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Adaptive State Observation of Linear Time-Varying Systems with Delayed Measurements and Unknown Parameters †

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Summary

In this paper we address the problem of adaptive state observation of linear time-varying systems with delayed measurements and *unknown* parameters. Our new developments extend the results reported in [1] and [2]. The case with known parameters has been studied by many researchers—see [3,4] and references therein. We show in this paper that the generalized parameter estimation-based observer design proposed in [5] provides a very simple solution for the unknown parameter case. Moreover, when this observer design technique is combined with the dynamic regressor extension and mixing estimation procedure [6,7], the estimated state and parameters converge in *fixed-time* imposing extremely weak excitation assumptions.

KEYWORDS:
Linear time-varying systems, nonlinear adaptive state observers, delay systems, parameter estimation

1 | INTRODUCTION AND PROBLEM FORMULATION

It is common in control applications that real sensor devices provide measurements with time-varying delays. This fact makes the task of the state estimation for a dynamical system more complicated. This problem has been explored by many authors. In the case of linear time invariant (LTI) systems, this issue is well understood and the observer convergence can be verified by checking the feasibility of a linear matrix inequality [8]. On the other hand, for linear time-varying (LTV) systems this problem is widely open—see the literature review and references in the recent papers [3,9,4]. In [1] and [2] LTV systems with constant unknown parameters entering in the state dynamics were considered. In [2] we treat the case when the output signal is not delayed. In [1] the case with delay in the output signal is considered, however it is assumed that the term containing the unknown parameters is available to the estimator *without delay*. This allows the designer to use this term for the key step of injecting this output signal to stabilize the observer dynamics. In many practical applications the term containing the unknown parameters is only measurable with a *time delay*, a scenario where the adaptive state estimation problem is much more complicated because of the fact that the delay hampers the output injection step mentioned above. Providing a solution to this more challenging problem is the main contribution of this paper.

We consider in the paper single-input-single output LTV systems of the form[1]

$$\begin{cases} \dot{x}(t) = A(t)x(t) + \kappa C^\top(t)x(t) + B(t)u(t),\ x(0) = x_0 \in \mathbb{R}^n,\ t \geq 0, \\ y(t) = C^\top(\phi(t))x(\phi(t)), \end{cases} \tag{1}$$

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[0]**Abbreviations:** LTV, linear time-varying; LTI, linear time invariant; GPEBO, generalized parameter estimation-based observer; DREM, dynamic regression extension and mixing; LRE, linear regression equation; IE, interval excitation

[1]To simplify the notation and without loss of generality we assume that the initial time is zero. Also, to simplify the presentation, we have assumed a *single-input-single-output* system. As will become clear below the extension to multivariable systems is straightforward.

where $x(t) \in \mathbb{R}^n$ is the *unmeasurable* state, $u(t) \in \mathbb{R}$ is a *known* input, $y(t) \in \mathbb{R}$ is the *measured* output and $\phi(t) \in \mathbb{R}$ is a function that defines the measurement delay. We assume that $A(t)$, $B(t)$ and $C(t)$ are *known*, continuous and bounded but the constant vector $\kappa \in \mathbb{R}^n$ is *unknown*. We bring to the readers attention the fact that the term that contains the unknown parameter, *i.e.*, $\kappa C^\top(t)x(t)$, is only available for measurement—via $y(t)$—with a time delay.

Following standard practice in observer theory we assume that the system (1) is bounded-input-bounded-output stable, that its autonomous part, that is, $\dot{x}(t) = [A(t) + \kappa C^\top(t)]x(t)$ is uniformly stable and that the input $u(t)$ is *bounded*—additional assumptions on the system are given below.

One can alternatively view the function $\phi(t)$ in (1) in the more standard form $\phi(t) = t - D(t)$, where $D(t) \geq 0$ is a time-varying delay. However, the formalism involving the function $\phi(t)$ turns out to be more convenient, particularly because our design requires the *existence* of the inverse function of $\phi(t)$, *i.e.*, a function $\phi^{\mathtt{I}}(\cdot)$ such that $\phi^{\mathtt{I}}(\phi(t)) = t$, so we will proceed with the model (1).

We assume that $\phi(t)$ satisfies the following.

