Local decay rates of best-approximation errors using vector-valued finite elements for fields with low regularity and integrable curl or divergence

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Zhaonan Dong* Alexandre Ern† Jean-Luc Guermond‡

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Abstract

We estimate best-approximation errors using vector-valued finite elements for fields with low regularity in the scale of fractional-order Sobolev spaces. By assuming that the target field enjoys an additional integrability property on its curl or its divergence, we establish upper bounds on these errors that can be localized to the mesh cells. These bounds are derived using the quasi-interpolation errors with or without boundary prescription derived in [A. Ern and J.-L. Guermond, ESAIM Math. Model. Numer. Anal., 51 (2017), pp. 1367–1385]. In the present work, a localized upper bound on the quasi-interpolation error is derived by using the face-to-cell lifting operators analyzed in [A. Ern and J.-L. Guermond, Found. Comput. Math., (2021)] and by exploiting the additional assumption made on the curl or the divergence of the target field. As an illustration, we show how to apply these results to the error analysis of the curl-curl problem associated with Maxwell's equations.

1 Introduction

A central question in the finite element approximation theory is to establish local upper bounds on the best-approximation error for functions that satisfy some minimal regularity assumptions typically quantified in the scale of fractional-order Sobolev spaces. The goal of the present work is to derive some novel results in this context when the approximation is realized using Nédélec finite elements and Raviart–Thomas finite elements. Most of our developments focus on the Nédélec finite elements since they require more elaborate arguments. The corresponding results for the Raviart–Thomas finite elements only improve marginally the state of the art from the literature, and only a short discussion is provided in a specific section of the paper.

We are interested in approximating fields with a smoothness Sobolev index that is so low that one cannot invoke the canonical interpolation operators associated with the considered finite elements. In this case, the best-approximation error can be bounded by considering quasi-interpolation errors, such as those derived in [10]. However, the resulting upper bound cannot be localized to the mesh cells if the regularity of the target function is only measured in the scale of fractional-order Sobolev spaces. The lack of localization is relatively mild if no boundary conditions are prescribed in the finite element spaces, since in this case the quasi-interpolation error can still be bounded by local contributions involving the fractional-order Sobolev seminorm of the target function over patches of mesh cells instead of just each mesh cell individually. The lack of localization is more significant if boundary conditions are additionally prescribed in the finite element spaces since in this case the upper bound on the quasi-interpolation error is global. This is not surprising as in this situation the target function has not enough smoothness to have a well-defined trace at the boundary. The main contribution of this work is to show that the localization becomes possible provided some (mild) additional assumptions are made on the integrability of the curl

*Inria Paris, 75589 Paris, France, and CERMICS, Ecole des Ponts, 77455 Marne-la-Vallée cedex 2, France. zhaonan.dong@inria.fr.

†CERMICS, Ecole des Ponts, 77455 Marne-la-Vallée cedex 2, France, and Inria Paris, 75589 Paris, France. alexandre.ern@enpc.fr.

‡Department of Mathematics, Texas A&M University, 3368 TAMU, College Station, TX 77843, USA. guermond@math.tamu.edu.

or the divergence of the target function. The main tool to achieve this result hinges on the face-to-cell lifting operators introduced in [15]. The additional assumptions on the curl or the divergence allow us to give a weak meaning to the trace of the target function in the dual space of a suitable fractional-order Sobolev space.

In this work, the space dimension is $d = 3$ for the Nédélec finite elements and $d \geq 2$ for the Raviart–Thomas finite elements. For $d = 2$, the results for the Raviart–Thomas elements can be transposed to the Nédélec elements by invoking a rotation of angle $\frac{\pi}{2}$; details are omitted for brevity (see, e.g., [13, Sec. 15.3.1]). We consider a polyhedral Lipschitz domain $D \subset \mathbb{R}^d$. Moreover, we use boldface for $\mathbb{R}^d$-valued fields and linear spaces composed of such fields. For instance, for real numbers $r \geq 0$ and $p \in [1, \infty]$ (we assume $p \in [1, \infty)$ if $r \notin \mathbb{N}$), $\boldsymbol{W}^{r,p}(D)$ denotes the (fractional-order) Sobolev space equipped with the Sobolev–Slobodeckij norm.

Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ denote a shape-regular family of affine, matching, simplicial meshes such that each mesh covers D exactly. Let $\boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h)$ denote the $\boldsymbol{H}(\mathrm{curl}; D)$-conforming finite element space built on the mesh $\mathcal{T}_h$ using the Nédélec finite element of degree $k \in \mathbb{N}$ (here, the superscript $^{\mathrm{c}}$ refers to the curl operator). Given a target field $\boldsymbol{v} \in \boldsymbol{W}^{r,p}(D)$, with $r > 0$ possibly very small, our goal is to establish localized upper bounds on the best-approximation error

$$\inf_{\boldsymbol{v}_h\in\boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_{\boldsymbol{L}^p(D)}. \tag{1}$$

The first natural idea is to invoke the canonical interpolation operator for Nédélec finite elements, say $\mathcal{I}_h^{\mathrm{c}}$. Since this operator can only act on those fields having an integrable tangential trace along all the mesh edges, invoking the standard trace theory in Sobolev spaces (see, e.g., [16]) shows that a suitable domain for the canonical interpolation operator $\mathcal{I}_h^{\mathrm{c}}$ is $\boldsymbol{W}^{r,p}(D)$ with $rp > d - 1 = 2$ (and $r \geq 2$ if $p = 1$). Assume that the polynomial degree is such that $k \geq 1$ if $p \in [1, 2]$ and $k \geq 0$ otherwise, and let $r \in (\frac{2}{p}, k + 1]$ if $p > 1$ or $r \in [2, k + 1]$ if $p = 1$. Then, it is well-known (see, e.g., [18] or [13, Sec. 16.2]) that there is c such that for all $\boldsymbol{v} \in \boldsymbol{W}^{r,p}(D)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$, we have

$$\|\boldsymbol{v} - \mathcal{I}_h^{\mathrm{c}}(\boldsymbol{v})\|_{\boldsymbol{L}^p(K)} \leq c\, h_K^r |\boldsymbol{v}|_{\boldsymbol{W}^{r,p}(K)}, \tag{2}$$

where h_K denotes the diameter of the mesh cell $K \in \mathcal{T}_h$. Here, the symbol c denotes a generic positive constant whose value can change at each occurrence provided it only depends on the mesh shape-regularity, the space dimension, and the polynomial degree k of the considered finite elements. Notice that c is unbounded as $r \downarrow \frac{2}{p}$ if $p > 1$. For instance, in the Hilbert setting where $p = 2$, the minimal regularity requirement is $\boldsymbol{v} \in \boldsymbol{H}^r(D)$ with $r > 1$, and c is unbounded as $r \downarrow 1$.

The requirement $r > \frac{2}{p}$ can be lowered to $r > \frac{1}{2}$ (with $p = 2$) by invoking more sophisticated results on traces derived in [2] which, however, hinge on some additional integrability assumption on $\nabla\times\boldsymbol{v}$. To use these results, the edge-based degrees of freedom of the Nédélec finite element are extended by defining them using edge-to-cell lifting operators and an integration by parts formula (see, e.g., [13, Sec. 17.3]). One can then show (see [6]; see also [1, 8, 3] for slight variants) that for all $r \in (\frac{1}{2}, 1]$ and all $p > 2$, there is c such that for all $\boldsymbol{v} \in \boldsymbol{H}^r(D)$ with $\nabla\times\boldsymbol{v} \in \boldsymbol{L}^p(D)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$, we have

$$\|\boldsymbol{v} - \mathcal{I}_h^{\mathrm{c}}(\boldsymbol{v})\|_{\boldsymbol{L}^2(K)} \leq c \left(h_K^r |\boldsymbol{v}|_{\boldsymbol{H}^r(K)} + h_K^{1+d(\frac{1}{2}-\frac{1}{p})} \|\nabla\times\boldsymbol{v}\|_{\boldsymbol{L}^p(K)} \right), \tag{3}$$

with c unbounded as $r \downarrow \frac{1}{2}$ or $p \downarrow 2$.

Unfortunately, the regularity assumption $r > \frac{1}{2}$ is often not realistic in applications. To go beyond this assumption, one can invoke the quasi-interpolation operators devised in [10]. Recall that the construction of these quasi-interpolation operators consists of first projecting the target field $\boldsymbol{v}$ onto a fully discontinuous finite element space and then stitching together the projected field to recover the desired conformity property by averaging the canonical degrees of freedom of the projected field. Since the projected field is always piecewise smooth, this construction is always meaningful, regardless of the regularity of the target field $\boldsymbol{v}$. Let $\mathcal{I}_h^{\mathrm{c,av}} : \boldsymbol{L}^1(D) \to \boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h)$ denote the quasi-interpolation operator thus constructed with the Nédélec finite elements. Then, [13, Thm. 22.6] shows that there is c such that for all $r \in [0, k + 1]$, all $p \in [1, \infty]$ if $r \in \mathbb{N}$ and $p \in [1, \infty)$ otherwise, all $\boldsymbol{v} \in \boldsymbol{W}^{r,p}(D)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$, we have

$$\|\boldsymbol{v} - \mathcal{I}_h^{\mathrm{c,av}}(\boldsymbol{v})\|_{\boldsymbol{L}^p(K)} \leq c\, h_K^r |\boldsymbol{v}|_{\boldsymbol{W}^{r,p}(D_K^{\mathrm{c}})}, \tag{4}$$

where $D_K^{\rm c} := \text{int}\left(\bigcup_{K'\in\mathcal{T}_K^{\rm c}} K'\right)$ and $\mathcal{T}_K^{\rm c}$ denotes the collection of mesh cells sharing at least one edge with K. We notice that a slight loss of localization occurs in (4) since the Sobolev–Slobodeckij seminorm on the right-hand side is evaluated over the macroelement $D_K^{\rm c}$ and not just over K. In the present work, we show that provided some (mild) additional integrability assumption is made on $\nabla{\times}\boldsymbol{v}$, the estimate (4) can be localized to the mesh cells in $\mathcal{T}_K^{\rm c}$; see Theorem 2.1 and Corollary 2.2.

