delta^{-\alpha}$$

so we have the pointwise bound

$$\begin{aligned}
\widetilde{N}_\delta f(x) &= \frac{1}{|\mathbb{S}_\delta|} \sup_{\theta \in \mathbb{S}^{n-1}} \int_{\mathbb{S}_\delta} |f(x - y - \theta)| dy \\
&\lesssim \delta^{-1} \sup_{\theta \in \mathbb{S}^{n-1}} \int_{\mathbb{S}_\delta} |f(x - y - \theta)| dy \\
&\lesssim_\alpha \delta^{\alpha - 1} \sup_{\theta \in \mathbb{S}^{n-1}} \int_{\mathbb{S}_\delta} |f(x - y - \theta)|(1 - |y|^2)^{-\alpha} dy \\
&\lesssim_\alpha \delta^{\alpha - 1} N_\alpha f(x).
\end{aligned}$$

Since N_α is bounded in L^2 for any $0 < \alpha < 1$, it follows that

$$\|\widetilde{N}_\delta f\|_2 \lesssim_\epsilon \delta^{-\epsilon} \|f\|_2, \qquad \text{for all } \epsilon > 0 \tag{5.18}$$

Using (5.18) and a standard pigeonhole on scales argument, we conclude that the Hausdorff dimension of any Nikodym set for spheres is n. See, e.g., [53, Proposition 10.2] or [38, Theorem 22.9] for the details. This completes the proof of Corollary 5.2. □

5.4 The L^1 Bound

Proof of Lemma 5.2. Let $T \subset \mathbb{R}^n$ be a compact set. Given $\epsilon > 0$, fix a large integer N such that $n/N < \epsilon$. In this proof, the implied constants can depend on T and N.

Let ψ be given by

$$\widehat{\psi}(\xi) = \rho_0(2^{-j}N^{-1}\xi).$$

We have

$$\begin{aligned}\sup_{\theta\in T}|\psi(x+\theta)| &\lesssim \frac{(2^jN)^n}{(1+2^jN\operatorname{dist}(x,T))^{100nN}} \\ &\lesssim \frac{2^{jn}}{(1+|x|)^{100nN}} \\ &=: \Psi(x),\end{aligned}$$

where the first inequality is analogous to (5.12), and the second is by the boundedness of T.

Observe that the Fourier transform of $(f*\sigma_j)^N$ is supported in $B(0,2^jN)$, so for any $t>0$

$$\begin{aligned}\sup_{\theta\in T}|(f*(\sigma_j)_t)(x+t\theta)|^N &= \sup_{\theta\in T}|((f*(\sigma_j)_t)^N*(\psi)_t)(x+t\theta)| \\ &\le (|f*(\sigma_j)_t|^N*(\Psi)_t)(x).\end{aligned}$$

Thus, by Minkowski's integral inequality

$$\begin{aligned}S_T^jf(x) &\le \sup_{t>0}(|f*(\sigma_j)_t)|^N*(\Psi)_t)(x)^{1/N} \\ &\le \sup_{t>0}\left(f*(|(\sigma_j)_t|^N*(\Psi)_t)^{1/N}\right)(x) \qquad (5.19)\\ &= \sup_{t>0}\big(f*(K_j)_t\big)(x).\end{aligned}$$

where

$$K_j = (|\sigma_j|^N * \Psi)^{1/N}.$$

Similarly,

$$M_T^j f(x) \leq \big(f * K_j\big)(x). \tag{5.20}$$

Next, we claim

$$K_j(x) \lesssim \frac{2^{j(1+\epsilon)}}{(1+|x|)^{100n}} \tag{5.21}$$

By [19, p. 480],

$$|\sigma_j(x)| = \big|(\rho_j^\vee * \sigma)(x)\big| \lesssim \frac{2^j}{(1+|x|)^{200n}}.$$

Therefore,

$$\begin{aligned}
\left(|\sigma_j|^N * \Psi\right)(x) &\lesssim \int \frac{2^{jN}}{(1+|y|)^{200nN}} \frac{2^{jn}}{(1+|x-y|)^{100nN}}\, dy \\
&\leq 2^{j(N+n)} \int \frac{1}{(1+|y|)^{200nN}} \frac{(1+|y|)^{100nN}}{(1+|x|)^{100nN}}\, dy \\
&\lesssim \frac{2^{j(N+n)}}{(1+|x|)^{100nN}}
\end{aligned}$$

which together with $n/N < \epsilon$ proves (5.21).

