Finite Developments in the λ-calculus

Part 1

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http://jeanjacqueslevy.net/talks/21isr

λ-calculus

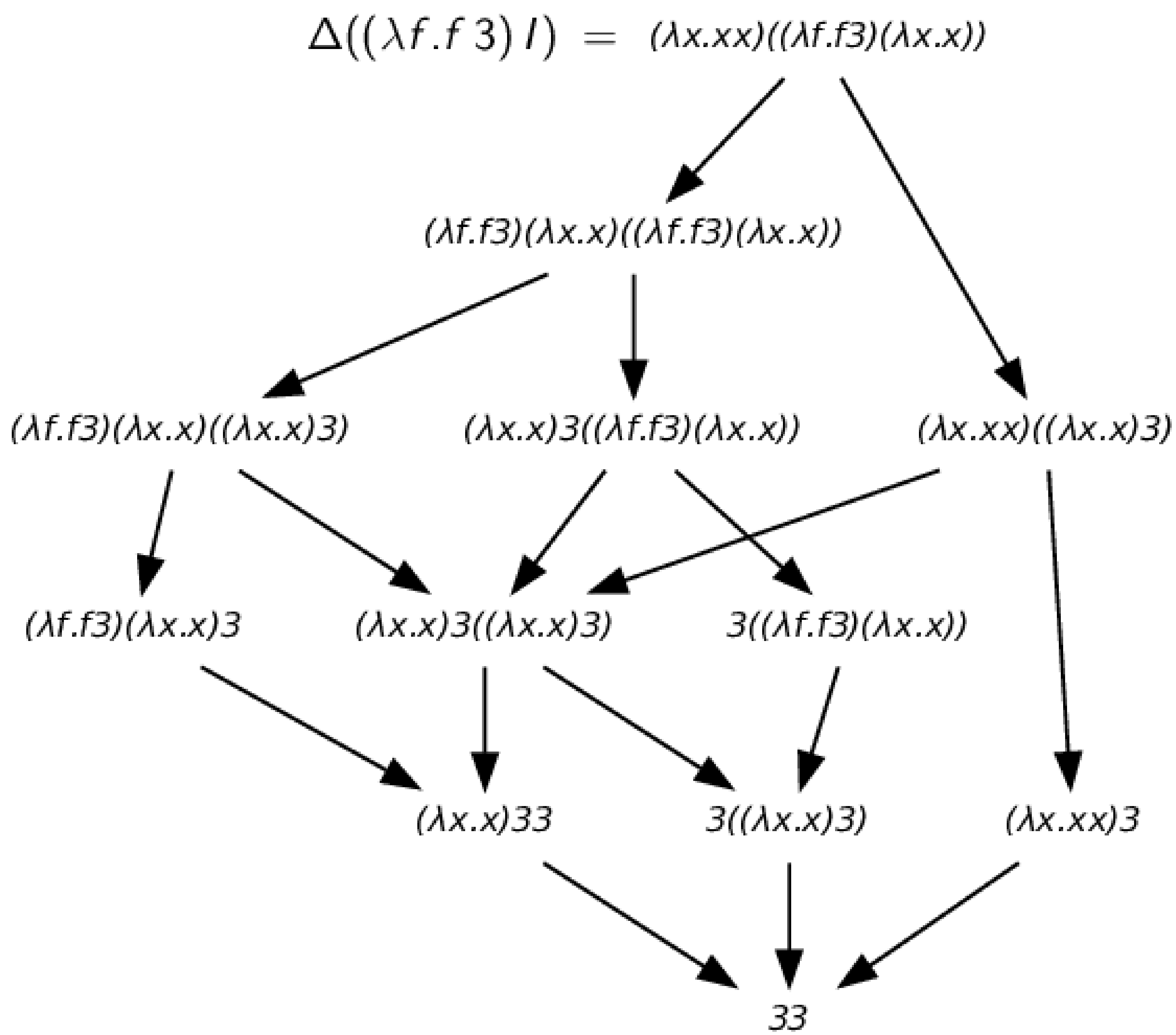
function	λ-term	β-reduction
$I\,x = x$	$I = \lambda x.x$	$I\,a \longrightarrow a$
$K\,x\,y = x$	$K = \lambda x.\lambda y.x$	$K\,a\,b \longrightarrow (\lambda y.a)\,b \longrightarrow a$
$\Delta\,x = x\,x$	$\Delta = \lambda x.x\,x$	$\Delta\,a \longrightarrow a\,a$
$\Omega = \Delta\,\Delta$		$\Omega \longrightarrow \Omega$

Exercise 1

$$\Delta(\lambda x.x\,x\,x) \longrightarrow \cdots$$

$$Y_f = (\lambda x.f\,(x\,x))(\lambda x.f\,(x\,x)) \longrightarrow \cdots$$

λ-calculus

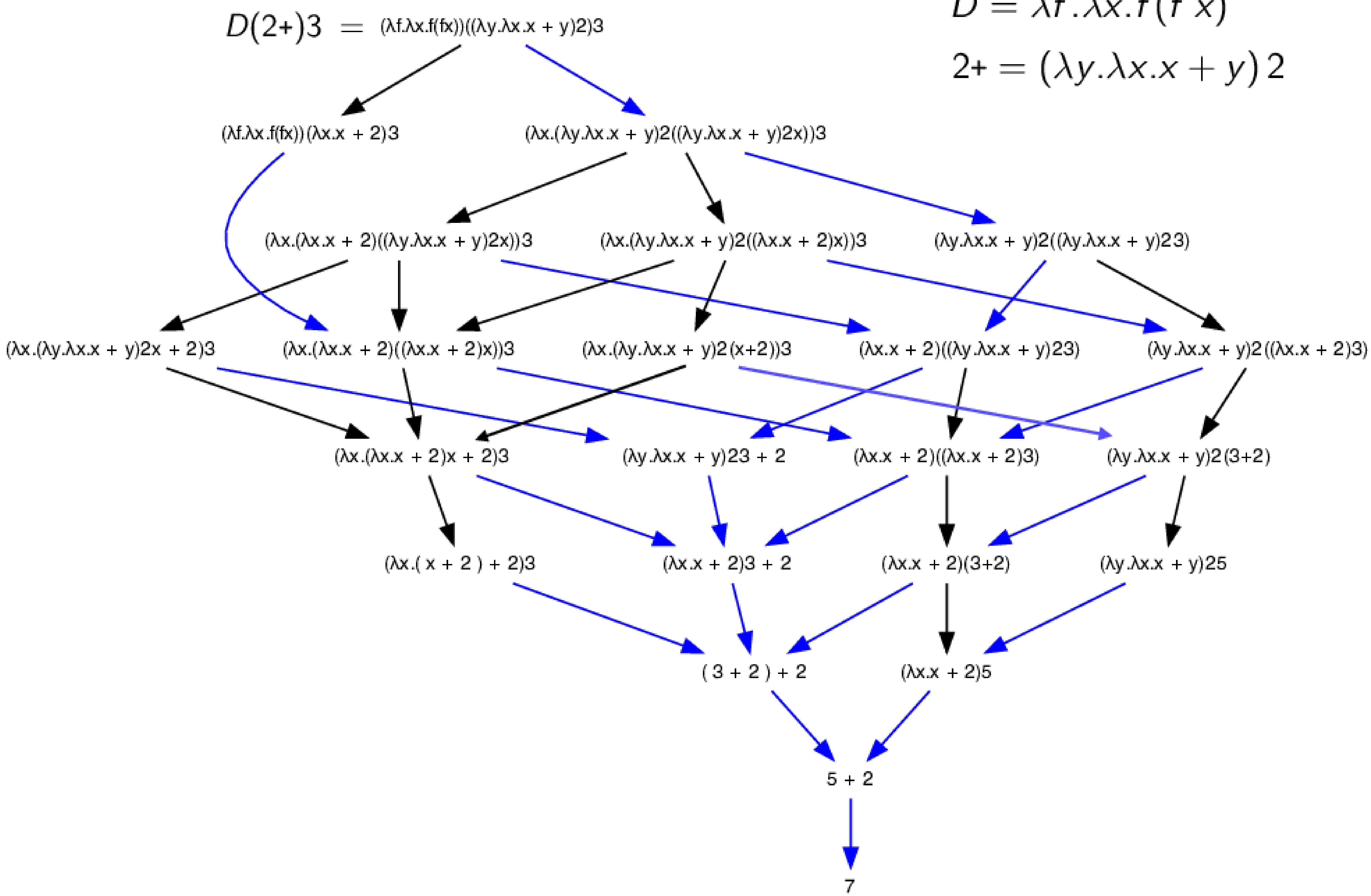


Δ((λf.f 3) I) = (λx.xx)((λf.f3)(λx.x))
(λf.f3)(λx.x)((λf.f3)(λx.x))
(λf.f3)(λx.x)((λx.x)3)
(λx.x)3((λf.f3)(λx.x))
(λx.xx)((λx.x)3)
(λf.f3)(λx.x)3
(λx.x)3((λx.x)3)
3((λf.f3)(λx.x))
(λx.x)33
3((λx.x)3)
(λx.xx)3
33

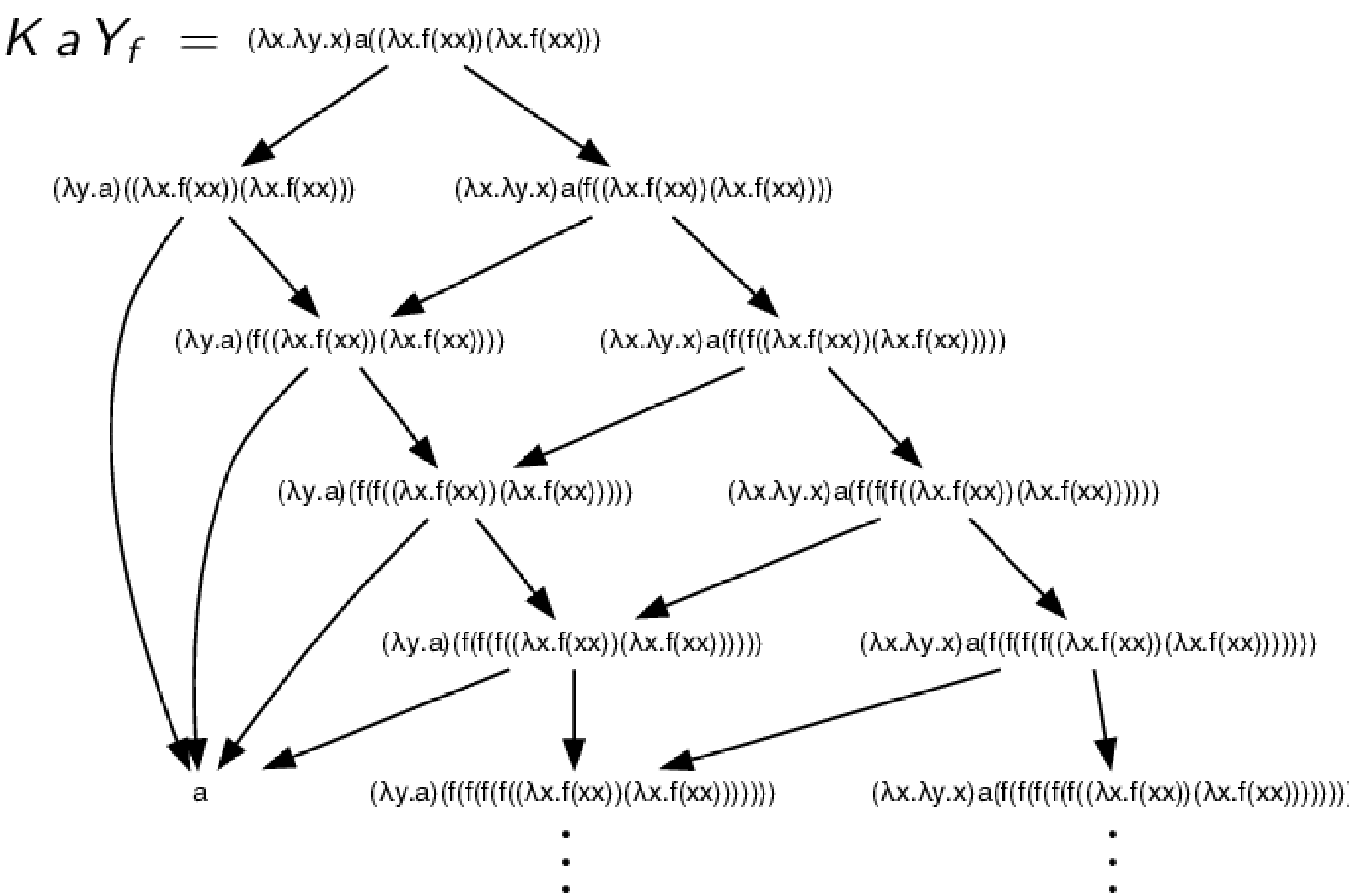
λ-calculus

$D = \lambda f.\lambda x.f(f\,x)$

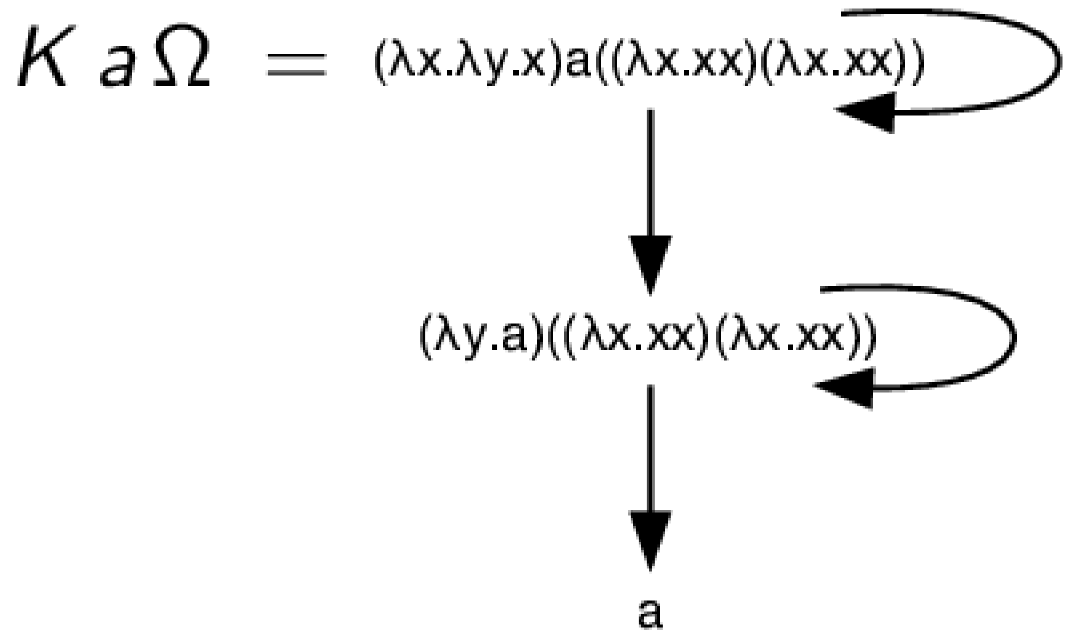
$2+ = (\lambda y.\lambda x.x + y)\,2$



λ-calculus


K a Y_f =
(λx.λy.x)a((λx.f(xx))(λx.f(xx)))
(λy.a)((λx.f(xx))(λx.f(xx)))
(λx.λy.x)a(f((λx.f(xx))(λx.f(xx))))
(λy.a)(f((λx.f(xx))(λx.f(xx))))
(λx.λy.x)a(f(f((λx.f(xx))(λx.f(xx)))))
(λy.a)(f(f((λx.f(xx))(λx.f(xx)))))
(λx.λy.x)a(f(f(f((λx.f(xx))(λx.f(xx))))))
(λy.a)(f(f(f((λx.f(xx))(λx.f(xx))))))
(λx.λy.x)a(f(f(f(f((λx.f(xx))(λx.f(xx)))))))
a
(λy.a)(f(f(f(f((λx.f(xx))(λx.f(xx)))))))
(λx.λy.x)a(f(f(f(f(f((λx.f(xx))(λx.f(xx))))))))

λ-calculus



Empirical facts

- **deterministic** result when it exists — **Church-Rosser**
- multiple reduction strategies — **CBN - CBV - ..**
- **terminating** strategy ? — **normalisation**
- **efficient** reduction strategy ? — **optimal reduction**
- **worst** reduction strategy ? — **perpetual reduction**
- when all reductions are finite ? — **strong normalisation**
- the reduction graph has a **lattice** structure ? — **NO!**

Redexes

- a redex is any reductible expression: $(\lambda x.M)N$
- the β-conversion rule is:

$$(\lambda x.M)N \longrightarrow M\{x := N\}$$

- a reduction step contracts a given redex $R = (\lambda x.A)B$

and is written: $M \xrightarrow{R} N$

- a reduction step contracts a singleton set of redexes $M \xrightarrow{\{R\}} N$

- a more precise notation would be with occurences of subterms. We avoid it here (but it is sometimes mandatory to avoid ambiguity)
- we replaced occurences by giving names (labels) to redexes.

Bound variables

$$(\lambda x.x\,(\lambda y.x\,y))y = (\lambda x.x\,(\lambda z.x\,z))y$$

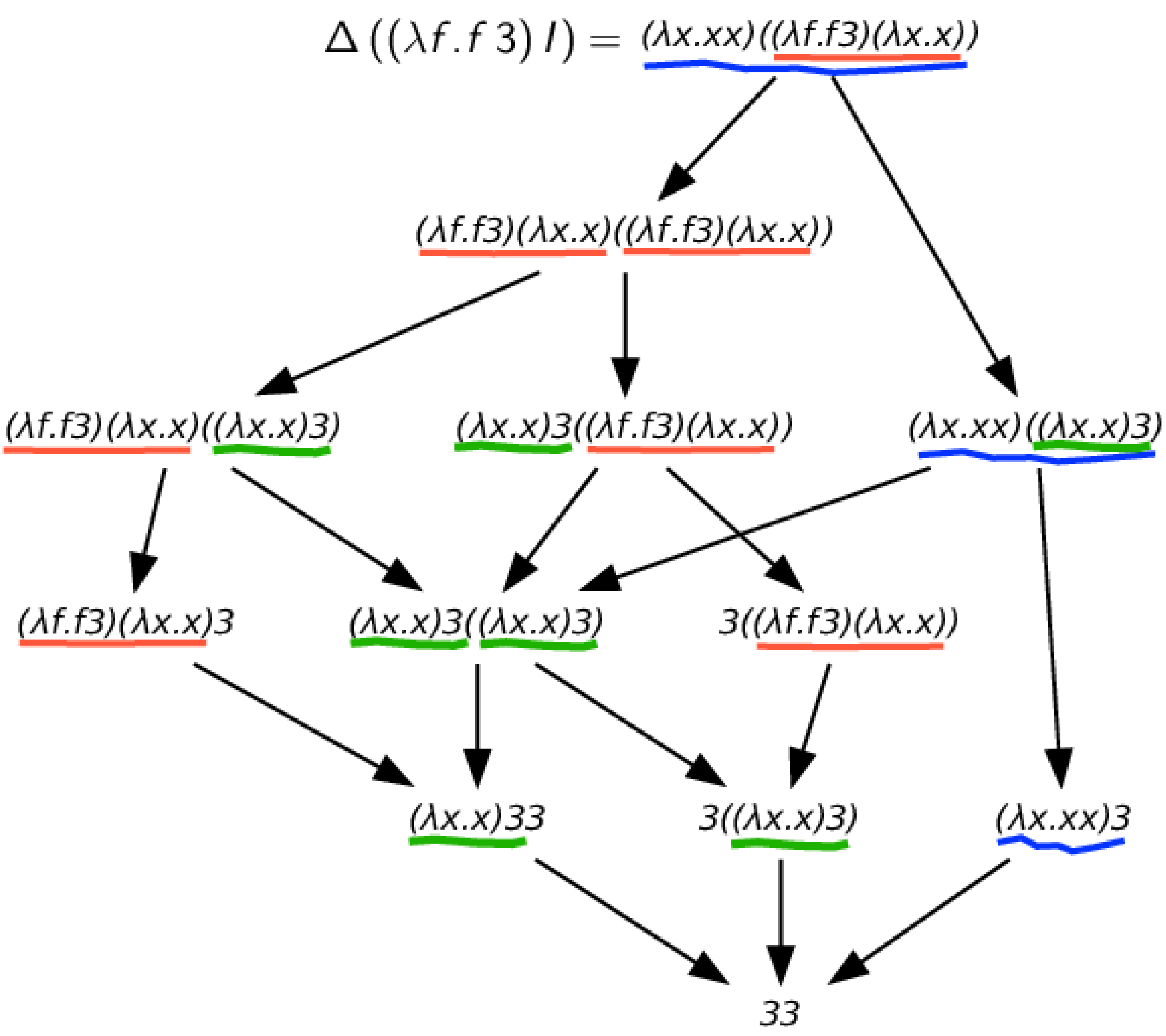
$$y\,(\lambda z.y\,z)$$

- names of bound variables are not important
- we consider λ-terms up-to renaming of bound variables (**α-conversion**)
- free variables of M are formally defined by:

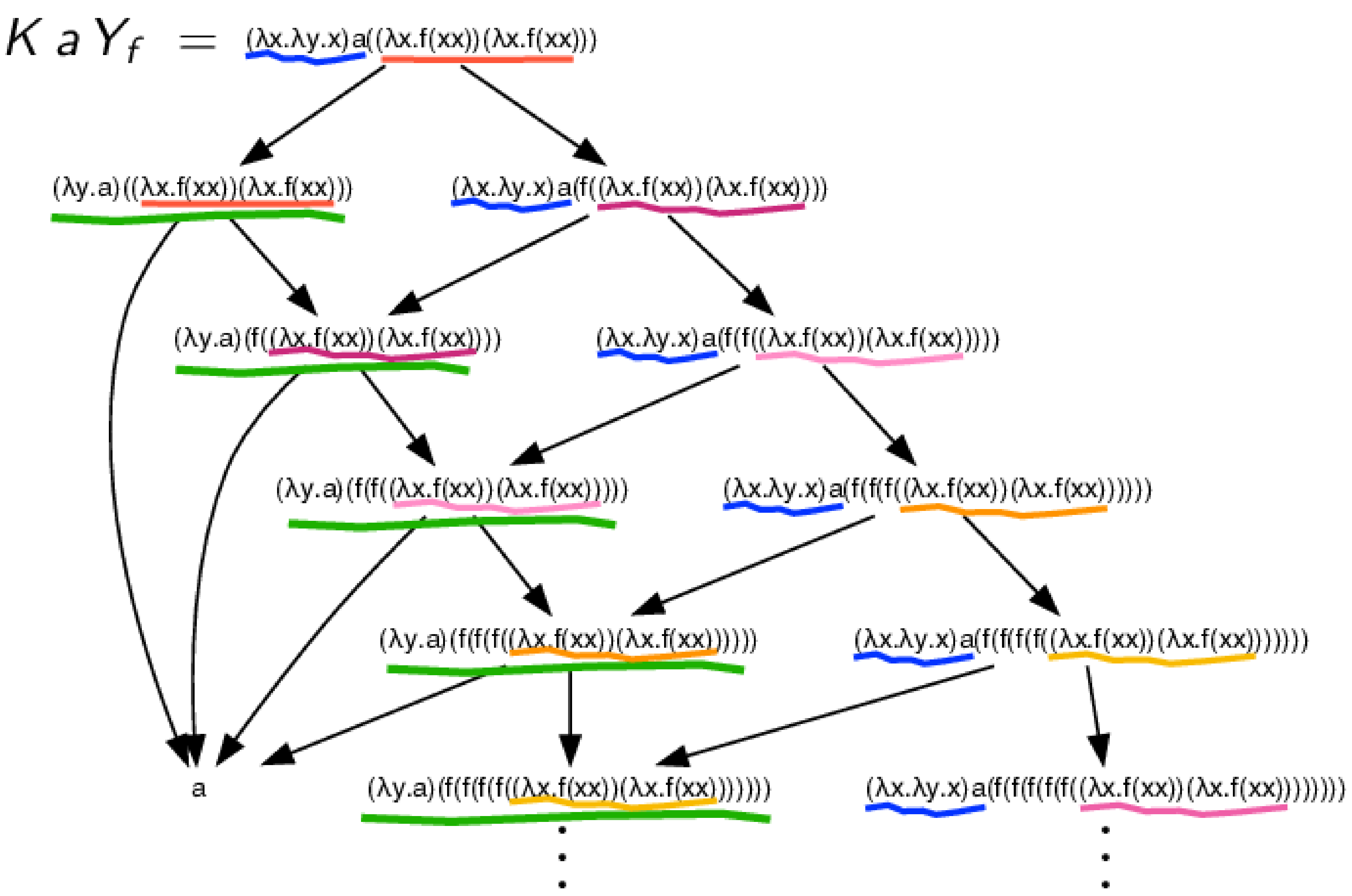
$$\mathsf{FV}(x) = \{x\}$$
$$\mathsf{FV}(\lambda x.M) = \mathsf{FV}(M) - \{x\}$$
$$\mathsf{FV}(MN) = \mathsf{FV}(M) \cup \mathsf{FV}(N)$$