Assumption 1. $\phi(t)$ is a continuous *known* function verifying

$$0 \leq \phi(t) \leq t,\ 0 < \dot{\phi}(t) \leq m,\ \forall t \geq 0, \tag{2}$$

with $\dot{\phi}(t)$ also *known* and $m \in \mathbb{R}$.

We note that Assumption 1 ensures that, via the Implict Function Theorem, the inverse function $\phi^{\mathtt{I}}(\cdot)$ exists—see [10,4] where similar assumptions on the delay function are made.

Problem Formulation Given the system (1), with the delay function $\phi(t)$ verifying Assumption 1, design an adaptive observer

$$\dot{\chi}(t) = F(\chi(t), u(t), y(t)),$$
$$\begin{bmatrix} \hat{x}(t) \\ \hat{\kappa}(t) \end{bmatrix} = H(\chi(t), u(t), y(t))$$

with $\chi(t) \in \mathbb{R}^{n_\chi}$ such that all signals are bounded, and *fixed-time convergence* (FTC) of the estimated state and parameters to their actual values is ensured, that is,

$$\hat{x}(t) = x(t),\ \hat{\kappa}(t) = \kappa,\ \forall t \geq t_c, \tag{3}$$

for some $t_c \in (0, \infty)$ and for all $x_0 \in \mathbb{R}^n$, $\chi(0) \in \mathbb{R}^{n_\chi}$.[2]

As indicated in the paper abstract the design of our adaptive observer relies on the use of two techniques: generalized parameter estimation-based observer (GPEBO) and the dynamic regression extension and mixing (DREM) estimator. GPEBO is a new technique to design observers for state-affine nonlinear systems reported in [5] , which generalizes the original PEBO introduced in [12]. The main novelty of PEBO and GPEBO[5], is that the state observation problem is reformulated as a problem of *parameter estimation*. Combining GPEBO with the recently introduced DREM parameter estimator[6]—in particular the *FTC* version reported in [7]—we propose an observer that converges in fixed time under a extremely weak assumption, namely *interval excitation* (IE) of the regressor vector. The interested reader is referred to the aforementioned papers for further details on GPEBO and DREM.

The remainder of the paper is organized as follows. In Section 2 we present a reparameterization of the system (1) that is used in the sequel. Section 3 is devoted to the application of GPEBO for the derivation of a linear regression equation (LRE) related to the state observation task and the design of the parameter estimator for the associated constant parameter vector. The final expression for the estimate of the state $x(t)$ is given in Section 4. Simulation results, which illustrate the performance of the proposed observer are presented in Section 5 and the paper is wrapped-up with concluding remarks in Section 6.

Notation. I_n is the $n \times n$ identity matrix and $0_{p\times q}$ is a $p \times q$ matrix of zeros. For $x \in \mathbb{R}^n$ we denote the Euclidean norm as $|x| := \sqrt{x^\top x}$. For a matrix $M \in \mathbb{R}^{n\times n}$, the symbol $\|M\|$ denotes the induced norm.

[2]We underscore the fact that we aim at *fixed*-time convergence, where there is an upper bound for the convergence time for arbitrary initial conditions—this is in contrast to *finite*-time convergence, where the convergence time depends on the initial condition [11].

2 | A REPARAMETERIZATION OF THE SYSTEM

In this section we propose a reparameterization of the system (1) that is instrumental for the design of the proposed observer.

Lemma 1. Consider the LTV system (1) with $\phi(t)$ verifying Assumption 1. The dynamics of the system may be rewritten as

$$\dot{z}(t) = \mathcal{A}(t)z(t) + \dot{\phi}(t)\kappa y(t) + \mathcal{B}(t)v(t), \tag{4a}$$

$$y(t) = \mathcal{C}^\top(t)z(t), \tag{4b}$$

where we defined the matrices

$$\mathcal{A}(t) := \dot{\phi}(t)A(\phi(t)),\ \mathcal{B}(t) := \dot{\phi}(t)B(\phi(t)),\ \mathcal{C}(t) := C(\phi(t)), \tag{5}$$

and the signals

$$z(t) := x(\phi(t)),\ v(t) := u(\phi(t)). \tag{6}$$

Proof. Due to the fact that $\kappa C^\top(t)x(t) = \kappa y(\phi^{\mathrm{I}}(t))$ we can rewrite (1) in the following form