The loss of localization is more striking if one wants to additionally enforce a homogeneous boundary condition on the tangential component of the target field. We assume for simplicity that the condition is enforced over the whole boundary ∂D of D. Recall that the tangential trace operator $\gamma^{\rm c} : \boldsymbol{H}(\text{curl}; D) \to \boldsymbol{H}^{-\frac12}(D)$ is defined through a global integration by parts formula (see, e.g., [13, Thm. 4.15]) and that we have $\gamma^{\rm c}(\boldsymbol{v}) := \boldsymbol{v}_{|\partial D} \times \boldsymbol{n}_D$ whenever the field $\boldsymbol{v}$ is smooth enough, where $\boldsymbol{n}_D$ denotes the unit outward normal to D. Then, setting $\boldsymbol{P}_{k,0}^{\rm c}(\mathcal{T}_h) := \{\boldsymbol{v}_h \in \boldsymbol{P}_k^{\rm c}(\mathcal{T}_h) \mid \gamma^{\rm c}(\boldsymbol{v}_h) = \boldsymbol{0}\}$, one is interested in establishing local upper bounds on the best-approximation error

$$\inf_{\boldsymbol{v}_h\in\boldsymbol{P}_{k,0}^{\rm c}(\mathcal{T}_h)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_{\boldsymbol{L}^p(D)}. \tag{5}$$

Let $\mathcal{I}_{h0}^{\rm c,av} : \boldsymbol{L}^1(D) \to \boldsymbol{P}_{k,0}^{\rm c}(\mathcal{T}_h)$ denote the quasi-interpolation operator with homogeneous boundary prescription associated with the Nédélec finite elements. Then, [13, Thm. 22.14]) shows that for all $r \in [0, \frac1p)$, there is c such that for all $\boldsymbol{v} \in \boldsymbol{W}^{r,p}(D)$, and all $h \in \mathcal{H}$, we have

$$\|\boldsymbol{v} - \mathcal{I}_{h0}^{\rm c,av}(\boldsymbol{v})\|_{\boldsymbol{L}^p(D)} \le c\, h^r \ell_D^{-r} \|\boldsymbol{v}\|_{\boldsymbol{W}^{r,p}(D)}, \tag{6}$$

where $h := \max_{K\in\mathcal{T}_h} h_K$, ℓ_D is a characteristic (global) length scale associated with D, and $\|\boldsymbol{v}\|_{\boldsymbol{W}^{r,p}(D)} = \|\boldsymbol{v}\|_{\boldsymbol{L}^p(D)} + \ell_D^r |\boldsymbol{v}|_{\boldsymbol{W}^{r,p}(D)}$. Notice that the target field $\boldsymbol{v}$ has not sufficient regularity to have a well-defined tangential trace on the boundary. The loss of localization in (6) arises when bounding the quasi-interpolation error over those mesh cells that have at least one edge located on the boundary ∂D (the upper bound (4) holds true for the other mesh cells). The presence of the global length scale ℓ_D and of the full Sobolev–Slobodeckij norm of $\boldsymbol{v}$ instead of just the seminorm in (6) comes from the need to invoke a Hardy inequality near the boundary (see the proof of [10, Thm. 6.4]). In the present work, we show that provided some (mild) additional integrability assumption is made on $\nabla{\times}\boldsymbol{v}$, the estimate (6) can be fully localized to the mesh cells; see again Theorem 2.1 and Corollary 2.2.

2 Main results on Nédélec finite elements

In this section, we first state our main results and then present their proofs.

2.1 Statement of the main results

Let us first observe that the domain of the tangential trace operator can be extended to $\boldsymbol{Y}^{\rm c}(D) := \{\boldsymbol{v} \in \boldsymbol{L}^2(D) \mid \nabla{\times}\boldsymbol{v} \in \boldsymbol{L}^q(D)\}$ for all $q \in (\frac{2d}{2+d}, 2]$. Indeed, for all $\boldsymbol{v} \in \boldsymbol{Y}^{\rm c}(D)$, $\gamma^{\rm c}(\boldsymbol{v}) \in \boldsymbol{H}^{-\frac12}(\partial D)$ can still be defined by duality by setting for all $\boldsymbol{w} \in \boldsymbol{H}^{\frac12}(\partial D)$,

$$\langle \gamma^{\rm c}(\boldsymbol{v}), \boldsymbol{w}\rangle_{\partial D} := \int_D \Big(\boldsymbol{v}{\cdot}\nabla{\times}\boldsymbol{l}(\boldsymbol{w}) - (\nabla{\times}\boldsymbol{v}){\cdot}\boldsymbol{l}(\boldsymbol{w})\Big)\,{\rm d}x, \tag{7}$$

where $\boldsymbol{l}(\boldsymbol{w})$ denotes a lifting of $\boldsymbol{w}$ in $\boldsymbol{H}^1(D)$. Indeed, owing to the Sobolev embedding theorem, and the fact that $q > \frac{2d}{2+d}$, we infer that $\boldsymbol{H}^1(D) \hookrightarrow \boldsymbol{L}^{q'}(D)$ with $\frac1q + \frac1{q'} = 1$, so that the second term on the right-hand side of (7) is meaningful owing to Hölder's inequality. We can now state our main result. For simplicity, we estimate the quasi-interpolation error only in the $\boldsymbol{L}^2$-norm.

Theorem 2.1 (Localized quasi-interpolation error estimate for Nédélec elements) *For all $r \in (0, 1]$ and all $q \in (\frac{2d}{2+d}, 2]$, there is c such that for all $\boldsymbol{v} \in \boldsymbol{H}^r(D)$ with $\nabla{\times}\boldsymbol{v} \in \boldsymbol{L}^q(D)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$, we have*

$$\|\boldsymbol{v} - \mathcal{I}_h^{\rm c,av}(\boldsymbol{v})\|_{\boldsymbol{L}^2(K)} \le c \sum_{K'\in\mathcal{T}_K^{\rm c}} \Big\{ h_{K'}^r |\boldsymbol{v}|_{\boldsymbol{H}^r(K')} + h_{K'}^{1+d(\frac12-\frac1q)} \|\nabla{\times}\boldsymbol{v}\|_{\boldsymbol{L}^q(K')} \Big\}. \tag{8}$$

Moreover, assuming that $\gamma^{\mathrm{c}}(\boldsymbol{v}) = \mathbf{0}$, *we also have*

$$\|\boldsymbol{v} - \mathcal{I}_{h0}^{\mathrm{c,av}}(\boldsymbol{v})\|_{\boldsymbol{L}^2(K)} \le c \sum_{K' \in \mathcal{T}_K^{\mathrm{c}}} \Big\{ h_{K'}^r |\boldsymbol{v}|_{\boldsymbol{H}^r(K')} + h_{K'}^{1+d(\frac{1}{2}-\frac{1}{q})} \|\nabla\times\boldsymbol{v}\|_{\boldsymbol{L}^q(K')} \Big\}. \tag{9}$$

Squaring the above inequalities, summing over the mesh cells, and observing that the cardinality of the set $\{K' \in \mathcal{T}_h \mid K \in \mathcal{T}_{K'}^{\mathrm{c}}\}$ is uniformly bounded for all $K \in \mathcal{T}_h$ and all $h \in \mathcal{H}$, we infer the following result.

Corollary 2.2 (Localized best-approximation error for Nédélec elements) *For all* $r \in (0,1]$ *and all* $q \in (\frac{2d}{2+d}, 2]$, *there is* c *such that for all* $\boldsymbol{v} \in \boldsymbol{H}^r(D)$ *with* $\nabla\times\boldsymbol{v} \in \boldsymbol{L}^q(D)$, *and all* $h \in \mathcal{H}$, *we have*

$$\inf_{\boldsymbol{v}_h \in \boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_{\boldsymbol{L}^2(D)} \le c \bigg\{ \sum_{K \in \mathcal{T}_h} \Big\{ h_K^{2r} |\boldsymbol{v}|_{\boldsymbol{H}^r(K)}^2 + h_K^{2+2d(\frac{1}{2}-\frac{1}{q})} \|\nabla\times\boldsymbol{v}\|_{\boldsymbol{L}^q(K)}^2 \Big\} \bigg\}^{\frac{1}{2}}. \tag{10}$$

Moreover, assuming that $\gamma^{\mathrm{c}}(\boldsymbol{v}) = \mathbf{0}$, *we also have*

$$\inf_{\boldsymbol{v}_h \in \boldsymbol{P}_{k,0}^{\mathrm{c}}(\mathcal{T}_h)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_{\boldsymbol{L}^2(D)} \le c \bigg\{ \sum_{K \in \mathcal{T}_h} \Big\{ h_K^{2r} |\boldsymbol{v}|_{\boldsymbol{H}^r(K)}^2 + h_K^{2+2d(\frac{1}{2}-\frac{1}{q})} \|\nabla\times\boldsymbol{v}\|_{\boldsymbol{L}^q(K)}^2 \Big\} \bigg\}^{\frac{1}{2}}. \tag{11}$$

Remark 2.3 (Exponent) *Observe that* $2 + 2d(\frac{1}{2} - \frac{1}{q}) = 2d(\frac{d+2}{2d} - \frac{1}{q}) > 0$ *since* $q \in (\frac{2d}{2+d}, 2]$. *Moreover, we have* $2 + 2d(\frac{1}{2} - \frac{1}{q}) = 2$ *for* $q = 2$.

