Dirty Paper Coding for Consecutive Messages with Heterogeneous Decoding Deadlines in the Finite Blocklength Regime

Homa Nikbakht, Malcolm Egan, Jean-Marie Gorce

▶ **To cite this version:**

Homa Nikbakht, Malcolm Egan, Jean-Marie Gorce. Dirty Paper Coding for Consecutive Messages with Heterogeneous Decoding Deadlines in the Finite Blocklength Regime. [Research Report] RR-9462, Inria - Research Centre Grenoble – Rhône-Alpes. 2022, pp.1-21. hal-03556888

HAL Id: hal-03556888

https://hal.inria.fr/hal-03556888

Submitted on 24 Feb 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Dirty Paper Coding for Consecutive Messages with Heterogeneous Decoding Deadlines in the Finite Blocklength Regime

Homa Nikbakht, Malcolm Egan, Jean-Marie Gorce

RESEARCH REPORT
N° 9462
February 2022
Project-Teams MARACAS

ISSN 0249-6399   ISRN INRIA/RR--9462--FR+ENG

Dirty Paper Coding for Consecutive Messages with Heterogeneous Decoding Deadlines in the Finite Blocklength Regime

Homa Nikbakht, Malcolm Egan, Jean-Marie Gorce

Project-Teams MARACAS

Research Report n° 9462 — version 1 — initial version February 2022 — revised version Février 2022 — 21 pages

Abstract: To improve reliability in latency-critical applications, a point-to-point communication system with heterogeneous decoding deadlines is considered. Unlike existing work, this system allows for a message to arrive before the decoding deadline of a prior message. A new coding scheme with finite blocklength codewords is introduced exploiting the dirty paper coding principle. Rigorous bounds are derived for achievable error probabilities. Moreover, numerical results illustrate that the proposed scheme outperforms time sharing for a wide range of blocklengths.

Key-words: URLLC, Finite Blocklength Regime, Heterogeneous Decoding Deadline, Dirty Paper Coding

Homa Nikbakht, Malcolm Egan and Jean-Marie Gorce are with the Laboratoire CITI, a joint laboratory between the Institut National de Recherche en Informatique et en Automatique (INRIA), the Université de Lyon and the Institut National de Sciences Apliquées (INSA) de Lyon. 6 Av. des Arts 69621 Villeurbanne, France. ({homa.nikbakht, malcom.egan, jean-marie.gorce}@inria.fr)

This work was funded by INRIA Nokia Bell Labs ADR “Network Information Theory" and by the French National Agency for Research (ANR) under grant ANR-16-CE25-0001 - ARBURST.

RESEARCH CENTRE
GRENOBLE – RHÔNE-ALPES

Inovallée
655 avenue de l'Europe Montbonnot
38334 Saint Ismier Cedex

Résumé : Pour améliorer la fiabilité des applications critiques en termes de latence, un système de communication point à point avec des délais de décodage hétérogènes est envisagé. Contrairement aux travaux existants, ce système permet à un message d'arriver avant la date limite de décodage d'un message précédent. Un nouveau schéma de codage avec des mots de code de longueur de bloc finie est introduit en exploitant le principe de codage du Dirty Paper Coding. Des limites rigoureuses sont dérivées pour les probabilités d'erreur atteignables. De plus, les résultats numériques montrent que le schéma proposé surpasse le partage de temps pour une large gamme de longueurs de bloc.

Mots-clés : URLLC, Régime de longueur de bloc fini, Date limite de décodage hétérogène

Contents

1 Introduction

Mobile wireless networks in 5G and proposals for 6G are increasing intended for use in latency-critical and high-reliability systems; notably in industrial control applications as well as for autonomous vehicles and remote surgery [1–4]. Commonly known as ultra-reliable low-latency communications (URLLC), packets are typically short. As a consequence, data transmission is no longer arbitrarily reliable and a key challenge is to design coding schemes that support high reliability requirements [5].

A standard assumption in the design of coding schemes is that consecutive messages have separated arrival times and decoding deadlines. As such, there is no choice but to encode the messages independently. On the other hand, this assumption is violated when a message arrives before the decoding deadline of a prior message. For example, a sensor in an unstable control system may send rapid measurements in order to stablize the system. In general, each of the messages will have a different decoding deadline.

In such a situation, code design must account for two issues: (i) messages with close arrival times; and (ii) messages with heterogeneous decoding deadlines. One approach is to begin encoding only after the second message has arrived. In this case, joint encoding is possible; however, messages must be decoded at different times.

The problem of code design with heterogeneous deadlines was first considered in the context of static broadcasting [6], where a single message is decoded at multiple receivers under different relative decoding delay constraints. The work in [6] was recently generalized to multi-source and multi-terminal networks by Langberg and Effros in [7]. In particular, the notion of a time-rate region was introduced, which accounted for different decoding delay constraints for each message at each receiver.

The work in both [6] and [7] focused on the asymptotic regime. In the finite blocklength regime, a coding scheme for the Gaussian broadcast channel with heterogeneous blocklength constraints depending on channel signal-to-noise ratio was introduced in [8]. By exploiting an early decoding scheme, the authors showed that significant improvements are possible over standard successive interference cancellation.

Nevertheless, the work in [6–8] all focused on the case where both messages are available at the time of encoding. In our previous work [9], we introduced a coding scheme for the Gaussian point-to-point channel where encoding begins for the first message before the second message arrives. This scheme exploited power sharing for symbols between the arrival time of the second message and the decoding deadline of the first message. Under a Gaussian interference assumption, bounds on the error probabilities for each message were established accounting for the message set size and finite decoding deadline constraints.

In this paper, we introduce a coding scheme for the Gaussian point-to-point channel model with heterogeneous decoding deadlines in [9] exploiting the dirty paper coding (DPC) principle. Accounting for finite decoding deadline constraints (corresponding to fixed blocklengths), we derive rigorous bounds on the achievable error probabilities for each of the messages. This is achieved by combining techniques to analyze the Gel'fand-Pinsker channel in the finite blocklength regime in [10] and multiple parallel channels in [11].

A natural question is under what conditions our joint encoding scheme outperforms time sharing with each message allocated a disjoint set of channel uses? We show via a numerical study that for sufficiently large power constraints and a wide range of finite blocklengths error probability of *both* messages can outperform time sharing.

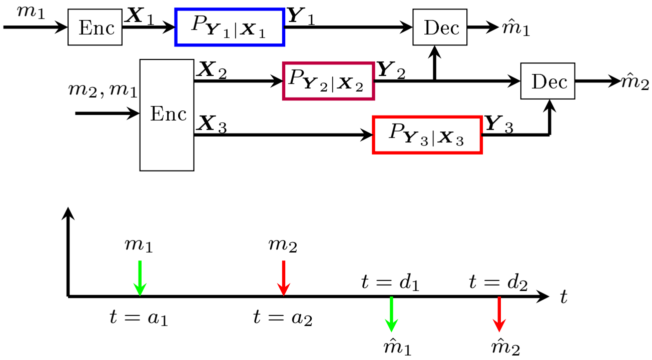

Figure 1: System model.

2 Problem Setup

Consider a sensor that seeks to send two messages, where each message lies in the set $\{1, \ldots, M\}$. At time $t = a_1$, transmission commences for the first message m_1. At time $t = a_2$, transmission commences for the second message m_2. The two messages m_1, m_2 are assumed to be drawn independently and uniformly on $\{1, \ldots, M\}$. For the sake of consistency we only consider the symmetric case where both messages are of same size. The study can be easily extended to asymmetric cases.

Each message is subject to different decoding delay constraints. In particular, at time d_1, the receiver attempts to reconstruct the message m_1. Similarly, at time $d_2 > d_1$, the receiver attempts to reconstruct the message m_2.

Under the assumption that $a_1 < a_2$ and $a_2 < d_1 < d_2$, the encoder outputs symbols at time $t \in \{a_1, \ldots, d_2\}$ as

$$X_t = \begin{cases} f_t(m_1), & t \in \{a_1, \ldots, a_2 - 1\} \\ \psi_t(m_1, m_2), & t \in \{a_2, \ldots, d_1\} \\ \phi_t(m_2), & t \in \{d_1 + 1, \ldots, d_2\}, \end{cases} \tag{1}$$

where f, ψ, ϕ are the encoding functions corresponding to the channel uses where only message m_1 has arrived but not m_2, where both m_1, m_2 are present, and m_1 has been decoded. We highlight that m_2 is not known before time $t = a_2$; i.e., encoding is causal.

The channel inputs can then be written as

$$\begin{aligned} \boldsymbol{X}_1 &= \{X_{a_1}, \ldots, X_{a_2-1}\} \\ \boldsymbol{X}_2 &= \{X_{a_2}, \ldots, X_{d_1}\} \\ \boldsymbol{X}_3 &= \{X_{d_1+1}, \ldots, X_{d_2}\}. \end{aligned} \tag{2}$$

We assume that the encoding functions satisfy an average block power constraint; namely,

$$\mathbb{E}[\|\boldsymbol{X}_i\|^2] \leq n_i P_i, \ i \in \{1, 2, 3\}, \tag{3}$$

where

$$n_1 = a_2 - a_1, \quad n_2 = d_1 - a_2 + 1, \quad n_3 = d_2 - d_1. \tag{4}$$

Denote the channel outputs by

$$\boldsymbol{Y}_1 = \{Y_{a_1}, \ldots, Y_{a_2-1}\}$$

$$\begin{aligned}\boldsymbol{Y}_2 &= \{Y_{a_2}, \dots, Y_{d_1}\}\\ \boldsymbol{Y}_3 &= \{Y_{d_1+1}, \dots, Y_{d_2}\}.\end{aligned} \tag{5}$$

The conditional distributions governing the three channels are then denoted by $P_{\boldsymbol{Y}_1|\boldsymbol{X}_1}$, $P_{\boldsymbol{Y}_2|\boldsymbol{X}_2}$, and $P_{\boldsymbol{Y}_3|\boldsymbol{X}_3}$. We assume that each channel is additive, memoryless, stationary, and Gaussian; that is,

$$\boldsymbol{Y}_i = \boldsymbol{X}_i + \boldsymbol{Z}_i, \tag{6}$$

where $\boldsymbol{Z}_i \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_{n_i})$. This setup is illustrated in Fig. 1.

At the receiver, the decoder attempts to reconstruct the two messages m_1, m_2 based on the channel outputs via the decoding functions g_1 and g_2; i.e.,

$$\hat{m}_1 = g_1(\boldsymbol{Y}_1, \boldsymbol{Y}_2) \quad \text{and} \quad \hat{m}_2 = g_2(\boldsymbol{Y}_2, \boldsymbol{Y}_3). \tag{7}$$

Observe that both decoders are causal.

The average probability of error for each of the messages is then

$$\epsilon_1 = \mathbb{P}[\hat{m}_1 \neq m_1], \quad \epsilon_2 = \mathbb{P}[\hat{m}_2 \neq m_2]. \tag{8}$$

The focus of the remainder of this paper is to characterize the tradeoff between the size of the message set M, the error probabilities ϵ_1, ϵ_2, and the decoding deadlines d_1, d_2. Formally, we study the achievable region defined as follows.