Observe that (5.20), (5.21), and Young's convolution inequality together imply (5.10). Moreover, since K_j has a radially decreasing majorant,

$$\sup_{t>0} |f * (K_j)_t|(x) \leq \|K_j\|_{L^1} Mf(x),$$

where Mf is the the Hardy-Littlewood maximal function of f. (See, e.g., [19, Theorem 2.1.10].) This, (5.19), (5.21), and the weak type (1,1) bound on the Hardy-Littlewood maximal function together imply (5.11). □

5.5 Lower bounds on the range of p

In this section we obtain lower bounds for the range of boundedness of $\mathcal{M}_T$ and S_T. Let $\dim_H T$ denote the Hausdorff dimension of T.

Lemma 5.3. *Let $k \in \mathbb{Z}$ such that $1 \leq k \leq n-1$. If $T \subset \mathbb{R}^{n-k} \times \{0\}^k$, $\overline{\dim}_B T = \dim_H T$, and $S = \{0\}^{n-k-1} \times \mathbb{S}^k$, then $\overline{\dim}_B(T+S) = k + \overline{\dim}_B T$.*

Lemma 5.3 is consequence of a Fubini-type argument.

Proof. The inequality

$$\overline{\dim}_B(T+S) \leq k + \overline{\dim}_B T$$

is immediate from the definition of the upper Minkowski dimension. On the other hand, by[37, Theorem 5.12],

$$\overline{\dim}_B(T+S) \geq \dim_H(T+S).$$

Since

$$\dim_H A \leq \dim_H B, \quad \text{if } A \subset B$$

it is enough to work with a single chart of $\mathbb{S}^k$. Since the Hausdorff dimension is preserved under diffeomorphisms, it is enough to show that

$$\dim_H \Big(T + \big(\{0\}^{n-k} \times B^k(0,1)\big)\Big) \geq k + \dim_H T,$$

or equivalently

$$\dim_H \Big(T \times B^k(0,1)\Big) \geq k + \dim_H T,$$

which is true by [37, Theorem 8.10]. $\square$

Theorem (Theorem 5.3). *Let $n \geq 2$ and let $0 \leq s \leq 1$. If $\mathcal{M}_T$ is bounded on $L^p(\mathbb{R}^n)$ for all $\overline{\dim}_B T = s$, then $p \geq 1 + \frac{s}{n-1}$.*

Proof. Let T be any subset of $\mathbb{R}^1 \times \{0\}^{n-1}$ with

$$\overline{\dim}_B T = \dim_H T = s$$

and $T = -T$. Suppose $\mathcal{M}_T$ is bounded on $L^p(\mathbb{R}^n)$. Note that

$$\mathcal{M}_T \chi_{B(0,2\delta)} \gtrsim \delta^{n-1}$$

on the set

$$\{x \in \mathbb{R}^n : \operatorname{dist}(x, T + \mathbb{S}^{n-1}) < \delta\}.$$

Thus

$$\begin{aligned} \delta^{n-1} |\{x : \operatorname{dist}(x, T + \mathbb{S}^{n-1}) < \delta\}|^{1/p} &\lesssim \|\mathcal{M}_T \chi_{B(0,2\delta)}\|_p \\ &\lesssim \|\chi_{B(0,2\delta)}\|_{L^p} \\ &\lesssim \delta^{n/p} \end{aligned}$$

so

$$|\{x : \operatorname{dist}(x, T + \mathbb{S}^{n-1}) < \delta\}| \lesssim \delta^{n - p(n-1)}.$$

By Lemma 5.3, $T + \mathbb{S}^{n-1}$ has upper Minkowski dimension $n - 1 + s$, so

$$p \geq 1 + \frac{s}{n-1}.$$

□

The following applies for $1 < s \leq n - 1$. It actually applies for $s \leq 1$ too, but the previous theorem gives a stronger bound in that case.