$$\dot{x}(t) = A(t)x(t) + \kappa y\left(\phi^{\mathrm{I}}(t)\right) + B(t)u(t), \tag{7a}$$

$$y(t) = C^\top(\phi(t))x(\phi(t)). \tag{7b}$$

After substituting $\phi(t)$ instead of t into (7a) we get

$$\begin{aligned}\frac{dx(\phi(t))}{d\phi(t)} &= A(\phi(t))x(\phi(t)) + \kappa y\left(\phi^{\mathrm{I}}(\phi(t))\right) + B(\phi(t))u(\phi(t)) \\ &= A(\phi(t))x(\phi(t)) + \kappa y(t) + B(\phi(t))u(\phi(t)).\end{aligned}$$

Noting that, in view of Assumption 1, $\dot{\phi}(t)$ is bounded away from zero, we multiply the left and right parts of the previous equation by $\dot{\phi}(t)$ to get

$$\begin{aligned}\frac{dx(\phi(t))}{d\phi(t)}\dot{\phi}(t) &= \dot{\phi}(t)A(\phi(t))x(\phi(t)) + \dot{\phi}(t)\kappa y(t) + \dot{\phi}(t)B(\phi(t))u(\phi(t)) \\ &= \dot{x}(\phi(t)).\end{aligned}$$

The proof is completed invoking the definitions (5) and (6). □□□

3 | APPLICATION OF GPEBO AND DREM-BASED PARAMETER ESTIMATOR

In this section we first apply GPEBO to the model (4) to derive a LRE containing the unknown, constant parameter vector κ and express the state $z(t)$, defined in (4), in terms of the constant parameter vector associated to this LRE. Then, we use this LRE to generate, via DREM, an estimate for them with FTC.

3.1 | Assumptions

To design the parameter estimator we need to impose some restrictions on the matrices $\mathcal{A}(t), \mathcal{B}(t)$ and $\mathcal{C}(t)$ defined in (5). To simplify the notation we introduce a new matrix

$$\mathbf{A}(t) := \mathcal{A}(t) - \mathcal{L}(t)\mathcal{C}^\top(t), \tag{8}$$

where $\mathcal{L}(t) \in \mathbb{R}^n$ is a continuous bounded vector satisfying the assumptions below.

Assumption 2. The state transition matrix associated to the matrix $\mathbf{A}(t)$, denoted $\Phi_{\mathbf{A}}(t,\tau)$, verifies

$$\|\Phi_{\mathbf{A}}(t,\tau)\| \leq c_1,\ \forall t \geq \tau \geq 0.$$

Assumption 3.

$$\int_\tau^t \|\Phi_{\mathbf{A}}(t,s)\mathcal{B}(s)\|ds \leq c_2,\ \forall t \geq \tau \geq 0.$$

Assumption 2 implies *uniform detectability* of the pair $(A(t), C(t))$ [13, Theorem 6.4] of the homogeneous part of the system (4), while Assumption 3 is a necessary and sufficient condition for *bounded-input-bounded-state* stability [13, Theorem 12.2] of the system (4) with an output injection $-\mathcal{L}(t)y(t)$ and $\kappa = 0$.

3.2 | Derivation of the LRE

Lemma 2. Consider the system (4). Introduce the following dynamic extension

$$\dot{\zeta}(t) = \mathbf{A}(t)\zeta(t) + \mathcal{L}(t)y(t) + \mathcal{B}(t)v(t), \quad \zeta(0) = 0_{n\times 1}, \tag{9a}$$

$$\dot{\Upsilon}(t) = \mathbf{A}(t)\Upsilon(t), \quad \Upsilon(0) = I_n, \tag{9b}$$

$$\dot{\chi}(t) = \mathbf{A}(t)\chi(t) + I_n\dot{\phi}(t)y(t), \quad \chi(0) = 0_{n\times n}. \tag{9c}$$

with $\mathbf{A}(t)$ defined in (8) and $\mathcal{L}(t)$ satisfying Assumptions 2 and 3. Define the *measurable* signals

$$\Psi(t) := \left[-\Upsilon(t) \mid \chi(t)\right] \in \mathbb{R}^{n\times 2n} \tag{10a}$$

$$\varphi(t) := C^\top(t)\Psi(t) \in \mathbb{R}^{1\times 2n} \tag{10b}$$

$$\rho(t) := y(t) - C^\top(t)\zeta(t) \in \mathbb{R}. \tag{10c}$$

(i) The state $z(t)$ satisfies the equation

$$z(t) = \Psi(t)\Theta + \zeta(t), \tag{11}$$

with the constant vector $\Theta \in \mathbb{R}^{2n}$ defined as

$$\Theta := \begin{bmatrix}\theta \\ \kappa\end{bmatrix}, \tag{12}$$

and $\theta \in \mathbb{R}^n$ a constant unknown vector.