Definition 1 *Given the power constraints P_1, P_2, P_3, a tuple $(a_1, a_2, d_1, d_2, M, \epsilon_1, \epsilon_2)$ is achievable if the messages m_1, m_2 of cardinality M arriving at the a_1-th and a_2-th channel uses can be decoded by the d_1-th and d_2-th channel uses with an average probability of error not exceeding ϵ_1, ϵ_2, respectively.*

3 Random Coding Scheme

In this section, we introduce a random coding scheme that is based on dirty paper coding [10], [13] and [15]. In Section 4, we provide upper bounds on ϵ_1, ϵ_2 for this scheme.

3.1 Encoding

Let $\beta_1, \beta_2 \in [0, 1]$ such that $\beta_1 + \beta_2 < 1$. The encoding process consists of four phases.

3.1.1 Channel 1

In the first channel, consisting of n_1 channel uses, only m_1 is known to the encoder. The channel input $\boldsymbol{X}_1$ corresponding to message m_1 is a codeword $\boldsymbol{X}_1(m_1) \in \mathbb{R}^{n_1}$, where $\boldsymbol{X}_1(m_1) \sim \mathcal{N}(0, P_1\mathbf{I}_{n_1})$ with $\boldsymbol{X}_1(m) \perp \boldsymbol{X}_1(m')$, $m \neq m'$.

3.1.2 Channel 2 − 1

In the second channel, both m_1 and m_2 are known to the encoder. The channel input $\boldsymbol{X}_2$ is based on the DPC scheme in [10], which makes use of the notion of a power type.

Definition 2 (Power Types) *For a given blocklength n, fix $\delta > 0$. For each $P_t = \frac{i\delta}{n}$ with $i \in \{0, 1, 2, \ldots\}$, define the type class*

$$T^n(P_t) = \{\boldsymbol{x} \in \mathbb{R}^n : nP_t \leq \|\boldsymbol{x}\|^2 < nP_t + \delta\}. \tag{9}$$

For each $\boldsymbol{x} \in T^n(P_t)$, we call P_t the type of $\boldsymbol{x}$ and i the index of the type. The set of all types is given by

$$\mathcal{P}_n = \left\{\frac{i\delta}{n} : i \in \{0, 1, 2, \ldots\}\right\}. \tag{10}$$

In general, the type P_t of $\boldsymbol{x}$ can be arbitrarily large with non-zero probability. It is therefore desirable to consider a *typical set of power types*, defined as

$$\tilde{\mathcal{P}}_n = \{P_t \in \mathcal{P}_n : P_t \leq \Pi\} \tag{11}$$

for some $\Pi > 0$. To produce the channel input $\boldsymbol{X}_2$, consider the first $n_{2,1} < n_2$ symbols of n_2 symbols allocated to the second channel. These $n_{2,1}$ symbols are utilized to send the power type P_t to the receiver. Over the remaining $n_{2,2} = n_2 - n_{2,1}$ symbols in the second channel, a codeword constructed from both m_1 and m_2 is sent to the receiver.

More precisely, in the second channel, the encoder first produces a codeword $\boldsymbol{X}_{2,2} \in \mathbb{R}^{n_{2,2}}$ drawn from a Gaussian codebook of power $\beta_2 P_2$, $\beta_2 \in [0, 1]$ to encode m_2. The encoder then computes the power type of $\boldsymbol{X}_{2,2}$.

Indeed, for a fixed δ_2, the power type of $\boldsymbol{X}_{2,2}$ is of the form

$$P_t = \frac{k_2 \delta_2}{n_{2,2}}, \; k_2 \in \{0, 1, 2, \ldots\}. \tag{12}$$

The set of all power types P_t is then given by

$$\mathcal{P}_{n_{2,2}} = \left\{\frac{k_2 \delta_2}{n_{2,2}} : k_2 \in \{0, 1, 2, \ldots\}\right\}. \tag{13}$$

The corresponding typical set is given by

$$\tilde{\mathcal{P}}_{n_{2,2}} = \{P_t \in \mathcal{P}_{n_{2,2}} : P_t \leq P_{\max}\} \tag{14}$$

for some $P_{\max} > 0$. A power type P_t belongs to $\tilde{\mathcal{P}}_{n_{2,2}}$ if its index belongs to the set $\{0, \ldots, \lfloor \frac{n_{2,2} P_{\max}}{\delta_2} \rfloor\}$. Denoting by P_t as the power type of the signal $\boldsymbol{X}_{2,2}$, we have the following error event:

$$\mathcal{E}_{1,1} \triangleq \{P_t \notin \tilde{\mathcal{P}}_{n_{2,2}}\}. \tag{15}$$

To send the power type, the index k_2 is encoded to produce a codeword $\boldsymbol{X}_T \in \mathbb{R}^{n_{2,1}}$ drawn from a Gaussian point-to-point codebook of power $(1 - \beta_1 - \beta_2) P_2$.

3.1.3 Channel $2 - 2$

Over the remaining $n_{2,2}$ symbols allocated to the second channel, called channel $2-2$, the encoder exploits DPC treating $\boldsymbol{X}_{2,2}$ associated to m_2 as the state. To employ DPC, we generate the following codebooks.

Codebook Generation: Denote by $L^{(P_t)}$ the random coding parameter illustrating the number of auxiliary codewords for the type $P_t \in \tilde{\mathcal{P}}_{n_{2,2}}$. For each message $m_1 \in \{1, \ldots, M\}$ and each power type P_t a random codebook $C_{\boldsymbol{U}}^{(P_t)}$ containing $ML^{(P_t)}$ auxiliary codewords $\{\boldsymbol{U}^{(P_t)}(m_1,\ell)\}$ with $\ell \in \{1,\dots,L^{(P_t)}\}$, where each codeword is independently distributed on the sphere $\mathbb{S}^{n_{2,2}-1}$ with power $n_{2,2}(\beta_1 P_2 + \alpha^2 P_t)$, $\alpha \in (0,1)$. That is, the probability density function of $\boldsymbol{U}$ is given by

$$f_{\boldsymbol{U}}^{(P_t)}(\boldsymbol{u}) = \frac{\delta\left(||\boldsymbol{u}||^2 - n_{2,2}(\beta_1 P_2 + \alpha^2 P_t)\right)}{S_{n_{2,2}}(\sqrt{n_{2,2}(\beta_1 P_2 + \alpha^2 P_t)})}, \tag{16}$$

where $\delta(\cdot)$ is the Dirac delta function, and $S_n(r)$ is the surface area of a sphere of radius r in n-dimensional space.

Given $\boldsymbol{X}_{2,2}$, the second channel input over the $n_{2,2}$ symbols allocated for transmission of m_1, m_2 is given by

$$\boldsymbol{X}_2 = \boldsymbol{X}_{2,1} + \boldsymbol{X}_{2,2}, \tag{17}$$

where

$$\boldsymbol{X}_{2,1} = \boldsymbol{U} - \alpha \boldsymbol{X}_{2,2}. \tag{18}$$

Here, $\boldsymbol{U} \in C_{\boldsymbol{U}}^{(P_t)}$ with ℓ chosen such that $\boldsymbol{X}_{2,1} \in \mathcal{D}$, where

$$\mathcal{D} = \{\boldsymbol{x}_{2,1} : n_{2,2}\beta_1 P_2 - \delta_1 \le \|\boldsymbol{x}_{2,1}\|^2 \le n_{2,2}\beta_1 P_2\} \tag{19}$$

for some $\delta_1 > 0$. If more that one index exists, then one is selected arbitrarily. We have the encoding error event:

$$\mathcal{E}_{1,2} \triangleq \{\text{no } \ell \text{ exists such that } \boldsymbol{U} - \alpha \boldsymbol{X}_{2,2} \in \mathcal{D}\}. \tag{20}$$

3.1.4 Channel 3

Over the last n_3 channel uses, the sensor encodes only m_2 with a codeword $\boldsymbol{X}_3(m_2) \sim \mathcal{N}(0, P_3 \mathbf{I}_{n_3})$ with $\boldsymbol{X}_3(m_2) \perp \boldsymbol{X}_3(m_2')$, $m_2 \neq m_2'$.

3.2 Decoding

Given the structure of the encoding functions, receiver observations can be viewed as arising from four channels: over the first channel of n_1 blocks only m_1 is transmitted; over the second channel of $n_{2,1}$ blocks the power type index k_2 is transmitted; over the third channel of $n_{2,2}$ blocks, both m_1 and m_2 are transmitted; and over the fourth channel of n_3 blocks only m_2 is transmitted. We define the outputs of these four channels by $\boldsymbol{Y}_1, \boldsymbol{Y}_T, \tilde{\boldsymbol{Y}}_2$ and $\boldsymbol{Y}_3$, respectively which are of the following forms

$$\boldsymbol{Y}_1 = \boldsymbol{X}_1 + \boldsymbol{Z}_1, \tag{21a}$$
$$\boldsymbol{Y}_T = \boldsymbol{X}_T + \boldsymbol{Z}_T, \tag{21b}$$
$$\tilde{\boldsymbol{Y}}_2 = \boldsymbol{U} - \alpha \boldsymbol{X}_{2,2} + \boldsymbol{X}_{2,2} + \tilde{\boldsymbol{Z}}_2, \tag{21c}$$
$$\boldsymbol{Y}_3 = \boldsymbol{X}_3 + \boldsymbol{Z}_3, \tag{21d}$$

where $\boldsymbol{Z}_1 \in \mathbb{R}^{n_1}$, $\boldsymbol{Z}_T \in \mathbb{R}^{n_{2,1}}$, $\tilde{\boldsymbol{Z}}_2 \in \mathbb{R}^{n_{2,2}}$ and $\boldsymbol{Z}_3 \in \mathbb{R}^{n_3}$ are i.i.d standard Gaussian noise sequences.

3.2.1 Decoding the power type

The receiver first decodes the power type based on the outputs of the second channel (channel $2-1$). Denoting by $\hat{k}_2$ the decoded index of the power type, then we have the following error event while decoding the power type

$$\mathcal{E}_{1,3} \triangleq \{\text{Decoder chooses an index } \hat{k}_2 \neq k_2\}. \tag{22}$$

In our scheme, we fix the maximum error probability of decoding the power type at $\epsilon_T \in [0,1]$. Then for a given $P_{\max}$, P_2, β_2, n_2 and δ_2, we fix $n_{1,2}$ at the smallest value such that the probability that event $\mathcal{E}_{1,3}$ occurs is less than ϵ_T.