Theorem (Theorem 5.4)**.** *Let $n \geq 2$ and let $1 < s \leq n-1$. If $\mathcal{M}_T$ is bounded on $L^p(\mathbb{R}^n)$ for all $\overline{\dim}_B T = s$, then*

$$p \geq 1 + \frac{s - \lceil s \rceil + 1}{n - \lceil s \rceil + 1}.$$

Proof. Let $k = \lceil s \rceil$, i.e., k is the integer such that $k - 1 < s \leq k$. Let k be any subset of $\mathbb{R}^k \times \{0\}^{n-k}$ with $\overline{\dim}_B T = \dim_H T = s$ and $T = -T$. Define

$$C = \frac{1}{\sqrt{2}}\mathbb{S}^{k-2} \times \{0\}^{n-k+1} = \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_{k-1}^2 = \tfrac{1}{2}, x_k = \cdots = x_n = 0\}$$
$$C' = \{0\}^{k-1} \times \frac{1}{\sqrt{2}}\mathbb{S}^{n-k} = \{x \in \mathbb{R}^n : x_1 = \cdots = x_{k-1} = 0, x_k^2 + \cdots + x_n^2 = \tfrac{1}{2}\}$$

and note that

$$C = \bigcap_{x \in C'} (x + \mathbb{S}^{n-1}).$$

This implies

$$\mathcal{M}_T \chi_{C_{2\delta}} \gtrsim \delta^{n-k+1} \quad \text{on } \{x : \operatorname{dist}(x, T + C') < \delta\},$$

where $C_{2\delta}$ denotes the 2δ-neighborhood of C. Thus

$$\begin{aligned}
&\delta^{n-k+1} |\{x : \operatorname{dist}(x, T + C') < \delta\}|^{1/p} \\
&\lesssim \|\mathcal{M}_T \chi_{C_{2\delta}}\|_{L^p} \\
&\lesssim \|\chi_{C_{2\delta}}\|_{L^p} \\
&\lesssim \delta^{(n-k+2)/p}
\end{aligned}$$

so

$$|\{x : \operatorname{dist}(x, T + C') < \delta\}| \lesssim \delta^{(n-k+2)-p(n-k+1)}$$

By Lemma 5.3, $T + C'$ has upper Minkowksi dimension $n - k + s$, so

$$p \geq 1 + \frac{s - k + 1}{n - k + 1}.$$ □

Since

$$M_T f(x) \leq S_T f(x)$$

pointwise for all $x \in \mathbb{R}^n$, the same lower bounds hold for S_T as well.

BIBLIOGRAPHY

[1] Anderson, T., Palsson, E., *Bounds for discrete multilinear spherical maximal functions in higher dimensions.* (2019), Preprint. arXiv:1911.00464 [math.CA].

[2] Anderson, T., Palsson, E., *Bounds for discrete multilinear spherical maximal functions.* (2019), Preprint. arXiv:1910.11409 [math.CA]

[3] Barrionuevo, J., Grafakos, L., He, D., Honzík, P., Oliveira, L., *Bilinear spherical maximal function.* Mathematics Research Letters **25**(5) (2018), 1369–1388.

[4] Bak, J.-K., Oberlin, D. M., Seeger, A., *Restriction of Fourier transforms to curves and related oscillatory integrals.* American Journal of Mathematics **131**(2) (2009), 277–311.

[5] Besicovitch, A., *On Kakeya's problem and a similar one.* Mathematische Zeitschrift **27** (1), 312–320.

[6] Bourgain, J., *Estimations de certaines fonctions maximales.* Comptes Rendus de l'Académie des Sciences - Series I - Mathematics. **301** (1985), 499–502.

