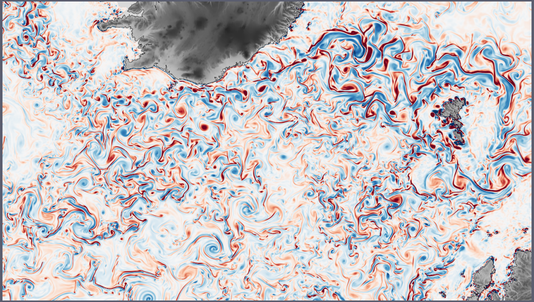
Consistent subgrid scale modeling for oceanic climate models

Florian Lemarié



Consistent subgrid scale modeling for oceanic climate models

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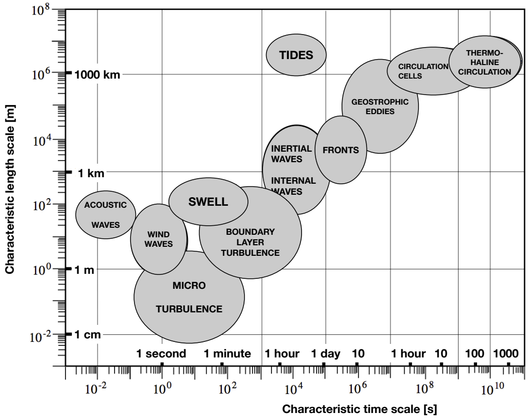
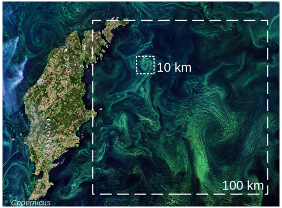
Context

Scales of motion and processes present in the ocean occur on a wide range of scales from global $(\mathcal{O}(10^7\ \mathrm{m}, 10^{10}\ \mathrm{s}))$ to micro $(\mathcal{O}(10^{-3}\ \mathrm{m}, 10^{-3}\ \mathrm{s}))$ scales



Various important processes occur on scales that are too fine to be resolved



Processes smaller than the grid size must be homogenized (i.e. averaged)

$\Rightarrow$ **subgrid models (a.k.a. "parameterizations")**

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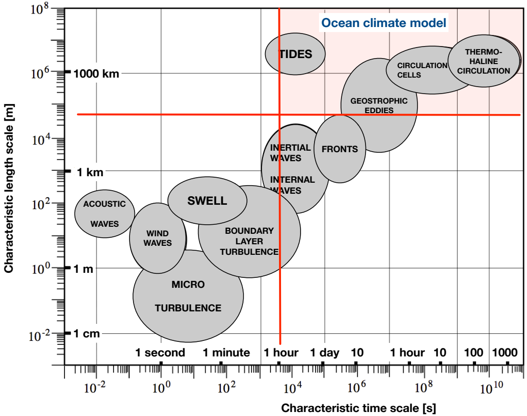

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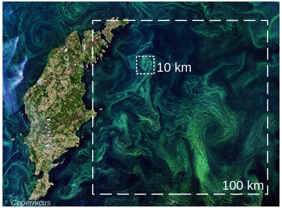

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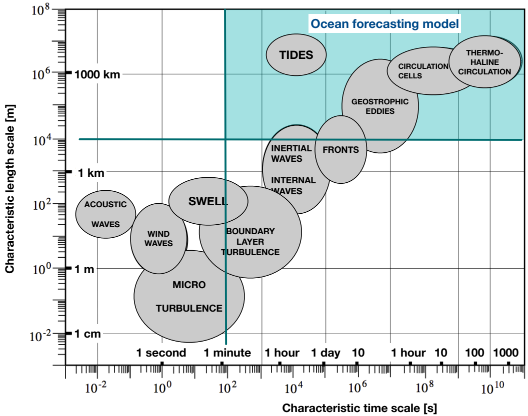

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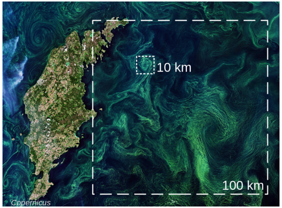


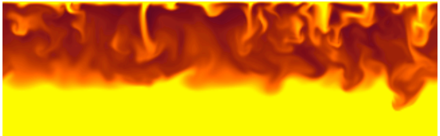
Processes smaller than the grid size must be homogenized (i.e. averaged)

$\Rightarrow$ **subgrid models (a.k.a. "parameterizations")**

Context

A great variety of processes need to be parameterized:

- Unresolved waves (e.g. breaking internal waves)
- Local turbulence (e.g. boundary layer turbulence)
- Mesoscale eddies
- Intermittent coherent structures (e.g. convective plumes)



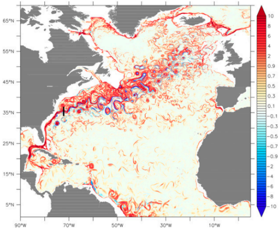
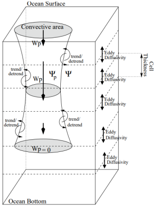


FIG. 2. Instantaneous energy transfer from mean flow to waves (10^{-6} W m^{-3}) at 300 m depth in a high-resolution model of the North Atlantic Ocean. The position of the section shown in Fig. 3 at