3.2.2 Decoding m_1

Given the power type P_t, the receiver decodes m_1 based on the outputs of the first and third channels (channel 1 and channel $2-2$). More specifically, given observations $\boldsymbol{y}_1, \tilde{\boldsymbol{y}}_2$, the receiver estimates m_1 according to the pair $(\hat{m}_1, \hat{\ell})$, such that the corresponding sequences $\boldsymbol{U}^{P_t}(\hat{m}_1, \hat{\ell})$ and $\boldsymbol{X}_1(\hat{m}_1)$ maximize

$$i_1(\boldsymbol{u}, \boldsymbol{x}_1; \boldsymbol{y}_1, \tilde{\boldsymbol{y}}_2) = \ln\left(\frac{f_{\boldsymbol{Y}_1|\boldsymbol{X}_1}(\boldsymbol{y}_1|\boldsymbol{x}_1) f_{\tilde{\boldsymbol{Y}}_2|\boldsymbol{U}}(\tilde{\boldsymbol{y}}_2|\boldsymbol{u})}{f_{\boldsymbol{Y}_1}(\boldsymbol{y}_1) f_{\tilde{\boldsymbol{Y}}_2}(\tilde{\boldsymbol{y}}_2)}\right) \tag{24}$$

over all pairs of $\boldsymbol{x}_1$ and $\boldsymbol{u} \in C_{\boldsymbol{U}}^{(P_t)}$. We have the following error event while decoding m_1:

$$\mathcal{E}_{1,4} \triangleq \{\text{Decoder chooses a message } \hat{m}_1 \neq m_1\}. \tag{25}$$

3.2.3 Decoding m_2

The receiver decodes m_2 based on the outputs of the third and fourth channels (channel $2-2$ and channel 3). After observing $\tilde{\boldsymbol{y}}_2$ and $\boldsymbol{y}_3$, the receiver estimates m_2 such that the corresponding sequences $\boldsymbol{x}_{2,2}, \boldsymbol{x}_3$ maximize

$$i_2(\boldsymbol{x}_{2,2}, \boldsymbol{x}_3; \tilde{\boldsymbol{y}}_2, \boldsymbol{y}_3) = \ln\left(\frac{f_{\tilde{\boldsymbol{Y}}_2|\boldsymbol{X}_{2,2}}(\tilde{\boldsymbol{y}}_2|\boldsymbol{x}_{2,2}) f_{\boldsymbol{Y}_3|\boldsymbol{X}_3}(\boldsymbol{y}_3|\boldsymbol{x}_3)}{f_{\tilde{\boldsymbol{Y}}_2}(\tilde{\boldsymbol{y}}_2) f_{\boldsymbol{Y}_3}(\boldsymbol{y}_3)}\right). \tag{26}$$

over all pairs of $\boldsymbol{x}_{2,2}$ and $\boldsymbol{x}_3$. We have the following error event while decoding m_2:

$$\mathcal{E}_2 \triangleq \{\text{Decoder chooses a message } \hat{m}_2 \neq m_2\}. \tag{27}$$

$$\tilde{\gamma}_1 \triangleq -2\gamma_1 + 2\ln(J_1) + n_1 \ln(1+P_1) + n_{2,2}\left(1 + \ln\left(\frac{\sigma^2_{y_2^*}}{\sigma^2_{z^*}}\right) - \frac{\delta_y}{\sigma^2_{y_2^*}}\right)$$
$$-\frac{n_{2,2}}{\sigma^2_{z^*}}\left(\sqrt{\sigma^2_{y_2^*} + \delta_y} + \sqrt{\sigma^2_{u^*} - 1}\right)^2, \tag{23a}$$
$$\tilde{\gamma}_2 \triangleq -2\gamma_2 + 2\ln(J_2) + n_3 \ln(1+P_3) + n_{2,2}\left(1 + \ln\left(\frac{\sigma^2_{y_2^*}}{\sigma^2_{u^*}}\right) - \frac{\delta_y}{\sigma^2_{y_2^*}}\right)$$
$$-\frac{n_{2,2}}{\sigma^2_{u^*}}\left(\sqrt{\sigma^2_{y_2^*} + \delta_y} + (1-\alpha)\sqrt{P_t + \delta_2/n_{2,2}}\right)^2. \tag{23b}$$

4 Main Result

Fix $\beta_1 \in [0,1]$ and $\beta_2 \in [0,1]$ such that $\beta_1 + \beta_2 < 1$ and define

$$Q_1 \triangleq v_1 - v_2 \quad \text{and} \quad Q_2 \triangleq v_1' - v_2', \tag{28}$$

with $v_1 \sim \mathcal{X}^2(n_1)$, $v_2 \sim \mathcal{X}^2(n_1)$, $v_1' \sim \mathcal{X}^2(n_3)$ and $v_2' \sim \mathcal{X}^2(n_3)$ where $\mathcal{X}^2(s)$ is central chi-squared distribution of degree s. Also define

$$\sigma^2_{y_2^*} \triangleq 1 + (\beta_1 + \beta_2)P_2, \tag{29a}$$
$$\sigma^2_{z^*} \triangleq 1 + (1-\alpha)^2\beta_2 P_2, \tag{29b}$$
$$\sigma^2_{u^*} \triangleq 1 + \beta_1 P_2 + \alpha^2 P_t. \tag{29c}$$

By employing the scheme proposed in Section 3 we have the following theorem on the upper bounds on the error probabilities ϵ_1 and ϵ_2.

Theorem 1 (Achievability bounds) *For a fixed message set size M, the error probabilities ϵ_1 and ϵ_2 are upper bonded by*

$$\begin{aligned}\epsilon_1 \leq\ & 1 - F_{Q_1}(\tilde{\gamma}_1) + ML^{(P_t)}e^{-\gamma_1}(1 - e^{-n_{2,2}^{1/3}\kappa}) + \epsilon_T \\ & + \Gamma(\frac{n_{2,2}}{2}, \frac{n_{2,2}P_{\max}}{2\beta_2 P_2})/\Gamma(\frac{n_{2,2}}{2}) + G + e^{-n_{2,2}^{1/3}\kappa},\end{aligned} \tag{30}$$

$$\begin{aligned}\epsilon_2 \leq\ & 1 - F_{Q_2}(\tilde{\gamma}_2) + Me^{-\gamma_2}(1 - e^{-n_{2,2}^{1/3}\kappa}) \\ & + \Gamma(\frac{n_{2,2}}{2}, \frac{n_{2,2}}{2})/\Gamma(\frac{n_{2,2}}{2}) + G + e^{-n_{2,2}^{1/3}\kappa},\end{aligned} \tag{31}$$

for any γ_1 and γ_2, where $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are defined in (23)*, and*

$$G \triangleq \left(1 - A\left(\beta_1 P_2/(\beta_1 P_2 + \alpha^2 P_t)\right)^{\frac{n_{2,2}}{2}}\right)^{L^{(P_t)}}, \tag{32a}$$

$$\begin{aligned}A \triangleq\ & \frac{\delta_1(\beta_1 P_2 + \alpha^2 P_t)}{2\alpha n_{2,2}\beta_1 P_2 \sqrt{\pi P_{\max}\beta_1 P_2}} \cdot \frac{\Gamma(\frac{n_{2,2}}{2})}{\Gamma(\frac{n_{2,2}-1}{2})} \\ & \times \left(1 - (c + \alpha^2\delta_2 + \delta_1\sqrt{P_t/P_{\max}})/n_{2,2}\beta_1 P_2\right)^{\frac{n_{2,2}-3}{2}}\end{aligned} \tag{32b}$$

$$c \triangleq \left(\alpha\delta_2/(2\sqrt{n_{2,2}P_t}) + \delta_1/(2\alpha\sqrt{n_{2,2}P_{\max}})\right)^2. \tag{32c}$$

Note that J_1, J_2, κ, δ_1, δ_2 and δ_y are constants and $\Gamma(\cdot)$ is the gamma function, $\Gamma(\cdot,\cdot)$ is the upper incomplete gamma function, $F_{Q_1}(\cdot)$ and $F_{Q_2}(\cdot)$ are the cumulative distribution functions of Q_1 and Q_2, respectively.

Proof: See Appendix A. ■

4.1 Numerical Results

In Figure 2, we evaluate the bounds in Theorem 1 for the case where the consecutive messages m_1 and m_2 are each of a transmission window of 20 channel uses, i.e., $|d_1 - a_1 + 1| = |d_2 - a_2 + 1| = 20$. The solid lines correspond to the error probabilities under our scheme and the dashed lines to time sharing scheme, where each message is allocated the same number of channel uses.

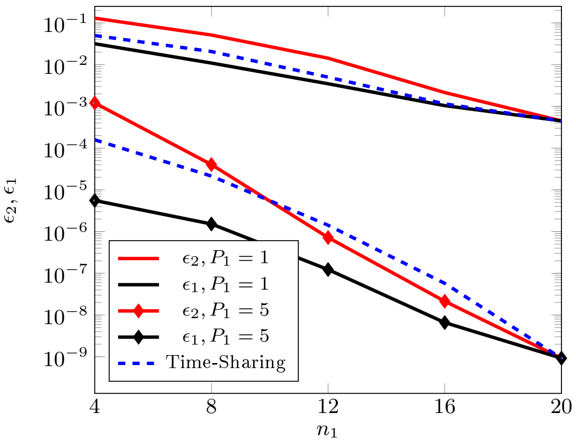

Figure 2: Upper bounds on ϵ_1 and ϵ_2 in terms of n_1 for $P_1 = P_2 = P_3$, $P_{\max} = 2P_1$, $\beta_1 = 0.4$, $\beta_2 = 0.4$, $R_1 = R_2 = 0.3466$ and $\epsilon_T = 10^{-10}$.

The bounds are evaluated for different values of n_1, different transmit power levels and for α set at $\frac{\beta_1 P_2}{1+\beta_2 P_2}$. Recall that n_1 is equal to the number of channel uses between the arrivals of m_1 and m_2. As n_1 increases, the transmissions of m_1 and m_2 overlap over a fewer number of channel uses and thus their corresponding error probabilities decrease. The minimum error probabilities occur at $n_1 = 20$. In the time-sharing scheme, for example when $n_1 = 4$, we assign 12 channel uses to the transmission of each message. Note that in this figure we assign $(1 - \beta_1 - \beta_2)P_2$ to the transmission of the power type. One can increase this power up to $\frac{n_2}{n_{2,1}}(1 - \frac{n_{2,2}}{n_2}(\beta_1 + \beta_2))P_2$ which still satisfies the power constraint in (3).

Observe that when the transmit power is small, the error probabilities obtained under the time sharing scheme are slightly lower than the bound on ϵ_2. For larger transmit powers, however, the error probabilities of *both* messages outperform time sharing over a wide range of n_1.

5 Conclusions

We considered a point-to-point communication where messages arrive at different times and are subject to heterogeneous decoding delay constraints. We proposed a coding scheme exploiting the dirty paper coding principle to jointly transmit consecutive messages with heterogeneous decoding deadlines. We derived rigorous bounds on the achievable error probabilities for each of the messages. We also numerically analyzed the obtained bounds in the finite block length regime and identified potential performance improvements over standard time sharing schemes.

Acknowledgment

This work was funded by INRIA Nokia Bell Labs ADR “Network Information Theory" and by the French National Agency for Research (ANR) under grant ANR-16-CE25-0001 - ARBURST.

