Parameter sensitivity for wave breaking closures in Boussinesq-type models

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Abstract

We consider the issue of wave-breaking closure for the well known Green-Naghdi model and attempt at providing some more understanding of the sensitivity of some closure approaches to the numerical set-up. More precisely and based on [16] we used two closure strategies for modelling wave-breaking of a solitary wave over a slope. The first one is the hybrid method consisting of suppressing the dispersive terms in a breaking region and the second one is an eddy viscosity approach based on the solution of a turbulent kinetic energy model. The two closures use the same conditions for the triggering of the breaking mechanisms. Both the triggering conditions and the breaking models themselves use case depended / ad/hoc parameters which are affecting the numerical solution wile changing. The scope of this work is to make use of sensitivity indices computed by means of Analysis of Variance (ANOVA) to provide the sensitivity of wave breaking simulation to the variation of parameters such as the mesh size and the breaking parameters specific to each breaking model. The sensitivity analysis is performed using the UQlab framework for Uncertainty Quantification [24].

Key words: keywords

Contents

1 Introduction

The solitary wave run up and run down over a slope is a classical hydrodynamic problem. It is highly relevant for the investigation of the characteristics of Tsunami waves, since solitary waves model some of the important aspects of Tsunami waves. We refer the readers to a series of works investigating the physics of solitary waves' propagation including shoaling, breaking and run up over a slope, by either an analytical approach (e.g [33]) or by a numerical one, see [12, 13, 20, 21, 33, 38] and references therein. Among the various available mathematical models for the propagating Tsunami waves, the depth-average models enjoy a wide popularity thanks to their ability to describe the physics with computational ease. The Boussinesq type (BT) models, which typically incorporate frequency dispersion, have been also used to study the physics of solitary waves, see for example [26, 28, 38]. Moreover, solitary wave propagation, including the breaking and run-up phases, is considered as a classical test case for the validation and calibration of BT models, see [11, 31, 34, 35] among others.

When using BT models, the treatment of wave-breaking necessitates deploying a wave-breaking closure model/mechanism in the surf zone in order to account for the dissipative dynamics associated with it. We can generally say that we can distinguish the wave-breaking treatment in three main categories. The hybrid models, the roller models and the eddy viscosity models. The philosophy of the wave-breaking closures is to mimic the energy dissipation either by adding extra terms to the equations (roller models and eddy viscosity models) or by a local coupling of the dispersive propagation model with the shallow water equations. In the later approach, the non linear shallow water equations (NSWE) are solved in the breaking regions, with the breaking wave modeled as a shock so that the total energy is dissipated. We have to be aware that all the wave-breaking closures introduce coefficients/parameters which need to be calibrated with experimental/analytical data. We also have to keep in mind that the calibration methodology may be specific to a particular set of equations. Extensive reviews on wave-breaking closures for BT models can be found in [15, 16] and [6].

The work of Zelt [37] is one of the earliest works that included a simple eddy viscosity wave-breaking mechanism into a Lagrangian finite element Boussinesq wave model to treat the wave-breaking, as well as study the run-up and breaking of solitary waves. In that work, a simple parameter analysis and calibration of the breaking model was performed, which in turn used the work of Synolakis [33]. Later, Grilli et al. [12] examined breaking and non breaking waves on a range of slopes using a fully non-linear potential flow model, with particular attention paid to the criterion that distinguishes breaking and non-breaking waves, and breaking indices. Zhou et al. [38] performed a parametric investigation of breaking solitary wave over fringing reefs using the weakly nonlinear weakly dispersive equations of Nwogu [27], and the hybrid wave-breaking closure. A more recent work that also performs parametric studies of solitary wave propagation and run-up over fringing reefs is the one of Ning et al. [26]. They use a fully non linear Boussinesq wave model with the hybrid wave-breaking closure. However, they don't account for the parameters of the breaking model.

Recently in [16] the authors considered the issue of wave-breaking closure when using weakly dispersive Boussinesq propagation models. They focused on the enhanced equations of Nwogu [27] and the enhanced equations of Green-Naghdi (GN) [5, 11], and considered two different wave-breaking closures. The first one being the hybrid closure proposed in [35] and the second one being an eddy viscosity approach based an adaptation of the turbulent kinetic energy (TKE) proposed by Nwogu [29]. The results clearly showed a reduced sensitivity to the mesh size when using the TKE eddy viscosity closure compared to the hybrid closure model. The above models also present a dependency on the parameters of the detection criteria, as well as on the coefficients of the TKE equation. The scope of this work is to quantify the sensitivity of these parameters when using the Green Naghdi equations.

1.1 A summary on the model and the numerical implementation

The GN equations are weakly non-linear fully dispersive. The range of validity of the model may vary over a wide range as far as the nonlinearity parameter (ϵ) is concerned but it requires that the shallowness parameter (μ) to be small (less than one). In this work the equations are used in the form proposed in [11]

$$\begin{aligned} h_t + (hu)_x &= 0 && (1)\\ (hu)_t + (hu^2)_x + gh\eta_x &= \phi + F_{br} + F_m \\ (I + \alpha\mathcal{T})\,\phi &= \mathcal{W} - \mathcal{R} && (2)\end{aligned}$$

having also defined $\mathcal{W} = g\mathcal{T}(d\eta_x)$ and $\mathcal{R} = h\mathcal{Q}(u)$, where $\mathcal{T}(\cdot)$ and $\mathcal{R}(\cdot)$ are the linear operators given as:

$$\mathcal{T}(\cdot) = -\frac{1}{3}h^2(\cdot)_{xx} - \frac{1}{3}hh_x(\cdot)_x + \frac{1}{3}\left[h_x^2 + hh_{xx}\right](\cdot) + \left[b_xh_x + \frac{1}{2}hb_{xx} + b_x^2\right](\cdot) \quad (3)$$

$$\mathcal{Q}(\cdot) = 2hh_x(\cdot)_x^2 + \frac{4}{3}h^2(\cdot)_x(\cdot)_{xx} + b_xh(\cdot)_x^2 + b_{xx}h(\cdot)(\cdot)_x + \left[b_{xx}h_x + \frac{1}{2}hb_{xxx} + b_xb_{xx}\right](\cdot)^2 \quad (4)$$

h being the water height, u the velocity, d the still water level, b the topography function, and F_m is the bottom friction. F_{br} accounts for the extra breaking terms , activated only when an eddy viscosity breaking closure is used.

The above system can be solved in two independent steps. Te first one is the elliptic step where we solve for the non-hydrostatic term ϕ and the second step is the hyperbolic in which we evolve the flow variables. We implement a standard C^0 Galerkin finite element approximation for the elliptic step and a high order finite volume method for the hyperbolic one. More precisely we use a third order MUSCL scheme for the reconstruction of the variables. The approximate Riemann solver of Roe [30] is used to evaluate the numerical fluxes at the interfaces. We also use an upwind discretization of the topography source. The discretized term ϕ is the link between the two steps and is integrated exactly over the cell C_i. A conservative computation of wet/dry fronts is used resulting in a well-balanced scheme. The time integrator that we use is the Adams-Bashforth/Adams Moulton predictor corrector scheme. An extensive description of the numerical scheme can be found in [11, 16].

The paper is organized as follows. Section 2 presents the two different wave-breaking closures used in this work. Section 3 discusses briefly the sensitivity analysis model used to perform the parameter sensitivity analysis. The numerical experiment and the results of the analysis, for both approaches, are presented in Section 4. Section 5 presents a summary of the results and a comparison of the two breaking models. The paper ends with conclusions and an outlook of the future developments.

2 Wave breaking closures

A wave breaking closure can be distinguished in two steps. The first one is the detection of the breaking wave in time and in space, including both an initiation and/or termination trigger. The second important aspect is an energy dissipation mechanism. The wave breaking trigger models can be classified to phase averaged and phase resolving models. The phase averaged models use wave characteristics which are representative of one wave's phase and its more difficult to compute locally, while phase resolving models use local information of the wave. The last ones rely on a wave-by-wave analysis and are more efficient to program in the context of phase resolved simulations, see the recent work [1, 9] and [2] on breaking onset criteria. For the triggering mechanism, in this work, we use a combination of local and non-local criteria proposed in [15] and used also in [11, 16]. Each one is computed and checked in each cell of our mesh and if one of them is satisfied then the cell is flagged as a breaking cell. The criteria are the following:

1. the surface variation criterion, satisfied if $|\eta_t| \geq \gamma\sqrt{gh}$;
2. the local slope angle criterion, satisfied if $||\nabla\eta|| \geq \tan\theta$ with θ being a critical angle;
3. the Froude number criterion, requiring that $\text{Fr} = \sqrt{H_{\max}\bar{H}}/H_{\min} > 1.3$. If this condition is not met, breaking is deactivated independently on the first two conditions.

In the last criterion $H_{\min\,/\,\max}$ are the depths in correspondence a wave's trough/peak, and $\bar{H}$ the average of the $\min$ and $\max$. The Froude criterion acts as a termination trigger. Flagged cells are grouped according to the above local criteria and form a breaking region which is enlarged to account for the typical roller length, as suggested in [15, 34]. The first two criteria involve the dimensionless parameter γ and the critical angle θ which are crucial for the flagging, and for which, to the authors knowledge, there is no general parametrisation.