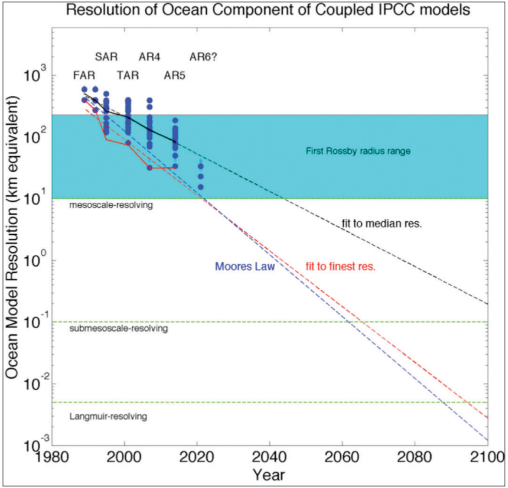
Challenges :

▷ Appropriate choice of an averaging operator to separate the resolved and unresolved scales

▷ Variety of processes $\rightarrow$ structurally different subgrid models (diffusive/viscous term, advective term, drag, etc.)

Context

Energy transfers from the thousand of kilometers scales, to the centimeters scales



In the case of the oceanic turbulence

1. **From the large-scale currents to the mesoscale eddies**. The large-scale oceanic currents are unstable which generate eddies with scales of 10 to 100 kilometers (the mesoscales).
2. **The mesoscale eddies then interact and generate submesoscale turbulent filaments** on scales from 10 kilometers to 100 meters.
3. Only at scales below approximately 10 meters, **the turbulence becomes three-dimensional** and it is described as stratified microscale turbulence

$\Rightarrow$ *discretizing fluid equations without further approximations and parameterizations would require computers about* 10^{10} ***times faster and bigger in storage*** *than present supercomputers*

(Fox-Kemper et al., 2014)

Content

1. Averaged oceanic primitive equations
2. Standard turbulent boundary layer closures
3. Coupling two turbulent Ekman boundary layers
4. Multi-fluid representation of oceanic convection

1

Averaged oceanic primitive equations

Un-averaged oceanic primitive equations

Energy consistent seawater Boussinesq system (e.g. Young, 2010)

In Cartesian geopotential coordinate

$$
\begin{aligned}
\partial_t \mathbf{v} + 2\boldsymbol{\Omega} \times \mathbf{u} + \nabla \cdot (\mathbf{u} \otimes \mathbf{v}) &= -\frac{1}{\rho_0}\nabla p - \frac{\rho}{\rho_0}\mathbf{g} \\
\nabla \cdot \mathbf{u} &= 0 \\
\partial_t \varphi + \nabla \cdot (\mathbf{u}\varphi) &= F_\varphi, \qquad (\varphi = \Theta, S_A) \\
\rho &= \rho_{\mathrm{eos}}(\Theta, S_A, p_0(z))
\end{aligned}
$$

$\mathbf{u} = (u, v, w)^T$

$\mathbf{v} = (u, v, 0)^T$

$\boldsymbol{\Omega} = (0, 0, f/2)^T$ shallow ocean traditional assumption

$p_0 = -\rho_0 g z$ background hydrostatic pressure

$p = p_h + \rho_0 g \eta$

$\mathbf{g} = (0, 0, g)$ spherical geoid assumption

Common approximations:

- Geometrical assumptions (spherical geoid approximation, traditional shallow-ocean approximation)
- Dynamical assumptions (Hydrostatic and Boussinesq assumptions)

Which filter is adequate to construct the coarse-grained equations ?

Scale separation operator $\mathcal{S}$ to obtain the following decomposition for any variable φ

$$\begin{aligned} \varphi &= \overline{\varphi} + \varphi' \\ \overline{\varphi} &= \mathcal{S}(\varphi) && \text{(resolved part)} \\ \varphi' &= (\mathrm{Id} - \mathcal{S})(\varphi) && \text{(unresolved part)} \end{aligned}$$

The scale separation operator should be chosen

- to reduce the complexity of the problem (limit the number of degrees of freedom)
- such that the averaged equations keep the same form as the un-averaged equations

Example of the continuity equation:

$$\nabla \cdot \overline{\mathbf{u}} = -\mathcal{S}(\nabla \cdot \mathbf{u}')$$

Scale separation operator must commute with differential operators otherwise the resolved field is no longer incompressible

Which filter $\mathcal{S}(\varphi)$ is adequate to construct the coarse-grained equations ?