A Proof of Theorem 1

A.1 Analyzing ϵ_1

The average error probability ϵ_1 is lower bounded based on the four error events $\mathcal{E}_{1,1}$, $\mathcal{E}_{1,2}$, $\mathcal{E}_{1,3}$ and $\mathcal{E}_{1,4}$ defined in (15), (20), (22) and (25), respectively and is as following:

$$\epsilon_1 \leq \mathbb{P}[\mathcal{E}_{1,1}] + \mathbb{P}[\mathcal{E}_{1,2}|\mathcal{E}_{1,1}^c] + \mathbb{P}[\mathcal{E}_{1,3}|\mathcal{E}_{1,1}^c, \mathcal{E}_{1,2}^c] + \mathbb{P}[\mathcal{E}_{1,4}|\mathcal{E}_{1,1}^c, \mathcal{E}_{1,2}^c, \mathcal{E}_{1,3}^c]. \tag{33}$$

A.1.1 Analyzing $\mathbb{P}[\mathcal{E}_{1,1}]$

We start by analyzing the probability that the power type of the signal $\boldsymbol{X}_{2,2}$ denoted by P_t does not belong to the typical set $\tilde{\mathcal{P}}_{n_{2,2}}$.

$$\begin{aligned}
\mathbb{P}[\mathcal{E}_{1,1}] &= \mathbb{P}[P_t \notin \tilde{\mathcal{P}}_{n_{2,2}}] && (34)\\
&= \mathbb{P}[||\boldsymbol{X}_{2,2}||^2 > n_{2,2}P_{\max}] && (35)\\
&= \mathbb{P}\left[||\frac{\boldsymbol{X}_{2,2}}{\sqrt{\beta_2 P_2}}||^2 > \frac{n_{2,2}P_{\max}}{\beta_2 P_2}\right] && (36)\\
&= 1 - F(\frac{n_{2,2}P_{\max}}{\beta_2 P_2}; n_{2,2}) && (37)\\
&= 1 - \frac{\gamma(\frac{n_{2,2}}{2}, \frac{n_{2,2}P_{\max}}{2\beta_2 P_2})}{\Gamma(\frac{n_{2,2}}{2})} && (38)\\
&= 1 - \frac{\Gamma(\frac{n_{2,2}}{2}) - \Gamma(\frac{n_{2,2}}{2}, \frac{n_{2,2}P_{\max}}{2\beta_2 P_2})}{\Gamma(\frac{n_{2,2}}{2})} && (39)\\
&= \frac{\Gamma(\frac{n_{2,2}}{2}, \frac{n_{2,2}P_{\max}}{2\beta_2 P_2})}{\Gamma(\frac{n_{2,2}}{2})} && (40)
\end{aligned}$$

where in (36), the variable $\frac{\boldsymbol{X}_{2,2}}{\sqrt{\beta_2 P_2}} \sim \mathcal{N}(0, I_{n_{2,2}})$ and therefore $||\frac{\boldsymbol{X}_{2,2}}{\sqrt{\beta_2 P_2}}||^2$ has a central chi-squared distribution of degree $n_{2,2}$. In (37), $F(\cdot;\cdot)$ is the cumulative distribution function (CDF) of the central chi-squared distribution and in (38) this CDF is replaced by its following form

$$F(x;k) = \frac{\gamma(\frac{k}{2}, \frac{x}{2})}{\Gamma(\frac{k}{2})}, \tag{41}$$

where $\gamma(\cdot,\cdot)$ is the lower incomplete gamma function and $\Gamma(\cdot)$ is the gamma function. In (39), the lower incomplete gamma function is replaced by

$$\gamma(s,x) = \Gamma(s) - \Gamma(s,x), \tag{42}$$

where $\Gamma(\cdot,\cdot)$ is the upper incomplete gamma function.

A.1.2 Analyzing $\mathbb{P}[\mathcal{E}_{1,2}|\mathcal{E}_{1,1}^c]$

To calculate this probability we follow the same argument provided in [10, Appendix E]. Recall that for a fixed power type P_t, we have $||\boldsymbol{U}||^2 = n_{2,2}(\beta_1 P_2 + \alpha^2 P_t)$ almost surely. From (19) we have that $\boldsymbol{U} - \alpha\boldsymbol{X}_{2,2} \in \mathcal{D}$ if and only if

$$n_{2,2}\beta_1 P_2 - \delta_1 \leq ||\boldsymbol{U}||^2 + \alpha^2||\boldsymbol{X}_{2,2}||^2 - 2\alpha\langle\boldsymbol{X}_{2,2}, \boldsymbol{U}\rangle \leq n_{2,2}\beta_1 P_2 \tag{43}$$

or equivalently

$$\frac{n_{2,2}\alpha P_t}{2} \leq \langle \boldsymbol{X}_{2,2}, \boldsymbol{U} \rangle - \frac{\alpha}{2}||\boldsymbol{X}_{2,2}||^2 \leq \frac{n_{2,2}\alpha P_t}{2} + \frac{\delta_1}{2\alpha}. \tag{44}$$

Since $\boldsymbol{U}$ is drawn uniformly from the sphere, the distribution of $\langle \boldsymbol{X}_{2,2}, \boldsymbol{U} \rangle$ depends on $\boldsymbol{X}_{2,2}$ only through its magnitude, this is seen by noting that the inner product of two vectors is unchanged when an orthogonal transformation is applied to both arguments, and the distribution of $\boldsymbol{U}$ is unchanged under any orthogonal transformation. Thus assume that $\boldsymbol{X}_{2,2} = (||\boldsymbol{X}_{2,2}||, 0, \ldots, 0)$. In this case,

$$\frac{n_{2,2}\alpha P_t}{2} \leq u_1||\boldsymbol{X}_{2,2}|| - \frac{\alpha}{2}||\boldsymbol{X}_{2,2}||^2 \leq \frac{n_{2,2}\alpha P_t}{2} + \frac{\delta_1}{2\alpha} \tag{45}$$

or equivalently

$$\frac{n_{2,2}\alpha P_t}{2||\boldsymbol{X}_{2,2}||} + \frac{\alpha}{2}||\boldsymbol{X}_{2,2}|| \leq u_1 \leq \frac{n_{2,2}\alpha P_t}{2||\boldsymbol{X}_{2,2}||} + \frac{\delta_1}{2\alpha||\boldsymbol{X}_{2,2}||} + \frac{\alpha}{2}||\boldsymbol{X}_{2,2}||. \tag{46}$$

According to the definition of the typical power set in (14), for $P_t \in \tilde{\mathcal{P}}_{n_{2,2}}$, we have $||\boldsymbol{X}_{2,2}|| \leq \sqrt{n_{2,2}P_{\max}}$, thus

$$\frac{n_{2,2}\alpha P_t}{2||\boldsymbol{X}_{2,2}||} + \frac{\alpha}{2}||\boldsymbol{X}_{2,2}|| \leq u_1 \leq \frac{n_{2,2}\alpha P_t}{2||\boldsymbol{X}_{2,2}||} + \frac{\alpha}{2}||\boldsymbol{X}_{2,2}|| + \frac{\delta_1}{2\alpha\sqrt{n_{2,2}P_{\max}}}. \tag{47}$$

We conclude that $\mathbb{P}[\boldsymbol{U} - \alpha\boldsymbol{X}_{2,2} \in \mathcal{D}]$ is lower bounded by the probability of the first entry of $\boldsymbol{U}$ falling with an interval of length $\frac{\delta_1}{2\alpha\sqrt{n_{2,2}P_{\max}}}$ starting at $\frac{n_{2,2}\alpha P_t}{2||\boldsymbol{X}_{2,2}||} + \frac{\alpha}{2}||\boldsymbol{X}_{2,2}||$. The distribution of a given symbol in a length-$n_{2,2}$ random sequence distributed uniformly on the sphere is [16]

$$\begin{aligned} f_{U_1}(u_1) = & \frac{1}{\sqrt{\pi n_{2,2}(\beta_1 P_2 + \alpha^2 P_t)}} \frac{\Gamma(\frac{n_{2,2}}{2})}{\Gamma(\frac{n_{2,2}-1}{2})} \left(1 - \frac{u_1^2}{n_{2,2}(\beta_1 P_2 + \alpha^2 P_t)}\right)^{\frac{n_{2,2}-3}{2}} \\ & \times \mathbb{1}\{u_1 \leq n_{2,2}(\beta_1 P_2 + \alpha^2 P_t)\}. \end{aligned} \tag{48}$$

This density function is decreasing in u_1^2, which implies that

$$\mathbb{P}[\boldsymbol{U} - \alpha\boldsymbol{X}_{2,2} \in \mathcal{D}] \geq \frac{\delta_1}{2\alpha\sqrt{n_{2,2}P_t}} f_{U_1}\left(\frac{n_{2,2}\alpha P_t}{2||\boldsymbol{X}_{2,2}||} + \frac{\alpha}{2}||\boldsymbol{X}_{2,2}|| + \frac{\delta_1}{2\alpha\sqrt{n_{2,2}P_{\max}}}\right) \tag{49}$$

Conditioning on $P_t \in \tilde{\mathcal{P}}_{n_{2,2}}$, then from (9) we have

$$\begin{aligned} \frac{n_{2,2}\alpha P_t}{2||\boldsymbol{X}_{2,2}||} + \frac{\alpha}{2}||\boldsymbol{X}_{2,2}|| \leq \alpha||\boldsymbol{X}_{2,2}|| \leq \alpha\sqrt{n_{2,2}P_t + \delta_2} & \quad (50) \\ \leq \alpha\sqrt{n_{2,2}P_t} + \frac{\alpha\delta_2}{2\sqrt{n_{2,2}P_t}} & \quad (51) \end{aligned}$$

where the last inequality is because $\sqrt{1+a} \leq 1 + \frac{a}{2}$. Thus

$$\begin{aligned} & \left(\frac{n_{2,2}\alpha P_t}{2||\boldsymbol{X}_{2,2}||} + \frac{\alpha}{2}||\boldsymbol{X}_{2,2}|| + \frac{\delta_1}{2\alpha\sqrt{n_{2,2}P_{\max}}}\right)^2 & \quad (52) \\ & \leq \left(\alpha\sqrt{n_{2,2}P_t} + \frac{\alpha\delta_2}{2\sqrt{n_{2,2}P_t}} + \frac{\delta_1}{2\alpha\sqrt{n_{2,2}P_{\max}}}\right)^2 & \quad (53) \end{aligned}$$

$$= n_{2,2}\alpha^2 P_t + 2\alpha\sqrt{n_{2,2}P_t}\left(\frac{\alpha\delta_2}{2\sqrt{n_{2,2}P_t}} + \frac{\delta_1}{2\alpha\sqrt{n_{2,2}P_{\max}}}\right)$$
$$+\left(\frac{\alpha\delta_2}{2\sqrt{n_{2,2}P_t}} + \frac{\delta_1}{2\alpha\sqrt{n_{2,2}P_{\max}}}\right)^2 \tag{54}$$
$$\le n_{2,2}\alpha^2 P_t + \alpha^2\delta_2 + \sqrt{\frac{P_t}{P_{\max}}}\delta_1 + \left(\frac{\alpha\delta_2}{2\sqrt{n_{2,2}P_t}} + \frac{\delta_1}{2\alpha\sqrt{n_{2,2}P_{\max}}}\right)^2 \tag{55}$$
$$= n_{2,2}\alpha^2 P_t + \alpha^2\delta_2 + \sqrt{\frac{P_t}{P_{\max}}}\delta_1 + c, \tag{56}$$

where

$$c \triangleq \left(\frac{\alpha\delta_2}{2\sqrt{n_{2,2}P_t}} + \frac{\delta_1}{2\alpha\sqrt{n_{2,2}P_{\max}}}\right)^2. \tag{57}$$

By substituting (56) into (49), and again using the fact that $f_{U_1}(u_1)$ is a decreasing function of u_1^2, we have

$$\mathbb{P}[\boldsymbol{U} - \alpha\boldsymbol{X}_{2,2} \in \mathcal{D}] \ge A \cdot \left(\frac{\beta_1 P_2}{\beta_1 P_2 + \alpha^2 P_t}\right)^{\frac{n_{2,2}}{2}} \tag{58}$$

where

$$A \triangleq \frac{\delta_1(\beta_1 P_2 + \alpha^2 P_t)}{2\alpha n_{2,2}\beta_1 P_2\sqrt{\pi P_{\max}\beta_1 P_2}} \cdot \left(1 - \frac{c + \alpha^2\delta_2 + \sqrt{\frac{P_t}{P_{\max}}}\delta_1}{n_{2,2}\beta_1 P_2}\right)^{\frac{n_{2,2}-3}{2}} \cdot \frac{\Gamma(\frac{n_{2,2}}{2})}{\Gamma(\frac{n_{2,2}-1}{2})}. \tag{59}$$

Since the $L^{(P_t)}$ codewords are generated independently, thus the probability of the error event $\mathcal{E}_1$ conditioning on that $\boldsymbol{X}_{2,2}$ has a given type P_t is equal to

$$\mathbb{P}[\mathcal{E}_{1,2}|\mathcal{E}_{1,1}^c] \le \left(1 - A\left(\frac{\beta_1 P_2}{\beta_1 P_2 + \alpha^2 P_t}\right)^{\frac{n_{2,2}}{2}}\right)^{L^{(P_t)}}. \tag{60}$$

A.1.3 Analyzing $\mathbb{P}[\mathcal{E}_{1,3}|\mathcal{E}_{1,1}^c, \mathcal{E}_{1,2}^c]$

In our scheme, we fix the error probability of decoding the power type at $\epsilon_T \in [0,1]$. Recall that the sensor transmits the power type over an AWGN channel during $n_{2,1}$ channel uses. Then for a given $P_{\max}$, P_2, β_2, n_2 and δ_2, we fix $n_{1,2}$ at the smallest value such that this probability is less than ϵ_T.