(ii) The vector Θ verifies the LRE

$$\rho(t) = \varphi(t)\Theta. \tag{13}$$

(iii) The signals $\zeta(t), \Upsilon(t)$ and $\chi(t)$ are *bounded*.

Proof. Define the error signal

$$e(t) := \chi(t)\kappa + \zeta(t) - z(t), \tag{14}$$

which satisfies

$$\dot{e}(t) = \mathbf{A}(t)e(t).$$

Hence, invoking the properties of the *principal matrix solution* of (9b) [13] we can write

$$e(t) = \Upsilon(t)\theta,$$

with $\theta := e(0)$. Replacing this equation in (14) we get

$$z(t) = \chi(t)\kappa + \zeta(t) - \Upsilon(t)\theta.$$

the proof of claim (i) is completed invoking (10) and (12).

Claim (ii) is established multiplying (11) by $C^\top(t)$ and using (7b) and (10).

Finally, the proof of boundedness of the signals $\zeta(t), \Upsilon(t)$ and $\chi(t)$ follows immediately from Assumptions 2 and 3 and boundendness of $v(t)$ and $\dot{\phi}(t)$. □□□

3.3 | Application of DREM estimator

To streamline the formulation of the main result of this subsection we recall the definition of IE of a bounded signal [14], [15, Definition 3.1].

Definition 1. A bounded signal $\Delta(t) \in \mathbb{R}$ is IE if there exists a time $t_{\text{IE}} \in (0, \infty)$ such that

$$\int_0^{t_{\text{IE}}} \Delta^2(s)ds \geq \beta, \tag{15}$$

for some $\beta > 0$.

Proposition 1. Consider the system (4a), (4b) and the LRE (13) derived in Lemma 2. Fix $\lambda > 0$ and introduce the filtered signals

$$\dot{Y}(t) = -\lambda Y(t) + \lambda \varphi^\top(t)\rho(t), \quad Y(0) = 0_{2n\times 1}, \tag{16a}$$

$$\dot{\Omega}(t) = -\lambda \Omega(t) + \lambda \varphi^\top(t)\varphi(t), \quad \Omega(0) = 0_{2n\times 2n}. \tag{16b}$$

Consider the DREM parameter estimator

$$\dot{\hat{\Theta}}(t) = \gamma\Delta(t)\left[Z(t) - \Delta(t)\hat{\Theta}(t)\right], \; \hat{\Theta}(0) = \Theta_0 \in \mathbb{R}^{2n}, \tag{17}$$

with $\gamma > 0$, and we introduced the definitions:

$$\Delta(t) := \det\{\Omega(t)\} \in \mathbb{R}, \tag{18a}$$

$$Z(t) := \operatorname{adj}\{\Omega(t)\}\, Y(t) \in \mathbb{R}^{2n}, \tag{18b}$$

with adj$\{\cdot\}$ is the adjugate matrix.

Define the estimates as

$$\begin{bmatrix} \hat{z}(t) \\ \hat{\kappa}(t) \end{bmatrix} = \begin{bmatrix} \Psi(t) \\ [0_{n\times n} \; I_n] \end{bmatrix} \hat{\Theta}^{\text{FTC}}(t) + \begin{bmatrix} \zeta(t) \\ 0_{n\times 1} \end{bmatrix}, \tag{19}$$

with

$$\hat{\Theta}^{\text{FTC}}(t) = \frac{1}{1 - w_c(t)}\left[\hat{\Theta}(t) - w_c(t)\Theta_0\right], \tag{20}$$

and $w_c(t)$ defined via the clipping function

$$w_c(t) = \begin{cases} w(t) & \text{if } w(t) \leq 1-\mu \\ 1-\mu & \text{if } w(t) > 1-\mu, \end{cases},$$

where

$$\dot{w}(t) = -\gamma\Delta^2(t)w(t), \; w(0) = 1, \tag{21}$$

and $\mu \in (0, 1)$ is a designer chosen parameter. Assume $\Delta(t)$ verifies (15) with

$$\beta = -\frac{1}{\gamma}\ln(1-\mu), \tag{22}$$

then

$$|z(t) - \hat{z}(t)| = 0, \; |\kappa - \hat{\kappa}(t)| = 0 \text{ for } t \geq t_c > 0. \tag{23}$$

is ensured for some $t_c \geq t_{\text{IE}}$, and all signals remain bounded.