- Ensemble average (statistical description of the flow)

$$\mathcal{S}_E(\varphi) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \varphi_n(\mathbf{x},t)$$

- Low-pass filter

$$\mathcal{S}_F(\varphi) = \sum_{|\mathbf{k}|\leq k_e} F(\varphi) e^{i\mathbf{k}\cdot\mathbf{x}}$$

- Space averaging ($B(x,r)$: ball of center x and radius r)

$$\mathcal{S}_B(\varphi) = \frac{1}{|B|} \int_{B(x,r)} \varphi(y,z,t) dy$$

- Time averaging

$$\mathcal{S}_T(\varphi) = \frac{1}{T} \int_{t-T}^{t} \varphi(\mathbf{x},\tau) d\tau$$

- Space-Time averaging

$$\mathcal{S}_T \circ \mathcal{S}_B(\varphi) = \frac{1}{T} \int_{t-T}^{t} \mathcal{S}_B(\varphi) d\tau$$

Filter properties

	$\mathcal{S}_E$	$\mathcal{S}_F$	$\mathcal{S}_B$	$\mathcal{S}_T$	$\mathcal{S}_T \circ \mathcal{S}_B$
Linearity $\mathcal{S}(u+v) = \mathcal{S}(u) + \mathcal{S}(v)$	X	X	X	X	X
Derivatives and averages commute	X	X	X	X	X
Double averages $\mathcal{S} \circ \mathcal{S}(u) = \mathcal{S}(u)$	X	X			
Product average $\mathcal{S}(v\mathcal{S}(u)) = \mathcal{S}(v)\mathcal{S}(u)$	X				

Only the ensemble averaging satisfies all properties

Link with observations and experiments: turbulence is assumed ergodic

$\Rightarrow$ *The ergodic theorem of probability theory (Neveu[1967]) says that, under certain conditions, statistical means can be replaced by time averages or spatial averages in the case of multidimensional processes.*

Averaging of nonlinear dynamics

Any physical variable designated by φ can be decomposed in ensemble mean and fluctuating components (**Reynolds decomposition**)

$$\varphi = \overline{\varphi} + \varphi', \qquad \overline{\varphi'} = 0$$

For any physical variable designated by φ

$$\partial_t \varphi + \nabla \cdot (\mathbf{u}_h \varphi) + \partial_z (w\varphi) = F_\varphi$$

Apply the decomposition and use properties of ensemble averaging

$$\partial_t \overline{\varphi} + \nabla \cdot (\overline{\mathbf{u}}_h \overline{\varphi}) + \partial_z (\overline{w}\,\overline{\varphi}) = \overline{F}_\varphi - \nabla \cdot (\overline{\mathbf{u}'_h \varphi'}) - \partial_z \overline{w'\varphi'}$$

- **Subgrid-scale parameterization problem** = find a closed expression for $\overline{\mathbf{u}'_h \varphi'}$ and $\overline{w'\varphi'}$ in terms of the known resolved-scale variables
- Because of the stratification, in many oceanic turbulent problems, the dominant term is the vertical eddy transport

Averaging of nonlinear EOS

For practical reasons a constraint is to treat the averaged variables in the same way as the un-averaged variables

Averaged equations traditionally use the hydrostatic assumption

$$\partial_z \overline{p}_h = -g\rho_{\text{eos}}\left(\overline{\Theta}, \overline{S}, p_0(z)\right)$$

but it can not be obtained from the average operation

$$\partial_z \overline{p}_h = -g\overline{\rho_{\text{eos}}\left(\overline{\Theta}+\Theta', \overline{S}+S', p_0(z)\right)} \approx -g\rho_{\text{eos}}\left(\overline{\Theta}, \overline{S}, p_0(z)\right) - g\left[(\partial_S^2\rho)\frac{\overline{(S')^2}}{2} + (\partial_{\Theta S}\rho)\overline{S'\Theta'} + (\partial_\Theta^2\rho)\frac{\overline{(\Theta')^2}}{2}\right] + \dots$$

Subgrid scale variability of S and Θ neglected in hydrostatic balance

Averaged primitive equations

$$
\begin{aligned}
\partial_t \overline{\mathbf{v}} + 2\boldsymbol{\Omega} \times \overline{\mathbf{u}} + \nabla \cdot (\overline{\mathbf{u}} \otimes \overline{\mathbf{v}}) &= -\frac{1}{\rho_0}\nabla \overline{p} - \frac{\overline{\rho}}{\rho_0}\mathbf{g} - \nabla \cdot (\overline{\mathbf{v}'\varphi'}) - \partial_z(\overline{w'\mathbf{v}'}) \\
\nabla \cdot \overline{\mathbf{u}} &= 0 \\
\partial_t \overline{\varphi} + \nabla \cdot (\overline{\mathbf{u}}\overline{\varphi}) &= \overline{F}_\varphi - \nabla \cdot (\overline{\mathbf{v}'\varphi'}) - \partial_z(\overline{w'\varphi'}), \qquad (\varphi = \Theta, S_A) \\
\overline{\rho} &= \rho_{\text{eos}}(\overline{\Theta}, \overline{S_A}, p_0(z))
\end{aligned}
$$

Notion of physics-dynamics coupling

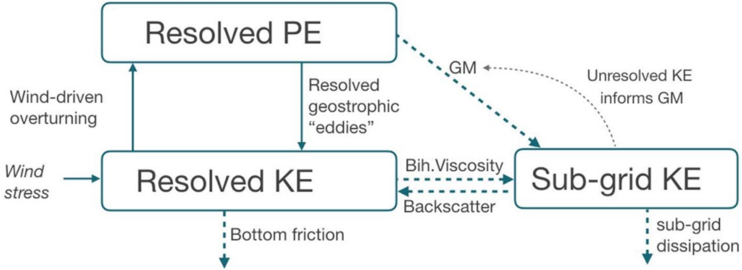
- Resolved fluid dynamics component = the dynamical core or simply “dynamics”
- Parameterizations that represent the unresolved and under-resolved processes = “physics”

Traditionally the 2 components are discretized mostly independently of each other (compartmentalization of the model codes)

Recent effort to address physics–dynamics coupling issues (Gross et al., 2018)

What is consistency?