A.1.4 Analyzing $\mathbb{P}[\mathcal{E}_{1,4}|\mathcal{E}_{1,1}^c, \mathcal{E}_{1,2}^c, \mathcal{E}_{1,3}^c]$

Note that the output sequence $\tilde{\boldsymbol{Y}}_2$ does not follow a Gaussian distribution due to the fact that the vector $\boldsymbol{U}$ is not Gaussian. Thus similar to [14, Section IV.B], for the outputs of the third channel, we define the following set of typical channel outputs

$$\mathcal{F} \triangleq \{\tilde{\boldsymbol{y}}_2 \in \mathbb{R}^{n_{2,2}} : \frac{1}{n_{2,2}}||\tilde{\boldsymbol{y}}_2||^2 \in [\sigma^2_{y_2^*} - \delta_y, \sigma^2_{y_2^*} + \delta_y]\}. \tag{61}$$

for a fixed $\delta_y > 0$ where $\sigma^2_{y_2^*}$ is defined in (29a). By *Cramer's theorem* in [12], we have

$$\mathcal{E}_y \overset{\triangle}{=} \mathbb{P}[\tilde{\boldsymbol{Y}}_2 \notin \mathcal{F}] < \exp(-n_{2,2}\kappa\delta_y^2) \tag{62}$$

for some constant $\kappa > 0$. By setting $\delta_y = n_{2,2}^{-1/3}$, then $\exp(-n_{2,2}\kappa\delta_y^2)$ decays faster than any polynomial.

To evaluate this error event, we use the threshold bound for maximum-metric decoding. I.e.,

$$\begin{aligned}\mathbb{P}[\mathcal{E}_{1,4}|\mathcal{E}^c_{1,1},\mathcal{E}^c_{1,2},\mathcal{E}^c_{1,3}] \leq \mathcal{E}_y + \mathbb{P}[i_1(\boldsymbol{U},\boldsymbol{X}_1;\boldsymbol{Y}_1,\tilde{\boldsymbol{Y}}_2) \leq \gamma_1 | \tilde{\boldsymbol{Y}}_2 \in \mathcal{F}]\mathcal{E}^c_y \\ +ML^{(P_t)}\mathbb{P}[i_1(\bar{\boldsymbol{U}},\bar{\boldsymbol{X}}_1;\boldsymbol{Y}_1,\tilde{\boldsymbol{Y}}_2) > \gamma_1 | \tilde{\boldsymbol{Y}}_2 \in \mathcal{F}]\mathcal{E}^c_y\end{aligned} \tag{63}$$

for any γ_1, where $\bar{\boldsymbol{U}} \sim f_{\boldsymbol{U}}$ and $\bar{\boldsymbol{X}}_1 \sim f_{\boldsymbol{X}_1}$ are independent of $(\boldsymbol{X}_{2,2},\boldsymbol{U},\boldsymbol{X}_1,\boldsymbol{Y}_1,\tilde{\boldsymbol{Y}}_2)$.

We start by bounding $\mathbb{P}[i_1(\bar{\boldsymbol{U}},\bar{\boldsymbol{X}}_1;\boldsymbol{Y}_1,\tilde{\boldsymbol{Y}}_2) > \gamma_1]$. By Bayes rule we have

$$\begin{aligned} f_{\boldsymbol{X}_1}(\bar{\boldsymbol{x}}_1)f_{\boldsymbol{U}}(\bar{\boldsymbol{u}}) &= \frac{f_{\boldsymbol{Y}_1}(\boldsymbol{y}_1)f_{\tilde{\boldsymbol{Y}}_2}(\tilde{\boldsymbol{y}}_2)f_{\boldsymbol{U}|\tilde{\boldsymbol{Y}}_2}(\bar{\boldsymbol{u}}|\tilde{\boldsymbol{y}}_2)f_{\boldsymbol{X}_1|\boldsymbol{Y}_1}(\bar{\boldsymbol{x}}_1|\boldsymbol{y}_1)}{f_{\boldsymbol{Y}_1|\boldsymbol{X}_1}(\boldsymbol{y}_1|\bar{\boldsymbol{x}}_1)f_{\tilde{\boldsymbol{Y}}_2|\boldsymbol{U}}(\tilde{\boldsymbol{y}}_2|\bar{\boldsymbol{u}})} &(64)\\ &= f_{\boldsymbol{U}|\tilde{\boldsymbol{Y}}_2}(\bar{\boldsymbol{u}}|\tilde{\boldsymbol{y}}_2)f_{\boldsymbol{X}_1|\boldsymbol{Y}_1}(\bar{\boldsymbol{x}}_1|\boldsymbol{y}_1)\exp\left(-i_1(\bar{\boldsymbol{u}},\bar{\boldsymbol{x}}_1;\boldsymbol{y}_1,\tilde{\boldsymbol{y}}_2)\right) &(65)\end{aligned}$$

For fixed sequences $\boldsymbol{y}_1$ and $\tilde{\boldsymbol{y}}_2 \in \mathcal{F}$, by multiplying both side of the above equation by $\mathbb{1}\{i_1(\bar{\boldsymbol{u}},\bar{\boldsymbol{x}}_1;\boldsymbol{y}_1,\tilde{\boldsymbol{y}}_2) > \gamma_1\}$ and integrating over all $\bar{\boldsymbol{x}}_1$ and $\bar{\boldsymbol{u}}$, we have

$$\begin{aligned} &\int_{\bar{\boldsymbol{x}}_1}\int_{\bar{\boldsymbol{u}}} \mathbb{1}\{i_1(\bar{\boldsymbol{u}},\bar{\boldsymbol{x}}_1;\boldsymbol{y}_1,\tilde{\boldsymbol{y}}_2) > \gamma_1\} f_{\boldsymbol{X}_1}(\bar{\boldsymbol{x}}_1)f_{\boldsymbol{U}}(\bar{\boldsymbol{u}})d\bar{\boldsymbol{u}}d\bar{\boldsymbol{x}}_1 &(66)\\ &= \int_{\bar{\boldsymbol{x}}_1}\int_{\bar{\boldsymbol{u}}} \exp\left(-i_1(\bar{\boldsymbol{u}},\bar{\boldsymbol{x}}_1;\boldsymbol{y}_1,\tilde{\boldsymbol{y}}_2)\right)\\ &\quad\times\mathbb{1}\{i_1(\bar{\boldsymbol{u}},\bar{\boldsymbol{x}}_1;\boldsymbol{y}_1,\tilde{\boldsymbol{y}}_2) > \gamma_1\} f_{\boldsymbol{U}|\tilde{\boldsymbol{Y}}_2}(\bar{\boldsymbol{u}}|\tilde{\boldsymbol{y}}_2)f_{\boldsymbol{X}_1|\boldsymbol{Y}_1}(\bar{\boldsymbol{x}}_1|\boldsymbol{y}_1)d\bar{\boldsymbol{u}}d\bar{\boldsymbol{x}}_1. &(67)\\ &= \int_{\bar{\boldsymbol{x}}_1}\int_{\bar{\boldsymbol{u}}} \exp\left(-i_1(\bar{\boldsymbol{u}},\bar{\boldsymbol{x}}_1;\boldsymbol{y}_1,\tilde{\boldsymbol{y}}_2)\right)\\ &\quad\times\mathbb{1}\left\{\frac{f_{\boldsymbol{Y}_1|\boldsymbol{X}_1}(\boldsymbol{y}_1|\bar{\boldsymbol{x}}_1)f_{\tilde{\boldsymbol{Y}}_2|\boldsymbol{U}}(\tilde{\boldsymbol{y}}_2|\bar{\boldsymbol{u}})}{f_{\boldsymbol{Y}_1}(\boldsymbol{y}_1)f_{\tilde{\boldsymbol{Y}}_2}(\tilde{\boldsymbol{y}}_2)} > e^{\gamma_1}\right\} f_{\boldsymbol{U}|\tilde{\boldsymbol{Y}}_2}(\bar{\boldsymbol{u}}|\tilde{\boldsymbol{y}}_2)f_{\boldsymbol{X}_1|\boldsymbol{Y}_1}(\bar{\boldsymbol{x}}_1|\boldsymbol{y}_1)d\bar{\boldsymbol{u}}d\bar{\boldsymbol{x}}_1. &(68)\\ &\leq \int_{\bar{\boldsymbol{x}}_1}\int_{\bar{\boldsymbol{u}}} \exp\left(-i_1(\bar{\boldsymbol{u}},\bar{\boldsymbol{x}}_1;\boldsymbol{y}_1,\tilde{\boldsymbol{y}}_2)\right)\\ &\quad\times\frac{f_{\boldsymbol{Y}_1|\boldsymbol{X}_1}(\boldsymbol{y}_1|\bar{\boldsymbol{x}}_1)f_{\tilde{\boldsymbol{Y}}_2|\boldsymbol{U}}(\tilde{\boldsymbol{y}}_2|\bar{\boldsymbol{u}})}{f_{\boldsymbol{Y}_1}(\boldsymbol{y}_1)f_{\tilde{\boldsymbol{Y}}_2}(\tilde{\boldsymbol{y}}_2)} e^{-\gamma_1} f_{\boldsymbol{U}|\tilde{\boldsymbol{Y}}_2}(\bar{\boldsymbol{u}}|\tilde{\boldsymbol{y}}_2)f_{\boldsymbol{X}_1|\boldsymbol{Y}_1}(\bar{\boldsymbol{x}}_1|\boldsymbol{y}_1)d\bar{\boldsymbol{u}}d\bar{\boldsymbol{x}}_1. &(69)\\ &\leq e^{-\gamma_1} &(70)\end{aligned}$$

Equivalently

$$\mathbb{P}[i_1(\bar{\boldsymbol{U}},\bar{\boldsymbol{X}}_1;\boldsymbol{Y}_1,\tilde{\boldsymbol{Y}}_2) > \gamma_1 | \tilde{\boldsymbol{Y}}_2 \in \mathcal{F}] \leq e^{-\gamma_1}. \tag{71}$$

Thus

$$\mathbb{P}[i_1(\bar{\boldsymbol{U}},\bar{\boldsymbol{X}}_1;\boldsymbol{Y}_1,\tilde{\boldsymbol{Y}}_2) > \gamma_1 | \tilde{\boldsymbol{Y}}_2 \in \mathcal{F}] \cdot \mathcal{E}^c_y \leq (1 - e^{-n_{2,2}^{1/3}\kappa})e^{-\gamma_1}. \tag{72}$$

To calculate $\mathbb{P}[i_1(\boldsymbol{U},\boldsymbol{X}_1;\boldsymbol{Y}_1,\tilde{\boldsymbol{Y}}_2) \leq \gamma_1 | \tilde{\boldsymbol{Y}}_2 \in \mathcal{F}]$, for a fixed P_t we first define the output $\boldsymbol{Y}_2^* \sim \mathcal{N}(0, I_{n_{2,2}}\sigma^2_{y_2^*})$. Then by [5, proof of Lemma 61], we have

$$\min_{\tilde{\boldsymbol{y}}_2 \in \mathcal{F}} \frac{f_{\tilde{\boldsymbol{Y}}_2}(\tilde{\boldsymbol{y}}_2)}{f_{\boldsymbol{Y}_2^*}(\tilde{\boldsymbol{y}}_2)} \leq j_1 \tag{73}$$

for a finite constant $j_1 > 0$.