forget **α**-conversion

Tracing redexes



Tracing redexes



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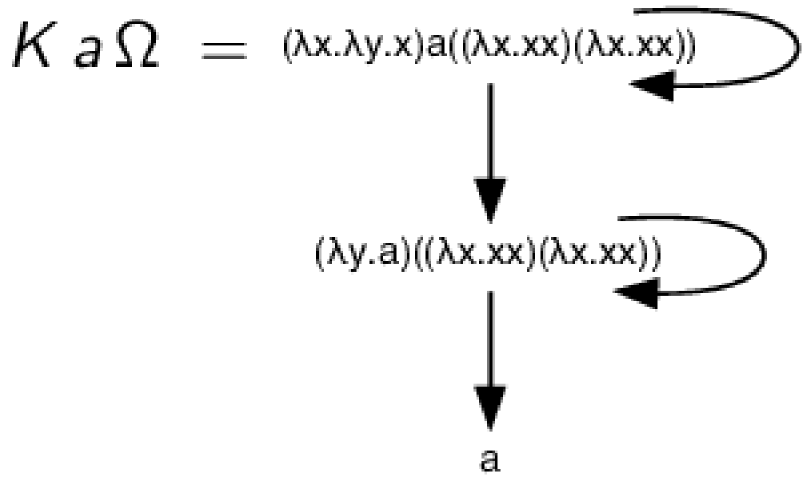
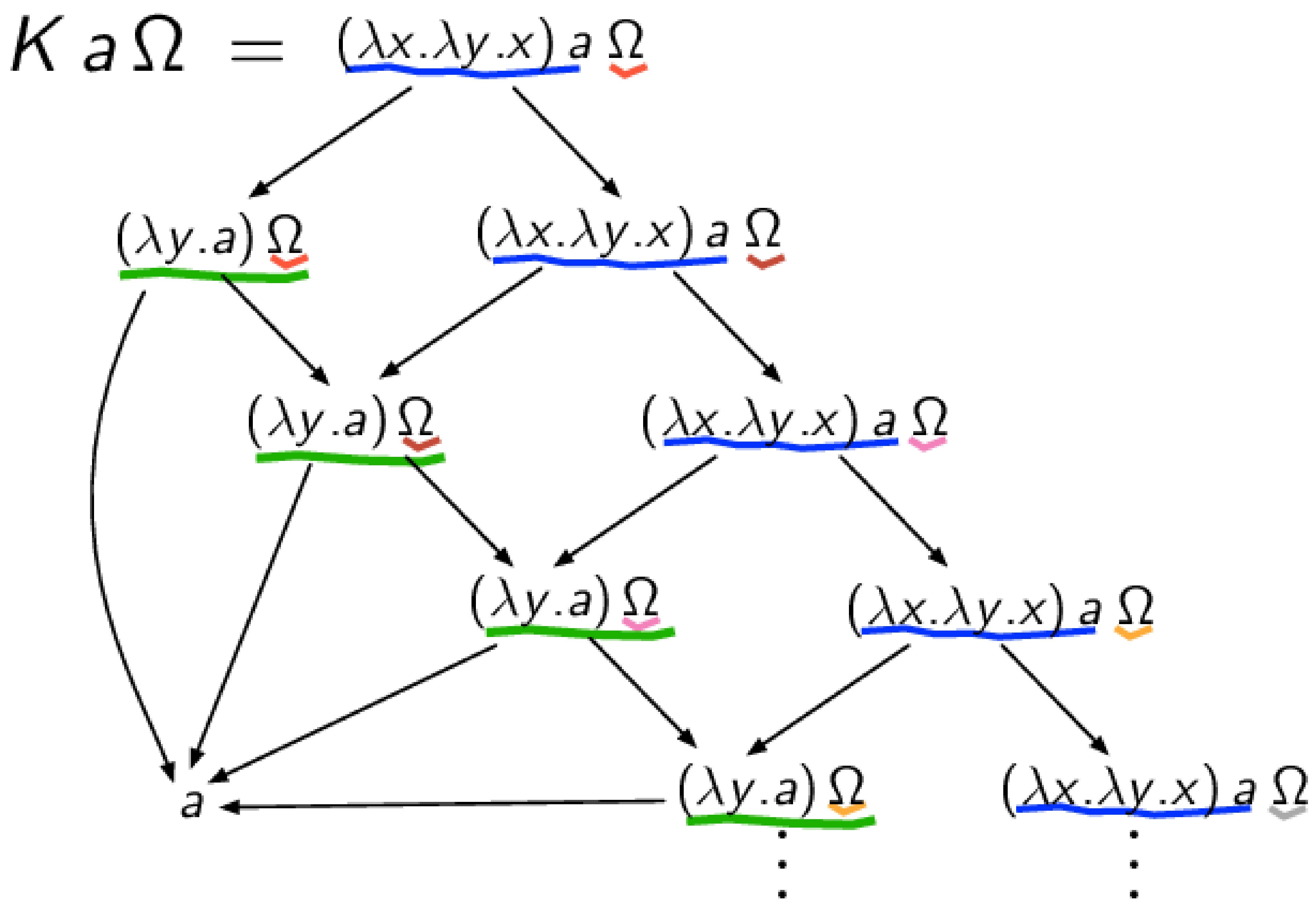
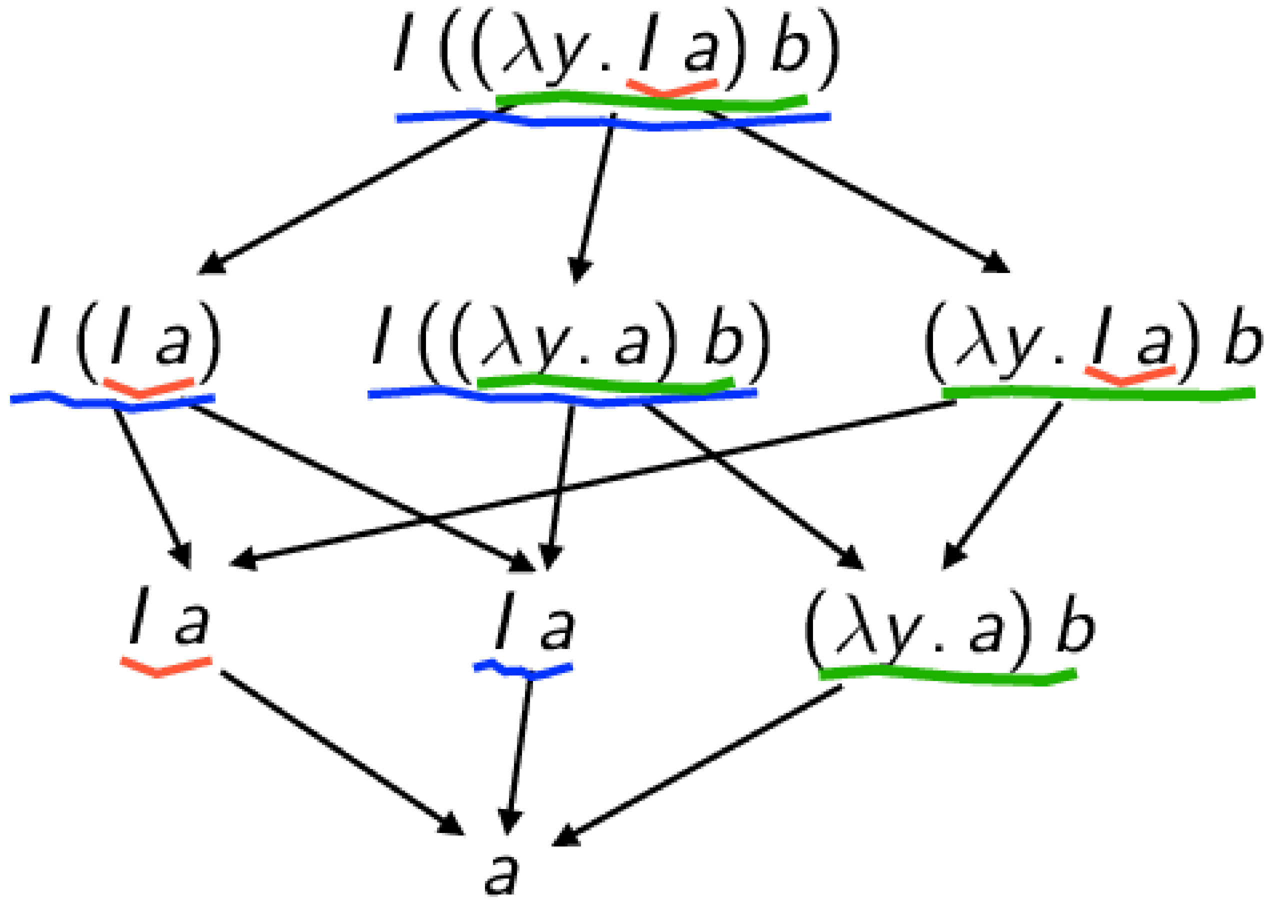
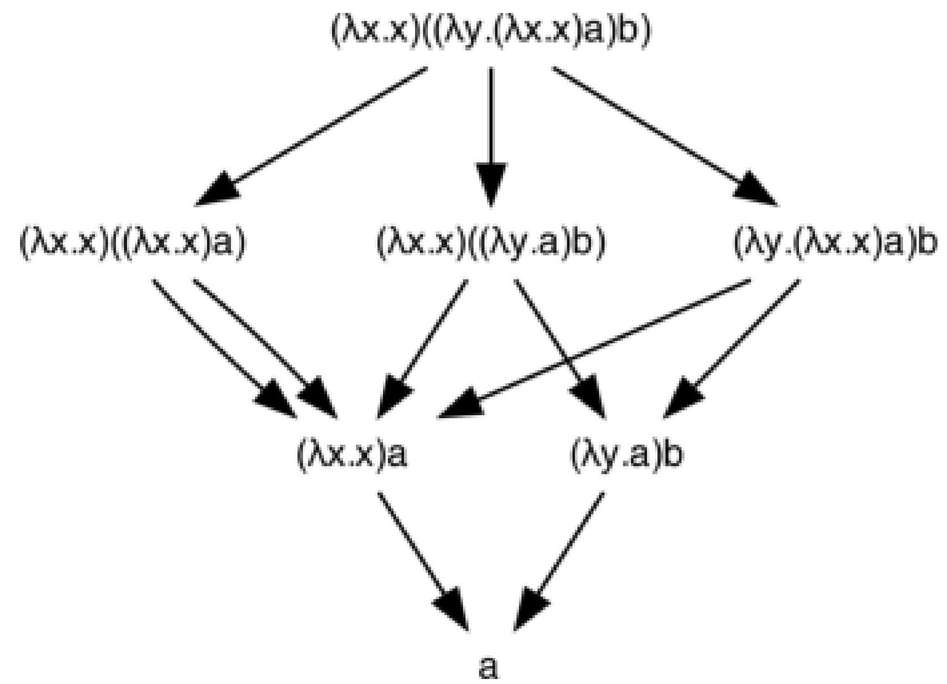




Tracing redexes





Empirical facts

- initial redexes in the initial term
- and **newly** created redexes along reductions
- **infinite** reduction iff length of creation is unbounded ?
- **deterministic** result when finite families of redexes are contracted ?

↓

Finite Developments Theorem

`Curry'50` `JJL'78`

Parallel reduction steps

Parallel reductions (1/3)

- permutation of reductions has to cope with copies of redexes

$$\begin{array}{ccc}
(\lambda x.xx)(Ia) & \longrightarrow & Ia\,(Ia) \\
\searrow & & \searrow^{\star} \\
& (\lambda x.xx)a \longrightarrow & a\,a
\end{array}$$

- in fact, a parallel reduction $Ia(Ia) \not\Longrightarrow aa$

- in λ-calculus, need to define parallel reductions for nested sets

Fact In the λ-calculus, disjoint redexes may become nested $(\lambda x.Ix)(\Delta y) \longrightarrow I(\Delta y)$

Parallel reductions (2/3)

- the axiomatic way (à la Martin-Löf)

[Var Axiom] $x \mathrel{+\!\!\!+\!\!\!\!\rightarrow} x$ [Const Axiom] $c \mathrel{+\!\!\!+\!\!\!\!\rightarrow} c$

[App Rule] $\dfrac{M \mathrel{+\!\!\!+\!\!\!\!\rightarrow} M' \quad N \mathrel{+\!\!\!+\!\!\!\!\rightarrow} N'}{MN \mathrel{+\!\!\!+\!\!\!\!\rightarrow} M'N'}$ [Abs Rule] $\dfrac{M \mathrel{+\!\!\!+\!\!\!\!\rightarrow} M'}{\lambda x.M \mathrel{+\!\!\!+\!\!\!\!\rightarrow} \lambda x.M'}$

[//Beta Rule] $\dfrac{M \mathrel{+\!\!\!+\!\!\!\!\rightarrow} M' \quad N \mathrel{+\!\!\!+\!\!\!\!\rightarrow} N'}{(\lambda x.M)N \mathrel{+\!\!\!+\!\!\!\!\rightarrow} M'\{x := N'\}}$

inside-out (possibly void) parallel reductions

- examples:

$(\lambda x.Ix)(Iy) \mathrel{+\!\!\!+\!\!\!\!\rightarrow} (\lambda x.x)y$

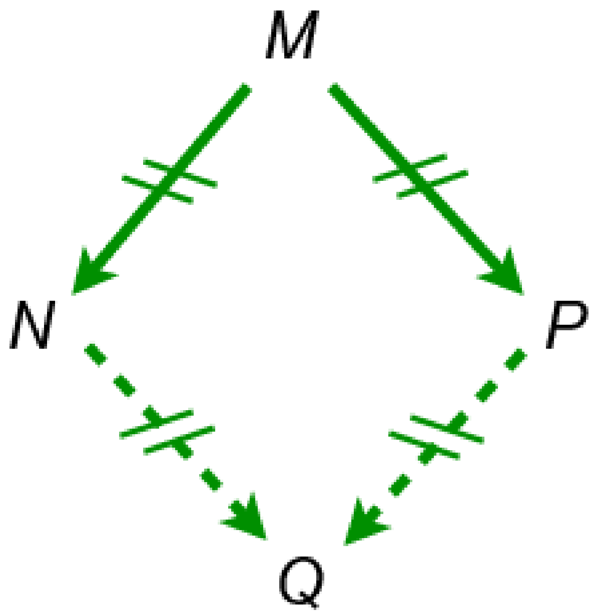
$(\lambda x.(\lambda y.yy)x)(Ia) \mathrel{+\!\!\!+\!\!\!\!\rightarrow} Ia(Ia)$

$(\lambda x.(\lambda y.yy)x)(Ia) \mathrel{+\!\!\!+\!\!\!\!\rightarrow} (\lambda y.yy)a$

Parallel reductions (3/3)

- **Parallel moves lemma** [Curry 50]

  If $M \twoheadrightarrow\!\!\!\!\!/\!\!/\ N$ and $M \twoheadrightarrow\!\!\!\!\!/\!\!/\ P$, then $N \twoheadrightarrow\!\!\!\!\!/\!\!/\ Q$ and $P \twoheadrightarrow\!\!\!\!\!/\!\!/\ Q$ for some Q.



lemma 1-1-1-1
(strong confluency)

Enough to prove Church Rosser theorem since $\longrightarrow \;\subset\; \twoheadrightarrow\!\!\!\!\!/\!\!/ \;\subset\; \xrightarrow{\star}$

[Tait--Martin Löf 60?]

Reduction of a set of redexes (1/4)

- Goal: parallel reduction of a given set of redexes

$$M, N ::= x \mid \lambda x.M \mid MN \mid (\lambda x.M)^a N$$

$$a, b, c, \ldots ::= \text{redex labels}$$

(labeled β-rule)

$$(\lambda x.M)^a N \longrightarrow M\{x := N\}$$

- Substitution as before with add-on:

$$((\lambda y.P)^a Q)\{x := N\} = (\lambda y.P\{x := N\})^a Q\{x := N\}$$

Reduction of a set of redexes (2/4)

- let $\mathcal{F}$ be a set of redex labels

[Var Axiom] $x \xrightarrow{\mathcal{F}} x$

[Const Axiom] $c \xrightarrow{\mathcal{F}} c$

[App Rule] $\dfrac{M \xrightarrow{\mathcal{F}} M' \quad N \xrightarrow{\mathcal{F}} N'}{MN \xrightarrow{\mathcal{F}} M'N'}$

[Abs Rule] $\dfrac{M \xrightarrow{\mathcal{F}} M'}{\lambda x.M \xrightarrow{\mathcal{F}} \lambda x.M'}$

[//Beta Rule] $\dfrac{M \xrightarrow{\mathcal{F}} M' \quad N \xrightarrow{\mathcal{F}} N' \quad a \in \mathcal{F}}{(\lambda x.M)^a N \xrightarrow{\mathcal{F}} M'\{x := N'\}}$

[Redex'] $\dfrac{M \xrightarrow{\mathcal{F}} M' \quad N \xrightarrow{\mathcal{F}} N' \quad a \notin \mathcal{F}}{(\lambda x.M)^a N \xrightarrow{\mathcal{F}} (\lambda x.M')^a N'}$

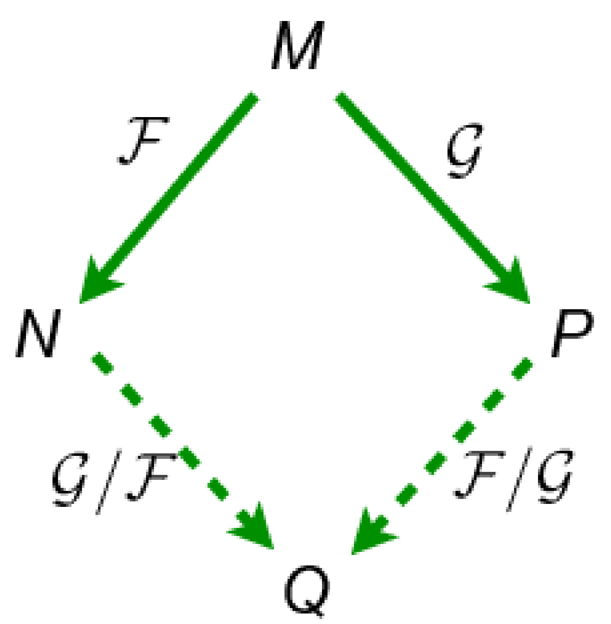
inside-out parallel reductions of redexes labeled in $\mathcal{F}$

- let $\mathcal{F}$, $\mathcal{G}$ be set of redexes in M and let $M \xrightarrow{\mathcal{F}} N$, then the set $\mathcal{G}/\mathcal{F}$ of residuals of $\mathcal{G}$ by $\mathcal{F}$ is the set of $\mathcal{G}$ redexes in N.

Reduction of a set of redexes (3/4)

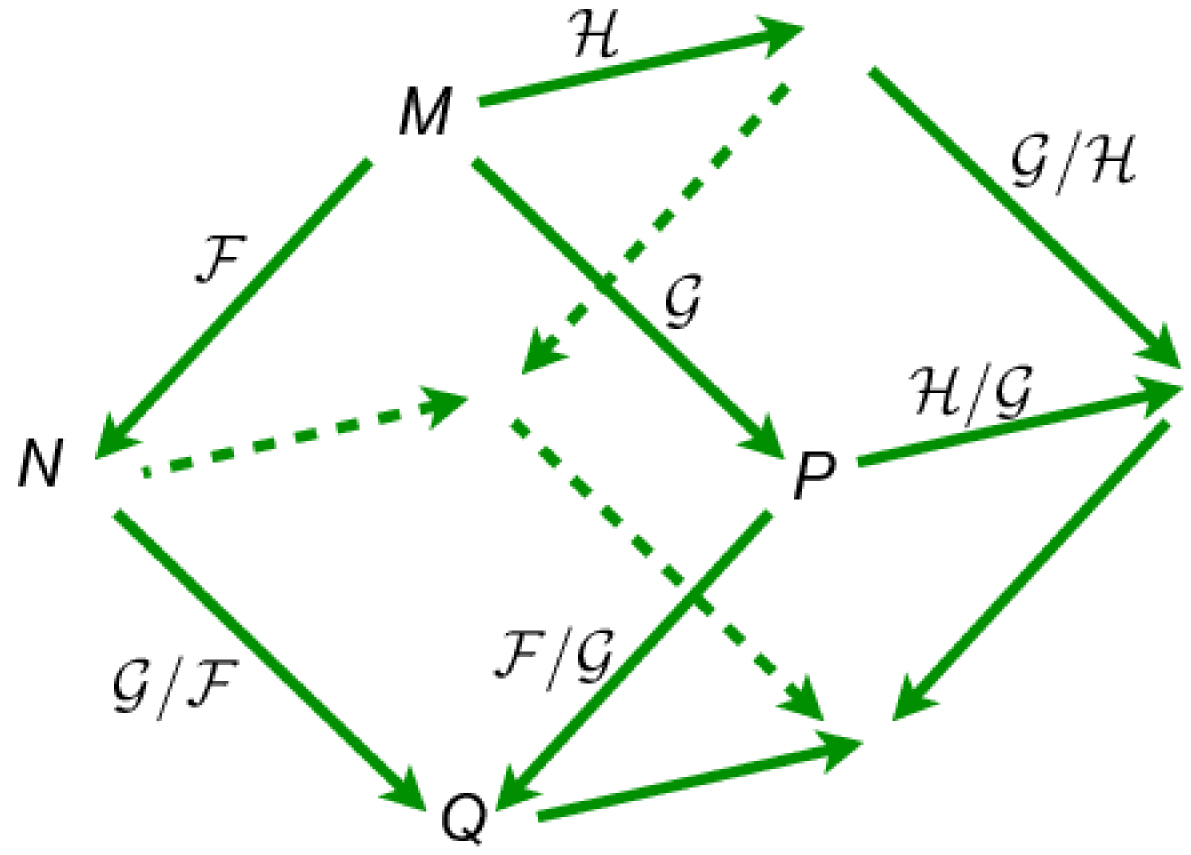
- **Parallel moves lemma+** [Curry 50]

  If $M \xrightarrow{\mathcal{F}} N$ and $M \xrightarrow{\mathcal{G}} P$, then $N \xrightarrow{\mathcal{G}/\mathcal{F}} Q$ and $P \xrightarrow{\mathcal{F}/\mathcal{G}} Q$ for some Q.



Reduction of a set of redexes (4/4)

- **Parallel moves lemma++** [Curry 50] The Cube Lemma



$$(\mathcal{H}/\mathcal{F})/(\mathcal{G}/\mathcal{F}) = (\mathcal{H}/\mathcal{G})/(\mathcal{F}/\mathcal{G})$$

Residuals of redexes

Redexes

- a redex is any reductible expression: $(\lambda x.M)N$

- a reduction step contracts a given redex $R = (\lambda x.A)B$

and is written: $M \xrightarrow{R} N$

- a reduction step contracts a singleton set of redexes $M \xrightarrow{\{R\}} N$

- a more precise notation would be with occurences of subterms. We avoid it here (but it is sometimes mandatory to avoid ambiguity)

- we replaced occurences by giving names (labels) to redexes.

Residuals of redexes (1/4)

- **residuals** of redexes were defined by considering **labels**

- residuals are redexes with **same labels**

- a closer look w.r.t. their relative positions give following cases:

let $R = (\lambda x.A)B$, let $M \xrightarrow{R} N$ and $S = (\lambda y.C)D$ be an other redex in M. Then:

Residuals of redexes (2/4)

Case 1:

$$M = \cdots\cdots R \cdots\cdots \underset{\smile}{S} \cdots\cdots \xrightarrow{R} \cdots\cdots R' \cdots\cdots \underset{\smile}{S} \cdots\cdots = N$$

or

$$M = \cdots\cdots \underset{\smile}{S} \cdots\cdots R \cdots\cdots \xrightarrow{R} \cdots\cdots \underset{\smile}{S} \cdots\cdots R' \cdots\cdots = N$$

Case 2:

$$M = \cdots\cdots \underset{\smile}{R} \cdots\cdots \xrightarrow{R} \cdots\cdots R' \cdots\cdots = N \quad (R \text{ and } S \text{ coincide})$$

Case 3:

$$M = \cdots \underbrace{(\lambda y. \cdots R \cdots)D} \cdots \xrightarrow{R} \cdots \underbrace{(\lambda y. \cdots R' \cdots)D} \cdots = N$$

Case 4:

$$M = \cdots \underbrace{(\lambda y.C)(\cdots R \cdots)} \cdots \xrightarrow{R} \cdots \underbrace{(\lambda y.C)(\cdots R' \cdots)} \cdots = N$$

Residuals of redexes (3/4)

Case 3:

$$M = \cdots(\lambda x.\cdots \underline{S} \cdots)B \cdots \xrightarrow{R} \cdots\cdots \underline{S\{x := B\}} \cdots\cdots = N$$

Case 4:

$$M = \cdots(\lambda x.\cdots x \cdots x \cdots)(\cdots \underline{S} \cdots)\cdots$$

$$\xrightarrow{R} \cdots\cdots(\cdots \underline{S} \cdots)\cdots(\cdots \underline{S} \cdots)\cdots\cdots = N$$

Residuals of redexes (4/4)

Examples: $\Delta = \lambda x.xx, I = \lambda x.x$

$$\Delta(\underbrace{I\,x}) \longrightarrow \underbrace{I\,x}(\underbrace{I\,x})$$

$$\underbrace{I\,x}\,(\Delta(I\,x)) \longrightarrow \underbrace{I\,x}\,(I\,x(I\,x))$$

$$\underbrace{I\,(\Delta(I\,x))} \longrightarrow \underbrace{I\,(I\,x(I\,x))}$$

$$\underbrace{\Delta(I\,x)} \longrightarrow I\,x(I\,x)$$

$$I\,x\,(\Delta(\underbrace{I\,x})) \longrightarrow I\,x\,(\underbrace{I\,x}(\underbrace{I\,x}))$$

$$\underbrace{\Delta\Delta} \longrightarrow \Delta\Delta$$

Residuals of reductions

Parallel reductions

- Consider reductions where each step is parallel

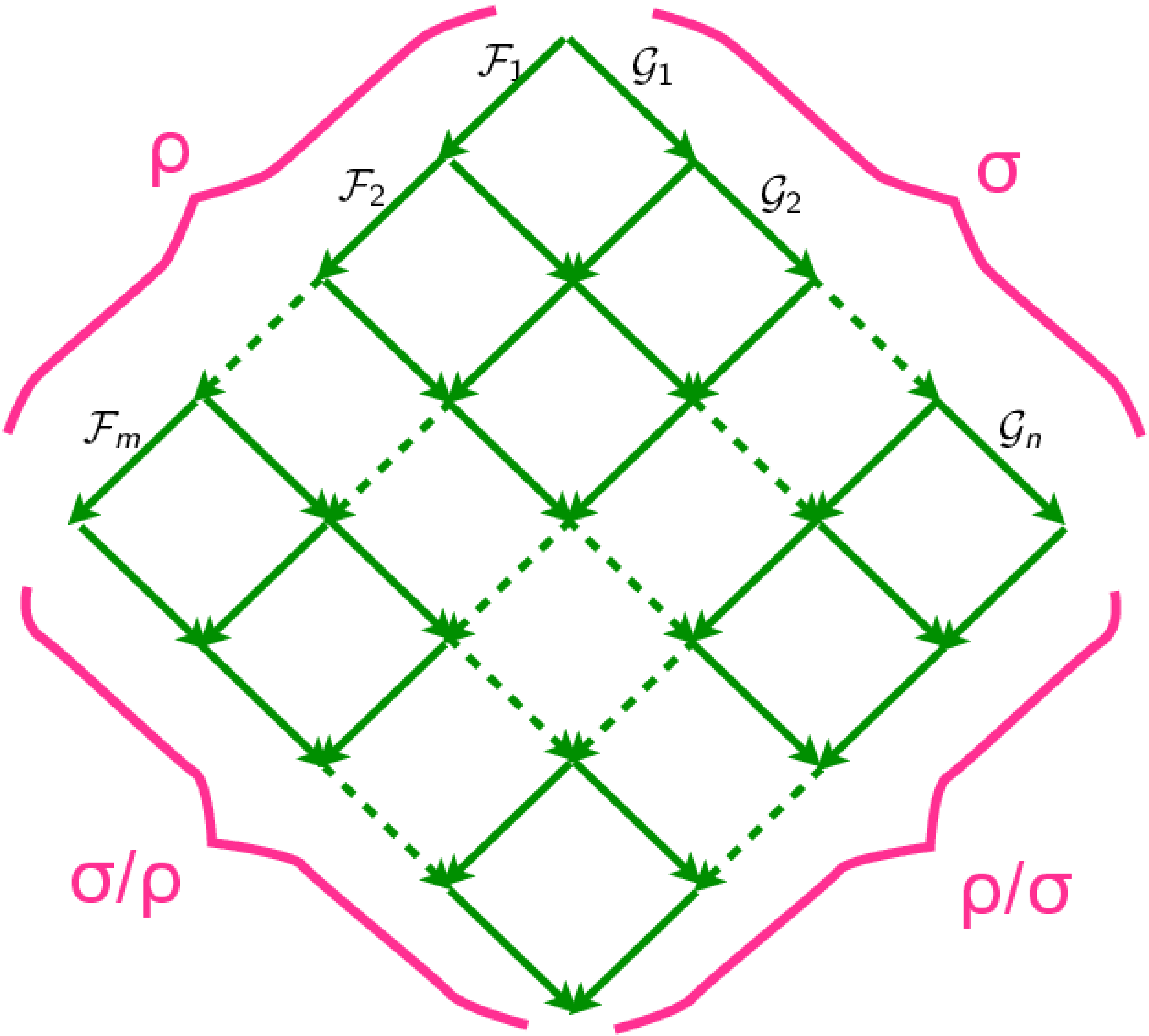
$$\rho : M = M_0 \xrightarrow{\mathcal{F}_1} M_1 \xrightarrow{\mathcal{F}_2} M_2 \cdots \xrightarrow{\mathcal{F}_n} M_n = N$$

- We also write

$$\rho = 0 \text{ when } n = 0$$

$$\rho = \mathcal{F}_1 \mathcal{F}_2 \cdots \mathcal{F}_n \text{ when } M \text{ clear from context}$$

Residuals of reductions (1/4)



Residuals of reductions (2/4)