2.1 Energy dissipation via bore capturing (hybrid wbc)

A common approach to deal with wave breaking for Boussinesq-type models , here referred to as the hybrid wave breaking closure (wbc) technique, relies on the deactivation of all the dispersive terms in the flagged region. This reduced the model to the shallow water equations, which exhibits the formation of a bore (or of a hydraulic jump) quite rapidly in the breaking region. Across the discontinuity, the classical Rankine-Hugoniot theory can be applied, and a net decrease in total energy is observed and can be readily estimated [4, 7]. In this closure there is no additional diffusion term and no tuning parameter, besides those already involved in the detection. On the other hand, it is known that at the interface of the closure region spurious oscillation may appear and grow as numerical dissipation is reduced using e.g. mesh refinement. This issue has been investigated in [1, 16].

2.2 Energy dissipation using PDE based eddy viscosity (TKE wbc)

One way to parametrize the effect of wave breaking is by the addition of an eddy viscosity term in the momentum equation [17, 29, 31]. This results in the explicit addition of the right hand side of the second equation in (1) of a dissipative term, of the form $\mathbf{F}_{br} = \partial_x(v_t h \partial_x u)$ where v_t is the turbulent eddy viscosity. In this work we consider an approach similar to the one studied in [16], in which the eddy viscosity is determined from an additional partial differential equation for the generation and transport of turbulent kinetic energy in breaking waves. The added viscosity is written as:

$$v_t = C_\nu \sqrt{\kappa_b k_b} A$$

where k_b is the kinetic energy resolved for a propagating bore and A is the bore's initial amplitude. Following the work of Nwogu [29], the turbulent kinetic energy for k_b is determined from a semi-empirical transport equation, with a source term for turbulent kinetic energy production by wave-breaking. Dropping the subscript b for simplicity, we write:

$$k_t + u k_x = \mathcal{D} + \mathcal{P} - \mathcal{E} \tag{5}$$

with $\mathcal{D},\ \mathcal{P},\ \mathcal{E}$ being diffusion, production and dissipation terms respectively.

$$\mathcal{D} = \sigma \nu_t k_{xx}, \quad \mathcal{E} = -C_\nu^3 \frac{k^{3/2}}{\ell_t} \tag{6}$$

The constant σ controls the smoothness of the TKE and hence of the breaking viscosity in the breaking region and, $\ell_t = \kappa A$.

$$\mathcal{P} = B(t,x) \frac{\ell_t}{\sqrt{C_\nu^3}} (u^s)^3 \tag{7}$$

The parameter $B(t,x)$ is the breaking flag, and is either 0 or 1 depending on the wave breaking criterion, while u_s is the velocity at the free surface, for the GN equations defined as

$$u_s = u - \left[\frac{\eta^2}{2} - \left(\frac{h^2}{6} - \frac{h(h-d)}{2} \right) \right] u_{xx} - \left[\eta - \left(\frac{h}{2} - d \right) \right] (du)_{xx}. \tag{8}$$

The fully discrete distribution of the nodal values of the TKE is obtained by integrating Eq. (5) with a semi implicit approach. Before the predictor time step is applied to the Boussinesq model, the nodal TKEs are evolved by first applying an explicit Euler update involving the same third order MUSCL upwind discretization of the transport operator $(uk)_x$ used for the shallow water equations. To avoid spurious negative values in this phase, the min-mod limiter is applied [19]. The predicted values k_i^* are then corrected by diagonally semi-implicit relaxation iterations as :

$$\left(\frac{\Delta t}{\Delta x} + \frac{2\sigma \nu_{t,i}^n}{\Delta x} \right) (k_i^{m+1} - k_i^m) = \Delta x \frac{k_i^m - k_i^*}{\Delta t} + \sigma \nu_{t,i}^n \frac{k_{i+1}^m - 2k_i^m + k_{i-1}^m}{\Delta x} + \left(\frac{B \ell_{t,i}^2}{\sqrt{C_{nu}^3}} u_s^3 \right)^n - C_\nu^3 \left(\frac{k_i^{3/2}}{\ell_{t,i}} \right)^n$$

with an initial condition $k^0 = k^*$. The number of relaxation iterations used in practice is 5.

3 Sensitivity Analysis Model

The Analysis of Variance (ANOVA), first proposed by Sobol et. al. [22, 23], is a powerful technique for global sensitivity analysis. Later, an improved algorithm was proposed by Saltelli et. al. to reduce the computational cost of ANOVA [32]. A comprehensive review of the global sensitivity analysis methods, including the ANOVA technique, is given by Iooss and Lemaítre in [3]. Here, we briefly describe the ANOVA method to find out Sobol indices, that characterize the sensitivity of the model output with respect to contribution of each of the input parameters. Consider $\mathbf{X} \in \Omega \subset \mathbb{R}^d,\ d \in \mathbb{N}$ to be a random variable and let $f(\mathbf{X}) \in L^2(\Omega)$ be a square integrable function on domain $\Omega = [0,1]^d$. ANOVA decomposition of f quantifies the variance of the function output in terms of the input parameters (or a combination of the input parameters), and is given as follows (see [3, 22] and the references therein).

$$f(\mathbf{X}) = f_0 + \sum_{1 \le i \le d} f_i(X_i) + \sum_{1 \le i < j \le d} f_{ij}(X_i, X_j) + ... + f_{12...d}(\mathbf{X}) \tag{9}$$

under the condition imposed by the following equation:

$$\int_0^1 f_{i_1, i_2, ..., i_s}(x_{i_1}, ..., x_{i_s})\, dx_{i_k} = 0, \quad 1 \le k \le s, \quad \{i_1, ..., i_s\} \subseteq \{1, ..., d\}. \tag{10}$$

Here, f_0 is the expected value of the function f, which is quantified by $\mathbb{E}[Y]$ for each random variable $Y = f(\mathbf{X})$. The univariate function $f_i(X_i) = \mathbb{E}[Y|X_i] - \mathbb{E}[Y]$ quantifies contribution due to each random variable $Y = f(\mathbf{X})$. The bivariate function $f_{ij}(X_i, X_j) = \mathbb{E}(Y|x_i, x_j) - F_0 - F_i - F_j$ quantifies the joint contribution of each pair of input parameters covering all possible combinations. Similarly, the rest of the terms quantify the higher-order effects. It is possible to get a similar decomposition of the variance of the function $f(\cdot)$ as follows [3],

$$Var(Y) = \sum_{i=1}^{d} D_i(Y) + \sum_{i<j}^{d} D_{ij}(Y) + ... + D_{12...d}(Y), \tag{11}$$

where, $D_i(Y) = Var[\mathbb{E}(Y|X_i)]$, $D_{ij}(Y) = Var[\mathbb{E}(Y|X_i, X_j)] - (D_i(Y) + D_j(Y))$ and so on for the higher order terms. This decomposition is known as ANOVA decomposition. Finally, the Sobol' indices are given as (refer [3, 22]),

$$S_i = \frac{D_i(Y)}{Var(Y)}, \quad S_{ij} = \frac{D_{ij}(Y)}{Var(Y)}, \quad \text{Higher Order Terms} \tag{12}$$

Further, the Total Order (TO) Sobol' indices combine the lower-order and higher-order terms and are given by:

$$S_{T_i} = S_i + \sum_{i<j} S_{ij} + \sum_{j\neq i, k\neq i, j<k} S_{ijk} + \quad \text{Higher Order Terms} \tag{13}$$

Note that two-parameters (as well as three-, four- etc.) interactions involving x_i are already accounted in S_i, the same way that three-parameters contributions to the variance are also hidden in the two-parameters contributions, and so on. For this reason, the sum of all the total indices usually larger than 1.

To perform the sensitivity analysis in this work we used the UQlab-The framework for Uncertainty Quantification [25]. More precisely, we use the sensitivity analysis module which contains the methods for global sensitivity analysis that quantitatively measure the importance of each input parameter. We construct a surrogate model (meta-model) of our computational model prior to the Sobol sensitivity analysis. The surrogate model is a functional approximation of our computational model and is faster to evaluate. It is constructed using a relatively small number of input parameters and corresponding output results from our computational model. This way, we reduce the total computational cost to a great extend. In this work we use the Kriging (or Gaussian process regression) metamodel. A review on Kriging metamodeling can be found in [10, 18]. The preliminary results of our analysis have been presented in [14].

4 Case study: propagation, breaking and runup of a solitary wave

We study a well known numerical experiment involving the propagation, breaking, and runup of a solitary wave over a slope (see sketch on Figure 1). It is based on the work of Synolakis [33], where solitary waves with various nonlinearities $\epsilon = A/h_0$ were studied experimentally and numerically. For simplicity, we only investigate the breaking and runup processes and their sensitivity to the different physical and model parameters. This already will provide a quite large spectrum of results and informations related to the closure models. The initial solution consists of a solitary wave of amplitude A, defined by

$$\eta(x) = A\,\text{sech}^2(k(x - x0)), \quad u(x) = c\left(1 - \frac{h_0}{\eta(x)}\right) \tag{14}$$

with $k = \sqrt{\frac{3A}{(h_0+A)*h_0^2}}$, $c = \sqrt{g(h_0+A)}$ and $h_0 = 1m$. The bathymetry has the form

$$b(x) = \begin{cases} \frac{-x}{s}, & \text{if } x \leq s \\ -1, & \text{elswere.} \end{cases} \quad (15)$$

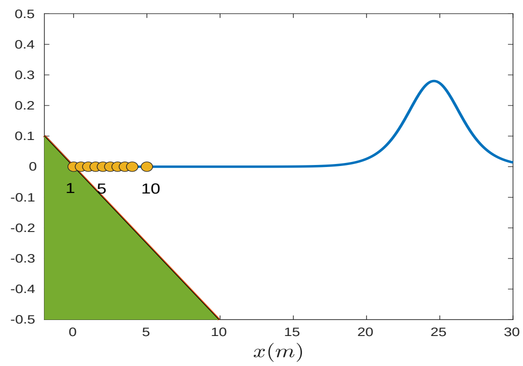

Figure 1: Initial conditions and position of the wave gauges

The crest of the wave is placed offshore at $x_0 = s + \frac{1}{\rho}2.17832722$ with $\rho = \sqrt{3A/4h_0}$. The computational domain is $[-20, 80]m$ and the numerical parameters are constant and set to $\Delta x = 0.05,\ \text{cfl} = 0.1$. They are intentionally kept relatively small as to limit the numerical diffusion in the results. Figure 2 confirms that the computed solution, presented at time $6.4sec$ (i.e. already breaking), does not change as Δx is refined and that we are still far from the fine mesh instabilities highlighted for the hybrid closure in [16]. We record time series of the surface elevation in 10 wave gauges every $0.5m$ from $x = 0$ to $x = 3.5$, plus two additional at $x = 4.0$, and $5.0m$. In this spatial window, we have observed wave-breaking occurs for all the examined waves.