To calculate $f_{\tilde{\boldsymbol{Y}}_2|\boldsymbol{U}}(\tilde{\boldsymbol{Y}}_2|\boldsymbol{U})$, we define vector $\boldsymbol{U}^*$ following the distribution $\mathcal{N}(0, (\beta_1 P_2 + \alpha^2 P_t)I_{n_{2,2}})$. Then by [5, proof of Lemma 61] and also by [17, Proposition 2]:

$$\min_{\boldsymbol{u}:||\boldsymbol{u}||^2 = n_{2,2}(\beta_1 P_2 + \alpha^2 P_t)} \frac{f_{\boldsymbol{U}}(\boldsymbol{u})}{f_{\boldsymbol{U}^*}(\boldsymbol{u})} \leq j_2 \tag{74}$$

where $j_2 > 0$ is a constant. We also define $f_{\boldsymbol{U}^*, \boldsymbol{Y}_2^*}(\boldsymbol{u}, \tilde{\boldsymbol{y}}_2)$ as the joint distribution of $\boldsymbol{Y}_2^*$ and $\boldsymbol{U}^*$. We have

$$\frac{f_{\boldsymbol{U}, \tilde{\boldsymbol{Y}}_2}(\boldsymbol{u}, \tilde{\boldsymbol{y}}_2)}{f_{\boldsymbol{U}^*, \boldsymbol{Y}_2^*}(\boldsymbol{u}, \tilde{\boldsymbol{y}}_2)} = D^*_{f,Q} \tag{75}$$

where it can be shown that there is a constant j_3 such that

$$D^*_{f,Q} \geq j_3. \tag{76}$$

By combining (73) and (74), we have

$$\begin{aligned}
\frac{f_{\tilde{\boldsymbol{Y}}_2|\boldsymbol{U}}(\tilde{\boldsymbol{y}}_2|\boldsymbol{u})}{f_{\tilde{\boldsymbol{Y}}_2}(\tilde{\boldsymbol{y}}_2)} &\geq \frac{f_{\boldsymbol{U}|\tilde{\boldsymbol{Y}}_2}(\boldsymbol{u}|\tilde{\boldsymbol{y}}_2)}{j_2 f_{\boldsymbol{U}^*}(\boldsymbol{u})} && (77)\\
&= \frac{f_{\boldsymbol{U}, \tilde{\boldsymbol{Y}}_2}(\boldsymbol{u}, \tilde{\boldsymbol{y}}_2)}{j_2 f_{\boldsymbol{U}^*}(\boldsymbol{u}) f_{\tilde{\boldsymbol{Y}}_2}(\tilde{\boldsymbol{y}}_2)} && (78)\\
&\geq \frac{f_{\boldsymbol{U}, \tilde{\boldsymbol{Y}}_2}(\boldsymbol{u}, \tilde{\boldsymbol{y}}_2)}{j_1 j_2 f_{\boldsymbol{U}^*}(\boldsymbol{u}) f_{\boldsymbol{Y}_2^*}(\tilde{\boldsymbol{y}}_2)} && (79)\\
&\geq \frac{j_3}{j_1 j_2} \cdot \frac{f_{\boldsymbol{U}^*, \boldsymbol{Y}_2^*}(\boldsymbol{u}, \tilde{\boldsymbol{y}}_2)}{f_{\boldsymbol{U}^*}(\boldsymbol{u}) f_{\boldsymbol{Y}_2^*}(\tilde{\boldsymbol{y}}_2)} && (80)\\
&= J_1 \frac{f_{\boldsymbol{Y}_2^*|\boldsymbol{U}^*}(\tilde{\boldsymbol{y}}_2, \boldsymbol{u})}{f_{\boldsymbol{Y}_2^*}(\tilde{\boldsymbol{y}}_2)} && (81)
\end{aligned}$$

where $J_1 \triangleq \frac{j_3}{j_1 j_2}$. We thus have

$$\begin{aligned}
&\mathcal{E}_y^c \cdot \mathbb{P}[i_1(\boldsymbol{U}, \boldsymbol{X}_1; \boldsymbol{Y}_1, \tilde{\boldsymbol{Y}}_2) \leq \gamma_1 | \tilde{\boldsymbol{Y}}_2 \in \mathcal{F}] && (82)\\
&\leq \mathbb{P}\left[\ln\left(\frac{f_{\boldsymbol{Y}_1|\boldsymbol{X}_1}(\boldsymbol{Y}_1|\boldsymbol{X}_1) f_{\boldsymbol{Y}_2^*|\boldsymbol{U}^*}(\tilde{\boldsymbol{Y}}_2|\boldsymbol{U})}{f_{\boldsymbol{Y}_1}(\boldsymbol{Y}_1) f_{\boldsymbol{Y}_2^*}(\tilde{\boldsymbol{Y}}_2)}\right) \leq \gamma_1 - \ln(J_1) | \tilde{\boldsymbol{Y}}_2 \in \mathcal{F}\right] \cdot \mathcal{E}_y^c && (83)\\
&= \mathbb{P}\Bigg[\ln\left(\frac{\frac{1}{(\sqrt{2\pi})^{n_1}} \exp\left(-\frac{||\boldsymbol{Y}_1 - \boldsymbol{X}_1||^2}{2}\right)}{\frac{1}{(\sqrt{2\pi(1+P_1)})^{n_1}} \exp\left(-\frac{||\boldsymbol{Y}_1||^2}{2(1+P_1)}\right)}\right)\\
&\qquad + \ln\left(\frac{\frac{1}{(\sqrt{2\pi\sigma_{z^*}^2})^{n_{2,2}}} \exp\left(-\frac{||\tilde{\boldsymbol{Y}}_2 - \boldsymbol{U}||^2}{2\sigma_{z^*}^2}\right)}{\frac{1}{(\sqrt{2\pi\sigma_{z^*}^2})^{n_{2,2}}} \exp\left(-\frac{||\tilde{\boldsymbol{Y}}_2||^2}{2\sigma_{y_2^*}^2}\right)}\right) \leq \gamma_1 - \ln(J_1) | \tilde{\boldsymbol{Y}}_2 \in \mathcal{F}\Bigg] \cdot \mathcal{E}_y^c && (84)\\
&= \mathbb{P}\Bigg[\frac{n_1}{2}\ln(1 + P_1) - \frac{||\boldsymbol{Y}_1 - \boldsymbol{X}_1||^2}{2} + \frac{||\boldsymbol{Y}_1||^2}{2(1 + P_1)} + \frac{n_{2,2}}{2}\ln\left(\frac{\sigma_{y_2^*}^2}{\sigma_{z^*}^2}\right)\\
&\qquad - \frac{||\tilde{\boldsymbol{Y}}_2 - \boldsymbol{U}||^2}{2\sigma_{z^*}^2} + \frac{||\tilde{\boldsymbol{Y}}_2||^2}{2\sigma_{y_2^*}^2} \leq \gamma_1 - \ln(J_1) | \tilde{\boldsymbol{Y}}_2 \in \mathcal{F}\Bigg] \cdot \mathcal{E}_y^c && (85)
\end{aligned}$$

$$= \mathbb{P}\left[||\boldsymbol{Y}_1 - \boldsymbol{X}_1||^2 - \frac{||\boldsymbol{Y}_1||^2}{(1+P_1)} + \frac{||\tilde{\boldsymbol{Y}}_2 - \boldsymbol{U}||^2}{\sigma^2_{z^*}} - \frac{||\tilde{\boldsymbol{Y}}_2||^2}{\sigma^2_{y_2^*}} \geq \gamma_1'' | \tilde{\boldsymbol{Y}}_2 \in \mathcal{F}\right] \cdot \mathcal{E}_y^c \tag{86}$$

$$\leq \mathbb{P}\left[||\boldsymbol{Z}_1||^2 - \frac{1}{1+P_1}||\boldsymbol{Y}_1||^2 + \frac{||\tilde{\boldsymbol{Y}}_2 - \boldsymbol{U}||^2}{\sigma^2_{z^*}} \geq \gamma_1'' + n_{2,2} - \frac{n_{2,2}\delta_y}{\sigma^2_{y_2^*}} | \tilde{\boldsymbol{Y}}_2 \in \mathcal{F}\right] \cdot \mathcal{E}_y^c \tag{87}$$

$$\leq \mathbb{P}\left[||\boldsymbol{Z}_1||^2 - \frac{1}{1+P_1}||\boldsymbol{Y}_1||^2 + \frac{||\tilde{\boldsymbol{Y}}_2||^2 + ||\boldsymbol{U}||^2 + 2||\boldsymbol{U}|| \cdot ||\tilde{\boldsymbol{Y}}_2||}{\sigma^2_{z^*}} \geq \gamma_1' | \tilde{\boldsymbol{Y}}_2 \in \mathcal{F}\right] \cdot \mathcal{E}_y^c \tag{88}$$

$$\leq \mathbb{P}\left[||\boldsymbol{Z}_1||^2 - \frac{1}{1+P_1}||\boldsymbol{Y}_1||^2 \geq \gamma_1' - n_{2,2}\frac{\sigma^2_{y_2^*}}{\sigma^2_{z^*}} - \frac{n_{2,2}\delta_y}{\sigma^2_{z^*}} - \frac{||\boldsymbol{U}||^2}{\sigma^2_{z^*}} - \frac{2||\boldsymbol{U}||\sqrt{n_{2,2}(\sigma^2_{y_2^*} + \delta_y)}}{\sigma^2_{z^*}}\right] \cdot \mathcal{E}_y^c \tag{89}$$

$$\leq \mathbb{P}\left[||\boldsymbol{Z}_1||^2 - ||\frac{\boldsymbol{Y}_1}{\sqrt{1+P_1}}||^2 \geq \tilde{\gamma}_1\right], \tag{90}$$

where

$$\gamma_1'' \triangleq -2\gamma_1 + 2\ln(J_1) + n_1 \ln(1+P_1) + n_{2,2}\ln\left(\sigma^2_{y_2^*}\right) - n_{2,2}\ln\left(\sigma^2_{z^*}\right) \tag{91}$$

$$\gamma_1' \triangleq \gamma_1'' + n_{2,2} - \frac{n_{2,2}\delta_y}{\sigma^2_{y_2^*}} \tag{92}$$

$$\tilde{\gamma}_1 \triangleq \gamma_1' - n_{2,2}\frac{\sigma^2_{y_2^*}}{\sigma^2_{z^*}} - \frac{n_{2,2}\delta_y}{\sigma^2_{z^*}} - \frac{||\boldsymbol{U}||^2}{\sigma^2_{z^*}} - \frac{2||\boldsymbol{U}||\sqrt{n_{2,2}(\sigma^2_{y_2^*} + \delta_y)}}{\sigma^2_{z^*}}. \tag{93}$$

