Towards less manipulable voting systems

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Towards Less Manipulable Voting Systems

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Abstract

We study coalitional manipulation of voting systems: can a subset of voters, by voting strategically, elect a candidate they all prefer to the candidate who would have won if all voters had voted truthfully?

From a theoretical point of view, we develop a formalism which makes it possible to study all voting systems, whether the ballots are orders of preference on the candidates (*ordinal* systems), ratings or approval values (*cardinal* systems), or even more general objects. We show that for almost all classical voting systems, their manipulability can be strictly reduced by adding a preliminary test aiming to elect the Condorcet winner if there is one. For the other voting systems, we define the *generalized Condorcification* which leads to similar results. Then we define the notion of *decomposable* culture, an assumption of which the probabilistic independence of voters is a special case. Under this assumption, we prove that, for each voting system, there exists a voting system which is ordinal, shares certain properties with the original voting system, and is at most as manipulable. Thus, the search for a voting system of minimal manipulability (in a class of reasonable systems) can be restricted to those which are ordinal and satisfy the Condorcet criterion.

In order to allow everyone to examine these phenomena in practice, we present SVVAMP, a Python package of our own dedicated to the study of voting systems and their manipulability. Then we use it to compare the coalitional manipulability of various voting systems in several types of cultures, i.e. probabilistic models that generate populations of voters equipped with random preferences. We then complete the analysis with elections from real experiments. Finally, we determine the voting systems with minimal manipulability for very low values of the number of voters and of the number of candidates, and we compare them with the classical voting systems of the literature. In general, we establish that Borda's method, Range voting, and Approval voting are particularly manipulable. Conversely, we show the excellent resistance to manipulation of the system called *IRV*, also known as *STV*, and of its variant *Condorcet-IRV*.

Note to the reader

This is a translation of the original French version of this memoir, entitled: "Vers des modes de scrutin moins manipulables". I apologize for any errors in language in this English version.

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Introduction

When voting systems are mentioned, we immediately think of the elections that punctuate our public life. However, the practice of elections is far from being limited to the political domain: they can be used in any situation where a certain number of agents, the *voters*, collectively wish to choose one option among several possibilities exposed to their sagacity, the *candidates*. Thus, elections can be used to designate the best restaurant or the best movie. They can be used in associations or professional organizations. And the development of structured communities crossing physical borders thanks to the Internet and also needing to appoint representatives makes public life and a form of democracy independent of States all the richer. For example, the Debian project,[1] the Ubuntu community,[2] the Wikipedia foundation France,[3] the website www.boardgamegeek.com, and many others use Schulze's method, a voting rule developed in 1997, and are thus at the forefront of the experimentation with innovative voting systems.

At a time when there is a permanent increase of abstention in the country where the Declaration of Human Rights originated, while in other countries people struggle to obtain the right to vote; at a time when voters of our countries are showing growing distrust of the political class, and when traditional voting methods do not seem to satisfy voters' desire to express themselves; at a time, finally, when the possibilities still in their infancy offered by electronic voting make it possible to consider, in the near future, the use of voting methods with more complex counting, it seems to us more urgent than ever to take the time to think about the procedures used to vote, that is to say the voting rules themselves.

Simple majority voting

In this thesis, we will always use the word *candidates* to refer to the various options that are available to voters, even when they are not persons running for public office. The simplest situation occurs when voters are asked to respond to a closed question with a *yes* or *no* answer.

Since at least the Athenian democracy of the century of Pericles (V$^{\text{e}}$ century B.C.), simple majority voting has been used for such decisions. It was the case in the principal Athenian assemblies:

- The *Ecclesia* (ἐκκλησία), the assembly of all citizens, gathered on Pnyx hill about 40 times a year;
- The *Boule* (βουλή), a small council made up of 50 members drawn by lot who met daily to deal with current affairs;

[1] www.debian.org/.
[2] www.ubuntu-fr.org/.
[3] fr.wikipedia.org/.

- And the *dikasteria* (δικαστήρια), popular jurors drawn by lot, responsible both for judging specific disputes and discussing the legality of decrees, thus prefiguring popular juries as well as institutions such as the current Constitutional Council (for France) or the European Court of Human Rights (for the European Union).

In the case of a closed question, simple majority voting therefore very quickly imposed itself as an obvious choice.

Of course, there is another case where there are only two candidates: when two real candidates in the usual sense, that is to say two human beings, are placed in competition and submitted to the opinion of the voters.

May (1952) formalized this superiority of simple majority voting by an axiomatic approach: in the case where there are two candidates, it is the only voting method that has the following properties.

1. It is *anonymous*, i.e. it treats all voters equally.

2. It is *neutral*, i.e. it treats both candidates equally.

3. It is *positively responsive*,[4] i.e. if a voter prefers candidate a to candidate b, there is no case in which it is in her interest to vote for b.

4. Implicitly, May assumes that the voting system is *ordinal*. This term means that the voter can establish an order of preference on the candidates: either she prefers candidate a to b, or the reverse, or she likes them equally. But she cannot express a more nuanced opinion: it is thus impossible for her to express herself differently depending on whether she strongly prefers a to b or slightly prefers a to b.

The first two assumptions, anonymity and neutrality, seem obvious in practice, at least in application contexts where voters, on the one hand, and candidates, on the other hand, are assumed to be equal in law. This is not always so, and sometimes in an arguable way, as in a shareholders' meeting or a federation of states of different sizes. However, there is a wide field of application where these assumptions are self-evident.

The third assumption, positive responsiveness, also seems to follow from common sense. It implies, in particular, that for each voter and whatever her opinions, she can issue a ballot that will best defend them, regardless of the ballots of the other voters: in the terminology of game theory, we say that she always has a *dominant strategy*. We will come back to this notion because it is deeply linked to manipulability, which will be the central theme of our study.

The fourth assumption, almost implicit in May's formulation, is ordinality. It seems quite intuitive, also for strategic reasons: imagine that a voter can reinforce her vote for a by stating that she strongly prefers her to b. If she only slightly prefers a to b, she may still claim to prefer her strongly, simply to give her opinion a better chance of winning. By using such a system, there is a risk of ending up in two situations.

- If all the voters reason in this way, we end up with a situation where the ballots declaring a slight preference are no longer used at all, and we are reduced to an ordinal system where the ballot "I strongly prefer" is simply used to mean "I prefer".

[4] In the French version of this dissertation, we say that the voting system "reacts positively" (*réagit positivement*).

- Alternatively, if some voters think like this but not all of them, the situation is even worse, since the former, who vote strategically, have more power than the latter, who vote naively. Such a situation compromises *de facto* the principle of equality between voters.

It is also possible to justify ordinality by other very deep arguments. We will come back to it.

Paradoxes of social choice

We have seen that when there are two candidates, there is a voting system, simple majority voting, which has satisfactory properties, and that this voting system is unique under fairly reasonable assumptions. But as soon as there are three candidates, things start to go wrong.

Condorcet's paradox

The question is: since we have a system with good properties for two candidates and it is to some extent unique, how can it be satisfactorily extended to cases where there are more candidates?

To answer this question, a natural idea is to require independence of irrelevant alternatives (IIA), which intuitively means that the presence or absence of losing candidates does not influence the outcome of the election. More precisely, this principle can be formulated in two equivalent ways.

1. If a losing candidate is removed and the election is rerun (with the same voters holding the same opinions), then the winner must not change.
2. If a candidate is added and the election is rerun (with the same voters holding the same opinions), then the new winner must be either the same as in the original election, or the added candidate.

Again, this principle seems to make perfect sense. If the community of voters believes that candidate a is the best option among a, b, and c, then it seems obvious that by removing the irrelevant alternative c, the community should consider that a is the best option if one has to choose between a and b.

To illustrate this principle, the following joke is often used. In a restaurant, the server informs a customer that she has a choice between chicken and beef. The customer then orders chicken. A few minutes later, the server comes back: "By the way, I forgot to tell you that there is also fish." And the customer replies: "Very well, then I'll have some beef." Here, the customer seems to prefer beef when all three options are available, but chicken when only beef and chicken are offered, thus violating the IIA assumption.

This joke serves both to show the relative naturalness of the IIA hypothesis and to discuss its exact meaning. In this particular case, one could, for example, imagine that the customer simply changed her mind during the five minutes the server was away, regardless of the addition of fish to the menu. In this case, the change of decision does not seem absurd. But the IIA concerns a more restricted and natural case: it simply asks that, if the voters retain the same opinions, the absence or presence of an unselected candidate does not influence the outcome.

One could also imagine that the presence of fish on the menu informs the customer that this is a restaurant of a higher category and that, in this case, she prefers to have beef. With this interpretation, the customer's opinion does not vary over time, it is the options under consideration that change: initially, she thinks she has the choice between "chicken in an average restaurant" and "beef in an average restaurant" but at a later time, she thinks she has the choice between "chicken in a superior restaurant", "beef in a superior restaurant", and "fish in a superior restaurant". Thus, there is nothing contradictory about her change of decision and she only apparently violates the IIA principle.

One could also explain the customer's behavior by various other explanations, involving, for example, preferences that are not transitive. In this way, we will see another possible interpretation of this thought experiment in Example 1.8. That said, IIA still seems to be a desirable property in general, which guarantees a certain consistency in the choices made.

In order to extend simple majority voting while respecting the IIA principle, it is necessary to elect a candidate w who, compared to any other candidate c, is preferred by a majority of voters. Indeed, if we remove all the other irrelevant candidates, then, by IIA principle, it is necessary that w wins the simple majority vote against c. When a candidate satisfies this condition, she is said to be a *Condorcet winner*.

Ramon Llull, a Majorcan scholar of the XIII^e century, seems to have been the first to describe voting rules that have the property of electing the Condorcet winner, as McLean (1990) analyzes based on the original writings of Llull (c. 1285, 1299). But, to our knowledge, it was Nicolas de Condorcet (1785) who was the first to explicitly formulate this guiding principle and above all to have noticed that such a candidate does not always exist. Indeed, consider three voters with the following preferences.

$$\begin{array}{c|c|c} a & b & c \\ b & c & a \\ c & a & b \end{array}$$

In the notation above, each column represents one voter. For example, the first voter prefers candidate a to candidate b, which she prefers to candidate c. With the above preferences, a majority of voters prefer a to b, a majority (not made up of the same voters) prefer b to c, and a majority prefer c to a. There is therefore no Condorcet winner: this is called the *Condorcet paradox*. This phenomenon is so important in social choice that it is sometimes simply called the *paradox of voting*.

In particular, the above example shows that it is impossible to extend simple majority voting while respecting the IIA principle (which was neither named nor formulated so explicitly in Condorcet's time). For example, if we decide that a is the winner, then this is not consistent with the result of an election between a and c, since candidate c would win this electoral duel. Similarly, no candidate is a consistent winner with simple majority voting and the IIA principle.

Arrow's theorem

Arrow (1950) kind of generalized Condorcet's observation. Although his original theorem deals with *social welfare functions*, which allow to completely rank the options available to agents, it has an immediate transcription for voting systems, which simply designate one option according to the preferences of the community.[5]

Being relatively modest in our demands, we can require that a voting system possess the following properties.

[5] We will not use the term *social choice function* because it is usually reserved for systems that are ordinal. As we will elaborate in Section 1.4, we use the term *voting system* in a broader sense, which includes non-ordinal systems.

1. It satisfies IIA.
2. It is *unanimous*: if all the voters prefer candidate a, then she must be elected.
3. It is not *dictatorial*: there is no voter who has the power to decide the outcome alone.
4. Implicitly, Arrow assumes that the voting system is *ordinal*.
5. Explicitly, Arrow assumes that the voting system is *universal*, in the sense that it identifies a winning candidate for every combination of allowed ballots. In this dissertation, we will always make this assumption implicitly.

Thus, we ask the same question as before: how to obtain a voting system that satisfies IIA? But, instead of assuming that the voting system reduces to simple majority voting when there are two candidates, we make the *a priori* less demanding assumptions that it is unanimous, non-dictatorial, and ordinal.[6]

Yet even with these weaker requirements, Arrow's theorem tells us that there is no voting system that satisfies them (for a number of candidates greater than or equal to 3).[7]

With regard to unanimity and non-dictatorship, these assumptions seem very difficult to abandon. For ordinality, we will come back to it. That leaves the IIA property, which therefore seems to have to be sacrificed to preserve the other assumptions. In practice, all the usual ordinal voting systems effectively violate the same assumption: IIA.[8]

It is important not to reduce Arrow's theorem to a mere procedural matter, precluding the existence of a perfectly satisfactory voting system in practice. The most profound consequence is that, in an ordinal approach, there does not exist a canonical notion of "candidate preferred by the population" if one makes the understandable wish that this notion has natural properties.

Interpersonal comparison of utilities

In order to solve the problems encountered above, it would seem that removing the ordinality assumption is a good angle of attack. A simple way to model the intensity of a voter's preferences is to use cardinal utilities: the interest of each voter in each candidate is represented by a real number. Here, the term *cardinal* means not only that the comparison between two utilities (which one is higher) reveals which candidate is preferred by the voter, but also that the numerical value of the utilities reflects a preference intensity.

Several variations of this model exist, but the most common is that of Von Neumann and Morgenstern (1944). In this model, the utilities of a voter are defined up to an additive constant and a positive multiplicative constant. Intuitively, if we make the analogy between the measurement of the position of a candidate on the abstract axis of a voter's preferences (i.e. the axis of utilities) and the measurement of the position of a concrete object on a straight line, the voter can

[6]These assumptions are *a priori* less demanding because we have assumed IIA. This is not the case otherwise.

[7]One can consult Geanakoplos (2005) for elegant proofs of Arrow's theorem, which have also inspired several variants of the proof of the Gibbard-Satterthwaite theorem, to which we will return in a moment. Furthermore, Mossel (2012) and Keller (2012) provide "quantitative" versions of Arrow's theorem that discuss the probability of observing a violation of the required properties.

[8]For a discussion of the assumptions of Arrow's theorem, see Gibbard (2014).

choose arbitrarily where she places the origin of the reference frame (hence the additive constant) and which unit of length she uses (hence the multiplicative constant).

In the case of a length measurement, one agent can lend her graduated ruler to another, which at least allows the same unit to be used. But in the case of preferences, the measuring instrument remains in the mental universe of each agent, and it is impossible to know whether one is using the same unit of length as another. The question does not even make sense, since it is impossible to place the preference axes of two agents in the same mental universe in order to compare the measured distances.

More generally, it is impossible to make an interpersonal comparison of utilities, i.e. of the preference intensities of the agents, without making an additional, necessarily arbitrary, assumption which, *in fine*, amounts to favoring such or such type of agent. We will not develop these complex questions any further: for a good overview, the reader is invited to consult Hammond (1991).[9]

We will keep in mind, in any case, that removing the ordinality assumption is far from being as innocuous as it seems. Moreover, we will see other reasons for favoring ordinal voting systems later in this manuscript.[10]

Gibbard-Satterthwaite theorem

From our point of view, the main conclusion of Arrow's theorem and of the fundamental problems posed by an interpersonal comparison of utilities is that there is no canonical and indisputable notion of "candidate preferred by the population." We can therefore ask the question from another perspective: in practice, how does the voting system behave? In particular, does it give the same power to all voters according to their level of information? From a game-theory point of view, is it easy to reach equilibrium situations?

More precisely, a voting system is said to be *manipulable* in a certain configuration of voter preferences if and only if a subset of the voters, by voting insincerely, can lead to the election of a candidate they prefer to the outcome of the sincere vote (assuming the other voters cast sincere ballots anyway). It seems intuitively obvious that manipulability is a bad property, and we will soon discuss why in more detail.

Unfortunately, Gibbard (1973) has shown that as soon as there are 3 eligible candidates or more, for each non-dictatorial voting system, there exists at least one configuration where the voting system is not only manipulable, but by a coalition made up of one single voter!

We will follow the tradition of calling this result the Gibbard-Satterthwaite theorem. However, Gibbard's result is both earlier and stronger than that of Satterthwaite (1975): indeed, it applies to any type of voting rule (or *game form*, cf. Section 1.4), whereas that of Satterthwaite applies only to ordinal voting systems.

Non-deterministic Gibbard's theorem

If one accepts without reservation the principle of resorting to chance, then there are satisfactory systems, contrary to the deterministic case where we have seen that the path is paved with impossibility theorems.

[9] Chichilnisky (1985) also proves, by other arguments, that it is essentially impossible to aggregate cardinal preferences by a method possessing reasonable properties.

[10] In particular, one can consult the theoretical results of Chapter 5, the simulations of Chapters 7 and 8, and the experiments of Chapter 9.

First, randomness can be used unconditionally, regardless of voter preferences. Again, examples can be found in ancient Athenian democracy, where many offices were assigned by lot from a pool of eligible candidates.

Second, voter preferences can be combined with chance. In this case, it is natural to require that the voting system be non-manipulable, unanimous, and anonymous. Gibbard (1977, 1978) has shown, first for ordinal voting systems and then in general, that the only voting system that satisfies these hypotheses is the *random dictatorship*: each voter votes for a candidate, then a ballot is drawn equiprobably and the candidate indicated on this ballot is elected.[11]

For some applications, this system can be interesting. In particular, if the collective decision entails measured consequences and/or if it will be renewed frequently, for example the choice of a restaurant by a group of friends, then it allows a balance of power while eliminating the issue of strategic voting.

In other contexts, however, such a use of chance will always remain debated, except as an additional rule to decide between candidates in the event of a perfect tie in the ballots of the voters. In this dissertation, we will focus exclusively on *deterministic* voting systems, where the winning candidate depends only on the voters' ballots, without the use of a random element.

Manipulation is good. Manipulability is bad.

We have quickly said that manipulability is a harmful property of a voting system. We will now discuss this fundamental question.

What we call *manipulation* is the practice by some voters of strategic voting. We think it is important to distinguish it from *manipulability*, which is the fact that manipulation can work, i.e. lead to an outcome different from sincere voting. In other words, the manipulability of a voting system in a certain configuration of voters' preferences is the fact that sincere voting is not a *strong Nash equilibrium* (SNE): a coalition of voters can deviate from sincere voting and obtain an outcome they deem preferable to the winner resulting from a sincere vote.

Defense of manipulation

Manipulation, that is, the practice of strategic voting, is sometimes viewed in a negative light. Here are the main arguments for this view, which we will discuss.

1. Manipulators are cheaters.
2. Manipulation leads to an "incorrect" election result.
3. Manipulation is to the detriment of the community.

Argument 1 is defensible if one attaches a moral dimension to sincere voting, but it goes against all modern views on *mechanism design* and *Nash-implementability*:[12] in general, it is nowadays considered that agents are strategic and that the problem is to find a rule of the game which leads to a satisfactory result for social welfare, by accepting — and most of the time, exploiting — this strategic behavior. In economics, such a point of view essentially goes back to the "invisible hand" of Adam Smith, and this idea can be naturally transposed into voting theory: from our point of view, the strategic

[11] More generally, Gibbard (1978) shows, by authorizing non-deterministic voting systems, which are all non-manipulable systems. The anonymous and unanimous case that we cite for simplicity is only a corollary of this general result.

[12] See for example Feldman (1980).

voter is thus not a cheater but an agent who contributes to seeking and perhaps obtaining a Nash equilibrium.

As for argument 2, one can argue against it using Arrow's theorem (in an ordinal framework) and the fundamental problems of interpersonal comparison of utilities (in a cardinal framework): there is no canonical notion of "candidate preferred by the population". So talking about an "incorrect" outcome does not really make sense.

Argument 3 has similarities to the previous one. But whereas argument 2 assumes the existence of some sort of higher truth (the "correct" result, given the sincere preferences of the population), argument 3 is more pragmatic and ultimately comes down to the following question: which voters prefer the sincere election winner, and which prefer the winner resulting from manipulation?

In the general case, it is clearly possible that some voters are less satisfied with the alternative winner than with the sincere winner. But this cannot be the case for all voters: indeed, at least the manipulators prefer the alternative winner by definition. On the other hand, we will see that the opposite can happen: it is possible that all the voters prefer the alternative winner to the sincere winner (that is to say that the former *Pareto-dominates* the latter).

Indeed, consider the following example. There are 26 candidates, designated by the letters of the alphabet. We use the voting rule called *Veto*, with the alphabetical tie-breaking rule.

1. Each voter casts a *veto*, i.e. votes against a single candidate.[13]

2. The candidate receiving the fewest vetoes is elected.

3. In the event of a tie, the first candidate in alphabetical order, among the tied ones, is declared the winner.

Suppose there are 25 voters and they unanimously prefer the candidates in reverse alphabetical order (they love z and they hate a). If they vote sincerely, they all veto a and, by the tie-breaking rule, b is elected: she is the second worst candidate for all voters!

If a voter is aware of this, she may decide to vote strategically against b and then c is elected, improving the lot of the whole community. This manipulation is therefore not harmful. One can even go further: if the voters skillfully coordinate to vote against all the candidates except z, then this one is elected, which satisfies all the voters at best.

To sum up, it is possible for a manipulation to benefit the whole community. Conversely, even if it is possible that it is exercised to the detriment of a part of the community, it cannot harm the whole community. Argument 3 is therefore quite circumstantial and debatable in general.[14]

Manipulability and strong Nash equilibria (SNE)

Let us continue with a more practical example, since it now belongs to history. The 2002 French presidential election was held using the Two-round system. In the first round, Jacques Chirac (right) received 19.9% of the votes, Jean-Marie Le

[13] In this dissertation, a negative vote against a candidate is called a *veto*, with a lower case letter. In contrast, the word *Veto*, capitalized, refers to the voting system described above, also known as *Antiplurality*.

[14] For a study of the potential negative impact of manipulation on social welfare, see Brânzei et al. (2013).

Pen (far right) 16.9%, Lionel Jospin (left) 16.2%, and 13 other candidates shared the rest. In the second round, Chirac won by 82.2% of the votes against Le Pen.

However, according to some opinion polls at the time, Jospin would have won the second round against any opponent, i.e. he was the Condorcet winner. It is impossible to verify whether this was indeed the case for the entire French electorate, but it does indicate that the reality was close enough to this scenario so that a situation in which the observed votes would have been sincere and Jospin would have been the Condorcet winner can be considered a realistic example.

Under this assumption, there was a possibility of manipulation: if all the voters preferring Jospin to Chirac had voted for Jospin in the first round, then Jospin would have been elected directly in the first round. In other words, sincere voting was not a strong Nash equilibrium (SNE). But the voters did not perform this manipulation, and this is a key observation to which we will return.

This situation poses two main types of problems:

1. The *a posteriori* feeling of the voters,
2. The legitimacy of the result.

On the one hand, after the election, some sincere voters may feel regret about their ballot choice and also a sense of injustice: since insincere ballots would have defended their opinion better, they may consider that their sincere ballots did not have the impact they deserved. They may draw from this a distrust of the voting system itself: the fact that a sincere ballot does not best defend the opinion it expresses appears to be a *bug* of the voting system.[15] This experience may even develop or reinforce their distrust of elective practice in general.

On the other hand, one may fear for the outcome of such an election. On this aspect, this example can be interpreted in two ways, but both lead to condemn manipulability. If the result of the sincere vote is deemed by definition to best represent the opinions of the voters, then Chirac was the legitimate winner. But in this case, the manipulability of the situation would have risked leading to the election of Jospin if the voters concerned had been more skillful. Conversely, if it is believed that a manipulated outcome can be better in terms of collective welfare (as the proponents of the Condorcet criterion would argue in this case), then the manipulation itself is not undesirable, but the *manipulability* of the situation is undesirable all the same: indeed, it makes this "better" outcome more difficult to identify and to produce. For example, if the entire population votes sincerely, it is not achieved.

Thus, it seems that in this case, there was a difference between the observed outcome of the election and that of a possible SNE. It is amusing to note that, in other contexts of game theory, it may happen to study SNE because they are thought to be the configurations towards which agents will naturally converge. In voting theory, it seems to us that the situation is slightly different: these are situations towards which it would be desirable to converge for the reasons explained above, but in practice, this goal may be difficult to achieve because sincere voting does not necessarily lead to an SNE. This problem corresponds precisely to the definition of manipulability.

Like any real example, the French presidential election of 2002 must be examined with more caution than an artificial example, which could be adjusted at will

[15] Please forgive me for a personal anecdote: when I began to take an interest in voting theory, following that famous April 21 of 2002, I was convinced that the Two-round system was a catastrophic voting system in terms of manipulability. However, we will see in Chapters 7, 8, and 9 that, even if it is not the least manipulable voting system in general, it is far from being the worst, even for about fifteen candidates.

to illustrate an argument in a paradigmatic way. In particular, it is necessary here to clearly distinguish the fact that the situation is not an SNE from the fact that the winner is not the Condorcet winner (although the two aspects are linked). The former can lead to various kinds of problems, which we have described. The latter can also pose a problem of legitimacy of the winner, since a majority of voters prefer the Condorcet winner to the elected candidate. Fortunately, we will see (notably in Chapter 2) that addressing these two types of problems is not incompatible, quite the contrary.

Moreover, the situation we have described also had an important symbolic dimension, linked to the practical course of the Two-round system and its analogy in principle with certain sports competitions. Thus, it seemed shocking to some voters that a far-right candidate reached the "final" of the competition and somehow appeared as second in the final "ranking". However, we will not develop this point too much: keeping in mind that the symbolic dimension and the public perception of events is always important, especially for a political election, we will consider in this dissertation that the main outcome of an election is still the choice of the winning candidate.

Manipulability and *straightforwardness*

By definition, the manipulability of a voting system means that the sincere voting situation is not necessarily an SNE. This poses several problems.

1. Before the election, the voter is faced with a dilemma: vote sincerely or strategically? If she grants a moral virtue to sincere voting, it can be a case of conscience. If not, there is still a practical problem: how to choose a strategic ballot best suited to the situation?

2. These strategic aspects lead *de facto* to an asymmetry of power between voters that are informed and strategic and those who are neither. As Dodgson puts it in a sentence made famous by Black (1958), voting then becomes "more of a game of skill than a real test of the wishes of the electors."[16]

3. As seen in the Veto voting example, if the situation is manipulable, then voters may need information, computational power, and coordination to reach an SNE (even in an *a priori* favorable situation where they all agree). Conversely, if the situation is non-manipulable, it suffices to vote sincerely to reach such an equilibrium. No information exchange, no calculation, and no coordination is then necessary.

4. This need for information, whether individual or collective, gives questionable power to sources of information, such as the media and polling institutes.

We will show that these problems are deeply linked to an issue intimately connected to manipulability, *straightforwardness*. A voting method is said to be *straightforward* (Gibbard, 1973) if any voter, whatever her opinions, has a dominant strategy. In order to fully understand the nuance between this notion and non-manipulability, consider the simple case where there are only two candidates and examine the following three voting systems. The last two are rather theoretical curiosities, but they will allow us to illustrate our point.

[16] In the French version of this dissertation, this sentence is translated to: *le vote devient alors « davantage un jeu d'habileté qu'un test réel des souhaits des électeurs ».*

Simple majority voting Each voter is expected to announce her preferred candidate, a or b, and the candidate receiving the most votes is elected.

Inverted majority voting Each voter is expected to announce her preferred candidate, a or b, but the vote is counted in favor of the other candidate.

Parity voting Each voter places a white ball or a black ball in the ballot box. If there is an odd number of black balls, then a is elected. Otherwise, b is elected.

Simple majority voting is non-manipulable. In particular, as we have already noticed, each voter always has a dominant strategy (which is simply the sincere vote): in other words, this voting system is *straightforward*.

Inverted majority voting, on the other hand, is generally manipulable: if a voter prefers a, it is in her interest to abandon a ballot with the label a in favor of a ballot with the label b, since this will be counted in favor of a. On the other hand, this voting method is *straightforward*: a voter who prefers a always has an interest in using a ballot b, and vice versa. Now, this voting method, even if it can cause some confusion to the voter, does not pose any of the other problems mentioned: it is easy to find SNE without any calculation, without any exchange of information, and without any particular coordination. These problems are therefore clearly related to the defects in *straightforwardness* and not to manipulability. We can also see from this example that the notion of sincere voting is partly conventional: one could, on the contrary, have interpreted the ballot with the label a as a veto against a. With this convention of language, we would have concluded that this voting system is never manipulable.

Now let us look at parity voting. If a voter knows the ballots of the other voters, then she can always decide which candidate wins. Most of the time, therefore, this voting system is manipulable. And it is particularly not *straightforward*: if a voter prefers a, she has essentially no clue as to whether a white or black ball will best defend her opinion. In this voting system, all the problems mentioned above are exacerbated: for example, even if all the voters prefer the same candidate, they need perfect coordination to elect her. Moreover, the balance of power between voters is particularly destroyed: a voter with perfect information has absolute power, while an ignorant voter has essentially no decision-making power.

To sum up, we have examined a non-manipulable (and therefore straightforward) voting system, a straightforward but manipulable voting system, and a non-straightforward (and therefore manipulable) voting system. And we found that the problems we mentioned are actually more related to the lack of straightforwardness than to manipulability, since these problems are absent from the second voting system we examined.