2.2 Preliminary: localizing the tangential trace to the mesh faces

This section collects some preliminary results needed in the proof of Theorem 2.1. These results are drawn from [15] and are briefly restated here for the reader's convenience.

Let $K \in \mathcal{T}_h$ be a mesh cell, let $\mathcal{F}_K$ be the collection of the faces of K, and let $F \in \mathcal{F}_K$. To define a tangential trace that is localized to the mesh face F, we introduce the local functional space $\boldsymbol{V}^{\mathrm{c}}(K) := \{\boldsymbol{v} \in \boldsymbol{L}^p(K) \mid \nabla\times\boldsymbol{v} \in \boldsymbol{L}^q(K)\}$ with $q \in (\frac{2d}{2+d}, 2]$ (as above) and $p > 2$. We equip this space with the (dimensionally consistent) norm

$$\|\boldsymbol{v}\|_{\boldsymbol{V}^{\mathrm{c}}(K)} := \|\boldsymbol{v}\|_{\boldsymbol{L}^p(K)} + h_K^{1+d(\frac{1}{p}-\frac{1}{q})} \|\nabla\times\boldsymbol{v}\|_{\boldsymbol{L}^q(K)}. \tag{12}$$

Let $\varrho \in (2,p]$ be such that $q \ge \frac{\varrho d}{\varrho + d}$ (this is indeed possible since the function $x \mapsto \frac{xd}{x+d}$ is increasing on $[2,\infty)$). Let $\varrho' \in [1,2)$ be such that $\frac{1}{\varrho} + \frac{1}{\varrho'} = 1$. We consider the (fractional-order) Sobolev space $\boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F)$, equipped with the (dimensionally consistent) norm

$$\|\boldsymbol{\phi}\|_{\boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F)} := \|\boldsymbol{\phi}\|_{\boldsymbol{L}^{\varrho'}(F)} + h_F^{\frac{1}{\varrho}} |\boldsymbol{\phi}|_{\boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F)}. \tag{13}$$

Let $(\boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F))'$ denote the dual space of $\boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F)$. It is shown in [15, Equ. (5.5)] that upon introducing suitable face-to-cell lifting operators, it is possible to define a tangential trace operator localized to the mesh face F through an integration by parts formula, namely $\gamma_{K,F}^{\mathrm{c}} : \boldsymbol{V}^{\mathrm{c}}(K) \to (\boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F))'$, such that the following two properties hold: (i) $\gamma_{K,F}(\boldsymbol{v}) = (\boldsymbol{v} \times \boldsymbol{n}_K)_{|F}$ whenever the field $\boldsymbol{v}$ is smooth, where $\boldsymbol{n}_K$ denotes the unit normal to ∂K pointing outward K; (ii) There is c such that for all $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{c}}(K)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$,

$$\|\gamma_{K,F}^{\mathrm{c}}(\boldsymbol{v})\|_{(\boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F))'} \le c\, h_K^{-\frac{1}{\varrho}+d(\frac{1}{\varrho}-\frac{1}{p})} \|\boldsymbol{v}\|_{\boldsymbol{V}^{\mathrm{c}}(K)}. \tag{14}$$

Let $\mathbb{N}^k(K)$ be composed of the restriction to K of the Nédélec polynomials of order $k \in \mathbb{N}$. Let us set $\mathbb{N}^k(F) := \gamma_{K,F}^{\mathrm{c}}(\mathbb{N}^k(K))$ for all $F \in \mathcal{F}_K$. In this work, we need to invoke the following inverse inequality.

Lemma 2.4 (Inverse inequality on F) *Let $t \in [1,\infty]$. There is c such that for all $\boldsymbol{\phi}_h \in \mathbb{N}^k(F)$, all $K \in \mathcal{T}_h$, all $F \in \mathcal{F}_K$, and all $h \in \mathcal{H}$,*

$$\|\boldsymbol{\phi}_h\|_{\boldsymbol{L}^t(F)} \le c\, h_K^{(d-1)(\frac{1}{t}-\frac{1}{\varrho})} \|\boldsymbol{\phi}_h\|_{(\boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F))'}. \tag{15}$$

Proof. Let $\boldsymbol{\xi}_h \in \mathbb{N}^k(F)$. Recalling the definition (13) of the $\|\cdot\|_{\boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F)}$-norm and invoking an inverse inequality on $\mathbb{N}^k(F)$ (see, e.g., [13, Sec. 12.1], and observe that $\mathbb{N}^k(F) \subset \mathbb{P}^{k+1}_{d-1} \circ \boldsymbol{T}_F^{-1}$, where $\boldsymbol{T}_F : \widehat{S}^{d-1} \to F$ is the geometric mapping from the reference $(d-1)$-dimensional simplex $\widehat{S}^{d-1}$ to F and where $\mathbb{P}^{k+1}_{d-1}$ is composed of $\mathbb{R}^d$-valued, $(d-1)$-variate polynomials of order at most $(k+1)$), we infer that

$$\|\boldsymbol{\xi}_h\|_{\boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F)} \le c\, h_K^{(d-1)(\frac{1}{\varrho'}-\frac{1}{t'})} \|\boldsymbol{\xi}_h\|_{\boldsymbol{L}^{t'}(F)},$$

with $t' \in [1,\infty]$ such that $\frac{1}{t}+\frac{1}{t'}=1$. This implies that

$$\|\boldsymbol{\phi}_h\|_{\boldsymbol{L}^t(F)} = \sup_{\boldsymbol{\xi}_h \in \mathbb{N}^k(F)} \frac{\int_F \boldsymbol{\phi}_h \cdot \boldsymbol{\xi}_h \,\mathrm{d}s}{\|\boldsymbol{\xi}_h\|_{\boldsymbol{L}^{t'}(F)}} \le c h_K^{(d-1)(\frac{1}{\varrho'}-\frac{1}{t'})} \sup_{\boldsymbol{\xi}_h \in \mathbb{N}^k(F)} \frac{\int_F \boldsymbol{\phi}_h \cdot \boldsymbol{\xi}_h \,\mathrm{d}s}{\|\boldsymbol{\xi}_h\|_{\boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F)}}.$$

Since $\int_F \boldsymbol{\phi}_h \cdot \boldsymbol{\xi}_h \,\mathrm{d}s = \langle \boldsymbol{\phi}_h, \boldsymbol{\xi}_h \rangle_{(\boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F))',\boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F)}$, the assertion follows from the definition of the dual norm in $(\boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F))'$ and the identity $\frac{1}{\varrho'}-\frac{1}{t'}=\frac{1}{t}-\frac{1}{\varrho}$. □

Let us set

$$\boldsymbol{V}^{\mathrm{c}}(D) := \{\boldsymbol{v} \in \boldsymbol{L}^p(D) \mid \nabla\times\boldsymbol{v} \in \boldsymbol{L}^q(D)\}, \tag{16}$$
$$\boldsymbol{V}^{\mathrm{c}}_0(D) := \{\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{c}}(D) \mid \gamma^{\mathrm{c}}(\boldsymbol{v}) = \boldsymbol{0}\}. \tag{17}$$

(Observe that the tangential trace operator γ^{c} is meaningful on $\boldsymbol{V}^{\mathrm{c}}(D)$ since $p > 2$.) Proceeding as in [13, Thm. 4.15], one can show that $\boldsymbol{V}^{\mathrm{c}}_0(D)$ coincides with the closure of $\boldsymbol{C}^\infty_0(D)$ in $\boldsymbol{V}^{\mathrm{c}}(D)$. We notice that for all $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{c}}(D)$ and all $K \in \mathcal{T}_h$, we have $\boldsymbol{v}_{|K} \in \boldsymbol{V}^{\mathrm{c}}(K)$. We also define the broken version of $\boldsymbol{V}^{\mathrm{c}}(D)$ as follows:

$$\boldsymbol{V}^{\mathrm{c}}(\mathcal{T}_h) := \{\boldsymbol{v} \in \boldsymbol{L}^p(D) \mid \boldsymbol{v}_{|K} \in \boldsymbol{V}^{\mathrm{c}}(K), \forall K \in \mathcal{T}_h\}. \tag{18}$$

The collection of the mesh faces, $\mathcal{F}_h$, is split into the collection of the mesh interfaces, $\mathcal{F}^\circ_h$, and the collection of the mesh boundary faces, $\mathcal{F}^\partial_h$. For all $F \in \mathcal{F}^\circ_h$, there are two distinct mesh cells $K_-, K_+ \in \mathcal{T}_h$ such that $F = \partial K_- \cap \partial K_+$. For all $F \in \mathcal{F}^\partial_h$, there is one mesh cell $K_- \in \mathcal{T}_h$ such that $F = \partial K_- \cap \partial D$. For every field $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{c}}(\mathcal{T}_h)$, the jump of the tangential component across the mesh interface $F = \partial K_- \cap \partial K_+ \in \mathcal{F}^\circ_h$ is defined as

$$[\![\gamma_{K,F}(\boldsymbol{v})]\!]_F := \gamma_{K_-,F}(\boldsymbol{v}_{|K_-}) + \gamma_{K_+,F}(\boldsymbol{v}_{|K_+}). \tag{19}$$