- **Energetic considerations**



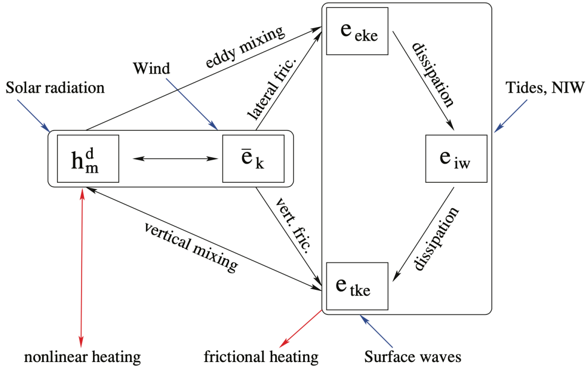
Solid arrows: explicitly resolved energy transfers
Dashed arrows: parameterized transfers

GM = adiabatic, potential-energy diminishing eddy transport scheme

What is consistency?

- **Energetic considerations**

Parameterizations are linked in an energetically consistent way



Eden (2016) "Closing the energy cycle in an ocean model", *Ocean Modelling*

What is consistency?

- Energetic considerations
- **Convergence in time and space (scale-awareness)**
  - Statistical assumptions of some parameterizations are satisfied only in specific range of scales
  - ”Grey zone”: resolution threshold where a process that was completely unresolved at larger scales becomes partially resolved

What is consistency?

- Energetic considerations
- Convergence in time and space (scale-awareness)
- **Compatibility with the laws of physics**

$\rightarrow$ Compatibility of the model formulation with the second law of thermodynamics: positive definiteness of the entropy production specifies the direction of the subgrid fluxes, but not their amount (this is the task of the physics parameterizations)

$\rightarrow$ Realizability conditions on Reynolds stress tensor

What is consistency?

- Energetic considerations
- Convergence in time and space (scale-awareness)
- Compatibility with the laws of physics
- The whole problem must be well-posed

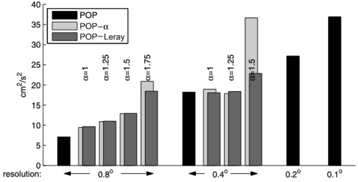
Alternatives to Reynolds averaging

- **LANS – α model** (Holm et al., 2005): extension of the Leray regularization of the NS equations

$$\begin{cases} \partial_t \mathbf{u}_h + \widetilde{\mathbf{u}}_h \cdot \nabla \mathbf{u}_h + \widetilde{w}\partial_z \mathbf{u}_h + (\nabla \widetilde{\mathbf{u}}_h)^T \cdot \mathbf{u}_h &= -f\mathbf{e}_z \times \widetilde{\mathbf{u}}_h - \frac{1}{\rho_0}\nabla p + \mathbf{F} \\ \partial_z p &= -\rho g \\ \nabla \cdot \widetilde{\mathbf{u}}_h + \partial_z \widetilde{w} &= 0 \\ \mathbf{u}_h &= \left(1 - \alpha^2(\nabla^2 + \partial_z^2)\right) \widetilde{\mathbf{u}}_h \qquad (\widetilde{\mathbf{u}}_h = G_\alpha * \mathbf{u}_h) \end{cases}$$

- converges toward NS when $\alpha \to 0$
- α = effective cutoff scale
- $\widetilde{\mathbf{u}}_h$ is smoother than $\mathbf{u}_h$
- α parameter controls the strength of smoothing in a Helmholtz inversion operator

Attempt to parameterize the effects of mesoscale eddies but requires closure schemes for other processes (e.g. vertical mixing parameterizations)



Globally averaged horizontal kinetic energy (Hecht et al., 2008)

Alternatives to Reynolds averaging

- **Fluid flow dynamics under location uncertainty** (Memin, 2014)

Deterministic transport: $d\mathbf{X}_t = \mathbf{u}dt$

$$D_t\varphi := \partial_t\varphi + \mathbf{u}\cdot\nabla\varphi = 0$$

Stochastic transport: $d\mathbf{X}_t = \mathbf{u}dt + \boldsymbol{\sigma}d\mathbf{B}_t$ $\quad (\mathbf{a}(\mathbf{x},t) = \boldsymbol{\sigma}\boldsymbol{\sigma}^T(\mathbf{x},t))$, $\quad \boldsymbol{\sigma}$: noise diffusion tensor

$$\mathbf{X}_t = \mathbf{X}_0 + \underbrace{\int_0^t \mathbf{u}(\mathbf{X}_s,s)ds}_{\text{slow}} + \underbrace{\int_0^t \boldsymbol{\sigma}(\mathbf{X}_s,s)d\mathbf{B}_s}_{\text{fast (Brownian)}}$$