Proof. Consider the LRE (13) and the following chain of implications

$$\begin{aligned} (13) \Rightarrow \; & \varphi^\top(t)\rho(t) = \varphi^\top(t)\varphi(t)\Theta \quad (\Leftarrow \varphi^\top(t)\times) \\ \Rightarrow \; & Y(t) = \Omega(t)\Theta \; \left(\Leftarrow \frac{\lambda}{\mathbf{p}+\lambda}[\cdot] \text{ and } (16)\right) \\ \Rightarrow \; & Z(t) = \Delta(t)\Theta, \; (\Leftarrow \operatorname{adj}\{\Omega\}\times \text{ and } (18)), \end{aligned}$$

with $\mathbf{p} = \frac{d}{dt}$, where we have used the fact that for any, *possibly singular*, $n \times n$ matrix M we have $\operatorname{adj}\{M\}M = \det\{M\}I_n$ in the last line. Replacing the new LRE of the last equation in (17) yields the error equation

$$\dot{\tilde{\Theta}}_i(t) = -\gamma\Delta^2(t)\tilde{\Theta}_i(t), \; i = 1, \ldots, 2n,$$

where $\tilde{\Theta}(t) := \hat{\Theta}(t) - \Theta$. The solution of this equation is given by

$$\tilde{\Theta}(t) = e^{-\gamma\int_0^t \Delta^2(s)ds}\tilde{\Theta}(0).$$

Notice that the solution of (21) is

$$w(t) = e^{-\gamma \int_0^t \Delta^2(s)ds},$$

hence, we have that

$$\tilde{\Theta}(t) = w(t)\tilde{\Theta}(0).$$

Clearly, this is equivalent to

$$[1 - w(t)]\Theta = \hat{\Theta}(t) - w(t)\Theta_0.$$

On the other hand, if $\Delta(t)$ is IE with β satisfying (22), we have that there exists a $t_c \geq 0$ such that

$$w(t) = w_c(t) < 1, \ \forall t \geq t_c.$$

Consequently, we conclude from (20) that

$$\hat{\Theta}^{\text{FTC}}(t) = \Theta, \ \forall t \geq t_c.$$

The proof is completed substituting the last identity in (19) and recalling (11). □□□

4 | RECONSTRUCTION OF THE UNDELAYED STATE $\hat{X}(T)$

As seen in (23) the adaptive state observer presented in Proposition 1 allows us to reconstruct in finite-time the state $z(t)$. Whence, from (6) we can generate an estimate of the delayed state $x(\phi(t))$. In most applications, for instance observer-based state feedback control of the original system (1), it is necessary to have an estimate of $x(t)$ not its delayed version. Although it is possible, in principle, to compute this estimate via

$$\hat{x}(t) := \hat{z}(\phi^{\mathrm{I}}(t)),$$

this operation requires the computation of the function $\phi^{\mathrm{I}}(\cdot)$. Except from some trivial cases like $\phi(t) = t - d$, with constant $d \geq 0$, for which $\phi^{\mathrm{I}}(s) = s + d$, it is not possible to get an analytic expression for this inverse. In general, its calculation requires the solution—in terms of t—of the nonlinear algebraic equation $\phi(t) = s$.[3]