[7] Bourgain, J., *Averages in the plane over convex curves and maximal operators.* Journal d'Analyse Mathématique **47**(1) (1986), 69–85.

[8] Calderon, C. P., *Lacunary spherical means.* Illinois Journal of Mathematics **23**(3) (1979), 476–484.

[9] Carbery, A., *Radial Fourier multipliers and associated maximal functions.* North–Holland Mathematics Studies **111** (1985), 49–56.

[10] Chang, A., Csörnyei, M., *The Kakeya needle problem and the existence of Besicovitch and Nikodym Sets for rectifiable sets.* Proceedings of the London Mathematical Society **118** (2019), 1084–1114.

[11] Chen, J., Grafakos, L., He, D. and Honzík, P., Slavíková, L., *Bilinear maximal functions associated with surfaces.* (2019), Submitted.

[12] Cowling, M., Mauceri, G., *On maximal functions.* Milan Journal of Mathematics **49**(1) (1979), 79–87.

[13] Cunningham, F. Jr., *The Kakeya problem for simply connected and for star-shaped sets*, American Mathematical Monthly **78** (1971), 114–129.

[14] Duoandikoetxea, J., Vega, L., *Spherical means and weighted inequalities.* Journal of the London Mathematical Society **53**(2) (1996), 343–353.

[15] Falconer, K. J., *Sets with prescribed projections and Nikodym sets*, Proceedings of the London Mathematical Society **53** (1986), 48–64.

[16] H. Federer, *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band **153**, Springer-Verlag New York Inc., New York, 1969.

[17] Fefferman, C., *The multiplier problem for the ball.* Annals of Mathematics, **94** (2) (1971) 330–336.

[18] Geba, D., Greenleaf, A., Iosevich, A., Palsson, E., Sawyer, E., *Restricted convolution inequalities, multilinear operators and applications.* Mathematical Research Letters **20**(4) (2013), 675–694.

[19] Grafakos, L., *Classical Fourier Analysis*, Third Edition. Graduate Texts in Mathematics, **249**, Springer, New York, 2014.

[20] Grafakos, L., *Modern Fourier Analysis*, Third Edition. Graduate Texts in Mathematics, **250**, Springer, New York, 2014.

[21] Grafakos, L., He, D., and Honzík P., *Rough bilinear singular integrals.* Advances in Mathematics, **326** (2018), 54–78.

[22] Grafakos, L., He, D., Honzík P., *Maximal operators associated with bilinear multipliers of limited decay.* Journal d'Analyse Mathématique (2018) accepted for publication.

[23] Grafakos, L., He, D., Honzík P., Park, B. J., *Initial $L^2 \times \cdots \times L^2$ bound for multilinear operators.* 2020, Preprint.

[24] Grafakos, L., He, D. and Slavíková, L., *$L^2 \times L^2 \to L^1$ boundedness criteria.* Mathematische Annalen, **376**(1-2) (2020), 431–455.

[25] Grafakos, L., Liu, L., Lu, S., Zhao, F., *The multilinear Marcinkiewicz interpolation theorem revisited: The behavior of the constant.* Journal of Functional Analysis **262**(5), (2012), 2289–2313.

[26] Greenleaf, A., *Principal curvature and harmonic-analysis.* Indiana University Mathematics Journal **30**(4) (1981), 519–537.

[27] Greenleaf, A., Iosevich, A., *On triangles determined by subsets of the Euclidean plane, the associated bilinear operators and applications to discrete geometry.* Annals of PDE **5**(2) (2012), 397–409.

[28] Héra, K., Laczkovich, M., *The Kakeya problem for circular arcs,* Acta Mathematica Hungarica **150**(2) (2016), 479–511.

[29] Heo, Y., Hong, S., Yang, C. W., *Improved bounds for the bilinear spherical maximal operators.* (2019), Preprint.

[30] Iosevich, A. and Sawyer, E., *Oscillatory integrals and maximal averages over homogeneous surfaces.* Duke Mathematical Journal **82**(1) (1996), 103–141.