- **Definition** [JJL 76]

$$\rho/0 = \rho$$

$$\rho/(\sigma\,\tau) = (\rho/\sigma)/\tau$$

$$(\rho\,\sigma)/\tau = (\rho/\tau)\,(\sigma/(\tau/\rho))$$

$\mathcal{F}/\mathcal{G}$ already defined

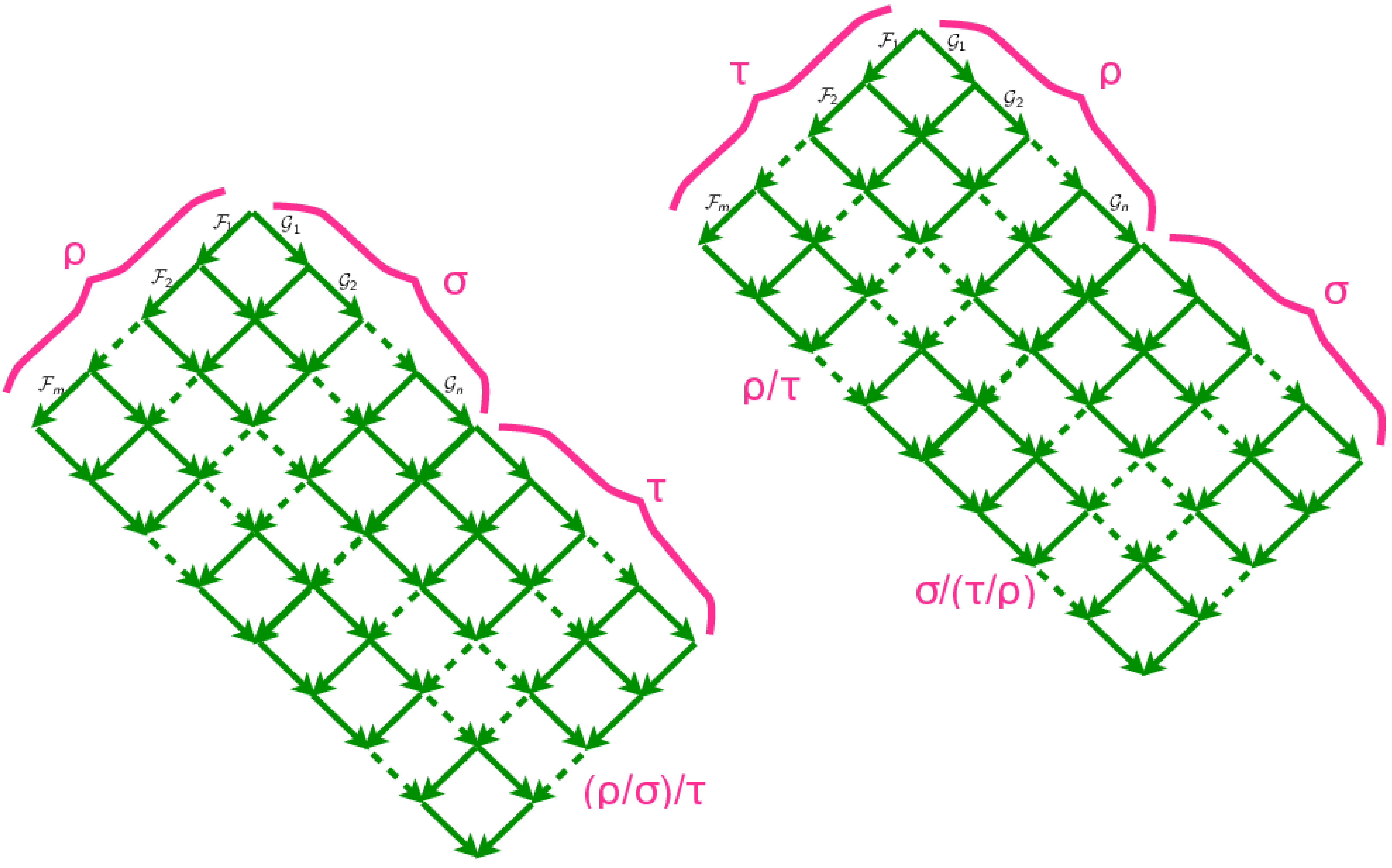
- **Notation**

$$\rho \sqcup \sigma = \rho\,(\sigma/\rho)$$

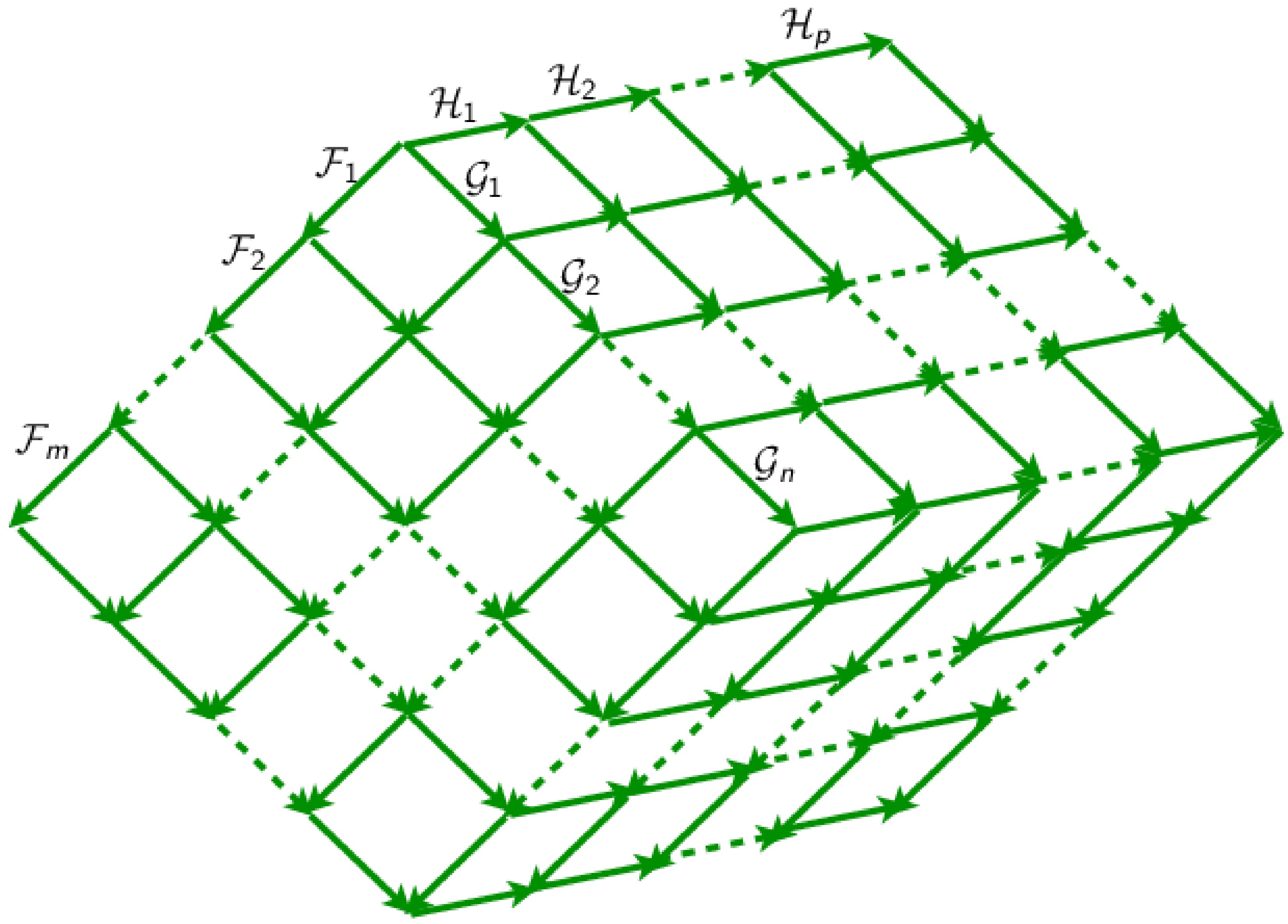
- **Proposition** [Parallel Moves +]:

$\rho \sqcup \sigma$ and $\sigma \sqcup \rho$ are cofinal

Residuals of reductions (3/4)



Residuals of reductions (4/4)



- **Proposition** `[Cube Lemma ++]`:

$$\tau/(\rho \sqcup \sigma) = \tau/(\sigma \sqcup \rho)$$

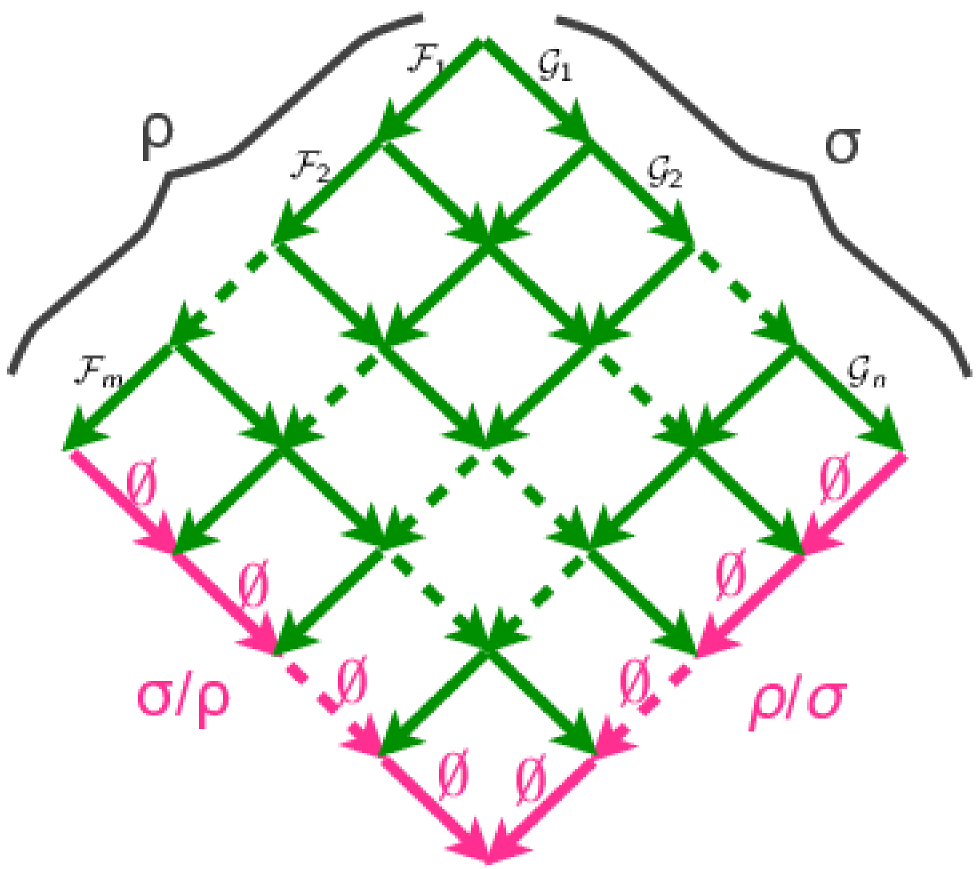
Equivalence by permutations

Equivalence by permutations (1/4)

- **Definition:**

  Let ρ and σ be 2 coinitial reductions. Then ρ is equivalent to σ by permutations, $\rho \simeq \sigma$, iff:

$$\rho/\sigma = \emptyset^m \quad \text{and} \quad \sigma/\rho = \emptyset^n$$



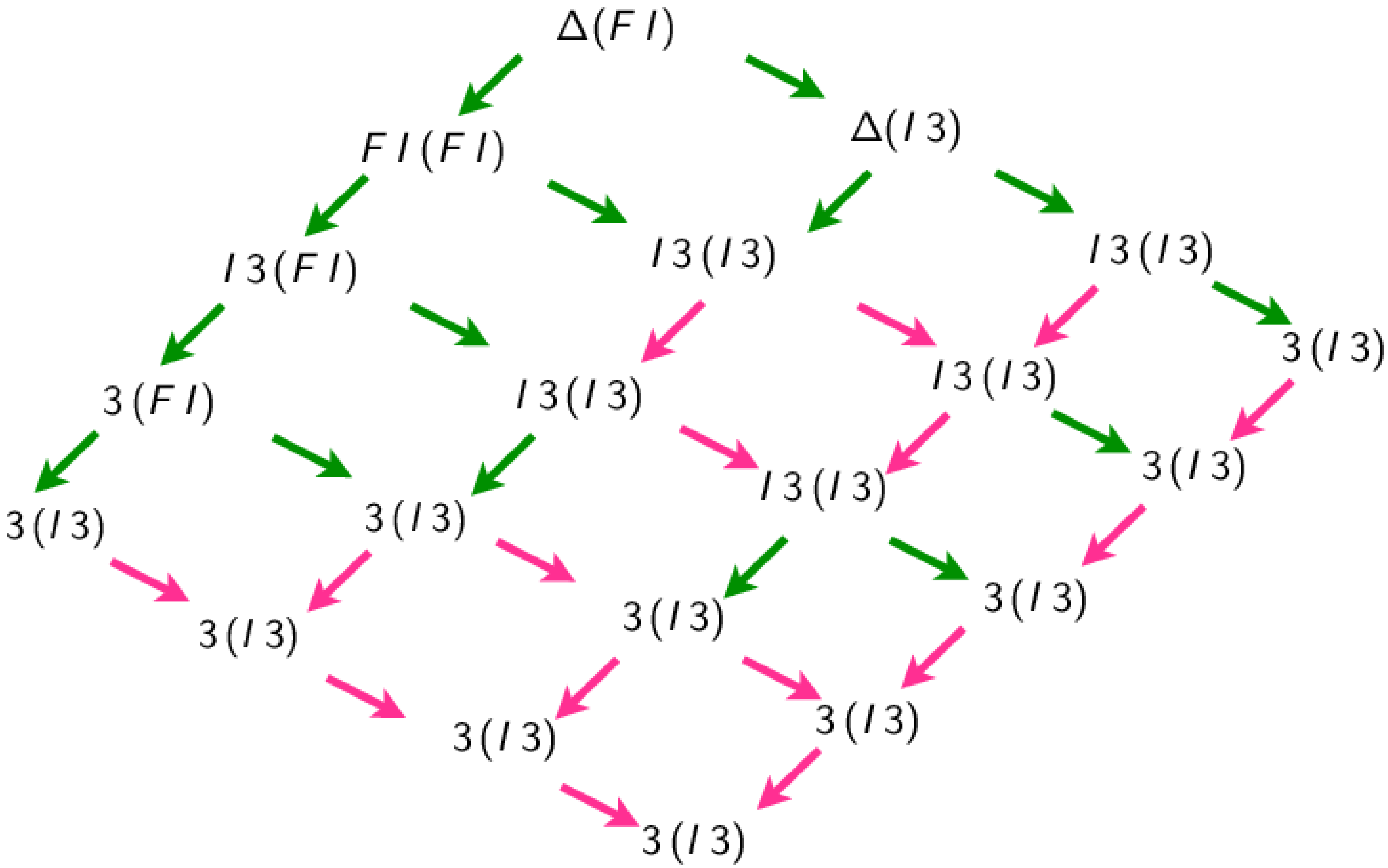
$\rho \simeq \sigma$ means that ρ and σ are coinitial and cofinal
but converse is not true (see later)

Equivalence by permutations (2/4)

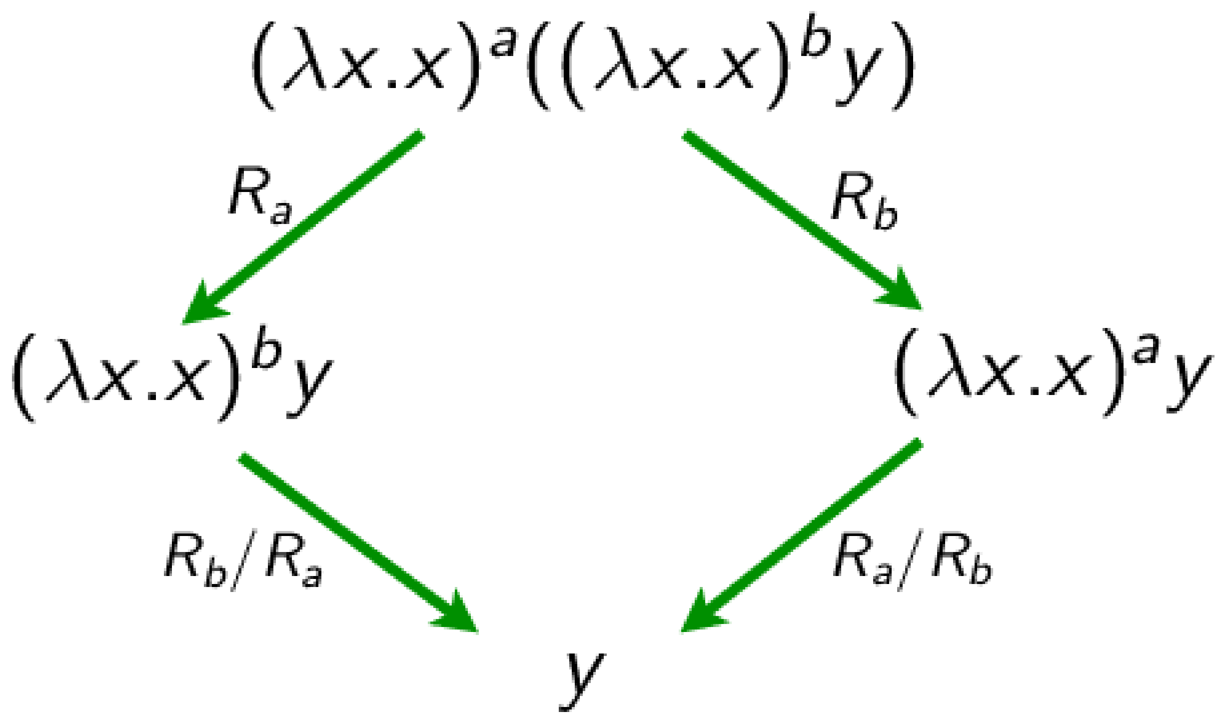
$\Delta = \lambda x.xx$

$F = \lambda f.f\,3$

$I = \lambda x.x$



Equivalence by permutations (3/4)



$$\rho : M = I^a(I^b y) \xrightarrow{R_a} I^b y$$
$$\sigma : M = I^a(I^b y) \xrightarrow{R_b} I^a y$$

- Here $\rho \not\simeq \sigma$ while ρ and σ are coinitial and cofinal in the calculus with no labels

Equivalence by permutations (4/4)

- Same with $0 \not\simeq \rho$ when $\rho : \Delta\Delta \longrightarrow \Delta\Delta$

  $\Delta = \lambda x.xx$

Exercise 1: Give other examples of non-equivalent reductions between same terms.

Exercise 2: Show following equalities

$$\rho/0 = \rho \qquad \emptyset^n/\rho = \emptyset^n$$

$$0/\rho = 0 \qquad 0 \simeq \emptyset^n$$

$$\rho/\emptyset^n = \rho \qquad \rho/\rho = \emptyset^n$$

Equivalence by permutations (4/4)

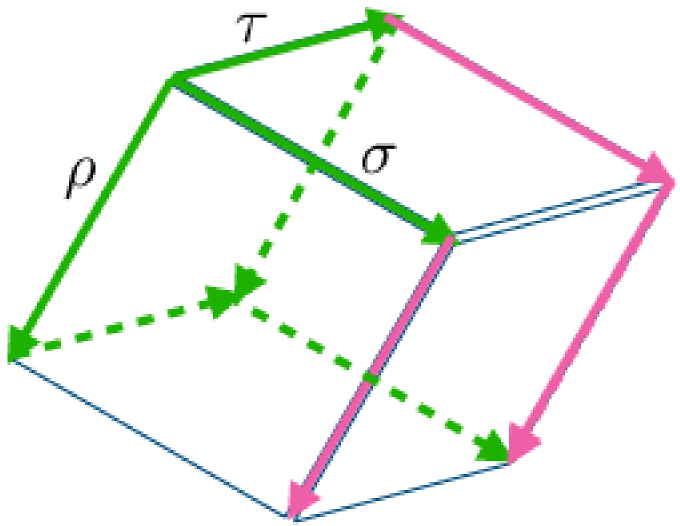
Exercise 3: Show that $\simeq$ is an equivalence relation.

Proof

(i) $\rho \simeq \rho$ obvious

(ii) same with $\rho \simeq \sigma$ implies $\sigma \simeq \rho$

(iii) $\rho \simeq \sigma \simeq \tau$ implies $\rho \simeq \tau$??



Properties of permutations (1/3)

- **Proposition**

$$(i) \quad \rho \simeq \sigma \ \text{ iff } \ \forall \tau.\ \tau/\rho = \tau/\sigma$$

$$(ii) \quad \rho \simeq \sigma \ \text{ implies } \ \rho/\tau = \sigma/\tau$$

$$(iii) \quad \rho \simeq \sigma \ \text{ iff } \ \tau\rho \simeq \tau\sigma$$

$$(iv) \quad \rho \simeq \sigma \ \text{ implies } \ \rho\tau \simeq \sigma\tau$$

$$(v) \quad \rho \sqcup \sigma \simeq \sigma \sqcup \rho$$

Proof

(i) $\rho \simeq \sigma$ implies $\sigma/\rho = \emptyset^n$ and $\rho/\sigma = \emptyset^m$.

Thus $\tau/(\rho \sqcup \sigma) = \tau/(\rho\,(\sigma/\rho)) = \tau/\rho/(\sigma/\rho) = \tau/\rho/\emptyset^m = \tau/\rho$

Similarly $\tau/(\sigma \sqcup \rho) = \tau/\sigma$

By cube lemma $\tau/\rho = \tau/\sigma$

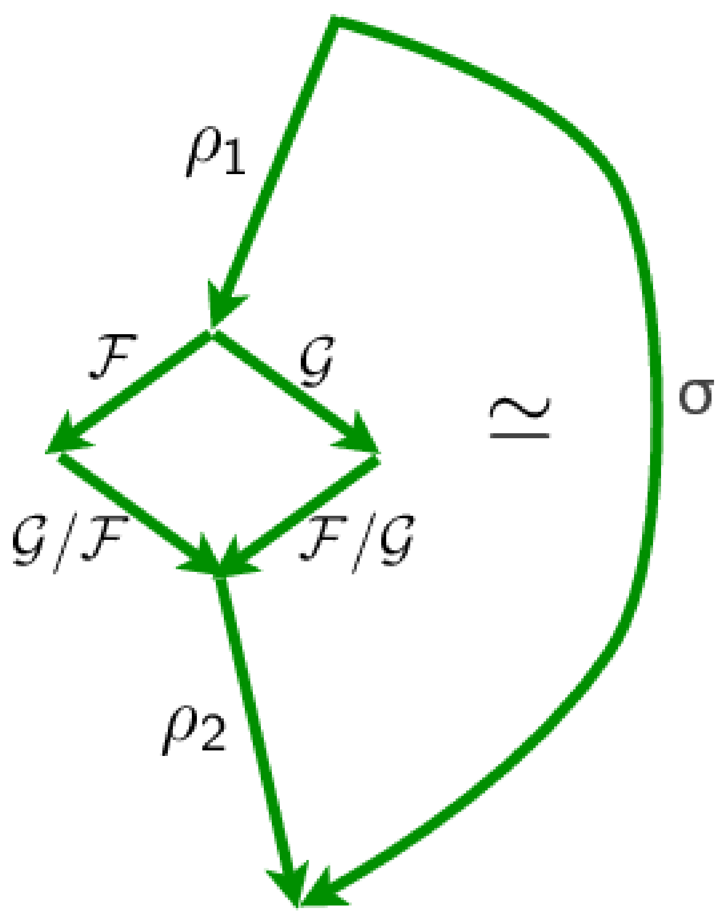
Conversely, take $\tau = \rho$ and $\tau = \sigma$.

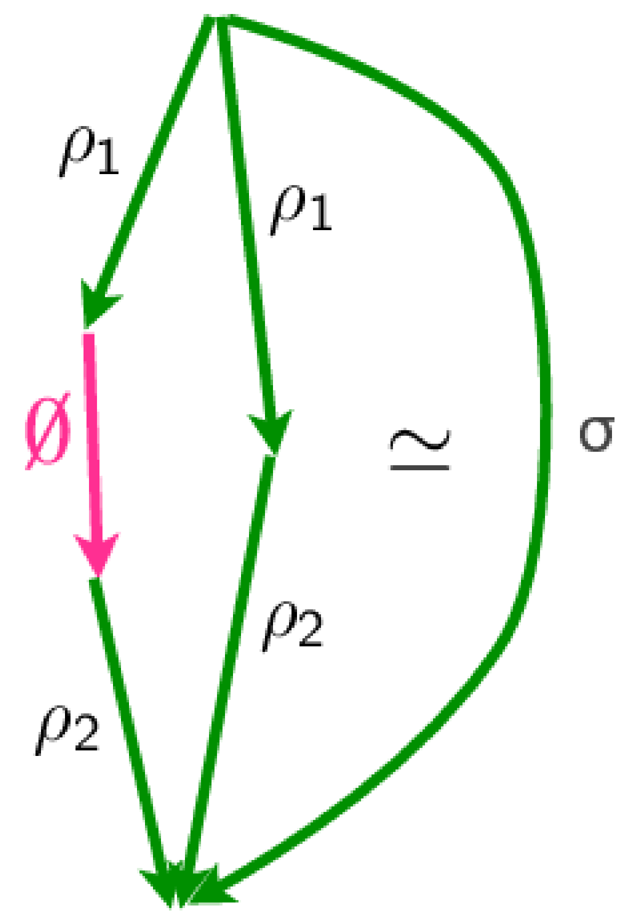
Properties of permutations (2/3)

- **Proposition** $\simeq$ is the smallest congruence containing

$$\mathcal{F}\,(\mathcal{G}/\mathcal{F}) \simeq \mathcal{G}\,(\mathcal{F}/\mathcal{G})$$

$$0 \simeq \emptyset$$



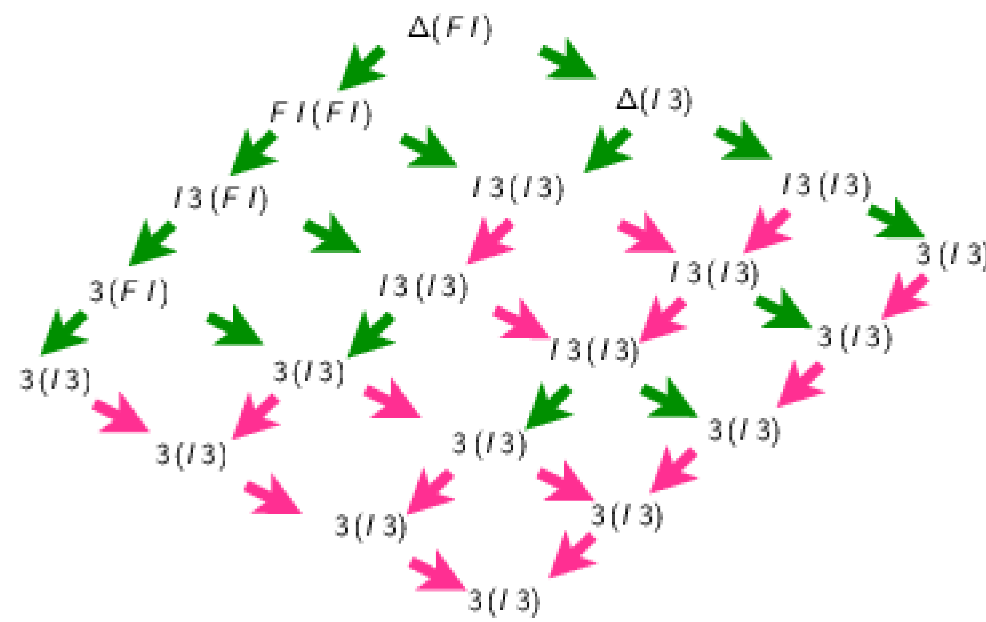


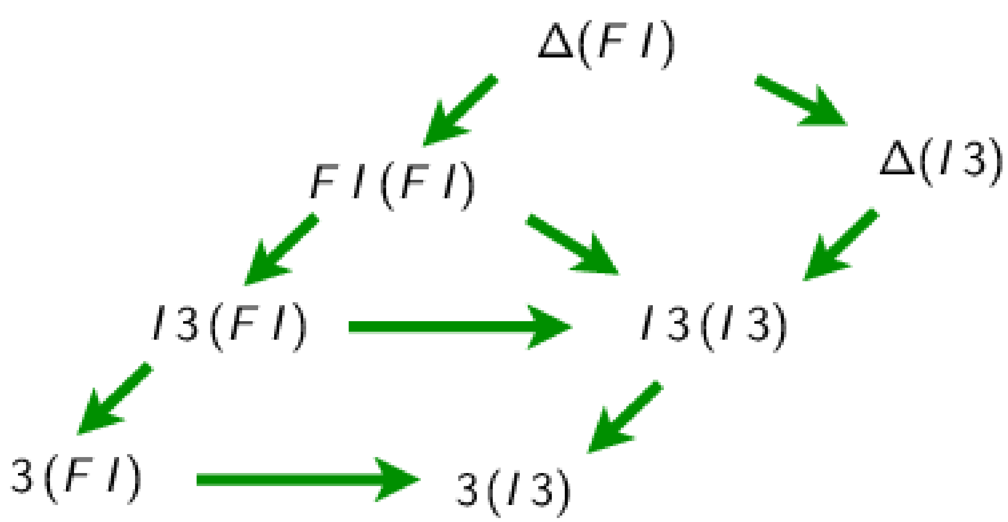
Properties of permutations (3/3)

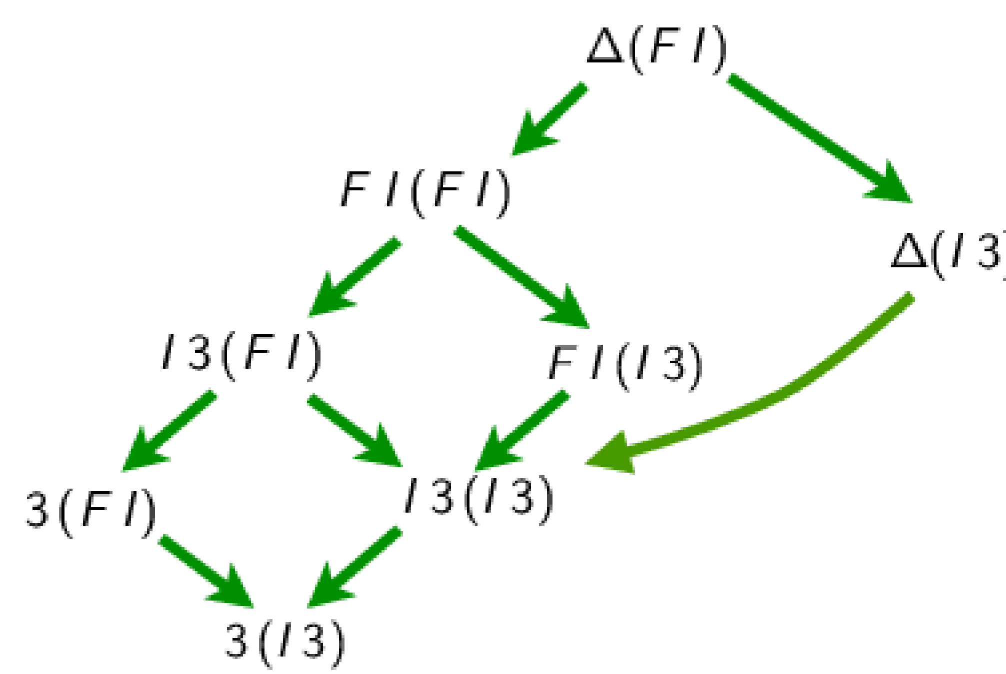
$\Delta = \lambda x.xx$

$F = \lambda f.f\,3$

$I = \lambda x.x$







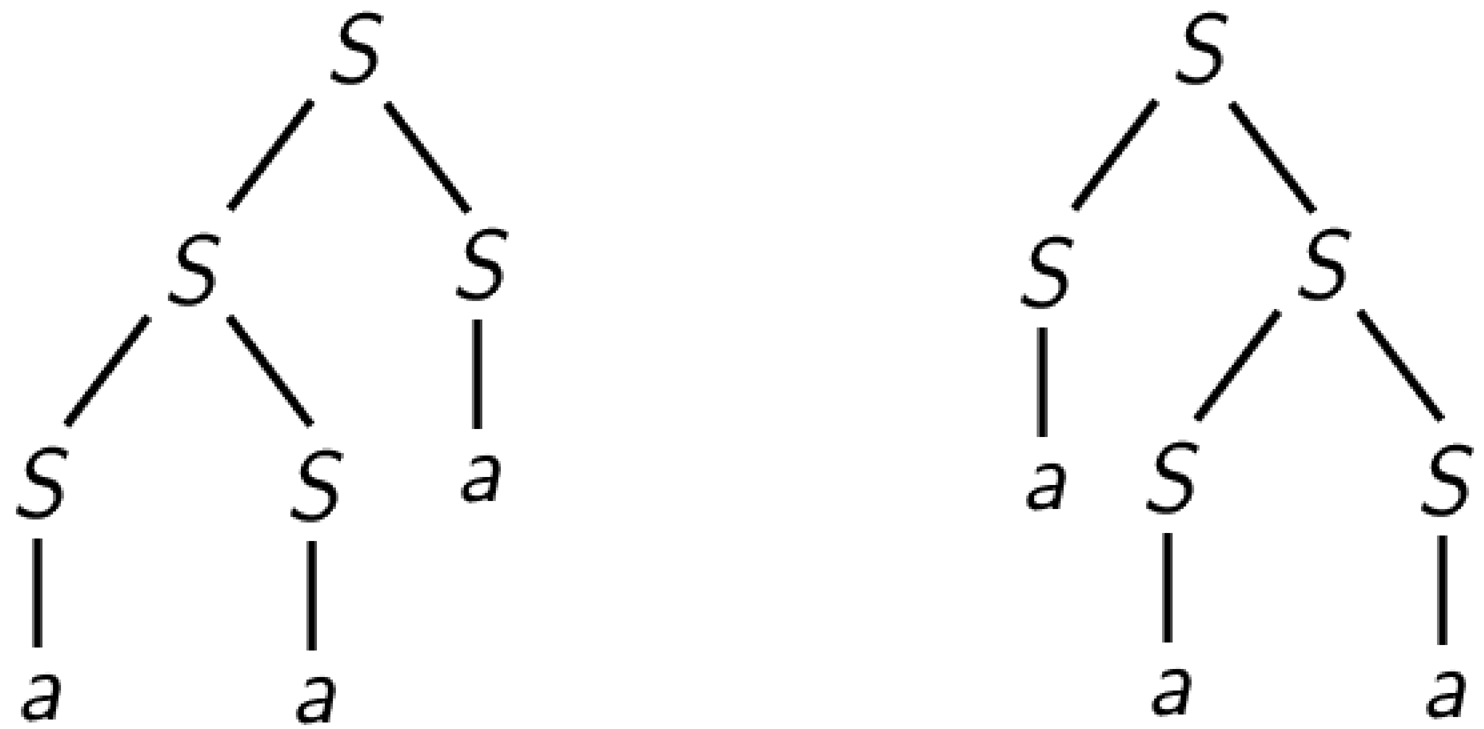
Beyond the λ-calculus

Context-free languages

- permutations of derivations in contex-free languages

$$S \to SS$$

$$S \to a$$



- each parse tree corresponds to an equivalence class

Term rewriting

- recursive program schemes [Berry-JJL'77]
- permutations of derivations in orthogonal TRS [Huet-JJL'81]
- permutations of derivations are defined with critical pairs
- critical pairs make conflicts
- only 2nd definition of equivalence works [Boudol'82]
- interaction systems [Asperti-Laneve'93]