In the proposition below we define a predictor for the calculation of $\hat{x}(t)$ from $\hat{z}(t)$ without the need of the inversion of $\phi(t)$. To simplify the presentation, and with some obvious abuse of notation, we look at the system for times $t \geq t_c$, that is, after convergence of the estimated parameters $\hat{\Theta}^{\text{FTC}}(t)$ to their true values Θ. For the practical implementation of the predictor given below it is necessary to replace θ and κ in the equations below by their FTC estimates given in Lemma 2. To simplify the notation we introduce a new matrix

$$\mathcal{A}_\kappa(t) := A(t) + \kappa C^\top(t). \tag{24}$$

Proposition 2. Consider the system (1) and its reparameterization (4a), (4b). Define the dynamic extension

$$\dot{\Phi}(t) = \mathcal{A}_\kappa(t)\Phi(t), \ \Phi(0) = I_n \tag{25a}$$

$$\dot{\xi}(t) = \mathcal{A}_\kappa(t)\xi(t) + B(t)u(t), \ \xi(0) = 0_{n\times 1} \tag{25b}$$

$$\dot{P}(t) = \mathcal{A}_\kappa(t)P(t) - \dot{\phi}(t)P(t)\mathcal{A}_\kappa(\phi(t)), \ P(0) = I_n. \tag{25c}$$

The estimate of the state $x(t)$ given as

$$\hat{x}(t) = P(t)\left[\Psi(t)\Theta + \zeta(t) - \xi\left(\phi(t)\right)\right] + \xi(t), \tag{26}$$

ensures that

$$\lim_{t\to\infty} |\hat{x}(t) - x(t)| = 0,$$

with $\Phi(t), \xi(t)$ and $P(t)$ bounded.

Proof. Define the error signal $\epsilon(t) := x(t) - \xi(t)$, which clearly satisfies

$$\dot{\epsilon}(t) = \mathcal{A}_\kappa(t)\epsilon(t).$$

[3] We underscore the fact that the construction of our state observer *does not* require the knowledge of the function $\phi^{\mathrm{I}}(\cdot)$, only its existence, which is guaranteed by Assumption 1. This is in contrast with other papers, like [10], where this inverse function is actually used in the construction of the predictor.

Consequently, taking into account (25a), we can write $\epsilon(t) = \Phi(t)\eta$, where $\eta \in \mathbb{R}^n$ is a new vector of unknown parameters defined as $\eta := \epsilon(0) = x(0)$. Hence, the vectors $x(t)$ can be written as

$$x(t) = \Phi(t)\eta + \xi(t), \tag{27}$$

and $z(t) = x(\phi(t))$ as

$$z(t) = \Phi(\phi(t))\,\eta + \xi(\phi(t)). \tag{28}$$

From (28) we can find the vector η:

$$\eta = \Phi^{-1}(\phi(t))\,[z(t) - \xi(\phi(t))].$$

After substituting the last equation into (27) we obtain

$$x(t) = \Phi(t)\Phi^{-1}(\phi(t))\,[z(t) - \xi(\phi(t))] + \xi(t). \tag{29}$$

To avoid the need of the computation of $\Phi^{-1}(\phi(t))$, define the matrix $P(t) := \Phi(t)\Phi^{-1}(\phi(t))$, whose time derivative satisfies

$$\begin{aligned}\dot{P}(t) &= \dot{\Phi}(t)\Phi^{-1}(\phi(t)) + \Phi(t)\dot{\Phi}^{-1}(\phi(t)) = \\ &= \dot{\Phi}(t)\Phi^{-1}(\phi(t)) - \Phi(t)\dot{\phi}(t)\Phi^{-1}(\phi(t))\frac{d\Phi(\phi(t))}{d\phi(t)}\Phi^{-1}(\phi(t)) = \\ &= \mathcal{A}_\kappa(t)\Phi(t)\Phi^{-1}(\phi(t)) - \dot{\phi}(t)\Phi(t)\Phi^{-1}(\phi(t))\,\mathcal{A}_\kappa(\phi(t))\Phi(\phi(t))\,\Phi^{-1}(\phi(t)) \\ &= \mathcal{A}_\kappa(t)P(t) - \dot{\phi}(t)P(t)\mathcal{A}_\kappa(\phi(t)).\end{aligned}$$

The proof of (26) is concluded substituting $P(t) = \Phi(t)\Phi^{-1}(\phi(t))$ into (29) and replacing $z(t)$ by its reparameterized version (19)

$$z(t) = \Psi(t)\Theta + \zeta(t).$$

The claim of boundedness of $\Phi(t)$, $\xi(t)$ and $P(t)$ follows from the standing assumptions that (1) is a bounded-input-bounded-output stable system, that its autonomous part is uniformly stable and that the input $u(t)$ and $\dot{\phi}(t)$ are *bounded*.