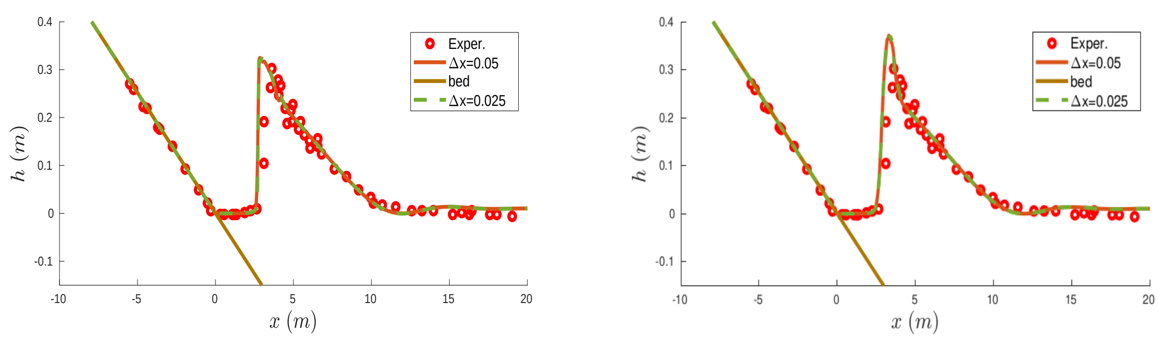

Figure 2: Computed solution, at early breaking for the hybrid(left) and the TKE (right) closures using $\Delta x = 0.05m$ and $\Delta x = 0.025m$.

We introduced uncertainties in both the model set up and the model parameters. In particular, wave amplitude, slope of the beach, and Manning friction coefficient are assumed to vary uniformly in the ranges $A \in [0.1,\ 0.6]$, slope $\in [-1/15,\ -1/25]$, $Nm \in [0.009,\ 0.075]$ respectively. All of the possible combinations of these parameters cover most of the practical scenarios. Uncertainty is introduced also in all the parameters of the breaking models.

For completeness, Figure 3 shows the evolution of the free surface, compared to the experimental data of [33], computed in the deterministic setting: slope $= -1/19.85$, $A = 0.28$, $Nm = 0.01$. For these computations, the wave breaking closure models have been parametrized using values proposed in literature [11, 16, 35]: $\gamma = 0.6$, $\theta = 0.53$ and $C_v = 10$ $\kappa_b = 2.5$ and $\sigma_b = 20$ for the TKE model. As the wave approximates the shore and shoals it is clear that the two models, as expected, give identical results since wave-breaking has not started yet. The breaking procedure starts around $t = 4.79sec$. Both the hybrid closure and the TKE model represent the solution as a triangular bore, albeit with a different resolution of the peak which is kept smoother and higher by the TKE approach. At time $t \sim 7.9sec$ the bore collapses on the shore and the wave starts to run-up. For this particular case both numerical solutions provide a good qualitative agreement with the data.

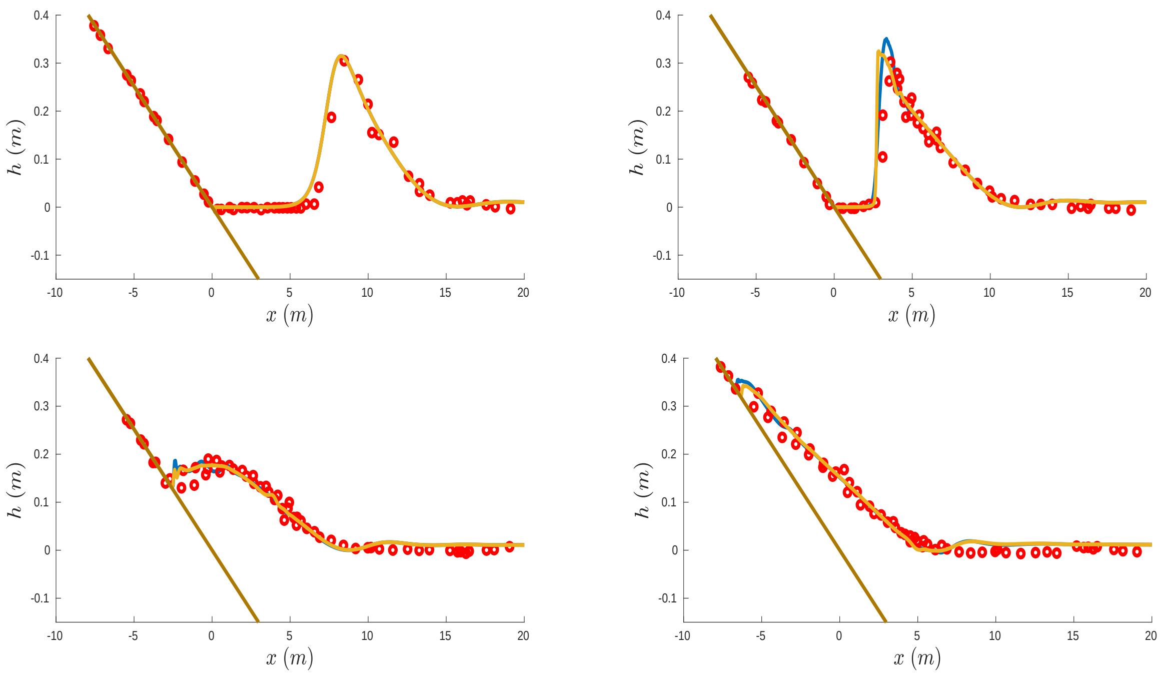

Figure 3: Time evolution of the free surface (deterministic). Yellow line: Hybrid wbc, blue line: TKE wbc.

4.1 Model outputs and post-processing

An important point is the definition of the output observables to be used in the analysis. We will start by considering the sensitivities of the location of the breaking point, the wave height at the breaking point, the maximum wave height at the wave gauges and the maximum run-up.

To have a more detailed picture of the evolution of the wave as it shoals and breaks, we will also consider a set of statistical outputs. There exist several works that examine them in the context of BT models, see for example [8, 17, 36]. A measure to examine the left-right differences in a wave is the wave asymmetry which is defines as

$$As = \frac{< H(\eta^3) >}{< \eta^2 >^{3/2}} \tag{16}$$

where H is the Hilbert transformation and $< \cdot >$ denotes the mean operator. A similar information is provided by wave skewness, which is a measure of crest-trough shape. It is defined as

$$Sk = \frac{< \eta^3 >}{< \eta^2 >^{3/2}}. \tag{17}$$

Finally, the kurtosis will allow to estimate whether the wave is heavy tailed or light tailed with respect to the normal distribution. It is defined as

$$Ks = \frac{\frac{1}{n} \sum (\eta_i - < \eta >)^4}{\left(\frac{1}{n} \sum (\eta_i - < \eta >)^2\right)^2} \tag{18}$$

where n is the length of the time series data in the wave gauge.

Finally, we will look in more detail into the impact of the physical set up parameters on the variability of maximum runup, allowing to compare with some of the data provided in [33].

4.2 Parameter sensitivity analysis using the hybrid wbc

First we perform a parameter sensitivity analysis using the hybrid wbc. Besides the 4 physical parameters, we include uncertainties on the two additional parameters involved in the detection of wave-breaking. In particular, we assume the uniform distributions $\gamma \in [0.3,\ 0.8]$ and $\theta \in [0.32,\ 0.6]$.

Figure 4 presents the First and Total order Sobol indices for the physical outputs, including the results in wave gauges. The results indicate that the amplitude of the wave is the dominant parameter for the maximum wave height at the wave gauges and on the breaking point. γ also affects the location of the breaking point and friction plays a dominant role on the maximum run-up and on the maximum wave height at $x = 0m$. All the second order Sobol indices are less that 0.1 (not shown). This indicates a small correlation between input parameters.

To go further we look at the statistical parameters of the water elevation signals in the gauges. The results are summarized in Figure 5. Wave amplitude, slope, and friction coefficient have the largest impact. The asymmetry is largely controlled by the amplitude, and to a smaller extent by the slope. This is especially true in the first gauges, close to the collapse of the wave (impact of amplitude), and to the runup/backwash (slope). Similarly, the skewness is highly affected by the amplitude as the wave breaks and collapses on the slope (first gauges). The impact of the slope itself dominates the shoaling phase (intermediate gauges). The thickness of the wave tails, measured by the kurtosis, is controlled mainly by slope and friction. The first has a much higher sensitivity in the intermediate gauges suggesting that its impact is related to the propagation, while the impact of the friction more to the backwash. The results show that for this closure model the wave-breaking parameters have considerably smaller contributions to the shape of the wave.