Itô integral

$$\mathbb{D}_t\varphi := d_t\varphi + (\mathbf{u} - \underbrace{\frac{1}{2}\nabla\cdot\mathbf{a}}_{\text{Spatial inhomogeneity}})\cdot\nabla\varphi dt + \underbrace{\boldsymbol{\sigma}d\mathbf{B}_t\cdot\nabla\varphi}_{\text{Energy backscattering}} - \underbrace{\frac{1}{2}\nabla\cdot(\mathbf{a}\nabla\varphi)}_{\text{Turbulent diffusion}}dt = 0$$

requires a smoothing operator in space for practical implementations

2

Standard turbulent boundary layer closures

Turbulence closure problem (vertical eddy transport)

- Most operationally used closures employ the concept of the downgradient turbulent transport

$$\overline{w'u'} = -K_m \partial_z \overline{u}, \qquad \overline{w'v'} = -K_m \partial_z \overline{v}, \qquad \overline{w'\Theta'} = -K_h \partial_z \overline{\Theta}$$

- K_m and K_h are the unknowns to be determined from the turbulence closure theory.

$$K_m, K_h \propto u' \, l'$$

  - u' : typical magnitude of the eddy velocity
  - l' : typical length scale of the eddies (the mixing length)

▷ Use turbulent KE $E_K = \frac{1}{2}\left(\overline{u'u'} + \overline{v'v'} + \overline{w'w'}\right)$ to quantity the intensity of turbulence

$$u' \sim \sqrt{E_K}$$

▷ Geophysical approximations to derive an evolution equation for E_K

$$\begin{aligned} (\partial_z \mathbf{u}, \partial_z \Theta) &\gg (\partial_{x,y} \mathbf{u}, \partial_{x,y} \Theta) \qquad \text{(horizontal homogeneity)} \\ \partial_z(\overline{w'\mathbf{u}'}, \overline{w'\Theta'}) &\gg \overline{w}\partial_z(\mathbf{u}, \Theta) \end{aligned}$$

Moment-based approach

Subgrid-scale boundary-layer turbulence

$$\frac{1}{n}\left(\partial_t\overline{(\varphi')^n} + \overline{\mathbf{u}}_h\cdot\nabla\overline{(\varphi')^n} + \overline{w}\partial_z\overline{(\varphi')^n}\right) + \overline{\mathbf{u}'_h(\varphi')^{n-1}}\cdot\nabla\overline{\varphi} + \overline{w'(\varphi')^{n-1}}\partial_z\overline{\varphi}$$
$$= -\frac{1}{n}\left(\nabla\cdot\overline{\mathbf{u}'_h(\varphi')^n} + \partial_z\overline{w'(\varphi')^n}\right) + \overline{(\varphi')^{n-1}}\left(\nabla\cdot\overline{\mathbf{u}'_h\varphi'} + \partial_z\overline{w'\varphi'}\right) + \overline{(\varphi')^{n-1}F'_\varphi}$$

From geophysical approximations the following prognostic equation for E_K is obtained:

$$\partial_t E_K + \partial_z\underbrace{\left(\overline{w'E_K} + \frac{1}{\rho_0}\overline{w'p'} - \nu\nabla E_K\right)}_{-K_e\partial_z E_K} = \underbrace{-\overline{w'\mathbf{u}'_h}\cdot\partial_z\overline{\mathbf{u}}_h}_{\text{Shear production}} + \underbrace{\frac{g}{\rho_0}\overline{w'\rho'}}_{\text{buoyancy}} - \underbrace{\frac{\nu}{2}\|\nabla\mathbf{u}'\|^2}_{\text{dissipation } \varepsilon}$$

→ ε: ultimate dissipation of KE of all motions

- Kolmogorov (1941) : $\varepsilon = {E_K}^{3/2}/l'$
- Evolution equation for ε (k-ε closure scheme)

Alternative approach (KPP scheme, Large et al., 1994)

Subgrid-scale boundary-layer turbulence

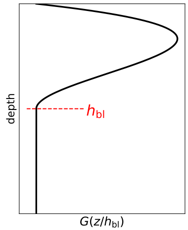
Parameterization formulated as a decision tree

$$K_{m,h}(z) = w_{m,h}\left(h_{\rm bl} G(z/h_{\rm bl})\right), \qquad w_{m,h} = \kappa\, u_\star\, \psi_{m,h}(z, u_\star, \partial_z \Theta)$$

- $h_{\rm bl}$: extent of the boundary layer determined from Richardson number

$$\mathrm{Ri} = \frac{(g/\rho_0)\partial_z \rho}{\|\partial_z \mathbf{u}_h\|^2}$$

- G : "universal" non-dimensional shape function
- $u_\star$: friction velocity determined from surface boundary condition
- $\psi_{m,h}$: "stability" functions



Surface boundary conditions for $\overline{w'\mathbf{u}'_h}$ and $\overline{w'\Theta'}$ terms

Monin-Obukhov similarity theory

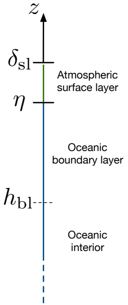
- Neutrally-stratified atmosphere $\rightarrow$ ”logarithmic boundary layer”
- During stratification conditions profiles significantly deviate from the logarithmic law