Process algebras

- similar to TRS [Boudol-Castellani'82]
- connection to event structures [Laneve'84]

PCF

- LCF considered as a programming language [Plotkin'74]

M, N, P ::=	x	variable
	$\mid$ λx.M $\mid$ M N	abstraction application
	$\mid$ $\underline{n}$	integer constant
	$\mid$ M ⊗ N	⊗ ∈ {+,−,×,÷}
	$\mid$ ifz M then N then N	conditionnal
	$\mid$ μx.M	recursive definition

β	(λx.M)N → M {x := N}
op	$\underline{m} \otimes \underline{n} \rightarrow \underline{m \otimes n}$
cond1	ifz $\underline{0}$ then M else N → M
cond2	ifz $\underline{n+1}$ then M else N → N
μ	μx.M → M {x := μx.M}

Exemples de termes

$\texttt{Fact}(3)$

$\texttt{Fact} = Y(\lambda f.\lambda x.\ \texttt{ifz}\ x\ \texttt{then}\ 1\ \texttt{else}\ x \star f(x-1))$

$Y = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$

s'écrit

$(\lambda\,\texttt{Fact}\,.\,\texttt{Fact}(3))$

$(\,(\lambda Y.Y(\lambda f.\lambda x.\ \texttt{ifz}\ x\ \texttt{then}\ 1\ \texttt{else}\ x \star f(x-1)))$

$(\lambda f.(\lambda x.f(xx))(\lambda x.f(xx)))\,)$

(\Fact.Fact3)((\y.y(\f.\x.ifz x then 1 else x * f(x-1)))(\f.Yf))

↓

(\y.y(\f.\x.ifz x then 1 else x * f(x-1)))(\f.Yf)3

↓

(\f.Yf)(\f.\x.ifz x then 1 else x * f(x-1))3

↓

(\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx))(\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx))3

↓

(\f.\x.ifz x then 1 else x * f(x-1))((\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx))(\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx)))3

↓

(\x.ifz x then 1 else x * (\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx))(\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx))(x-1))3

↓

ifz 3 then 1 else 3 * (\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx))(\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx))(3-1)

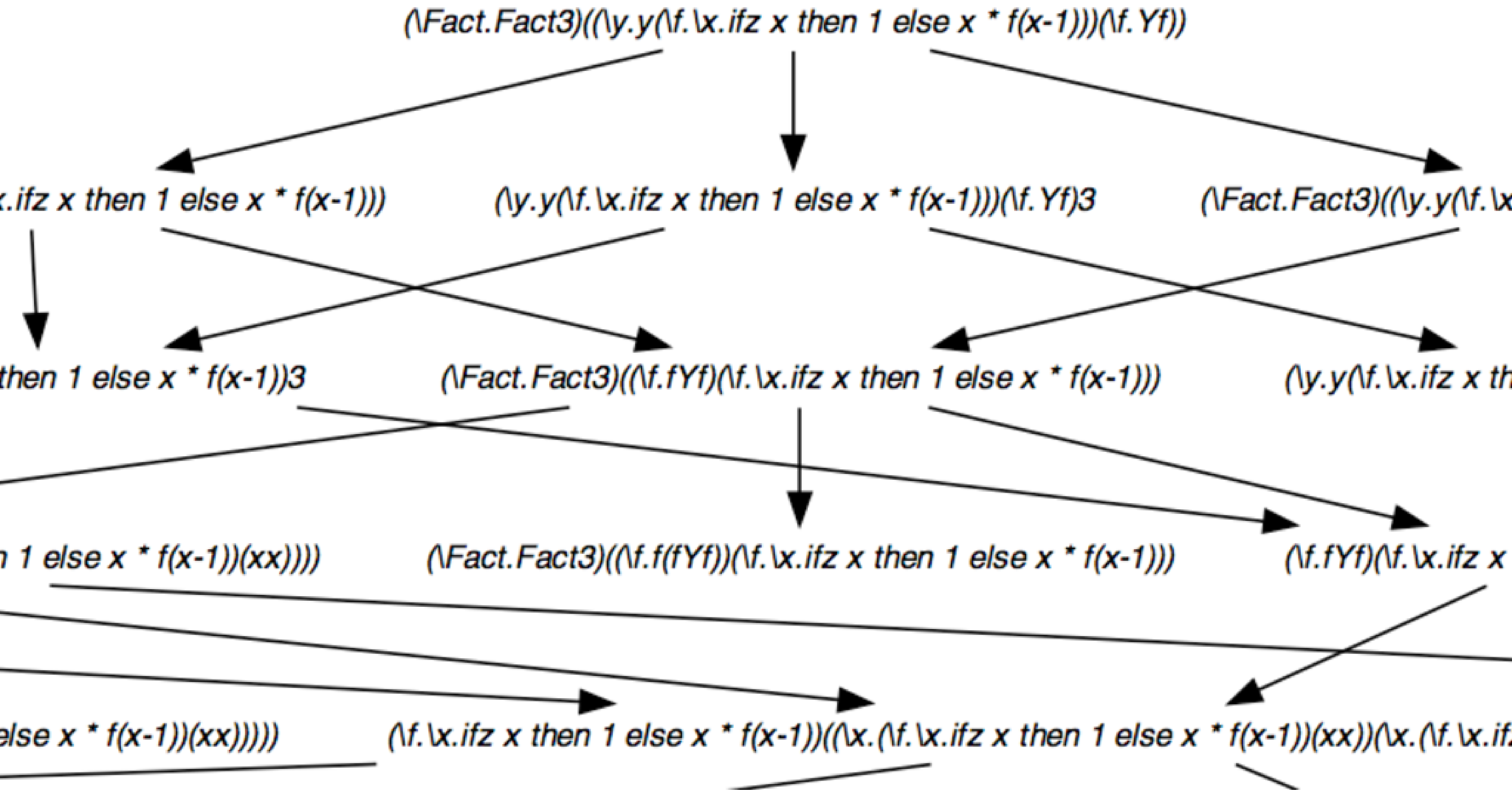
↓

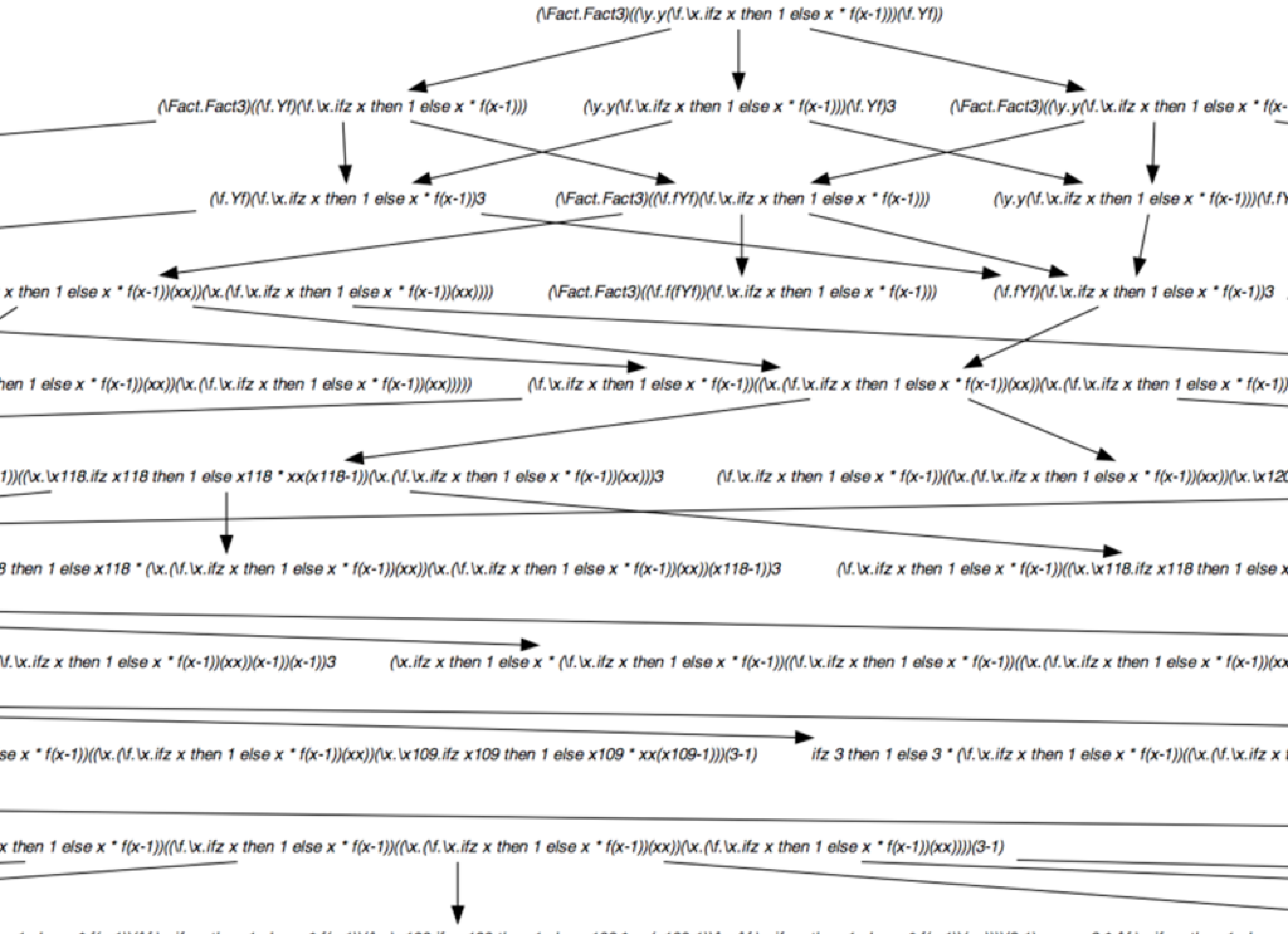
3 * (\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx))(\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx))(3-1)

↓

3 * (\f.\x.ifz x then 1 else x * f(x-1))((\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx))(\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx)))(3-1)

↓


(\Fact.Fact3)((\y.y(\f.\x.ifz x then 1 else x * f(x-1)))(\f.Yf))
x.ifz x then 1 else x * f(x-1)))
(\y.y(\f.\x.ifz x then 1 else x * f(x-1)))(\f.Yf)3
(\Fact.Fact3)((\y.y(\f.\x
then 1 else x * f(x-1))3
(\Fact.Fact3)((\f.fYf)(\f.\x.ifz x then 1 else x * f(x-1)))
(\y.y(\f.\x.ifz x th
n 1 else x * f(x-1))(xx))))
(\Fact.Fact3)((\f.f(fYf))(\f.\x.ifz x then 1 else x * f(x-1)))
(\f.fYf)(\f.\x.ifz x
else x * f(x-1))(xx)))))
(\f.\x.ifz x then 1 else x * f(x-1))((\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx))(\x.(\f.\x.if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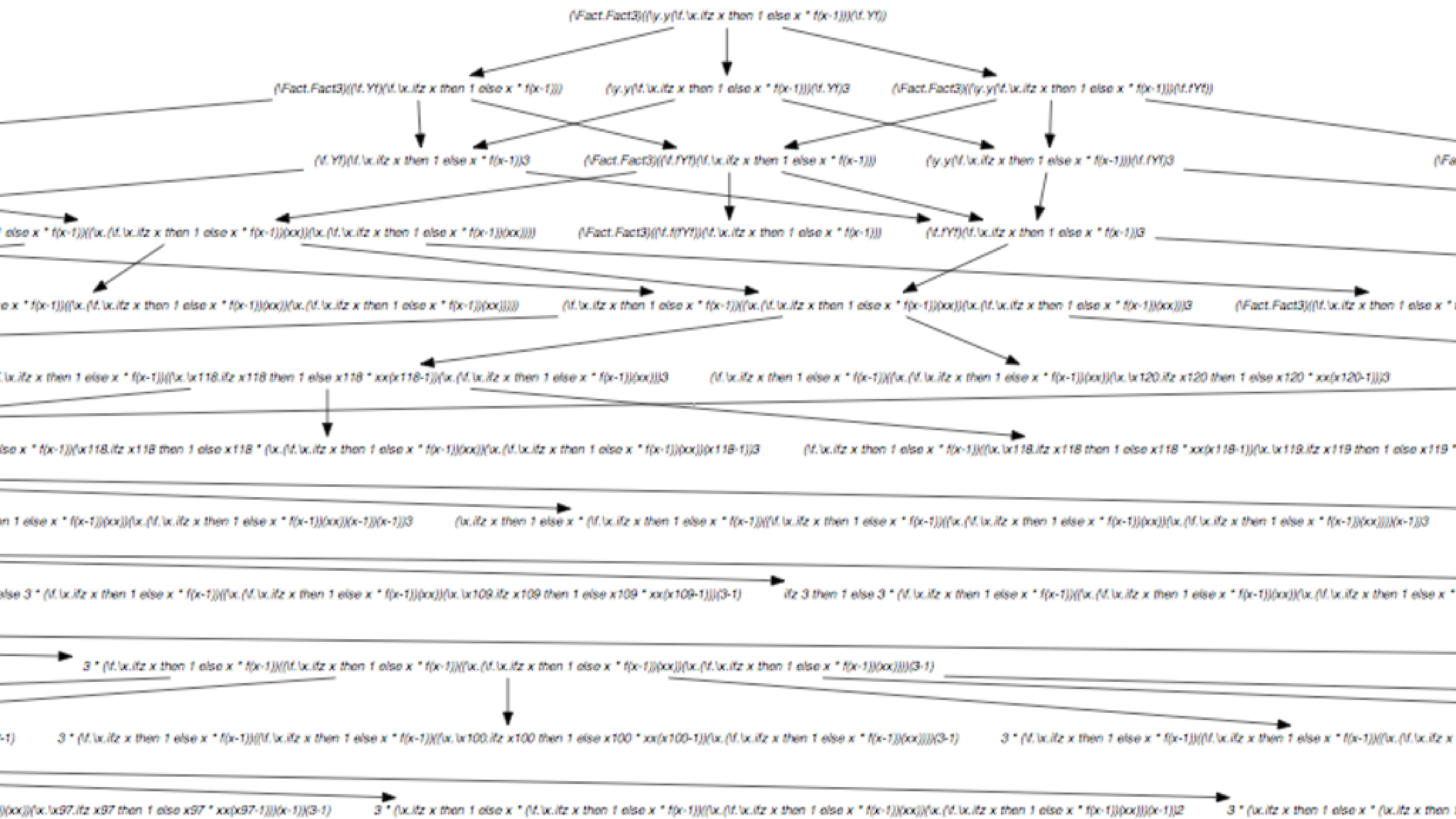
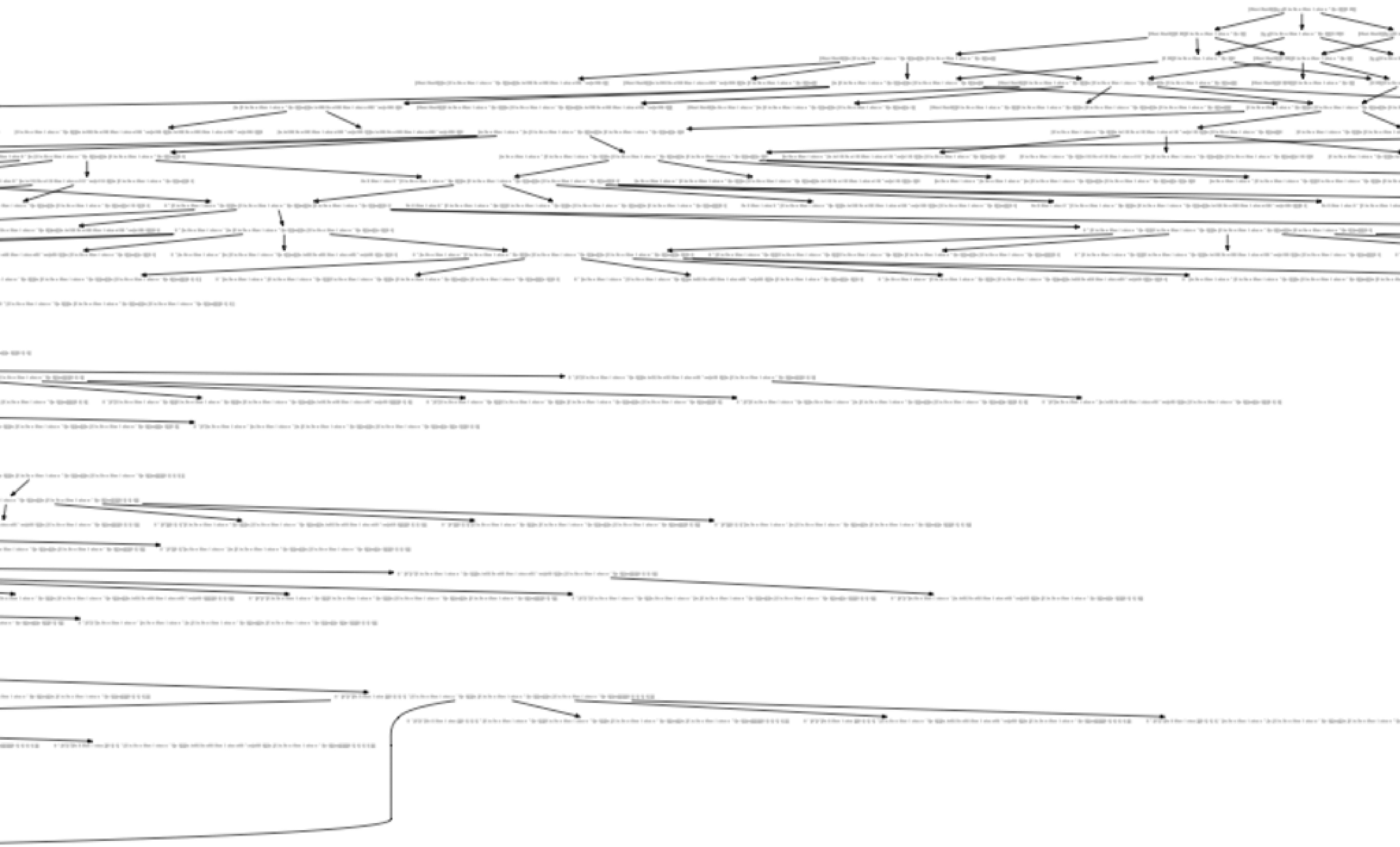
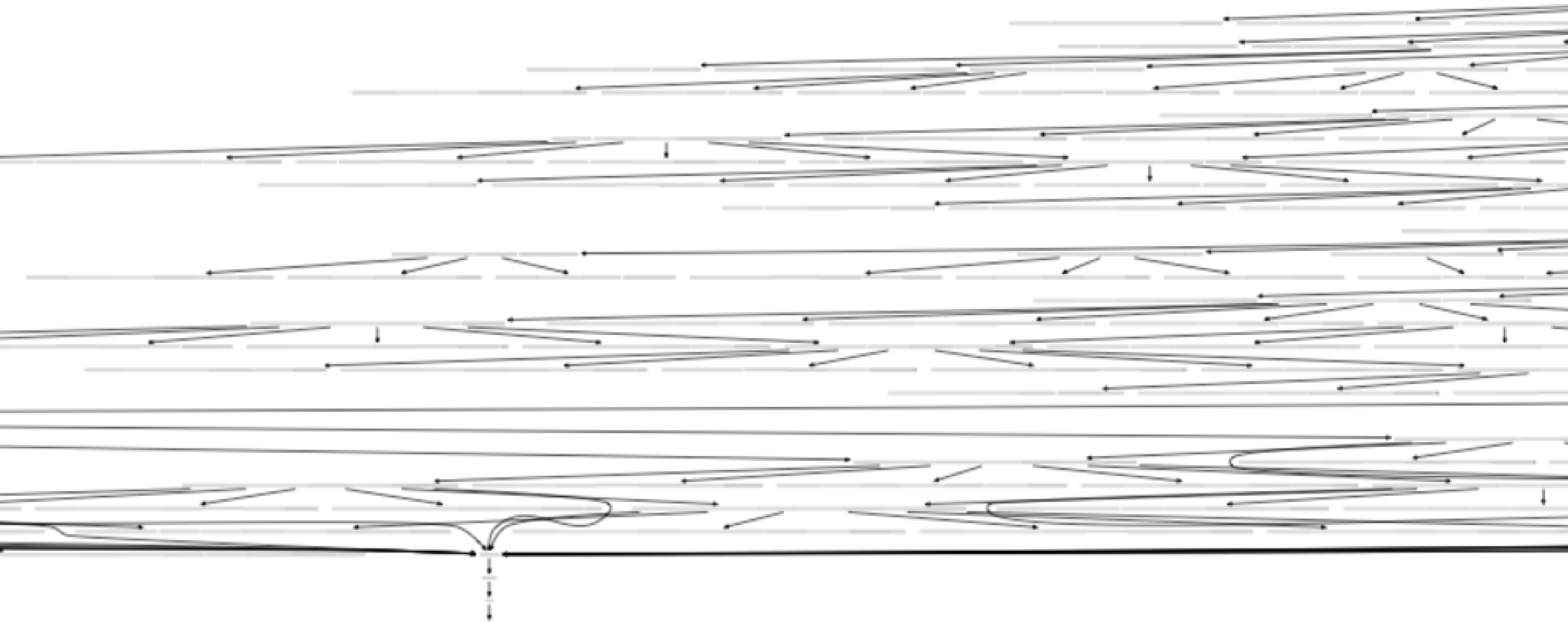
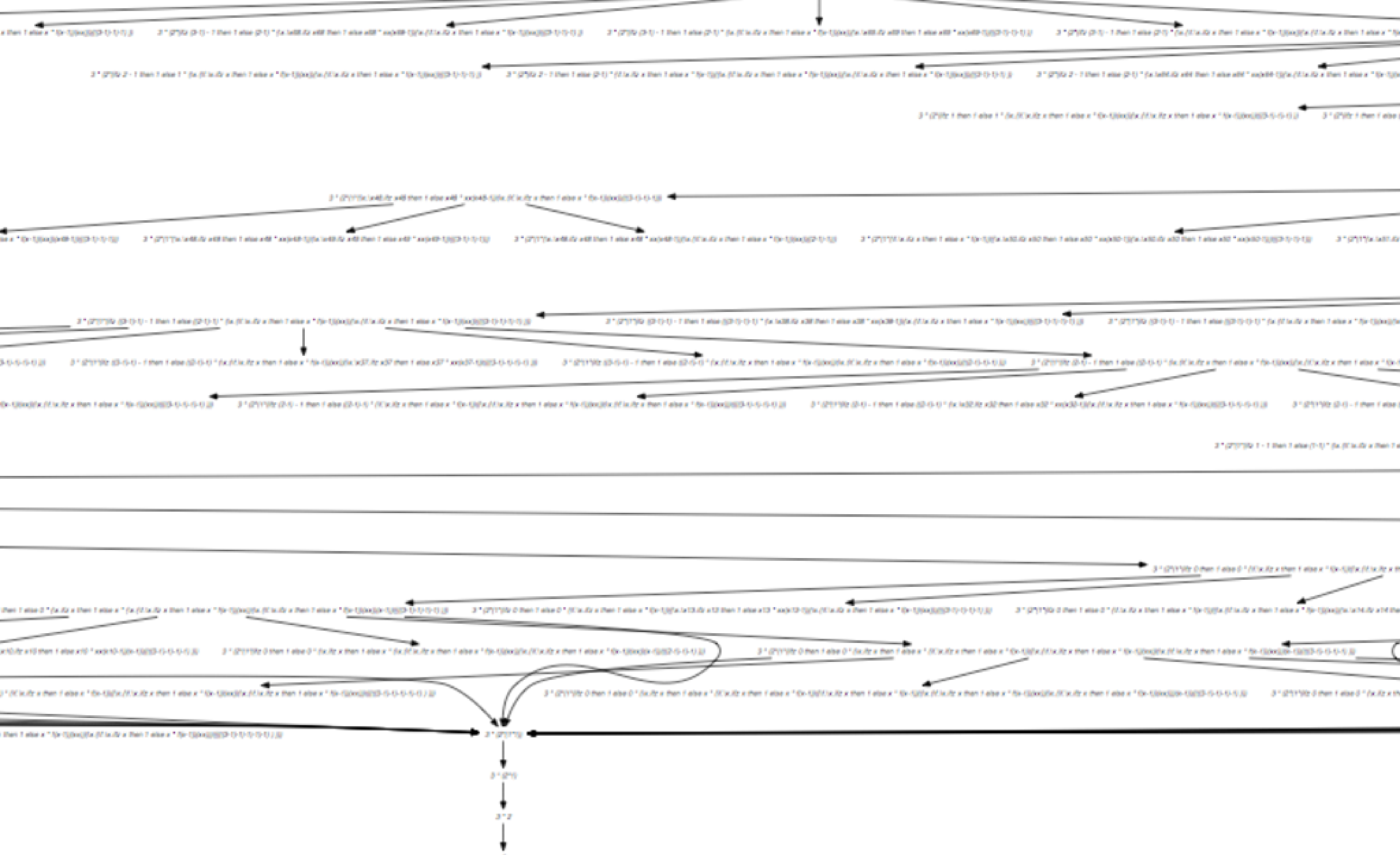
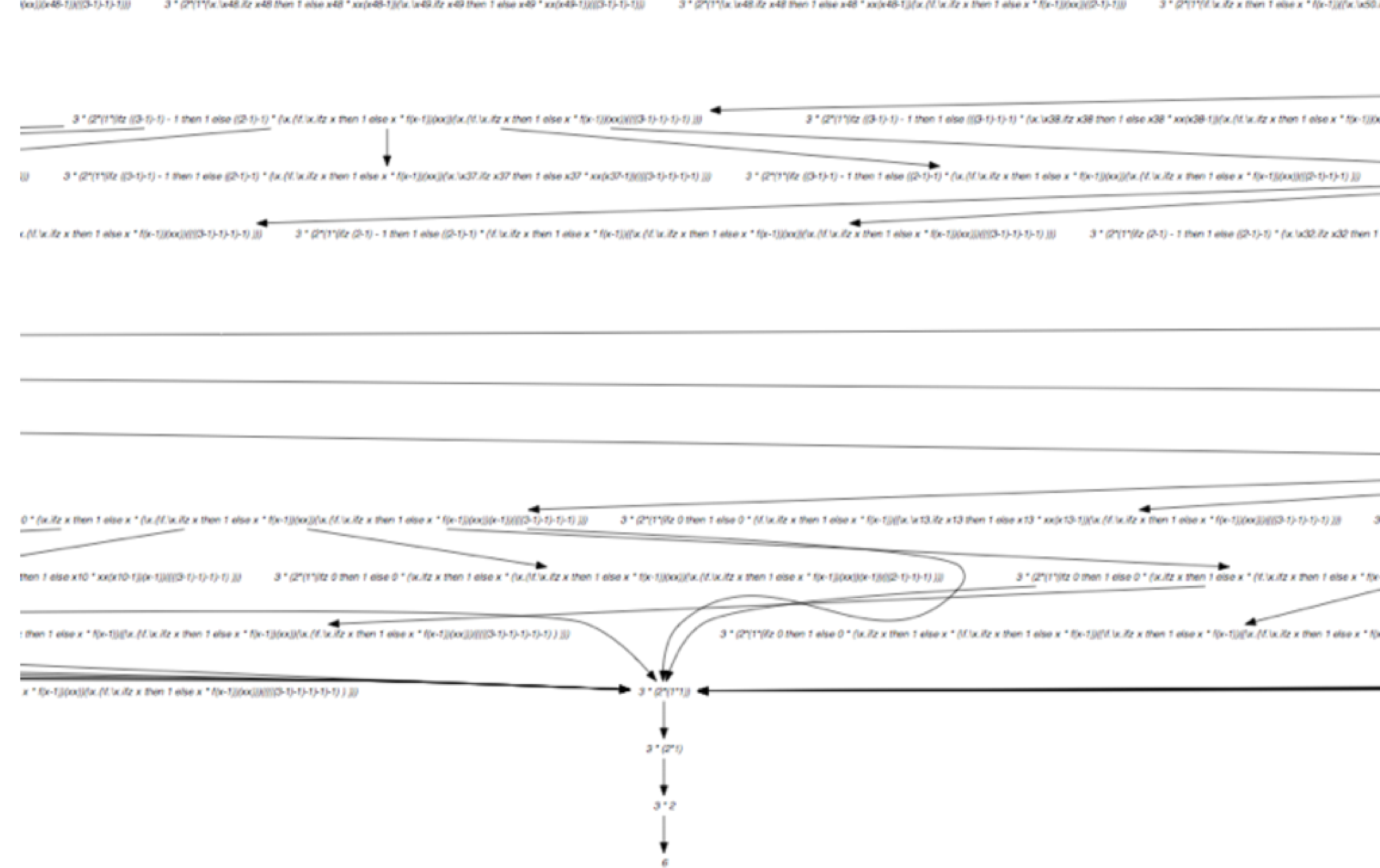
(\Fact.Fact3)((\f.Yf)(\f.\x.ifz x then 1 else x * f(x-1)))
(\y.y(\f.\x.ifz x then 1 else x * f(x-1)))(\f.Yf)3
(\Fact.Fact3)((\y.y(\f.\x.ifz x then 1 else x * f(x-
(\f.Yf)(\f.\x.ifz x then 1 else x * f(x-1))3
(\Fact.Fact3)((\f.fYf)(\f.\x.ifz x then 1 else x * f(x-1)))
(\y.y(\f.\x.ifz x then 1 else x * f(x-1)))(\f.fY
x then 1 else x * f(x-1))(xx))(\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx))))
(\Fact.Fact3)((\f.f(fYf))(\f.\x.ifz x then 1 else x * f(x-1)))
(\f.fYf)(\f.\x.ifz x then 1 else x * f(x-1))3
hen 1 else x * f(x-1))(xx))(\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx)))))
(\f.\x.ifz x then 1 else x * f(x-1))((\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx))(\x.(\f.\x.ifz x then 1 else x * f(x-1))
1))((\x.\x118.ifz x118 then 1 else x118 * xx(x118-1))(\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx)))3
(\f.\x.ifz x then 1 else x * f(x-1))((\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx))(\x.\x12
8 then 1 else x118 * (\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx))(\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx))(x118-1))3
(\f.\x.ifz x then 1 else x * f(x-1))((\x.\x118.ifz x118 then 1 else x
\f.\x.ifz x then 1 else x * f(x-1))(xx))(x-1))(x-1))3
(\x.ifz x then 1 else x * (\f.\x.ifz x then 1 else x * f(x-1))((\f.\x.ifz x then 1 else x * f(x-1))((\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx
se x * f(x-1))((\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx))(\x.\x109.ifz x109 then 1 else x109 * xx(x109-1)))(3-1)
ifz 3 then 1 else 3 * (\f.\x.ifz x then 1 else x * f(x-1))((\x.(\f.\x.ifz x t
x then 1 else x * f(x-1))((\f.\x.ifz x then 1 else x * f(x-1))((\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx))(\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx))))(3-1)