□□□

5 | SIMULATION RESULTS

To illustrate the performance of the observer proposed in the paper we consider the LTV dinamical system (1) with corresponding matrixes

$$A(t) = \begin{bmatrix} 0 & 0.5 \\ -0.5 & 0.25 + 0.5\sin t \end{bmatrix},\ B(t) = \begin{bmatrix} 1 \\ 0.5\cos 0.5t \end{bmatrix},\ C(t) = \begin{bmatrix} 0.75\sin 2t \\ 0 \end{bmatrix},\ \kappa = \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

For simulations we used $\mathcal{L}(t) = 0$ and the delay function $D(t)$ was given by $D(t) = 1 + 0.5\sin t$. Parameter λ for Kreisselmeier's scheme (16a) – (16b) is equal to $\lambda = 1$. Parameter $\overline{w}$ of FTC algorithm (17) –(20) is equal to $\overline{w} = 0.9$. Initial conditions of Θ estimations are equal to $\Theta(0) = 0_{4\times 1}$.

Simulation results are presented in Figs. 1-4. Figs. 1 – 2 demonstrate transients of state and unknown parameter estimation errors respectively for different values of adaptation parameter γ. For simulation we used initial condition of state $x(0) = \begin{bmatrix} 2 & -1 \end{bmatrix}^\top$ and adaptation parameter γ was changed in the range from 1 to 100. Fig. 3 –4 demonstrate transients of state and unknown parameter estimation errors respectively for three sets of initial conditions: IC1: $x(0) = \begin{bmatrix} 1 & 20 \end{bmatrix}^\top$, IC2: $x(0) = \begin{bmatrix} 2 & -1 \end{bmatrix}^\top$ and IC3: $x(0) = \begin{bmatrix} -5 & 2 \end{bmatrix}^\top$, in this case we used the parameter $\gamma = 10$. Simulation results demonstrate convergence of estimation errors $\tilde{x}_i$ and $\tilde{\kappa}_i$ to zero uniformly in time and initial conditions.

6 | CONCLUDING REMARKS

We have presented in this paper an adaptive observers for systems with measurement delays and unknown parameters. The main novelty of our result is the fact that the term in the state dynamics that depends on the unknown constant parameter is proportional to the systems output, hence is available for measurement with a delay. This situation stymies the possibility to carry out an output injection in the observer, which is a fundamental step to ensure stability of the observation error dynamics. This scenario should be compared with the one studied in [1], where this uncertain term is measurable on line.

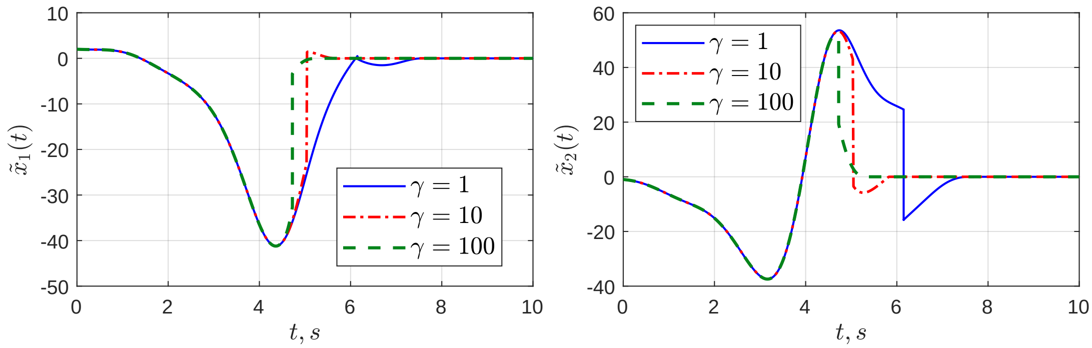

FIGURE 1 Transients of states estimation errors $\tilde{x}(t) = x(t) - \hat{x}(t)$ for different values of parameter γ