Monin-Obukhov theory : basis to derive turbulent quantities in the surface layer from the mean variables

$\rightarrow$ Generalization of the classical logarithmic boundary layer to stratified conditions

$$
\begin{aligned}
-\overline{w'\mathbf{u}'_h}_{\mathrm{sfc}} &= C_D \left\| \overline{\mathbf{u}^{\mathrm{a}}_h}(z=\delta_{\mathrm{sl}}) - \overline{\mathbf{u}_h}(z=\eta) \right\| \left(\overline{\mathbf{u}^{\mathrm{a}}_h}(z=\delta_{\mathrm{cls}}) - \overline{\mathbf{u}_h}(z=\eta) \right) \\
-\overline{w'\Theta'}_{\mathrm{sfc}} &= u_\star \Theta_\star = C_H \left\| \overline{\mathbf{u}^{\mathrm{a}}_h}(z=\delta_{\mathrm{sl}}) - \overline{\mathbf{u}_h}(z=\eta) \right\| \left(\overline{\Theta}^{\mathrm{a}}(z=\delta_{\mathrm{sl}}) - \overline{\Theta}(z=\eta) \right) \\
{u_\star}^2 &= \left\| \overline{w'\mathbf{u}'_h}_{\mathrm{sfc}} \right\|
\end{aligned}
$$

C_D and C_H are solution of a nonlinear system (bulk algorithm)



Examples of unsteady boundary layer models

External data: $\mathbf{u}_h^{\mathrm{a}}(z=\delta_{\mathrm{sl}},t)$

$$\begin{cases} \partial_t \mathbf{u}_h + f\mathbf{e}_z \times \mathbf{u}_h = \partial_z \left(K_m \partial_z \mathbf{u}_h \right) + \mathbf{F} & \text{for } t>0 \text{ and } H<z<0 \\ K_m \partial_z \mathbf{u}_h(0,t) = \dfrac{\rho_a}{\rho_o} C_D \| \mathbf{u}_h^{\mathrm{a}}(t) - \mathbf{u}_h(z=0,t) \| \left(\mathbf{u}_h^{\mathrm{a}}(t) - \mathbf{u}_h(z=0,t) \right) & \text{for } t>0 \\ \mathbf{u}_h(-H,t) = \mathbf{u}_G & \text{for } t>0 \\ \mathbf{u}_h(z,t=0) = \mathbf{u}_h^0(z) & \text{for } H<z<0 \end{cases}$$

KPP model (unstratified case)

$$\begin{aligned} K_m(z) &= \kappa u_\star h_{\mathrm{bl}} G(z/h_{\mathrm{bl}}) + \nu_m \\ h_{\mathrm{bl}} &= c\, u_\star / f \\ u_\star^2 &= \| K_m \partial_z \mathbf{u}_h(0,t) \| \\ h_{\mathrm{bl}} G(z/h_{\mathrm{bl}}) &= z \left(1 - \frac{z}{h_{\mathrm{bl}}} \right)^2 \times H \left(1 - \frac{z}{h_{\mathrm{bl}}} \right) \end{aligned}$$

Examples of unsteady boundary layer models

External data: $\mathbf{u}_h^{\mathrm{a}}(z = \delta_{\mathrm{sl}}, t)$

$$\begin{cases} \partial_t \mathbf{u}_h + f\mathbf{e}_z \times \mathbf{u}_h = \partial_z \left(K_m \partial_z \mathbf{u}_h \right) + \mathbf{F} & \text{for } t > 0 \textit{ and } H < z < 0 \\ K_m \partial_z \mathbf{u}_h(0,t) = \dfrac{\rho_a}{\rho_o} C_D \| \mathbf{u}_h^{\mathrm{a}}(t) - \mathbf{u}_h(z=0,t) \| \left(\mathbf{u}_h^{\mathrm{a}}(t) - \mathbf{u}_h(z=0,t) \right) & \text{for } t > 0 \\ \mathbf{u}_h(-H,t) = \mathbf{u}_G & \text{for } t > 0 \\ \mathbf{u}_h(z, t=0) = \mathbf{u}_h^0(z) & \text{for } H < z < 0 \end{cases}$$

TKE model (unstratified case) + appropriate mixing length model to specify $l(z)$

$$\begin{aligned} K_m(z) &= c_m l \sqrt{E_K} + \nu_m \\ \partial_t E_K &= K_m \| \partial_z \mathbf{u}_h \|^2 + \partial_z \left(c_e K_m \partial_z E_K \right) - \frac{c_\epsilon E_K^{3/2}}{l} \\ E_K(z=0,t) &= \frac{u_\star^2}{\sqrt{c_m c_\varepsilon}} \\ E_K(z,t=0) &= E_K^0(z) \end{aligned}$$