One of the most relevant outputs for risk assessment is the maximum runup. It is well understood that waves with higher amplitudes have higher impact on the coast, or in other words higher maximum runup. To investigate this, we look at the variability of the latter with respect to physical parameters. The maximum run-up as a function of the wave's amplitudes (for fixed slope=$1/19.85$, $Nm = 0.01$) is reported in the leftmost picture in Figure 6. For comparison, the experimental data [33] is also plotted. We also report in the figure the dependence of the maximum runup on the slope (for fixed $A = 0.28$ and $Nm = 0.01$), and on the Manning coefficient (for fixed slope=$1/19.85$ and $A = 0.28$). The metamodel predictions for different values of the wave breaking parameter γ are plotted.

Above the amplitude ~ 0.15 the maximum run-up increases almost linearly with A. The metamodel predictions are quite close to the data. The impact of the wave breaking parameter γ is not uniform over

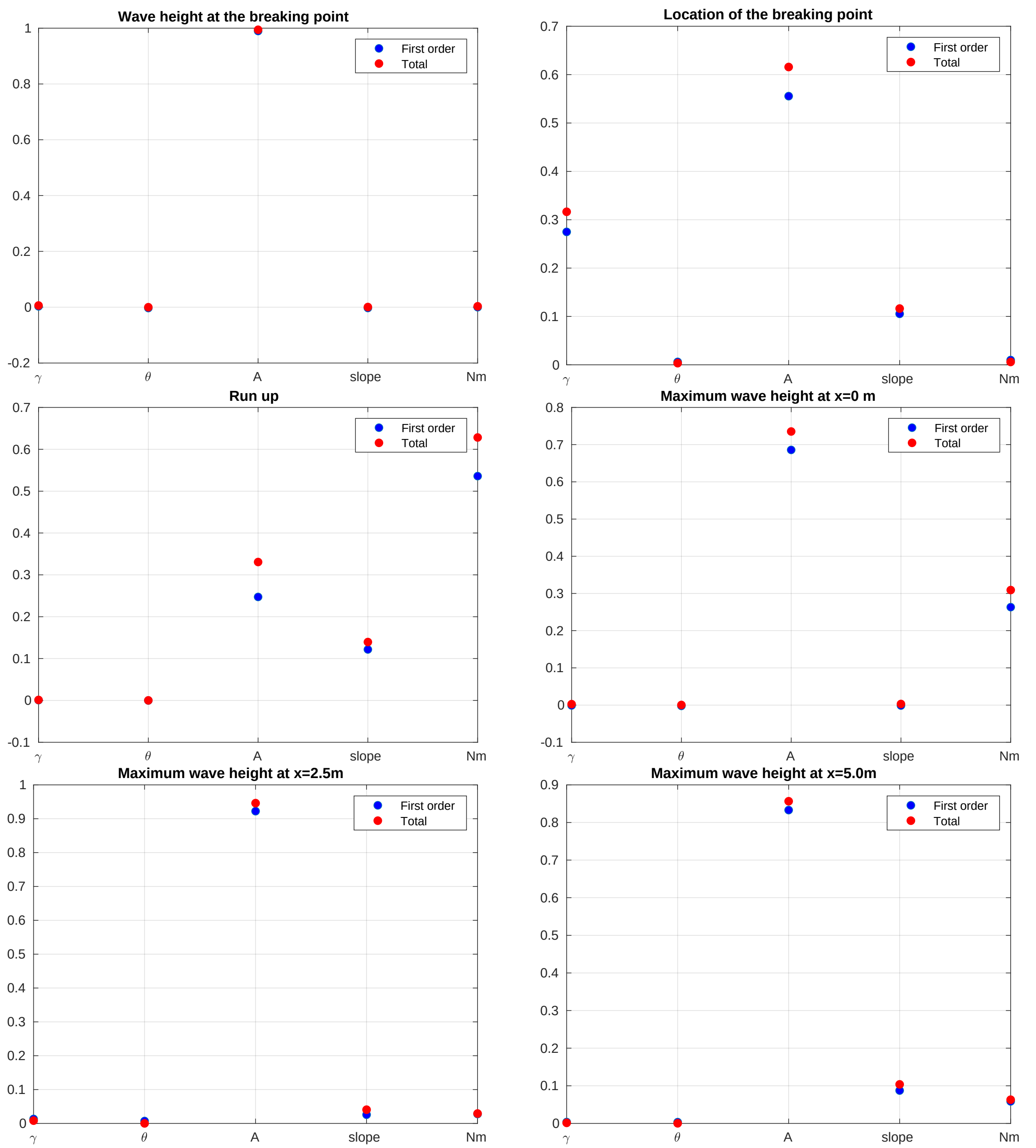

Figure 4: First (blue) and Total (red) Sobol Indices for the output parameters using the hybrid wbc.

the range of amplitudes. Relevant variations of runup are only observed for intermediate values of A, when sensibly reducing γ (early breaking), and for the highest amplitudes when delaying too much the detection (highest values of γ). The local maximum variations of runup observed are of the order of $\sim$5-7%, the predictions remaining relatively close to the experimental data.

The variability of the runup with the slope and the Manning has however a different behaviour. When lowering the value of the detection parameter (early breaking), we see a systematic impact of the order of $\sim$

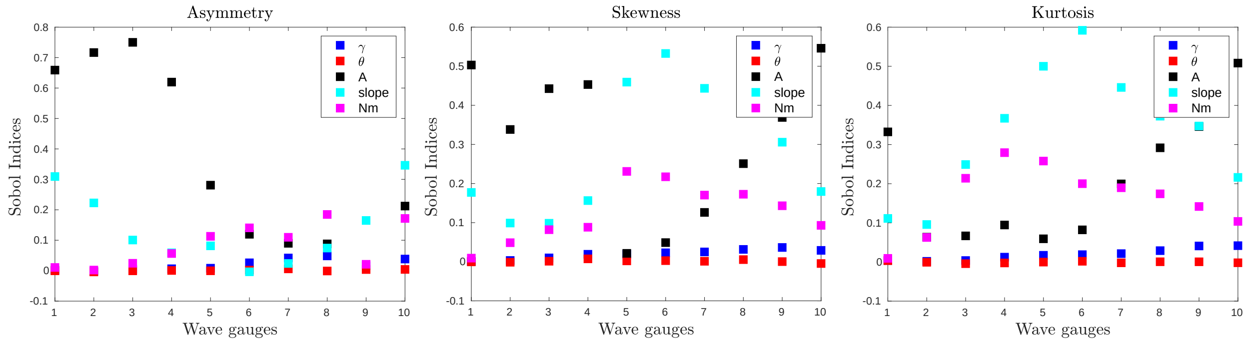

Figure 5: First order Sobol indices for Asymmetry, Kurtosis and Skewness measured on the wave gauges

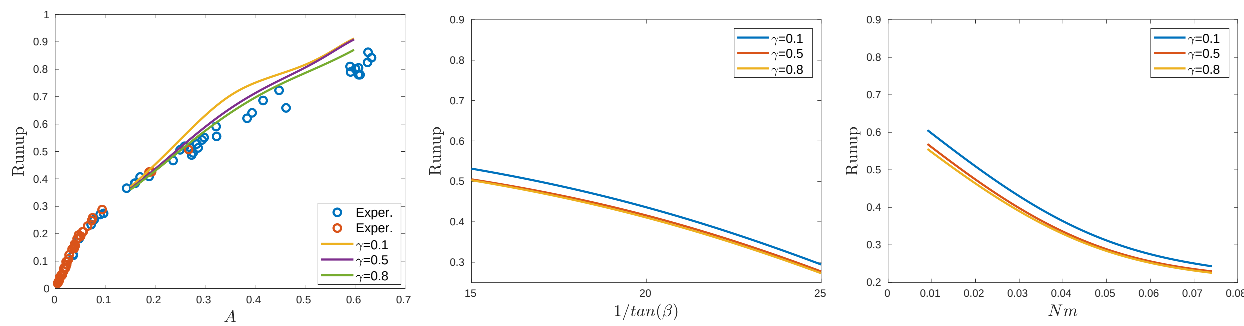

Figure 6: Run up as a function of wave's amplitude were slope=$1/19.85,\ Nm = 0.01$ (left), of slope were $A = 0.28,\ Nm = 0.01$ (center) and of the Nm number were slope=$1/19.85,\ A = 0.28$ (right)

9-10% over the entire the range of slopes. A similar trend is observed when looking at the dependence with the Manning coefficient, the lowest value of the parameter γ leading consistently to variations of the runup of the order of the maximum observed when changing the amplitude. This suggests that the runup may be more tolerant to some input error on the wave amplitude, while the correct parametrization of the breaking detection for the hybrid closure is more crucial when we have uncertainties in the slope and friction.

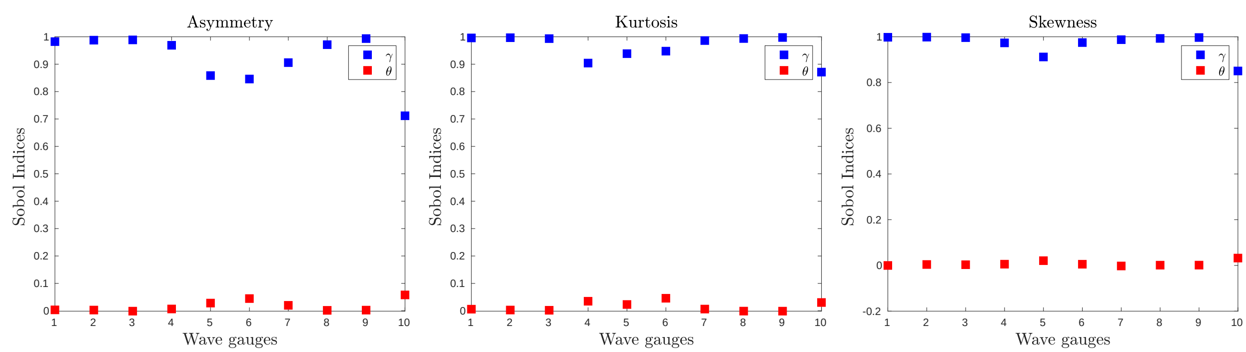

Figure 7: First order Sobol indices for Asymmetry, Kurtosis and Skewness measured on the wave gauges, when $A = 0.28m$.