However, there is a deep connection between manipulability and straightforwardness. Indeed, up to defining sincere voting, conventionally, as the use of the dominant strategy (as we did by transforming inverted majority voting into Veto), a straightforward voting system is non-manipulable. Conversely, if a voting system is not straightforward, then there is no way to define sincere voting that makes it non-manipulable.

In short, by an adequate choice of the *sincerity function*, i.e. of the conventional way in which a vote deemed sincere is associated with an opinion, the question of straightforwardness, which is basically the fundamental point, can therefore be reduced to the question of manipulability, which is more convenient to grasp in practice.

Manipulability indicators

The theorem of Gibbard (1973) teaches us that no non-trivial voting system, whether ordinal or cardinal, can be straightforward.[17] In other words, whatever the sincerity function used, a non-trivial voting system is necessarily manipulable.

All that can be hoped for is therefore to limit the magnitude of the problem, by studying the extent to which classical voting systems are manipulable and by identifying processes for designing less manipulable voting systems.

To this end, our reference indicator will be the manipulability rate of a voting system: depending on the *culture*, i.e. the probabilistic distribution of the population's preferences, this rate is defined as the probability that the voting system is manipulable in a randomly drawn configuration. In other words, it is the probability that a vote without information exchange will lead to an SNE.

In the literature, there are mainly two other ways of estimating the manipulability of a voting system:

1. The number of manipulators required, as well as other similar types of indicators that quantify manipulability when possible.[18]

2. The algorithmic complexity of manipulation.[19]

In both cases, it is generally considered that the more difficult the manipulation, the more it is a laudable property of the voting system. While understanding and respecting this point of view, we find it interesting and relevant to defend precisely the opposite.

Indeed, we have argued that easily reaching an SNE is a good property for a voting system. For this, the best case is the one where the voting system is not manipulable, since this means that an equilibrium can be reached without exchanging information and without any particular calculation. But in the other cases, to have the best chance of reaching an equilibrium, it is better if strategic voting is inexpensive in terms of number of voters, information, computational complexity, and communication.

Let us think back to the Veto example cited earlier. In the situation we described, in order to achieve an equilibrium, 24 out of 25 voters must vote strategically. Some might consider this a good thing, since this manipulation is very difficult to perform. On the contrary, we believe it is a harmful property, since it takes a lot of effort to achieve the only reasonable result!

Consider another example, Approval voting. Each voter votes for as many candidates as she wants, and the candidate receiving the most votes is declared the winner. In practice, we will see in this thesis that this voting method is often manipulable. However, it has at least the advantage that the strategic question

[17] Here, we use the shorthand *non-trivial* to mean: not reduced to two eligible candidates and non-dictatorial.

[18] See Chamberlin et al. (1984); Saari (1990); Lepelley and Valognes (1999); Slinko (2004); Favardin and Lepelley (2006); Pritchard and Slinko (2006); Pritchard and Wilson (2007); Xia and Conitzer (2008); Lepelley et al. (2008); Reyhani et al. (2009); Reyhani (2013); Green-Armytage (2014). For various indicators of manipulability related to single voter manipulation, see Aleskerov and Kurbanov (1999).

[19] This line of research was initiated by Bartholdi et al. (1989a); Bartholdi and Orlin (1991). Since then, various manipulation complexity results have been proved for classical voting systems, to which we will return. In the same spirit, Conitzer and Sandholm (2003), Elkind and Lipmaa (2005b), Elkind and Lipmaa (2005a) propose methods to transform a voting system in order to increase the complexity of manipulation. However, Conitzer and Sandholm (2006); Procaccia and Rosenschein (2006, 2007); Faliszewski and Procaccia (2010) show that it is essentially impossible to have a voting system that has reasonable properties and is algorithmically difficult to manipulate on average.

is relatively simple: as explained by Laslier (2009), it is enough to have access to polls giving the two favorite candidates to be able to use an efficient strategy, the *Leader rule*. Moreover, using this strategic behavior leads to electing the Condorcet winner if she exists. Thus, even in cases where sincere voting does not lead to an equilibrium, it is possible to find one with a relatively low cost.[20]

In summary, it seems to us that a good property is that the manipulation is easy. And the ideal manipulation, since it requires no information at all, is simply the sincere vote.

Our main objectives

With these considerations in mind, it is time to present the main goals that guided our research.

First, we want to quantify manipulability. How common is it in practice? Which of the voting systems are the least manipulable? Are the differences significant?

Second, in studying manipulability rates, we quickly realized the following problem: we do not know the minimum rate that can be achieved in a given class of voting systems.[21] Consequently, we can compare voting systems with each other, but we cannot say whether the observed manipulability rates are far from an optimum. Ideally, we would like to identify a voting system of minimal manipulability, or at least to estimate the corresponding manipulability rate: even if the resulting voting system were too complex to be used in practice, it would provide us with a yardstick for gauging the manipulability rates of other voting systems.

Contributions and road map

Theoretical study of manipulability

Chapter 1 develops the formalism of *electoral spaces*, which makes it possible to apprehend all types of voting systems, including cardinal systems and even other types of systems. We take advantage of this chapter to recall the formal definition of Condorcet notions and to present the voting systems studied in this thesis. We introduce in particular the *IRV* system, also known as *STV*, which plays an important role in the following chapters. We also introduce the *IRVD* system, a variant of IRV suggested by Laurent Viennot. Finally, the *IB* system is, to our knowledge, an original contribution.

In Chapter 2, we show that, for all classical voting systems except Veto, we can make the system less manipulable by adding a preliminary test aiming at electing the Condorcet winner if there is one.[22] We call this system the *Condorcification* of the initial voting system. We discuss precisely with which Condorcet notions this result is valid and we show that the obtained manipulability decrease is strict for all classical voting systems except Veto, using a notion we introduce and characterize, the *resistant Condorcet winner*.

[20] One may also want to check out Myerson (1999).

[21] Whenever we mention the notion of minimal manipulability, it is in a certain class of "reasonable" voting systems, which we will define a bit later. Indeed, if we were to consider all voting systems, the question would be trivial, since dictatorship is not manipulable at all.

[22] Another slightly less classical voting system is not affected by this result, just like Veto: it is the Kim-Roush method, which is inspired by Veto and whose definition we will recall. In the following, we will see that most of the theorems that do not apply to Veto do not apply to the Kim-Roush method either.

Condorcification theorems assume that the voting system under study satisfies a property that we call the *informed majority coalition criterion* (**InfMC**), which means that a strict majority of voters always have the power to choose the outcome of the election, if they know the ballots of the other voters. In practice, all classical voting systems satisfy this assumption, except Veto. In Chapter 3, we define several other majoritarian criteria, some of which are original contributions, and we study their relationship to the difficulty of finding SNE. For all the classical voting systems, we study which criteria are satisfied by each.

Although a large class of systems satisfy the **InfMC** hypothesis, this is not always the case. In Chapter 4, we study how to decrease manipulability in general, through a process we call *generalized Condorcification*, using an approach inspired by the theory of *simple games*. For usual systems satisfying **InfMC**, we use this formalism to show that Condorcification in the usual sense, i.e. based on the Condorcet winner, is in a certain sense optimal.

In Chapter 5, we study the influence of the ordinality of a voting system on its manipulability. We show that, for each non-ordinal voting system, there exists an ordinal voting system that has certain properties in common with the original voting system and that is at most as manipulable, provided that the culture satisfies a condition that we introduce, *decomposability*. In particular, we show that this theorem is applicable when the voters are independent in the probabilistic sense. Combining this result with the Condorcification theorems, we conclude that the search for a voting system of minimal manipulability (within the class of systems satisfying **InfMC**) can be restricted to voting systems that are ordinal and that satisfy the Condorcet criterion.

Computer-aided study of manipulability

From Chapter 6 onward, we temporarily put aside the search for a voting system of minimal manipulability and we try to quantify the manipulability of classical voting systems. For that, we present SVVAMP, a Python package of our own dedicated to the study of voting systems and their manipulation. Its software architecture is modular, allowing the rapid implementation of new voting systems. Using the criteria defined in Chapter 3, SVVAMP has generic manipulation algorithms and specific algorithms for some voting systems, either taken from the literature or designed specifically for this software package. SVVAMP will be used in all subsequent chapters.

In Chapter 7, we study the manipulability rate of different voting systems in *spheroidal* cultures, which generalize the model usually known as *impartial culture* (cf. for example Nitzan, 1985). In particular, we are the first to use the Von Mises–Fisher model to generate preferences and we explain the reasons for this choice. We study the effect of the variation of the different parameters and we introduce *meta-analysis diagrams*, which make it possible to compare the manipulability of the voting systems studied. In particular, we show that the voting system *CIRV*, obtained by Condorcification of IRV, is generally the least manipulable.

In Chapter 8, we study another model, based on an abstract political spectrum and generalizing the notion of *single-peaked* culture introduced by Black (1958). This different framework allows us to qualify our conclusion about the CIRV system: indeed, its supremacy is then more questionable than in spheroidal cultures. In particular, other voting systems show interesting performances, such as CSD, IB, the Two-round system, and some others.

In Chapter 9, we analyze real experiments from a wide variety of contexts, including preference-revealing settings that are not elections. These experiments allow us to confirm previous results that there is frequently a Condorcet winner (for example Tideman, 2006) and to establish that the CIRV system is distinguished by its low manipulability.

Chapter 10 takes up the search for a voting system with minimal manipulability (in the class of systems satisfying **InfMC**). We define the *opportunity graph* of an electoral space and we show that the question can be reduced to an integer linear programming problem, which can be studied for very modest values of the parameters by combining a theoretical approach and the use of dedicated software (CPLEX). Finally, we use SVAMP to compare the classical voting systems with the optimum.

Appendices

In Appendix A, we study measurability issues, i.e. a technical point related to the use of probability spaces, mainly useful for Chapter 5 on slicing.

In Appendix B, we present a work carried out in collaboration with Benoît Kloeckner on the geometry in the utility space. In particular, we motivate the use of spheroidal cultures in Chapter 7.

In Appendix C, we illustrate how the study of voting systems and their manipulability can provide answers to telecommunication questions, using a model initially designed by Ludovic Noirie. In particular, we show that the IRV system can reconcile non-manipulability and economic efficiency.

In the glossary (page 305), the reader can find a summary of the main notations and acronyms used in this thesis.

We wish you a good reading!

Part I

Theoretical Study of Manipulability

Chapter 1

Framework

In the Arrovian social choice literature, a voter's preferences are usually represented by a strict order of preference over the candidates (Arrow, 1950). A voting rule (or *social choice function*) is a function which, to the preferences of all voters, associates a winning candidate. However, with this formalism, it is impossible to correctly apprehend voting rules such as *Approval voting* or *Range voting*.

We develop the framework of *electoral spaces* (Section 1.1), a class of models able to use arbitrary mathematical objects to represent the preferences of the voters, while allowing to prove general theorems about these models. In this framework, Section 1.2 defines *state-based voting systems* (SBVS), which will be the main focus of our study, and Section 1.3 translates the usual definition of *manipulability* into this model. Section 1.4 then defines *general voting systems*, which make it possible to represent all the voting rules imaginable in practice. We outline how one can navigate between general voting systems and SBVS, and we show why the study can be restricted to the latter.

We then take advantage of this first general chapter to recall the notions of weighted majority matrix and Condorcet winner in Section 1.5. We take care to clearly distinguish the different variants of this notion, linked to the fact that we authorize preferences which are arbitrary binary relations, for example weak orders. In particular, we generalize the usual notion of *weak Condorcet winner* by defining the notion of *Condorcet-admissible* candidate, which the following chapters will reveal to be a central notion for the manipulability of almost all usual voting systems.

Finally, in Section 1.6, we present the voting systems that we will study in this thesis. Among them, the *IB* system is, to our knowledge, an original contribution. The same is true of the systems *Condorcet-dean* and *Condorcet-dictatorship*, which serve above all as convenient raw material for theoretical examples. We also introduce the *IRVD* system. It was suggested to us by Laurent Viennot, whom we thank.

1.1 Electoral spaces

In this section, we introduce the framework of *electoral spaces*, which will allow us to represent both the preferences and the ballots of voters.

1.1.1 Binary relations

First of all, a few reminders about binary relations are necessary. Let E be a set and $\mathrm{P}_0 \in \mathcal{P}(E^2)$ a binary relation on E. We say that P_0 is:

- *Reflexive* iff $\forall c \in E, c \,\mathrm{P}_0\, c$;
- *Irreflexive* iff $\forall c \in E, \mathrm{non}(c \,\mathrm{P}_0\, c)$ (i.e. iff non P_0 is reflexive);
- *Weakly complete*, or simply *complete*,[1] iff $\forall (c, d) \in E^2, c \neq d \Rightarrow c \,\mathrm{P}_0\, d$ or $d \,\mathrm{P}_0\, c$;
- *Antisymmetric* iff $\forall (c, d) \in E^2, c \,\mathrm{P}_0\, d$ and $d \,\mathrm{P}_0\, c \Rightarrow c = d$ (i.e. iff not P_0 is complete);
- *Transitive* iff $\forall (c, d, e) \in E^3, c \,\mathrm{P}_0\, d$ and $d \,\mathrm{P}_0\, e \Rightarrow c \,\mathrm{P}_0\, e$;
- *Negatively transitive* iff not P_0 is transitive.

We say that P_0 is a *strict weak order* iff it is negatively transitive, irreflexive, and antisymmetric; a *strict total order* iff it is transitive, irreflexive, and weakly complete.

1.1.2 Profiles

We can now represent the preferences of the voters on the candidates.

Let V and C be two positive integers. Let $\mathcal{V} = [\![1, V]\!]$ be the set of indices of the *voters* and $\mathcal{C} = [\![1, C]\!]$ the set of indices of the *candidates*, where the notation with open bracket denotes the set of integers included in the closed interval. The candidates can be voters themselves, without altering our results.[2]

We denote $\mathcal{R}_\mathcal{C}$ the set of binary relations on $\mathcal{C}$: an element of $\mathcal{R}_\mathcal{C}$ represents the binary relation of preference of a voter on the candidates. Let $\mathcal{W}_\mathcal{C}$ be the set of strict weak orders and $\mathcal{L}_\mathcal{C}$ the set of strict total orders on $\mathcal{C}$.

Let $\mathcal{R} = (\mathcal{R}_\mathcal{C})^V$. As it is usual in social choice, we call an element of $\mathcal{R}$ a *profile*. For each voter, it gives her binary relation of preference on the candidates.

When the relations of preference of the voters are strict orders (total or weak), we represent a profile as in the following example.

44	32	24
a	b	c
b, c	c	a
	a	b

In this case, the first 44 voters prefer a and are indifferent between b and c, the next 32 prefer b to c and c to a, etc. In the general case, the header row shows the number of voters in each column. If each column corresponds to a single voter, then the header row is omitted. The number of voters in each column can also be given as a percentage of the total number of voters V.

[1] We say that P_0 is *strongly complete* when it is also reflexive. Depending on the literature, the qualifier *complete*, used alone, can refer to either notion. Since we will generally use the word *complete* for irreflexive relations, it will necessarily mean *weakly complete* so there will be no possible confusion.

[2] On the other hand, we consider that the set of candidates is fixed *a priori*. An interesting extension of the model, which we will not discuss in this dissertation, is to allow potential candidates to enter or withdraw for strategic reasons (Dutta et al., 2001, 2002; Lang et al., 2013; Brill and Conitzer, 2015).

As an abbreviation, we can also write that the weak order of preference of the first 44 voters is $(a \succ b \sim c)$, where the symbol $\succ$ represents a strict preference and the symbol $\sim$ an indifference.

1.1.3 Definition of an electoral space

Before defining the notion of *electoral space* in general, we present a model that makes it possible to study all common voting systems and that we will use frequently to illustrate our results.

Definition 1.1 (reference electoral space)
In the model that we call *reference electoral space*, each voter v is able to mentally establish:

- A strict weak order of preference $p_v \in \mathcal{W}_\mathcal{C}$ on the candidates,
- A vector $u_v \in [0,1]^C$ of *grades* on the candidates,
- And a vector $a_v \in \{0,1\}^C$ of *approval values* on the candidates.

The triple $\omega_v = (p_v, u_v, a_v)$ will be called her sincere *state* and we will denote P_v the function that extracts the first element of this triple: $\mathrm{P}_v(\omega_v)= p_v$.

In general, we will note Ω_v the set of possible states for voter v. In the first model above, we have $\Omega_v = \mathcal{W}_\mathcal{C} \times [0,1]^C \times \{0,1\}^C$. We suggest to the reader to see this set as the analog of a universe in probability theory: for most problems, it is not necessary to specify exactly its content. The main point is the possibility to define functions that extract this or that information about voter v, as a random variable would do in probability theory: for example, the function P_v gives access to her binary relation of preference.

By analogy with the usual notations for random variables, we will frequently write P_v as a shortcut for $\mathrm{P}_v(\omega_v)$: therefore, the expressions $c \, \mathrm{P}_v(\omega_v) \, d$ and $c \, \mathrm{P}_v \, d$ are synonyms, both meaning that voter v (in the state ω_v) prefers candidate c to candidate d. For a certain binary relation of preference on the candidates p_v, the notation $\mathrm{P}_v(\omega_v) = p_v$ or, in short, $\mathrm{P}_v = p_v$, means that the variable P_v has the value p_v (in state ω_v).

We call *configuration* a V-tuple $\omega = (\omega_1, \ldots, \omega_V)$ giving the state of each voter, as illustrated in Figure 1.1. We note $\Omega = \prod_{v \in \mathcal{V}} \Omega_v$ the Cartesian product that contains all the possible configurations and $\mathrm{P} = (\mathrm{P}_1, \ldots, \mathrm{P}_V)$ the function with V variables that, to a configuration ω, associates the profile that corresponds to it: $(\mathrm{P}_1(\omega_1), \ldots, \mathrm{P}_V(\omega_V))$.

We now have all the necessary elements to define an *electoral space*, i.e. a mathematical model representing the preferences of voters.

Definition 1.2 (electoral space)
An *electoral space* is given by:

- Two positive integers V and C,
- For each voter $v \in \mathcal{V}$, a nonempty set Ω_v of her possible states,
- For each voter $v \in \mathcal{V}$, a function $\mathrm{P}_v : \Omega_v \to \mathcal{R}_\mathcal{C}$, whose result is her binary relation of preference.

Such an electoral space is denoted $(V, C, \Omega, \mathrm{P})$, or simply Ω when there is no ambiguity.

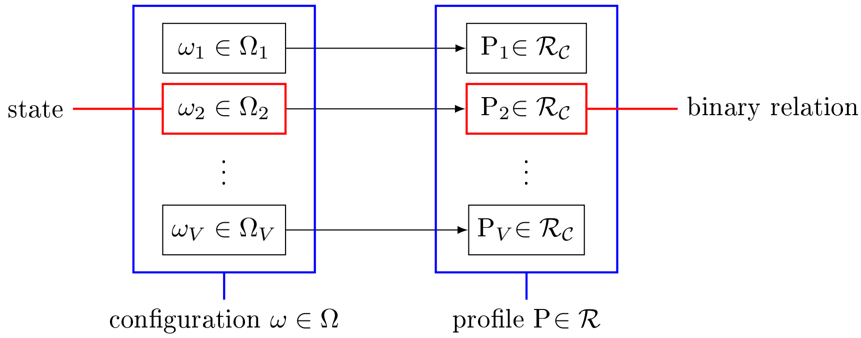


Figure 1.1 – Configuration and profile of the voters.

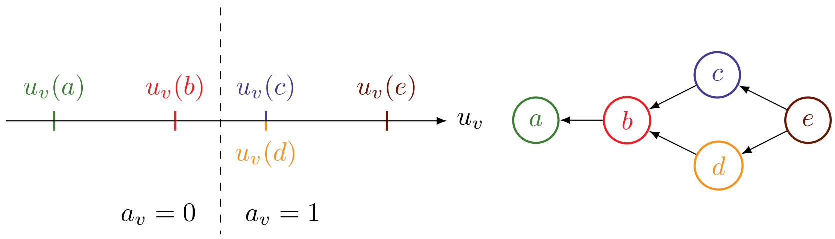


Figure 1.2 – State of a voter in the reference electoral space.

Let us return to the example of the reference electoral space, where $\omega_v = (p_v, u_v, a_v)$. For more realism, the social planner[3] has the option of adopting the assumption that each voter's grades and approval values are consistent with her binary relation of preference: if $c\ p_v\ d$, then $u_v(c) > u_v(d)$ and $a_v(c) \geq a_v(d)$. Figure 1.2 illustrates the state of a voter satisfying these assumptions. The left part of the figure represents the positions of the candidates on her utility axis u_v, as well as the boundary between the candidates she disapproves of ($a_v = 0$) and those she approves of ($a_v = 1$). The right-hand side represents the graph of her binary relation of preference. In this case, since it is a strict weak order by assumption, transitivity is implied in the figure.

It is easy to include this assumption in the model, by defining Ω_v as the set of triples (p_v, u_v, a_v) that satisfy the requested conditions. Other hypotheses can be added in the same way, by choosing a suitable set Ω_v for each voter.

1.1.4 Examples of electoral spaces

Another model is very important for theory: in traditional Arrovian social choice, it is common to represent each voter's opinions only by a strict total order on the candidates. This practice corresponds to the following electoral space.

[3]In the economic and social sciences, the social planner is a decision maker who tries to obtain the best possible result for all the actors involved. In our case, it is an abstract person who can represent both social choice researchers and all the people who can have an influence in choosing the voting system used in a given human organization. By hypothesis, the social planner tries to establish a voting system with as good properties as possible. To do this, it is first necessary to identify the preferred application domain by defining the electoral space used.

Definition 1.3 (electoral space of strict total orders)
For all $v \in \mathcal{V}$, let $\Omega_v = \mathcal{L}_\mathcal{C}$ and P_v the identity function.[4] This model is called *electoral space of strict total orders for V and C.*

The model above is sometimes extended by allowing strict weak orders. We can also consider arbitrary binary relations.

Definition 1.4 (electoral space of strict weak orders)
For all $v \in \mathcal{V}$, let $\Omega_v = \mathcal{W}_\mathcal{C}$ and P_v the identity function. This model is called *electoral space of strict weak orders for V and C.*

Definition 1.5 (electoral space of binary relations)
For all $v \in \mathcal{V}$, let $\Omega_v = \mathcal{R}_\mathcal{C}$ and P_v be the identity function. This model is called *electoral space of binary relations for V and C.*

The choice of the electoral space does not guarantee anything about the expressiveness of a possible voting system that we would plan to study later. For example, if we choose the electoral space of strict weak orders, it does not mean that the studied voting system will allow a voter to transmit a strict weak order in her ballot. It just means that the social planner admits that a voter may have a weak order of preference on the candidates and considers it impossible for her to have any other form of preference (*descriptive* approach). Or, more reasonably, it means that the good properties that would be proven later for the studied voting system are guaranteed if the voters have strict weak orders of preference, but *a priori* not in the other cases (*normative* approach).

Let us continue to explore the possibilities offered by electoral spaces with some examples.

Example 1.6 (utility with margin of uncertainty)
Suppose that each voter v is able to mentally establish a vector of scores $u_v \in [0, 1]^C$ and a real number $\varepsilon \geq 0$, interpreted as some uncertainty, such that she prefers a candidate c to a candidate d iff $u_v(c) > u_v(d) + \varepsilon$. Her state space is then $\Omega_v = [0, 1]^C \times \mathbb{R}$ and her function P_v is defined by the previous inequality.[5]

Example 1.7 (utility intervals)
It is easy to generalize the previous example. To each candidate c, voter v associates a nonempty interval $[\underline{u_v}(c), \overline{u_v}(c)]$. It is interpreted as follows: v situates her utility for candidate c in this interval, but not more precisely (because of an inability to be more precise, because of a lack of interest or because it represents a too high cognitive cost). She prefers candidate c to d iff $\underline{u_v}(c) > \overline{u_v}(d)$. Figure 1.3 represents an example of state for such a voter. The left side represents her utility interval for each candidate. The right side displays the graph of her binary relation of preference. This one is irreflexive, antisymmetric, and transitive. This is why we did not represent explicitly the edges from d or e to a, which are deduced from the other edges by transitivity. On the other hand, the relation is not negatively transitive: indeed, the voter prefers neither e to d nor d to c, and yet she prefers e to c. So it is not a strict weak order.

Example 1.8 (multi-criteria preferences)
Now, each voter mentally rates each candidate in the interval $[0, 1]$ according to three criteria: her state space Ω_v is the set of matrices of size $3 \times C$ with values

[4] This is an abuse of language that we will commit without scruples: to be quite rigorous, P_v is the inclusion map from $\mathcal{L}_\mathcal{C}$ into $\mathcal{R}_\mathcal{C}$.

[5] This idea can be compared to the work of Ehlers et al. (2004), in which manipulability with a threshold is considered, i.e. where voters wish to improve their utility by at least a certain amount.

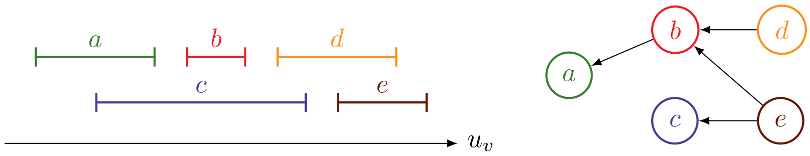


Figure 1.3 – State of a voter in the electoral space of utility intervals.

		Candidate		
		Chicken (a)	Beef (b)	Fish (c)
Criterion	Cheap	1	0,5	0
	Healthy	0,5	0	1
	Soft	0	1	0,5

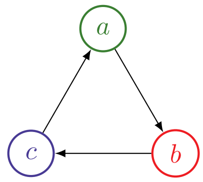


Figure 1.4 – State of a voter with multi-criteria preferences.

in $[0, 1]$. We suppose that she prefers a certain candidate to another one iff the first one is strictly better rated than the second one according to at least two criteria: this defines the function P_v. Let us consider a voter in the state ω_v represented by Figure 1.4. She prefers a to b, b to c, and c to a: her binary relation of preference $\mathrm{P}_v(\omega_v)$ is not transitive. In some sense, such a voter realizes a Condorcet paradox on her own: the multiplicity of criteria she considers, and the fact that she decides her preferences according to the majority of the criteria, mimics the behavior of three distinct voters using simple majority voting for each pair of candidates.

If such a voter is asked to designate a favorite among the three candidates, then she necessarily violates the property of independence of irrelevant alternatives (IIA) that we mentioned in the introduction: for example, if she designates beef (b) as her favorite candidate among the three, then this seems to contradict the fact that she chooses chicken when it comes to choosing between chicken and beef.

Example 1.9 (decentralized random generator)
Each voter has a binary relation of preference p_v on the candidates and owns a coin. After flipping the coin, her (honest) state is (p_v, x_v), where $x_v \in \{\text{Heads}, \text{Tails}\}$. It is assumed that nobody else knows the result of her random draw: therefore, if she has to communicate her state to a voting system, then she can lie about the result of the draw as well as about her order of preference.

1.1.5 Basic properties of an electoral space

Although we interpret P_v as *strict* preferences in most of our examples,[6] antisymmetry is not required by the definition 1.2 of an electoral space. We will discuss in Section 1.3 (page 34) what interpretation can be given to relation P_v when it is not antisymmetric. If the reader is troubled by this possibility, she can read all that follows with the addition of an antisymmetry hypothesis. However, in all generality, we will note:

- $c \,\mathrm{I}_v\, d$ iff not $c \,\mathrm{P}_v\, d$ and not $d \,\mathrm{P}_v\, c$ (indifference),

[6]Until now, the only case where antisymmetry was not satisfied was the electoral space of binary relations (Definition 1.5).

- $c \,\mathrm{PP}_v\, d$ iff $c \,\mathrm{P}_v\, d$ and not $d \,\mathrm{P}_v\, c$ (antisymmetric preference),
- $c \,\mathrm{MP}_v\, d$ iff $c \,\mathrm{P}_v\, d$ and $d \,\mathrm{P}_v\, c$ (mutual preference).

If relation P_v is antisymmetric, which is a common assumption, then there are only three mutually exclusive possibilities: $c \,\mathrm{P}_v\, d$ (equivalent to $c \,\mathrm{PP}_v\, d$ in this case), $d \,\mathrm{P}_v\, c$, and $c \,\mathrm{I}_v\, d$.

For some of our results, we will assume that voters have a certain freedom of opinion (without prejudging the opportunities to express that opinion in their ballots).

Definition 1.10 (richness of an electoral space)
We say that:

1. Ω *includes all strict total orders* iff any voter can have any strict total order as her binary relation of preference, that is, $\forall (v, p_v) \in \mathcal{V} \times \mathcal{L}_\mathcal{C}, \exists \omega_v \in \Omega_v$ s.t. $\mathrm{P}_v(\omega_v) = p_v$;
2. Ω *allows any candidate as most liked* iff any voter can strictly prefer any candidate over all others, that is, $\forall (v, c) \in \mathcal{V} \times \mathcal{C}, \exists \omega_v \in \Omega_v$ s.t. $\forall d \in \mathcal{C} \setminus \{c\}, c \,\mathrm{PP}_v\, d$.