Moreover, for every mesh boundary face $F = \partial K_- \cap \partial D \in \mathcal{F}^\partial_h$, we conventionally set

$$[\![\gamma_{K,F}(\boldsymbol{v})]\!]_F := \gamma_{K_-,F}(\boldsymbol{v}_{|K_-}). \tag{20}$$

Lemma 2.5 (Vanishing jumps and boundary traces) (i) *For all $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{c}}(D)$, we have $[\![\gamma_{K,F}(\boldsymbol{v})]\!]_F = \boldsymbol{0}$ for all $F \in \mathcal{F}^\circ_h$.* (ii) *For all $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{c}}_0(D)$, we additionally have $[\![\gamma_{K,F}(\boldsymbol{v})]\!]_F = \boldsymbol{0}$ for all $F \in \mathcal{F}^\partial_h$.*

Proof. Let $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{c}}(D)$ and let $F \in \mathcal{F}_h$. For all $K \in \mathcal{T}_h$ such that $F \in \mathcal{F}_K$, $\gamma_{K,F}(\boldsymbol{v})$ is defined in [15, Equ. (5.5)] so that

$$\langle \gamma_{K,F}(\boldsymbol{v}), \boldsymbol{\phi} \rangle_{(\boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F))',\boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F)} := \int_K \Big(\boldsymbol{v} \cdot \nabla\times E^K_F(\boldsymbol{\phi}) - (\nabla\times\boldsymbol{v}) \cdot E^K_F(\boldsymbol{\phi}) \Big) \,\mathrm{d}x,$$

for all $\boldsymbol{\phi} \in \boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F)$, where $E^K_F : \boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F) \to \boldsymbol{W}^{1,\varrho'}(K)$ is the face-to-cell lifting operator from [15, Def. 5.1].

(i) Let $F = \partial K_- \cap \partial K_+ \in \mathcal{F}^\circ_h$. We define the global lifting operator $E^D_F : \boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F) \to \boldsymbol{W}^{1,\varrho'}(D)$ such that $E_F^D(\phi)_{|K_\pm} := E_F^{K_\pm}(\phi)$ and $E_F^D(\phi) = \mathbf{0}$ otherwise. Summing the above identity for $K \in \{K_-, K_+\}$, we infer that

$$\langle [\![\gamma_{K,F}(\boldsymbol{v})]\!]_F, \phi \rangle_{(\boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F))',\boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F)} := \int_D \Big(\boldsymbol{v}\cdot\nabla\times E_F^D(\phi) - (\nabla\times\boldsymbol{v})\cdot E_F^D(\phi) \Big)\,\mathrm{d}x, \tag{21}$$

for all $\phi \in \boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F)$. Since, by construction, $E_F^D(\phi)$ has a zero trace at the boundary of D and since $\boldsymbol{V}_0^{\mathrm{c}}(D)$ coincides with the closure of $\boldsymbol{C}_0^\infty(D)$ in $\boldsymbol{V}^{\mathrm{c}}(D)$, we conclude that

$$\langle [\![\gamma_{K,F}(\boldsymbol{v})]\!]_F, \phi \rangle_{(\boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F))',\boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F)} = 0.$$

Since ϕ is arbitrary in $\boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F)$, this implies that $[\![\gamma_{K,F}(\boldsymbol{v})]\!]_F = \mathbf{0}$.
(ii) If $F = \partial K_- \cap \partial D \in \mathcal{F}_h^\partial$, we set $E_F^D(\phi)_{|K_-} := E_F^{K_-}(\phi)$ and $E_F^D(\phi) = \mathbf{0}$ otherwise. This yields again (21), and invoking that $\gamma^{\mathrm{c}}(\boldsymbol{v}) = \mathbf{0}$ still gives $\langle [\![\gamma_{K,F}(\boldsymbol{v})]\!]_F, \phi \rangle_{(\boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F))',\boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F)} = 0$, whence the assertion. □

2.3 Proof of (8)

Let us start with a preliminary result of independent interest. For all $K \in \mathcal{T}_h$, let $\Pi_K^0 : \boldsymbol{L}^1(K) \to \mathbb{P}^0(K)$ denote the (local) $\boldsymbol{L}^2$-orthogonal projection onto $\mathbb{P}^0(K)$ (that is, $\Pi_K^0(\boldsymbol{v})$ is the mean value of $\boldsymbol{v}$ over K).

Lemma 2.6 (Localized quasi-interpolation error in $\boldsymbol{V}^{\mathrm{c}}(D)$) *For all $p > 2$ and all $q \in (\frac{2d}{2+d}, 2]$, there is c such that for all $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{c}}(D)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$, we have*

$$\|\boldsymbol{v} - \mathcal{I}_h^{\mathrm{c,av}}(\boldsymbol{v})\|_{\boldsymbol{L}^2(K)} \le c\, h_K^{d(\frac12-\frac1p)} \sum_{K'\in\mathcal{T}_K^{\mathrm{c}}} \Big(\|\boldsymbol{v} - \Pi_{K'}^0(\boldsymbol{v})\|_{\boldsymbol{L}^p(K')} + h_{K'}^{1+d(\frac1p-\frac1q)} \|\nabla\times\boldsymbol{v}\|_{\boldsymbol{L}^q(K')} \Big). \tag{22}$$

Proof. Let $\Pi_h^{\mathrm{c}} : \boldsymbol{L}^1(D) \to \boldsymbol{P}_k^{\mathrm{c,b}}(\mathcal{T}_h) := \{\boldsymbol{v}_h \in \boldsymbol{L}^1(D) \mid \boldsymbol{v}_{h|K} \in \mathbb{N}^k(K), \forall K \in \mathcal{T}_h\}$ denote the global $\boldsymbol{L}^2$-orthogonal projection onto the broken Nédélec finite element space $\boldsymbol{P}_k^{\mathrm{c,b}}(\mathcal{T}_h)$. Let $\Pi_K^{\mathrm{c}} : \boldsymbol{L}^1(K) \to \mathbb{N}^k(K)$ denote the local $\boldsymbol{L}^2$-orthogonal projection onto $\mathbb{N}^k(K)$, so that we have $\Pi_h^{\mathrm{c}}(\boldsymbol{v})_{|K} = \Pi_K^{\mathrm{c}}(\boldsymbol{v}_{|K})$ for all $\boldsymbol{v} \in \boldsymbol{L}^1(D)$ and all $K \in \mathcal{T}_h$. Recall (see [10, Sec. 5]) that we have $\mathcal{I}_h^{\mathrm{c,av}} := \mathcal{J}_h^{\mathrm{c,av}} \circ \Pi_h^{\mathrm{c}}$, where the operator $\mathcal{J}_h^{\mathrm{c,av}} : \boldsymbol{P}_k^{\mathrm{c,b}}(\mathcal{T}_h) \to \boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h)$ is built by averaging the canonical degrees of freedom, see [10, Sec. 4.2]. The triangle inequality implies that

$$\|\boldsymbol{v} - \mathcal{I}_h^{\mathrm{c,av}}(\boldsymbol{v})\|_{\boldsymbol{L}^2(K)} \le \|\boldsymbol{v} - \Pi_K^{\mathrm{c}}(\boldsymbol{v})\|_{\boldsymbol{L}^2(K)} + \|(I - \mathcal{J}_h^{\mathrm{c,av}})(\Pi_h^{\mathrm{c}}(\boldsymbol{v}))\|_{\boldsymbol{L}^2(K)} =: \mathfrak{T}_1 + \mathfrak{T}_2.$$

Since $\mathbb{P}^0(K) \subset \mathbb{N}^k(K)$, standard properties of the $\boldsymbol{L}^2$-orthogonal projection and Hölder's inequality (recall that $p > 2$) imply that

$$\mathfrak{T}_1 \le \|\boldsymbol{v} - \Pi_K^0(\boldsymbol{v})\|_{\boldsymbol{L}^2(K)} \le c\, h_K^{d(\frac12-\frac1p)} \|\boldsymbol{v} - \Pi_K^0(\boldsymbol{v})\|_{\boldsymbol{L}^p(K)}.$$

Moreover, [10, Lem. 4.3] followed by Lemma 2.4 (with $t = 2$) give (recall that the value of c can change at each occurrence)

$$\mathfrak{T}_2 \le c\, h_K^{\frac12} \sum_{F\in\check{\mathcal{F}}_K^\circ} \|[\![\gamma_{K,F}^{\mathrm{c}}(\Pi_h^{\mathrm{c}}(\boldsymbol{v}))]\!]_F\|_{\boldsymbol{L}^2(F)} \le c\, h_K^{\frac12+(d-1)(\frac12-\frac1\varrho)} \sum_{F\in\check{\mathcal{F}}_K^\circ} \|[\![\gamma_{K,F}^{\mathrm{c}}(\Pi_h^{\mathrm{c}}(\boldsymbol{v}))]\!]_F\|_{(\boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F))'},$$

where $\check{\mathcal{F}}_K^\circ$ denotes the collection of the mesh interfaces sharing at least an edge with K. Observing that $\gamma_{K,F}^{\mathrm{c}}(\boldsymbol{v}) = \mathbf{0}$ since $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{c}}(D)$ (see Lemma 2.5(i)), we infer that

$$\mathfrak{T}_2 \le c\, h_K^{\frac12+(d-1)(\frac12-\frac1\varrho)} \sum_{F\in\check{\mathcal{F}}_K^\circ} \|[\![\gamma_{K,F}^{\mathrm{c}}(\boldsymbol{v} - \Pi_h^{\mathrm{c}}(\boldsymbol{v}))]\!]_F\|_{(\boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F))'}.$$