For completeness we now focus on the impact of the model parameters, fixing all the physical constants to $A = 0.28$, slope=$-1/19.85$ and $Nm = 0.01$. We are now left only with the γ and θ parameters controlling wave breaking detection, and we have seen already that the first has a dominant contribution to output variations in the general case. For fixed values of the problem parameters, we see a similar behaviour of the Sobol indices in Figure 7. The γ parameter is responsible for almost 100% of the variance, except in the intermediate regions (breaking area), and in the last gauge.

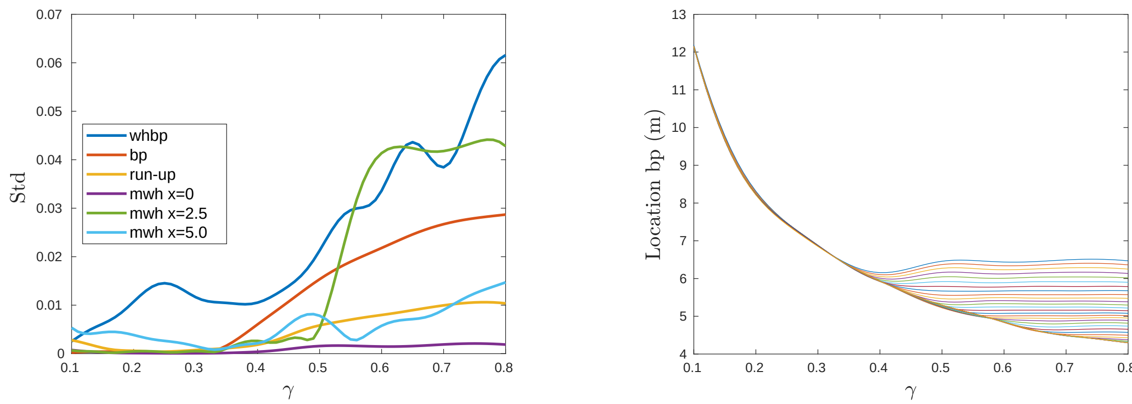

Figure 8: Standard deviation for the output parameters in respect to gamma (left) and evolution of the location of the bp.

To try to understand how θ still affects the solution, we report on Figure 8 the variance of dimensionless outputs for fixed values of γ. More precisely, the wave height at the breaking point (whbp) and the maximum height for the wave (mwh) at $x = 0,\ 2.5,$ and $3m$ are scaled by the initial amplitude of the wave, and the location of the breaking point (bp) is scaled by the initial position of the solitary. Finally, the run-up values are scaled by the experimental values for the specific case. We see that variations above 1% are only observed for the position of the breaking point, and for the maximum wave height at breaking point and at $x = 2.5$, corresponding to the initial shoreline. In particular, for all these outputs we see a clear increase variance above a certain threshold of γ. This is the sign that above this value breaking detection is related to the second criterion. This is visualized in the right picture on the figure showing the variation of the position of the breaking point with γ for different samples of the metamodel. Above a critical value of γ this position only depends on θ.

Finally, we examine the First order Sobol indices for the asymmetry, skewness and kurtosis of the wave in the wave gauges. The behavior observed confirms previous observations: γ is most dominant parameter, but some impact of θ is observed in the wave gauges close and after the breaking point. This is further confirmed when looking the spatial evolution of the sensitivity indices of the free surface elevation, shown in the left down plot in Figure 10. The spatial distribution is computed at $t = 6.0sec$, i.e. during the breaking process. Red line denotes the location of the closest wave gauge, i.e. WG 10. Since γ is the dominant parameter in the whole domain we want to check where exactly it plays a crucial role, that is why we plot right next to the figure the absolute mean deviation of the water depth over the water depth, at $t = 6sec$. We observe that the maximum mean deviation is 25%, compered to the mean water depth, and occurs between $3m$ and $4m$. After this region the deviation is less than 5%, so γ is still the dominant parameter but in negligible changes of the water depth. The temporal distribution 9 of the sensitivity indices is computed for the free surface elevation recorded at WG10 ($x = 5.0m$) and WG6 ($x = 2.5m$). Like before, θ is not the dominant parameter while the bore is breaking. To confirm how much it contributes we plot again the deviation of the free surface elevation. We can see that the maximum deviation grows as the wave propagates on shore reaching more than 90%, meaning that mainly γ is affecting the bore front.

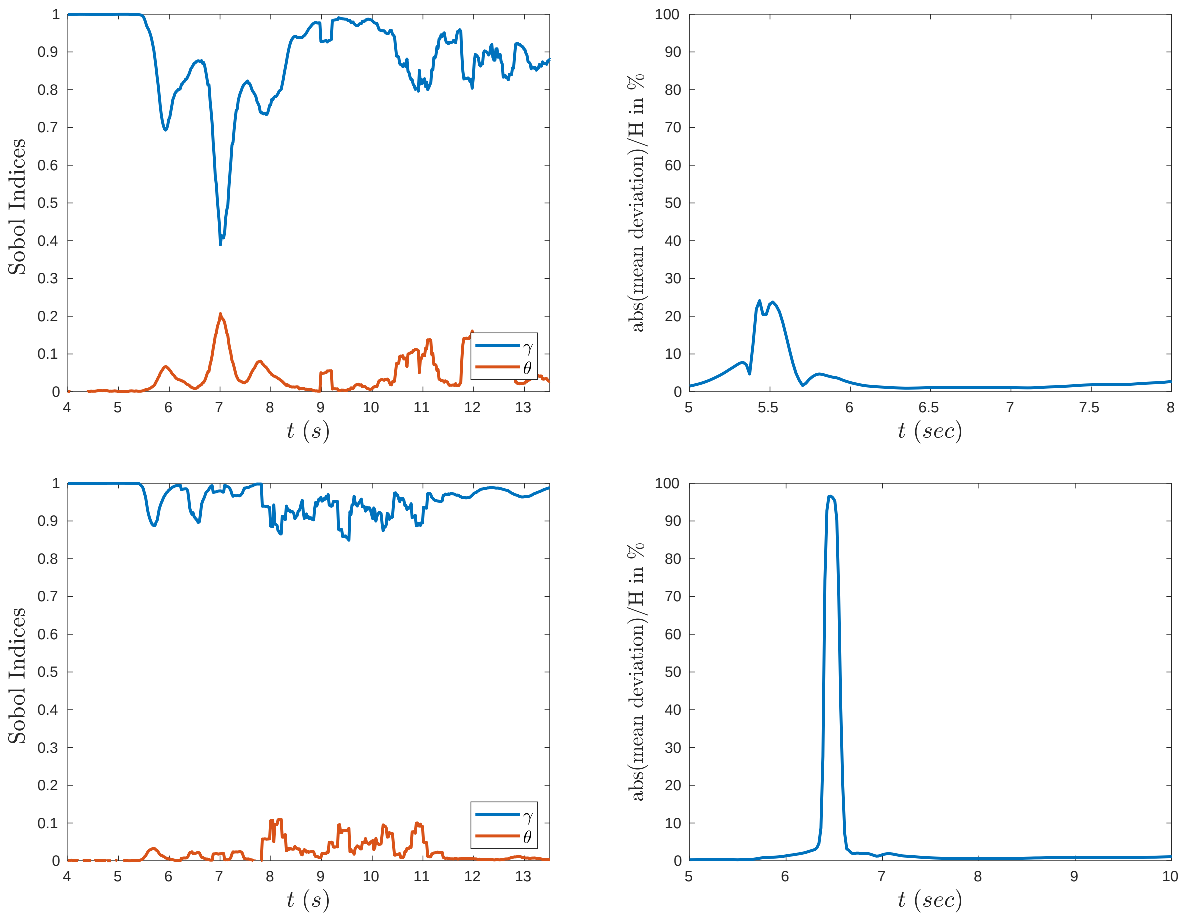

Figure 9: Temporal evolution of the First Order SI and deviation of the free surface elevation, at WG10 and WG 6.