The implication 1 $\Rightarrow$ 2 is trivial.

For example, in the reference electoral space, where $\omega_v = (p_v, u_v, a_v)$, the binary relation of preference p_v of a voter v can be any strict weak order. In particular, any strict total order is possible. Therefore, this electoral space satisfies both properties 1 and 2.

We insist again on the fact that these properties have nothing to do with the expressiveness of a possible voting system we would like to study. For example, if it is assumed that the electoral space includes all strict total orders, this does not mean that the voting system allows a voter to express her strict total order of preference if she has one. It just means that such an opinion is considered as *a priori* possible.

While limiting an electoral space to strict total orders is debatable, the weaker assumption that it includes all strict total orders seems quite natural in general. However, it is easy to devise reasonable models that do not satisfy this property. For example, if there are a very large number of candidates, then one can argue that it is cognitively impossible for a voter to establish a strict total order of preference on the candidates. We nevertheless address a criticism to this type of model: even if the idea is interesting from a descriptive point of view, it is legitimate to add the possibility of establishing a total order for normative reasons. Indeed, even if we consider that a voter cannot reach this state of complete knowledge about her preferences, we have no reason (and no practical way) to forbid her to do so *a priori*.

Before we look at another example of electoral space that does not include all strict total orders, let us recall the classical notion of *single-peakedness* (Black, 1958). The intuition is as follows: suppose there is a "natural" way to place all candidates on an abstract axis, for example a left-right axis in politics. This gives a reference order on the candidates that we will denote $\mathrm{P}_{\mathrm{ref}}$.

Imagine that this axis has the following property: each voter has her maximum utility value for a certain candidate (her *peak*) and her utility decreases when moving away from this candidate, both to the right and to the left. Then her order of preference P_v starts with the candidate that corresponds to her peak utility, then she places all the candidates further to the left in the order of distance from the peak, and similarly for the candidates to the right. It is possible that she prefers all the candidates to the left of the peak and then all those to the right, or the reverse, or that these candidates are interleaved in the order P_v.

The characteristic property of such a preference order P_v is that between a certain candidate d and two other candidates c and e which are respectively to her left and right (in the sense of P_{ref}), it is impossible to appreciate d less than c and e at the same time: indeed, this would imply that the voter's preference peak is both to the left and to the right of d, whereas this peak is unique by assumption. This characterization has the advantage of not using the notion of utility and this is why it is commonly used as a definition of *single-peakedness.*

Definition 1.11 (single-peakedness)

For P_v and P_{ref} two strict total orders on the candidates, we say that P_v is *single-peaked* with respect to P_{ref} iff:

$$c\,\mathrm{P}_{\text{ref}}\ d \text{ and } d\,\mathrm{P}_{\text{ref}}\ e \Rightarrow \text{not}(c\,\mathrm{P}_v\ d \text{ and } e\,\mathrm{P}_v\ d).$$

For P a profile and P_{ref} a strict total order, we say that P is *single-peaked* with respect to P_{ref} iff for every voter v, P_v is a strict total order that is *single-peaked* with respect to P_{ref}.

We say that a profile P is *single-peaked* iff there exists a strict total order P_{ref} such that P is *single-peaked* with respect to P_{ref}.

We say that an electoral space Ω is *single-peaked* iff any profile P of this electoral space is *single-peaked.*

Example 1.12 (room temperature)

The occupants of a room have to choose the temperature of the thermostat among the candidate options $\{17°, 18°, 19°, 20°\}$ (in degrees Celsius). The social planner assumes that each voter can have any order of preference as long as it is single-peaked with respect to the natural order on temperatures. This electoral space allows any candidate as most liked but does not include all strict total orders: for example, the preference order $20° \succ 17° \succ 19° \succ 18°$ is excluded by hypothesis. Indeed, in this framework, if the ideal temperature of a voter is 20°, then she cannot estimate that 17° is preferable to 19°.

1.1.6 Probabilized electoral space

Now we will equip Ω with a probability measure, or *culture.* Theoretically, in order to handle probabilistic notions in a rigorous way, we must consider sigma-algebras, measurable sets, and probabilistic events. However, measurability is not a crucial problem in practice: for example, without the axiom of choice, any subset of $\mathbb{R}^C$ is Lebesgue-measurable. This is why these technical questions will only be discussed in Appendix A.

Definition 1.13 (probabilized electoral space)

A *probabilized electoral space*, or *PES*, is defined as an electoral space $(V, C, \Omega, \mathrm{P})$ equipped with a probability measure π on Ω, called *culture.*

Such a PES is denoted $(V, C, \Omega, \mathrm{P}, \pi)$, or simply (Ω, π).

We denote μ the distribution of the random variable P according to culture π.

For example, consider the reference electoral space. Independently for each voter v:

- Draw a vector of scores u_v uniformly in $[0, 1]^C$;

- Define p_v as the strict weak order naturally induced by u_v, in the sense that $c\ p_v\ d \Leftrightarrow u_v(c) > u_v(d)$;
- For each candidate c, define the approval value $a_v(c)$ as the rounding of $u_v(c)$ to the nearest integer, 0 or 1.

Then we have defined an example of culture π, that is, a probability measure on the electoral space Ω. Implicitly, this defines a distribution μ for profile P.

A classic example of culture is the *impartial culture*, whose definition we now recall.

Definition 1.14 (impartial culture)
According to the *impartial culture*, which is defined on the electoral space of strict total orders, the preference order of each voter is drawn independently and uniformly in $\mathcal{L}_\mathcal{C}$.

Let's finish with a more complex example. Suppose we have 3 voters and 3 candidates. We would like to study an electoral space where each voter is characterized by a strict total order of preference and where all *single-peaked* profiles, and only those, are possible, whatever the reference order $\mathrm{P}_{\mathrm{ref}}$. We will see that this is impossible, why it is desirable that it be impossible, and how we can transform the problem in order to solve it.

Given any strict total order of preference p_0, the profile where all voters have this order of preference is obviously *single-peaked* (with respect to p_0, at least), so it is allowed. Therefore, any order p_0 belongs to the set of possible states Ω_v of any voter v. The following profile is thus allowed.

$$\begin{array}{c|c|c} a & b & c \\ b & c & a \\ c & a & b \end{array}$$

But this profile is not *single-peaked*:[7] indeed, the second candidate in the order $\mathrm{P}_{\mathrm{ref}}$ cannot be c (resp. a, b) because of the voter 1 (resp. 2, 3).

This is due to the fact that we have defined Ω as a Cartesian product: if a particular opinion ω_v is *a priori* possible for voter v, then we consider that it remains possible, authorized, whatever the opinions of the other voters. But *it's not a bug, it's a feature*: this is a desirable property of the model.

On the other hand, the probability of a certain opinion ω_v can vary according to the state of the other voters and even become zero. In order to study all the *single-peaked* configurations, we can therefore use a PES in order to reduce to zero the probability of the other configurations: for example, we can consider the electoral space of strict total orders equipped with a culture π whose support is equal to the set of *single-peaked* configurations.

1.2 State-based voting systems (SBVS)

1.2.1 Definition

In this section, we model voting rules by defining state-based voting systems (SBVS), which will be the focal point of our study. At first glance, this model does not seem to be able to represent all conceivable voting rules. But we will see

[7]The reader familiar with these notions may also notice that the profile cannot be *single-peaked* because there is no Condorcet winner. We will come back to this.

in Section 1.4 that, in order to limit manipulability, we can restrict the study to SBVS, which allows to lighten the burden of formalism.[8]

Definition 1.15 (state-based voting system)
A *state-based voting system* over the electoral space Ω, or *SBVS*, is a function $f : \Omega \to \mathcal{C}$.
In the following, unless explicitly stated otherwise, f will always denote an SBVS on an electoral space Ω.

For example, consider one of the possible variants of the voting system called *Range voting*, in the reference electoral space, where $\omega_v = (p_v, u_v, a_v)$.

- Each voter v communicates a state belonging to Ω_v.
- She is said to vote *sincerely* iff she communicates her true state ω_v.
- The function f takes into account only the vectors of scores u_v communicated by the voters and returns the candidate with the highest total score.

To finish the description of this SBVS, it is necessary to give a *tie-breaking rule*, i.e. a procedure which solves the cases of tie. Saying that we have defined an SBVS is therefore a slight abuse of language: it is rather a class of SBVS, each member of which gives the same result when there is no tie.

In the literature, it is sometimes allowed for a voting rule to output a subset of the candidates. In that framework, it is said to be *resolute* iff it always outputs a single candidate. On the contrary, in this thesis, the resoluteness assumption is an integral part of the definition of an SBVS. We agree with the point of view of Gibbard (1973): if the goal of an election is to choose a unique item among several options, then the system is only fully defined if we include the possible tie-breaking rule in its description. This may be important for the properties studied: in particular, it may have an influence on the possible manipulations.[9]

Moreover, to implement in practice a voting system such as Range voting, presented above, it is sufficient that the ballots contain only the information actually used by the function f, i.e. the grades in this example. We will come back to this point in Section 1.4. But this state-based formalism facilitates general analysis, independently of the practical implementation of the voting system. On the one hand, it avoids the need for a tedious *sincerity function* which associates to a state of opinion the corresponding sincere ballot (as we mentioned in the introduction and as we will see more precisely in Section 1.4). On the other hand, it makes it possible to easily define transformations of voting systems, such as the Condorcification that we will see in Chapter 2 and the slicing in Chapter 5.

1.2.2 Basic criteria for an SBVS

First, we say that a system is *unanimous* iff it satisfies the following property: for every candidate c, if all voters strictly prefer c to the other candidates and vote sincerely, then c is elected.

We will also define *anonymity* and *neutrality*. Since the model of electoral spaces is more abstract and general than the simple use of strict total orders of

[8] This notion of *state-based voting system* is a generalization of what is called *elementary voting procedure* by Moulin (1978, chapter II, definition 2). The author already notes that considering such a procedure can only decrease the strategic possibilities of the voters.

[9] We will come back to the complex issue of defining manipulability for irresolute or non-deterministic voting rules in Section 6.3.3.

preference, these definitions require a bit more care than the informal definition that we gave in the introduction to this dissertation.

The electoral space Ω itself is said to be *anonymous* iff all voters have the same state space: $\forall (v, v') \in \mathcal{V}^2, \Omega_v = \Omega_{v'}$ and $\mathrm{P}_v = \mathrm{P}_{v'}$. Note that this is an equality of P_v and $\mathrm{P}_{v'}$ as functions, not an equality between two relations of preference.

We say that an SBVS f on the electoral space Ω is *anonymous* iff:

1. Ω is anonymous;
2. For every state $\omega \in \Omega$, for every permutation $\sigma \in \mathfrak{S}_{\mathcal{V}}$ of the voters, noting $\omega_\sigma = (\omega_{\sigma(1)}, \ldots, \omega_{\sigma(V)})$, we have $f(\omega) = f(\omega_\sigma)$.

In other words, the winner depends only on the ballots received, not on the identity of the voters who cast them.

To define neutrality, let Φ be an *action* of the permutation group of the candidates $\mathfrak{S}_{\mathcal{C}}$ on the state space Ω_v of each voter v. This means that to any state ω_v and to any permutation σ of the candidates, we associate a state $\Phi_\sigma(\omega_v)$, which we simply denote $\sigma(\omega_v)$; and that we require that this transformation be compatible with the group structure of $\mathfrak{S}_C$: this means, for example, that if we denote σ' another permutation and $\circ$ the composition operator, then we require that $\sigma(\sigma'(\omega_v)) = (\sigma \circ \sigma')(\omega_v)$.

We require, moreover, that Φ be compatible with P_v, in the sense that $\mathrm{P}_v(\sigma(\omega_v)) = \sigma(\mathrm{P}_v(\omega_v))$. In the right-hand side of this equality, the relation $\sigma(\mathrm{P}_v(\omega_v))$ is naturally defined by: $c\ \mathrm{P}_v(\omega_v)\ d \Leftrightarrow \sigma(c)\ \sigma(\mathrm{P}_v(\omega_v))\ \sigma(d)$, i.e. we consider the same preference relation by simply permuting the names of the candidates.

When such a group action is defined on the state space of each voter, it induces a group action on the whole electoral space Ω: for all configurations $\omega \in \Omega$, it suffices to pose $\sigma(\omega) = (\sigma(\omega_1), \ldots, \sigma(\omega_V))$.

We say that f is *neutral* (with respect to the group action Φ) iff $\forall \sigma \in \mathfrak{S}_{\mathcal{C}}, \forall \omega \in \Omega, f(\sigma(\omega)) = \sigma(f(\omega))$. In other words, the winning candidate does not depend on the labels used to name the candidates: if we change the labels of all the candidates, then there is a way to re-label the ballots (Φ) that yields the same voting system.

In general, the group action Φ that we consider is intuitively obvious: for example, in the reference electoral space, we will apply σ to the order of preference p_v, we will permute the coordinates of the vector of scores u_v, and we will do the same for the vector of approval values a_v.

However, in all generality, it may be necessary to specify the group action Φ used. Consider again Example 1.9 of decentralized random generator, where each voter has the result of a coin toss. We consider a case where there are $C = 2$ candidates. Therefore, to define the group action Φ, it is sufficient to give its effect for the only non-trivial permutation $\sigma = (1 \leftrightarrow 2)$, which consists in exchanging the two candidates. Its effect on the orders of preference is automatic, since we have required that Φ be compatible with the functions P_v. It only remains to define the effect of σ on Heads and Tails. Moreover, we assume that there are 3 voters (to avoid questions of ties).

First of all, consider the following SBVS: if there are more Tails, then candidate 1 wins; otherwise, candidate 2 wins. This system is particularly manipulable: if a voter prefers candidate 1 and her coin has fallen on Heads, then it is always in her interest to lie and to announce Tails. But it is the neutrality of this system that interests us for the moment. If we define the action of $\sigma = (1 \leftrightarrow 2)$ as reversing Heads and Tails (in addition to its effect on binary relations of preference), then this voting system is neutral with respect to Φ.

Now consider another SBVS. Each voter communicates her order of preference and the state of her coin. If it is Tails, then 1 point is counted in favor of her preferred candidate. If it is Heads, then 3 points are counted in favor of her preferred candidate. Then the candidate with the most points is elected. This system is similar to a variant of Range voting where Tails means "prefer slightly" and Heads, "prefer strongly". Moreover, as we have already noticed, a voter who has Tails in her state always has an interest in lying and announcing Heads. As far as neutrality is concerned, if we define the action of $\sigma = (1 \leftrightarrow 2)$ as leaving Tails and Heads unchanged (in addition to its effect on the binary relations of preference), then this SBVS is neutral with respect to Φ.

Thus, in the general case, it is possible to have various voting systems that are neutral by considering different group actions that are not made obvious *a priori* by the sole definition of the electoral space under consideration. That said, in most cases of practical study, the group action Φ will be intuitively obvious, as we have seen in the reference electoral space.

Later on (notably in Chapter 10), we will sometimes focus on the electoral space of strict total orders. A natural question then arises: under what condition does an anonymous and neutral voting system exist?

Proposition 1.16
We place ourselves in the electoral space of strict total orders.

1. *The following conditions are equivalent.*

   (a) *There exists an anonymous and neutral SBVS.*

   (b) *It is impossible to write C as a sum of divisors of V greater than 1. In other words, there is no list of natural numbers $(k_1, \dots, k_n)$ such that:*

   $$\left[\forall i \in [\![1, n]\!], k_i > 1 \text{ and } k_i \text{ divides } V\right] \text{ and } \sum_{i=1}^{n} k_i = C.$$

   (c) *It is impossible to write C as a sum of prime factors of V. In other words, there is no list of natural numbers $(k_1, \dots, k_n)$ such that:*

   $$\left[\forall i \in [\![1, n]\!], k_i \text{ is prime and } k_i \text{ divides } V\right] \text{ and } \sum_{i=1}^{n} k_i = C.$$

2. *For there to exist an anonymous and neutral SBVS, it is necessary but not sufficient that V and C are relatively prime.*

Proof. We will first assume 1 and deduce 2, which will give a first intuition for the proof of 1 that we will give next.

2. First, we show that this simplified condition is not sufficient: consider $V = 6$ and $C = 7$, which are indeed prime. It is possible to write C as a sum of factors of V greater than 1: $C = 3 + 2 + 2$, where 3 divides V and 2 divides V. So condition 1b is not satisfied, which implies that there is no anonymous and neutral SBVS.

To be convinced intuitively, examine the following profile, which gives a clue to the proof of the equivalence stated in point 1.

1	2	3	1	2	3
2	3	1	2	3	1
3	1	2	3	1	2
4	5	4	5	4	5
5	4	5	4	5	4
6	7	6	7	6	7
7	6	7	6	7	6

This profile is obtained by starting from the ranking of the first voter and applying circular permutations to the first three candidates, the next two, and the last two. In particular, it should be noticed that the permutations of the last two pairs are always simultaneous (if 4 is above 5, then 6 is above 7), which makes it possible to have 3×2 voters only and not $3 \times 2 \times 2 = 12$ (which would be the case if we considered the images of the first voter by all the mentioned circular permutations, independently of one another). We can also note that we have, in this case, $V = \mathrm{lcm}(3, 2, 2)$.

It is easy to see that, whatever the winning candidate in this profile, the SBVS cannot be anonymous and neutral: for example, if 1 is elected, then 2 and 3 should win as well, by anonymity and neutrality.

Now we prove that it is necessary that V and C are relatively prime. If this is not the case, then let k be a common prime factor of C and V. Let $n = \frac{C}{k}$ and $k_i = k$ for each $i \in [\![1, n]\!]$. Then $\sum_{i=1}^{n} k_i = C$, where all terms k_i are prime and divide V. So condition 1c is not satisfied, which implies that there is no anonymous and neutral SBVS.

1a $\Rightarrow$ 1b. If condition 1b is not satisfied, then we proceed by generalizing the previous example. Consider a list $(k_1, \ldots, k_n)$ such that $\sum_{i=1}^{n} k_i = C$, all k_i are greater than 1 and divide V. This last property is equivalent to: $\mathrm{lcm}_i(k_i)$ divides V. Consider any order of preference, for example $1 \succ 2 \succ \ldots \succ C$, and all its images by applying simultaneously a circular permutation on the first k_1 candidates, on the following k_2 candidates, etc. The orbit, that is, the set of image rankings obtained, is of size $\mathrm{lcm}_i(k_i)$. By copying these images enough times, we obtain a profile with V voters. If one of the first k_1 candidates is the winner, then by anonymity and neutrality, each of the first k_1 candidates should be the winner as well, which contradicts the uniqueness of the result because $k_1 > 1$. The same is true for the following k_2 candidates, etc. Therefore, there is no anonymous and neutral SBVS: condition 1a is not satisfied.

1b $\Rightarrow$ 1a. Let us assume that condition 1b is satisfied. Let P be a profile. Two candidates are said to be equivalent in P iff there exists a permutation of the candidates that sends one to the other and leaves P stable, up to a permutation of the voters. Then let $k_1, \ldots, k_n$ be the cardinalities of the equivalence classes: we have $\sum_{i=1}^{n} k_i = C$. Consider a certain equivalence class of size k_i: since any candidate of the class takes the best rank (within the class) among the same number of voters, k_i divides V. Since condition 1b is satisfied, there exists a k_1 equal to 1. We can then choose the corresponding candidate as the winner without violating anonymity and neutrality (which fixes the winner in any other profile obtained from P by permuting candidates and/or voters). Performing the same reasoning for each profile P whose winner is not yet chosen, we obtain an anonymous and neutral SBVS.

1b $\Rightarrow$ 1c. This is immediate because any prime natural number is greater than 1.

1c $\Rightarrow$ 1b. If condition 1b is not satisfied, then it is possible to write C as a sum $C = \sum_{i=1}^{n} k_i$, where all k_i divide V and are different from 1. For each k_i, let k'_i be an arbitrary prime factor of k_i, which is therefore also a prime factor of V, and let $n_i = \frac{k_i}{k'_i}$. Then we have $C = \sum_{i=1}^{n} \sum_{j=1}^{n_i} k'_i$, which is a sum of prime factors of V. Hence condition 1c is not satisfied. □

This proposition is to be put in parallel with a classical result (Moulin, 1978), which concerns voting systems which are not only anonymous and neutral, but also *efficient*. This qualifier means that we have the following property: if all voters strictly prefer a certain candidate c to a certain d (i.e. if d is *Pareto-dominated* by c), then d cannot be elected. The previously known result is that there exists a neutral, anonymous, and efficient SBVS (on the electoral space of strict total orders) iff $C!$ and V are relatively prime, i.e. iff any integer less than or equal to C is prime with V.

In order to check the consistence between this result and Proposition 1.16, we can assume that this condition is satisfied and show that it implies condition 1b. If we write C as a sum of natural numbers greater than 1, then each of them is less than or equal to C, so it is prime with V. Since it is greater than 1, this means that it does not divide V. So condition 1b is satisfied.

To the best of our knowledge, if we remove the efficiency assumption, then there is no previous result in the literature giving a necessary and sufficient condition on the pair (V, C) for there to be an anonymous and neutral SBVS (on the electoral space of strict total orders), as Proposition 1.16 does.

Erratum added to the English version of this dissertation After the final version of the original French dissertation, I was informed that this result is actually already given by Moulin (1989) in his book *Axioms of Cooperative Decision Making*, Cambridge Books, Exercise 9.9, pp. 252–253. Thanks to an anonymous (and neutral) reviewer of *Social Choice and Welfare* for pointing out this reference.

1.3 Manipulability

Now we translate the usual definition of manipulability into the framework of electoral spaces.

For two candidates w and c, we note:

$$\mathrm{Manip}_\omega(\mathrm{w} \to c) = \{v \in \mathcal{V} \text{ s.t. } c \, \mathrm{P}_v(\omega_v) \, \mathrm{w}\}.$$

This is the set of voters who prefer c to w. If w wins the sincere vote, then these voters are interested in a manipulation in favor of c. Conversely, we note:

$$\mathrm{Sinc}_\omega(\mathrm{w} \to c) = \{v \in \mathcal{V} \text{ s.t. } \mathrm{non}(c \, \mathrm{P}_v(\omega_v) \, \mathrm{w})\}.$$

These voters are not interested in manipulating to make c win instead of w. For these two notions, we generally imply the dependence in ω and we simply note $\mathrm{Manip}(\mathrm{w} \to c)$ and $\mathrm{Sinc}(\mathrm{w} \to c)$.

Definition 1.17 (manipulability)

For $(\omega, \psi) \in \Omega^2$, a subset of voters $M \in \mathcal{P}(\mathcal{V})$, and a candidate $c \in \mathcal{C}$, we say that f is *manipulable* in configuration ω by coalition M to configuration ψ in favor of candidate c iff:

$$\begin{cases} c \neq f(\omega), \\ f(\psi) = c, \\ M \subseteq \mathrm{Manip}(f(\omega) \to c), \\ \forall v \in \mathcal{V} \setminus M, \psi_v = \omega_v. \end{cases}$$

In the rest of this thesis, we say either *coalition-manipulable* (CM), *coalitionally manipulable*, *manipulable by coalition* or, simply, *manipulable* without further precision: indeed, it is the most general notion of manipulability, in the sense that it is implied by all the other forms of manipulability that we will mention later, notably in Chapter 6. We will use the acronym CM indifferently as an adjective (coalition-*manipulable*) or as a noun (coalition-*manipulation*).

When we say that f is manipulable in ω without specifying ψ (resp. M, c), it means that there exists ψ (resp. M, c) which satisfies the previous definition.
Thus, for $(\omega, \psi) \in \Omega^2$ and a candidate $c \in \mathcal{C}$, we say that f is *manipulable* in configuration ω to configuration ψ in favor of candidate c iff there exists a coalition M such that the previous conditions are satisfied, which translates into the following relations.

$$\begin{cases} c \neq f(\omega), \\ f(\psi) = c, \\ \forall v \in \mathrm{Sinc}(f(\omega) \to c), \psi_v = \omega_v. \end{cases}$$

Similarly, for $(\omega, \psi) \in \Omega^2$, we say that f is *manipulable* in configuration ω to configuration ψ iff there exists a candidate c such that the previous relations are satisfied, which is written:

$$\begin{cases} f(\psi) \neq f(\omega), \\ \forall v \in \mathrm{Sinc}(f(\omega) \to f(\psi)), \psi_v = \omega_v. \end{cases}$$

Finally, for $\omega \in \Omega$, we say that f is *manipulable* in configuration ω iff there exists a configuration $\psi \in \Omega$ such that f is manipulable in ω to ψ. As a convenience of language, we will sometimes say that it is the configuration ω which is manipulable (in the context of the voting system f).

We denote CM_f the set of configurations ω where f is manipulable and we use the same notation for the indicator function of this set:

$$\mathrm{CM}_f : \left| \begin{array}{lll} \Omega & \to & \{0, 1\} \\ \omega & \to & \left| \begin{array}{l} 1 \text{ if } f \text{ is manipulable in } \omega, \\ 0 \text{ otherwise.} \end{array} \right. \end{array} \right.$$

For a culture π, we call *CM rate of f for π* (provided that CM_f is measurable, cf. Appendix A):

$$\begin{aligned} \tau_{\mathrm{CM}}^{\pi}(f) &= \pi(f \text{ is manipulable in } \omega) \\ &= \int_{\omega \in \Omega} \mathrm{CM}_f(\omega) \pi(\mathrm{d}\omega). \end{aligned}$$

When there is no ambiguity about the culture π used, this rate is simply denoted $\tau_{\mathrm{CM}}(f)$.

For two SBVS f and g, we say that f is *at most as manipulable as g in the set-theoretic sense*[10] iff $\mathrm{CM}_f \subseteq \mathrm{CM}_g$, and we say that f is *at most as manipulable*

[10] This notion is defined by Lepelley and Mbih (1994).

as g in the probabilistic sense iff $\tau^{\pi}_{\mathrm{CM}}(f) \leq \tau^{\pi}_{\mathrm{CM}}(g)$. The first property is very strong since it implies the second one for every culture π: if f is less manipulable than g in the set-theoretic sense, then, in any culture, f is less manipulable than g in the probabilistic sense.[11]

This definition of manipulability is the first one where we really exploit the binary relations of preference P_v. We seize this opportunity to come back to the fact that we did not assume them to be antisymmetric.

Consider two candidates a and b. Suppose that for each voter v, her state ω_v is a pair (t_v, x_v) where:

- $t_v \in \{a, b, \varnothing\}$ is equal to either her favorite candidate (her "top", which can be a or b), or $\varnothing$ if she supports neither of them;
- x_v is a Boolean variable that represents whether the voter is corruptible or not, typically in the case where a candidate would offer her a bribe.

We ask the following question: given the voting system used, is the voting outcome robust to the combined effects of bribery and strategic voting in the usual sense?

If a (resp. b) is the winner of the sincere vote, then the voters that may be interested in changing the outcome are those who are corruptible, regardless of their favorite candidate, and those whose favorite candidate is really b (resp. a). Note, then, that this problem can be treated with our manipulation formalism using preferences that are not antisymmetric. Indeed, for each voter v, define the binary relation P_v as in the following table.

		t_v (favorite candidate)		
		a	$\varnothing$	b
x_v (corruptible)	False	$a \ \mathrm{PP}_v \ b$	$a \ \mathrm{I}_v \ b$	$b \ \mathrm{PP}_v \ a$
	True	$a \ \mathrm{MP}_v \ b$		

In the electoral space defined in this way, manipulability (in the formal sense of Definition 1.17) coincides with vulnerability to the combined effects of corruption and strategic voting in the usual sense.

As another illustration, the absence of antisymmetry hypothesis makes it possible to consider a voter who is *never happy*, for whom we have $c \ \mathrm{P}_v \ d$ for each pair of candidates (c, d). Such a voter is ready to participate in any manipulation attempt, thus introducing an interesting instability in the system. From this pleasant example, we can see that, when we want to study manipulation, the relevant question for a given voter can be formulated as follows: if she knows that the default winner is a certain w, then is she willing to act so that a certain candidate c is the winner instead? While requiring that the resulting relation have interesting properties (antisymmetry, transitivity) may be of interest for attributing some consistency to a voter's actions, such rationality assumptions are not necessary for studying manipulation.

We have two other reasons for not assuming antisymmetry in general. On the one hand, most of our results do not require this assumption, so it would be a shame to prove theorems in a particular case when they are true in the general case. On the other hand, removing this assumption allows us to have definitions and statements that lend themselves more easily to other types of generalization, notably those of Chapter 4.

[11] If Ω is equipped with a sigma-algebra such that any singleton is measurable (which is generally the case), then it is not only an implication but an equivalence.

For readers who might be confused by such a model, everything that follows can be read with the addition of the antisymmetry assumption and the usual interpretation of manipulability.

The theorem of Gibbard (1973), which we already mentioned in the introduction, implies that in an electoral space that includes all strict total orders, for each non-trivial SBVS f (i.e., which is not dictatorial and whose image contains at least 3 candidates), we have $\mathrm{CM}_f \neq \varnothing$. The theorem of Satterthwaite (1975) states essentially the same result in the electoral space of strict weak orders.[12] Moreover, we will see that Gibbard's theorem also concerns general voting systems, which we will define in Section 1.4.