By definition of the jump operator, invoking the triangle inequality and recalling the definition of the set $\mathcal{T}_K^{\mathrm{c}}$ yields

$$\mathfrak{T}_2 \le c\, h_K^{\frac12+(d-1)(\frac12-\frac1\varrho)} \sum_{K'\in\mathcal{T}_K^{\mathrm{c}}} \sum_{F\in\mathcal{F}_{K'}\cap\mathcal{F}_h^\circ} \|\gamma_{K',F}^{\mathrm{c}}(\boldsymbol{v} - \Pi_{K'}^{\mathrm{c}}(\boldsymbol{v}))\|_{(\boldsymbol{W}^{\frac{1}{\varrho},\varrho'}(F))'}.$$

Owing to the bound (14) and the shape-regularity of the mesh sequence, we infer that

$$\mathfrak{T}_2 \le c\, h_K^{d(\frac{1}{2}-\frac{1}{p})} \sum_{K'\in\mathcal{T}_K^{\rm c}} \|\boldsymbol{v} - \Pi_{K'}^{\rm c}(\boldsymbol{v})\|_{\boldsymbol{V}^{\rm c}(K')}. \tag{23}$$

Invoking the triangle inequality and recalling the definition (12) of the norm equipping $\boldsymbol{V}^{\rm c}(K')$ gives for all $K' \in \mathcal{T}_K^{\rm c}$,

$$\|\boldsymbol{v} - \Pi_{K'}^{\rm c}(\boldsymbol{v})\|_{\boldsymbol{V}^{\rm c}(K')} \le \|\boldsymbol{v} - \Pi_K^0(\boldsymbol{v})\|_{\boldsymbol{L}^p(K')} + h_{K'}^{1+d(\frac{1}{p}-\frac{1}{q})}\|\nabla\times\boldsymbol{v}\|_{\boldsymbol{L}^q(K')} + \|(\Pi_{K'}^{\rm c} - \Pi_{K'}^0)(\boldsymbol{v})\|_{\boldsymbol{V}^{\rm c}(K')},$$

where we used that $\Pi_K^0(\boldsymbol{v})$ is a constant field in K'. Invoking inverse inequalities on the first line, the triangle inequality and standard properties of the $\boldsymbol{L}^2$-orthogonal projection on the second line, and, finally, Hölder's inequality (recall that $p > 2$) and the regularity of the mesh sequence on the third line, we also have

$$\begin{aligned}\|(\Pi_{K'}^{\rm c} - \Pi_{K'}^0)(\boldsymbol{v})\|_{\boldsymbol{V}^{\rm c}(K')} &\le c\, h_{K'}^{d(\frac{1}{p}-\frac{1}{2})}\|(\Pi_{K'}^{\rm c} - \Pi_{K'}^0)(\boldsymbol{v})\|_{\boldsymbol{L}^2(K')} \\ &\le c\, h_{K'}^{d(\frac{1}{p}-\frac{1}{2})}\|(I - \Pi_{K'}^0)(\boldsymbol{v})\|_{\boldsymbol{L}^2(K')} \\ &\le c\, \|(I - \Pi_{K'}^0)(\boldsymbol{v})\|_{\boldsymbol{L}^p(K')}.\end{aligned}$$

Combining this bound with (23) and recalling the bound on $\mathfrak{T}_1$ completes the proof. □

Proof of (8). Let $r \in (0, 1]$ and $q \in (\frac{2d}{2+d}, 2]$. Consider a field $\boldsymbol{v} \in \boldsymbol{H}^r(D)$ with $\nabla\times\boldsymbol{v} \in \boldsymbol{L}^q(D)$. The Sobolev embedding theorem implies that $\boldsymbol{v} \in \boldsymbol{V}^{\rm c}(D)$ (indeed, we can take $p := \frac{2d}{d-2r} > 2$ if $r < \frac{d}{2}$ and any $p \in (2, \infty)$ otherwise). Applying Lemma 2.6, re-arranging the factors involving the local mesh sizes on the right-hand side of (22) and using the shape-regularity of the mesh sequence, we infer that

$$\|\boldsymbol{v} - \mathcal{I}_h^{\rm c,av}(\boldsymbol{v})\|_{\boldsymbol{L}^2(K)} \le c \sum_{K'\in\mathcal{T}_K^{\rm c}} \Big(h_{K'}^{d(\frac{1}{2}-\frac{1}{p})}\|\boldsymbol{v} - \Pi_{K'}^0(\boldsymbol{v})\|_{\boldsymbol{L}^p(K')} + h_{K'}^{1+d(\frac{1}{2}-\frac{1}{q})}\|\nabla\times\boldsymbol{v}\|_{\boldsymbol{L}^q(K')}\Big).$$

Owing to [13, Equ. (17.19)] and since $\Pi_{K'}^0(\boldsymbol{v})$ is a constant field in K', we infer that

$$h_{K'}^{d(\frac{1}{2}-\frac{1}{p})}\|\boldsymbol{v} - \Pi_{K'}^0(\boldsymbol{v})\|_{\boldsymbol{L}^p(K')} \le c\,\big(\|\boldsymbol{v} - \Pi_{K'}^0(\boldsymbol{v})\|_{\boldsymbol{L}^2(K')} + h_{K'}^r|\boldsymbol{v}|_{\boldsymbol{H}^r(K')}\big),$$

so that invoking the (fractional) Poincaré–Steklov inequality in K' (see, e.g., [13, Sec. 12.3.1]) gives

$$h_{K'}^{d(\frac{1}{2}-\frac{1}{p})}\|\boldsymbol{v} - \Pi_{K'}^0(\boldsymbol{v})\|_{\boldsymbol{L}^p(K')} \le c\, h_{K'}^r|\boldsymbol{v}|_{\boldsymbol{H}^r(K')}.$$

Combining the above bounds completes the proof. □

2.4 Proof of (9)

The proof of (9) follows from minor adaptations of the arguments to prove (8). The most relevant change hinges on the following result.

Lemma 2.7 (Localized quasi-interpolation error in $\boldsymbol{V}_0^{\rm c}(D)$) *For all $p > 2$ and all $q \in (\frac{2d}{2+d}, 2]$, there is c such that for all $\boldsymbol{v} \in \boldsymbol{V}_0^{\rm c}(D)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$, we have*

$$\|\boldsymbol{v} - \mathcal{I}_{h0}^{\rm c,av}(\boldsymbol{v})\|_{\boldsymbol{L}^2(K)} \le c\, h_K^{d(\frac{1}{2}-\frac{1}{p})} \sum_{K'\in\mathcal{T}_K^{\rm c}} \Big(\|\boldsymbol{v} - \Pi_{K'}^0(\boldsymbol{v})\|_{\boldsymbol{L}^p(K')} + h_{K'}^{1+d(\frac{1}{p}-\frac{1}{q})}\|\nabla\times\boldsymbol{v}\|_{\boldsymbol{L}^q(K')}\Big). \tag{24}$$

Proof. The only relevant difference with the proof of Lemma 2.6 is in the upper bound estimate for $\mathfrak{T}_2$. Owing to the homogeneous boundary prescription, we now have $\mathcal{I}_{h0}^{\rm c,av} := \mathcal{J}_{h0}^{\rm c,av} \circ \Pi_h^{\rm c}$, where the averaging operator $\mathcal{J}_{h0}^{\rm c,av} : \boldsymbol{P}_k^{\rm c,b}(\mathcal{T}_h) \to \boldsymbol{P}_{k,0}^{\rm c}(\mathcal{T}_h)$ is built as $\mathcal{J}_h^{\rm c,av}$ with the additional requirement that all the degrees of freedom attached to the mesh edges and faces located on the boundary ∂D are zero (see [10, Sec. 6.2]). As a result, we now have

$$\mathfrak{T}_2 \le c\, h_K^{\frac{1}{2}} \sum_{F\in\check{\mathcal{F}}_K} \|[\![\gamma^{\mathrm{c}}_{K,F}(\Pi^{\mathrm{c}}_h(\boldsymbol{v}))]\!]_F\|_{\boldsymbol{L}^2(F)},$$

where $\check{\mathcal{F}}_K$ denotes the collection of the mesh faces (and not only the mesh interfaces) sharing at least an edge with K. The bound on $\mathfrak{T}_2$ is obtained by using the same arguments as above and the fact that $[\![\gamma_{K,F}(\boldsymbol{v})]\!]_F = \boldsymbol{0}$ for every mesh boundary face $F \in \mathcal{F}_h^{\partial}$ since $\boldsymbol{v} \in \boldsymbol{V}_0^{\mathrm{c}}(D)$ (see Lemma 2.5(ii)). □

3 Raviart–Thomas finite elements

The discussion for the Raviart–Thomas finite elements goes along the same lines as for the Nédélec finite elements, except that the minimal regularity requirements on the target field are less demanding when using the canonical interpolation operator (or any extension thereof) since one only needs to give a meaning to the trace of the normal component of the target field on the mesh faces.