Note that $\frac{\boldsymbol{Y}_1}{\sqrt{1+P_1}} \sim \mathcal{N}(0, I_{n_1})$. Define

$$v_1 \triangleq ||\boldsymbol{Z}_1||^2 : \quad v_1 \sim \mathcal{X}^2(n_1) \tag{94}$$

$$v_2 \triangleq ||\frac{\boldsymbol{Y}_1}{\sqrt{1+P_1}}||^2 : \quad v_2 \sim \mathcal{X}^2(n_1) \tag{95}$$

where $\mathcal{X}^2(s)$ is central chi-squared distribution of degree s. We define

$$Q_1 \triangleq v_1 - v_2. \tag{96}$$

Thus

$$\mathcal{E}_y^c \cdot \mathbb{P}[i_1(\boldsymbol{U}, \boldsymbol{X}_1; \boldsymbol{Y}_1, \tilde{\boldsymbol{Y}}_2) \leq \gamma | \tilde{\boldsymbol{Y}}_2 \in \mathcal{F}] \tag{97}$$

$$\leq \mathbb{P}[Q_1 \geq \tilde{\gamma}_1] = 1 - F_{Q_1}(\tilde{\gamma}_1) \tag{98}$$

where F_{Q_1} is the CDF of Q_1. To calculate this CDF we follow the argument in [18] on the difference of correlated chi-squared random variables.

To sum up,

$$\epsilon_1 \leq 1 - F_{Q_1}(\tilde{\gamma}_1) + \left(1 - A\left(\frac{\beta_1 P_2}{\beta_1 P_2 + \alpha^2 P_t}\right)^{\frac{n_{2,2}}{2}}\right)^{L^{(P_t)}} + \epsilon_T + e^{-n_{2,2}^{1/3}\kappa}$$
$$+ \frac{\Gamma(\frac{n_{2,2}}{2}, \frac{n_{2,2}P_{\max}}{2\beta_2 P_2})}{\Gamma(\frac{n_{2,2}}{2})} + ML^{(P_t)}e^{-\gamma}(1 - e^{-n_{2,2}^{1/3}\kappa}). \tag{99}$$

A.2 Analyzing ϵ_2

Recall the definition of $\mathcal{E}_2$ from (27). We upper bound the average error probability ϵ_2 as

$$\epsilon_2 \leq \mathbb{P}[\mathcal{E}_2] + \mathbb{P}[||\boldsymbol{X}_{2,2}||^2 > n_{2,2}\beta_2 P_2] + \mathbb{P}[\mathcal{E}_{1,2}|\mathcal{E}_{1,1}^c]. \tag{100}$$

A.2.1 Analyzing $\mathbb{P}[\mathcal{E}_2]$

Similar to Section A.1.4, to evaluate this error event, we use the threshold bound for maximum-metric decoding and we condition on $\tilde{\boldsymbol{y}}_2 \in \mathcal{F}$. I.e.,

$$\begin{aligned}\mathbb{P}[\mathcal{E}_2] \leq \mathcal{E}_y^c \cdot \mathbb{P}[i_2(\boldsymbol{X}_{2,2}, \boldsymbol{X}_3; \tilde{\boldsymbol{Y}}_2, \boldsymbol{Y}_3) \leq \gamma_2 | \tilde{\boldsymbol{Y}}_2 \in \mathcal{F}] \\ + M\mathbb{P}[i_2(\bar{\boldsymbol{X}}_{2,2}, \bar{\boldsymbol{X}}_3; \tilde{\boldsymbol{Y}}_2, \boldsymbol{Y}_3) > \gamma_2 | \tilde{\boldsymbol{Y}}_2 \in \mathcal{F}] \cdot \mathcal{E}_y^c + \mathcal{E}_y\end{aligned} \tag{101}$$

for any γ_2, where $\bar{\boldsymbol{X}}_{2,2} \sim f_{\boldsymbol{X}_{2,2}}$ and $\bar{\boldsymbol{X}}_3 \sim f_{\boldsymbol{X}_3}$ are independent of $(\boldsymbol{X}_{2,2}, \boldsymbol{X}_3, \boldsymbol{Y}_3, \tilde{\boldsymbol{Y}}_2)$.

For given channel outputs $\tilde{\boldsymbol{Y}}_2$ and $\boldsymbol{Y}_3$, we follow the same argument proposed in Section A.1.4 to bound $P[i_2(\bar{\boldsymbol{X}}_{2,2}, \bar{\boldsymbol{X}}_3; \tilde{\boldsymbol{Y}}_2, \boldsymbol{Y}_3) > \gamma_2]$. Thus

$$P[i_2(\bar{\boldsymbol{X}}_{2,2}, \bar{\boldsymbol{X}}_3; \tilde{\boldsymbol{Y}}_2, \boldsymbol{Y}_3) > \gamma_2 | \tilde{\boldsymbol{Y}}_2, \boldsymbol{Y}_3] \leq e^{-\gamma_2}. \tag{102}$$

We continue by calculating $\mathbb{P}[i_2(\boldsymbol{X}_{2,2}, \boldsymbol{X}_3; \tilde{\boldsymbol{Y}}_2, \boldsymbol{Y}_3) \leq \gamma_2 | \tilde{\boldsymbol{Y}}_2 \in \mathcal{F}]$. First note that similar to the argument provided for bonding ϵ_1, here also $\tilde{\boldsymbol{Y}}_2$ does not follow a Gaussian distribution. We again bound $f_{\tilde{\boldsymbol{Y}}_2}$ as in (73). To calculate $f_{\tilde{\boldsymbol{Y}}_2|\boldsymbol{X}_{2,2}}(\tilde{\boldsymbol{y}}_2|\boldsymbol{x}_{2,2})$, note that $\tilde{\boldsymbol{Y}}_2|\boldsymbol{X}_{2,2}$ is equivalent to the output of a Gaussian channel with a Gaussian noise $\tilde{Z} \sim \mathcal{N}((1-\alpha)\boldsymbol{X}_{2,2}, I_{n_{2,2}})$ and with the input vector $\boldsymbol{U}$ that is on the power-shell. According to [17, equation 109], $f_{\tilde{\boldsymbol{Y}}_2|\boldsymbol{X}_{2,2}}(\tilde{\boldsymbol{y}}_2|\boldsymbol{x}_{2,2})$ is of the following form:

$$\begin{aligned} & f_{\tilde{\boldsymbol{Y}}_2|\boldsymbol{X}_{2,2}}(\tilde{\boldsymbol{y}}_2|\boldsymbol{x}_{2,2}) && (103)\\ & = \frac{1}{2(\sqrt{\pi})^{n_{2,2}}}\Gamma(\frac{n_{2,2}}{2})\exp\left(-||\boldsymbol{U}||^2 - ||\tilde{\boldsymbol{y}}_2 - (1-\alpha)\boldsymbol{x}_{2,2}||^2\right) \\ & \times\frac{\mathcal{I}_{n_{2,2}/2-1}\left(||\tilde{\boldsymbol{y}}_2 - (1-\alpha)\boldsymbol{x}_{2,2}|| \cdot ||\boldsymbol{u}||\right)}{\left(||\tilde{\boldsymbol{y}}_2 - (1-\alpha)\boldsymbol{x}_{2,2}|| \cdot ||\boldsymbol{U}||\right)^{n_{2,2}/2-1}}, && (104)\end{aligned}$$

where $\mathcal{I}_a(\cdot)$ is the modified Bessel function of the first kind and a-th order. Define $Q_{\boldsymbol{Y}^*|\boldsymbol{X}_{2,2}}$ as the output distribution of the Gaussian channel with noise $\tilde{\boldsymbol{Z}}$ but where the input sequence $\boldsymbol{U}$ follows the Gaussian distribution $\mathcal{N}(0, (\beta_1 P_2 + \alpha^2 P_t)I_{n_{2,2}})$. Then according to [17, Appendix B]:

$$\frac{f_{\tilde{\boldsymbol{Y}}_2|\boldsymbol{X}_{2,2}}(\tilde{\boldsymbol{y}}_2|\boldsymbol{x}_{2,2})}{Q_{\boldsymbol{Y}^*|\boldsymbol{X}_{2,2}}(\tilde{\boldsymbol{y}}_2|\boldsymbol{x}_{2,2})} = D_{f,Q,2} \tag{105}$$

where one can prove that for all $\tilde{\boldsymbol{y}}_2$ and $\boldsymbol{x}_{2,2}$

$$D_{f,Q,2} \geq j_4 \tag{106}$$

where $j_4 > 0$. Define $J_2 \triangleq \frac{j_4}{j_1}$, then by employing above arguments, we have

$$\begin{aligned} & \mathcal{E}_y^c \cdot \mathbb{P}[i_2(\boldsymbol{X}_{2,2}, \boldsymbol{X}_3; \tilde{\boldsymbol{Y}}_2, \boldsymbol{Y}_3) \leq \gamma_2 | \tilde{\boldsymbol{Y}}_2 \in \mathcal{F}] && (107)\\ & \leq \mathbb{P}\left[\ln\left(\frac{Q_{\boldsymbol{Y}^*|\boldsymbol{X}_{2,2}}(\tilde{\boldsymbol{Y}}_2|\boldsymbol{X}_{2,2}) f_{\boldsymbol{Y}_3|\boldsymbol{X}_3}(\boldsymbol{Y}_3|\boldsymbol{X}_3)}{f_{\boldsymbol{Y}_2^*}(\tilde{\boldsymbol{Y}}_2) f_{\boldsymbol{Y}_3}(\boldsymbol{Y}_3)}\right) \leq \gamma_2 - \ln(J_2) | \tilde{\boldsymbol{Y}}_2 \in \mathcal{F}\right] \cdot \mathcal{E}_y^c && (108)\end{aligned}$$

$$= \mathbb{P}\left[\ln\left(\frac{\frac{1}{(\sqrt{2\pi\sigma^2_{u^*}})^{n_{2,2}}}\exp\left(-\frac{||\tilde{\boldsymbol{Y}}_2-(1-\alpha)\boldsymbol{X}_{2,2}||^2}{2\sigma^2_{u^*}}\right)}{\frac{1}{(\sqrt{2\pi\sigma^2_{y_2^*}})^{n_{2,2}}}\exp\left(-\frac{||\tilde{\boldsymbol{Y}}_2||^2}{2\sigma^2_{y_2^*}}\right)}\right)\right.$$

$$\left. +\ln\left(\frac{\frac{1}{(\sqrt{2\pi})^{n_3}}\exp\left(-\frac{||\boldsymbol{Z}_3||^2}{2}\right)}{\frac{1}{(\sqrt{2\pi(1+P_3)})^{n_3}}\exp\left(-\frac{||\boldsymbol{Y}_3||^2}{2(1+P_3)}\right)}\right) \leq \gamma_2 - \ln(J_2)\,|\tilde{\boldsymbol{Y}}_2 \in \mathcal{F}\right]\cdot \mathcal{E}_y^c \tag{109}$$