If the electoral space does not include all strict total orders, then it is possible to have a non-trivial, non-manipulable SBVS. For example, in the temperature space of Example 1.12, and more generally in a *single-peaked* space (Definition 1.11), it is classical and easy to prove that the following voting rule is not manipulable: each voter indicates her preferred temperature, and then the median temperature is elected.[13] Thus, when the reference order $\mathrm{P}_{\mathrm{ref}}$ is fixed and known at the time when the voting rule is set, there exists at least one non-manipulable system.

However, for many applications, such an assumption is unreasonable, both for descriptive and normative reasons. Consider the example of a political election. From a descriptive point of view, there may not be an obvious and canonical way to place candidates on an axis (typically left-right, in the political sense), and it is far from guaranteed that every voter has preferences that are *single-peaked* with respect to this reference order. From a normative point of view, using a voting system that presupposes a fixed reference order necessarily violates neutrality: a voter is forbidden to prefer a left-wing candidate, then a right-wing candidate, then a centrist, while the reverse order is allowed.

For such applications, even if we consider, as a first approximation, that preferences are *single-peaked* with probability 1, the order $\mathrm{P}_{\mathrm{ref}}$ is not fixed *a priori*, and it is impossible to take advantage of it in the choice of the voting system, as we could do in the example of temperatures. Under these assumptions, we see that the conclusion of the Gibbard-Satterthwaite theorem remains true: any non-trivial system is manipulable (Penn et al., 2011). Contrary to what one can sometimes read, it is therefore false to say that in single-peaked cultures, any

[12] For alternative proofs of the Gibbard-Satterthwaite theorem, see Gärdenfors (1977); Barberá (1983); Benoît (2000); Sen (2001). Several approaches, notably that of Reny (2001), help to understand the deep connection between this result and Arrow's theorem. Muller and Satterthwaite (1977) show that it is also intimately related to the impossibility that a non-trivial voting rule satisfies the *strong positive association*, a monotonicity condition with respect to preferences that generalizes the *positive response* property that we have seen for 2 candidates. Weber (2009) stresses, as did Gibbard and Satterthwaite, that it is not necessary to assume that the voting system is surjective (a frequently encountered formulation of the theorem), but simply that at least 3 candidates are in its image, i.e. are eligible. Duggan and Schwartz (2000) generalize the result to an irresolute voting system, i.e. without an explicit tie-breaking rule. Brandt and Brill (2011) discuss what happens when considering less demanding notions of non-manipulability. Slinko and White (2008) show that the theorem remains true even when restricted to so-called *safe* manipulations. Barberá and Peleg (1990) propose a variant of the theorem for an infinite number of candidates and a continuity hypothesis on the preferences of the voters. See also Taylor (2005) for an overview of various impossibility results of the same type.

[13] Moulin (1980) shows an extension of this result, which is itself generalized by Border and Jordan (1983) and Barberà (2001) in a multidimensional setting.

Condorcet system is non-manipulable: this is true only if the reference order is fixed, both in sincere preferences and in authorized ballots.[14]

The question arises, then, whether there is a simple necessary and sufficient condition on the electoral space for the conclusions of the Gibbard-Satterthwaite theorem to hold, i.e., that there is no non-trivial, non-manipulable voting system. Aswal et al. (2003) partly answer this question by providing a (rather technical) sufficient condition, but to our knowledge the problem remains open. In this dissertation, we will often consider electoral spaces that include all strict total orders, where the Gibbard-Satterthwaite theorem applies. This said, most of our theoretical results will concern any electoral space.

1.4 General voting systems

So far, we have defined state-based voting systems, or SBVS. However, real voting systems are sometimes more complex: in particular, they may have several rounds and the shape of the ballots may not coincide with the mathematical object used to model the preferences of the voters. We now present a formalism to study any kind of voting system and of which state-based voting systems (SBVS) are only special cases. Then we prove that, in order to limit manipulability, we can restrict the study to SBVS.

1.4.1 Definition

To begin with, let us consider one of the possible variants of the voting method called *Range voting*. Each voter v can choose from a set of strategies $\mathcal{S}_v = [0, 1]^C$: she must assign a score to each candidate. Once these scores are communicated, we use a counting rule f which returns the candidate with the best average score (and uses, in case of a tie, a deterministic rule chosen in advance). Let us generalize this particular case to define a *game form*, as in the seminal paper by Gibbard (1973).

Definition 1.18 (game form)
A *game form* (for V and C) is defined by giving:

- For each voter $v \in \mathcal{V}$, a nonempty set $\mathcal{S}_v$, whose elements are called *strategies*;[15]
- And a function $f : \mathcal{S}_1 \times \ldots \times \mathcal{S}_V \to \mathcal{C}$, which is called *counting rule*.

When a voter v is in the state $\omega_v \in \Omega_v$, one may wonder which strategy $S_v \in \mathcal{S}_v$ should be considered as sincere voting. As Gibbard (1973) notes, there is no general, canonical way to define sincere voting based only on the game form.[16] Let's see why with two examples.

Example 1.19 (parity voting)
We have already mentioned this system in the introduction. We consider $C = 2$ candidates. Each voter puts a white ball or a black ball in the ballot

[14] With a slightly different framework, Blin and Satterthwaite (1976) discuss precisely the assumptions involved, and show in particular that it is crucial that not only ballots can express only single-peaked preferences with respect to the reference order, but also that voters can sincerely have only such preferences.

[15] Here, strategies correspond to what is called "pure strategies" in game theory. They can be ballots, decision trees in a multi-step process, or more generally any kind of object.

[16] He presents an illuminating example, the *non-alcoholic party*.

box. If the number of black balls is odd, then candidate 1 is elected. If not, then candidate 2 is elected. Given a voter's preferred candidate, it is difficult to say that she has a natural honest ballot at her disposal!

One might object that this example is based on a counter-intuitive way of voting (although it is a game form that is entirely consistent with Definition 1.18). But the next example is based on a very classical voting system.

Example 1.20 (plurality without preference ranking)
In the electoral space of strict weak orders, consider the game form *Plurality* (first-past-the-post): each voter casts a ballot for one candidate, and the candidate who receives the most votes is elected. If a voter does not prefer one candidate to all the others, then there is no obvious, canonical way to define her sincere ballot.

These examples show that the notion of sincere strategy is more evanescent than it first appears. Consequently, it must be defined extrinsically to the game form by functions $s_v : \Omega_v \to \mathcal{S}_v$ which, to each state of opinion ω_v, associate a ballot $S_v = s_v(\omega_v)$.

Definition 1.21 (voting system)
A *voting system* F (on the electoral space Ω) is defined by giving:

- A game form $((\mathcal{S}_v)_{v\in\mathcal{V}}, f)$;
- And for each voter $v \in \mathcal{V}$, a function $s_v : \omega_v \to \mathcal{S}_v$, which is called *sincerity function*.

According to one interpretation, the function s_v represents a social consensus on the spirit in which the game form, that is, the voting rule, should be used, or an instruction manual for the system provided by the social planner.

From a slightly different perspective, consider a voter who always chooses her ballot deterministically, based only on her state of opinion, without any information about the opinions and strategies of other voters. This may be because she does not have access to this type of information before the election, or because she refuses to depend on her sources of information, such as polling organizations. Then, by definition, she uses precisely a function $s_v : \Omega_v \to \mathcal{S}_v$, which can be seen, in this case, as a heuristic to choose her ballot according to her own opinion, but without external information.

In the example of Range voting, it is possible to choose, for example, the sincerity function defined by $s_v(p_v, u_v, a_v) = u_v$ (in the reference electoral space). Following the first interpretation, the social planner conveys the message to voters that their sincere vector u_v is considered an appropriate ballot. According to the second interpretation, s_v is a heuristic way to choose one's ballot in the absence of information about the other voters. Whatever interpretation we choose, other choices are possible: for example, we could decide that a voter's sincere ballot consists in applying a positive linear transformation to her score vector such that its minimum is 0 and its maximum is 1. In this case, we would study another *voting system* based on the same *game form*.

The formal distinction between *game form* and *voting system* is therefore important when one considers using a certain game form and varying the sincerity function. In other cases, this distinction is not necessary and, for convenience of language, we may use the terms *game form* and *voting system* interchangeably.

In the real world, more often than not, only the game form is defined, without explicitly defining a sincerity function. In the example of France, the following information can be found on governmental sites.[17]

[17] http://www.interieur.gouv.fr/Elections/Les-elections-en-France/Les-modalites-d-elections/Les-differents-modes-de-scrutins.

> The principle of majority voting is simple. The candidate or candidates who obtain a majority of the votes cast are elected. [...] The ballot may be single-member if there is one seat to be filled per constituency. In this case, electors vote for a single candidate.

To the best of our knowledge, no official text specifies that it is recommended to vote for one's preferred candidate: the choice of the possible sincerity function or of a strategic behavior to adopt is left to the voters' entire discretion. Thus, the texts define a game form, but not a voting system.

Now we define the manipulability of a voting system in general.

For $\omega \in \Omega$ and $S = (S_1, \ldots, S_V) \in \mathcal{S}_1 \times \ldots \times \mathcal{S}_V$, we say that F is *manipulable in ω to S* iff:

$$\begin{cases} f(S_1, \ldots, S_V) \neq f\big(s_1(\omega_1), \ldots, s_V(\omega_V)\big), \\ \forall v \in \mathrm{Sinc}\Big(f\big(s_1(\omega_1), \ldots, s_V(\omega_V)\big) \to f(S_1, \ldots, S_V)\Big), S_v = s_v(\omega_v). \end{cases}$$

We say that F is *manipulable in ω* iff there exists $S \in \mathcal{S}_1 \times \ldots \times \mathcal{S}_V$ such that F is manipulable in ω to S. The set of manipulable configurations of F is denoted CM_F.

If F is not manipulable in ω, then it means that with the heuristic s_v (which can be provided by the social planner or chosen individually by each voter), voters are able to find a strong Nash equilibrium without any preliminary exchange of information. Therefore, even if all ballots are revealed after the election, they will have no regrets about the ballot they chose.

In his seminal article, Gibbard (1973) presents a first fundamental theorem which can be rephrased as follows. We say that a game form is *straightforward* iff for each strict total order on the candidates, each voter has a dominant strategy when she has this order of preference. The result is this: there is no non-trivial (i.e., non-dictatorial and with at least 3 candidates in the image) game form that is *straightforward*. In other words, it is impossible for all voters to develop a sincerity function that will allow them to always defend their opinions to the best of their ability (even without any information about the ballots of the other voters).

From this first theorem, Gibbard deduces a second result which can be rephrased as follows: in an electoral space which includes all strict total orders, for each non-trivial voting system F (i.e. which is not dictatorial and whose image contains at least 3 candidates), we have $\mathrm{CM}_F \neq \varnothing$. When we restrict this second theorem to SBVS in the electoral space of strict weak orders, we obtain the Gibbard-Satterthwaite theorem in its most often quoted version.

1.4.2 Reduction of a general voting system to an SBVS

Now we will prove that, in the interest of reducing manipulability, our study can be restricted to SBVS. The following proposition is essentially a result of Moulin (1978, chapitre II). Adapting it to the formalism of electoral spaces allows us to generalize it to voters whose preferences are represented by any mathematical objects, and not necessarily by order relations.

Let us start with an example. In the electoral space of strict total orders, consider the Two-round system (TR) with its most intuitive sincerity function: in each round, each voter casts a ballot for her preferred candidate among those still in the race. If a voter casts a ballot for candidate c in the first round and for candidate d in the second round while c is still in the race, then she is obviously not sincere.

To prevent this type of behavior, we can modify the voting system: each voter communicates her order of preference; then, the Two-round system is simulated with the corresponding sincere strategies. This does not prevent voters from lying about their order of preference, but they can no longer vote for c in the first round and for d against c in the second round. We call the resulting system the Instant two-round system (ITR). It is this simple idea that we now generalize.

Proposition 1.22 (state-based version)
Given a voting system F, consider the voting system $F' = ((\mathcal{S}'_v)_{v \in \mathcal{V}},\ f',\ (s'_v)_{v \in \mathcal{V}})$ defined as follows.

1. *Each voter communicates a state: $\forall v \in \mathcal{V}, \mathcal{S}'_v = \Omega_v$.*
2. *Sincerity consists in giving one's true state: $\forall v \in \mathcal{V}, s'_v = \mathrm{Id}$.*
3. *To obtain the result, we use the original counting rule f, considering that each voter uses the sincere strategy corresponding to the state she communicated: $f'(\omega) = f(s_1(\omega_1), \ldots, s_V(\omega_V))$.*

Then the voting system F', called the state-based version *of F, is at most as manipulable as F: $\mathrm{CM}_{F'} \subseteq \mathrm{CM}_F$.*

Proof. In F', sincere voting leads to the same result as in F, but manipulators have access to at most the same ballots. ☐

A voting system F is called a state-based voting system (SBVS) iff it is equal to its state-based version, i.e., iff, for every voter v, we have $\mathcal{S}_v = \Omega_v$ and $s_v = \mathrm{Id}$. As a shorthand, an SBVS can thus be denoted only by its counting function f, as we have done so far.

Considering the state-based version of a voting system has various advantages.

- It may prevent voters from using strategies that are clearly insincere, as in the Two-round system example.
- It simplifies formalism by identifying the opinion states $\omega_v \in \Omega_v$ and the allowed strategies $S_v \in \mathcal{S}_v$, thus omitting the sincerity functions s_v.
- In an SBVS, the ballot includes the binary relation of preference, even though this was not the case in the original game form. For voting systems like Approval voting, this step is necessary before defining the *Condorcification*, as we will do in the next chapter.

The ballot used in this framework must be seen as a theoretical abstraction, more than a real ballot. For a practical implementation of the considered voting system, we will be able to simplify the used ballot, as we will see in Section 1.4.3.

Thus, Proposition 1.22 tells us that considering the state-based version cannot increase manipulability. In Table 1.1, we show that this procedure can even strictly decrease it. We consider the TR and ITR systems. In sincere voting, the second round opposes w to c, and w is elected.

In TR or ITR, it is impossible to manipulate for d : indeed, the manipulators concerned cannot avoid that d is eliminated at the end of the first round. So, examine the manipulability in favor of c. The manipulators must avoid a second round against w, otherwise she is elected. So they need a runoff between c and d. For this, they must give at least one vote to d. In ITR, this means that in the

1	4	8	4
w	w	c	d
c	d	w	w
d	c	d	c

Table 1.1 – An example where TR is manipulable, but not ITR.

second round, d has at least 8 votes from sincere voters (those who do not prefer c to w), and at least 1 vote from manipulators, so d is elected and the manipulation fails.

On the other hand, in TR, 6 manipulators vote for c and 2 vote for d in the first round, which makes 6 votes in total for c and d, against 5 for w. In the second round between c and d, all the manipulators can vote for c, including those who voted for d in the first round. Then c is elected by 9 votes against 8. In conclusion, TR is manipulable in favor of c, while ITR is not manipulable in the studied configuration.

1.4.3 Canonical implementation of an SBVS

In the definition 1.15 of SBVS, we saw that a voter is supposed to communicate all information about her state. For the example of Range voting in the reference electoral space, this means that she must communicate her weak order and approval values, which is clearly unnecessary since this information has no influence on the voting outcome. We will simply formalize that, in order to implement such a voting system in practice, one will not ask for this unnecessary information.

For the following definition, we need a notation to describe the states of a subset of voters $M \in \mathcal{P}(\mathcal{V})$. We note $\Omega_M = \prod_{v \in M} \Omega_v$ the set of all possible states for these voters. The notation ω_M will designate an element of this set. If a configuration $\omega \in \Omega$ has been previously defined, then ω_M designates the restriction of the configuration ω to the voters of M.

For $M \in \mathcal{P}(\mathcal{V})$, $\omega_{\mathcal{V}\setminus M} \in \Omega_{\mathcal{V}\setminus M}$, and $\psi_M \in \Omega_M$, the notation $(\omega_{\mathcal{V}\setminus M}, \psi_M)$ designates the configuration obtained by gathering the voters of $\omega_{\mathcal{V}\setminus M}$ and ψ_M. By notation shortcut, we will write $f(\omega_{\mathcal{V}\setminus M}, \psi_M)$ without adding a second pair of parentheses.

Proposition 1.23 (canonical implementation)

Let f be an SBVS and v a voter. We say that two states $(\omega_v, \psi_v) \in (\Omega_v)^2$ are indistinguishable *for f, and we note $\omega_v \equiv_f \psi_v$, iff for every state of the other voters $\phi_{\mathcal{V}\setminus\{v\}} \in \Omega_{\mathcal{V}\setminus\{v\}}$, we have $f(\phi_{\mathcal{V}\setminus\{v\}}, \omega_v) = f(\phi_{\mathcal{V}\setminus\{v\}}, \psi_v)$. In all rigor, the relation $\equiv_f$ should also be indexed by voter v, but in practice there is no ambiguity.*

We call canonical implementation *of f the voting system $G = ((\mathcal{S}_v)_{v\in\mathcal{V}},\ g,\ (s_v)_{v\in\mathcal{V}})$ defined as follows.*

1. *For every voter $v \in \mathcal{V}$, her strategy set $\mathcal{S}_v$ is the quotient of Ω_v by the equivalence relation $\equiv_f$.*

2. *For every voter $v \in \mathcal{V}$, her sincerity function s_v is the canonical projection of Ω_v to the quotient $\mathcal{S}_v = (\Omega_v / \equiv_f)$.*

3. *The counting function $g : \mathcal{S}_1 \times \ldots \times \mathcal{S}_V \to \mathcal{C}$ is obtained by passing f to the quotient.*

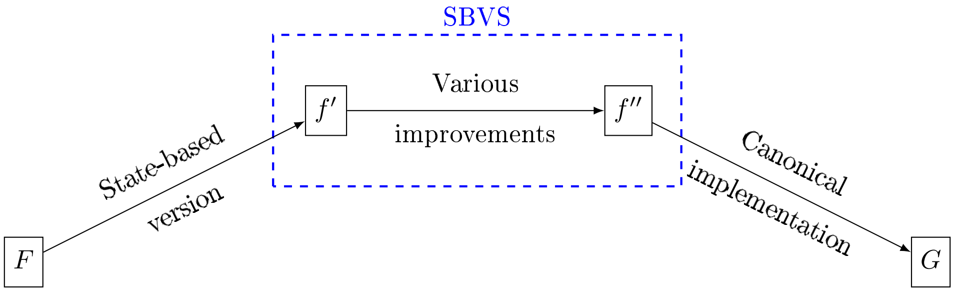


Figure 1.5 – Transformations of voting systems.

An SBVS f and its canonical implementation G have the same manipulability: $\mathrm{CM}_G = \mathrm{CM}_f$.

Proof. It is enough to notice that the definition of manipulability is unchanged by passing to the quotient. □

Consider another example. In the electoral space of strict total orders (with $V \geq 3$), consider Plurality, equipped with a tie-breaking rule that uses no other state information than the voters' preferred candidate. Here are two examples of such rules.

1. Among the tied candidates, the winner is the one who is first in the order of the indices.
2. If the first voter in the order of indices votes for one of the tied candidates, then that candidate is elected. If not, the second voter's ballot is examined, then the third, and so on. In other words, the voter with the lowest index among those who vote for one of the tied candidates breaks the tie.

In the SBVS representing this voting system, each voter communicates her order of preference. In the canonical implementation, she communicates only her preferred candidate.

When we study a voting system F, we almost always (and usually implicitly) use the mechanism shown in Figure 1.5: first, we consider its state-based version, denoted here as f', which cannot increase manipulability. Then we apply various transformations to it to decrease the manipulability and we obtain a new SBVS f''. If we want to put this voting system into practice, then we use its canonical implementation G, which does not change the manipulability.

1.5 Weighted majority matrix and Condorcet notions

At this point, we recall the classic notions of weighted majority matrix, Condorcet winner, and weak Condorcet winner. Since we allow for binary relations of preference that can be arbitrary, we pay special attention to different variants of these notions, even though some of them may coincide in some electoral spaces, e.g., when only strict total orders are allowed. In particular, we introduce the notion of *Condorcet-admissible candidate*, which will be a central notion to study the manipulability of a large class of voting systems.

When $\mathcal{A}(v)$ is an assertion that depends on voter v, we note $|\mathcal{A}(v)| = \text{card}\{v \in \mathcal{V} \text{ s.t. } \mathcal{A}(v)\}$. For example, for a pair of candidates (c, d), the notation $|c \mathrm{~P}_v~ d|$ designates the number of voters who prefer c to d.

Definition 1.24 (weighted majority matrix)
For each pair of candidates (c, d), we denote $D_{cd}(\omega) = |c \mathrm{~P}_v~ d|$, i.e. the number of voters who prefer c to d. The matrix $D(\omega)$ is called the *weighted majority matrix* of ω.

The relation of *relative victory* $\mathrm{P}_{\text{rel}}(\omega)$ is defined by:

$$c \mathrm{~P}_{\text{rel}}~ d \Leftrightarrow D_{cd} > D_{dc}.$$

It is easy to check that this relation is irreflexive, antisymmetric but not necessarily complete. However, if there is an odd number of voters and if their relations of preference P_v are strict total orders, then we cannot have $D_{cd} = D_{dc}$ so one of the candidates is necessarily the winner: therefore, P_{rel} is complete. In the same way as for individual preferences, we define I_{rel} (absence of relative victory). The relations PP_{rel} and MP_{rel} are of little interest: since P_{rel} is antisymmetric, PP_{rel} is the same as P_{rel}, and MP_{rel} is the empty relation.

The relation of *absolute victory* $\mathrm{P}_{\text{abs}}(\omega)$ is defined by:

$$c \mathrm{~P}_{\text{abs}}~ d \Leftrightarrow D_{cd} > \frac{V}{2}.$$

When this relation is satisfied, we say indifferently that c has an *absolute victory* against d in ω, or that d has an *absolute defeat* against c in ω. When we speak of victory (resp. defeat) without further precision, it is always an absolute victory (resp. defeat).

From $\mathrm{P}_{\text{abs}}(\omega)$, we define the relations I_{abs} (no victory), PP_{abs} (strict victory), and MP_{abs} (mutual victory) as we did for individual preferences.

Please remark that when we say that c has an absolute victory against d, which is denoted $c \mathrm{~P}_{\text{abs}}~ d$, the inequality:

$$D_{cd} > \frac{V}{2}$$

is already strict by definition. So, when we say that c has a **strict** absolute victory against d, which is denoted $c \mathrm{~PP}_{\text{abs}}~ d$, it does not concern the strictness of this inequality but it adds the condition that d has no absolute victory against c, which means that:

$$D_{dc} \leq \frac{V}{2}.$$

Fortunately, we will see these subtleties dissipate as soon as the voters have antisymmetric preferences. However, formalizing things with this level of rigor is interesting because it will facilitate their generalization in the context of generalized Condorcification (Chapter 4).

We will often use the following result (which is trivial).

Proposition 1.25 (total number of points in a duel)

$$D_{cd} + D_{dc} = V + |c \mathrm{~MP}_v~ d| - |c \mathrm{~I}_v~ d|.$$

It immediately follows that, if relations P_v are antisymmetric (which is a common assumption), then relation P_{abs} is antisymmetric: two candidates c and d

cannot have mutual victories. In other words, under the classical antisymmetry assumption, any absolute victory is strict.

Moreover, it also follows from this proposition that, if relations P_v are complete and if the number of voters V is odd, then relation $\mathrm{P}_{\mathrm{abs}}$ is complete: between two distinct candidates c and d, there cannot be an absence of victory.

Now we will define several variants of the notion of Condorcet winner.

Definition 1.26 (absolute Condorcet winner)
Let $\omega \in \Omega$ and $c \in \mathcal{C}$. We say that c is an *absolute Condorcet winner*, or simply a *Condorcet winner*, in ω, iff c has a strict absolute victory against any other candidate. That is, for every other candidate d, we have $c \, \mathrm{PP}_{\mathrm{abs}} \, d$, which is also written:

$$\begin{cases} |c \, \mathrm{P}_v \, d| > \frac{V}{2}, & (1.1) \\ |d \, \mathrm{P}_v \, c| \leq \frac{V}{2}. & (1.2) \end{cases}$$

If relations P_v are antisymmetric (which is a common assumption), then we have noticed that any absolute victory is strict. In this case, c is a Condorcet winner iff for every other candidate d:

$$|c \, \mathrm{P}_v \, d| > \frac{V}{2}.$$

Definition 1.27 (relative Condorcet winner)
Let $\omega \in \Omega$ and $c \in \mathcal{C}$. We say that c is a *relative Condorcet winner* in ω iff c has a relative victory against any other candidate. That is, for every other candidate d, we have $c \, \mathrm{P}_{\mathrm{rel}} \, d$, which is also written:

$$|c \, \mathrm{P}_v \, d| > |d \, \mathrm{P}_v \, c|.$$

Definition 1.28 (weak Condorcet winner)
Let $\omega \in \Omega$ and $c \in \mathcal{C}$. We say that c is a *weak Condorcet winner* in ω iff c has no relative defeat. That is, for every other candidate d, we have $\mathrm{not}(d \, \mathrm{P}_{\mathrm{rel}} \, c)$, which is also written:

$$|c \, \mathrm{P}_v \, d| \geq |d \, \mathrm{P}_v \, c|.$$

Definition 1.29 (Condorcet-admissible candidate)
Let $\omega \in \Omega$ and $c \in \mathcal{C}$. We say that c is *Condorcet-admissible* in ω iff c has no absolute defeat. That is, for every other candidate d, we have $\mathrm{not}(d \, \mathrm{P}_{\mathrm{abs}} \, c)$, which is also written:

$$|d \, \mathrm{P}_v \, c| \leq \frac{V}{2}.$$

The following proposition indicates the links between these different Condorcet notions. All the implications mentioned in the proposition, as well as those that immediately follow, are represented in Figure 1.6. In particular, we note that, in the case of antisymmetric preferences, the notion of absolute Condorcet winner is stronger than that of relative Condorcet winner, which is stronger than that of weak Condorcet winner, which is stronger than that of Condorcet-admissible candidate.

Proposition 1.30 (implications between the Condorcet notions)
If c is an (absolute) Condorcet winner, then c is a relative Condorcet winner. If the relations of preference P_v are antisymmetric and complete, then the converse is true.

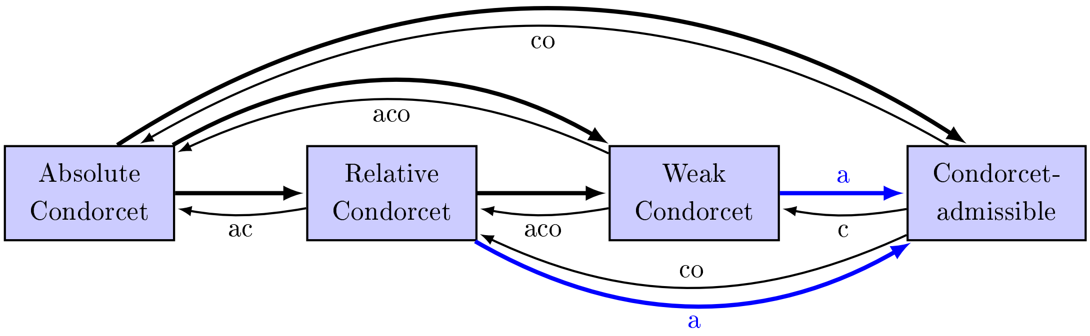


Figure 1.6 – Implication between the Condorcet notions (a: antisymmetric preferences, c: complete preferences, o: odd number of voters).

If c is a relative Condorcet winner, then c is a weak Condorcet winner. If the relations of preference P_v are antisymmetric and complete and if the number of voters is odd, then the converse is true.

If c is a weak Condorcet winner and if the relations of preference P_v are antisymmetric, then c is Condorcet-admissible. If c is Condorcet-admissible and if the relations of preference P_v are complete, then c is a weak Condorcet winner.

If c is an (absolute) Condorcet winner, then c is Condorcet-admissible. If c is Condorcet-admissible, if the relations of preference P_v are complete, and if the number of voters is odd, then c is an (absolute) Condorcet winner.

Proof. Each of these properties is trivial or follows immediately from Proposition 1.25. □

If the relations of preference P_v are antisymmetric and complete (in particular, if they are strict total orders), then these four notions reduce to only two: on the one hand, absolute and relative Condorcet winners are equivalent; on the other hand, weak Condorcet winner and Condorcet-admissible candidate are also equivalent. If, in addition, there is an odd number of voters, then all four notions are equivalent.

In the literature, the expression *Condorcet winner* (without precision) often refers to the relative Condorcet winner. In this thesis, however, we use this expression to designate the absolute Condorcet winner, since we deal with this notion much more often.

Clearly, the absolute or relative Condorcet winner, if it exists, is unique. We also have the following property, equally obvious but extremely useful.

Proposition 1.31

If a candidate is an (absolute) Condorcet winner, then no other candidate is Condorcet-admissible.

Now we can formally define the Condorcet criterion.