Let $\boldsymbol{P}_k^{\mathrm{d}}(\mathcal{T}_h)$ denote the $\boldsymbol{H}(\mathrm{div}; D)$-conforming finite element space built on the mesh $\mathcal{T}_h$ using the Raviart–Thomas finite element of degree $k \in \mathbb{N}$ (here, the superscript $^{\mathrm{d}}$ refers to the divergence operator) and consider the best-approximation error

$$\inf_{\boldsymbol{v}_h\in\boldsymbol{P}_k^{\mathrm{d}}(\mathcal{T}_h)} \|\boldsymbol{v}-\boldsymbol{v}_h\|_{\boldsymbol{L}^p(D)}, \tag{25}$$

for an arbitrary target field $\boldsymbol{v} \in \boldsymbol{W}^{r,p}(D)$, with $r > 0$ possibly very small. Let $\mathcal{I}_h^{\mathrm{d}}$ be the canonical interpolation operator associated with the Raviart–Thomas finite elements. It is well-known (see, e.g., [5] or [13, Sec. 16.1]) that for all $r \in (\frac{1}{p}, k+1]$ if $p > 1$ and $r \in [1, k+1]$ if $p = 1$, there is c such that for all $\boldsymbol{v} \in \boldsymbol{W}^{r,p}(D)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$, we have

$$\|\boldsymbol{v}-\mathcal{I}_h^{\mathrm{d}}(\boldsymbol{v})\|_{\boldsymbol{L}^p(K)} \le c\, h_K^r |\boldsymbol{v}|_{\boldsymbol{W}^{r,p}(K)}, \tag{26}$$

with c unbounded as $r \downarrow \frac{1}{p}$ if $p > 1$. For instance, in the Hilbert setting where $p = 2$, the minimal regularity requirement is $\boldsymbol{v} \in \boldsymbol{H}^r(D)$ with $r > \frac{1}{2}$, and c is unbounded as $r \downarrow \frac{1}{2}$. Moreover, by invoking more sophisticated results on traces and assuming some additional integrability property on $\nabla\cdot\boldsymbol{v}$, one can show (see, e.g., [13, Sec. 17.2]) that for all $r \in (0, 1]$ and all $q > \frac{2d}{2+d}$, there is c such that for all $\boldsymbol{v} \in \boldsymbol{H}^r(D)$ with $\nabla\cdot\boldsymbol{v} \in L^q(D)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$, we have

$$\|\boldsymbol{v}-\mathcal{I}_h^{\mathrm{d}}(\boldsymbol{v})\|_{\boldsymbol{L}^2(K)} \le c\left(h_K^r |\boldsymbol{v}|_{\boldsymbol{H}^r(K)} + h_K^{1+d(\frac{1}{2}-\frac{1}{q})} \|\nabla\cdot\boldsymbol{v}\|_{L^q(K)}\right), \tag{27}$$

with c unbounded as $r \downarrow 0$ or $q \downarrow \frac{2d}{2+d}$.

It is also possible to consider the quasi-interpolation operators devised in [10]. Let $\mathcal{I}_h^{\mathrm{d,av}} : \boldsymbol{L}^1(D) \to \boldsymbol{P}_k^{\mathrm{d}}(\mathcal{T}_h)$ denote the quasi-interpolation operator associated with the Raviart–Thomas finite element. Then, [13, Thm. 22.6] shows that there is c such that for all $r \in [0, k+1]$, all $p \in [1, \infty]$ if $r \in \mathbb{N}$ and $p \in [1, \infty)$ otherwise, all $\boldsymbol{v} \in \boldsymbol{W}^{r,p}(D)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$, we have

$$\|\boldsymbol{v}-\mathcal{I}_h^{\mathrm{d,av}}(\boldsymbol{v})\|_{\boldsymbol{L}^p(K)} \le c\, h_K^r |\boldsymbol{v}|_{\boldsymbol{W}^{r,p}(D_K^{\mathrm{d}})}, \tag{28}$$

$D_K^{\mathrm{d}} := \mathrm{int}\left(\bigcup_{K'\in\mathcal{T}_K^{\mathrm{d}}} K'\right)$ and $\mathcal{T}_K^{\mathrm{d}}$ denotes the collection of mesh cells sharing at least one face with K. We notice again that a slight loss of localization occurs in (28) since the Sobolev–Slobodeckij seminorm on the right-hand side is evaluated over the macroelement D_K^{d} and not just over K. One can also enforce a homogeneous condition on the normal component of the target field over the whole boundary ∂D of D (for simplicity). Recall that the normal trace operator $\gamma^{\mathrm{d}} : \boldsymbol{H}(\mathrm{div}; D) \to \boldsymbol{H}^{-\frac{1}{2}}(D)$ is defined through a global integration by parts formula (see, e.g., [13, Thm. 4.15]) and that we have $\gamma^{\mathrm{d}}(\boldsymbol{v}) := \boldsymbol{v}_{|\partial D}\cdot\boldsymbol{n}_D$ whenever the field $\boldsymbol{v}$ is smooth enough. Then, setting $\boldsymbol{P}_{k,0}^{\mathrm{d}}(\mathcal{T}_h) := \{\boldsymbol{v}_h \in \boldsymbol{P}_k^{\mathrm{d}}(\mathcal{T}_h) \mid \gamma^{\mathrm{d}}(\boldsymbol{v}_h) = 0\}$, one is interested in establishing local upper bounds on the best-approximation error

$$\inf_{\boldsymbol{v}_h\in\boldsymbol{P}_{k,0}^{\mathrm{d}}(\mathcal{T}_h)} \|\boldsymbol{v}-\boldsymbol{v}_h\|_{\boldsymbol{L}^p(D)}. \tag{29}$$

Let $\mathcal{I}_{h0}^{\mathrm{d,av}} : \boldsymbol{L}^1(D) \to \boldsymbol{P}_{k,0}^{\mathrm{d}}(\mathcal{T}_h)$ denote the quasi-interpolation operator with homogeneous boundary prescription associated with the Raviart–Thomas finite elements. Then, [13, Thm. 22.14] shows again that for all $r \in [0, \frac{1}{p})$, there is c such that for all $\boldsymbol{v} \in \boldsymbol{W}^{r,p}(D)$, and all $h \in \mathcal{H}$, we have

$$\|\boldsymbol{v} - \mathcal{I}_{h0}^{\mathrm{d,av}}(\boldsymbol{v})\|_{\boldsymbol{L}^p(D)} \leq c\, h^r \ell_D^{-r} \|\boldsymbol{v}\|_{\boldsymbol{W}^{r,p}(D)}. \tag{30}$$

We now show that the estimates (28) and (30) can be localized under suitable additional assumptions on the divergence of the field $\boldsymbol{v}$. Let us first observe that the domain of the normal trace operator can be extended to $\boldsymbol{Y}^{\mathrm{d}}(D) := \{\boldsymbol{v} \in \boldsymbol{L}^2(D) \mid \nabla{\cdot}\boldsymbol{v} \in L^q(D)\}$ for all $q \in (\frac{2d}{2+d}, 2]$. Indeed, for all $\boldsymbol{v} \in \boldsymbol{Y}^{\mathrm{d}}(D)$, $\gamma^{\mathrm{d}}(\boldsymbol{v}) \in H^{-\frac{1}{2}}(\partial D)$ can still be defined by duality by setting for all $w \in H^{\frac{1}{2}}(\partial D)$,

$$\langle \gamma^{\mathrm{d}}(\boldsymbol{v}), w \rangle_{\partial D} := \int_D \Big(\boldsymbol{v}{\cdot}\nabla l(w) + (\nabla{\cdot}\boldsymbol{v}) l(w) \Big) \,\mathrm{d}x, \tag{31}$$

where $l(w)$ denotes a lifting of w in $H^1(D)$. This is again a consequence of $q > \frac{2d}{2+d}$ and Hölder's inequality. We can now state our main result. For simplicity, we estimate the quasi-interpolation error only in the $\boldsymbol{L}^2$-norm.

Theorem 3.1 (Localized quasi-interpolation error estimate for Raviart–Thomas elements) *For all $r \in (0,1]$ and all $q \in (\frac{2d}{2+d}, 2]$, there is c such that for all $\boldsymbol{v} \in \boldsymbol{H}^r(D)$ with $\nabla{\cdot}\boldsymbol{v} \in L^q(D)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$, we have*

$$\|\boldsymbol{v} - \mathcal{I}_{h}^{\mathrm{d,av}}(\boldsymbol{v})\|_{\boldsymbol{L}^2(K)} \leq c \sum_{K' \in \mathcal{T}_K^{\mathrm{d}}} \Big\{ h_{K'}^r |\boldsymbol{v}|_{\boldsymbol{H}^r(K')} + h_{K'}^{1+d(\frac{1}{2}-\frac{1}{q})} \|\nabla{\cdot}\boldsymbol{v}\|_{L^q(K')} \Big\}. \tag{32}$$

Moreover, assuming that $\gamma^{\mathrm{d}}(\boldsymbol{v}) = 0$, we also have

$$\|\boldsymbol{v} - \mathcal{I}_{h0}^{\mathrm{d,av}}(\boldsymbol{v})\|_{\boldsymbol{L}^2(K)} \leq c \sum_{K' \in \mathcal{T}_K^{\mathrm{d}}} \Big\{ h_{K'}^r |\boldsymbol{v}|_{\boldsymbol{H}^r(K')} + h_{K'}^{1+d(\frac{1}{2}-\frac{1}{q})} \|\nabla{\cdot}\boldsymbol{v}\|_{L^q(K')} \Big\}. \tag{33}$$

Using the same arguments as above, we readily infer the following result.