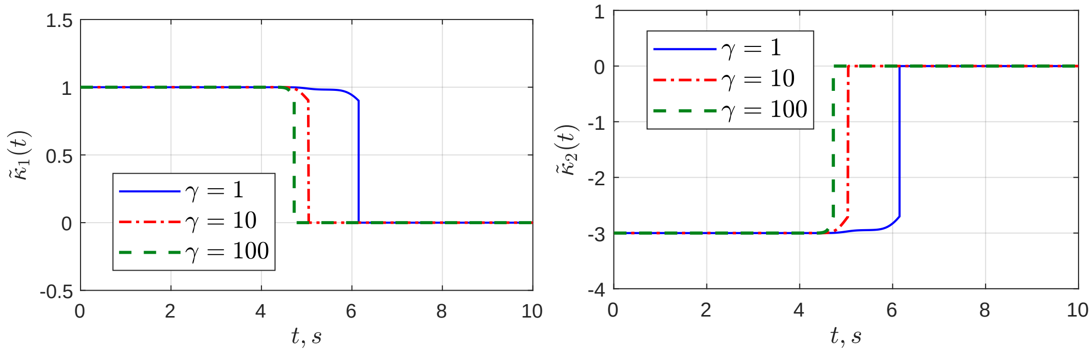

FIGURE 2 Transients of unknown parameter κ estimation errors $\tilde{\kappa}(t) = \kappa - \hat{\kappa}^{\mathrm{FTC}}(t)$ for different values of parameter γ

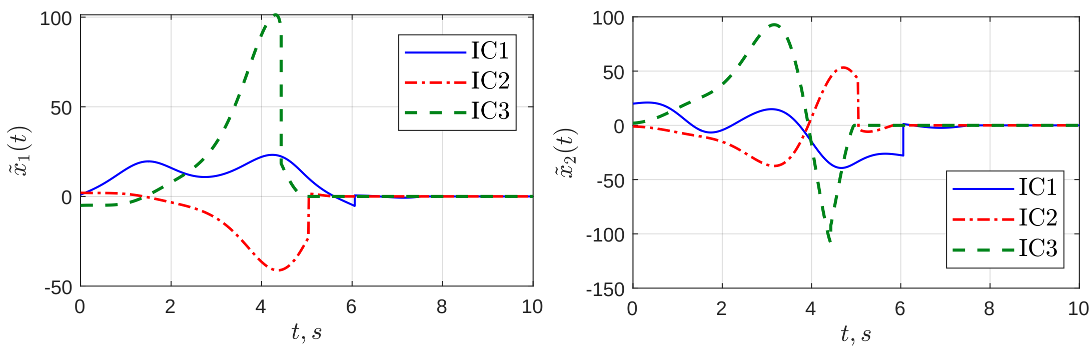

FIGURE 3 Transients of states estimation errors $\tilde{x}(t) = x(t) - \hat{x}(t)$ for different values of initial conditions $x(0)$

An additional novelty of our work is the development of a prediction-based technique to reconstruct the state on-line from its measurements subject to a time-varying delay. To the best of our knowledge such a construction has not been reported before.

Conflict of interest

The authors declare no potential conflict of interests.

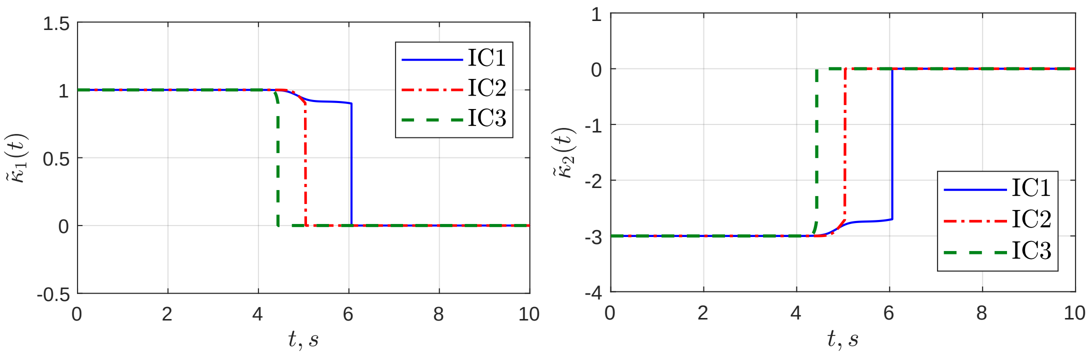

FIGURE 4 Transients of unknown parameter κ estimation errors $\tilde{\kappa}(t) = \kappa - \hat{\kappa}^{\mathrm{FTC}}(t)$ for different values of initial conditions $x(0)$

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