Definition 1.32 (Condorcet criterion)

An SBVS f is said to satisfy the *Condorcet criterion* iff, for every configuration $\omega \in \Omega$ and for every candidate $c \in \mathcal{C}$, if c is Condorcet winner in ω, then $f(\omega) = c$. We note indifferently $\mathbf{Cond}(\Omega)$ or simply $\mathbf{Cond}$ the set of SBVS on Ω that satisfy this criterion, as well as the criterion itself.

For the sake of convenience, we use also the word *Condorcet* as an adjective: thus, we will say that f is a Condorcet voting system. In the same spirit, when speaking of a configuration ω, we will say that it is:

- *Condorcet* iff there is a Condorcet winner in ω,
- *Semi-Condorcet* iff there is at least one Condorcet-admissible candidate but no Condorcet winner in ω,
- *Non-admissible* iff there is no Condorcet-admissible candidate in ω.

Similarly, we will say that the configuration ω is *admissible* iff there is a Condorcet-admissible candidate (i.e., it is Condorcet or semi-Condorcet) and say that it is *non-Condorcet* iff there is no Condorcet winner (i.e., it is semi-Condorcet or non-admissible).

1.6 Zoology of voting systems

To finish this first general chapter, we will present the voting systems frequently used in this thesis. In order to avoid heavy formalism, we do not always give a complete mathematical definition of each of these systems. For more details, the interested reader can refer for example to Tideman (2006). To each voting system, we also assign a short name (acronym or abbreviation), which will be used in the tables and figures in the second part of this dissertation.

In the following definitions, whenever a voting system defines a score, the candidate with the highest score is elected. In case of a tie, a deterministic tie-breaking rule is used to choose a candidate among the tied ones. As we have noticed before in the particular case of Range voting, what we define each time is not strictly a voting system, but a class of voting systems that return the same result except in a certain set of cases (reasonably limited in general), for which it is necessary to specify the tie-breaking rule.

1.6.1 Cardinal voting systems

A voting system is said to be *ordinal* iff the knowledge of the binary relation of preference of a voter is sufficient to determine her sincere strategy, i.e. we have the implication $[\mathrm{P}_v(\omega_v) = \mathrm{P}_v(\omega'_v)] \Rightarrow [s_v(\omega_v) = s_v(\omega'_v)]$. Otherwise, we say that it is *non-ordinal*. For convenience of language, and slightly informally, we say that a *game form* is non-ordinal iff there is no relatively natural sincerity function for which the resulting voting system is ordinal. Non-ordinal systems include cardinal systems, where each voter's ballot consists of assigning a score (or a similar mathematical object) to each candidate. In particular, we have the following three systems.

***Range voting* (RV)** Each voter assigns a score to each candidate from a set of allowed scores, which are real numbers. The score of a candidate is her average score.

***Majority judgment* (MJ)** Each voter assigns a rating to each candidate, from a set of allowed ratings. The ratings are objects of any mathematical nature, with a total order, for example real numbers. In practice, in this thesis, we will represent these appreciations by real numbers that we will call *scores*, as in Range voting, even if only the relative positions are important and not the numbers themselves.[18] The score of a candidate is its median score. If the number of voters is even, then we consider the unfavorable median: thus, if the two median voters for a certain candidate give her the scores 0.4 and 0.5, then her score is 0.4. For the rule used in the case where several candidates have the same median score, see Balinski and Laraki (2010).

For both of the above voting methods, we always assume that the set of allowed scores or ratings contains at least two distinct elements.

***Approval voting* (AV)** When the allowed scores are only 0 and 1, the two previous systems become equivalent.[19] The resulting voting system is called *Approval voting*. For more details, see, for example, Brams and Fishburn (1978).

In these cardinal voting systems, we will represent a configuration in the following form (provided that the binary relations of preference are strict weak orders that are consistent with the scores).

2	3	4
$a : 1$	$a : 1$	$c : 1$
$b : 0$	$b, c : 0$	$b : 1$
$c : 0$		$a : 0$

In this example, the first two voters (left column) prefer candidate a, then b, then c. They assign a score (or approval value) of 1 to a and 0 to both candidates b and c. The next three voters (middle column) prefer candidate a, to whom they assign a score of 1. They are indifferent between candidates b and c, to whom they assign a score of 0. Etc.

We will now turn our attention to ordinal voting systems. In general, we will assume that each voter provides a strict total order of preference. If the electoral space allows preferences that are not strict total orders, then a simple way to adapt these rules is to decide that if a voter does not provide a total order, then her ballot is not counted at all. In some cases, however, we will give other possible generalizations.

1.6.2 Positional scoring rules (PSR)

Definition 1.33 (positional scoring rule)
Let $\mathbf{x} = (x_1, \ldots, x_C) \in \mathbb{R}^C$ be a list of real numbers called *weights*. We always assume that such a vector is nonincreasing and that it is not constant, i.e. $x_1 > x_C$.

For each voter v and each candidate c, let $r(v, c)$ be the rank of c in the order of preference of v (1 for the best candidate, C for the worst). The score of candidate c is defined by:

$$\text{score}(c) = \sum_{v \in \mathcal{V}} x_{r(v,c)}.$$

The voting system that elects the candidate with the best score in the above sense is called the *positional scoring rule (PSR)* of weights $\mathbf{x}$.

[18] We thus consider the set $\mathbb{R}$ as equipped with its topological structure of linear order, but not as a metric space.

[19] To prove this, it is necessary to specify the tie-breaking rule of Majority judgment. For more details on this point, see Balinski and Laraki (2010).

PSR have been the subject of many characterizations, of which an overview is given by Merlin (2003). The following three particular cases of PSR are particularly famous.

***Plurality* (Plu.)** RPS of weights $\mathbf{x} = (1, 0, \ldots, 0)$.

Veto* or *Antiplurality RPS of weights $\mathbf{x} = (0, \ldots, 0, -1)$.

***Borda's method* (Bor.)** RPS of weights $\mathbf{x} = (C-1, C-2, \ldots, 0)$. This procedure has been the subject of countless studies. See Fishburn and Gehrlein (1976); Saari (2012).

These three particular systems have fairly natural generalizations to the case where a voter does not provide a strict total order.

For Plurality, if v strictly prefers a candidate c to any other candidate d, in the sense that $c \,\mathrm{PP}_v\, d$, then she gives a point to c. Otherwise, she gives a point to nobody. This way, if she strictly prefers c but has a preference cycle among the other candidates, then her preference for c is still taken into account. For Veto, it is easy to propose a similar adaptation, *mutatis mutandis*.

When voters have weak orders of preference, another possibility for Plurality (resp. Veto) is to divide the vote of a voter equally between the candidates that she puts at the top (resp. bottom) of her ballot.

For Borda's method, unless explicitly stated otherwise, we will always use the following generalization. Each voter v contributes to the score of candidate c:

- 1 point for each candidate d such that $c \,\mathrm{PP}_v\, d$,
- And 0.5 point for each candidate d such that $c \,\mathrm{I}_v\, d$ or $c \,\mathrm{MP}_v\, d$.

This generalization is quite natural insofar as the total number of points given by a voter is constant, whether her preferences are a strict total order or not. In particular, some properties involving the average score of a candidate will remain true with this generalization.

1.6.3 Bucklin's method

In *Bucklin's method* (Buck.), the counting process is conducted in rounds. In round t, the provisional score of a candidate c is the number of voters who place c between rank 1 and rank t on their ballot. If at least one candidate has a score strictly greater than $\frac{V}{2}$, then the candidate with the highest score is elected. Otherwise, the round $t+1$ of counting is done.

This system elects the candidate with the highest median rank, provided that the unfavorable median is considered when the number of voters is even. For example, if the two median ranks for a certain candidate c are ranks 2 and 3, then the median rank, in sense of Bucklin, is 3.

In principle, Bucklin's method is to Borda's method what Majority judgment is to Range voting, as illustrated by Table 1.2: indeed, Bucklin's method elects the candidate with the best median rank, while Borda's method elects the one with the best average rank. Majority judgment designates the candidate with the best median rating, while Range voting elects the one with the best average rating.

The rule stated above is more precise than simply choosing the candidate with the best median rank, since it allows, in general, to decide between two candidates

	Average	Median
Rank	Borda's method	Bucklin's method
Rating	Range voting	Majority Judgment

Table 1.2 – Principled comparison of the methods of Borda and Bucklin, Range voting, and Majority judgment.

with the same median rank. To see this, we need only observe that the following definition is equivalent to the one that we have given.

For each candidate c, let m_c be her median rank in the Bucklin sense (i.e. unfavorable when the number of voters is even) and x_c the number of voters who give her an equal or better (i.e. lower) rank. Denote:

$$\text{score}(c) = (m_c, x_c).$$

The set of median ranks m_c is equipped with the descending order relation (a lower rank is better) and the set of values x_c, with the ascending order relation (a higher number is better). Then it is easy to show that the candidate with the best score in the lexicographic sense (lower rank and, in case of a tie, higher value of x_c) is indeed the winner by Bucklin's method, in the sense of the first definition that we gave.

This variant of the definition allows us to naturally extend Bucklin's method to the case where the binary relations of preference are not strict total orders. Indeed, an elegant possibility consists in conventionally generalizing the rank $r(v, c)$ assigned by voter v to candidate c by $r(v, c) = C - B(v, c)$, where $B(v, c)$ is the Borda score as we have generalized it. Thus, if the voter places two candidates equally ahead of the others, then she is considered as putting them at rank 1.5. If she places three candidates in a non-transitive preference cycle $a \text{ PP}_v\ b \text{ PP}_v\ c \text{ PP}_v\ a$ above all the other candidates, then we consider that she puts them at rank 2. We can then compute the scores as we defined them above.

1.6.4 Iterated PSR with simple elimination

In this section and the next two, we will focus on elimination methods. Informally, these systems work as follows. Each candidate is assigned a score according to some given rule. Then a number of candidates with the worst scores are eliminated. This process is iterated until there is only one candidate left, who is then declared the winner.

In general, we will consider the case where the rounds are virtual: the voters give all useful information in their ballot from the beginning and the rounds are emulated automatically during the counting process (as for the ITR system, mentioned in Section 1.4.2). But the rounds can also be real (as for TR), i.e. the voters return to the polls each time. In this case, the system cannot usually be described as an SBVS, but by the formalism of a general voting system.

Among the elimination methods, we distinguish in particular the *iterated PSR* or *IPSR*: we use a positional scoring rule and we eliminate a certain number of candidates at each round. Two ways of doing this are usual:[20] the simple elimination and the one based on the average score.

[20] For IPSR, we use in part a classification proposed by Favardin and Lepelley (2006) for the case $C = 3$.

Definition 1.34 (IPSR with simple elimination)
Let $\mathbf{x}$ be a function which, to each integer k between 2 and C, associates a vector $\mathbf{x}^k = (x_1^k, \ldots, x_k^k) \in \mathbb{R}^k$.

The counting process takes place in rounds numbered from 1 to $C - 1$. During round t, the score of each non-eliminated candidate is calculated using the PSR of vector $\mathbf{x}^{C+1-t}$ and the candidate with the lowest score is eliminated. The candidate who is not eliminated at the end of the last round is declared the winner.

For convenience, the score of a candidate c in round t is written $\text{score}_k(c)$, where k is the number of candidates who are still running. This writing convention allows $\text{score}_k(c)$ to be computed using the weight vector $\mathbf{x}^k$ which has k components.

This system is called the *iterated PSR with simple elimination (iterated PSR-SE)* of weight vectors $\mathbf{x}$.

Within this family of voting systems, the following three systems are particularly studied.

Instant-runoff voting (IRV) The IPSR-1 of Plurality is known as *Instant-runoff voting*, *Single transferable voting*, *Hare method*, or *Alternative vote*.[21]

***Coombs' method* (Coo.)** IPSR-1 of Veto.

***Baldwin's method* (Bald.)** IPSR-1 of Borda's method. The interest of this system lies in the fact that a Condorcet winner always has a Borda score strictly higher than the average score (because it is the sum of her row in the weighted majority matrix). Therefore, this system satisfies the Condorcet criterion.

1.6.5 Iterated PSR with elimination based on the average

The principle of these systems is similar to IPSR-1 but, during each round of counting, all candidates with a score below the average score are eliminated at the same time.

Definition 1.35 (IPSR with elimination based on the average)
Let $\mathbf{x}$ be a function which, to each integer k between 2 and C, associates a vector $\mathbf{x}^k = (x_1^k, \ldots, x_k^k) \in \mathbb{R}^k$.

The counting process is conducted in numbered rounds. The total number of rounds is not fixed in advance. Noting k the number of candidates who are still running at the beginning of a round, the score for this round of each non-eliminated candidate is computed using the PSR of vector $\mathbf{x}^k$. Each candidate with a score strictly lower than the average score is eliminated. The candidate who is not eliminated at the end of the last round is declared the winner.

In the same way as for IPSR-1, the score of a candidate c at a certain round of counting is denoted $\text{score}_k(c)$, where k is the number of candidates who are still in the race. Thus, $\text{score}_k(c)$ is computed using the weight vector $\mathbf{x}^k$ which has k components.

This system is called the *iterated PSR with elimination based on the average (IPSR-A)* of weight vectors $\mathbf{x}$.

Two examples in particular received special attention.

[21] The name *Single transferable vote* may lead to confusion with its multi-winner version, used for example in Australia, whereas, to our knowledge, the terminology *Instant-runoff voting* is only used for the single-winner version. This is why we prefer to use the latter.

***Nanson's method* (Nan.)** IPSR-A of Borda's method. For the same reasons as Baldwin's method, this system satisfies the Condorcet criterion. Historically, this was the initial motivation for defining Nanson's method, which then inspired the IPSR-A class by generalization.

***Kim-Roush method* (KR)** IPSR-A of Veto. See in particular Kim and Roush (1996) and Lepelley and Valognes (1999).

In the same spirit, it is natural to consider the following system.

***IRV based on the average* (IRVA)** IPSR-A of Plurality.

In our computer simulations (Chapter 6 and following), we will not include KR and IRVA, although these systems are available in our software package SVVAMP. Indeed, they were implemented during the final phase of writing, when the simulations had already been performed and the corresponding chapters written. For reasons of computation time and in order to avoid adding last minute typos in these chapters, we preferred to stick to the simulations already carried out. We will nevertheless give some indications on the first simulation results obtained with these two systems in Section 7.2 and in the conclusion of this thesis.

1.6.6 Various elimination methods

The following voting methods do not fall into the IPSR-1 or IPSR-A families but they still use successive eliminations.

***Exhaustive ballot* (EB)** This is the variant of IRV with actual rounds. In each round, each voter casts a ballot for a candidate and the candidate who receives the fewest votes is eliminated. If the voters are sincere, then this voting system is equivalent to IRV. When this voting system is used in practice, the following clause is usually added: as soon as a candidate has an absolute majority, she is declared the winner. If the voters are sincere, then this clause does not change the result and it saves unnecessary rounds.

***Two-round system* (TR)** We have already mentioned this system several times. When there are three candidates, it is equivalent to Exhaustive ballot. And its instantaneous version, which we have already named ITR (Section 1.4.2), is then equivalent to IRV. In particular, the example of Table 1.1 in Section 1.4.2, which involved TR and ITR with 3 candidates, also demonstrates that IRV is in general strictly less manipulable than EB.

***Iterated Bucklin's method* (IB)** The worst candidate in the Bucklin sense, i.e., the one with the worst median rank, is eliminated, and in the case of a tie, the one with the fewest voters assigning her median rank or a better rank is eliminated. Then we iterate, reassessing the ranks in each round of elimination because of the eliminated candidates.

1.6.7 Condorcet methods

We already mentioned the methods of Baldwin and Nanson, which belong to the IPSR-1 and IPSR-A families respectively. Here are, now, various other methods which satisfy the Condorcet criterion.

Condorcet-dean **(CDean)** If there is a Condorcet winner, then she is elected. If not, then a predetermined candidate, called the *dean*, is declared the winner.

Condorcet-dictatorship **(CDict.)** If there is a Condorcet winner, then she is elected. If not, then the candidate who is at the top of the ballot of a predetermined voter is declared the winner.

Black's method (Condorcet-Borda or CBor.) If there is a Condorcet winner, then she is elected. Otherwise, the candidate with the highest Borda score is elected. This voting system was proposed by Black (1958). One can also consult Blin and Satterthwaite (1976).

IRV with duels **(IRVD)** The principle of this system is inspired by IRV and was suggested by Laurent Viennot, whom we thank. In each round of counting, the two candidates who are at the top in the least number of ballots are selected for a duel. The one who loses the duel (according to the weighted majority matrix) is eliminated. Then the next round of counting is performed.

Copeland's method **(Cop.)** The score of a candidate c is equal to her number of victories in the weighted majority matrix. The main drawback of this method is that it very often leads to ties: for example, whatever the number of voters, if there are 3 or 4 candidates, then as soon as there is no Condorcet winner, there is a tie. Thus, Copeland's method covers an important variety of voting systems, depending on the tie-breaking rule used (for example, by best Borda score, then by alphabetical order on the candidates).

Maximin **(Max.)** The score of a candidate c is equal to the minimum non-diagonal coefficient of her row in the weighted majority matrix; in other words, her score in her worst duel. If ballots are not strict total orders, then several variants are natural, depending on whether one considers the weighted majority matrix in expressed percentages or in number of voters, the raw score of each duel or the difference with the opponent. We will not dwell further on these variants.

Kemeny's method **(Kem.)** *Kendall's tau distance* between two preference orders in $\mathcal{L}_\mathcal{C}$ consists in counting 1 point for each pair of candidates (c, d) such that the first order places c ahead of d and the second order does the opposite.

For each order $p_0 \in \mathcal{L}_\mathcal{C}$ on the candidates, its Kemeny score is:

$$\mathrm{score}(p_0) = -\sum_{v \in \mathcal{V}} \delta(\mathrm{P}_v, p_0),$$

where δ denotes Kendall's tau distance. The order with the best score is chosen, and the first candidate of this order is declared the winner of the election. See in particular Kemeny (1959); Young and Levenglick (1978); Saari and Merlin (2000).

We can also express the problem in the following way: we have to find a permutation p_0 such that by applying it simultaneously to the rows and columns of the weighted majority matrix, the sum of the coefficients below the diagonal is minimal. This reformulation gives, moreover, a natural generalization of the method when the preferences are not strict total orders.

***Dodgson's method* (Dodg.)** An *elementary action* is defined as swapping two consecutive candidates in the order of preference of a voter. The score of a candidate c is, in negative, the minimal number of elementary actions that must be applied to the profile of the population for c to become the Condorcet winner.

In other words, we look for the profile closest to the population profile (in the sense of Kendall's tau distance) among those who have a Condorcet winner and whose preferences are strict total orders.

Charles Lutwidge Dodgson, who conceived this voting system, is better known under his writer's name, Lewis Carroll, with whom he signed *Alice in Wonderland*.

The methods of Kemeny and Dodgson have an important drawback: Bartholdi et al. (1989b) showed that it is $\mathcal{NP}$-difficult to determine the winner of an election in these voting systems.[22]

***Condorcet's method with sum of defeats* (CSD)** Now, what we call *elementary action* consists in reversing the preferences of a voter about a pair of candidates (but without requiring that her relation of preference remain transitive). The score of a candidate c is, in negative, the minimal number of elementary actions that must be applied to the population profile for c to become the Condorcet winner.

In other words, we look for the profile closest to the population profile (in the sense of Kendall's tau distance) among those who have a Condorcet winner, but without imposing that the preferences are strict total orders.

In practice, the score of a candidate c is:

$$\text{score}(c) = -\sum_{c \text{ does not beat } d} \left(\left\lfloor \frac{V}{2} \right\rfloor + 1 - D_{cd} \right).$$

For each duel that is not a victory, candidate c loses the number of points she needs to win. This is why, by way of shorthand, we call this system: *Condorcet's method with sum of defeats (CSD)*.

In its approach, this method is similar to Dodgson's method, but it has the advantage that the winner can be determined in polynomial time.[23]

***Ranked pairs* (RP)** We construct a directed graph whose vertices are the candidates. To do this, we examine all the duels between candidates (in the weighted majority matrix) by decreasing amplitude. For example, we start with the duel (c, d) for which $D_{cd} - D_{dc}$ is maximal. For each duel, we add an edge to the graph in the winning direction, unless adding this edge creates a cycle in the graph.

At the end of the process, we obtain a transitive directed graph whose adjacency relation is included in the victory relation. The candidate who is the maximal vertex of this graph by topological order is declared the winner (Tideman, 1987). When several duels have the same amplitude, a given tie-breaking rule is generally used to know which one should be examined first. This is what we will do in this dissertation, in particular for the simulations of the second part, because the determination of the winner is then clearly feasible in polynomial

[22] One may also wish to consult Caragiannis et al. (2009) and Betzler et al. (2010) for the complexity of determining the winner in these voting systems.

[23] The CSD system corresponds to the system denoted V by Caragiannis et al. (2014) (this paper uses a different notation for the number of voters). The authors note that CSD approximates the Dodgson score up to a multiplicative factor equal to the number of candidates. For an odd number of voters, CSD is equivalent to the *simplified Dodgson rule* of Tideman (2006). Thanks to Jérôme Lang for pointing out these antecedents.

time. On the other hand, if one wants to compute the set of all possible winners for all tie-breaking rules, then the problem is $\mathcal{NP}$-complete (Brill and Fischer, 2012).

***Schulze's method* (Sch.)** Consider the capacity graph defined by the weighted majority matrix: for each pair of candidates (c, d), there is an edge whose weight is D_{cd}.[24] It is advisable to think of an edge as a one-way pipe from c to d and the weight as the width of this pipe. We define the width of a path as the weight of the minimum edge of this path. We denote $\text{score}(c, d)$ the width of the widest path from candidate c to candidate d. Candidate c is said to be *better* than d iff $\text{score}(c, d) \geq \text{score}(d, c)$. Candidate c is a *potential winner* iff no candidate d is better than c.

Schulze (2011) proves that the set of potential winners is always nonempty. It is easy to see that if there is a Condorcet winner, then she is the only potential winner.

In Schulze's method as promoted by its inventor, the winner is drawn at random from the potential winners. In this dissertation, however, we will consider that the tie-breaking rule is deterministic.

[24] It is also possible to take the weights $D_{cd} - D_{dc}$. Since we are considering strict total orders here, we have $D_{cd} - D_{dc} = 2D_{cd} - V$, which is equivalent to D_{cd} after multiplication by the positive constant 2 and subtraction of the constant V. For this reason, the two definitions lead to the same voting system.

Chapter 2

Condorcification

Some authors, such as Chamberlin et al. (1984), Smith (1999), Favardin et al. (2002), Lepelley and Valognes (2003), Favardin and Lepelley (2006), or Tideman (2006), have expressed the intuition that the voting systems which satisfy the Condorcet criterion have a general tendency to be less manipulable than the others. In this chapter, we examine this aspect and we show that, if clarified, this intuition is justified. We take up and develop here the work presented by Durand et al. (2012, 2014b,d).

First (Section 2.1), we define the *informed majority coalition criterion* (**InfMC**), which will be used as a hypothesis for the main theorems of this chapter. Since this criterion is satisfied by most classical voting systems, with the notable exception of Veto, this will give some generality to our results.

In Section 2.2, we present a series of simple lemmas that we will often use later, especially to prove the theorems of this chapter. They make it possible to establish links between manipulability and the results of electoral duels, before and after manipulation. These results are well known in the literature dedicated to manipulation. We simply adapt them to the general case where the binary relations of preference are arbitrary.

Then, we define the *Condorcification* of a voting system (Section 2.3): in the new system, we simply add a preliminary test on the existence of a Condorcet winner and, in this case, she is declared the winner; otherwise, the original voting system is used. We thus systematize the process used to define the method of Black (1958) from that of Borda.

This hybrid system could be seen as an artificial construction and, as such, it is not obvious, *a priori*, that it has good properties. But, surprisingly enough, we show in Section 2.4 that, if a voting system satisfies the common criterion **InfMC**, then its Condorcification is at most as manipulable as the original system: this is the *weak Condorcification theorem* (Theorem 2.9).

In the general case, this theorem assumes that the Condorcification of a voting system is achieved using the notion of absolute Condorcet winner. We examine what happens with less demanding notions, such as the relative Condorcet winner (Section 2.5). We show that, even if these Condorcification variants do not decrease manipulability in the general case, the one based on the relative Condorcet winner does decrease manipulability for Plurality and ITR, but not for IRV or Approval voting for example.

Our goal, then, is to examine whether Condorcification *strictly* reduces manipulability. To this end, we define the notion of *resistant Condorcet winner*, a candidate who, for each pair of other candidates, has a majority of voters who simultaneously prefer her to both members of the pair (Section 2.6). We show that this definition is equivalent to the following property: in any voting system respecting the Condorcet criterion, this candidate is elected and the configuration is non-manipulable.

Using the notion of resistant Condorcet winner, we show in Section 2.7 that for all classical voting systems except Veto, their Condorcification is *strictly* less manipulable than the original voting system: this is the *strong Condorcification theorem* (Theorem 2.20).

In Section 2.8, we exploit the notion of resistant Condorcet winner to give an upper bound of manipulability for Condorcet systems, and we show that, in the electoral space of strict total orders with $C \geq 6$, this upper bound is reached.

Finally, in Section 2.9, we underline an important consequence of the Condorcification theorems: the search for a voting system with minimal manipulability (within **InfMC**) can be restricted to Condorcet systems.

Independently of our research, Green-Armytage et al. (2014) also introduced, in a research report available online, the informed majority coalition criterion under the name of *conditional majority determination* (CMD) and stated a version of the weak Condorcification theorem. However, on the one hand, we allow ourselves to claim anteriority (Durand et al., 2012). On the other hand, their proof is not correct if the preference relations are not strict total orders, an assumption which is however absent from their formulation. Moreover, when the preferences are not strict total orders, it is not clear which notion of Condorcet winner they are using. The most natural interpretation is to understand it as the relative Condorcet winner,[1] but we will see in Section 2.5 that the result is then wrong. In defense of the authors, the central object of their report is not Condorcification but the study by simulation of the manipulability of various voting systems for $C = 3$ candidates. We had cordial and rewarding exchanges with the authors of this report in June 2014, but, to our knowledge, the version of their report that is online at the time of writing this dissertation has not yet included these corrections and the reference to our work. Finally, we go further than this result in Section 2.5 and the following ones, in particular by examining Condorcification variants and by proving the strong version of the theorem.

2.1 Informed majority coalition criterion (InfMC)

We have already presented the Condorcet criterion (Definition 1.32). We now present a weaker criterion, the *informed majority coalition criterion* (**InfMC**).

Definition 2.1 (informed majority coalition criterion)

Let f be an SBVS.

We say that f satisfies the *informed majority coalition criterion* (**InfMC**) iff any majority coalition that is informed about what other voters are doing can decide the outcome.[2] Formally, $\forall M \in \mathcal{P}(\mathcal{V})$, if $\text{card}(M) > \frac{V}{2}$, then: $\forall c \in \mathcal{C}$, $\forall \omega_{\mathcal{V}\setminus M} \in \Omega_{\mathcal{V}\setminus M}$, $\exists \omega_M \in \Omega_M$ s.t. $f(\omega_M, \omega_{\mathcal{V}\setminus M}) = c$.

Like for the Condorcet criterion, the notation **InfMC** designates indifferently the criterion itself or the set of SBVS (on Ω) that satisfy it.

[1] Their definition is: “A *Condorcet winner* is a candidate who, according to the ballot data, would win a two-person race against any other candidate.”

[2] This notion is based on the same idea as what Peleg (1984) calls the *third simple game* associated with a voting system.

Proposition 2.2
If the electoral space allows any candidate as most liked (which is a common assumption, cf. Definition 1.10), then **Cond** $\subseteq$ **InfMC**.

Proof. If a voting system satisfies the Condorcet criterion, it is sufficient for a majority coalition to claim to prefer a candidate c over all the other candidates for c to appear as a Condorcet winner and to be elected. □

It is easy to extend the definitions of **InfMC** and **Cond** to a general voting system (Section 1.4). In this case, one could see that **InfMC** is a property of the *game form*, expressing the power granted by the counting rule to a strict majority of voters, while **Cond** is a property of the *voting system*, since it gives a relation between the preferences and the winner of sincere voting. Therefore, a game form can be said to satisfy **InfMC** without it being crucial to specify the electoral space. But, in order to define **Cond**, it is necessary to explicitly define the electoral space and the sincerity functions.

Proposition 2.3
The following voting systems satisfy **InfMC**: *Approval voting, IB, IRV, IRVA, ITR, Majority judgment, Plurality, Range voting, the methods of Borda, Bucklin, and Coombs, and all Condorcet systems.*

This property is not difficult to prove, but we will wait until Chapter 3 to include this result in more detailed propositions about each of these voting systems.

We will also see in Chapter 3 that Veto is one of the few often-studied voting systems that does not satisfy **InfMC**.[3] However, it is hardly used in practice.

Among common voting systems, it is interesting to note that even those whose usual arguments do not rely on the notion of majority (AV, MJ, or RV for instance) satisfy **InfMC**, which is clearly a majority property.