Corollary 3.2 (Localized best-approximation error for Raviart–Thomas elements) *For all $r \in (0,1]$ and all $q \in (\frac{2d}{2+d}, 2]$, there is c such that for all $\boldsymbol{v} \in \boldsymbol{H}^r(D)$ with $\nabla{\cdot}\boldsymbol{v} \in L^q(D)$, and all $h \in \mathcal{H}$, we have*

$$\inf_{\boldsymbol{v}_h \in \boldsymbol{P}_k^{\mathrm{d}}(\mathcal{T}_h)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_{\boldsymbol{L}^2(D)} \leq c \Big\{ \sum_{K \in \mathcal{T}_h} \Big\{ h_K^{2r} |\boldsymbol{v}|_{\boldsymbol{H}^r(K)}^2 + h_K^{2+2d(\frac{1}{2}-\frac{1}{q})} \|\nabla{\cdot}\boldsymbol{v}\|_{L^q(K)}^2 \Big\} \Big\}^{\frac{1}{2}}. \tag{34}$$

Moreover, assuming that $\gamma^{\mathrm{d}}(\boldsymbol{v}) = 0$, we also have

$$\inf_{\boldsymbol{v}_h \in \boldsymbol{P}_{k,0}^{\mathrm{d}}(\mathcal{T}_h)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_{\boldsymbol{L}^2(D)} \leq c \Big\{ \sum_{K \in \mathcal{T}_h} \Big\{ h_K^{2r} |\boldsymbol{v}|_{\boldsymbol{H}^r(K)}^2 + h_K^{2+2d(\frac{1}{2}-\frac{1}{q})} \|\nabla{\cdot}\boldsymbol{v}\|_{L^q(K)}^2 \Big\} \Big\}^{\frac{1}{2}}. \tag{35}$$

Remark 3.3 (Comparison with canonical interpolation) *The canonical Raviart–Thomas interpolation operator and the quasi-interpolation operator achieve the same convergence rates under the same smoothness assumptions. Notice though that the estimates (32)-(33) are localized over $\mathcal{T}_K^{\mathrm{d}}$ whereas the estimate (27) is localized over the single cell K. The upper bounds in Corollary 3.2 can also be derived by invoking the canonical Raviart–Thomas interpolation operator. This is in contrast with the Nédélec elements where the estimates (8)-(9) on the quasi-interpolation error are essential to prove Corollary 2.2. These estimates hold for the canonical Nédélec interpolation operator only if $r > \frac{1}{2}$.*

4 Application to Maxwell's equations

In this section, we briefly present two applications of the above results for the approximation of Maxwell's equations using Nédélec finite elements. We focus on simplified forms of Maxwell's equations obtained, e.g., in the time-harmonic regime and in the eddy currents approximation.

4.1 Model problem

Given a source term $\boldsymbol{f} \in \boldsymbol{L}^q(D)$ with $q \in (\frac{2d}{2+d}, 2]$, the model problem we consider consists of seeking $\boldsymbol{A} \in \boldsymbol{H}_0(\mathrm{curl}; D) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}; D) \mid \gamma^{\mathrm{c}}(\boldsymbol{v}) = \boldsymbol{0}\}$ such that $a(\boldsymbol{A}, \boldsymbol{b}) = \ell(\boldsymbol{b})$ for all $\boldsymbol{b} \in \boldsymbol{H}_0(\mathrm{curl}; D)$, with the following sesquilinear and antilinear forms:

$$a(\boldsymbol{a}, \boldsymbol{b}) := \int_D (\nu \boldsymbol{a}{\cdot}\bar{\boldsymbol{b}} + \kappa \nabla{\times}\boldsymbol{a}{\cdot}\nabla{\times}\bar{\boldsymbol{b}})\,\mathrm{d}x, \qquad \ell(\boldsymbol{b}) := \int_D \boldsymbol{f}{\cdot}\bar{\boldsymbol{b}}\,\mathrm{d}x. \tag{36}$$

We assume that the model parameters ν, κ are both bounded in D and we set $\nu_\sharp := \|\nu\|_{L^\infty(D)}$, $\kappa_\sharp := \|\kappa\|_{L^\infty(D)}$. We also assume that there are real numbers θ, $\nu_\flat > 0$, $\kappa_\flat > 0$ such that

$$\operatorname*{ess\,inf}_{\boldsymbol{x}\in D} \Re\big(e^{i\theta}\nu(\boldsymbol{x})\big) \ge \nu_\flat \quad \text{and} \quad \operatorname*{ess\,inf}_{\boldsymbol{x}\in D} \Re\big(e^{i\theta}\kappa(\boldsymbol{x})\big) \ge \kappa_\flat. \tag{37}$$

This condition ensures the coercivity of the sesquilinear form a. Therefore, the model problem is well-posed owing to the Lax–Milgram lemma, and its unique weak solution is such that $\nu\boldsymbol{A} + \nabla{\times}(\kappa\nabla{\times}A) = \boldsymbol{f}$ in D together with $\gamma^{\mathrm{c}}(\boldsymbol{A}) = \boldsymbol{0}$. To obtain a regularity result on the weak solution, we assume that there is a partition $\{D_m\}_{m\in\{1:M\}}$ of D into $M \ge 1$ disjoint polyhedral Lipschitz subsets such that $\nu_{|D_m}$ and $\kappa_{|D_m}$ are constant for all $m \in \{1{:}M\}$. One can then show (see [17, 7], see also [4, 9, 2]) that there is $r \in (0, \frac{1}{2})$ such that

$$\boldsymbol{A} \in \boldsymbol{H}^r(D), \qquad \nabla{\times}\boldsymbol{A} \in \boldsymbol{H}^r(D). \tag{38}$$

4.2 Classical approximation with prescribed boundary conditions

In this setting, the discrete problem approximated with Nédélec finite elements consists of seeking $\boldsymbol{A}_h \in \boldsymbol{P}^{\mathrm{c}}_{k,0}(\mathcal{T}_h)$ such that $a(\boldsymbol{A}_h, \boldsymbol{b}_h) = \ell(\boldsymbol{b}_h)$ for all $\boldsymbol{b}_h \in \boldsymbol{P}^{\mathrm{c}}_{k,0}(\mathcal{T}_h)$. Recall that $k \in \mathbb{N}$ denotes the polynomial degree and that $(\mathcal{T}_h)_{h\in\mathcal{H}}$ is a shape-regular family of affine, matching, simplicial meshes such that each mesh covers D exactly. Notice that the homogeneous boundary condition on the tangential component of $\boldsymbol{A}_h$ at the boundary is explicitly enforced in the discrete problem.

The error analysis is performed by establishing suitable stability, consistency, and boundedness properties. To avoid distracting technicalities, we do not track the dependency of the constants in the error analysis on the nondimensional factors $\frac{\nu_\sharp}{\nu_\flat}$, $\frac{\kappa_\sharp}{\kappa_\flat}$, and $\nu_\sharp\kappa_\sharp^{-1}\ell_D^2$ (recall that ℓ_D denotes a characteristic (global) length scale associated with D). Referring to [12] (see also [14, Chap. 44]) for further insight, the main error estimate states that there is c such that we have

$$\|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{H}(\mathrm{curl};D)} \le c \inf_{\boldsymbol{b}_h\in\boldsymbol{P}^{\mathrm{c}}_{k,0}(\mathcal{T}_h)} \|\boldsymbol{A} - \boldsymbol{b}_h\|_{\boldsymbol{H}(\mathrm{curl};D)}. \tag{39}$$

Moreover, invoking commuting quasi-interpolation operators, it is also shown therein that there is c such that we have

$$\|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{H}(\mathrm{curl};D)} \le c\Big(\inf_{\boldsymbol{b}_h\in\boldsymbol{P}^{\mathrm{c}}_{k,0}(\mathcal{T}_h)} \|\boldsymbol{A} - \boldsymbol{b}_h\|_{\boldsymbol{L}^2(D)} + \ell_D \inf_{\boldsymbol{d}_h\in\boldsymbol{P}^{\mathrm{d}}_{k,0}(\mathcal{T}_h)} \|\nabla{\times}\boldsymbol{A} - \boldsymbol{d}_h\|_{\boldsymbol{L}^2(D)}\Big). \tag{40}$$

The analysis performed in this paper allows us to derive the following result where the best approximation error is fully localized even when $r \in (0, \frac{1}{2})$.