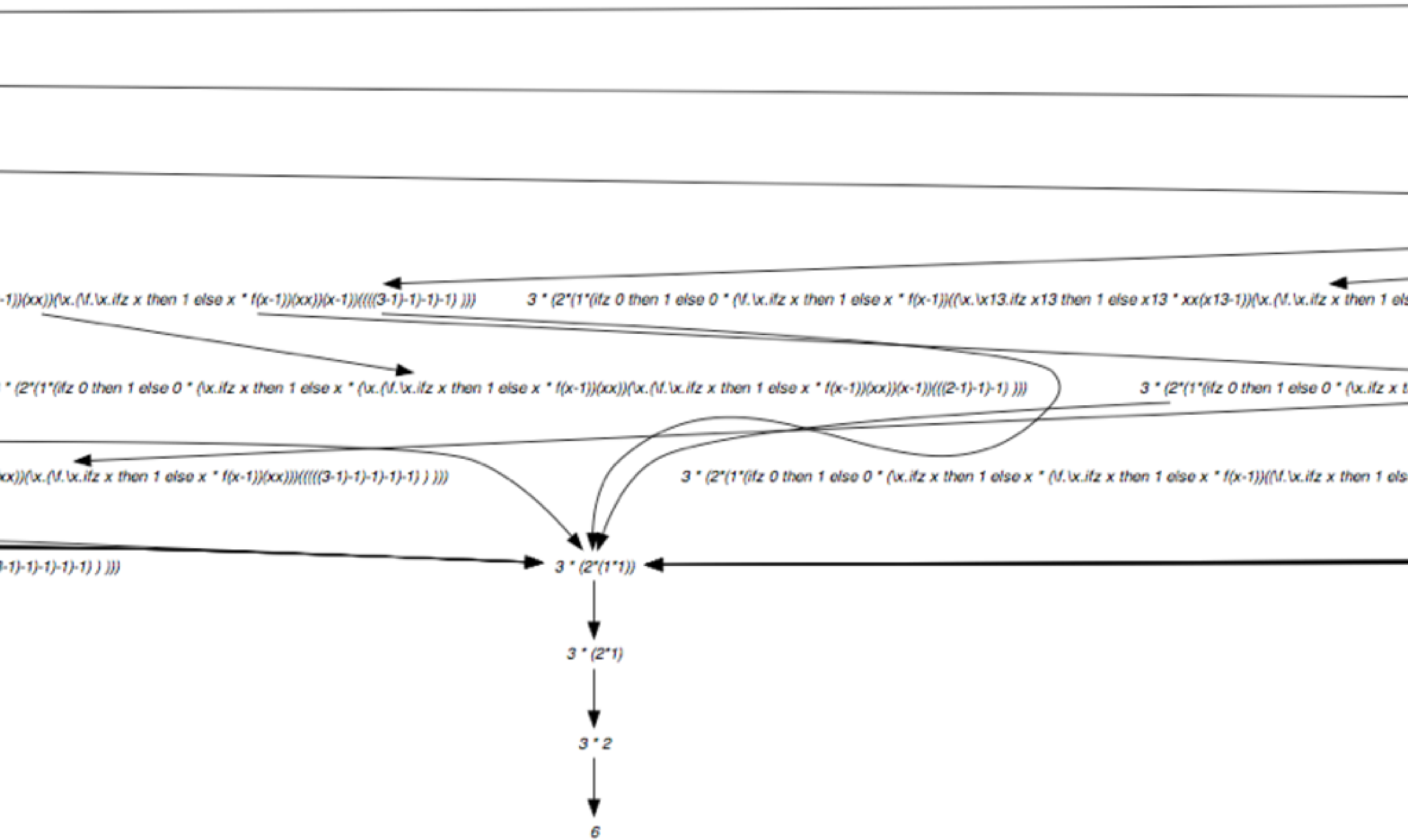

(\Fact.Fact3)((\y.y(\f.\x.ifz x then 1 else x * f(x-1)))(\f.Yf))

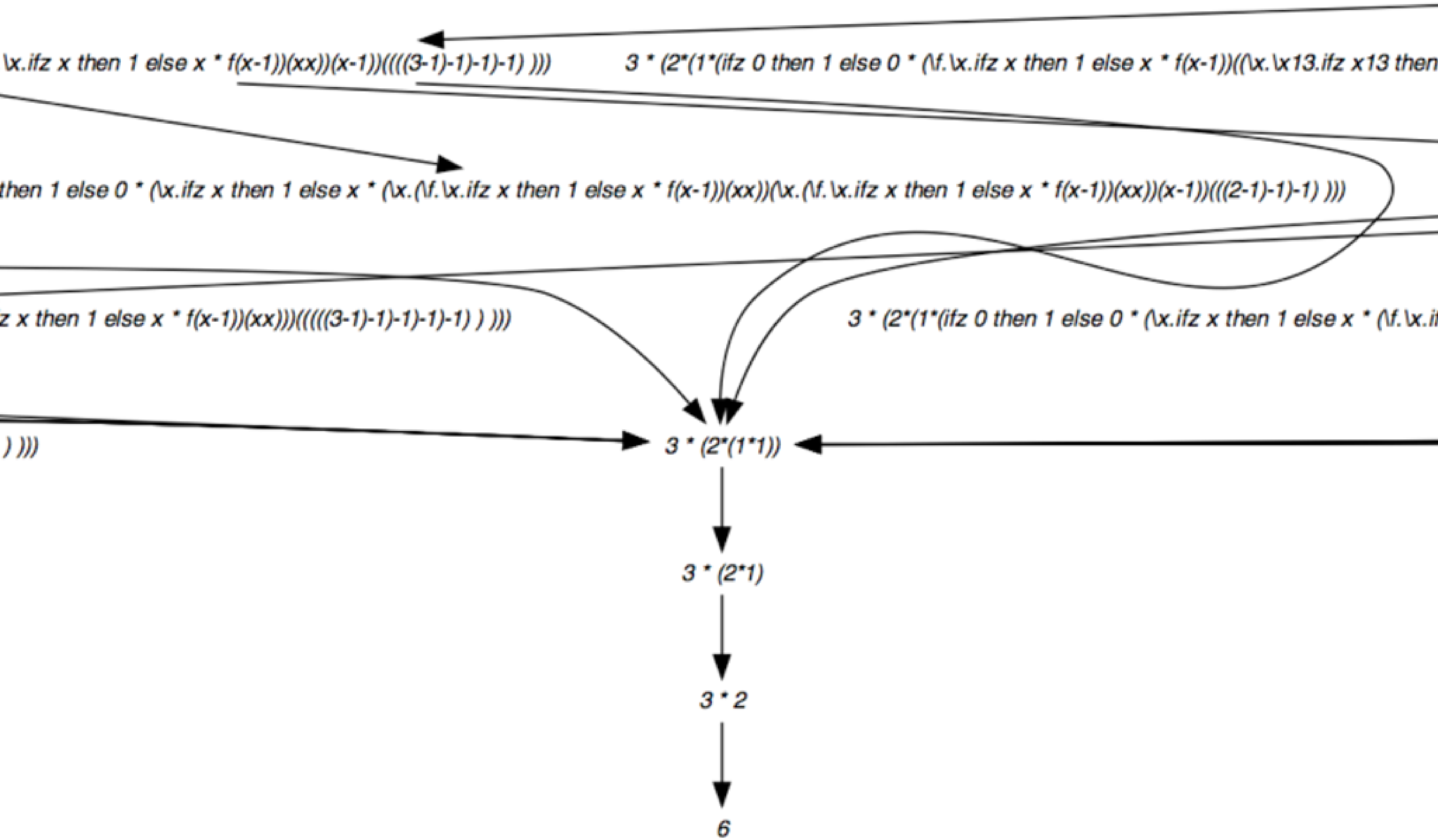







3 * (2*(1*1))
3 * (2*1)
3 * 2
6


-1))(xx))(\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx))(x-1))((((3-1)-1)-1)-1))))
3 * (2*(1*(ifz 0 then 1 else 0 * (\f.\x.ifz x then 1 else x * f(x-1))((\x.\x13.ifz x13 then 1 else x13 * xx(x13-1))(\x.(\f.\x.ifz x then 1 els
* (2*(1*(ifz 0 then 1 else 0 * (\x.ifz x then 1 else x * (\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx))(\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx))(x-1))(((2-1)-1)-1))))
3 * (2*(1*(ifz 0 then 1 else 0 * (\x.ifz x t
xx))(\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx)))(((((3-1)-1)-1)-1)-1)))))
3 * (2*(1*(ifz 0 then 1 else 0 * (\x.ifz x then 1 else x * (\f.\x.ifz x then 1 else x * f(x-1))((\f.\x.ifz x then 1 els
-1)-1)-1)-1)-1)))))
3 * (2*(1*1))
3 * (2*1)
3 * 2
6


\x.ifz x then 1 else x * f(x-1))(xx))(x-1))((((3-1)-1)-1)-1))))
3 * (2*(1*(ifz 0 then 1 else 0 * (\f.\x.ifz x then 1 else x * f(x-1))((\x.\x13.ifz x13 then
then 1 else 0 * (\x.ifz x then 1 else x * (\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx))(\x.(\f.\x.ifz x then 1 else x * f(x-1))(xx))(x-1))(((2-1)-1)-1))))
z x then 1 else x * f(x-1))(xx)))(((((3-1)-1)-1)-1)-1)))))
3 * (2*(1*(ifz 0 then 1 else 0 * (\x.ifz x then 1 else x * (\f.\x.i
))))
3 * (2*(1*1))
3 * (2*1)
3 * 2
6

Exercises

Parallel moves

- **Lemma** $M \xrightarrow{\mathcal{F}} N,\ M \xrightarrow{\mathcal{G}} P \ \Rightarrow\ N \xrightarrow{\mathcal{G}} Q,\ P \xrightarrow{\mathcal{F}} Q$

Proof

Case 1: $M = x = N = P = Q$. Obvious.

Case 2: $M = \lambda x.M_1,\ N = \lambda x.N_1,\ P = \lambda x.P_1$. Obvious by induction on M_1

Case 3: (App-App) $M = M_1 M_2,\ N = N_1 N_2,\ P = P_1 P_2$. Obvious by induction on $M_1,\ M_2$.

Case 4: (Red'-Red') $M = (\lambda x.M_1)^a M_2,\ N = (\lambda x.N_1)^a N_2,\ P = (\lambda x.P_1)^a P_2,\ a \notin \mathcal{F} \cup \mathcal{G}$

Then induction on $M_1,\ M_2$.

Case 4: (beta-Red') $M = (\lambda x.M_1)^a M_2,\ N = N_1\{x := N_2\},\ P = (\lambda x.P_1)^a P_2,\ a \in \mathcal{F}, a \notin \mathcal{G}$

By induction $N_1 \xrightarrow{\mathcal{G}} Q_1,\ P_1 \xrightarrow{\mathcal{F}} Q_1$. And $N_2 \xrightarrow{\mathcal{G}} Q_2,\ P_1 \xrightarrow{\mathcal{F}} Q_2$.

By lemma, $N_1\{x := N_2\} \xrightarrow{\mathcal{G}} Q_1\{x := Q_2\}$. And $(\lambda x.P_1)^a P_2 \xrightarrow{\mathcal{F}} Q_1\{x := Q_2\}$

Case 5: (beta-beta) $M = (\lambda x.M_1)^a M_2,\ N = N_1\{x := N_2\},\ P = P_1\{x := P_2\},\ a \in \mathcal{F} \cap \mathcal{G}$

As before with same lemma.

Parallel moves

- Lemma $M \xrightarrow{\mathcal{F}} N, P \xrightarrow{\mathcal{F}} Q \Rightarrow M\{x := P\} \xrightarrow{\mathcal{F}} N\{x := Q\}$

  Proof: exercise!

- Lemma [subst] $M\{x := N\}\{y := P\} = M\{y := P\}\{x := N\{y := P\}\}$

  when x not free in P

  Proof: exercise!

  this lemma about distribution of substitution is critical for the Church-Rosser property.

Finite Developments in the λ-calculus

Part 2

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ISR 2021

Madrid

July 6, 2021

http://jeanjacqueslevy.net/talks/21isr

A labeled lambda-calculus (1/3)

- Give names to redexes and to (some) subterms
- make names consistent with permutation equivalence.

$$M, N, \ldots ::= x \mid MN \mid \lambda x.M \mid M^\alpha$$

- Conversion rule is:

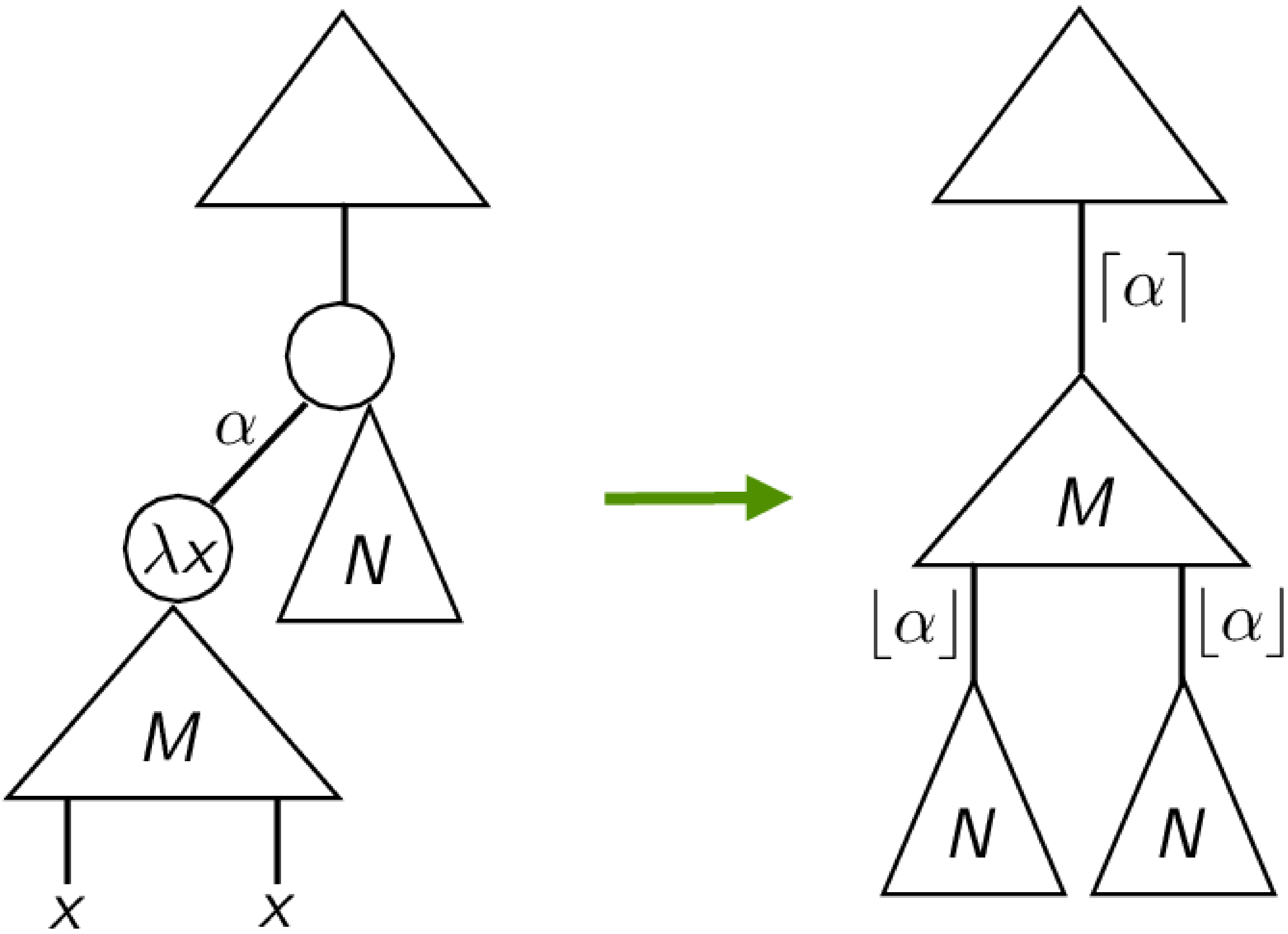
$$(\lambda x.M)^\alpha N \longrightarrow M^{\lceil \alpha \rceil}\{x := N^{\lfloor \alpha \rfloor}\}$$

α is the **name** of that redex

where

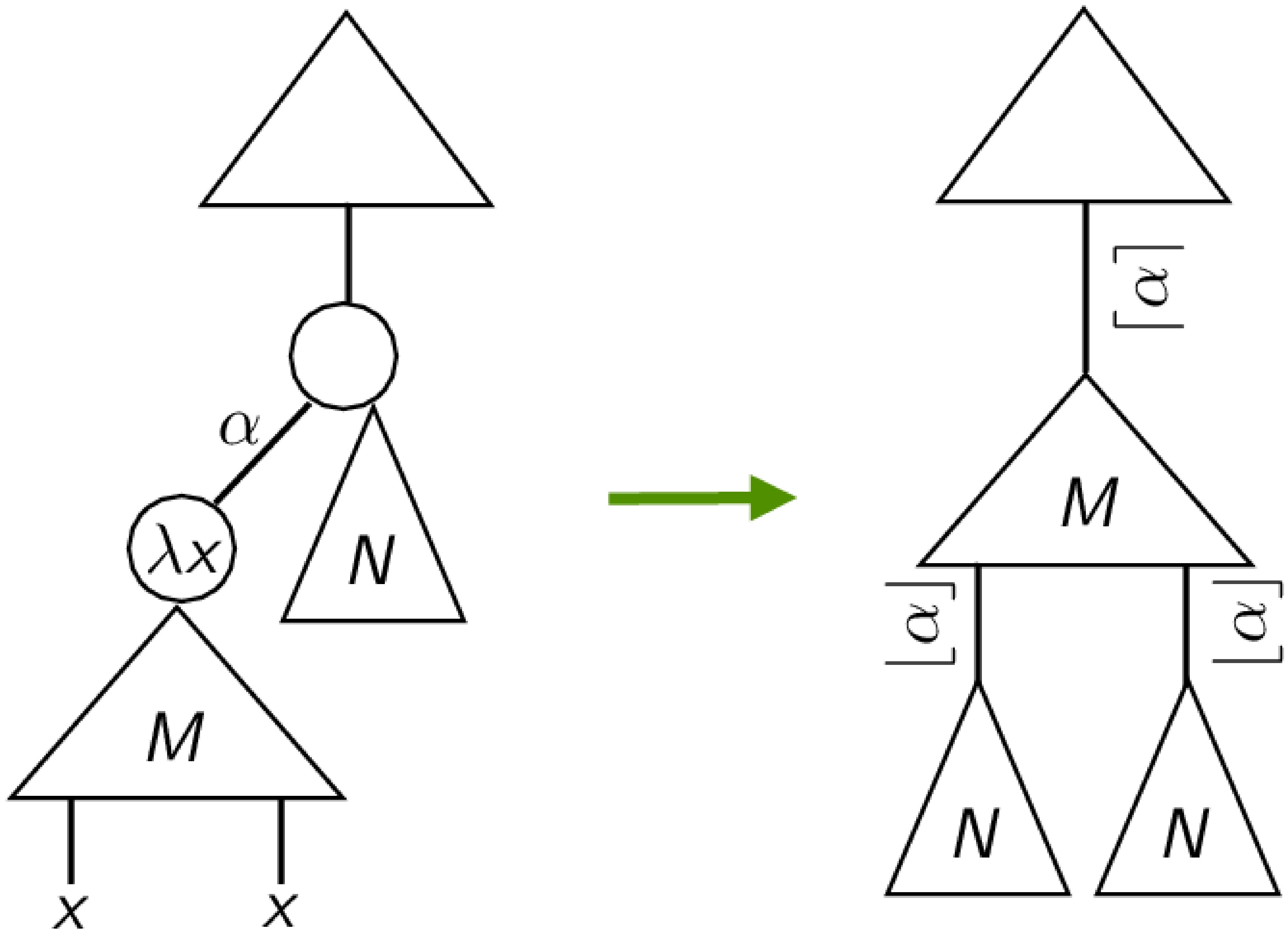
$(M^\alpha)^\beta = M^{\alpha\beta}$ and $(M^\alpha)\{x := N\} = (M\{x := N\})^\alpha$

A labeled lambda-calculus (2/3)



abstract syntax trees of labeled λ-terms

A labeled lambda-calculus (2/3)



A labeled lambda-calculus (3/3)

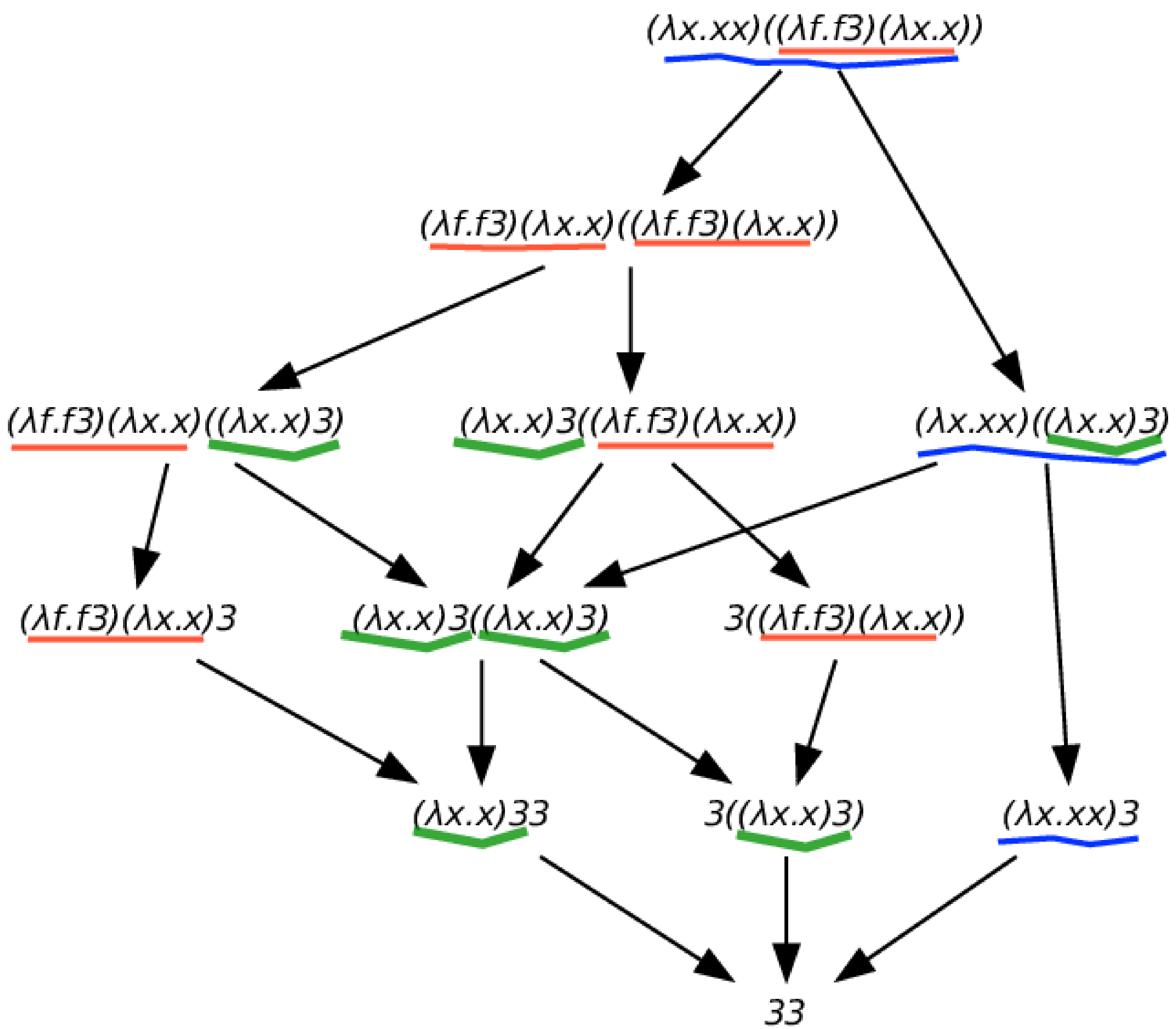
- Labels are strings of atomic labels:

$$\alpha, \beta, \ldots \quad ::= \quad \underbrace{a, b, c, \ldots \mid \lceil\alpha\rceil \mid \lfloor\alpha\rfloor}_{\texttt{atomic labels}} \mid \alpha\beta \mid \epsilon$$

- Labels are strings of atomic labels:

$a, b, c, \ldots$	atomic letters
$\lceil\alpha\rceil, \lfloor\alpha\rfloor, \ldots$	overlined, underlined labels
$\alpha\beta$	compound labels
$\epsilon = \lfloor\epsilon\rfloor = \lceil\epsilon\rceil$	empty label

Example



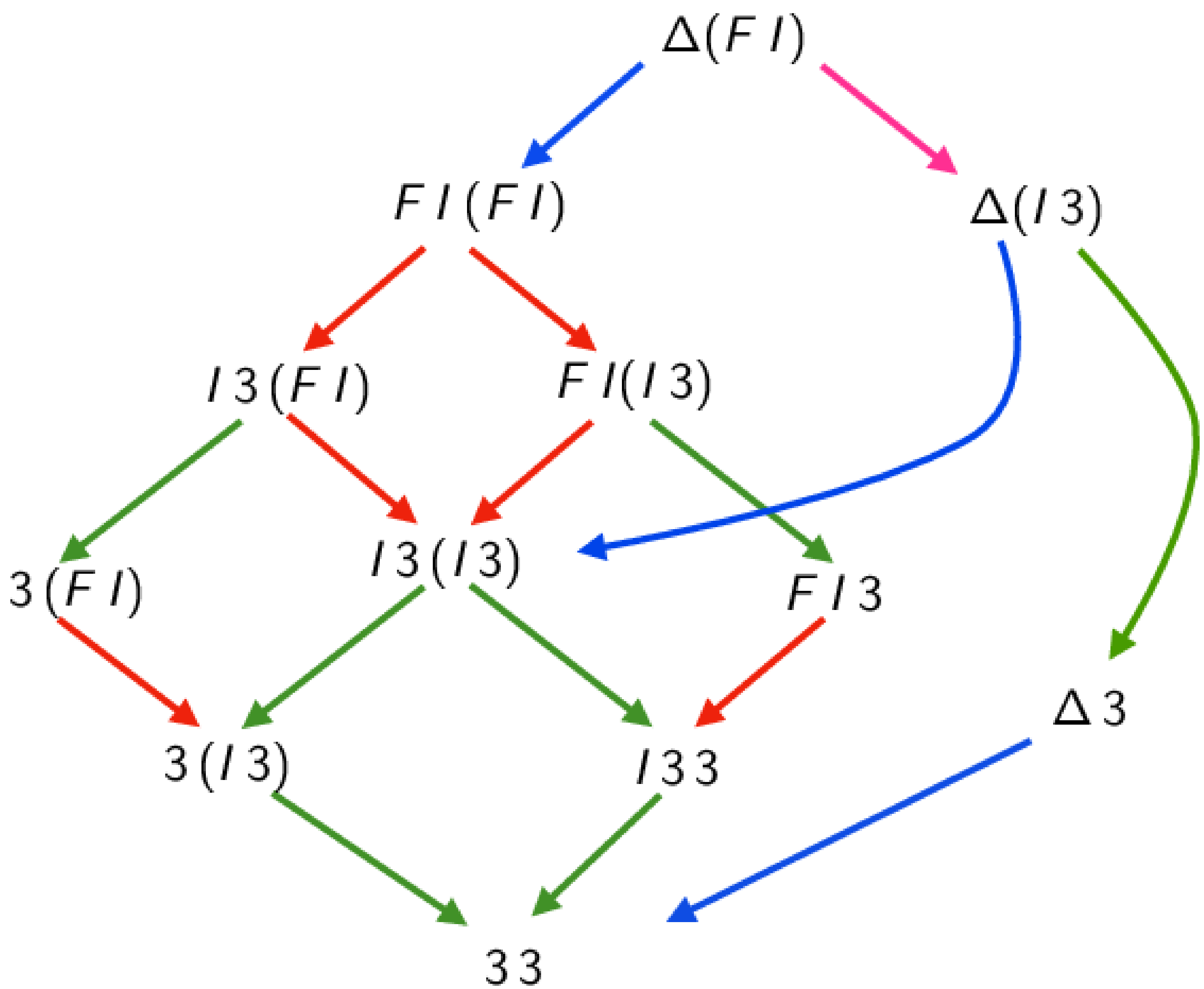
- 3 redex families: **red**, **blue**, **green**.