In practice, these observations give a wide range of applications to the results of this chapter. From a theoretical point of view, one can wonder whether there is a deep reason why most classical voting systems satisfy **InfMC**.[4] It seems to us that this question is an interesting issue for future work, whether with a hard science or human science approach. We will also come back to this in Chapter 3, where we will study more generally some criteria related to the notion of majority.

2.2 Links between manipulability and results in the electoral duels

Now we present a series of very simple lemmas that relate manipulability to the outcome of electoral duels, before and after manipulation. The proofs follow immediately from the definitions and are omitted. All of these results are part of standard gymnastics in the manipulation literature. We simply adapt them to the general case where binary preference relations are not necessarily strict total orders.

Lemma 2.4
Let $(\omega, \psi) \in \Omega^2$. *Let* $\mathrm{w} = f(\omega)$ *and* $c = f(\psi)$. *Assume that* f *is manipulable in* ω *to* ψ.

[3] More generally, Proposition 3.13 will give a necessary condition and a sufficient condition for an PSR to satisfy **InfMC**.

[4] At least, this is the case for voting systems used in practice that treat voters and candidates symmetrically, except in cases of a tie. We will discuss some deliberately asymmetric voting systems in Chapter 4.

The duel result of c versus w cannot be better in ψ than in ω, i.e. we have:

$$D_{\mathrm{cw}}(\psi) \leq D_{\mathrm{cw}}(\omega).$$

Therefore, if w is Condorcet-admissible in ω, then c cannot have a victory against w in ψ; in particular, c is not a Condorcet winner in ψ.

Note that for Lemma 2.4, no assumption is made on the voting system f: in particular, it is not necessary that it satisfies **InfMC** or **Cond**. Moreover, this lemma immediately leads to the following one.

Lemma 2.5
If the voting system meets the Condorcet criterion, then a Condorcet configuration cannot be manipulated to another Condorcet configuration.

On the other hand, an admissible configuration ω can be manipulated to a Condorcet configuration ψ, but only if a non-admissible candidate is elected in ω.
Lemmas 2.4 and 2.5 generalize a classical result of Moulin (1978, Chapter I, Theorem 1). Contrary to what is written here and there, this result absolutely does not prevent a Condorcet configuration from being manipulable.

In the same spirit, we have the following lemma, which applies when preferences are antisymmetric (which is a common assumption) and which focuses on the result D_{wc} of the duel of w against c (instead of D_{cw} in Lemma 2.4).
This result will not be needed to prove the Condorcification theorems 2.9 and 2.20, but it would have been a shame to state Lemma 2.4 without taking the opportunity to mention Lemma 2.6, which is extremely similar and which we will also use on several occasions.

Lemma 2.6
Let $(\omega, \psi) \in \Omega^2$. Let $\mathrm{w} = f(\omega)$ and $c = f(\psi)$. Assume that f is manipulable in ω to ψ. It is further assumed that the binary relations of preference are antisymmetric in ω.
The result of the duel of w against c cannot be worse in ψ than in ω, i.e.:

$$D_{\mathrm{wc}}(\psi) \geq D_{\mathrm{wc}}(\omega).$$

Therefore, if w is the Condorcet winner in ω, then c keeps a defeat against w in ψ; in particular, c is not Condorcet-admissible in ψ.

Finally, here is the other lemma that will be used to prove the Condorcification theorems. Its proof is just as immediate.

Lemma 2.7
Let $\omega \in \Omega$. Suppose that f satisfies **InfMC**.
If $f(\omega)$ is not Condorcet-admissible in ω, then f is manipulable in ω.

An immediate consequence of this lemma is that, if a configuration ω is non-admissible, then any voting system f satisfying **InfMC** is manipulable in ω.

2.3 Definition of Condorcification

The idea of Condorcification goes back to Black (1958), who applied it to Borda's method to define the rule that now bears his name. However, it seems that, since then, this principle has neither been exploited for other voting methods nor studied in general before the works of Green-Armytage et al. (2014) and

Durand et al. (2012). It seems to us that this is because this procedure, producing hybrid voting rules that seem artificial and perhaps lack mathematical elegance, did not have any known good property until now. We hope to remedy this in this chapter.

Definition 2.8 (Condorcification)
We call the following SBVS the *absolute Condorcification of* f, or simply *Condorcification of* f:

$$f^* : \left| \begin{array}{lll} \Omega & \to & \mathcal{C} \\ \omega & \to & \left| \begin{array}{l} \text{if } \omega \text{ has an absolute Condorcet winner } c, \text{then } c, \\ \text{otherwise, } f(\omega). \end{array} \right. \end{array} \right.$$

By definition, f^* satisfies the Condorcet criterion. Therefore, if we make the common assumption that the electoral space allows any candidate as most liked, then f^* satisfies **InfMC** (Proposition 2.2).

In order to understand how this definition applies in the case of a cardinal voting system, let us examine the Condorcification of Range voting, in the reference electoral space where $\omega_v = (p_v, u_v, a_v)$.

- Each voter communicates a state.
- If there is a Condorcet winner (calculated with the communicated relations p_v), then she is elected.
- Otherwise, the candidate with the maximum average score (computed with the communicated score vectors u_v) is elected.

To implement this system in practice, the ballot now need to include not only scores (as with the original Range voting, cf. Sections 1.2.1 and 1.4.3) but also orders of preference. However, if the social planner has adopted the assumption that $c\, p_v\, d \Leftrightarrow u_v(c) > u_v(d)$ (which we mentioned in Section 1.1.3), then it is sufficient to ask for the scores, since the orders of preference can then be immediately deduced.

For each voting system, we will prefix its name with "Condorcet" and its abbreviation with the letter C to denote its Condorcification. For example, the Condorcification of Bucklin's method will be called Condorcet-Bucklin and abbreviated as CBuck. We will also study in depth the Condorcification of IRV, denoted CIRV.

In fact, we already used this convention implicitly to define the Condorcet-dean system from a voting system with a constant outcome (which always elects the same candidate, called the *dean*) and to define the Condorcet-dictatorship system from a dictatorial voting system.

2.4 Weak Condorcification theorem

The following theorem shows an important advantage of Condorcification: for a system satisfying **InfMC**, its Condorcification is at most as manipulable as the original system.

Theorem 2.9 (weak Condorcification theorem)
Let f be an SBVS. Assume that f satisfies **InfMC**.
Then its Condorcification f^* is at most as manipulable as f:

$$\mathrm{CM}_{f^*} \subseteq \mathrm{CM}_f.$$

To give the intuition of the proof, we first recall that the configurations ω are of three types.

Condorcet There is a Condorcet winner.

Semi-Condorcet There is at least one Condorcet-admissible candidate but no Condorcet winner.

Non-admissible No candidate is Condorcet-admissible.

If the voting system satisfies **InfMC**, we will prove that, when a configuration is Condorcet, then one cannot worsen the manipulability of the system by choosing this one as the winner of the election. Indeed, on the one hand, we will see that this cannot worsen the manipulability of the considered configuration itself: if the winner was different in the original system, then the configuration was manipulable in favor of the Condorcet winner (this will be a consequence of Lemma 2.7), so we cannot worsen the situation. On the other hand, this cannot make another Condorcet configuration manipulable (Lemma 2.5, which follows from Lemma 2.4); it can only make a semi-Condorcet configuration manipulable if it was already manipulable in the first place (Lemma 2.4); and non-admissible configurations are doomed to be manipulable anyway (Lemma 2.7).

Proof. Suppose that f^* is manipulable in ω to ψ, but that f is not manipulable in ω.

Since f is not manipulable in ω, Lemma 2.7 ensures that $f(\omega)$ is Condorcet-admissible in ω. If she is a Condorcet winner in ω, then $f^*(\omega) = f(\omega)$. Otherwise, there is no Condorcet winner in ω (Proposition 1.31) so, by definition of f^*, we also have $f^*(\omega) = f(\omega)$.

Now let $\mathrm{w} = f^*(\omega) = f(\omega)$ and $c = f^*(\psi)$. Since w is Condorcet-admissible in ω, Lemma 2.4 (applied to f^*) ensures that c is not Condorcet-winner in ψ. So, by definition of f^*, we have $f^*(\psi) = f(\psi)$.

Therefore, we have $f(\omega) = f^*(\omega)$ and $f(\psi) = f^*(\psi)$, so f is manipulable in ω: a contradiction. $\square$

The demonstration of Green-Armytage et al. (2014) is articulated in 5 numbered points. At the end of point 2, the authors establish that there is no candidate B such that a strict absolute majority of voters prefers B to some candidate A; that is, candidate A is Condorcet-admissible, in our terminology. At point 3, they deduce that no candidate B distinct from A can be a (seemingly relative) Condorcet-winner.

However, in general, this implication is not correct: indeed, a candidate B can be a relative Condorcet winner by being preferred to A by 45 % of the voters while A is preferred to B by 40 % of the voters. In order for their demonstration to be correct, we must either restrict it to strict total orders (which seems implicitly to be their case), or use the notion of absolute Condorcet winner.

In fact, we will show in Section 2.5 that, if one performs the Condorcification using the notion of relative Condorcet winner, then not only does their proof become incorrect, but the Condorcification theorem itself becomes false in the general case: for some voting systems, it is possible to make some configurations manipulable whereas they were not so initially.

A classical objection to Condorcet voting systems is that they all suffer from the *no-show paradox* (Moulin, 1988; Pérez, 2001): there is at least one configuration where a voter can manipulate by abstention.[5] However, the weak Condorcification theorem 2.9 nuances this criticism: when comparing a system f and its Condorcification f^*, if we are in a configuration where manipulators can exploit strategic abstention in f^*, it means that the original voting system f is manipulable anyway (not necessarily by abstention). To see this, consider an electoral space where abstention is a possible sincere state and therefore an available ballot in an SBVS.[6]

2.5 Condorcification variants

Now we show that the weak Condorcification theorem 2.9 is sharp in the following sense: it does not generalize when using an extended version of Condorcification involving relative or weak Condorcet winners, or Condorcet-admissible candidates.

2.5.1 Definition of the Condorcification variants

Since the beginning of this chapter, we focused on the notion of absolute Condorcet winner, which is based on absolute victories: in the duel of such a candidate c against any opponent d, we have $|c \mathrm{P}_v d| > \frac{V}{2}$ and $|d \mathrm{P}_v c| \leq \frac{V}{2}$, these two conditions boiling down to the first one when the preferences are antisymmetric (which is a common assumption).

Instead, we could have considered weaker variants of this notion, whose definitions can be informally recalled.

Relative Condorcet winner $|c \mathrm{P}_v d| > |d \mathrm{P}_v c|$.

Weak Condorcet winner $|c \mathrm{P}_v d| \geq |d \mathrm{P}_v c|$.

Condorcet-admissible candidate $|d \mathrm{P}_v c| \leq \frac{V}{2}$.

For each SBVS f, in the same way that we defined its Condorcification f^*, we define the following voting systems.

f^{rel} : If there is a relative Condorcet winner, then she is elected, otherwise we apply f. This system is called the *relative Condorcification* of f.

$f^{!\text{weak}}$: If there is a unique weak Condorcet winner, then she is elected, otherwise we apply f.

f^{weak} : If there is at least one weak Condorcet winner, then one is chosen arbitrarily, otherwise we apply f.

[5] Many other voting systems suffer from this paradox, for example the IPSR-1. But in these, a situation that can be manipulated by abstention is always also manipulable by using strategic ballots that are not abstentions. On the other hand, in the Kim-Roush method for example, there are profiles which can only be manipulated by abstention. For more details, see Lepelley and Merlin (2001).

[6] For example, we can consider an electoral space where the state of a voter is a pair made up of a preference relation on the candidates and a Boolean variable which represents whether she abstains or not. Another possibility consists in considering an electoral space where the set of possible states of a voter is the set of strict total orders, to which we add the empty binary relation (perfect indifference), which represents sincere abstention but which can can also be used as a strategic behavior. Finally, it is possible to assume that sincere abstention does not exist but that strategic abstention is possible. For this, it is necessary to use a general voting system (and not an SBVS) where abstention is one of the authorized strategies.

$f^{!\text{adm}}$: If there is a unique Condorcet-admissible candidate, then she is elected, otherwise we apply f.

f^{adm} : If there is at least one Condorcet-admissible candidate, then we choose one arbitrarily, otherwise we apply f.

In the case of f^{weak} or f^{adm}, this does not define an SBVS *per se* but rather a class of SBVS, since the definition leaves arbitrary choices to be made. This notation shortcut will not be a problem in practice because the results that we prove are true for each SBVS meeting this definition, regardless of the choices made.

We will examine whether any of these notions leads to a result that is similar to the weak Condorcification theorem 2.9. However, we will see that this is not the case.

At all times in this section, we consider an electoral space where the set of possible binary relations of preference for each voter is equal to the set of strict weak orders (in order to avoid, for example, that the notions of absolute and relative Condorcet winner are identical, which would preclude revealing which notion makes the theorem valid). In particular, we consider a favorable case where the binary relations of preference are antisymmetric, which simplifies reasoning (whereas we recall that this assumption is not necessary for the weak Condorcification theorem 2.9).

2.5.2 Condorcification variants: general result

Let us start with the example of a voting system which is, admittedly, a bit artificial, but which makes it possible to prove in a rather concise way that none of the above notions leads to generalizing the weak Condorcification theorem 2.9.

Proposition 2.10
Let f be Condorcet-dean.

There exists at least one value of (V, C) such that for each voting system g chosen among the variants f^{rel}, $f^{!weak}$, f^{weak}, $f^{!adm}$, or f^{adm}, it does not hold that $\text{CM}_g \subseteq \text{CM}_f$.

Proof. Consider $V = 5$ voters, $C = 3$ candidates who are denoted a, b, c. Assume that the dean is a. Consider the following configuration ω and its weighted majority matrix $D(\omega)$.

a	a, c	b	b	c
b	b	a, c	c	a
c			a	b

$D(\omega)$	a	b	c
a	–	3	1
b	2	–	3
c	2	2	–

There is no absolute Condorcet winner: indeed, a candidate should have at least 3 points in each duel of her row in $D(\omega)$, which is not the case. So the dean is elected: $f(\omega) = a$. It is easy to see that f is not manipulable in ω: if it were the case, according to Lemma 2.5, it would be to a configuration without Condorcet winner. But, in all such configurations, a is elected, hence the manipulation fails.

The candidates b and c are not even Condorcet-admissible (each of them has an absolute defeat in her column). Therefore, whatever the variant g studied, we have $g(\omega) = a$.

Now consider the following configuration ψ, where the last two voters try to make c win instead of a.

a	a, c	b	$\mathbf{c}$	c
b	b	a, c	$\mathbf{b}$	$\mathbf{b}$
c			a	$\mathbf{a}$

$D(\psi)$	a	b	c
a	–	2	1
b	3	–	2
c	2	3	–

Now c is a relative Condorcet winner, hence also a weak Condorcet winner and a Condorcet-admissible candidate. Candidates a and b are not even Condorcet-admissible (each of them has a defeat in her column). Therefore, whatever variant g is used, we have $g(\psi) = c$.

In conclusion, g is manipulable in ω to ψ in favor of c, while f is not manipulable in ω. □

This proposition explains why the Condorcification that we used in the weak Condorcification theorem 2.9 relies on the notion of absolute Condorcet winner. Indeed, if we keep the same assumption $f \in$ **InfMC** (which is trivially satisfied by Condorcet-dean), then the theorem does not generalize to relative Condorcet winners, weak Condorcet winners, or Condorcet-admissible candidates.

To generalize the theorem using the notion of relative Condorcet winner, weak Condorcet winner, or Condorcet-admissible candidate, another option would consist in replacing the assumption $f \in$ **InfMC** by another one. This said, for the obtained theorem to have a domain of validity as interesting as Theorem 2.9, it would have to be at least applicable to usual voting systems, such as Plurality, ITR, IRV, or Approval voting.

We will therefore examine these particular voting systems. We will see that in general, one cannot have such a generalization of Theorem 2.9; but that for some of these systems, the relative Condorcification f^{rel} may be less manipulable than the original system f, and even than its Condorcification f^*.

2.5.3 Condorcification variants for Plurality

First, we clarify how we generalize Plurality, ITR, and IRV if voters can have strict weak orders of preference. We consider the two rules that we presented in Section 1.6.2: if a voter has several equally preferred candidates (among the non-eliminated candidates), we can decide that her vote is divided equally among them, or that it is not counted at all. The following results are valid in both cases.

Proposition 2.11

Let f be Plurality.

1. *We have:* $\text{CM}_{f^{rel}} \subseteq \text{CM}_{f^*}$.
2. *There exists (V, C) such that the above inclusion is strict.*
3. *There exists (V, C) such that for each voting system g chosen among the variants $f^{!weak}$, f^{weak}, $f^{!adm}$, or f^{adm}, it does not hold that* $\text{CM}_g \subseteq \text{CM}_f$.

In order to prove point 1, we first prove the following lemma, which we will then apply to $h = f^*$.

Lemma 2.12

Let h be an SBVS. Assume that, for every configuration ω and every candidate $d \neq h(\omega)$, if $h(\omega)$ has a relative defeat against d, then h is manipulable in ω.

Then $\text{CM}_{h^{rel}} \subseteq \text{CM}_h$.

Proof. Suppose that there exists a configuration ω where h is not manipulable but where h^{rel} is manipulable to some configuration ψ.

Let $c = h(\omega)$. Since h is not manipulable in ω, the assumption of the lemma implies that c has no relative defeat, so no other candidate can be a relative Condorcet winner. Therefore, $h^{\text{rel}}(\omega) = c$.

Let $d = h^{\text{rel}}(\psi)$. Since the relative result of the duel of d against c cannot have been improved by manipulation (Lemmas 2.4 and 2.6), d still has no relative victory against c, so she is not a relative Condorcet winner. By definition of h^{rel}, we deduce that $h(\psi) = d$.

Thus, $h(\omega) = h^{\text{rel}}(\omega) = c$ and $h(\psi) = h^{\text{rel}}(\psi) = d$, so h is manipulable from ω to ψ: a contradiction. □

We can now prove Proposition 2.11.

Proof. 1. We consider applying Lemma 2.12 to $h = f^*$ and $h^{\text{rel}} = (f^*)^{\text{rel}} = f^{\text{rel}}$. It is sufficient to show that $h = f^*$ satisfies the hypothesis of the lemma to conclude that $\text{CM}_{f^{\text{rel}}} \subseteq \text{CM}_{f^*}$.

Let ω be a configuration, $c = f^*(\omega)$ and d another candidate. Suppose that c has a relative defeat against d. Our goal is then to show that f^* is manipulable in ω. Since c has a relative defeat, she is not an absolute Condorcet winner. And since she is a winner by f^*, no other candidate is an absolute Condorcet winner in ω.

Define ψ as follows: all the voters who preferred d to c now say they prefer d to all the other candidates, without changing their other preferences.

So now d is the winner in Plurality. Indeed, we can reason by excluding the voters who place c and d equally on top: depending on the Plurality generalization chosen, they split their votes equally between c and d or give them no point. Except for these voters, we have $\text{score}(d) = |d \text{ P}_v\ c| > |c \text{ P}_v\ d| \geq \text{score}(c)$. Now the score of c has not decreased with respect to the sincere vote, so $f(\omega) = d$.

Moreover, the only duels whose scores have improved are those of d, so no other candidate can be an absolute Condorcet winner. Therefore $f^*(\psi) = d$. We conclude that f^* is manipulable in ω to ψ in favor of d.

2. We will exhibit a configuration where f^{rel} is not manipulable but where f^* is. Consider the following configuration ω.

24	19	19	19	19
a	d_1	d_2	d_3	d_4
c	c	c	a, c	a, c
Others	Others	Others	Others	Others

In Plurality, we have $f(\omega) = a$. Moreover, c has a relative victory against a (38 votes against 24) and an absolute victory against each d_i (81 votes against 19), hence she is a relative but not an absolute Condorcet winner. Therefore, $f^*(\omega) = a$ and $f^{\text{rel}}(\omega) = c$.

Let us show that f^{rel} is not manipulable in ω. It is impossible for the new winner to be a relative Condorcet winner (since she cannot improve her duel against c), hence she must be a Plurality winner and there must be no relative Condorcet winner left. Only a can become (in fact, remain) a Plurality winner. But, in case of manipulation for a, we cannot avoid that c remains a relative Condorcet winner, so the manipulation fails.

It remains to show that f^* is manipulable in ω. Consider the following configuration ψ, which is an attempt to manipulate in favor of c.

24	19	19	19	19
a	**c**	**c**	d_3	d_4
c			a, c	a, c
Others	Others	Others	Others	Others

In Plurality, we have $f(\omega) = c$. Moreover, c still is a relative Condorcet winner, so no other candidate is an absolute Condorcet winner and we have $f^*(\omega) = c$. Thus, f^* is manipulable in ω to ψ in favor of c.

3. First, we show that for $g = f^{!\mathrm{adm}}$ or $g = f^{\mathrm{adm}}$, we can exhibit a configuration where g is manipulable, but f is not. Consider the following configuration ω.

4	3	2
a	b	d
d	a, c	c
c	d	a, b
b		

$D(\omega)$	a	b	c	d
a	–	4	4	7
b	3	–	3	3
c	2	6	–	3
d	2	6	6	–

In Plurality, candidate a is elected and it is easy to see that this is not manipulable. Moreover, since a is a relative Condorcet winner, she remains elected whatever the variant g used.

Now consider the following configuration ψ, where the last 2 voters have changed their ballots in an attempt to make c win.

4	3	2
a	b	**c**
d	a, c	**d**
c	d	**b**
b		**a**

$D(\omega)$	a	b	c	d
a	–	4	4	7
b	5	–	3	3
c	2	6	–	5
d	2	6	4	–

Here, c is the only Condorcet-admissible candidate (every other candidate has an absolute defeat in her column), hence we have $g(\psi) = c$, both for $g = f^{!\mathrm{adm}}$ and for $g = f^{\mathrm{adm}}$.

Therefore, f^{adm} and $f^{!\mathrm{adm}}$ are manipulable in ω to ψ in favor of c, while f is not manipulable in ω.

In order to show the result for $f^{!\mathrm{weak}}$ and f^{weak}, since we will resort to ties, we must specify the tie-breaking rule.[7] We can decide, for example, that in case of equality in f, the candidates are ranked by alphabetical order; or by the

[7] It is a necessary evil here. Indeed, in all generality, we want to exhibit a configuration ω where f is not manipulable but where g is manipulable to a certain configuration ψ. Let $a = f(\omega)$.

If a candidate $b \neq a$ is a weak Condorcet winner, consider the configuration ϕ where her supporters use the trivial strategy (b on top): then b receives at least as many votes as a. But as f is supposed to be non-manipulable in ω, this requires to specify the tie-breaking rule (favorable to a in ϕ).

Conversely, suppose that no candidate different from a is a weak Condorcet winner. Then $g(\omega) = a$. For the counterexample to work, then, noting $c = g(\psi)$, we must have $c \neq f(\psi)$. So c is a weak Condorcet winner in ψ. As the manipulation cannot have improved her duel against a, she already had a relative non-defeat against a in ω. Now consider the situation χ where voters preferring c to a use the trivial strategy (c on top). Then c has at least as many votes as a in f. But since f is supposed to be non-manipulable in ω, this requires to specify the tie-breaking rule (favorable to a in χ).

order given by the ballot of a voter chosen in advance, even if it means using a complementary rule if her ballot is a weak order.

Let us redefine the configurations ω and ψ given above, but with 3 voters in each group. If we have chosen to break the ties by a preferred voter, we consider the case where she belongs to the first group, i.e. the supporters of a. If we have chosen to break the tie by alphabetical order, no precaution is necessary, since a is favored anyway. By the same reasoning as before, we have $f(\omega) = a$ and this is not manipulable. But $g(\omega) = a$ and $g(\psi) = c$ (since c is the only weak Condorcet winner in ψ), so g is manipulable in ω. □

For Plurality, we thus saw that the weak Condorcification theorem 2.9 does not generalize to weak Condorcet winners or Condorcet-admissible candidates. On the other hand, it works for the relative Condorcet winner, and even better: the relative Condorcification is less manipulable than the absolute Condorcification in that case.

2.5.4 Condorcification variants for ITR

Since the relative Condorcification works for Plurality, one may wonder if the Instant two-round system (ITR) is able to provide us with a counterexample of a possible generalization of Theorem 2.9 based on the notion of relative Condorcet winner. However, we will see that this is not the case: indeed, the relative Condorcification also works for ITR.

Proposition 2.13
Let f be ITR.

1. *We have:* $\mathrm{CM}_{f^{rel}} \subseteq \mathrm{CM}_{f^*}$.
2. *There exists (V, C) such that the above inclusion is strict.*
3. *There exists (V, C) such that for each voting system g chosen among the variants $f^{!weak}$, f^{weak}, $f^{!adm}$, or f^{adm}, it does not hold that $\mathrm{CM}_g \subseteq \mathrm{CM}_f$.*

Proof. 1. Suppose there exists a configuration ω where f^* is non-manipulable but f^{rel} is manipulable to some configuration ψ. Let $a = f^*(\omega)$.

Case 1 Suppose $f(\omega) \neq a$.
By definition of f^*, this means that a is the absolute Condorcet winner, hence the relative Condorcet winner (Proposition 1.30), which leads to $f^{\text{rel}}(\omega) = f^*(\omega) = a$. Let $d = f^{\text{rel}}(\psi)$. It must be the case that $d = f^*(\psi)$, otherwise f^* would also be manipulable in ω. Therefore, d is the relative (but not absolute) Condorcet winner in ψ. But a was the relative Condorcet winner in ω and the manipulation cannot have improved the result of the duel of d against a: a contradiction. It is worth noting that this case 1 does not use any particular property of ITR: it applies in the same way to any other voting system f.

Case 2 Suppose $f(\omega) = a$.
To show that we also have $f^{\text{rel}}(\omega) = a$, we only need to show that no other candidate can be a relative Condorcet winner. Denote b the opponent of a in the second round of f. We know that she is not a relative Condorcet winner, otherwise she would win the second round against a. Suppose that a candidate $c \notin \{a, b\}$ is the relative Condorcet winner in ω. Consider then the situation ϕ where each voter preferring c to a tries to manipulate for c by using the *compromise strategy* (put c in the first position, without changing the rest of her ballot).

- If we use f, candidate c is selected for the second round. Indeed, we can reason by excluding the voters who place a and c tied on top: depending on the generalization of ITR chosen, they distribute their votes equally between a and c or give them no point at all. Except for these voters, we have $\text{score}(c) = |c \text{ P}_v\ a| > |a \text{ P}_v\ c| \geq \text{score}(a)$. Now the score of a has not decreased compared to the sincere vote, hence c is selected for the second round.

- Only the duels of c were improved compared to configuration ω, so c remains the relative Condorcet winner. Therefore, she wins the second round, which leads to $f(\phi) = c$. Moreover, no other candidate can be the absolute Condorcet winner, hence $f^*(\phi) = c$.

- Therefore f^* is manipulable in ω to ϕ in favor of c, which is excluded.

Thus, we have: $a = f(\omega) = f^*(\omega) = f^{\text{rel}}(\omega)$.

Let $d = f^{\text{rel}}(\psi)$. It is necessary that $d = f^*(\psi)$, otherwise f^* would also be manipulable in ω (cf. case 1). Therefore, d is the relative (but not absolute) Condorcet winner in ψ. We deduce that d had a relative victory against a in ω: indeed, manipulation cannot have improved the result of this duel.

Now consider the configuration χ where the voters who prefer d to a use the compromise strategy (put d on top, without changing the rest of their ballot). By the same reasoning as above, we have $\text{score}(d) = |d \text{ P}_v\ a| > |a \text{ P}_v\ d| \geq \text{score}(a)$, so d is selected for the second round. Now, in χ, candidate d appears as the relative Condorcet winner: indeed, all her duels are at least as good as in ψ. So she wins the second round, which leads to $f(\chi) = d$. Since d is the relative Condorcet winner, no other candidate can be the absolute Condorcet winner and we have $f^*(\chi) = d$. In conclusion, f^* is manipulable in ω to χ in favor of d: a contradiction.

2. Consider the following configuration ω.

23	20	19	19	19
a	d_1	d_2	d_3	d_4
c	c	c	a, c	a, c
	a	a		
Others	Others	Others	Others	Others

In the second round, a faces d_1 and wins (80 votes against 20): we have $f(\omega) = a$.

Candidate c has a relative victory against a (39 votes against 23) and an absolute victory against any candidate d_i (by 80 or 81 votes). So c is the relative Condorcet winner, but not an absolute Condorcet winner. Consequently, $f^*(\omega) = a$ and $f^{\text{rel}}(\omega) = c$.

Let us show that f^{rel} is not manipulable in ω.

- In favor of a: candidate c remains the relative Condorcet winner, hence manipulation fails.

- In favor of d_i: it is easy to see that neither of them can win in ITR. Indeed, even with manipulation, only d_1 can be selected for the second round, but then she loses to a. Moreover, no d_i can become the relative Condorcet winner, because she will always have a relative (and even absolute) defeat against c.