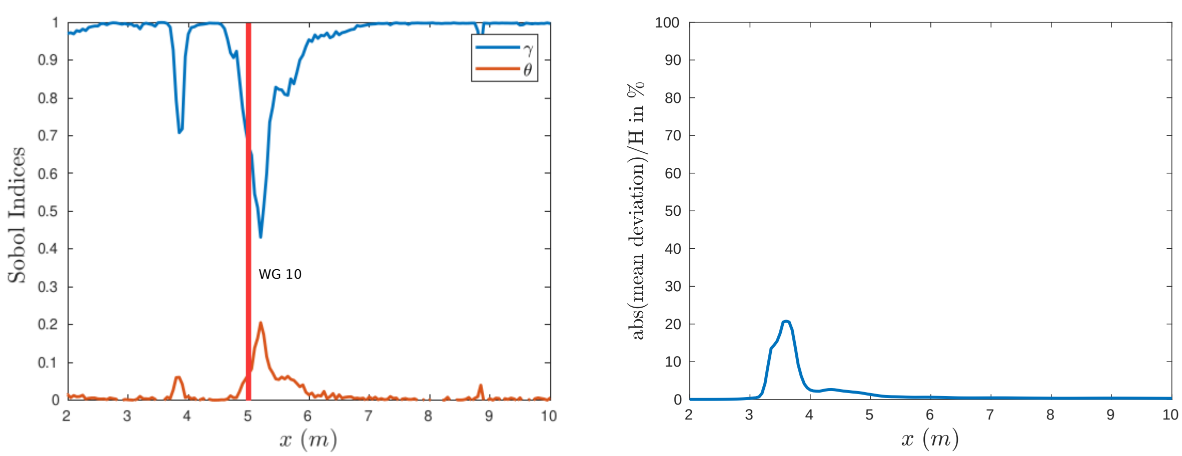

Figure 10: Spatial evolution of the First order SI and the deviation of water depth at $t = 6.0s$

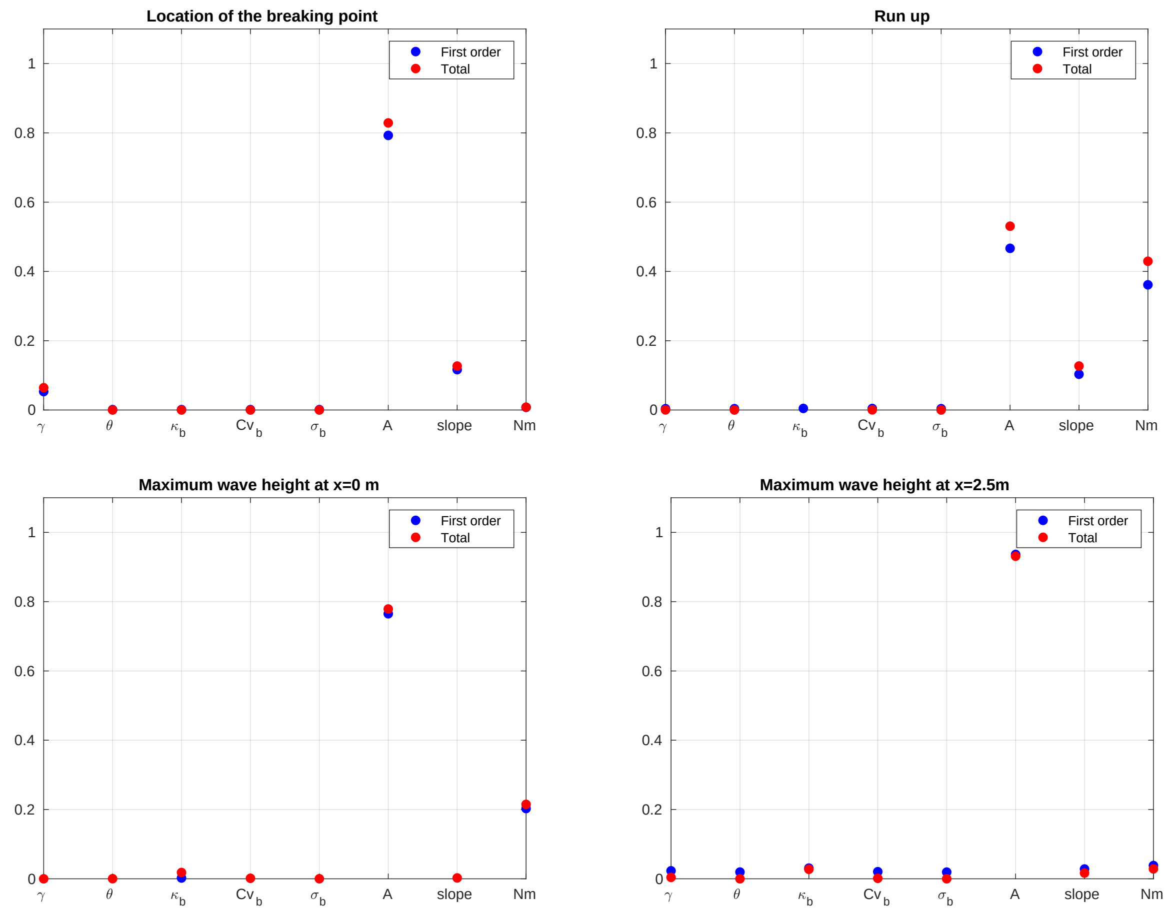

Figure 11: First (blue) and Total (red) Indices for three output parameters using the TKE wbc.

4.3 Parameter sensitivity analysis using the TKE wbc

We perform now a similar exercise for the TKE wave-breaking closure. In this case, in addition to the previous five parameters $(\gamma,\ \theta,\ A,\ \text{slope},\ Nm)$, we have three more parameters associated with the TKE model: κ, C_v and σ. For the analysis of this section, we assume the same uniform distribution for $(\gamma,\ \theta,\ A,\ \text{slope},\ Nm)$ already used in section §4.2. We moreover assume uniform distributions for the TKE mode parameters as $\kappa \in [0.5,\ 2.5]$, $C_v \in [5,\ 15]$, and $\sigma \in [10,\ 25]$.

Figure 11 presents the first order Sobol indices for the location of the breaking point, the run-up and the maximum wave height at the wave gauges 1 and 6. The other outputs, namely the maximum wave height at breaking point and wave gauge 10 are essentially only impacted by the amplitude. For conciseness we leave them out. From the figure we reach similar conclusions as for the hybrid closure: the model parameters have significantly smaller impact on the outputs than the problem setup parameters, with the exception of γ that does affect the breaking point location. Among the problem parameters, the amplitude is again the dominating one, with friction following. The slope has some smaller impact on the runup.

As before, we compute the First order Sobol indices for the asymmetry, kurtosis and skewness of the wave. Besides some impact of κ on the kurtosis, model parameters provide negligible contributions to the variance of the statistical moments. The wave's amplitude is once more the dominant parameter, with

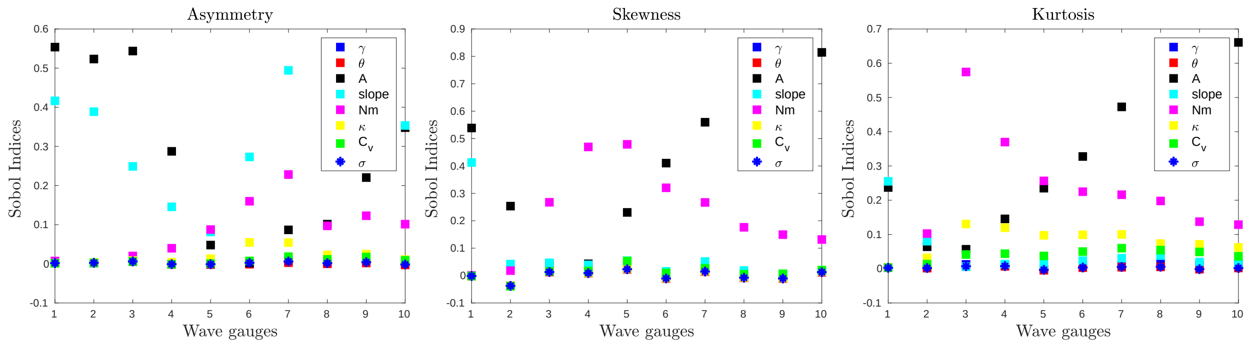

Figure 12: First order Sobol indices for Asymmetry, Kurtosis and Skewness measured on the wave gauges

friction also providing a significant contribution to kurtosis and skewness. The slope mostly affects the asymmetry of the wave before and after breaking. We recall that κ controls both the magnitude of the eddy viscosity as well as the dissipation of turbulent kinetic energy. The latter may affect the rate at which the eddy viscosity is reduced in space, thus affecting the wave tail.

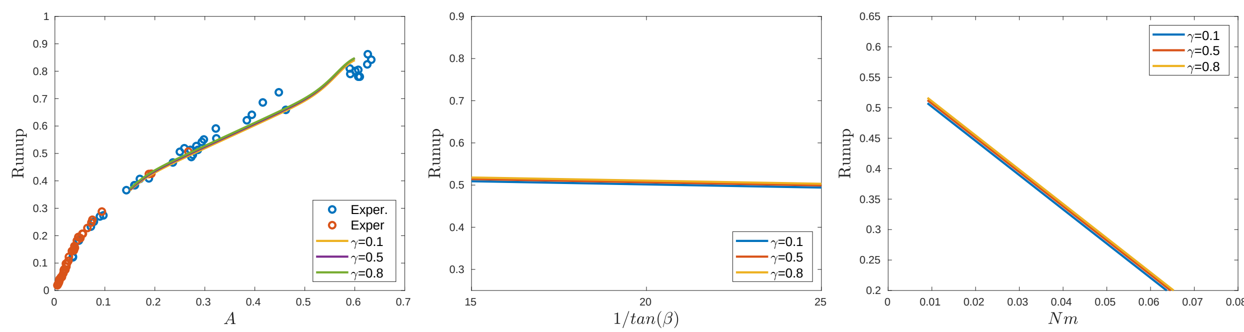

Figure 13: Run up as a function of wave's amplitude were slope=$1/19.85$, $Nm = 0.01$ (left), of slope were $A = 0.28$, $Nm = 0.01$ (center) and of the Nm number were slope=$1/19.85$, $A = 0.28$ (right)

Let us now focus on the run-up process. As a first exercise, we fix the TKE closure setting $\kappa = 0.5$, $C_\nu = 5.0$, $\sigma = 10.0$, and evaluate the impact of γ, of the slope and of the friction coefficient as done in the previous section. The plots obtained are reported on Figure 13. For the amplitude dependence, we see that the match with the data in the plot against the dependence is very good. We also note a behaviour much different than the one of the hybrid closure. For the present closure the wave breaking detection has virtually no effect. Also, while the runup dependence on the Manning is similar to the one of Figure 6, here there is a weaker dependence on the slope, contrary to what is observed for the hybrid closure.