Examples of unsteady boundary layer models

External data: $\mathbf{u}_h^{\mathrm{a}}(z=\delta_{\mathrm{sl}},t)$

$$
\begin{cases}
\partial_t \mathbf{u}_h + f\mathbf{e}_z \times \mathbf{u}_h = \partial_z\left({\color{red}K_m} \partial_z \mathbf{u}_h \right) + \mathbf{F} & \text{for } t>0 \; and \; H<z<0 \\
K_m \partial_z \mathbf{u}_h(0,t) = \dfrac{\rho_a}{\rho_o} C_D \| \mathbf{u}_h^{\mathrm{a}}(t) - \mathbf{u}_h(z=0,t) \| \left(\mathbf{u}_h^{\mathrm{a}}(t) - \mathbf{u}_h(z=0,t) \right) & \text{for } t>0 \\
\mathbf{u}_h(-H,t) = \mathbf{u}_G & \text{for } t>0 \\
\mathbf{u}_h(z,t=0) = \mathbf{u}_h^0(z) & \text{for } H<z<0
\end{cases}
$$

Stationary TKE model (unstratified case) + appropriate mixing length model to specify $l(z)$

$$
{\color{red}K_m}(z) = \beta\, l^2(z) \| \partial_z \mathbf{u}_h \| + \nu_m, \qquad \beta = c_m \sqrt{\frac{c_m}{c_\epsilon}}
$$

Mathematical stability of closure models

- **An example :** analogy with a local Ri-dependent model

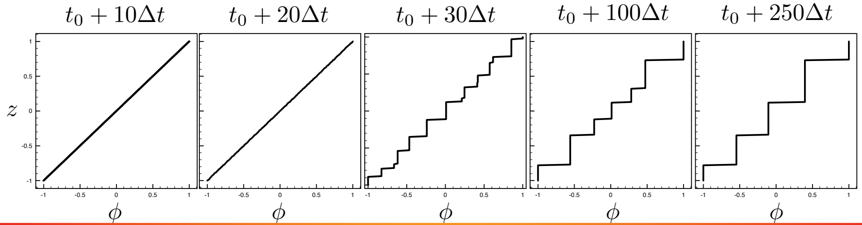
$$\partial_t \phi = \partial_z \left(K(z) \partial_z \phi \right), \qquad K(z) = (\partial_z \phi)^{-2}$$

  - ▷ $K(z) > 0 \;\rightarrow\; \phi$ remains bounded
  - ▷ Original equation can be reexpressed as

$$\partial_t \left(\partial_z \phi \right) = \partial_z \left(\widetilde{K}(z) \partial_z \left(\partial_z \phi \right) \right), \qquad \widetilde{K}(z) = -(\partial_z \phi)^{-2}$$

$\rightarrow$ the gradient can grow unbounded

- **Numerical test :** $\phi(z, t = 0) = z$, $\phi(z = -1, t) = -1$, $\phi(z = 1, t) = 1$



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- Ill-behaved solution due to the continuous formulation of the closure model and not to the details of its numerical discretisation

  $\rightarrow$ 0-equation closures are hard to study since it can change the diffusive nature of the equation

- More generally, spurious oscillations generally noticed are of a mathematical or a numerical nature ?

Energetic consistency – mixing terms vs turbulent closure

$$
\begin{array}{rcl} \partial_t u - \partial_z (K_m \partial_z u) &=& 0 \\ \partial_t b - \partial_z (K_s \partial_z b) &=& 0 \end{array}
\rightarrow
\begin{array}{rclcl} \partial_t \mathrm{KE} - \partial_z (K_m \partial_z \mathrm{KE}) &=& -K_m (\partial_z u)^2 &=& -P \\ \partial_t \mathrm{PE} - \partial_z ((-z) K_s \partial_z b) &=& K_s \quad \partial_z b &=& -B \end{array}
$$

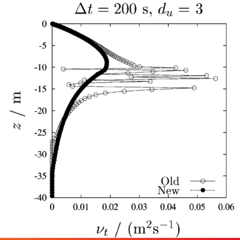
$$\partial_t \mathrm{TKE} - \partial_z (K_e \partial_z \mathrm{TKE}) = P + B - \varepsilon$$

Energy budget in a water column (ignoring the contribution of B.C.) :

$$E = \int_{z_{\mathrm{bot}}}^{z_{\mathrm{top}}} (\mathrm{KE} + \mathrm{PE} + \mathrm{TKE}) dz \quad \rightarrow \quad \partial_t E = -\int_{z_{\mathrm{bot}}}^{z_{\mathrm{top}}} \varepsilon dz$$

- The discrete counterpart of it tells you unambiguously how to discretize forcing terms in the TKE equation

Burchard H. (2002). *Energy-conserving discretisation of turbulent shear and buoyancy production*. Ocean Modelling



3

Coupling two turbulent Ekman boundary layers

Classical Ekman boundary layer equations

Ekman boundary layer theory's three-term balance equation for horizontal momentum

$$f\mathbf{e}_z \times (\mathbf{u}_h - \mathbf{u}_G) + \partial_z \overline{w'\mathbf{u}_h'} = 0$$

See Klein (2004) for a rigorous multiple scales derivation from the 3D compressible equations

Well posedness with KPP viscosities

z_∞^{a} $\quad \mathbf{u}_h^{\mathrm{a}} = \mathbf{u}_G^{\mathrm{a}}$

$$f\mathbf{e}_z \times (\mathbf{u}_h^{\mathrm{a}} - \mathbf{u}_G^{\mathrm{a}}) = \partial_z \left(K_m^{\mathrm{a}}(z, u_\star) \partial_z \mathbf{u}_h^{\mathrm{a}} \right)$$

$$K_m^{\mathrm{a}} \partial_z \mathbf{u}_h^{\mathrm{a}}(\delta^{\mathrm{a}}) = u_\star^2 \frac{\delta \mathbf{u}}{\|\delta \mathbf{u}\|}$$