Now consider the following configuration ψ, which is an attempt to manipulate f^* in favor of c.

23	20	19	19	19
a	$\mathbf{c}$	$\mathbf{c}$	d_3	d_4
c	$\mathbf{d_1}$	$\mathbf{d_2}$	a, c	a, c
	a	a		
Others	Others	Others	Others	Others

In ITR, it is easy to see that $f(\psi) = c$. Moreover, since the duels of c only improved, she remains the relative Condorcet winner and no other candidate is an absolute Condorcet winner. Hence $f^*(\psi) = c$.

Therefore, f^* is manipulable in ω to ψ in favor of c, while f^{rel} is not manipulable in ω.

3. We can use the same example as for the point 3 of Proposition 2.11. For $f^{!\text{weak}}$ or f^{weak}, here again, we consider 3 voters per group and it is necessary to specify the tie-breaking rule, for example alphabetical on the candidates. □

Thus, if f is the Instant two-round system (ITR), we have the same result as for Plurality: the voting system f^{rel} is less manipulable than f, and even than the Condorcification f^*.

2.5.5 Relative Condorcification for IRV

At this point, one might wonder whether the failure of relative Condorcification is limited to exotic voting systems like Condorcet-dean, since it works for Plurality and ITR. But IRV will stop us in this fortunate series: in this case, the relative Condorcification is not less manipulable than the original voting system.

Proposition 2.14
Let f be IRV.
There exists (V, C) such that it does not hold that $\text{CM}_{f^{\text{rel}}} \subseteq \text{CM}_f$.

Proof. Consider the following configuration ω.

12	11	25	12	12	28
a	a	b	b	c	d
b	d	a, c	c	a	a, c
c	c	d	a	d	b
d	b		d	b	

$D(\omega)$	a	b	c	d
a	–	63	23	72
b	37	–	49	49
c	24	51	–	61
d	28	51	39	–

In IRV, candidates c, d, and b are successively eliminated hence $f(\omega) = a$. We first show that f is not manipulable in ω.

- In favor of b: even if b made it to the last round, she would lose to any other candidate (by 63 or 51 votes).
- In favor of c: the fourth and fifth groups are interested ($12 + 12$ voters). For c not to be eliminated in the first round, 23 or 24 manipulators must put c at the top of their ballot and a must be eliminated (it is not possible to eliminate b or d). In the second round, since b and d each have more than a third of the votes (37 and 39 respectively), c is eliminated.

- In favor of d: only the sixth group (28 voters) is interested. In the first round, they cannot save both c and d: indeed, since a receives 23 votes, the manipulators and the voters who sincerely vote for c and d would have to total at least $2 \times 23 = 46$ votes, but they only have $12 + 28 = 40$ votes. Since d must remain in the race, c must be eliminated in the first round. In the second round, each of a and b has more than a third of the votes (35 and 37 respectively), so d is eliminated.

Since c is the relative Condorcet winner, $f^{\text{rel}}(\omega) = c$. Consider the following configuration ψ, which is an attempt to manipulate in favor of a.

12	11	25	12	12	28
a	a	b	b	c	d
b	d	a, c	c	a	a, c
d	**b**	d	a	d	b
c	**c**		d	b	

$D(\omega)$	a	b	c	d
a	–	63	23	72
b	37	–	60	49
c	24	40	–	49
d	28	51	51	–

From the point of view of IRV, the counting proceeds in the same way, leading to $f(\psi) = a$. And since the other candidates are not even Condorcet-admissible (they all have a defeat in their column), we also have $f^{\text{rel}}(\psi) = a$. Therefore, f^{rel} is manipulable in ω to ψ in favor of a. $\square$

2.5.6 Relative Condorcification for cardinal systems

We now show that the result we just saw for IRV is also valid for Approval voting, Range voting, and Majority judgment: for these systems, relative Condorcification does not decrease manipulability. In all three cases, the extreme scores allowed will be 0 and 1 by convention.

In a configuration where all scores are 0 and 1 (even if other scores are allowed), we will notice that these three voting systems return the same winner, i.e. the candidate who has the score 1 in the largest number of ballots (cf. Section 1.6.1). This will make it possible to deal with these three systems in a unified proof.

Proposition 2.15
Let f be Approval voting, Range voting, or Majority Judgment.
There exists (V, C) such that it does not hold that $\text{CM}_{f^{\text{rel}}} \subseteq \text{CM}_f$.

Proof. Consider the following configuration ω.

2	3	4
$a : 1$	$c : 1$	$b : 1$
$c : 0$	$a : 1$	$a, c : 0$
$b : 0$	$b : 0$	

$D(\omega)$	a	b	c
a	–	5	2
b	4	–	4
c	3	5	–

We have $\text{score}(a) = 5$, $\text{score}(b) = 4$, and $\text{score}(c) = 3$, hence $f(\omega) = a$. In the case of an attempt to manipulate for c, the interested voters (second group) can do no better than lower the score of a to 0, but in this case, b wins. As for a manipulation for b, the interested voters (third group) cannot do better than their current ballot. Therefore, f is not manipulable in ω.

Candidate c is the relative Condorcet winner, so $f^{\text{rel}}(\omega) = c$. Now consider the following configuration ψ, where the first group of voters is trying to make a win.

2	3	4
$a : 1$	$c : 1$	$b : 1$
$\mathbf{b} : 0$	$a : 1$	$a, c : 0$
$\mathbf{c} : 0$	$b : 0$	

$D(\omega)$	a	b	c
a	–	5	2
b	4	–	6
c	3	3	–

We have $f(\psi) = a$, and the other candidates are not even Condorcet-admissible (they all have a defeat in their column), hence $f^{\text{rel}}(\omega) = a$. Therefore, f^{rel} is manipulable in ω to ψ in favor of a, while f is not manipulable in ω. □

2.6 Resistant Condorcet winner

The weak Condorcification theorem 2.9 states that the Condorcification f^* is at most as manipulable as the original system f (if the latter satisfies **InfMC**). In the strong Condorcification theorem 2.20, we will show that, in fact, this decrease in manipulability is strict for many voting systems. In order to give us the tools to demonstrate this, we are now going to define the original notion of *resistant Condorcet winner* and we are going to prove that such a candidate is characterized by a form of immunity to manipulation.

Here is the intuition leading to the notion of *resistant Condorcet winner*. If f satisfies the Condorcet criterion and if there is a Condorcet winner c, then the manipulators in favor of another candidate d need to prevent c from appearing as the Condorcet winner. Therefore, they need to prevent a strict victory of c against some candidate $e \neq c$. However, this plan is doomed to failure if sincere voters (those who do not prefer d to c) already ensure: (1) a victory of c over e; and (2) a non-victory of e over c. This observation leads to the following definition.

Definition 2.16 (resistant Condorcet winner)
Let $\omega \in \Omega$ and $c \in \mathcal{C}$.
We say that c is *resistant Condorcet winner* in ω iff $\forall (d, e) \in (\mathcal{C} \setminus \{c\})^2$:

$$\begin{cases} |\text{not}(d \text{ P}_v\ c) \text{ and } c \text{ P}_v\ e| > \dfrac{V}{2}, & (2.1) \\ |\text{not}(d \text{ P}_v\ c) \text{ and not}(e \text{ P}_v\ c)| \geq \dfrac{V}{2}, & (2.2) \end{cases}$$

We say that a configuration ω is *resistant* iff it has a resistant Condorcet winner.

If all binary relations P_v are antisymmetric (which is a common assumption), then condition (2.2) becomes redundant. In this framework, c is a resistant Condorcet winner iff $\forall (d, e) \in (\mathcal{C} \setminus \{c\})^2$:

$$|\text{not}(d \text{ P}_v\ c) \text{ and } c \text{ P}_v\ e| > \frac{V}{2}.$$

If, in addition, all the binary relations P_v are complete, then the definition is even simpler since the above relation becomes symmetric with respect to d and e:

$$|c \text{ P}_v\ d \text{ and } c \text{ P}_v\ e| > \frac{V}{2}.$$

In other words, for each pair of other candidates (d, e), there is a strict majority of voters who simultaneously prefer c to d and c to e.

Clearly, a resistant Condorcet winner is also a Condorcet winner. Therefore, if there exists a resistant Condorcet winner, then she is unique.

We have chosen to define the resistant Condorcet winner in terms of the preferences of the voters in configuration ω. The following theorem formally takes up the property of immunity to manipulation that we mentioned to introduce the notion and shows that this property is characteristic (i.e. it is satisfied only by a resistant Condorcet winner), under the common assumption that the electoral space includes all strict total orders.

Theorem 2.17 (characterization of the resistant Condorcet winner) *Let $\omega \in \Omega$ and $c \in \mathcal{C}$. Consider the following conditions.*

1. *Candidate c is a resistant Condorcet winner in ω.*

2. *For each SBVS f meeting the Condorcet criterion, c is elected by the sincere vote and f is not manipulable in ω.*

We have the implication 1 $\Rightarrow$ 2.

If the electoral space includes all strict total orders (which is a common assumption, cf. Definition 1.10), then the converse 2 $\Rightarrow$ 1 is true.

Proof. 1 $\Rightarrow$ 2: since c is a resistant Condorcet winner, even after an attempted manipulation in favor of another candidate d, sincere voters ensure that c still has a strict victory over any candidate $e \neq c$; therefore, c still appears as the Condorcet winner and she is elected.

We now show the reciprocal 2 $\Rightarrow$ 1. Suppose that c is not a resistant Condorcet winner. If c is not a Condorcet winner, it is immediate: we necessarily have $C \geq 2$ (because there exists a candidate $d \neq c$ such that c does not have a strict victory against d) and so there exists a Condorcet system where c is not the winner by sincere voting. We can therefore focus on the case where c is a Condorcet winner (but not resistant). We will then prove that there exists an SBVS f respecting the Condorcet criterion and manipulable in ω.

Since c is not a resistant Condorcet winner, at least one of conditions (2.1) or (2.2) of Definition 2.16 is not satisfied. We distinguish three cases: condition (2.1) is not satisfied for some $e = d$; condition (2.2) is not satisfied; or condition (2.1) is not satisfied with $e \neq d$.

In each of these three cases, the principle of the proof is the same: we exhibit a configuration ψ which has no Condorcet winner and which differs from ω only by voters who prefer d to c. Therefore, it is possible to choose an SBVS f that satisfies the Condorcet criterion and such that $f(\psi) = d$. We deduce that f is manipulable in ω to ψ in favor of candidate d.

Case 1 If condition (2.1) is not satisfied for a certain $e = d$, it means that $|\text{not}(d \operatorname{P}_v c) \text{ and } c \operatorname{P}_v d| \leq \frac{V}{2}$. Let p_0 be a strict total order of the form: $(d \succ c \succ \text{other candidates})$. Since the electoral space includes all strict total orders, for each manipulator $v \in \text{Manip}(c \to d)$, we can choose a ballot ψ_v such that $\operatorname{P}_v(\psi_v) = p_0$. For each sincere voter $v \in \text{Sinc}(c \to d)$, we define $\psi_v = \omega_v$. In the new configuration ψ, candidate c cannot appear as a Condorcet winner because $|c \operatorname{P}_v(\psi_v)\ d| = |\text{not}(d \operatorname{P}_v(\omega_v)\ c) \text{ and } c \operatorname{P}_v(\omega_v)\ d| \leq \frac{V}{2}$. Candidate d cannot appear as a Condorcet winner (Lemma 2.4) and neither can any other candidate, because the number of voters who claim they prefer c to them has not decreased.

Case 2 If condition (2.2) is not satisfied for some d and some e, it means that $|\text{not}(d \mathrm{P}_v c) \text{ and not}(e \mathrm{P}_v c)| < \frac{V}{2}$. Note that $e \neq d$, because otherwise c would not be the Condorcet winner. Up to exchanging roles between d and e, we can suppose that e does not have a strict victory against d. Let p_0 be a strict total order of the form: ($d \succ e \succ c \succ$ other candidates). For each manipulator $v \in \mathrm{Manip}(c \to d)$, we can choose a ballot ψ_v such that $\mathrm{P}_v(\psi_v) = p_0$. For each sincere voter $v \in \mathrm{Sinc}(c \to d)$, we define $\psi_v = \omega_v$. In the new configuration ψ, candidate c cannot appear as the Condorcet winner because she has a defeat against e: indeed, $|\text{not}(e \mathrm{P}_v(\psi_v)\ c)| = |\text{not}(d \mathrm{P}_v(\omega_v)\ c) \text{ and not}(e \mathrm{P}_v(\omega_v)\ c)| < \frac{V}{2}$. Candidate d cannot appear as Condorcet winner (Lemma 2.4), neither can candidate e because she still does not have a strict victory against d, and neither does any other candidate, because the number of voters who say they prefer c to them has not decreased.

Case 3 There remains the case where condition (2.1) is not satisfied, with $e \neq d$. By denoting $B_{de} = |\text{not}(d \mathrm{P}_v c) \text{ and } c \mathrm{P}_v e|$, it means that $B_{de} \leq \frac{V}{2}$. Using the previous case, we can assume, however, that condition (2.2) is satisfied.

We will see that in the final configuration ψ, we can ensure that there is neither a victory of c against e, nor a victory of e against c.

Let p_0 be a strict total order of the form: ($d \succ e \succ c \succ$ other candidates).
Let p'_0 be a strict total order of the form: ($d \succ c \succ e \succ$ other candidates).
Since c is the Condorcet winner, we have $|c \mathrm{P}_v e| > \frac{V}{2}$, therefore:

$$|d \mathrm{P}_v c \text{ and } c \mathrm{P}_v e| > \frac{V}{2} - B_{de} \geq 0.$$

Therefore, we can choose $\lfloor \frac{V}{2} \rfloor - B_{de}$ voters among the manipulators (the voters who prefer d to c); for each of them, denoted v, let us choose ψ_v such that $\mathrm{P}_v(\psi_v) = p'_0$. For each other manipulator v, let us choose ψ_v such that $\mathrm{P}_v(\psi_v) = p_0$. Finally, for each sincere voter $v \in \mathrm{Sinc}(c \to d)$, let $\psi_v = \omega_v$.

Then we have:

$$D_{ce}(\psi) = B_{de} + \left(\left\lfloor \frac{V}{2} \right\rfloor - B_{de} \right) = \left\lfloor \frac{V}{2} \right\rfloor,$$

hence c has no victory against e.

Moreover, condition (2.1) is not satisfied for this pair (d, e) but condition (2.2) is satisfied. We thus have:

$$\begin{cases} |\text{not}(d \mathrm{P}_v c) \text{ and } c \mathrm{PP}_v e| + |\text{not}(d \mathrm{P}_v c) \text{ and } c \mathrm{MP}_v e| \leq \left\lfloor \frac{V}{2} \right\rfloor, \\ |\text{not}(d \mathrm{P}_v c) \text{ and } c \mathrm{PP}_v e| + |\text{not}(d \mathrm{P}_v c) \text{ and } c \mathrm{I}_v e| \quad \geq \left\lceil \frac{V}{2} \right\rceil, \end{cases}$$

therefore, by subtraction:

$$|\text{not}(d \mathrm{P}_v c) \text{ and } c \mathrm{MP}_v e| - |\text{not}(d \mathrm{P}_v c) \text{ and } c \mathrm{I}_v e| \leq \left\lfloor \frac{V}{2} \right\rfloor - \left\lceil \frac{V}{2} \right\rceil.$$

Using Proposition 1.25, we deduce:

$$\begin{aligned} D_{ec}(\psi) &= V + |c \mathrm{MP}_v(\psi_v)\ e| - |c \mathrm{I}_v(\psi_v)\ e| - D_{ce}(\psi), \\ &\leq V + \left\lfloor \frac{V}{2} \right\rfloor - \left\lceil \frac{V}{2} \right\rceil - \left\lfloor \frac{V}{2} \right\rfloor = \left\lfloor \frac{V}{2} \right\rfloor, \end{aligned}$$

hence e has no victory against c.

In summary, neither c nor e can be the Condorcet winner. For the same reasons as in the previous cases, neither can d nor the other candidates. ☐

In condition 2, it is necessary to require that all Condorcet voting systems f have the same result c or, equivalently, that c is Condorcet winner. Otherwise, the reciprocal $2 \Rightarrow 1$ is not true. Indeed, consider a configuration where each voter is indifferent between all the candidates: then any voting system is non-manipulable, but there is no resistant Condorcet winner.

If we suppose that the preferences are complete and antisymmetric (in particular if they are strict total orders), then the proof is simplified: conditions (2.1) and (2.2) of Definition 2.16 are equivalent, hence the proof is limited to case 2.

Definition 2.18 (resistant-Condorcet criterion[8])

We say that f satisfies the *resistant-Condorcet criterion* (**rCond**) iff, for every configuration $\omega \in \Omega$ and every candidate $c \in \mathcal{C}$, if c is a resistant Condorcet winner in ω, then $f(\omega) = c$.

It is clear that satisfying the Condorcet criterion implies satisfying the resistant-Condorcet criterion.

Proposition 2.19

We consider electoral spaces that include all strict total orders. For each voting system in the following list, there are values of V and C for which the system does not satisfy the resistant-Condorcet criterion: Approval voting, IB, IRV, IRVA, ITR, Majority judgment, Plurality, Range voting, Veto, and the methods of Borda, Bucklin, Coombs, and Kim-Roush.

Proof. We will give a single counterexample that covers all these voting systems except IB.

17	13	14	14	14	14	14
a	c	d_1	d_2	d_4	d_7	d_{11}
	a	c	d_3	d_5	d_8	d_{12}
		a	c	d_6	d_9	d_{13}
			a	c	d_{10}	d_{14}
				a	c	d_{15}
					a	c
Others	Others	Others	Others	Others	Others	Others
c	d_1	d_2	d_4	d_7	d_{11}	a

Candidate c is preferred to any pair of candidates (a, d_i) by 69 voters out of 100, and to any pair of candidates (d_i, d_j) by at least 55 voters, hence c is the resistant Condorcet winner.

In Majority judgment, Approval voting, or Range voting, consider the case where the sincere ballot of each voter consists in giving the maximum rating to her favorite candidate and the minimum rating to the other candidates. Then a is elected (like in Plurality).

In a PSR with weight vector $\mathbf{x}$, we have $\text{score}(a) - \text{score}(c) = 3(x_1 - x_C) + 1(x_1 - x_2) > 0$, hence c cannot be elected. In particular, this is the case for Plurality, Veto, and Borda's method.

[8] We write this expression with a hyphen to insist on the fact that the adjective *resistant* applies to the Condorcet notion, not to *criterion*.

In ITR, IRV, or IRVA, c is eliminated during the first round.

In Bucklin's method, $\text{score}(a) = (4, 58)$ and $\text{score}(c) = (4, 55)$, hence c cannot be elected.

In the method of Coombs or Kim-Roush, c is eliminated during the first round.

In Chapter 3, we will see more detailed results for each of these voting systems, depending on the number of candidates. We will also prove that, in general, IB does not satisfy **rCond**, that the only IPSR-1 that satisfies **rCond** is Baldwin's method, and that the only IPSR-A that satisfies **rCond** is Nanson's method. $\square$

To the best of our knowledge, there is no classical voting system (in the literature or in practical applications) that satisfies the resistant-Condorcet criterion but not the Condorcet criterion. It is easy to define artificial counterexamples, like the system *Resistant-Dean* (RDean): elect the resistant Condorcet winner when she exists, and a candidate fixed in advance otherwise. But this observation tends to show that a voting system that was not designed to elect all Condorcet winners generally has no "natural" reason to elect those who are resistant.

2.7 Strong Condorcification theorem

Thanks to the notion of resistant Condorcet winner, we have the necessary tool to show that, for many voting systems, the Condorcification f^* is *strictly* less manipulable than the original voting system f.

Theorem 2.20 (strong Condorcification theorem)

Let Ω be an electoral space and f an SBVS. Suppose that f satisfies **InfMC** *but not* **rCond**.

Then its Condorcification f^* is strictly less manipulable than f:

$$\mathrm{CM}_{f^*} \subsetneq \mathrm{CM}_f.$$

Proof. The Condorcification theorem 2.9 guarantees inclusion.

Since f does not satisfy **rCond**, there exists a configuration $\omega \in \Omega$ and a candidate $c \in \mathcal{C}$, resistant Condorcet winner in ω, such that $f(\omega) \neq c$. By Proposition 1.31, we know that $f(\omega)$ is not Condorcet-admissible in ω, so Lemma 2.7 ensures that f is manipulable in ω. In contrast, Theorem 2.17 ensures that f^* is not manipulable in ω. Therefore, the inclusion is strict. $\square$

Corollary 2.21

We consider electoral spaces that include all strict total orders. For each voting system f in the following list, there exist values of V and C such that the Condorcification f^* is strictly less manipulable than f: Approval voting, IB, IRV, IRVA, ITR, Majority judgment, Plurality, Range voting, and the methods of Borda, Bucklin, and Coombs.

Proof. Since these systems satisfy **InfMC** (Proposition 2.3) but not **rCond** (Proposition 2.19), we can apply the strong Condorcification theorem 2.20. $\square$

2.8 Upper bound of manipulability for Condorcet voting systems

Since any Condorcet SBVS is non-manipulable in the resistant configurations, this provides us with an upper bound on the manipulability rate. More precisely, denote τ^{π}_{RC} the probability that a configuration is resistant in culture π. Then, for each SBVS f satisfying the Condorcet criterion, we have:

$$\tau^{\pi}_{\mathrm{CM}}(f) \leq 1 - \tau^{\pi}_{\mathrm{RC}}.$$

We can therefore wonder whether this upper bound is tight, i.e. whether there exists a Condorcet voting system that is manipulable in all non-resistant configurations. We now show that, in an important class of electoral spaces, such a system exists.

Consider the electoral space of strict total orders for $C \geq 6$ and an arbitrary value of V. The voting system f is defined as follows. First, we choose an arbitrary bijection between the permutations of $C-3$ elements and the integers of the interval $[\![1,(C-3)!]\!]$. Then the voters provide their ballots and we compute the result as follows.

- If there is a Condorcet winner, then she is elected.
- Otherwise, for each voter v, denote x_v the integer associated to her order of preference on the last $C-3$ candidates of her ballot. We declare candidate $\sum x_v$ as the winner, counting modulo C to bring it back in the interval $[\![1, C]\!]$.

Consider a non-resistant configuration ω. We are going to show that it is manipulable. Let $\mathrm{w} = f(\omega)$ be the sincere winner.

Case 1 If w is a Condorcet winner but not a resistant Condorcet winner, then there are two distinct candidates d and e such that the voters simultaneously preferring w to d and to e do not make a strict majority (indeed, w cannot violate the definition of a resistant Condorcet winner for $d = e$, otherwise c would not be a Condorcet winner at all). Up to exchanging the roles of d and e, we can assume that d has a victory against e. We will build a manipulation in favor of d.

All manipulators (voters preferring d to c) put d on top of their ballots, then e, then c. This ensures that no candidate is a Condorcet winner (as in case 2 of the proof of Theorem 2.17, we use the fact that e has no victory against d). All but one of the manipulators put the other candidates at the bottom of their ballots in an arbitrary order. For the last manipulator v, by appropriately choosing the order of her last three candidates, she can make her value x_v equal to any integer in the interval $[\![1,(C-3)!]\!]$ and since $(C-3)! \geq C$, she can choose any winner, and in particular d.

Case 2 If w is Condorcet-admissible but not a Condorcet-winner, the process is similar. There exists a candidate c such that $|c \ \mathrm{P}_v \ \mathrm{w}| = \frac{V}{2}$. We construct a manipulation in favor of c. The manipulators place c on top, which ensures that no other candidate is a Condorcet winner (and c herself neither, since she still has no victory against w). Then all the manipulators arbitrarily choose the order in which they place the other candidates, except the last manipulator v whose choice of x_v allows to choose any winner, and in particular c.

Case 3 Finally, if w is not Condorcet-admissible, then we know that ω is manipulable (Lemma 2.7).

Therefore, the voting system f is as manipulable as possible for a Condorcet system: indeed, it is non-manipulable only in the resistant configurations! Thus, in any culture π, it reaches the upper bound of manipulability that we gave: $\tau^\pi_{\mathrm{CM}}(f) = 1 - \tau^\pi_{\mathrm{RC}}$.

We will see, however, that exhibiting a Condorcet voting system which is manipulable in all non-resistant configurations is not possible for all values of the pair (V, C): in the electoral space of strict total orders with $V = 3$ voters and $C = 3$ candidates, Section 10.2.1 will reveal that such a system does not exist.

2.9 Condorcification and optimal systems

Until now, we generally considered a given voting system f and we compared the set of manipulable configurations for f and for its Condorcification f^*. We showed that, under certain assumptions, f^* is at most as manipulable as f (weak Condorcification theorem 2.9), or even strictly less manipulable (strong Condorcification theorem 2.20). These results might suggest to use systems like the Condorcifications of Plurality, IRV, etc.

If one does not focus on a particular voting system, the Condorcification theorems also have profound consequences that concern any social planner willing to find an acceptable voting system with minimal manipulability.

Corollary 2.22

Consider the function:

$$\mathrm{CM} : \left| \begin{array}{ccc} \mathbf{InfMC} & \to & \mathcal{P}(\Omega) \\ f & \to & \mathrm{CM}_f, \end{array} \right.$$

which returns, for each SBVS f satisfying **InfMC**, *the set of its manipulable configurations.*

Let $A \in \mathcal{P}(\Omega)$ be a minimal value of CM *(if any), i.e. a subset of Ω such that at least one system $f \in$* **InfMC** *satisfies $\mathrm{CM}_f = A$, but no system $f \in$* **InfMC** *satisfies $\mathrm{CM}_f \subsetneq A$. Then:*

- *Any voting system $f \in$* **InfMC** *satisfying $\mathrm{CM}_f = A$ satisfies* **rCond**,
- *There exists $f \in$* **Cond** *s.t. $\mathrm{CM}_f = A$.*

To understand the significance of this corollary, note that the function CM can have several minima which are non-comparable, since the inclusion relation on $\mathcal{P}(\Omega)$ is not a total order. Thus there may exist two systems f and g such that neither system is less manipulable than f or g, but whose sets of manipulable configurations are non-inclusive of each other.

Corollary 2.22 can be summarized as follows: if we are looking for a voting system satisfying **InfMC** and whose manipulability is minimal (in the set-theoretic sense), then the study *must* be restricted to **rCond** and *can* be restricted to **Cond**.

Similarly, the Condorcification theorems lead to the following corollary.

Corollary 2.23

For a given culture π, consider the function:

$$\tau^\pi_{\mathrm{CM}} : \left| \begin{array}{ccc} \mathbf{InfMC} & \to & [0, 1] \\ f & \to & \tau^\pi_{\mathrm{CM}}(f). \end{array} \right.$$

*Let τ_0 be the lower bound of τ_{CM}^{π}. If it is reached, then there exists $f \in$ **Cond** s.t. $\tau_{\mathrm{CM}}^{\pi}(f) = \tau_0$.*

Contrary to what happens in Corollary 2.22, it is not necessary that a system of minimal manipulability (in the probabilistic sense) satisfies **rCond**: indeed, if some resistant configurations are of measure zero, then they can be manipulable without altering the manipulability rate. However, for each voting system of minimal manipulability rate (within **InfMC**), one can consider its Condorcification to obtain an optimum that satisfies **Cond** and *a fortiori* **rCond**.

Corollary 2.23 can be summarized as follows: if we are looking for a voting system satisfying **InfMC** and whose manipulability rate is minimal, then the study can be restricted to **Cond**.

In Corollaries 2.22 and 2.23, one can wonder whether an optimum such as the one mentioned exists. If we consider a finite electoral space (in particular an ordinal electoral space, such as that of strict total orders or that of strict weak orders), then there exists a finite number of possible SBVS, *a fortiori* if we require that they satisfy **InfMC**; therefore, an optimum exists.

On the contrary, if we consider an infinite electoral space (thus necessarily not limited to the ordinal aspect, like the reference electoral space), then the existence of an optimum is not guaranteed *a priori*. In Chapter 5 devoted to the study of non-ordinal systems, we will address this question for the manipulability rate: we will give a sufficient condition for there to exist an SBVS which (within the class **InfMC**) minimizes the manipulability rate in a given culture.

Chapter 3

Majoritarian Criteria

In the previous chapters, we recalled the Condorcet criterion (**Cond**), then we presented the informed majority coalition criterion (**InfMC**) and the resistant-Condorcet criterion (**rCond**). We will now see how these properties fit into a larger family of majoritarian criteria and develop links with game theory concepts, such as the set of strong Nash equilibria (SNE) and the ability to reach them.

In Section 3.1, we define other criteria related to the notion of majority. In voting theory, it is usual to consider the "majority criterion", whose definition will be recalled, and which we call the *majority favorite criterion* (**MajFav**) in order to distinguish it from other majoritarian criteria. We also introduce the *ignorant majority coalition criterion* and the *majority ballot criterion*. The initial motivation of these definitions is simply practical: provide easy-to-test criteria for proving that a given voting system meets **InfMC**.