In the same spirit, we fix $\gamma = 0.6$ and vary the TKE parameters. The resulting behaviour is the one reported on Figure 14. Again we see a secondary and almost negligible impact of the model parameters on the runup. This is quite interesting as it means that the value predicted is quite robust with respect to the parametrization of the wave breaking, and mostly controlled by the problem setup.

To have a closer look at this issue, as before we fix the problem setup, namely we set $A = 0.28$, $slope = -1/19.85$, $Nm = 0.01$. We repeat the sensitivity study with the remaining subset of parameters, all related to breaking detection and dissipation. The first order (blue) and total (red) Sobol indices are reported in Figure 15 for the same outputs of Figure 11.

We can see that breaking detection, and in particular γ, has a major impact on the maximum wave

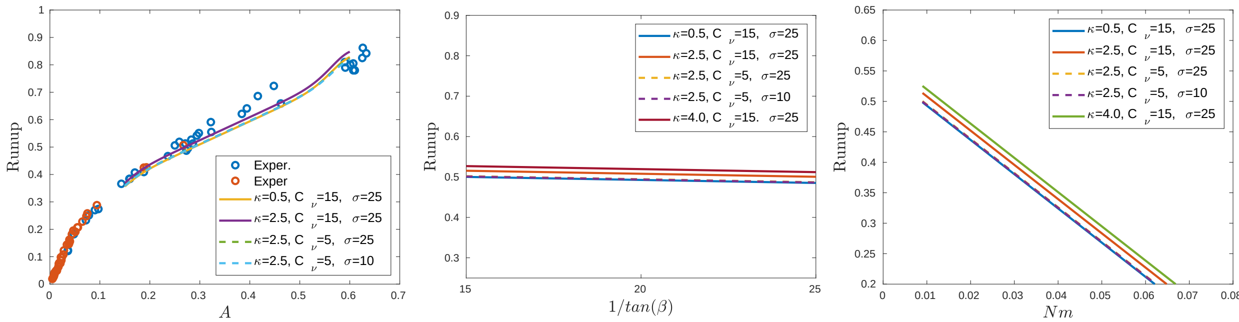

Figure 14: Run up as a function of wave's amplitude were slope=$1/19.85$, $Nm = 0.01$ (left), of slope were $A = 0.28$, $Nm = 0.01$ (center) and of the Nm number were slope=$1/19.85$, $A = 0.28$ (right)

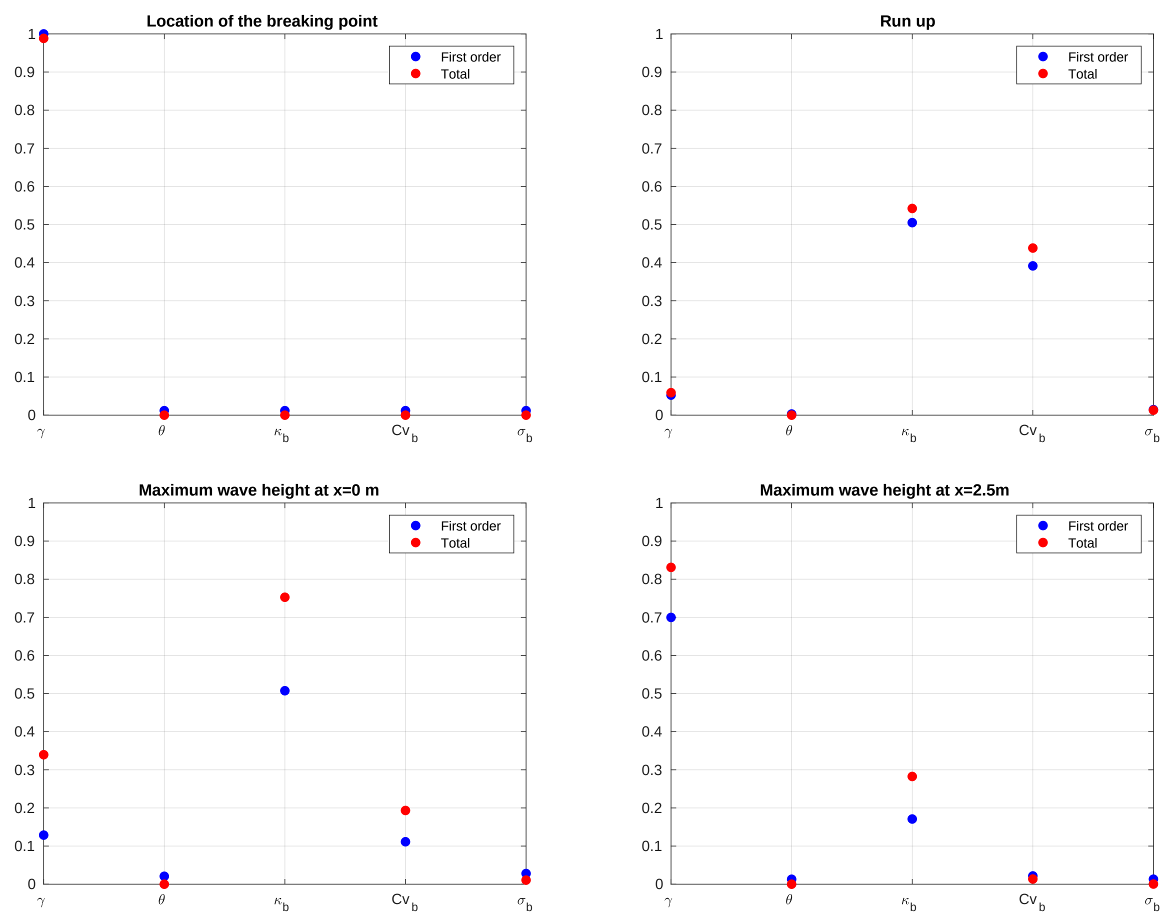

Figure 15: First (blue) and Total (red) Sobol Indices for the run-up and the maximum wave height at WG1 and WG6 using the TKE wbc.

heights. Large contributions to the variance also come from κ and C_ν, which in particular are the only dominating parameters for the run-up. We see here also a non-negligible difference between some first and Total indices, indicating the presence of some important interactions on the maximum wave's height.

More precisely at WG 1 ($x = 0m$) and WG 6 ($x = 2.5m$) the second order index for $\gamma\kappa$ is 0.17 and 0.109 respectively. To have a better insight we study the evolution of the water depth in the whole domain in space by freezing the time in specific characteristic time spots.

In 16 we report the spatial evolution of the water depth, sensitivity indices and absolute mean deviation of the water depth, at times representative of incipient $t = 6sec.$, and after collapsing on the shore ($t = 8.2sec$). Its clear that in the early breaking time γ and κ are the predominant parameters affecting the water depth until 6m in the domain. More precisely the deviation of the water depth compared to the mean water depth is maximum 2.5%. This shows the robustness of the TKE model in contrast to the hybrid where the maximum deviation at the same time is 25%. After the collapse of the bore on the shore κ and C_v are taking the lead and strongly affect the free surface elevation especially, the front of the wave. This is confirmed by the absolute mean deviation of the water depth shown in the low right figure of 16.