Example

$\Delta = \lambda x.x\,x$

$F = \lambda f.f\,3$

$I = \lambda x.x$



Example

$\Delta = \lambda x.(x^c\, x^d)^b$

$F = \lambda f.(f^k\, 3^\ell)^j$

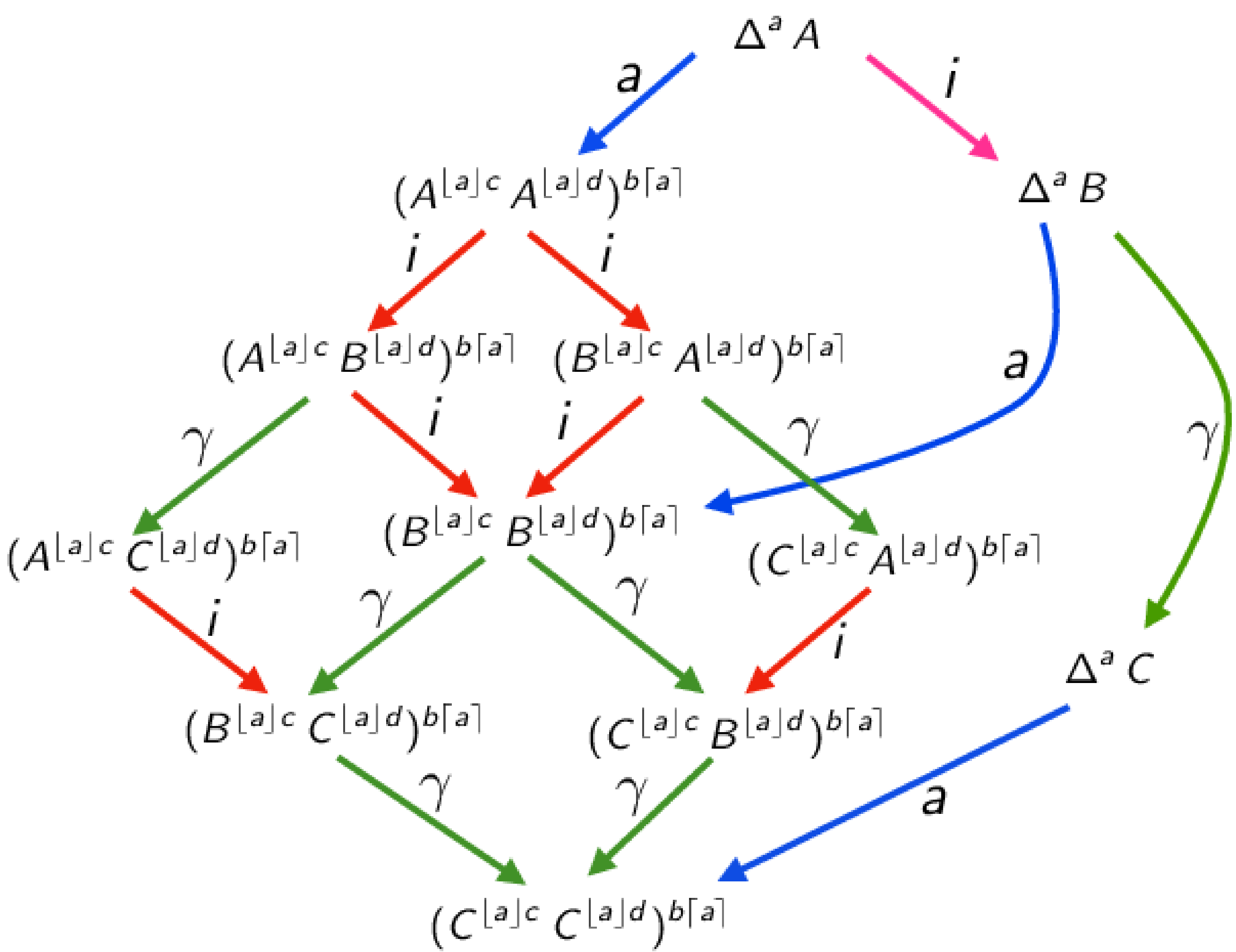
$I = \lambda x.x^v$

$A = (F^i\, I^u)^q$

$B = (I^\gamma\, 3^\ell)^q$

$C = 3^{\ell\lfloor\gamma\rfloor v\lceil\gamma\rceil q}$

$\gamma = u\lfloor i\rfloor k$



3 redexes names: $a, i, \gamma = u\lfloor i\rfloor k$

Example

$$D = \lambda x.(x^c\, x^d)^b$$
$$\Delta = \lambda x.(x^g\, x^h)^f$$

$$\Omega = D^a\, \Delta^e$$
$$\downarrow a$$
$$\Omega_1 = (\Delta^{\gamma_1}\ \Delta^{\delta_1})^{b\lceil a \rceil}$$
$$\downarrow \gamma_1$$
$$\Omega_2 = (\Delta^{\gamma_2}\ \Delta^{\delta_2})^{f\lceil \gamma_1 \rceil b \lceil a \rceil}$$
$$\downarrow \gamma_2$$
$$\Omega_3 = (\Delta^{\gamma_3}\ \Delta^{\delta_3})^{f\lceil \gamma_2 \rceil f\lceil \gamma_1 \rceil b \lceil a \rceil}$$
$$\downarrow \gamma_3$$
$$\Omega_4 = (\Delta^{\gamma_4}\ \Delta^{\delta_4})^{f\lceil \gamma_3 \rceil f\lceil \gamma_2 \rceil f\lceil \gamma_1 \rceil b \lceil a \rceil}$$
$$\downarrow \gamma_4$$

$$\gamma_1 = e\lfloor a \rfloor c$$
$$\gamma_2 = \delta_1\lfloor \gamma_1 \rfloor g$$
$$\gamma_3 = \delta_2\lfloor \gamma_2 \rfloor g$$
$$\gamma_4 = \delta_3\lfloor \gamma_3 \rfloor g$$

$$\delta_1 = e\lfloor a \rfloor d$$
$$\delta_2 = \delta_1\lfloor \gamma_1 \rfloor h$$
$$\delta_3 = \delta_2\lfloor \gamma_2 \rfloor h$$
$$\delta_4 = \delta_2\lfloor \gamma_2 \rfloor h$$

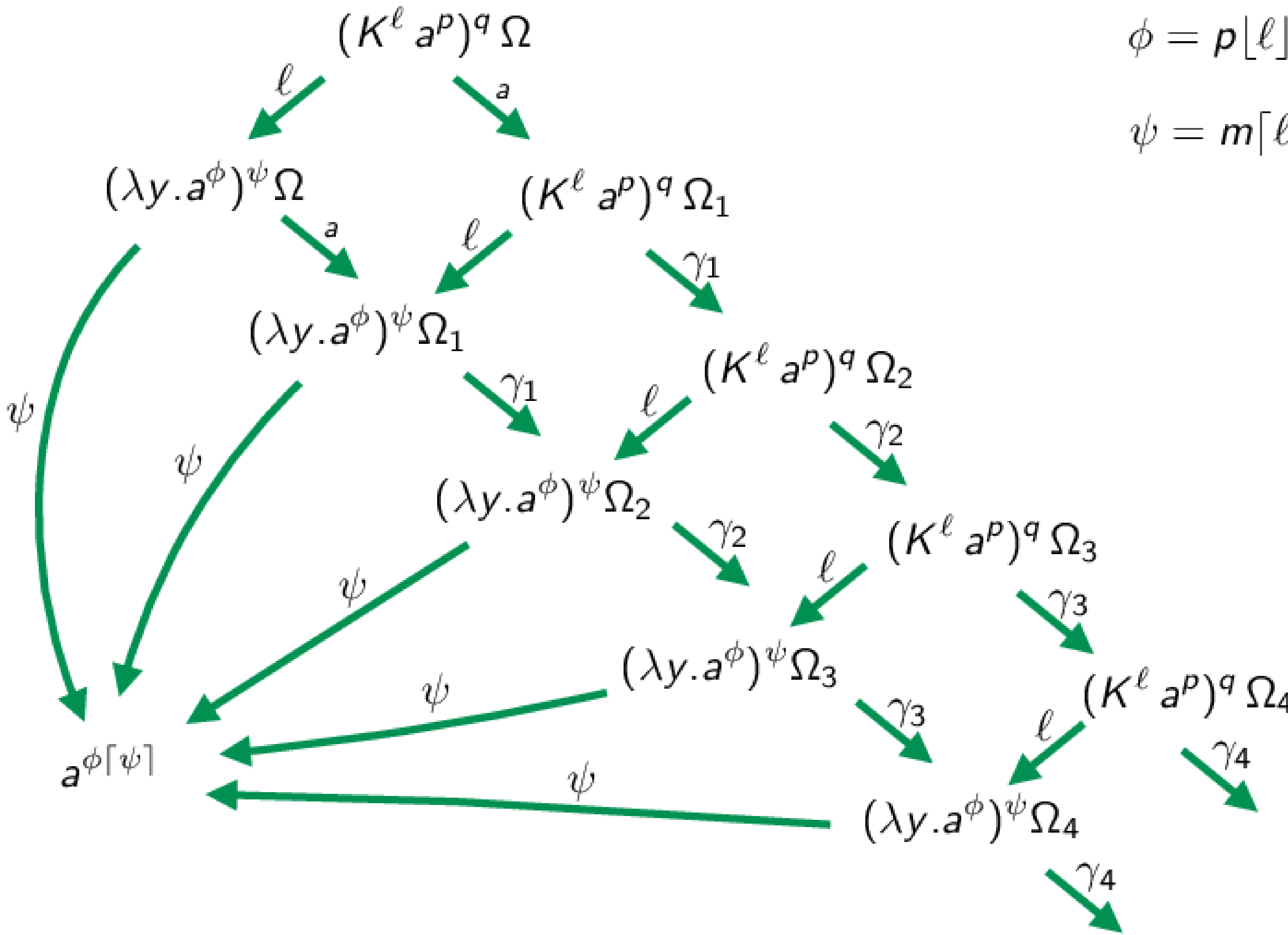
redexes names: $a, \gamma_1, \gamma_2, \gamma_3, \ldots$

Example

$K = \lambda x.(\lambda y.x^n)^m$

$\phi = p\lfloor \ell \rfloor n$

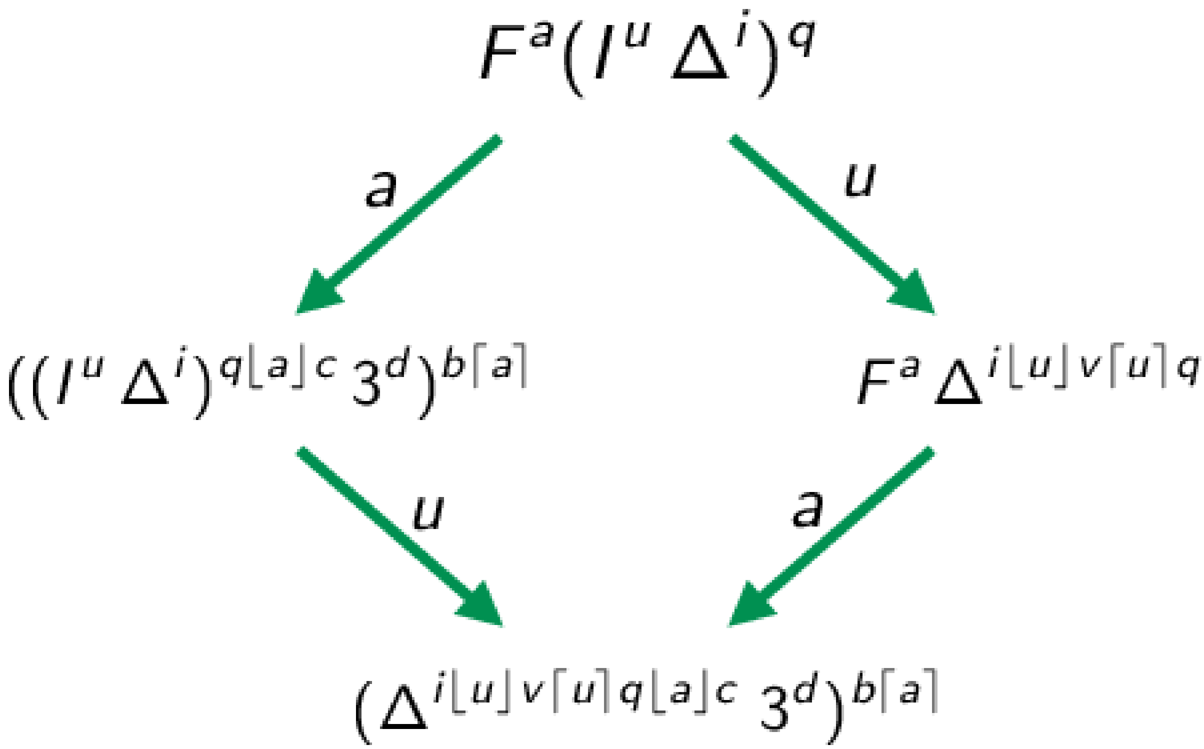
$\psi = m\lceil \ell \rceil q$



redexes names: $\ell, \psi, a, \gamma_1, \gamma_2, \gamma_3, \ldots$

Example

$$F = \lambda f.(f^c\, 3^d)^b$$
$$I = \lambda x.x^v$$
$$\Delta = \lambda x.(x^k\, x^\ell)^j$$



2 independent redexes a and u creates the new one $i\lfloor u\rfloor v\lceil u\rceil q\lfloor a\rfloor c$

Empirical facts (bis)

- deterministic result when it exists — Church-Rosser
- multiple reduction strategies
- terminating strategy ?
- efficient reduction strategy ? — optimal reduction
- worst reduction strategy ?
- when all reductions are finite ? — strong normalisation
- when finite, the reduction graph has a lattice structure ? — YES!

Permutation equivalence (1/7)

- **Proposition** [residuals of labeled redexes]
  $S \in R/\rho$ implies $\text{name}(R) = \text{name}(S)$

- **Definition** [created redexes] Let $\rho : M \xrightarrow{*} N$
  we say that ρ **creates** R in M when $\nexists R', R \in R'/\rho$.

- **Proposition** [created labeled redexes]
  If S creates R, then $\text{name}(S)$ is strictly contained in $\text{name}(R)$.

Permutation equivalence (2/7)

Proof (cont'd) Created redexes contains names of creator

$$\underbrace{(\lambda x.\cdots(x^{\beta}\, N)\cdots)^{\alpha}}_{\alpha}\,(\lambda y.M)^{\gamma} \rightarrow \cdots\underbrace{((\lambda y.M)^{\gamma\lfloor\alpha\rfloor\beta}\, N')}_{\gamma\lfloor\alpha\rfloor\beta}\cdots$$

α — creates → $\gamma\lfloor\alpha\rfloor\beta$

$$\underbrace{((\lambda x.(\lambda y.M)^{\gamma})^{\alpha} N)^{\beta}}_{\alpha}\, P \rightarrow \underbrace{(\lambda y.M')^{\gamma\lceil\alpha\rceil\beta}\, P}_{\gamma\lceil\alpha\rceil\beta}$$

α — creates → $\gamma\lceil\alpha\rceil\beta$

$$\underbrace{((\lambda x.\, x^{\gamma})^{\alpha}\,(\lambda y.M)^{\delta})^{\beta}}_{\alpha}\, N \rightarrow \underbrace{(\lambda y.M)^{\delta\lfloor\alpha\rfloor\gamma\lceil\alpha\rceil\beta}\, N}_{\delta\lfloor\alpha\rfloor\gamma\lceil\alpha\rceil\beta}$$

α — creates → $\delta\lfloor\alpha\rfloor\gamma\lceil\alpha\rceil\beta$

Permutation equivalence (3/7)

- **Labeled laws** $M^\alpha\{x := N\} = (M\{x := N\})^\alpha \qquad (M^\alpha)^\beta = M^{\alpha\beta}$

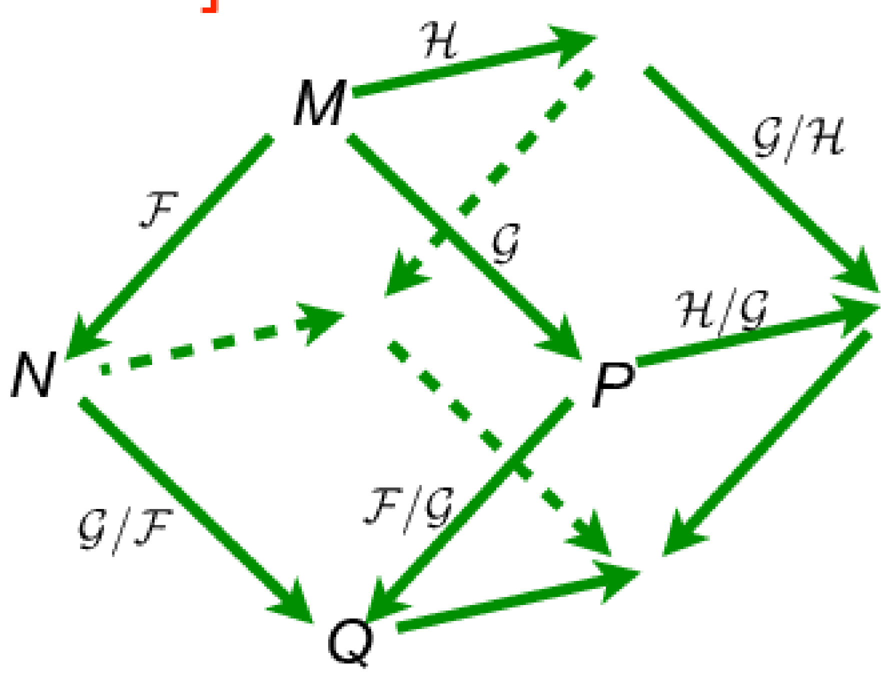
  If $M \longrightarrow N$, then $M^\alpha \longrightarrow N^\alpha$

- **Labeled parallel moves lemma+** [74]

  If $M \xrightarrow{\mathcal{F}} N$ and $M \xrightarrow{\mathcal{G}} P$, then $N \xrightarrow{\mathcal{G}/\mathcal{F}} Q$ and $P \xrightarrow{\mathcal{F}/\mathcal{G}} Q$ for some Q.

- **Parallel moves lemma++** [The Cube Lemma]

  still holds.



Permutation equivalence (4/7)

- Labels do not break Church-Rosser, nor residuals
- Labels refine λ-calculus:
  - any unlabeled reduction can be performed in the labeled calculus
  - but two cofinal unlabeled reductions may no longer be cofinal

  Take $I(I3)$ with $I = \lambda x.x$.

$$I^b(J^e 3^g)^d$$

$$I = \lambda x.x^c \qquad J = \lambda x.x^f$$

$$(J^e\, 3^g)^{d\lfloor b\rfloor c\lceil b\rceil} \qquad\qquad I^b 3^{g\lfloor e\rfloor f\lceil e\rceil d}$$

Permutation equivalence (5/7)

- **Definition** [pure labeled calculus]

  Pure labeled terms are labeled terms where all subterms have non empty labels.

- **Theorem** [labeled permutation equivalence, 76]

  Let ρ and σ be coinitial pure labeled reductions.
  Then $\rho \simeq \sigma$ iff ρ and σ are labeled cofinal.

  Proof Let $\rho \simeq \sigma$. Then obvious because of labeled parallel moves lemma.
  Conversely, we apply standardization thm and following lemma.

Permutation equivalence (6/7)

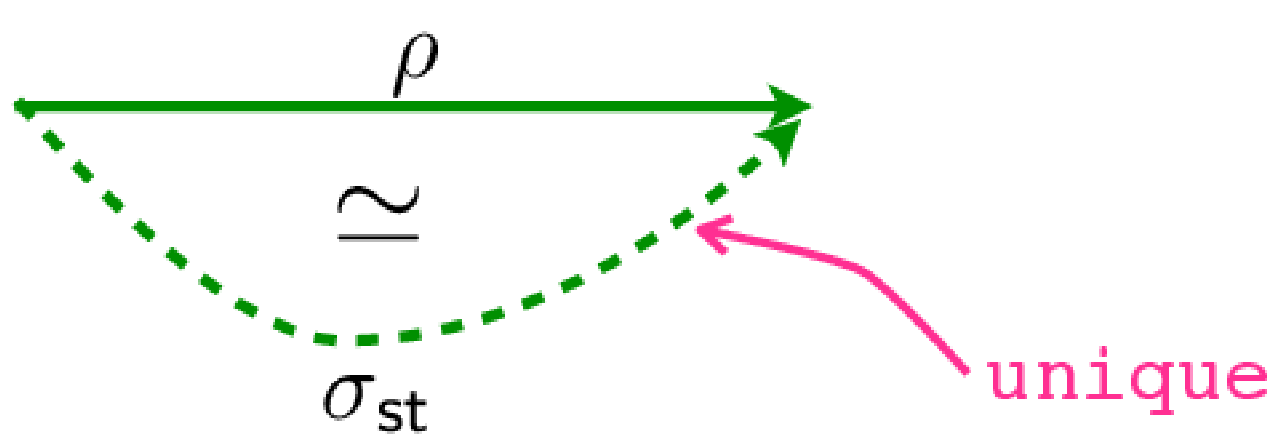
- **Definition:** The following reduction is **standard**

$$\rho : M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \cdots \xrightarrow{R_n} M_n = N$$

iff for all i and j, $i < j$, then R_j is not residual along ρ of some R'_j to the left of R_i in M_{i-1}.

- **Standardization** [Curry 50] Let $M \xrightarrow{\star} N$. Then $M \xrightarrow[\text{st}]{\star} N$.

- **Labeled standardization** $\forall \rho,\ \exists! \, \sigma_{\text{st}},\ \rho \simeq \sigma_{\text{st}}$



Permutation equivalence (7/7)

- **Notation** [prefix ordering] $\rho \sqsubseteq \sigma$ for $\exists \tau. \rho\tau \simeq \sigma$

- **Corollary** [labeled prefix ordering]

  Let $\rho : M \xrightarrow{*} N$ and $\sigma : M \xrightarrow{*} P$ be coinitial pure labeled reductions.
  Then $\rho \sqsubseteq \sigma$ iff $N \xrightarrow{*} P$.

- **Corollary** [lattice of labeled reductions]

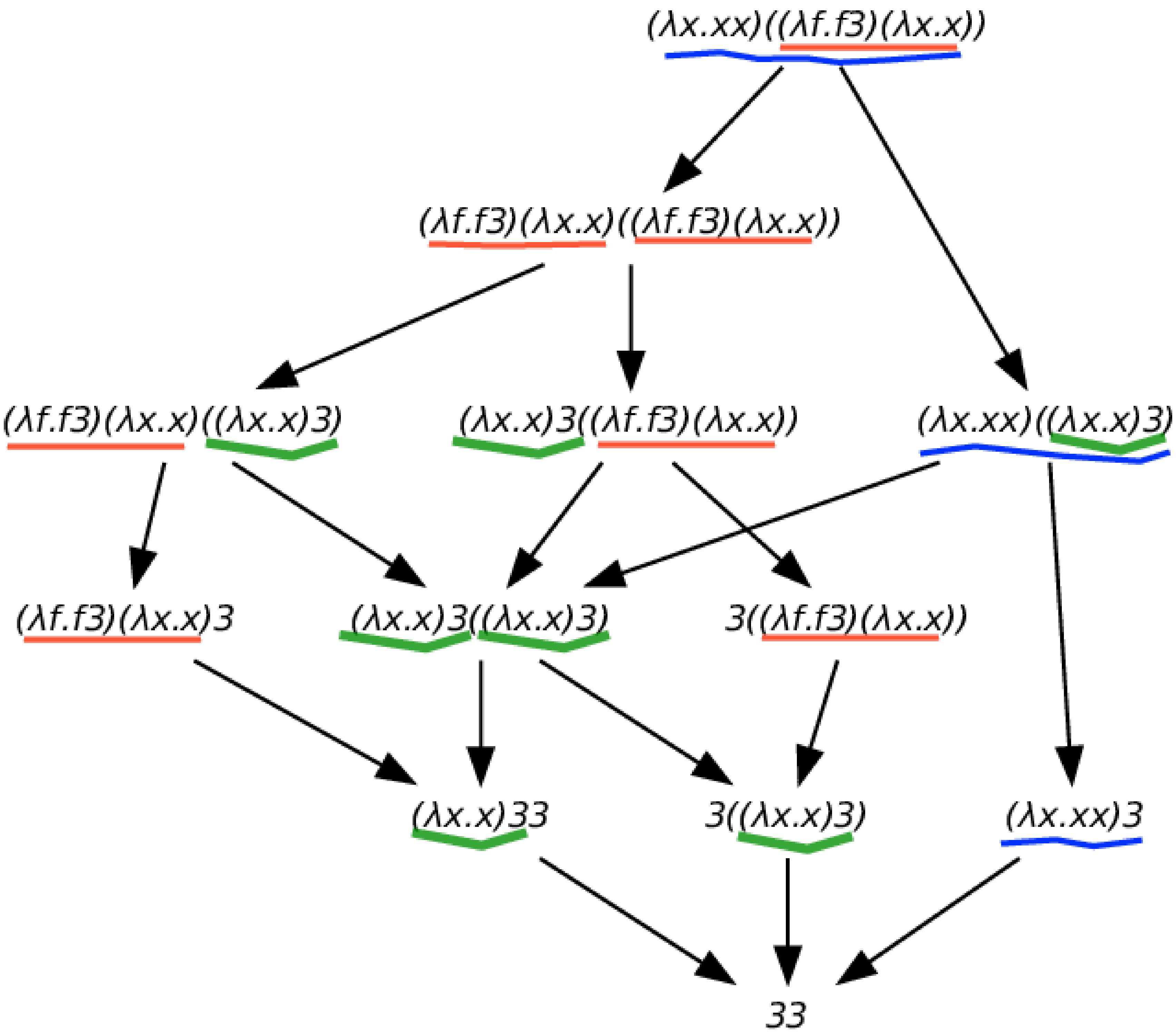
  Labeled reduction graphs are upwards semi lattices for any pure labeling.

In other terms, reductions up-to permutation equivalence is a push-out category.