δ^{a}

$$u_\star^2 = C_D \|\mathbf{u}_h^{\mathrm{a}}(\delta^{\mathrm{a}}) - \mathbf{u}_h^{\mathrm{o}}(\delta^{\mathrm{o}})\|^2 = C_D \|\delta \mathbf{u}\|^2$$

δ^{o}

$$K_m^{\mathrm{o}} \partial_z \mathbf{u}_h^{\mathrm{o}}(\delta^{\mathrm{o}}) = \frac{\rho^{\mathrm{a}}}{\rho^{\mathrm{o}}} u_\star^2 \frac{\delta \mathbf{u}}{\|\delta \mathbf{u}\|}$$

$$f\mathbf{e}_z \times (\mathbf{u}_h^{\mathrm{o}} - \mathbf{u}_G^{\mathrm{o}}) = \partial_z \left(K_m^{\mathrm{o}}(z, u_\star) \partial_z \mathbf{u}_h^{\mathrm{o}} \right)$$

z_∞^{o} $\quad \mathbf{u}_h^{\mathrm{o}} = \mathbf{u}_G^{\mathrm{o}}$

Under the assumption that $\delta_\alpha \leq h_{\mathrm{bl}}^\alpha \leq z_\infty^\alpha, (\alpha = \mathrm{a}, \mathrm{o})$ there is a unique solution to the problem (Thery, 2021, PhD; in coll. with V. Martin)

4

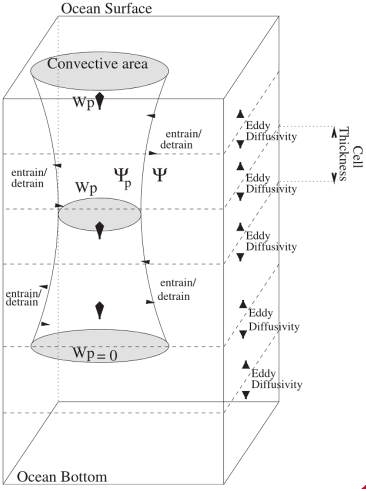
Multi-fluid representation of oceanic convection

Work undertaken as part of the AAP ”Changement climatique, Défis environnementaux et Mathématiques”

Beyond the downgradient flux assumption

$$\overline{w'\Theta'} = \underbrace{\overline{w'\Theta'}_{\rm local}}_{\text{local turbulence}} + \underbrace{\overline{w'\Theta'}_{\rm non-local}}_{\text{coherent convection}}$$

In convective conditions, turbulent fluxes are dominated by processes unrelated to local gradients (strong fluxes exist even when $\partial_z \overline{\Theta} = 0$)



Usual approaches :

- Increase vertical diffusion to 'mix' unstable density profiles
- Mass flux convection scheme

$$\overline{w'\Theta'}_{\rm non-local} = M(\Theta_p - \overline{\Theta}), \qquad M = \underbrace{a_p}_{\text{area fraction}} \underbrace{w_p}_{\text{plume velocity}}$$

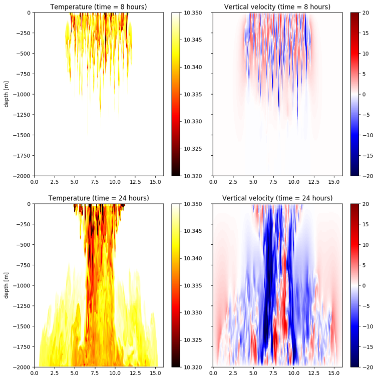
Mass-flux scheme = ODEs on a_p, w_p, Θ_p controlled by entrainment/detrainment rates (e.g. Giordani et al., 2020)

Limitations of typical mass flux convection schemes

- The plume equations are often not consistent with the equations used in the dynamical core
- Based on the assumption that $a_p \ll 1$, (what to do when $\Delta x \sim L_{\mathrm{conv}}$?)
- Stationarity and horizontal homogeneity (no dynamical memory, no horizontal propagation)

Objectives:

- Re-derive rigorously mass flux scheme from a multi-fluid approach (Yano 2014; Thuburn et al., 2018)
- Apply the modelling under location uncertainty approach to represent uncertainties
- Use LES simulations (Croco ocean model) to ”tune” the parameterization



Conclusion

Some basic principles to approach to the parameterization problem in a consistent manner

- Energetic considerations
- Convergence in time and space (scale-awareness)
- Compatibility with the laws of physics
- The whole problem must be well-posed

Remarks

- ▷ Moment-based approach is extensively used to represent local turbulence
- ▷ Representing intermittent coherent structures is an important challenge for ocean climate models
- ▷ Much needed efforts for ”deconstructing” a particular parameterization formulation to list the approximations made during its design
- ▷ Close link between subgrid-scale parameterization and numerical algorithm (physics/dynamics coupling)
- ▷ Going beyond the Reynolds decomposition ? Alternative clear guiding principle to pursue ?

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