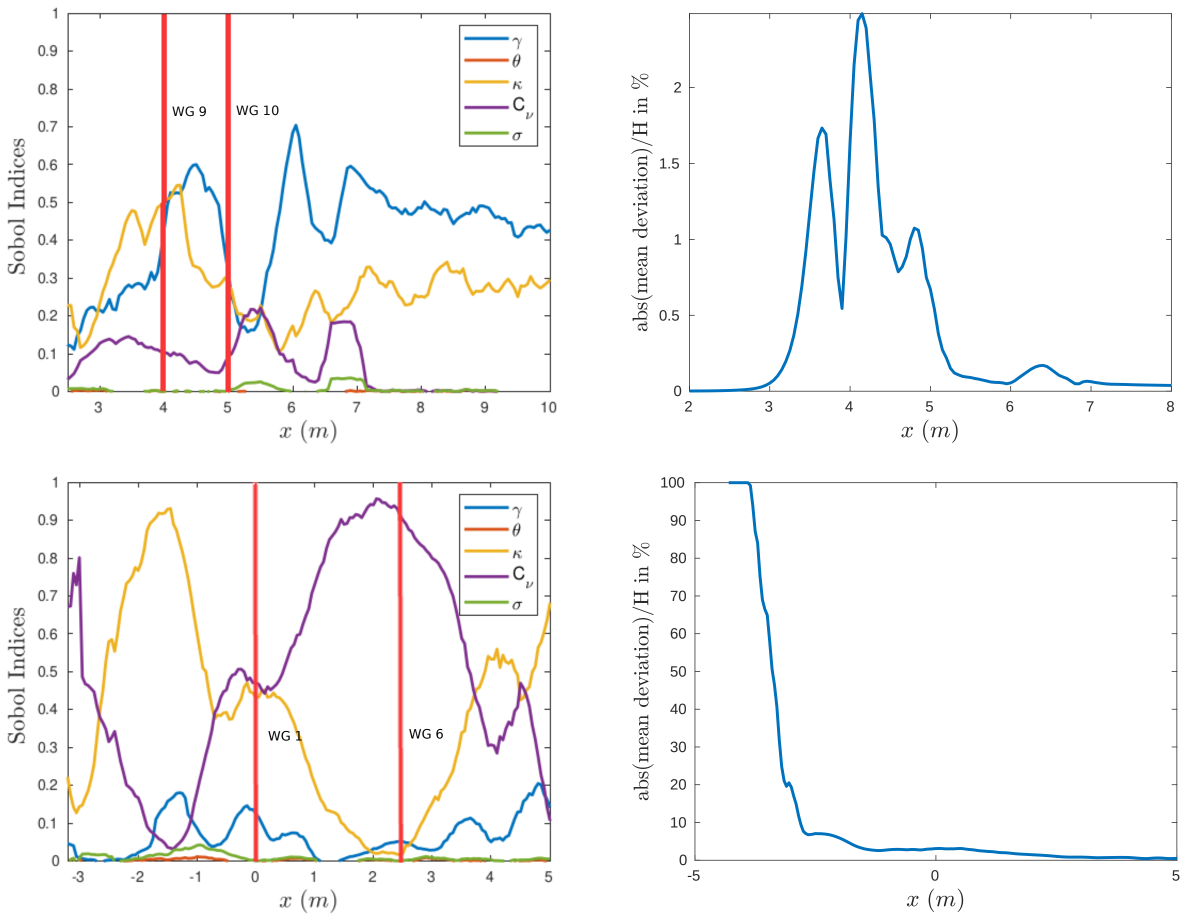

Figure 16: Spatial evolution of the Sobol indices and the absolute mean deviation for $t = 6,\ 8.2sec.$

Similarly and for the sake of completeness, we examine the temporal evolution of the SI and the absolute mean deviation of the free surface elevation in the wave gauges. For WG10 the dominating parameters are $\gamma,\ \kappa$ and C_ν but mean deviation is close to zero during all the time. The predominant parameters in WG 6 are the same as in WG 10 and the absolute mean deviation is maximum 40% within the time interval 6 and 7 seconds.

Next we study their temporal evolution in the wave gauges. For WG 10 the absolute mean deviation is almost zero, meaning that the parameters are not affecting the evolution of the free surface elevation. We report in Figure 17 , and for the sake of completeness the results for WG6. The absolute mean deviation reaches almost 40% within the time interval 6 to 7seconds where the most important parameters are κ and C_ν. A consistent behavior is observed in all the WG's verifying that the wave-breaking mechanism is "following" the wave as it breaks until its collapse on the shore.

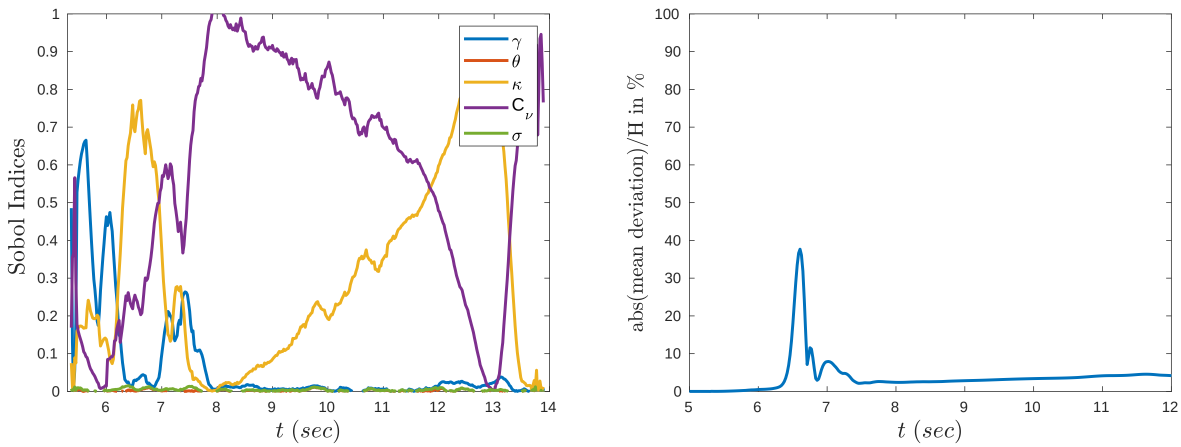

Figure 17: Temporal evolution of the Sobol indices and the absolute mean deviation of the free surface at WG6.

Finally, we plot in Figure 18 the first order Sobol indices for the wave moments in the wave gauges. Wave's asymmetry and kurtosis are affected mainly by the parameters κ and C_ν, i.e. from the dissipation and production term in the transport equation 5. On the other hand, skewness, i.e the crest to trough shape, is mainly affected by C_ν which probably controls the kinetic energy cutoff before and after the breaking front.

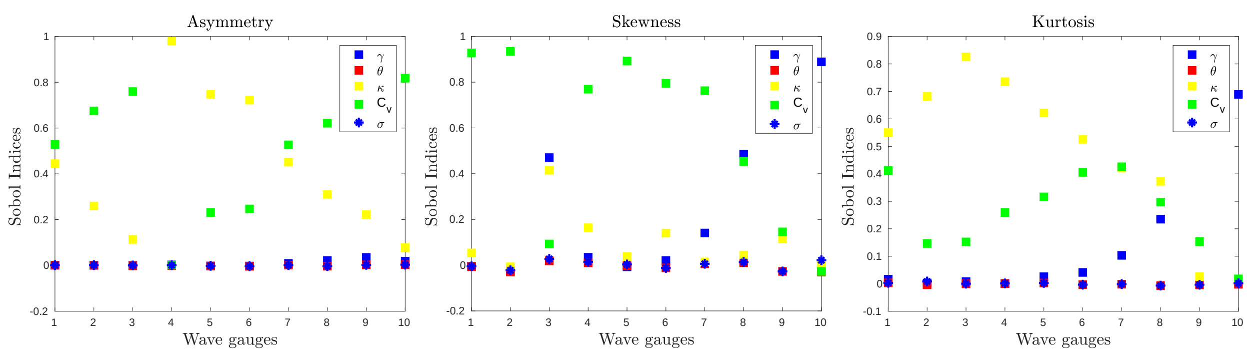

Figure 18: First order Sobol indices for Asymmetry, Kurtosis and Skewness measured on the wave gauges

5 Summary of the results and comparison of the two breaking closures

We conclude that, for the hybrid model, the initial amplitude of the wave is the most dominant parameter corresponding to almost every output metric before the run-up. The shape of the wave depends on the initial amplitude, and also on the slope of the beach during the run-up. The kurtosis of the wave is affected by mostly the slope of the beach and the friction coefficient. For the hybrid model, the statistical metrics are most affected by the wave-dependent input parameters. Among the model-specific parameters such as γ and θ, the former one has the most effect on the statistical metrics, more so after the value of $\gamma > 0.3$. θ, although not a dominant parameter for wave breaking, does contribute to the tail of the wave post breaking.

For the TKE model, the Sobol indices indicate that the wave amplitude is the most dominant parameter for the deterministic metrics, similar to the hybrid model. However, when the amplitude and the slope values are fixed, it emerges that κ and Cv are the parameters which most affect the output metrics.

It is also seen that, among the two candidate models, the hybrid model shows a significant effect of γ on the run-up of the wave, in contrast to the TKE model, which shows almost no effect of changes in γ. Further, in the case of the hybrid model, the effect of γ is less sensitive to changes in the initial amplitude when the slope and the friction parameters are fixed. This implies that, when using the hybrid model, small errors in the slope or the friction parameters may manifest into considerable errors in the prediction of the maximum run-up. On the other hand, the TKE model doesn't show a similar effect of γ on the run-up process over a range of values for the initial amplitude, friction and the slope parameters. However, once we fix the value of γ, the predicted run-up of the waves reflects a larger influence of the friction and the slope parameters compared to the initial amplitude of the wave when the TKE model parameters are varied. Thus, it can be concluded that the run-up predicted by either of the models may be more sensitive to the errors in the measurement of the friction or the slope parameters, in comparison to the initial amplitude of the waves, and will depend on the choice of γ for the hybrid model and the model parameters for the TKE model.

Finally, the absolute mean deviation of the free surface elevation measured at the wave gauges reveals that the hybrid model can lead to large differences between the observed values of the free surface elevation and the mean free surface elevation, up to 90%. On the other hand TKE model is more robust mainly on the off-shore region.

We have to note the differences of the Sobol indices for the statistical parameters between the two wave-breaking closures, see Figure 5 and 12. This is due to the fact that the wave-breaking closures simulate the breaking wave in a different way. The hybrid closure simulates the breaking wave as a traveling bore while the one obtained using the tke closure has a smoother propagating front.

6 Conclusion and perspectives

In this work, we systematically analyse the effect of various parameters on the performance of two models for wave breaking. In particular, we perform the Analysis of Variance (ANOVA) which gives First and Total-order Sobol's indices, which in turn quantify the sensitivities of specified output metric on the input parameters. We first obtain a Gaussian Process Regression metamodel of the high-fidelity CFD model prior to performing the ANOVA analysis. We quantify the sensitivities using several deterministic metrics such as the maximum height of the wave, the location of the breaking of the wave, the maximum run up, the time-series values at several gauges, as well as stochastic metrics such as the skewness, asymmetry and kurtosis of the time-series of the wave at several gauge locations. We study different configurations of the classical problem of the solitary wave run-up on a slope. We also study the time-evolution of the Sobol indices at specific gauge locations. The importance of this analysis is twofold. First, it helps us gain insights into physics of nonlinear and dispersive waves, specifically during the breaking and the run-up phases. Secondly, it quantitatively distinguishes between the two prominent approaches for modeling the breaking of the waves, viz. the hybrid method and the TKE model for wave breaking.

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