[31] Iosevich, A., Sawyer, E., *Three problems motivated by the average decay of the Fourier transform.* Contemporary Mathematics **320** (2003), 205–216.

[32] Iosevich, A., Sawyer, E. and Seeger, A., *Mean square discrepancy bounds for the number of lattice points in large convex bodies.* Journal d'Analyse Mathématique **87**(1) (2002), 209–230.

[33] Jeong, E., Lee, S., *Maximal estimates for the bilinear spherical averages and the bilinear Bochner-Riesz.* (2019), Preprint. arXiv:1903.07980 [math.CA].

[34] Lerner, A., Ombrosi, S., Pérez, C., Torres, R., Trujillo–González, R., *New maximal functions and multiple weights for the multilinear Calderón–Zygmund theory.* Advances in Mathematics **220** (4) (2009), 1222–1264.

[35] Magyar, A., Stein, E., and Wainger, S., *Discrete analogues in harmonic analysis: spherical averages.* Annals of Mathematics (2nd Ser.) **155**(1) (2002), 189–208.

[36] Marstrand, J. M., *Packing Circles in the Plane.* Proceedings of the London Mathematical Society **s3-55**(1) (1987), 37–58,

[37] Mattila, P., *Geometry of sets and measures in Euclidean spaces*, Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, 1995.

[38] Mattila, P., *Fourier analysis and Hausdorff dimension.* Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2015.

[39] Mockenhaupt, G., Seeger, A., and Sogge, C. D., *Wave front sets, local smoothing and Bourgain's circular maximal theorem.* Annals of Mathematics (2nd Ser.) **136**(1) (1992), 207–218.

[40] Nagel, A., Seeger, A., Wainger, S., *Averages over convex hypersurfaces.* American Journal of Mathematics **115**(4) (1993), 903–927.

[41] Neuman, M., *The pyramid averaging operator.* (2020), Preprint. arXiv:2005.06365 [math.CA].

[42] Nikodym, O., *Sur la mesure des ensembles plans dont tous les points sont rectilinéairement accessibles*, Fundamenta Mathematicae **10** (1927) (1), 116–168.

[43] Oberlin, D., *Multilinear convolutions defined by measures on spheres.* Transactions of the American Mathematical Society **310** (1988), 821–835.

[44] Palsson, E., Sovine, S., *The Triangle Averaging Operator.* Journal of Functional Analysis **279**(8) (2020).

[45] Rubio de Francia, J.-L., *Maximal functions and Fourier transforms.* Duke Mathematical Journal **53**(2) (1986), 395–404.

[46] Schlag, W., *A geometric proof of the circular maximal theorem.* Duke Mathematical Journal **93** (1998), 505–534.

[47] Seeger, A., Tao, T., Wright, J., *Endpoint mapping properties of spherical maximal operators.* Journal of the Institute of Mathematics of Jussieu **2**(1) (2003), 109–144.

[S91] Sogge, C.D., *Propagation of singularities and maximal functions in the plane.* Inventiones Mathematicae **104** (1991), 349–376.

[48] Sogge, C. D., Stein, E. M., *Averages of functions over hypersurfaces in* $\mathbb{R}^n$. Inventiones mathematicae **82**(3) (1985), 543–556.

[49] Sogge, C. D., Stein, E. M., *Averages over hypersurfaces: II.* Inventiones mathematicae, **86**(2) (1986), 233–242.

[50] Stein, E. M., *Maximal functions: spherical means.* Proceedings of the National Academy of Sciences **73**(7) (1976), 2174–2175.

[51] Stein, E. M., *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals.* Princeton Mathematical Series, **43**, Springer, New Jersey, 1993.

[52] Stein, E. M., Strömberg, J. O., *Behavior of maximal functions in R^n for large n.* Arkiv för Matematik **21**(1–2) (1983), 259–269.

[53] Wolff, T.H., *Lectures on harmonic analysis.* University Lecture Series, American Mathematical Society, 2003.

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