$$= \mathbb{P}\left[\frac{n_{2,2}}{2}\ln\left(\sigma^2_{y_2^*}\right) - \frac{n_{2,2}}{2}\ln\left(\sigma^2_{u^*}\right) - \frac{||\tilde{\boldsymbol{Y}}_2-(1-\alpha)\boldsymbol{X}_{2,2}||^2}{2\sigma^2_{u^*}} + \frac{||\tilde{\boldsymbol{Y}}_2||^2}{2\sigma^2_{y_2^*}}\right.$$

$$\left. +\frac{n_3}{2}\ln(1+P_3) - \frac{||\boldsymbol{Y}_3-\boldsymbol{X}_3||^2}{2} + \frac{||\boldsymbol{Y}_3||^2}{2(1+P_3)} \leq \gamma_2 - \ln(J_2)\,|\tilde{\boldsymbol{Y}}_2 \in \mathcal{F}\right]\cdot \mathcal{E}_y^c \tag{110}$$

$$= \mathbb{P}\left[\frac{||\tilde{\boldsymbol{Y}}_2-(1-\alpha)\boldsymbol{X}_{2,2}||^2}{\sigma^2_{u^*}} - \frac{||\tilde{\boldsymbol{Y}}_2||^2}{\sigma^2_{y_2^*}} + ||\boldsymbol{Y}_3-\boldsymbol{X}_3||^2 - \frac{||\boldsymbol{Y}_3||^2}{1+P_3} \geq \gamma_2''|\tilde{\boldsymbol{Y}}_2 \in \mathcal{F}\right]\cdot \mathcal{E}_y^c \tag{111}$$

$$\leq \mathbb{P}\left[||\boldsymbol{Z}_3||^2 - \frac{1}{1+P_3}||\boldsymbol{Y}_3||^2 + \frac{||\tilde{\boldsymbol{Y}}_2-(1-\alpha)\boldsymbol{X}_{2,2}||^2}{\sigma^2_{u^*}} \geq \gamma_2'' + n_{2,2} - \frac{n_{2,2}\delta_y}{\sigma^2_{y_2^*}}|\tilde{\boldsymbol{Y}}_2 \in \mathcal{F}\right]\cdot \mathcal{E}_y^c \tag{112}$$

$$\leq \mathbb{P}\left[||\boldsymbol{Z}_3||^2 - \frac{1}{1+P_3}||\boldsymbol{Y}_3||^2\right.$$

$$\left. +\frac{||\tilde{\boldsymbol{Y}}_2||^2 + (1-\alpha)^2||\boldsymbol{X}_{2,2}||^2 + 2(1-\alpha)||\boldsymbol{X}_{2,2}||\cdot||\tilde{\boldsymbol{Y}}_2||}{\sigma^2_{u^*}} \geq \gamma_2'|\tilde{\boldsymbol{Y}}_2 \in \mathcal{F}\right]\cdot \mathcal{E}_y^c \tag{113}$$

$$\leq \mathbb{P}\left[||\boldsymbol{Z}_3||^2 - \frac{1}{1+P_3}||\boldsymbol{Y}_3||^2 \geq \gamma_2' - n_{2,2}\frac{\sigma^2_{y_2^*}}{\sigma^2_{u^*}} - \frac{n_{2,2}\delta_y}{\sigma^2_{u^*}}\right.$$

$$\left. -\frac{n_{2,2}P_t+\delta_2}{\sigma^2_{u^*}} - \frac{2\sqrt{n_{2,2}P_t+\delta_2}\sqrt{n_{2,2}(\sigma^2_{y_2^*}+\delta_y)}}{\sigma^2_{u^*}}\right]\cdot \mathcal{E}_y^c \tag{114}$$

$$\leq \mathbb{P}\left[||\boldsymbol{Z}_3||^2 - ||\frac{\boldsymbol{Y}_3}{\sqrt{1+P_3}}||^2 \geq \tilde{\gamma}_2\right], \tag{115}$$

where

$$\gamma_2'' \triangleq -2\gamma_2 + 2\ln(J_2) + n_3\ln(1+P_3) + n_{2,2}\ln\left(\sigma^2_{y_2^*}\right) - n_{2,2}\ln\left(\sigma^2_{u^*}\right) \tag{116}$$

$$\gamma_2' \triangleq \gamma_2'' + n_{2,2} - \frac{n_{2,2}\delta_y}{\sigma^2_{y_2^*}} \tag{117}$$

$$\tilde{\gamma}_2 \triangleq \gamma_2' - n_{2,2}\frac{\sigma^2_{y_2^*}}{\sigma^2_{u^*}} - \frac{n_{2,2}\delta_y}{\sigma^2_{u^*}} - \frac{n_{2,2}P_t+\delta_2}{\sigma^2_{u^*}} - \frac{2\sqrt{n_{2,2}P_t+\delta_2}\sqrt{n_{2,2}(\sigma^2_{y_2^*}+\delta_y)}}{\sigma^2_{u^*}} \tag{118}$$

Define

$$v_1' \triangleq ||\boldsymbol{Z}_3||^2 : \quad v_1' \sim \mathcal{X}^2(n_3), \tag{119}$$

$$v_2' \triangleq ||\frac{\boldsymbol{Y}_3}{\sqrt{1+P_3}}||^2 : \quad v_2' \sim \mathcal{X}^2(n_3). \tag{120}$$

Also define

$$Q_2 \triangleq v_1' - v_2'. \tag{121}$$

Thus

$$\mathcal{E}_y^c \cdot \mathbb{P}[i_2(\boldsymbol{X}_{2,2}, \boldsymbol{X}_3; \tilde{\boldsymbol{Y}}_2, \boldsymbol{Y}_3) \leq \gamma_2] \leq \mathbb{P}[Q_2 > \tilde{\gamma}_2] = 1 - F_{Q_2}(\tilde{\gamma}_2). \tag{122}$$

where $F_{Q_2}(\cdot)$ is the CDF of Q_2.

A.2.2 Analyzing $\mathbb{P}[||\boldsymbol{X}_{2,2}||^2 > n_{2,2}\beta_2 P_2]$

We calculate $\mathbb{P}[||\boldsymbol{X}_{2,2}||^2 > n_{2,2}\beta_2 P_2]$ by following the same argument as in that of Section A.1.1. We can prove that

$$\mathbb{P}[||\boldsymbol{X}_{2,2}||^2 > n_{2,2}\beta_2 P_2] = \frac{\Gamma(\frac{n_{2,2}}{2}, \frac{n_{2,2}}{2})}{\Gamma(\frac{n_{2,2}}{2})}. \tag{123}$$

To sum up

$$\begin{aligned} \epsilon_2 \leq\ & 1 - F_{Q_2}(\tilde{\gamma}_2) + Me^{-\gamma_2}(1 - e^{-n_{2,2}^{1/3}\kappa}) + \frac{\Gamma(\frac{n_{2,2}}{2}, \frac{n_{2,2}}{2})}{\Gamma(\frac{n_{2,2}}{2})} \\ & + e^{-n_{2,2}^{1/3}\kappa} + \left(1 - A\left(\frac{\beta_1 P_2}{\beta_1 P_2 + \alpha^2 P_t}\right)^{\frac{n_{2,2}}{2}}\right)^{L^{(P_t)}}. \end{aligned} \tag{124}$$

References

[1] P. Popovski, Č. Stefanović, J. J. Nielsen, E. de Carvalho, M. Angjelichinoski, K. F. Trillingsgaard and A. Bana, “Wireless access in ultra-reliable low-latency communication (URLLC)," in *IEEE Trans. on Communications*, vol. 67, no. 8, pp. 5783–5801, Aug. 2019.

[2] X. Ge, “Ultra-reliable low-latency communications in autonomous vehicular networks," in *IEEE Trans. on Vehicular Technology*, vol. 68, no. 5, pp. 5005–5016, May 2019.

[3] M. Shirvanimoghaddam et al., “Short block-length codes for ultra-reliable low latency communications," in *IEEE Communications Magazine*, vol. 57, no. 2, pp. 130–137, Feb. 2019.

[4] X. Zhang, J. Wang and H. V. Poor, “Statistical QoS-driven energy-efficiency optimization for URLLC over 5G mobile wireless networks in the finite blocklength regime," in *Info. Sciences and Systems,* pp. 1–6, 2019.

[5] Y. Polyanskiy, H. V. Poor and S. Verdu, “Channel coding rate in the finite blocklength regime," in *IEEE Trans. on Info. Theory*, vol. 56, no. 5, pp. 2307–2359, May, 2010.

[6] N. Shulman and M. Feder. “Static broadcasting”, *IEEE International Symposium on Info. Theory (ISIT)*, 2000.

[7] M. Langberg and M. Effros,“ Beyond capacity: The joint time-rate region", *arXiv:2101.12236v1*, Jan. 2021.

[8] P. H. Lin, S. C. Lin and E. A. Jorswieck, “Early decoding for Gaussian broadcast channels with heterogeneous blocklength constraints," in *IEEE International Symposium on Information Theory (ISIT),* pp. 3243–3248, 2021.

[9] H. Nikbakht, M. Egan and J-M. Gorce, " Joint channel coding of consecutive packets with heterogeneous decoding deadlines in the finite blocklength regime", [Research Report] Inria - Research Centre Grenoble – Rhône-Alpes. 2021. *Available: https://hal.inria.fr/hal-03360567*

[10] J. Scarlett, "On the dispersions of the Gel'fand - Pinsker channel and dirty paper coding," in *IEEE Trans. on Info Theory*, vol. 61, no. 9, pp. 4569-4586, Sept. 2015.

[11] T. Erseghe, "Coding in the finite-blocklength regime: Bounds based on Laplace integrals and their asymptotic approximations," in *IEEE Trans. on Info. Theory*, vol. 62, no. 12, pp. 6854–6883, Dec. 2016.

[12] I. Csiszàr and J. Körner, "Information theory", *New York: Academic Press*, 1981.

[13] G. Caire and S. Shamai, "On the achievable throughput of a multiantenna Gaussian broadcast channel," in *IEEE Transactions on Information Theory*, vol. 49, no. 7, pp. 1691-1706, July 2003.

[14] V. Y. F. Tan and M. Tomamichel, "The third-order term in the normal approximation for the AWGN channel," in *IEEE Trans. on Info. Theory*, vol. 61, no. 5, pp. 2430–2438, May 2015.

[15] M. H. M. Costa, "Writing on dirty paper (Corresp.),' *IEEE Trans. Inf. Theory*, vol. 29, no. 3, pp. 439–441, May 1983.

[16] A. J. Stam, "Limit theorems for uniform distributions on spheres in high-dimensional Euclidean spaces," *J. Appl. Probab.*, vol. 19, no. 1, pp. 221–228, Mar. 1982.

[17] E. MolavianJazi and J. N. Laneman, "A second-order achievable rate region for Gaussian multi-access channels via a central limit theorem for functions," in *IEEE Transactions on Information Theory*, vol. 61, no. 12, pp. 6719–6733, Dec. 2015.

[18] A. Ferrari, "A note on sum and difference of correlated chi-squared variables", [Online]. Available:*arXiv:1906.09982*, 2019.

RESEARCH CENTRE
GRENOBLE – RHÔNE-ALPES

Inovallée
655 avenue de l'Europe Montbonnot
38334 Saint Ismier Cedex

Publisher
Inria
Domaine de Voluceau - Rocquencourt
BP 105 - 78153 Le Chesnay Cedex
inria.fr

ISSN 0249-6399