Exercise Try on $(\lambda x.x)((\lambda y.(\lambda x.x)a)b)$ or $(\lambda x.xx)(\lambda x.xx)$

Redex families

Example



- 3 redex families: **red**, **blue**, **green**.

hRedexes

- **Definition** `[hRedex]`

  hRedex is a pair $\langle \rho, R \rangle$ where R is a redex in final term of ρ

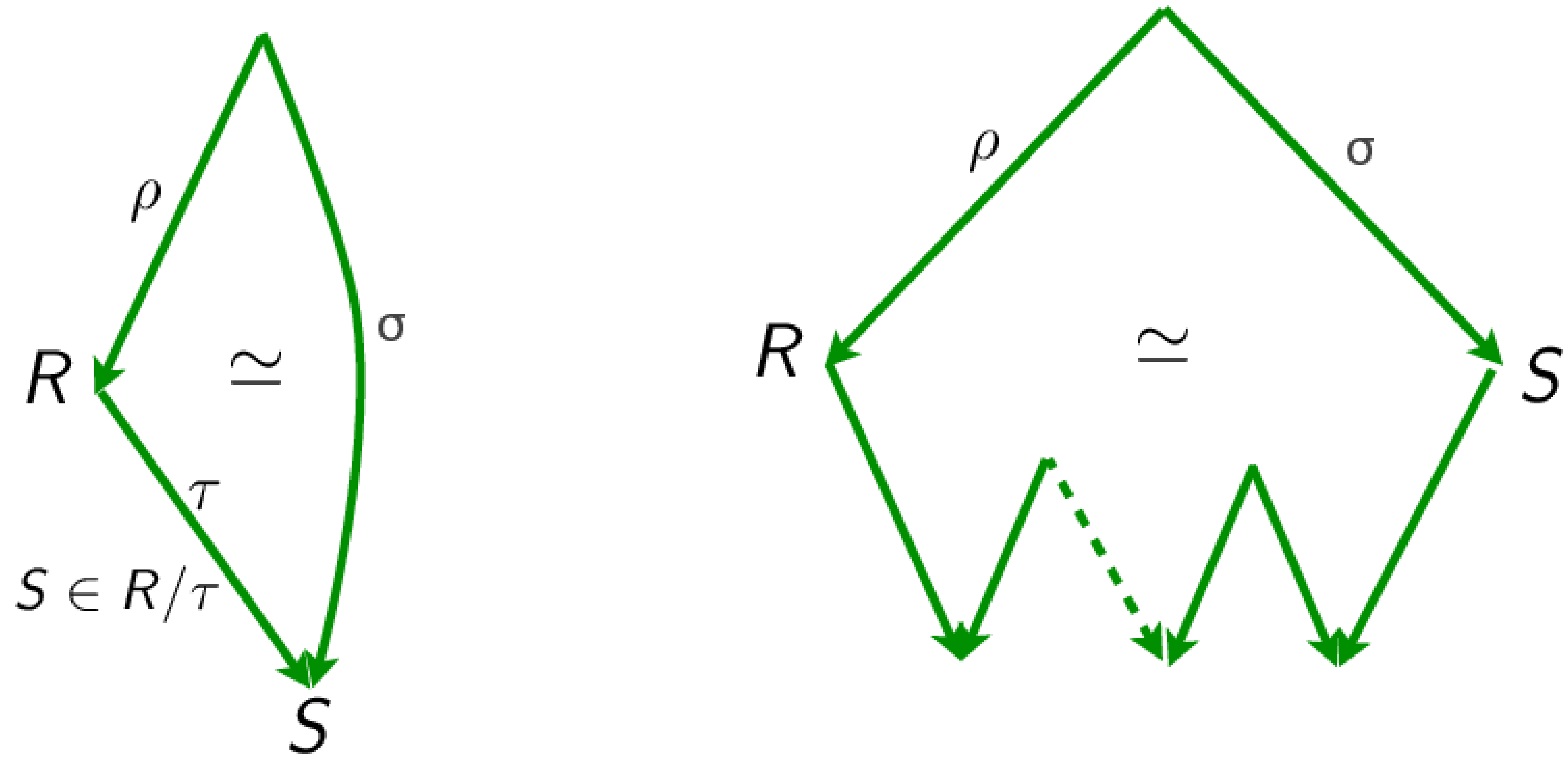
- **Definition** `[copies of hRedex]`

  $\langle \rho, R \rangle \leq \langle \sigma, S \rangle$ when $\exists \tau.\, \rho\tau \simeq \sigma$ and $S \in R/\tau$

- **Definition** `[families of hRedexes]`

  $\langle \rho, R \rangle \sim \langle \sigma, S \rangle$ for reflexive, symmetric, transitive closure of the copy relation.

Labels and history (1/4)



$\langle \rho, R \rangle \leq \langle \sigma, S \rangle$ $\langle \rho, R \rangle \sim \langle \sigma, S \rangle$

$\text{name}(R) = \text{name}(S)$

Labels and history (2/4)

- **Proposition** [same history → same name]

  In the labeled λ-calculus, for any labeling, we have:

  $\langle \rho, R \rangle \sim \langle \sigma, S \rangle$ implies $\mathsf{name}(R) = \mathsf{name}(S)$

- The opposite direction is clearly not true for any labeling

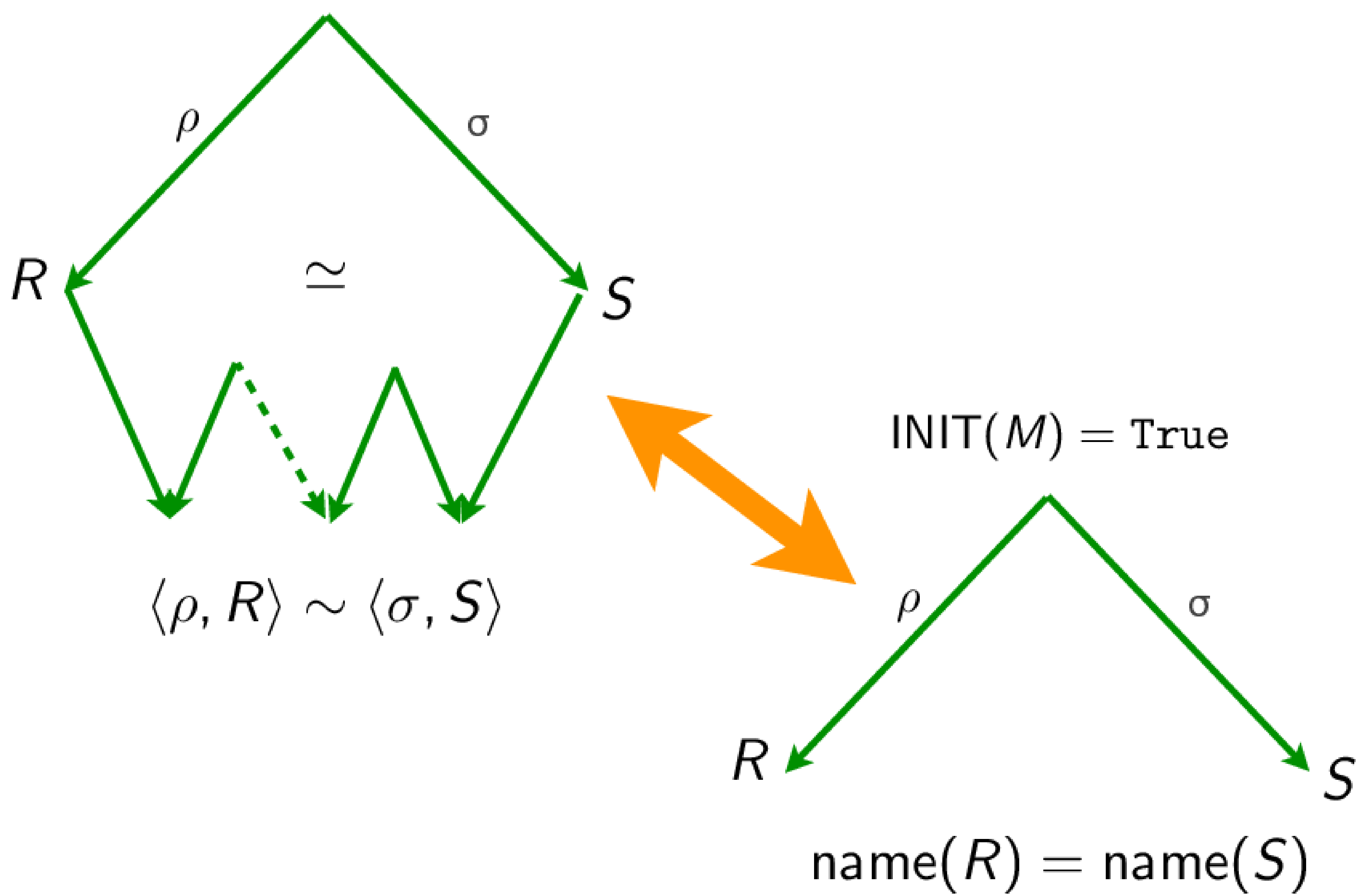
  (For instance, take all labels equal)

- But it is true when all labels are distinct atomic letters in the initial term.

- **Definition** [all labels distinct letters]

  $\mathsf{INIT}(M) = \mathtt{True}$ when all labels in M are distinct letters.

Labels and history (3/4)



Labels and history (4/4)

- **Theorem** `[same history = same name, 76]`

  When $\mathsf{INIT}(M)$ and reductions ρ and σ start from M :

  $$\langle \rho, R \rangle \sim \langle \sigma, S \rangle \quad \text{iff} \quad \mathsf{name}(R) = \mathsf{name}(S)$$

- **Corollary** `[decidability of family relation]`

  The family relation is decidable (although complexity is proportional to length of standard reduction).

Finite developments

Parallel steps revisited (1/3)

- parallel steps were defined with inside-out strategy
  `[à la Martin-Löf]`

  Can we take any order as a reduction strategy ?

- **Definition** A **reduction relative** to a set $\mathcal{F}$ of redexes in M is any reduction contracting only residuals of $\mathcal{F}$.
  A **development** of $\mathcal{F}$ is any maximal relative reduction of $\mathcal{F}$.

Parallel steps revisited (2/3)

- **Theorem** `[Finite Developments, Curry, 50]`

  Let $\mathcal{F}$ be set of redexes in M.

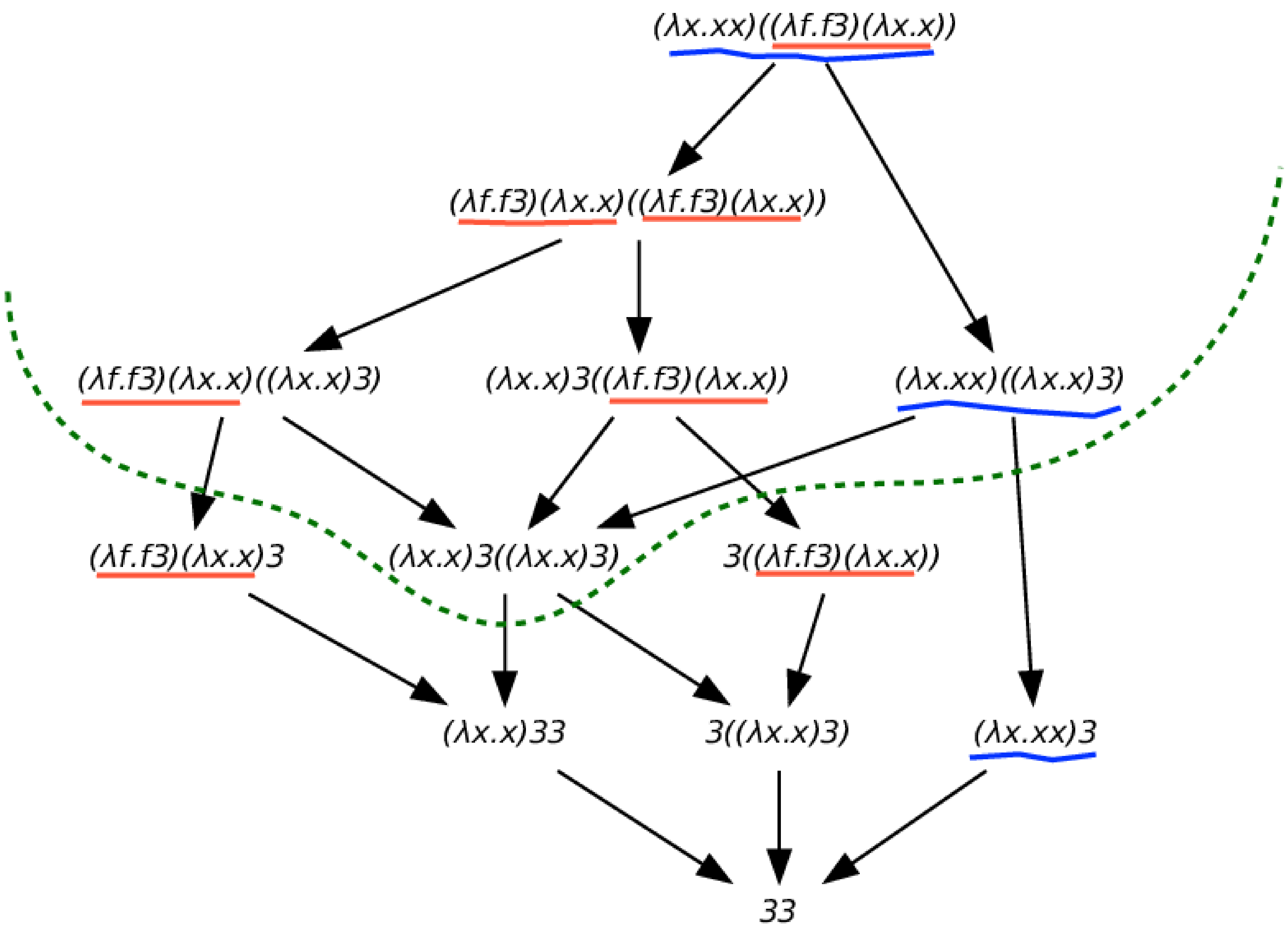
  (1) there are no infinite relative reductions of $\mathcal{F}$,

  (2) they all finish on same term N

  (3) Let R be redex in M. Residuals of R by all finite developments of $\mathcal{F}$ are the same.

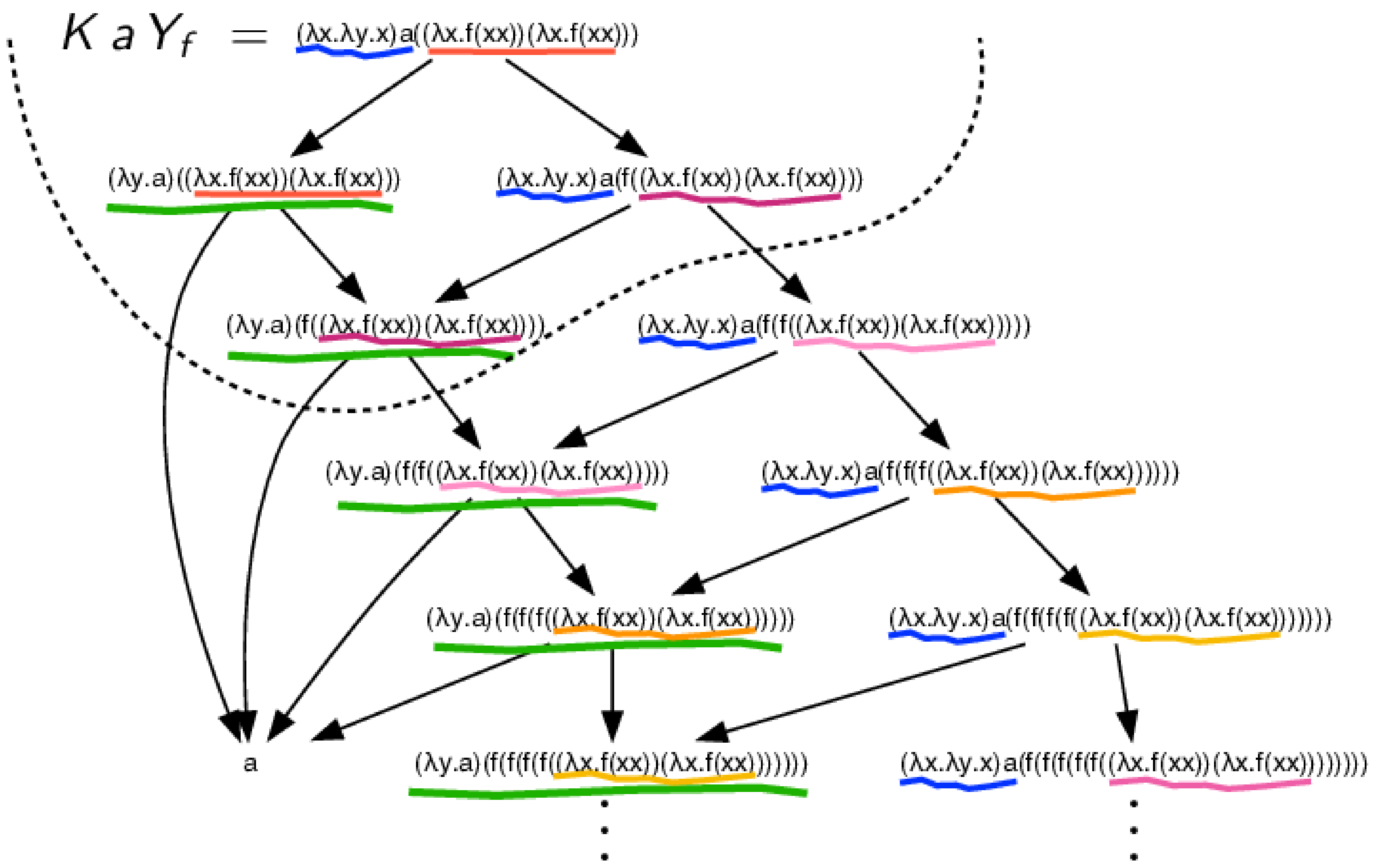
- Similar to the parallel moves lemma, but we considered a particular inside-out reduction strategy.

Example



developments of **red**, **blue**.

Example



developments of **red**, **blue**.

Parallel steps revisited (3/3)

- **Notation** `[parallel reduction steps]`
  Let $\mathcal{F}$ be set of redexes in M. We write $M \xrightarrow{\mathcal{F}} N$
  if a development of $\mathcal{F}$ connects M to N.

- This notation is consistent with previous definition
  (since inside-out parallel step is a particular development)

- Corollaries of FD thm are also parallel moves + cube lemmas

Finite and infinite reductions (1/3)

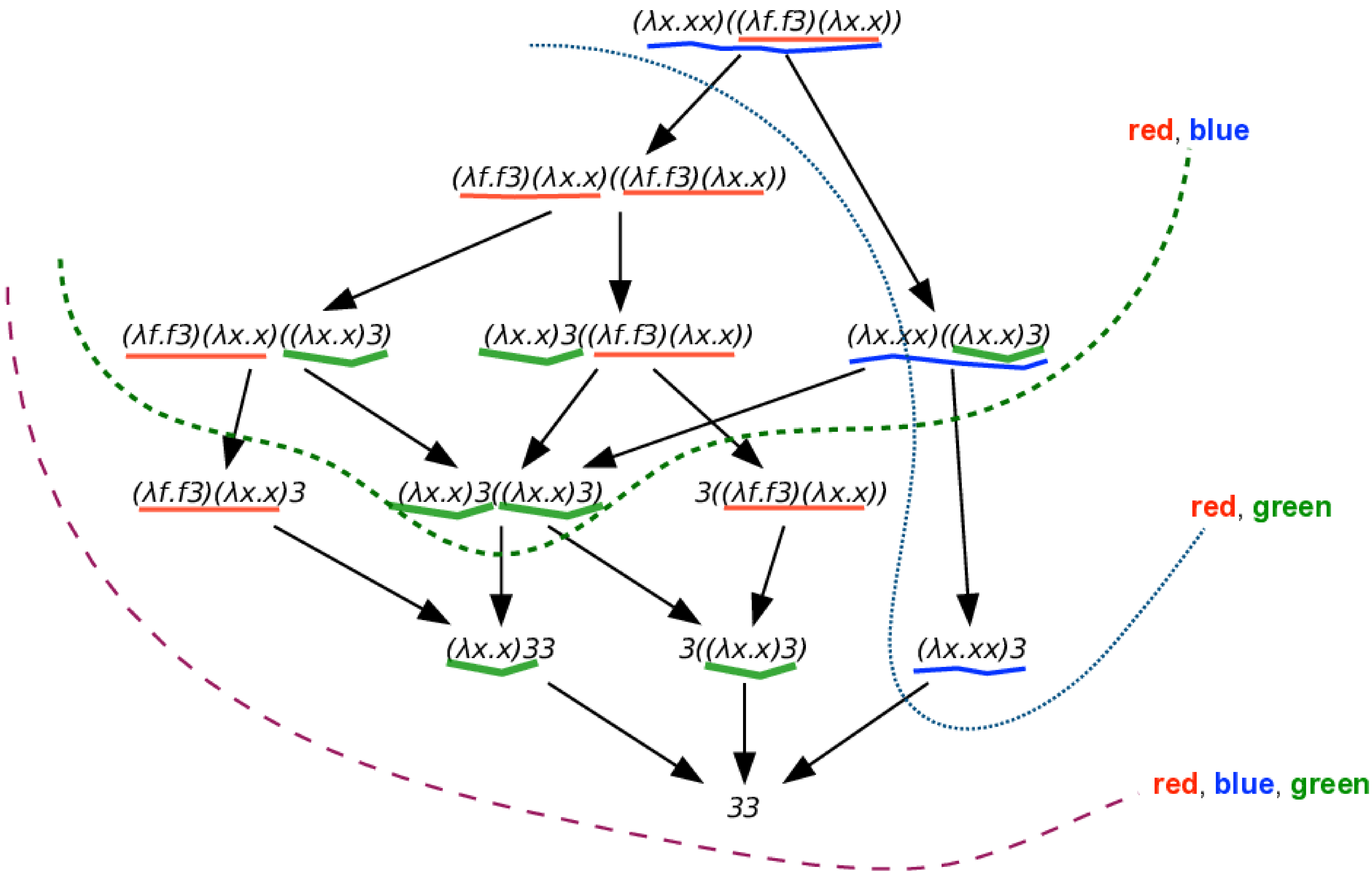
- **Definition** A **reduction relative** to a set $\mathcal{F}$ of redex families is any reduction contracting redexes in families of $\mathcal{F}$.

  A **development** of $\mathcal{F}$ is any maximal relative reduction.

- **Theorem** [Generalized Finite Developments+, 76]
  Let $\mathcal{F}$ be a finite set of redex families.

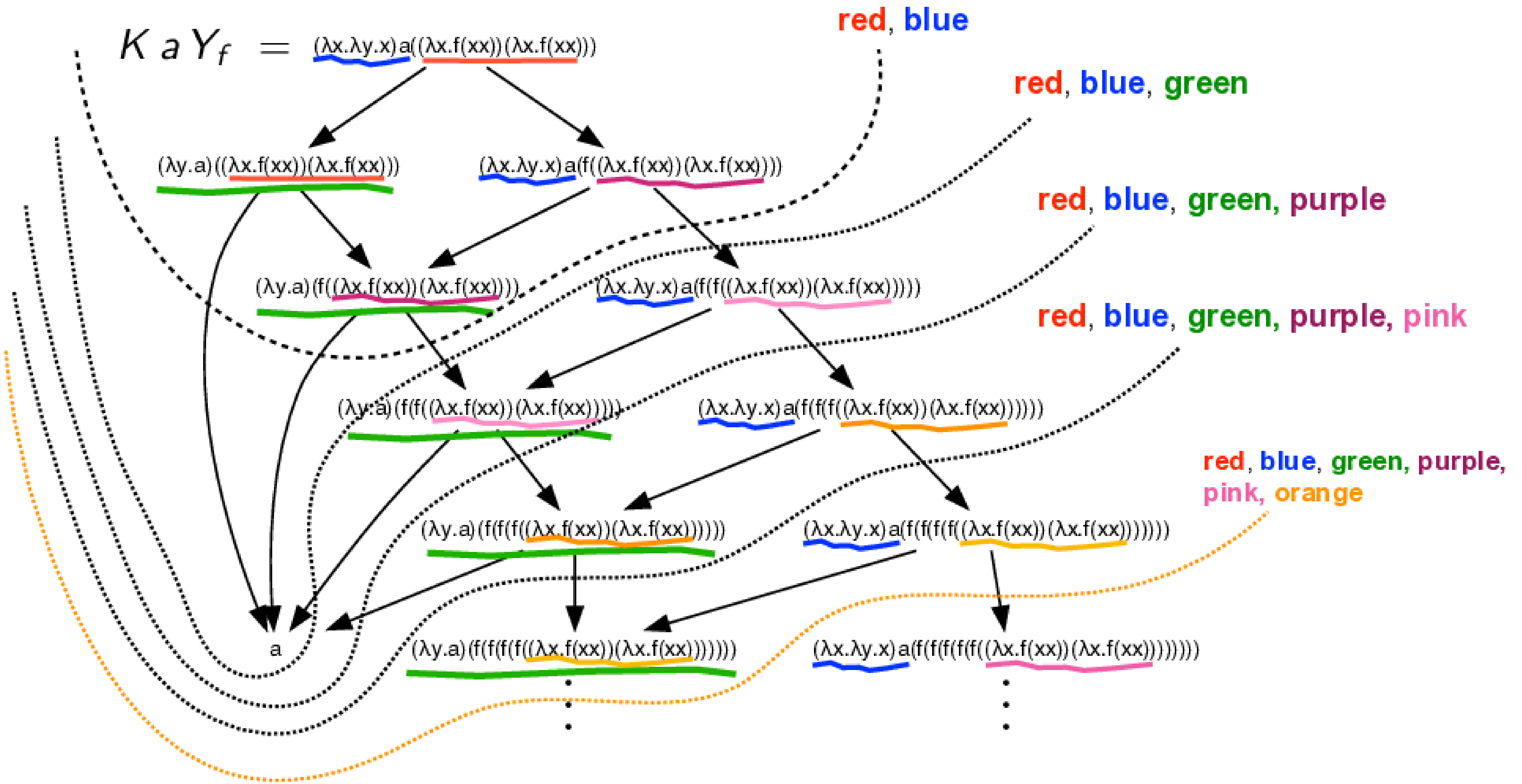
  (1) there are no infinite reductions relative to $\mathcal{F}$,
  (2) they all finish on same term N
  (3) All developments are equivalent by permutations.

Example



- 3 redex families: red, blue, green.

Example



developments of families.

Finite and infinite reductions (2/3)

- **Corollary** An **infinite reduction** contracts an **infinite set of redex families.**

- **Corollary** Any term generating a finite number of redex families strongly normalizes

finite number of redex families

strong normalization

Proof of the GFD thm

Bound on heights of labels

- **Definition** The height of a label is its nesting of underlines and overlines

$$h(a) = 0$$
$$h(\lceil \alpha \rceil) = h(\lfloor \alpha \rfloor) = 1 + h(\alpha)$$
$$h(\alpha\beta) = \max\{\alpha, \beta\}$$

- **Fact** Let $\mathcal{F}$ be a finite set of redex families, then there is an upper bound $H(\mathcal{F})$ on labels of subterms in reductions relative to $\mathcal{F}$.

When initial term is labeled with atomic letters, we have

$$H(\mathcal{F}) = \max\{h(\alpha) \mid \alpha \in \mathcal{F}\}$$

Proof of finite developments

- **Notation** $\tau(M^\alpha) = \alpha$ when M has an empty external label

- **Lemma 1** Let $M \xrightarrow{\star} M'$, then $h(\tau(M)) \leq h(\tau(M'))$

- **Lemma 2** Let $(\cdots((M\, M_1)^{\beta_1}\, M_2)^{\beta_2} \cdots M_n)^{\beta_n} \xrightarrow{\star} (\lambda x.N)^\alpha$
  Then $h(\tau(M)) \leq h(\alpha)$

- **Lemma 3** [Barendregt] Let $M\{x := N\} \xrightarrow{\star} (\lambda y.P)^\alpha$
  There are 2 cases:

  $M \xrightarrow{\star} (\lambda y.M')^\alpha$ and $M'\{x := N\} \xrightarrow{\star} P$

  $M \xrightarrow{\star} M' = (\cdots((x^\beta\, M_1)^{\beta_1}\, M_2)^{\beta_2} \cdots M_n)^{\beta_n}$ and $M'\{x := N\} \xrightarrow{\star} (\lambda y.P)^\alpha$

Proof of finite developments

- **Notation** Let $\mathcal{SN}_{\mathcal{F}}$ be the set of strongly normalizable terms w.r.t. reductions relative to $\mathcal{F}$.

- **Lemma** `[subst]` Let $\mathcal{F}$ be a finite set of redex families.
  $M, N \in \mathcal{SN}_{\mathcal{F}}$ implies $M\{x := N\} \in \mathcal{SN}_{\mathcal{F}}$

  Proof `[van Daalen]` by induction on $\langle H(\mathcal{F}) - h(\tau(N)), \mathrm{depth}(M), \|M\| \rangle$

- **Theorem** `GFD` Let $\mathcal{F}$ be a finite set of redex families.
  Then $M \in \mathcal{SN}_{\mathcal{F}}$ for all M.

  Proof by induction on $\|M\|$

Strong normalization

1st-order typed λ-calculus (1/2)

Residuals of redexes keep their types (of names)

Created redexes have lower types

$$\underbrace{(\lambda x.\cdots xN\cdots)}_{s \to t}\underbrace{(\lambda y.M)}_{s} \to \cdots\underbrace{(\lambda y.M)N'}_{s}\cdots$$

creates

Finite number of redexes families

$$\underbrace{(\lambda x.\underbrace{\lambda y.M}_{t})}_{s \to t}NP \to \underbrace{(\lambda y.M')P}_{t}$$

creates

$$\underbrace{(\lambda x.x)}_{s \to s}\underbrace{(\lambda y.M)}_{s}N \to \underbrace{(\lambda y.M)N}_{s}$$

creates

Strong normalization

1st-order typed λ-calculus (2/2)

- **Typed λ-calculus** as a specific labeled calculus

$$s,\ t \ ::= \ \mathbb{N}, \mathbb{B} \ \mid \ s \to t$$

Decorate subterms with their types

$$(\lambda f.\, (f^{\mathbb{N}\to\mathbb{N}} 3^{\mathbb{N}})^{\mathbb{N}})^{(\mathbb{N}\to\mathbb{N})\to\mathbb{N}} I^{\mathbb{N}\to\mathbb{N}}$$

$$\searrow$$

$$(I^{\mathbb{N}\to\mathbb{N}} 3^{\mathbb{N}})^{\mathbb{N}} \longrightarrow 3^{\mathbb{N}}$$

Apply following rules to labeled λ-calculus

$$\lceil s \to t \rceil = t$$

$$\lfloor s \to t \rfloor = s$$

$$s\, t = s$$

Scott D-infinity model (1/2)

- Another labeled λ-calculus was considered to study Scott D-infinity model [Hyland-Wadsworth, 74]

- D-infinity projection functions on each subterm (n is any integer):

$$M, N, \ldots ::= x^n \mid (MN)^n \mid (\lambda x.M)^n$$

- Conversion rule is:

$$((\lambda x.M)^{n+1} N)^p \longrightarrow M\{x := N_{[n]}\}_{[n][p]}$$

$n+1$ is **degree** of redex

$$U_{[m][n]} = U_{[p]} \quad \text{where} \quad p = \min\{m, n\}$$

$$x^n \{x := M\} = M_{[n]}$$

Scott D-infinity model (2/2)

- **Proposition** Hyland-Wadsworth calculus is derivable from labeled calculus by simple homomorphism on labels.