Corollary 4.1 (Localized error estimate) *There is c such that we have*

$$\|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{H}(\mathrm{curl};D)} \le c\bigg\{ \sum_{K\in\mathcal{T}_h} h_K^{2r}\Big(|\boldsymbol{A}|^2_{\boldsymbol{H}^r(K)} + \ell_D^{2(1-r)}\|\nabla{\times}\boldsymbol{A}\|^2_{\boldsymbol{L}^2(K)} + \ell_D^2|\nabla{\times}\boldsymbol{A}|^2_{\boldsymbol{H}^r(K)}\Big)\bigg\}^{\frac{1}{2}}. \tag{41}$$

Proof. We combine the error estimate (40) with the results from Corollary 2.2 and Corollary 3.2. In particular, we apply the estimate (11) with $\boldsymbol{v} := \boldsymbol{A}$ (notice that indeed $\gamma^{\mathrm{c}}(\boldsymbol{A}) = \boldsymbol{0}$) and $q := 2$, and the estimate (35) with $\boldsymbol{v} := \nabla{\times}\boldsymbol{A}$ (notice that indeed $\gamma^{\mathrm{d}}(\nabla{\times}\boldsymbol{A}) = 0$ and that $\nabla{\cdot}\boldsymbol{v} = 0$). This yields

$$\|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{H}(\mathrm{curl};D)} \le c\bigg\{ \sum_{K\in\mathcal{T}_h} \Big(h_K^{2r}|\boldsymbol{A}|^2_{\boldsymbol{H}^r(K)} + h_K^2\|\nabla{\times}\boldsymbol{A}\|^2_{\boldsymbol{L}^2(K)} + \ell_D^2 h_K^{2r}|\nabla{\times}\boldsymbol{A}|^2_{\boldsymbol{H}^r(K)}\Big)\bigg\}^{\frac{1}{2}},$$

whence the assertion follows from $h_K^2 \le h_K^{2r}\ell_D^{2(1-r)}$. $\square$

4.3 Nitsche's boundary penalty technique

Let us now consider a variant of the above discrete problem where the boundary condition is enforced by Nitsche's boundary penalty technique; see [11] and also [14, Chap. 45]. For all $K \in \mathcal{T}_h$, we set $\nu_K := |\nu_{|K}|$, $\kappa_K := |\kappa_{|K}|$, $\nu_{r,K} := \Re(e^{i\theta}\nu_K)$, and $\kappa_{r,K} := \Re(e^{i\theta}\kappa_K)$. To avoid distracting technicalities, we do not track the dependency of the constants in the error analysis on the nondimensional factors $\frac{\nu_K}{\nu_{r,K}}$ and $\frac{\kappa_K}{\kappa_{r,K}}$. Recall that for every mesh boundary face $F \in \mathcal{F}_h^\partial$, there is a mesh cell $K_- \in \mathcal{T}_h$ such that $F = \partial K_- \cap \partial D$. The discrete problem now consists of seeking $\boldsymbol{A}_h \in \boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h)$ such that $a_h(\boldsymbol{A}_h, \boldsymbol{b}_h) = \ell(\boldsymbol{b}_h)$ for all $\boldsymbol{b}_h \in \boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h)$, with $a_h(\cdot,\cdot) = a(\cdot,\cdot) - n_h(\cdot,\cdot) + s_h(\cdot,\cdot)$, with the following sesquilinear forms defined on $\boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h) \times \boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h)$:

$$n_h(\boldsymbol{a}_h, \boldsymbol{b}_h) := \sum_{F\in\mathcal{F}_h^\partial} \int_F ((\kappa\nabla\times\boldsymbol{a}_h)\times\boldsymbol{n}_F)\cdot\bar{\boldsymbol{b}}_h \,\mathrm{d}s, \qquad s_h(\boldsymbol{a}_h, \boldsymbol{b}_h) := \sum_{F\in\mathcal{F}_h^\partial} \eta_0 e^{-i\theta}\frac{\lambda_F}{h_F}\int_F (\boldsymbol{a}_h\times\boldsymbol{n}_F)\cdot(\bar{\boldsymbol{b}}_h\times\boldsymbol{n}_F)\,\mathrm{d}s, \tag{42}$$

where η_0 is a nondimensional parameter that must be chosen large enough so that the sesquilinear form a_h is coercive on $\boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h)$ (see [11, Lem. 3.3] or [14, Lem. 45.1]).

To perform the error analysis, it is convenient to introduce the space $\boldsymbol{V}_{\mathrm{S}} := \{\boldsymbol{a} \in \boldsymbol{H}_0(\mathrm{curl}; D) \mid \kappa\nabla\times\boldsymbol{a} \in \boldsymbol{V}^{\mathrm{c}}(D)\}$, where the space $\boldsymbol{V}^{\mathrm{c}}(D)$ is defined in (16). We observe that the exact solution satisfies $\boldsymbol{A} \in \boldsymbol{V}_{\mathrm{S}}$; indeed, $\boldsymbol{A} \in \boldsymbol{H}_0(\mathrm{curl}; D)$ by construction, $\nabla\times(\kappa\nabla\times\boldsymbol{A}) = \boldsymbol{f} - \nu\boldsymbol{A} \in \boldsymbol{L}^q(D)$ owing to our assumption on the source term $\boldsymbol{f}$, and $\kappa\nabla\times\boldsymbol{A} \in \boldsymbol{L}^p(D)$ for some $p > 2$ owing to the Sobolev embedding theorem (recall that $\nabla\times\boldsymbol{A} \in \boldsymbol{H}^r(D)$ by (38) and that κ satisfies a suitable multiplier property as shown in [17, 7]). Notice that the approximation error $\boldsymbol{A} - \boldsymbol{A}_h$ belongs to the space $\boldsymbol{V}_\sharp := \boldsymbol{V}_{\mathrm{S}} + \boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h)$ which we equip with the norm

$$\begin{aligned}\|\boldsymbol{b}\|_{\boldsymbol{V}_\sharp}^2 := &\sum_{K\in\mathcal{T}_h}\Big\{\nu_K\|\boldsymbol{b}\|_{\boldsymbol{L}^2(K)}^2 + \kappa_K\|\nabla\times\boldsymbol{b}\|_{\boldsymbol{L}^2(K)}^2\Big\} + \sum_{F\in\mathcal{F}_h^\partial}\frac{\kappa_{K_-}}{h_{K_-}}\|\boldsymbol{b}_h\times\boldsymbol{n}\|_{\boldsymbol{L}^2(F)}^2 \\ &+ \sum_{F\in\mathcal{F}_h^\partial}\kappa_{K_-}\Big\{h_{K_-}^{2d(\frac12-\frac1p)}\|\nabla\times\boldsymbol{b}\|_{\boldsymbol{L}^p(K_-)}^2 + h_{K_-}^{2+2d(\frac12-\frac1q)}\|\nabla\times(\nabla\times\boldsymbol{b})\|_{\boldsymbol{L}^q(K_-)}^2\Big\},\end{aligned} \tag{43}$$

for all $\boldsymbol{b} := \boldsymbol{b}_{\mathrm{S}} + \boldsymbol{b}_h \in \boldsymbol{V}_\sharp$ with $\boldsymbol{b}_{\mathrm{S}} \in \boldsymbol{V}_{\mathrm{S}}$ and $\boldsymbol{b}_h \in \boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h)$. The following error estimate is established in [11, Sec. 7.2] and [14, Thm. 45.6] (with $q = 2$): There is c such that we have

$$\|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{V}_\sharp} \le c \inf_{\boldsymbol{b}_h\in\boldsymbol{P}_k^c(\mathcal{T}_h)}\|\boldsymbol{A} - \boldsymbol{b}_h\|_{\boldsymbol{V}_\sharp}. \tag{44}$$

The analysis performed above allows us to derive the following error estimate where the best approximation error is fully localized even when $r \in (0, \frac12)$.

Corollary 4.2 (Localized error estimate) *There is c such that we have*

$$\begin{aligned}\|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{V}_\sharp} \le c\,\kappa_\sharp^{\frac12}\Big\{&\sum_{K\in\mathcal{T}_h} h_K^{2r}\Big(\ell_D^{-2}|\boldsymbol{A}|_{\boldsymbol{H}^r(K)}^2 + \ell_D^{-2r}\|\nabla\times\boldsymbol{A}\|_{\boldsymbol{L}^2(K)}^2 + |\nabla\times\boldsymbol{A}|_{\boldsymbol{H}^r(K)}^2\Big) \\ &+ \sum_{F\in\mathcal{F}_h^\partial} h_{K_-}^{2d(\frac{d+2}{2d}-\frac1q)}\|\boldsymbol{f} - \nu\boldsymbol{A}\|_{\boldsymbol{L}^q(K_-)}^2\Big\}^{\frac12}.\end{aligned} \tag{45}$$

Proof. Adapting the arguments from the proof of [14, Thm. 45.6] (setting $t = r$ therein) where the infimum is realized by using a commuting quasi-interpolant with prescribed boundary conditions, we infer that

$$\begin{aligned}\inf_{\boldsymbol{b}_h\in\boldsymbol{P}_k^c(\mathcal{T}_h)}\|\boldsymbol{A} - \boldsymbol{b}_h\|_{\boldsymbol{V}_\sharp} \le c\Big\{&\nu_\sharp \inf_{\boldsymbol{b}_h\in\boldsymbol{P}_{k,0}^c(\mathcal{T}_h)}\|\boldsymbol{A} - \boldsymbol{b}_h\|_{\boldsymbol{L}^2(D)}^2 + \kappa_\sharp \inf_{\boldsymbol{d}_h\in\boldsymbol{P}_{k,0}^{\mathrm{d}}(\mathcal{T}_h)}\|\nabla\times\boldsymbol{A} - \boldsymbol{d}_h\|_{\boldsymbol{L}^2(D)}^2 \\ &+ \kappa_\sharp\sum_{F\in\mathcal{F}_h^\partial}\Big(h_{K_-}^{2r}|\nabla\times\boldsymbol{A}|_{\boldsymbol{H}^r(K_-)}^2 + h_{K_-}^{2d(\frac{d+2}{2d}-\frac1q)}\|\boldsymbol{f} - \nu\boldsymbol{A}\|_{\boldsymbol{L}^q(K_-)}^2\Big)\Big\}^{\frac12}.\end{aligned}$$

Combining this estimate with the estimates (11) and (35) proves the assertion (as above, we hide the nondimensional factor $\nu_\sharp\kappa_\sharp^{-1}\ell_D^2$ in the generic constant c). □

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