Proof Assign an integer to any atomic letter and take:

$$\mathsf{h}(\alpha\beta) = \min\{\mathsf{h}(\alpha), \mathsf{h}(\beta)\}$$

$$\mathsf{h}(\lceil\alpha\rceil) = \mathsf{h}(\lfloor\alpha\rfloor) = \mathsf{h}(\alpha) - 1$$

- Redex degrees are bounded by maximum of labels in initial term. therefore a finite number of redex families

- **Proposition** Hyland-Wadsworth calculus strongly normalizes.

Conclusion

- **many** proofs of strong normalization for various calculi

- these proofs look often **magic**

- but intuition is

  GFD theorem $\equiv$ strong normalization

- more properties on redex families + labeled calculus
  - standardization theorem
  - completeness of inside-out reductions
  - compactness of main theorems about syntax
  - stability of redexes and sequentiality
  - optimal reductions and relation to Girard's GOI

Finite Developments in the λ-calculus

Part 3

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ISR 2021

Madrid

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http://jeanjacqueslevy.net/talks/21isr

Labeled lambda-calculus

Exercise 1 Show that residuals of redexes keep same names by case inspection on occurrences of redexes.

Exercise 2 Show that $M \longrightarrow N$ implies $M^\alpha \longrightarrow N^\alpha$

Exercise 3 Show the parallel moves lemma (with Martin-Löf way)

If $M \xrightarrow{\mathcal{F}} N$ and $M \xrightarrow{\mathcal{G}} P$, then $N \xrightarrow{\mathcal{G}/\mathcal{F}} Q$ and $P \xrightarrow{\mathcal{F}/\mathcal{G}} Q$ for some Q.

Exercise 4 Label Y_f, draw its reduction graph and show redexes families when $Y_f = (\lambda x.f(xx))(\lambda x.f(xx))$

Exercise 5 Same with KaY_f

Inside-out reductions

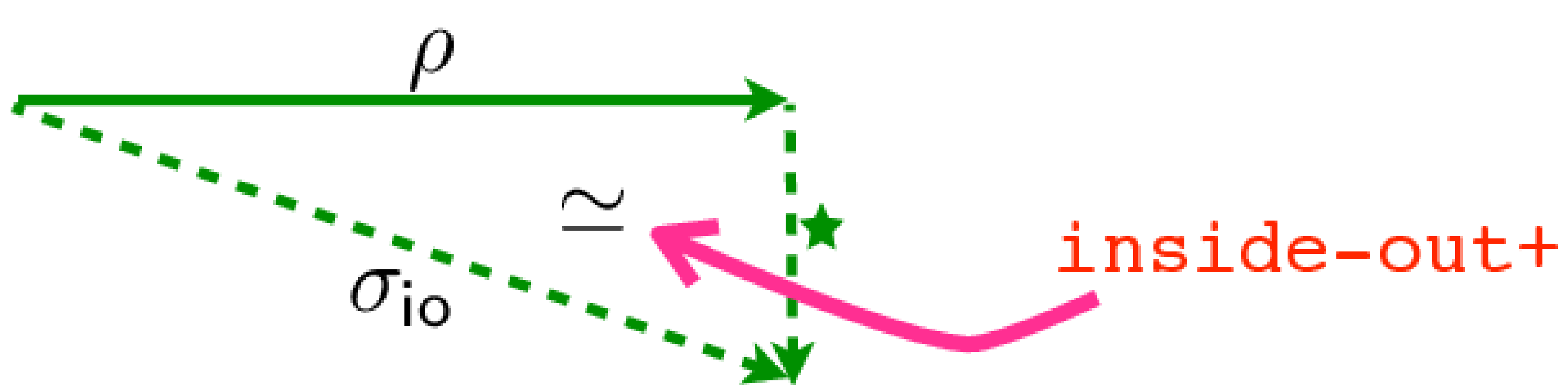
- **Definition:** The following reduction is **inside-out**

$$\rho : M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \cdots \xrightarrow{R_n} M_n = N$$

iff for all i and j, $i < j$, then R_j is not residual along ρ of some R'_j inside R_i in M_{i-1}.

- **Theorem** [Inside-out completeness, 74]

Let $M \xrightarrow{\star} N$. Then $M \xrightarrow[\text{io}]{\star} P$ and $N \xrightarrow{\star} P$ for some P.

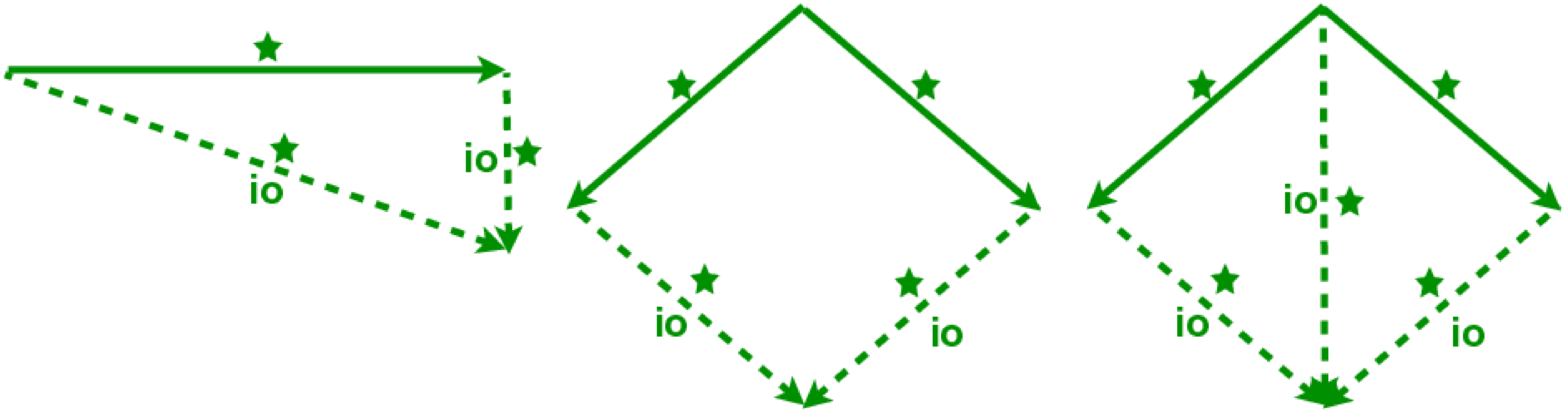


Exercises

Exercise 6 Prove inside-out completeness

Hint: use Finite Development theorem.

Exercise 7 Prove the following diagrams

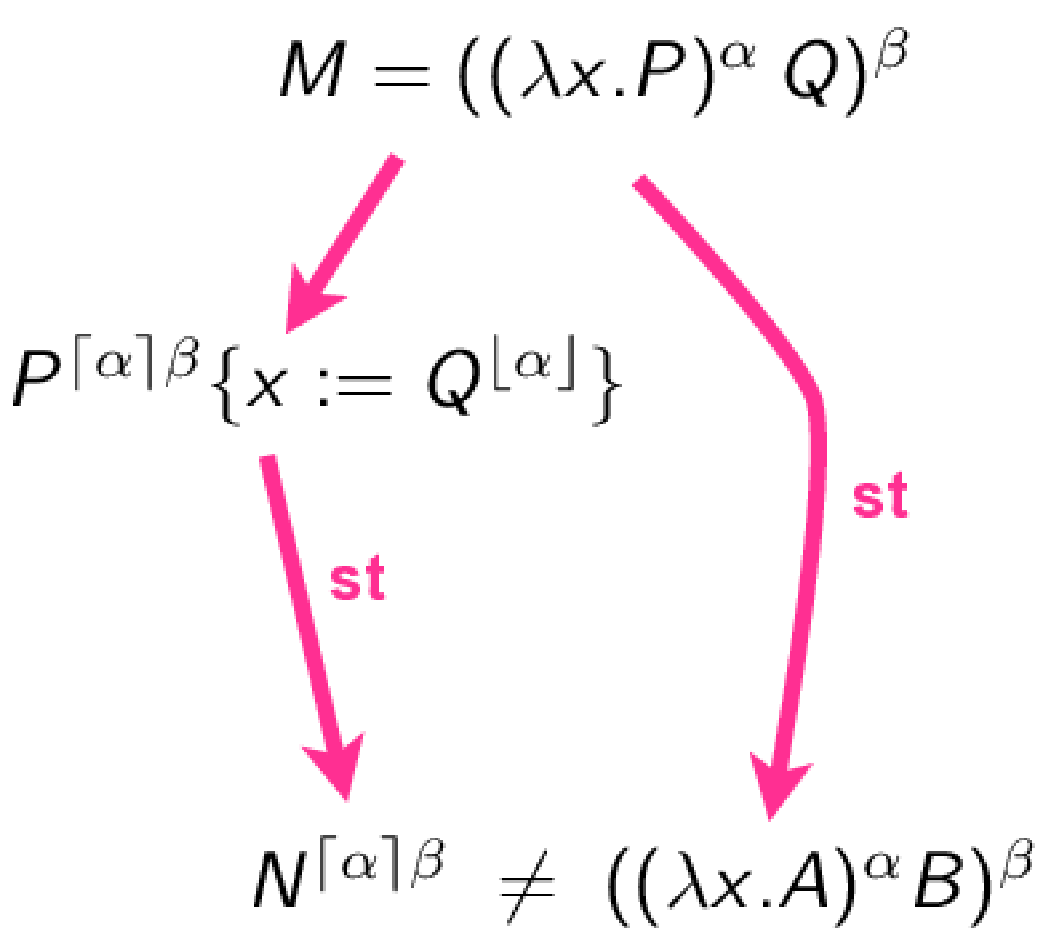


Permutation equivalence

Proof [uniqueness of labeled standard]

Let ρ and σ be 2 distinct coinitial pure labeled standard reductions.

Take first step when they diverge. Call M that term.
We make structural induction on M. Say ρ is more to the left.
If first step of ρ contracts an internal redex, we use induction.
If first step of ρ contracts an external redex, then:



Permutation equivalence

- **Corollary** `[labeled prefix ordering]`
  Let $\rho : M \xrightarrow{\star} N$ and $\sigma : M \xrightarrow{\star} P$ be coinitial pure labeled reductions.
  Then $\rho \sqsubseteq \sigma$ iff $N \xrightarrow{\star} P$.

- **Exercise 8** Show the following properties

(i) $\rho \sqsubseteq \rho$

(ii) $\rho \sqsubseteq \sigma \sqsubseteq \rho$ implies $\rho \simeq \sigma$

(iii) $\rho \sqsubseteq \sigma \sqsubseteq \tau$ implies $\rho \simeq \tau$

(iv) $\rho \sqsubseteq \sigma$ implies $\rho/\tau \sqsubseteq \sigma/\tau$

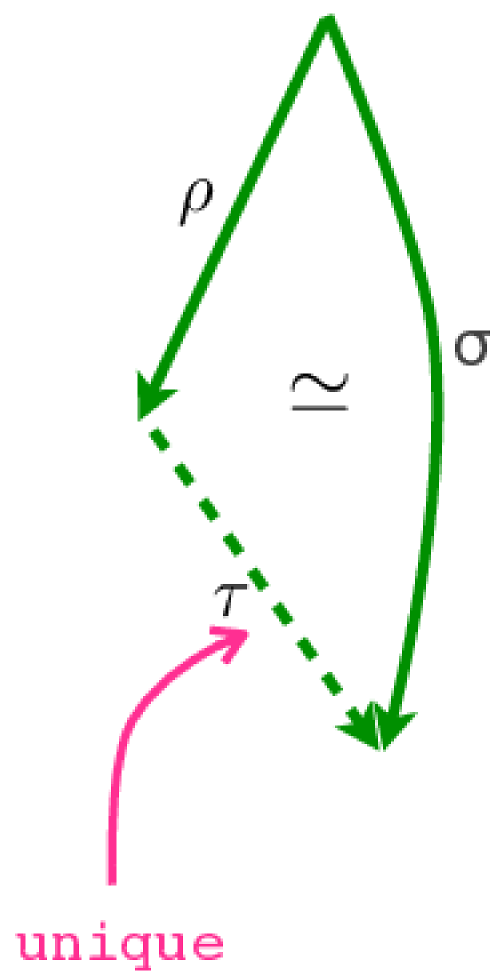
(v) $\rho \sqsubseteq \sigma$ iff $\exists \tau,\ \rho\tau \simeq \sigma$

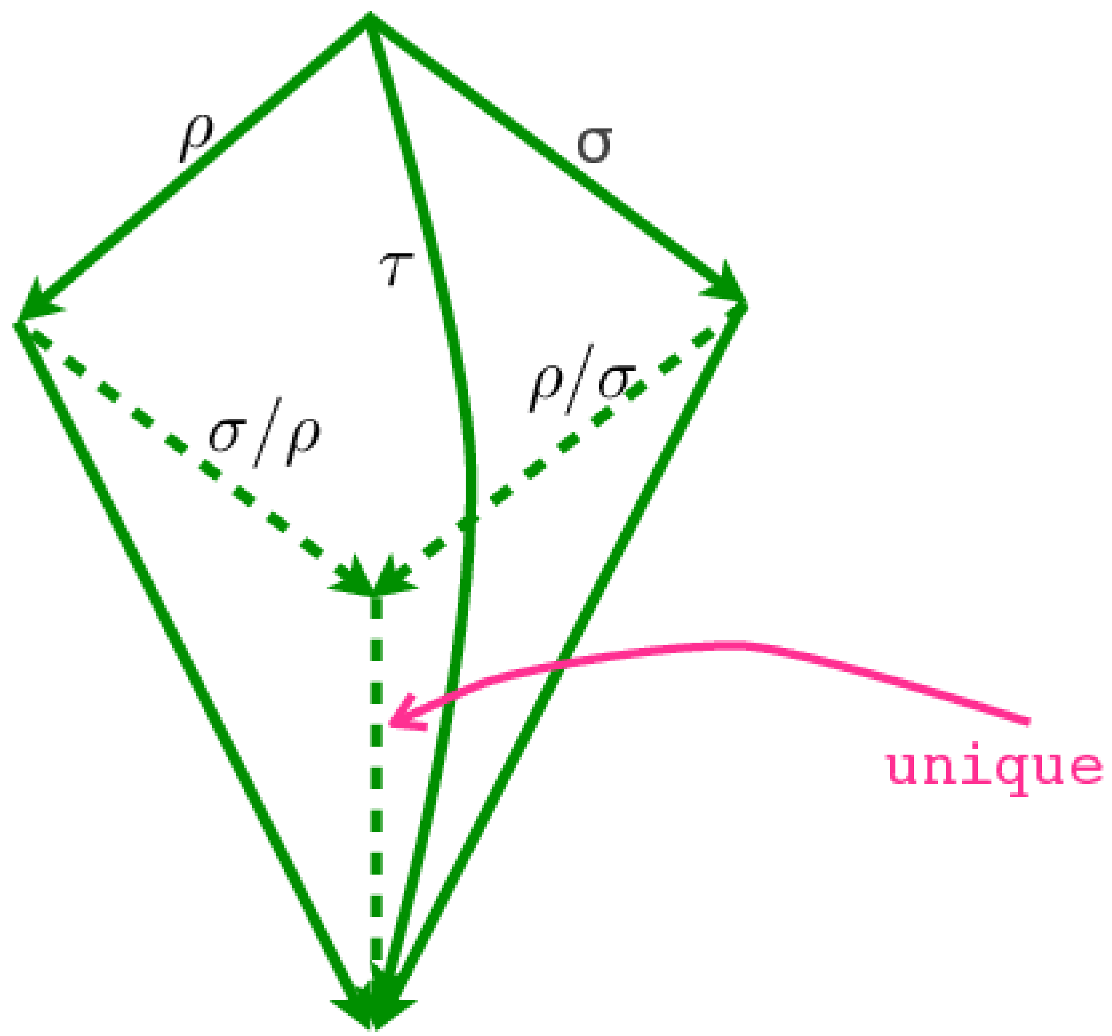
(vi) $\rho \sqsubseteq \rho \sqcup \sigma$, $\sigma \sqsubseteq \rho \sqcup \sigma$

(vii) $\rho \sqsubseteq \tau$, $\sigma \sqsubseteq \tau$ implies $\rho \sqcup \sigma \sqsubseteq \tau$

Permutation equivalence

- **Exercise 9** Show the following diagrams





Permutation equivalence

- **Corollary** `[lattice of labeled reductions]`

  Labeled reduction graphs are upwards semi lattices for any pure labeling.

- **Corollary** `[push-out category]`

  Prefix ordering on reductions is a push-out.

- **Exercise 10** Try on $(\lambda x.x)((\lambda y.(\lambda x.x)a)b)$ or $(\lambda x.xx)(\lambda x.xx)$

- **Exercise 11** Show that prefix ordering on reductions is not a pull-back.

Proof of the GFD theorem

Bound on heights of labels

- **Definition** The height of a label is its nesting of underlines and overlines

$$h(a) = 0$$
$$h(\lceil\alpha\rceil) = h(\lfloor\alpha\rfloor) = 1 + h(\alpha)$$
$$h(\alpha\beta) = \max\{\alpha, \beta\}$$

- **Fact** Let $\mathcal{F}$ be a finite set of redex families, then there is an upper bound $H(\mathcal{F})$ on labels of subterms in reductions relative to $\mathcal{F}$.

When initial term is labeled with atomic letters, we have

$$H(\mathcal{F}) = \max\{h(\alpha) \mid \alpha \in \mathcal{F}\}$$

Proof of finite developments

- **Notation** $\tau(M^\alpha) = \alpha$ when M has an empty external label

- **Lemma 1** Let $M \xrightarrow{*} M'$, then $h(\tau(M)) \leq h(\tau(M'))$

- **Lemma 2** Let $(\cdots((M\, M_1)^{\beta_1}\, M_2)^{\beta_2} \cdots M_n)^{\beta_n} \xrightarrow{*} (\lambda x.N)^\alpha$
  Then $h(\tau(M)) \leq h(\alpha)$

- **Lemma 3** [Barendregt] Let $M\{x := N\} \xrightarrow{*} (\lambda y.P)^\alpha$
  There are 2 cases:

  $M \xrightarrow{*} (\lambda y.M')^\alpha$ and $M'\{x := N\} \xrightarrow{*} P$

  $M \xrightarrow{*} M' = (\cdots((x^\beta\, M_1)^{\beta_1}\, M_2)^{\beta_2} \cdots M_n)^{\beta_n}$ and $M'\{x := N\} \xrightarrow{*} (\lambda y.P)^\alpha$

Proof of finite developments

- **Lemma 1** Let $M \xrightarrow{*} N$, then $h(\tau(M)) \leq h(\tau(N))$

Proof by induction on length of reduction. Let $M \xrightarrow{R} N$, $R = ((\lambda x.A)^\alpha B)^\beta$

If R is internal in M, then $\tau(M) = \tau(N)$.

If $M = R = ((\lambda x.A)^\alpha B)^\beta \longrightarrow A\{x := B^{\lfloor \alpha \rfloor}\}^{\lceil \alpha \rceil \beta} = N$,

then $h(\tau(M)) = h(\beta) \leq h(\gamma\beta) = h(\tau(N))$ for some γ.

- **Lemma 2** Let $(\cdots((M\, M_1)^{\beta_1}\, M_2)^{\beta_2} \cdots M_n)^{\beta_n} \xrightarrow{*} (\lambda x.N)^\alpha$

Then $h(\tau(M)) \leq h(\alpha)$

Proof by induction on n.

When $n = 0$, obvious by lemma 1.

Otherwise $(\cdots((M\, M_1)^{\beta_1}\, M_2)^{\beta_2} \cdots M_{n-1})^{\beta_{n-1}} \xrightarrow{*} (\lambda y.P)^\gamma$

and $((\lambda y.P)^\gamma Q)^{\beta_n} \longrightarrow P\{y := Q^{\lfloor \gamma \rfloor}\}^{\lceil \gamma \rceil \beta_n} \xrightarrow{*} (\lambda x.N)^\alpha$

So $h(\tau(M)) \leq h(\gamma) < h(\delta \lceil \gamma \rceil \beta_n) \leq h(\alpha)$ by induction and lemma 1.

Proof of finite developments

- **Lemma 3** [Barendregt] Let $M\{x := N\} \xrightarrow{*} (\lambda y.P)^\alpha$

  There are 2 cases:

  $$M \xrightarrow{*} (\lambda y.M')^\alpha \text{ and } M'\{x := N\} \xrightarrow{*} P$$

  $$M \xrightarrow{*} M' = (\cdots((x^\beta\, M_1)^{\beta_1}\, M_2)^{\beta_2} \cdots M_n)^{\beta_n} \text{ and } M'\{x := N\} \xrightarrow{*} (\lambda y.P)^\alpha$$

Proof Let $M^* = M\{x := N\}$. There are 3 cases on weak head reduction of M : it reaches an abstraction or a head variable which has to be x.
More precisely, we consider the standard reduction from M^* to $(\lambda y.P)^\alpha$.

Case 1: $M = (\lambda y.M')^\alpha$ and we are done since $M^* = (\lambda y.M'^*)^\alpha$.

Case 2: $M = ((\cdots((y^\beta\, M_1)^{\beta_1}\, M_2)^{\beta_2})\cdots M_n)^{\beta_n}$. Then $y = x$ and $M' = M$.

Case 3: $M = (\cdots((((\lambda z.A)^\beta\, B)^\gamma\, C_1)^{\beta_1}\, C_2)^{\beta_2} \cdots C_n)^{\beta_n}$

Let $M_1 = (\cdots((A\{z := B^{\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\gamma}\, C_1)^{\beta_1}\, C_2)^{\beta_2} \cdots C_n)^{\beta_n}$

Then $M^* = (\cdots((((\lambda z.A^*)^\beta\, B^*)^{\beta_1}\, C_1^*)^{\beta_1}\, C_2^*)^{\beta_2} \cdots C_n^*)^{\beta_n} \longrightarrow M_1^*$ is the first step of the standard reduction from M^* to $(\lambda y.P)^\alpha$. By induction on its length, we are done.

Proof of finite developments

- **Notation** Let $\mathcal{SN}_{\mathcal{F}}$ be the set of strongly normalizable terms w.r.t. reductions relative to $\mathcal{F}$.

- **Lemma** `[subst]` Let $\mathcal{F}$ be a finite set of redex families.
  $M, N \in \mathcal{SN}_{\mathcal{F}}$ implies $M\{x := N\} \in \mathcal{SN}_{\mathcal{F}}$

  Proof `[van Daalen]` by induction on $\langle H(\mathcal{F}) - h(\tau(N)), \mathrm{depth}(M), \|M\| \rangle$

- **Theorem** `GFD` Let $\mathcal{F}$ be a finite set of redex families.
  Then $M \in \mathcal{SN}_{\mathcal{F}}$ for all M.

  Proof by easy induction on $\|M\|$

Proof of finite developments

- **Lemma** [subst] Let $\mathcal{F}$ be a finite set of redex families.
  $M, N \in \mathcal{SN}_{\mathcal{F}}$ implies $M\{x := N\} \in \mathcal{SN}_{\mathcal{F}}$

Proof [van Daalen] by induction on $\langle H(\mathcal{F}) - h(\tau(N)), \text{depth}(M), \|M\| \rangle$

Cases $M = x$, $M = y$, $M = \lambda y.M_1$ are obvious or easy by induction on $\|M\|$.

Write M^* for $M\{x := N\}$ and consider case $M = (M_1 M_2)^\alpha$.

If all reductions are internal to M_1^* and M_2^*, then easy induction on $\|M\|$.

Otherwise, let $M_1^* \xrightarrow{*} (\lambda y.P)^\beta$ and $M_2^* \xrightarrow{*} Q$ and $((\lambda y.P)^\beta Q)^\alpha \longrightarrow P\{y := Q^{\{\lfloor\beta\rfloor\}}\}^{\lceil\beta\rceil\alpha}$

Then M_1^* and M_2^* are in $\mathcal{SN}_{\mathcal{F}}$ by induction on $\|M\|$,

and $M_1^* \xrightarrow{*} (\lambda y.P)^\beta$ and $M_2^* \xrightarrow{*} Q$. So P and Q are in $\mathcal{SN}_{\mathcal{F}}$.

How is $P\{y := Q^{\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha}$??

By lemma 3, we have 2 cases:

Proof of finite developments

Case 1:

Then $M_1 \xrightarrow{*} (\lambda y.M_1')^\beta$ and $M_1'^* \xrightarrow{*} P$.

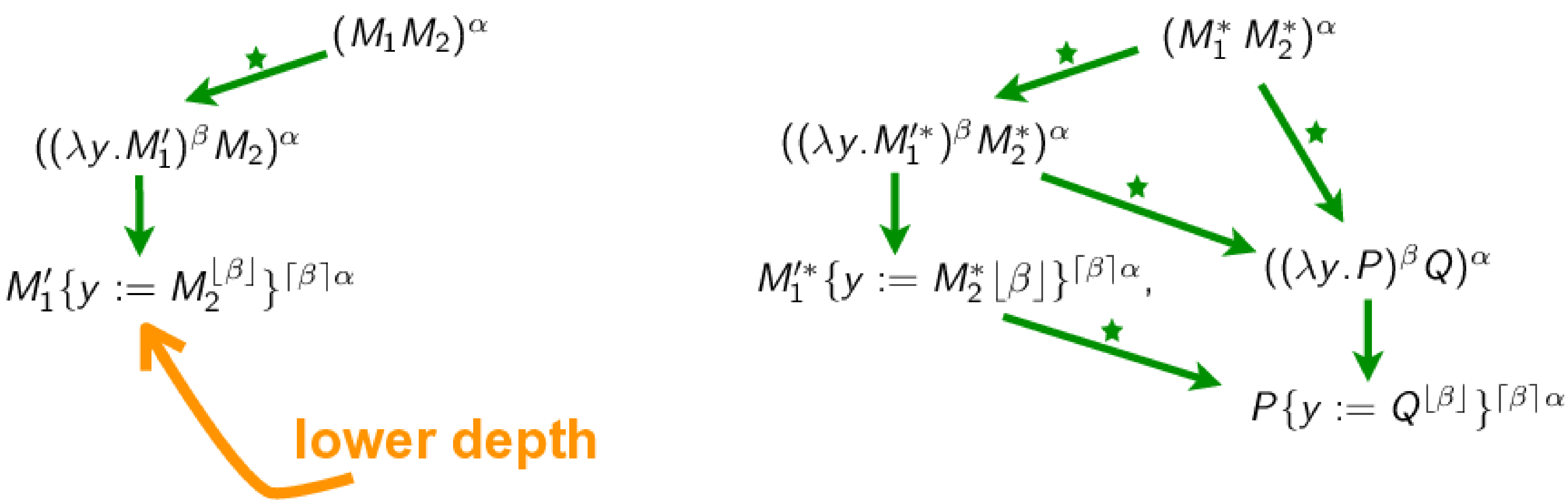
Therefore $M_1'^*\{y := M_2^{*\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha} \xrightarrow{*} P\{y := Q^{\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha}$.

But as $M = (M_1M_2)^\alpha \xrightarrow{*} ((\lambda y.M_1')^\beta M_2)^\alpha \longrightarrow M' = M_1'\{y := M_2^{\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha}$,

we have $\text{depth}(M') < \text{depth}(M)$.

Thus by induction $M'^* = M_1'^*\{y := M_2^{*\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha} \in \mathcal{SN}_\mathcal{F}$

and $P\{y := Q^{\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha} \in \mathcal{SN}_\mathcal{F}$.



Proof of finite developments

Case 2:

$M_1 \xrightarrow{*} M_1' = (\cdots((x^\gamma A_1)^{\gamma_1} A_2)^{\gamma_2}) \cdots A_n)^{\gamma_n}$ and

$$M_1'^* = (\cdots((N^\gamma A_1^*)^{\gamma_1} A_2^*)^{\gamma_2}) \cdots A_n^*)^{\gamma_n} \xrightarrow{*} (\lambda y.P)^\beta$$

Therefore $h(\tau(N)) \leq h(\tau(N^\gamma)) \leq h(\beta)$ by lemma 2.

So $M^* = (M_1^* M_2^*)^\alpha \xrightarrow{*} ((\lambda y.P)^\beta Q)^\alpha \xrightarrow{*} P\{y := Q^{\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha}$

and $h(\tau(N)) \leq h(\beta) < h(\lfloor\beta\rfloor) \leq h(\tau(Q^{\lfloor\beta\rfloor}))$.

We get by induction $P\{y := Q^{\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha} \in \mathcal{SN}_\mathcal{F}$.