Indeed, we show in Section 3.2 that all the other criteria mentioned imply **InfMC**. Furthermore, she show that they form a chain of implications,[1] from the strongest criterion (**Cond**) to the weakest one (**InfMC**).

In the introduction, we already mentioned strong Nash equilibria (SNE) and discussed the interest of such a notion.[2] In particular, the non-manipulability of a configuration means precisely that the sincere vote is an SNE for the corresponding preferences. Brill and Conitzer (2015) showed that, for a voting system satisfying **InfMC**, if there exists a Condorcet winner for the sincere preferences of the voters, then she is the only one who can win in a strong Nash equilibrium. This already known connection between a majority criterion and a notion of equilibrium leads us, in Section 3.3, to consider several equilibrium criteria for a voting system: the fact that the existence of an SNE is guaranteed by the existence of a Condorcet winner (**XSNEC**) or by that of a Condorcet-admissible candidate (**XSNEA**) and the restriction of SNE to the Condorcet winners (**RSNEC**) or to the Condorcet-admissible candidates (**RSNEA**). If the first two are existence criteria, the last two may be seen as weak versions of uniqueness criteria. We reveal the implication relationships between these equilibrium criteria and the majoritarian criteria. In particular, not only defining the criteria explicitly makes it possible to extend the result of Brill and Conitzer by showing that **InfMC** implies **RSNEA**, but we also show that these two criteria are actually equivalent.

In Proposition 2.3, we stated without proof that almost all classical voting systems meet **InfMC**. In Section 3.4, we prove and specify this result by studying

[1] In all rigor, we will see that this chain of implications is complete if the electoral space allows any candidate as most liked, which is a common assumption (Definition 1.10).

[2] About SNE and several variants of this concept, see Bernheim et al. (1987).

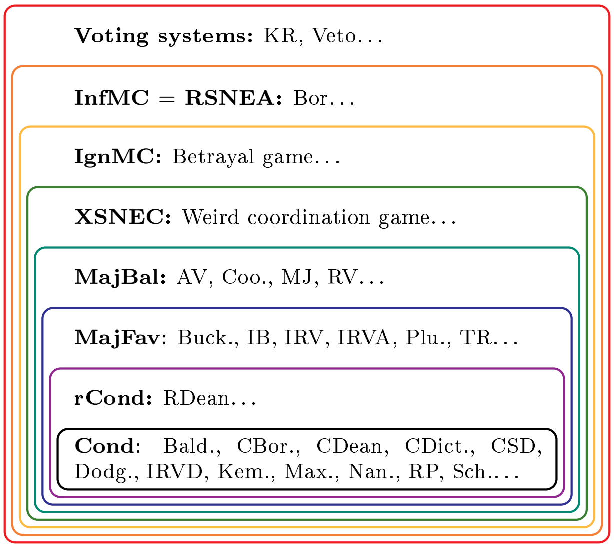


Figure 3.1 – Inclusion diagram of majoritarian criteria. We assume that the preferences are antisymmetric and that the electoral space allows any candidate as most liked.

which criteria are satisfied by the classical voting systems and we detail this study according to the number of candidates C. This will allow us to gradually establish the inclusion diagram of Figure 3.1, which has the disadvantage of spoiling a breathtaking suspense but the advantage of providing the reader with a road map for this chapter. This diagram reads as follows. For example, the set of voting systems meeting **MajFav** is included (in general, strictly) in the set of those meeting **MajBal**. Coombs' method belong to the latter, but generally not to the former (except in particular electoral spaces, such as a space of strict total orders with 2 candidates).

This section 3.4, devoted to the criteria met by different voting systems, operates the synthesis of classic results from the literature and original contributions. To the best of our knowledge, the results on **InfMC**, **IgnMC**, **MajBal**, and **rCond** are original since we introduced those criteria. That said, the results about **IgnMC** and **MajBal** essentially follow from the definitions, since these criteria are precisely designed to be easy to test. The results on **InfMC** and **rCond** usually require more effort. The results on **MajFav** and **Cond** are classic, with the following nuances. First, we found no trace of an exhaustive study of **MajFav** for all IPSR-A in the literature. That said, it would be surprising that the results we are presenting were not stated, because it is a classic criterion and classic voting systems. Second, the results on the Iterated Bucklin's method are all original, since this voting system is a contribution of this memoir.

Finally, in Section 3.5, we propose a reflection on the criteria studied, in terms of the quantity of information necessary to coordinate manipulation strategies and to reach SNE. This leads to discussing why these criteria can be considered desirable for a voting system.

3.1 Definition of majoritarian criteria

In the previous chapters, we have already recalled the definition of the Condorcet criterion (Definition 1.32) and introduced the informed majority coalition criterion (Definition 2.1) as well as the resistant-Condorcet criterion (Definition 2.18). Now, we define three other criteria, which will prove convenient tools to prove that a given voting meets **InfMC** (and that it is, in consequence, concerned by the weak Condorcification theorem 2.9).

Before moving on to the criteria themselves, we define the notion of *majority favorite*.

Definition 3.1 (majority favorite)
For a configuration $\omega \in \Omega$ and a candidate $c \in \mathcal{C}$, we say that c is a *majority favorite* in ω iff a strict majority of voters strictly prefer c to any other candidate: $|\forall d \in \mathcal{C} \setminus \{c\}, c \, \mathrm{PP}_v \, d| > \frac{V}{2}$. When preferences are strict orders (weak or total), it simply means that more than half of the voters put c in first position in their order of preference, without tie with other candidates.

If c is a majority favorite, than c is obviously a resistant Condorcet winner.

Definition 3.2 (majority criteria)
We say that that f meets the *majority favorite criterion* (**MajFav**) iff, for every configuration $\omega \in \Omega$ and for every candidate $c \in \mathcal{C}$, if c is majority favorite in ω, then $f(\omega) = c$.

We say that f meets the *majority ballot criterion* (**MajBal**) iff, for every candidate c, there exists an assignment of ballots to the voters that meets the following property: if it is respected by a strict majority of voters, then c is declared the winner. Formally, this condition reads: $\forall c \in \mathcal{C}, \exists \psi^c \in \Omega$ s.t. $\forall \omega \in \Omega, \left[|\omega_v = \psi^c_v| > \frac{V}{2} \Rightarrow f(\omega) = c \right]$.

We say that f meets the *ignorant majority coalition criterion* (**IgnMC**) iff any majority coalition can choose the result, whatever the other voters do. Formally, $\forall M \in \mathcal{P}(\mathcal{V})$, if $\operatorname{card}(M) > \frac{V}{2}$ then: $\forall c \in \mathcal{C}, \exists \omega_M \in \Omega_M$ s.t. $\forall \omega_{\mathcal{V} \setminus M} \in \Omega_{\mathcal{V} \setminus M}, f(\omega_M, \omega_{\mathcal{V} \setminus M}) = c$.

Like for **InfMC**, **Cond**, or **rCond**, each notation **MajFav**, **MajBal**, or **IgnMC** designates the criterion itself or the set of SBVS (on Ω) that satisfy it.

It is quite easy to extend the majoritarian criteria to general voting systems, defined in Section 1.4. In that case, it appears that **InfMC**, **IgnMC**, and **MajBal** are properties of the *game form*: these criteria describe the power given by the counting rule to a strict majority of voters. In contrast, **MajFav**, **rCond**, and **Cond** are properties of the *voting system*: they establish a link between the preferences of the voters and the sincere outcome. For these last criteria, it is necessary to explicit the electoral space and the sincerity functions.

In an anonymous electoral space (Section 1.2.2), we can also define the *majority unison ballot criterion* (**MajUniBal**) by the following property: for every candidate c, there exists a ballot ψ^c_0 (belonging to any Ω_v, since they are all identical) such that, if it is used by a strict majority of voters, then c is declared the winner. We have chosen not to include this notion in the inclusion diagram of Figure 3.1 because it is not defined in all electoral spaces.

The more general definition of **MajBal** avoids the need for an anonymous electoral space. Its wording makes it also possible to apply the criterion to non-anonymous voting systems (even if the electoral space itself is so): indeed, it is possible to give a different instruction to each voter; the criterion only requires that, if the instructions are followed by a strict majority of voters, candidate c is elected. In the case of a general voting system (Section 1.4), this definition has also the advantage that it is stable by *isomorphism of voting system*, in the following sense: if we change the labels of the ballots and the counting rule accordingly, it has no impact on whether the criterion is met or not.

The criterion **IgnMC** is very similar to **InfMC** (Definition 2.1): the only difference resides in the exchange of quantifiers "$\exists \omega_M \in \Omega_M$" and "$\forall \omega_{\mathcal{V}\setminus M} \in \Omega_{\mathcal{V}\setminus M}$". In practice, if quantifiers are in this order, as in **IgnMC**, manipulation is more difficult because manipulators vote first and other voters can reply. In the inverse order, which is the one used in **InfMC**, the ballots of sincere voters are known first, and manipulators can choose their ballots depending on them. Therefore, **IgnMC** is a more demanding criterion than **InfMC**.

3.2 Implications between majoritarian criteria

The following proposition establishes a hierarchy between the six majoritarian criteria that we have defined. We present these results in the form of inclusions in order to show the connection with Figure 3.1, but it is equivalent to present them as implications. For example, the inclusion **Cond** $\subseteq$ **rCond** is another way of stating that for each SBVS f, we have: $f \in$ **Cond** $\Rightarrow f \in$ **rCond**.

Proposition 3.3
We have the following inclusions.

1. **Cond** $\subseteq$ **rCond** $\subseteq$ **MajFav**.
2. **MajBal** $\subseteq$ **IgnMC** $\subseteq$ **InfMC**.

If the electoral space allows any candidate as most liked (which is a common assumption, cf. Definition 1.10), then we have also **MajFav** $\subseteq$ **MajBal**. *Under this assumption, we thus have:*

$$\textbf{Cond} \subseteq \textbf{rCond} \subseteq \textbf{MajFav} \subseteq \textbf{MajBal} \subseteq \textbf{IgnMC} \subseteq \textbf{InfMC}.$$

All these inclusions are strict in general, i.e. all converse implications are false.

Proof. **Cond** $\subseteq$ **rCond** $\subseteq$ **MajFav**: This chain of inclusions is immediately deduced from the facts that a majority favorite is a resistant Condorcet winner and that a resistant Condorcet winner is a Condorcet winner.

MajBal $\subseteq$ **IgnMC**: For every candidate c, **MajBal** ensure that there exists a assignment of ballots ψ^c that makes it possible to elect c if it is used by a strict majority of voters. If a majority coalition wishes to elect c, they just have to use these ballots so that c wins, whatever the other voters reply. So, the voting system meets **IgnMC**.

IgnMC $\subseteq$ **InfMC**: This inclusion follows immediately from the remark we made on the order of the quantifiers in the definitions of these criteria.

MajFav $\subseteq$ **MajBal**: Consider an assignment of ballots where each voter claims that she strictly prefers a certain candidate c to all other candidates, which is possible because the electoral space allows any candidate as most liked. If the considered voting system meets **MajFav**, then it is sufficient that a strict majority of voters respect this assignment for c to be elected. Hence, the voting system meets **MajBal**.

The fact that all converse implications are false will be proven when studying the classic voting systems in Section 3.4. As for now, we can have an overview of counter-examples in Figure 3.1. □

In an anonymous electoral space, it is clear that we have also **MajUniBal** $\subseteq$ **MajBal**. If, moreover, the electoral space allows any candidate as most liked, then we have also **MajFav** $\subseteq$ **MajUniBal**.

3.3 Connection with the strong Nash equilibria (SNE)

As Gibbard (1973) notices, the pair of a voting system and a configuration of preferences ω for the voters defines a game, in the usual sense of game theory: each player/voter has a set of strategies, well-determined objectives, and there exists a rule that decides the result, depending on the strategies of the players. As a consequence, it is immediate to adapt the usual notion of *strong Nash equilibrium* in this context.

Moreover, we can remark that when using the framework of general voting systems (Section 1.4), any game can be expressed as a game form and a state of preferences for the voters. Hence the terminology of *game form*: it is a proto-game defining the procedure to follow and which only lacks the data of the preferences of the players on the possible outcomes of the game.

Definition 3.4 (strong Nash equilibrium)

Let f be an SBVS and $(\omega, \psi) \in \Omega^2$.

We say that ψ is a *strong Nash equilibrium* (SNE) for preferences ω in system f iff ψ is a strong Nash equilibrium in the game defined by (f, ω). I.e., there exists no configuration ϕ such that:

$$\begin{cases} f(\phi) \neq f(\psi), \\ \forall v \in \mathrm{Sinc}_\omega(f(\psi) \to f(\phi)), \phi_v = \psi_v. \end{cases}$$

In the case of a general voting system, using notations from Section 1.4, we could adapt the definition by saying that a vector $S = (S_1, \ldots, S_V)$ of strategies for the voters (i.e. ballots) is an SNE for a configuration of preference ω. As we have already noticed, the fact that a configuration ω is not manipulable simply means that the sincere vote $s(\omega)$ is an SNE for ω.

For an SBVS, the formalism is simplified because notions of authorized ballot and possible states of preference are identified. The fact that a configuration ω is not manipulable means that configuration ω (seen as ballots) is an SNE for ω (seen as preferences).

As we mentioned in the introduction of this chapter, Brill and Conitzer (2015) showed that, in a voting system meeting **InfMC**, if a configuration is Condorcet, then only the Condorcet winner can be the winner of an SNE. This result suggests a deep connection between some of the majoritarian criteria, not only with manipulability, but with the notion of SNE in general. After reading this article and thanks to fruitful discussions in front of a probably non-alcoholic beverage with Markus Brill, whom we thank here, we examined a few questions connected to this issue. We defined the four following equilibrium criteria and we studied their connection with the majoritarian criteria.

Definition 3.5 (equilibrium criteria)

We say that f meets the criterion of *restriction of possible SNE to Condorcet-admissible candidates* (**RSNEA**) iff for all $(\omega, \psi) \in \Omega^2$: if ψ is an SNE for preferences ω, then $f(\psi)$ is Condorcet-admissible in ω.

We say that f meets the criterion of *restriction of possible SNE to Condorcet winners* (**RSNEC**) iff for all $(\omega, \psi) \in \Omega^2$: if ψ is an SNE for preferences ω, then $f(\psi)$ is Condorcet winner in ω.

We say that f meets the criterion of *existence of an SNE for any Condorcet winner* (**XSNEC**) iff $\forall(\omega, c) \in \Omega \times \mathcal{C}$: if c is Condorcet winner in ω, then there exists ψ that is an SNE for ω and such that $f(\psi) = c$.

We say that f meets the criterion of *existence of an SNE for any Condorcet-admissible candidate* (**XSNEA**) iff $\forall(\omega, c) \in \Omega \times \mathcal{C}$: if c is Condorcet-admissible in ω, then there exists ψ that is an SNE for ω and such that $f(\psi) = c$.

As we noted in the introduction of this chapter, the **RSNEC** criterion can be seen as a uniqueness criterion, not for the SNE, but for the possible winner in an SNE. In an electoral space where there exists at least a semi-Condorcet configuration (i.e. if it is possible to have a Condorcet-admissible candidate who is not a Condorcet winner, as in the electoral space of strict total orders with an even number of voters), the **RSNEA** criterion is a weaker version, which we will prove is met by most usual voting systems. If there is no semi-Condorcet configuration (for example in the electoral space of strict total orders with an odd number of voters), the two notions are equivalent. Similarly, in that case, existence criteria **XSNEC** and **XSNEA** are equivalent.

The **RSNEA** criterion implies, in particular, the criterion of Brill and Conitzer: when it is met, if ω is a Condorcet configuration, then the Condorcet winner is the only Condorcet-admissible candidate (Proposition 1.31), so she is the only possible winner of an SNE.

We now study the connections between these equilibrium notions and the majoritarian criteria. We focus first on **RSNEA** and **XSNEC**, which will see fit naturally into the chain of inclusions of Proposition 3.3, which justifies their position in the inclusion diagram of Figure 3.1.

Proposition 3.6

Assume that the electoral space allows any candidate as most liked.
Then **InfMC** $=$ **RSNEA**.

In the same spirit, Sertel and Sanver (2004) show that in the electoral space of strict weak orders, if a voting system meets **MajFav**, then the winner of an SNE is necessarily a weak Condorcet winner (which is equivalent to a Condorcet-admissible candidate in this context): in other words, **MajFav** $\subseteq$ **RSNEA**. Proposition 3.6 therefore extends this result. The authors, for their part, generalize it in another direction by considering a family of variants of the criteria **MajFav** and **RSNEA**.

Proof. If a voting system f meets **InfMC**, then it meets **RSNEA**: indeed, if the winning candidate of a configuration ψ is not Condorcet-admissible in ω, then a strict majority of voters prefer another candidate (in the sense of ω) and they can make her win, by virtue of **InfMC**. Hence ψ cannot be an SNE for ω.

Now, consider a voting system that does not meet **InfMC**. We are going to show that it does not meet **RSNEA**. By definition, there exists a candidate c, a coalition with a strict majority M, and a configuration $\psi_{\mathcal{V}\setminus M}$ of other voters such that, whatever the ballots ψ_M chosen by the coalition, $f(\psi_M, \psi_{\mathcal{V}\setminus M}) \neq c$.

Consider the state $\psi_{\mathcal{V}\setminus M}$ mentioned above (such that the minority makes the election of c impossible) and an arbitrary state ψ_M of the majority coalition M. We denote $a = f(\psi_M, \psi_{\mathcal{V}\setminus M})$, which is by assumption distinct of c.

Now, consider the following profile of preferences ω.

Voters in M (majority)	Voters in $\mathcal{V} \setminus M$ (minority)
c	a
a	c
Others	Others

Clearly, c is the Condorcet winner, hence she is the only Condorcet-admissible candidate.

If ballots are ψ, then candidate a is winner by assumption. The only possible deviations come from voters in M, and their only wish is to make c win, but it is impossible. As a consequence, ψ is an SNE for ω whose winner is not Condorcet-admissible in ω. As a consequence, the voting system does not meet **RSNEA**. □

Proposition 3.7

Assume that the binary relations of preference are antisymmetric.
We have: **MajBal** $\subseteq$ **XSNEC**. *The converse is false in general.*

Proof. Let f be an SBVS that meets **MajBal** and ω a configuration. If there is a Condorcet winner c in ω, then consider the configuration ψ^c whose existence is guaranteed by **MajBal**: if a strict majority of voters respect this ballot assignment, then c is elected. It is easy to see that this is an SNE for ω: indeed, if a subset of voters wishes to elect a certain candidate d instead of c, then since c is the Condorcet winner and since preferences are antisymmetric, these voters are strictly in the minority. So, by definition of ψ^c, candidate c remains elected. Thus, f meets **XSNEC**.

In order to show that the converse is false, consider the electoral space of strict total orders for $V = 3$ voters and $C = 3$ candidates. We are about to define a voting system that we call the *weird coordination game* and prove that it meets **XSNEC** but not **MajBal**.

Here is the rule of the game.

1. If there exists at least one pair of voters whose both members put the same candidate on top of their ballots but their two last candidates in different orders, then their preferred candidate is elected.

2. In all other cases, the dean (i.e. a candidate fixed in advance) is elected.

Thus, if two voters wish to elect a candidate c who is not the dean, they must coordinate skillfully to put c on top and above all to ensure that their ballots are not identical.

Let us show that this system meets **XSNEC**. Consider a Condorcet configuration ω. Since $V = 3$ and $C = 3$, a Condorcet winner is necessarily the most liked candidate for at least one voter (otherwise, another candidate is on top of at least two ballots, so she is a majority favorite and *a fortiori* a Condorcet winner).

If the Condorcet winner is the most liked candidate for at least two voters, then they just need to de-synchronize the bottom of their ballots so that she is elected, and the configuration is an SNE: they are fully satisfied, and if the third voter is not, well, too bad for her.

If the Condorcet winner is the most liked candidate for exactly one voter, then it is easy to show that, up to permuting voters and/or candidates, profile ω is of the following type.

c	a	b
Others	c	c
	b	a

Candidate c is the Condorcet winner. Up to exchanging a and b and the two last voters, we can assume that the dean is not b: so, it is c or a.

Then consider the following profile ψ.

c	a	$\mathbf{c}$
a	c	$\mathbf{b}$
b	b	a

Candidate c wins and this is an SNE for ω: indeed, the second voter cannot change the result and the third voter cannot make b win (she cannot exploit neither rule 1 because b cannot become a majority favorite, nor rule 2 because b is not the dean). Hence, the weird coordination game meets **XSNEC**.

It remains to prove that it does not meet **MajBal**. Consider a candidate c, distinct from the dean. Let us try to conceive an assignment of ballots ψ^c ensuring a victory of c as soon as it is respected by a strict majority of voters. For each pair of voters, there must be exactly one ballot $c \succ a \succ b$ and one ballot $c \succ b \succ a$, which is impossible: indeed, we authorize only two different ballots for three voters, so there exists two voters with the same ballot. If two voters use this identical ballot, the other voter just needs not to put c on top for the dean to be elected, which contradicts the definition of ψ^c. Thus, the weird coordination game does not meet **MajBal**. ☐

Proposition 3.8

Assume that the electoral space allows any candidate as most liked.
Then **XSNEC** $\subseteq$ **IgnMC**. *The converse is false in general.*

Proof. Consider a voting system that does not meet **IgnMC**. By definition, there exists a candidate c and a coalition with a strict majority M such that for every assignment ψ_M of ballots for the coalition, there exists a response $\psi_{\mathcal{V}\setminus M}$ from other voters such that $f(\psi_M, \psi_{\mathcal{V}\setminus M}) \neq c$.

Consider a profile of preference ω of the following type.

Voters in M (majority)	Voters in $\mathcal{V} \setminus M$ (minority)
c	Others
Others	c

Clearly, c is the Condorcet winner.

In a state ψ that is an SNE for ω, it is impossible that c wins: indeed, members of $\mathcal{V} \setminus M$ can prevent a victory for c and they want to do so, whatever the alternate result. Thus, the voting system does not meet **XSNEC**, which proves the inclusion **XSNEC** $\subseteq$ **IgnMC** by contraposition.

In order to prove that the converse is false, we are going to define a voting system that we call the *betrayal game* and show that it satisfies **IgnMC** but not **XSNEC**. We will first use a non-ordinal voting system in order to better reveal the underlying intuition, then we will show that, with a little effort, this counterexample can be adapted to an ordinal system.

Here is the rule of the game. Each voter v casts a ballot of the type $(M_v, c_v) \in \mathcal{P}(\mathcal{V}) \times \mathcal{C}$: she announces a coalition, i.e. a set of voters, and the index of a candidate. In order to form an intuition, we can interpret M_v as the set of voters that v considers as "friends". We could demand that M_v contains v herself, but this additional assumption has no impact on our demonstration. Now, here is how the winning candidate is determined.

1. If there exists a strict majority coalition M and a candidate c such that each voter in M casts the ballot (M, c), then c is elected (there is no ambiguity because such a coalition is necessarily unique).
2. In all other cases, the result is $\sum_{v \in \mathcal{V}} c_v$, reduced modulo C to the interval $[\![1, C]\!]$.

Clearly, this voting system meets **IgnMC**: indeed, if a strict majority coalition M wishes to have a certain candidate c elected, they can simply coordinate to announce the ballot (M, c), which guarantees the election of c, regardless of the ballots of the other voters.

In order to show that the betrayal game does not meet **XSNEC**, consider the following profile, where candidate c is the Condorcet winner.

$$\begin{array}{c|c|c} c & a & b \\ a & c & c \\ b & b & a \end{array}$$

Can there be an SNE where c is elected? If c is elected by virtue of rule 1, then the second voter, the third voter, or both participate to coalition M since it is a majority. Any one of them can betray coalition M by naming the coalition consisting of only herself and make a or b win by choosing the adequate modulo.

The other possibility is that c is elected by virtue of rule 2. Then the second voter can name the coalition consisting of herself and make a win by choosing the adequate modulo.

As a consequence, there is no SNE where c is elected,[3] which proves that the betrayal game does not meet **XSNEC**.

For those who prefer a purely ordinal example, it is sufficient to consider additional candidates $d_1, \ldots, d_4$ and to code coalition M in the order given on the last four candidates of a ballot (4 candidates provide a sufficient expressiveness because $4! = 24 > 2^3 = 8$), in a way similar to what we did in Section 2.8 to define a very manipulable Condorcet system. Then, we consider a variant of the above profile, where candidates $d_1, \ldots, d_4$ are last for all voters and we show similarly that there is no SNE where c is elected. □

Until now, we focused on the criteria **RSNEA** and **XSNEC** and we proved their position in the inclusion diagram of Figure 3.1. The criteria **RSNEC** and **XSNEA** are obviously stronger versions (respectively), in the sense that **RSNEC** $\subseteq$ **RSNEA** and **XSNEA** $\subseteq$ **XSNEC**. If the electoral space contains no semi-Condorcet configuration (such as the electoral space of strict total orders with an odd number of voters), then it is immediate that these inclusions are actually equalities: **RSNEC** $=$ **RSNEA** and **XSNEA** $=$ **XSNEC**.

As a consequence, for all the criteria inside the zone **RSNEA** (resp. **XSNEC**), and in particular **Cond**, there exists some SBVS that also meet

[3]In fact, with this configuration of preference, it is even possible to prove that there is no SNE at all.

RSNEC (resp. **XSNEA**): indeed, we can simply consider an electoral space with no semi-Condorcet configuration. In that case, any system meeting a criterion stronger than **RSNEA** (resp. **XSNEC**) meets also **RSNEC** (resp. **XSNEA**), since these notions are equivalent.

On the other side, for all criteria inside the zone **RSNEA** (resp. **XSNEC**), and in particular **Cond**, there exists some SBVS that do not meet **RSNEC** (resp. **XSNEA**). Indeed, consider the electoral space of strict total orders with an even number of voters and the voting system Condorcet-dean, where the dean is denoted a.

1. If half of the voters prefer a candidate c, distinct from the dean, to all other candidates and if the other half prefer all the other candidates to c, then c is Condorcet-admissible but there is no SNE where she is elected (there is always a possible deviation in favor of a). So, this system does not meet **XSNEA**.

2. If half of the voters prefer the dean a to all other candidates and if the other half have a last in their order of preference, then the sincere vote is an SNE where a is elected, although she is only Condorcet-admissible but not a Condorcet winner. So, this system does not meet **RSNEC**.

As a consequence, **XSNEA** is included in **XSNEC** but it has, in general, no simple relation of inclusion with the criteria inside the zone **XSNEC**: for example, it may have a nonempty intersection with **Cond** but also with its complement. It is similar for **RSNEC**, which is included in **RSNEA** but has no simple relation of inclusion with criteria stronger than **RSNEA**.

We can remark than, in an electoral space where there exists at least one semi-Condorcet configuration, criteria **XSNEA** and **RSNEC** are incompatible: indeed, in a semi-Condorcet configuration, **XSNEA** demands that there exists an SNE where a Condorcet-admissible candidate is elected, but since this candidate is not a Condorcet winner, this contradicts **RSNEC**.

For all these reasons, it seems that the criteria **XSNEA** and **RSNEC** are less "natural" than their respective weaker versions **XSNEC** and **RSNEA**. We have then chosen not to include them in the inclusion diagram of Figure 3.1 for the sake of readability.

To sum up this section, we studied the "uniqueness" and the existence of an SNE, in connection with Condorcet notions.

For the "uniqueness" of the equilibrium, we defined the criterion **RSNEC**, which demands than only a Condorcet winner can benefit from an SNE. But eventually, it seems that **RSNEA** is a more natural property than its weak version **RSNEC**, in the sense that it is more frequently met by usual systems, knowing that these notions are of course equivalent when there is no semi-Condorcet configuration. Whereas it was quite clear that **InfMC** implies **RSNEA**, we showed that, actually, these two criteria are equivalent.

We also wanted to know what guarantees the existence of an SNE when preferences are Condorcet, i.e. the criterion **XSNEC**. We saw that this criterion is more natural than its strong version **XSNEA**. We proved that **MajBal** is sufficient but not necessary (as illustrated by the weird coordination game) and that **IgnMC** is necessary but not sufficient (as shown by the betrayal game).

3.4 Majoritarian criteria met by the usual voting systems

We now understand better the position of the majoritarian and equilibrium criteria in the inclusion diagram of Figure 3.1. Before proposing a reflection about the motivation and the consequences of these criteria, we now study which criteria are satisfied by the usual voting systems. In this whole section, we consider a fixed number of candidates C but a variable number of voters V: we thus say that a voting system[4] satisfies a given criterion for a given value of C iff it satisfies it for every number of voters; and that it does not satisfy it iff there exists a configuration that contradicts the criterion, whatever the number of voters.

The motivation of this choice is two-sided. One the one hand, when an election is organized in real life, the number of candidates is almost always known (at least at the moment where ballots must be chosen by the voters), whereas it is very frequent that the number of voters in unknown in advance. On the other hand, fixing the number of voters may lead to difficult questions of quantification: so, it may happen that a given system meets some criterion for a value of (V, C) for backpacking-type reasons, whereas it does not meet the criterion in general for this value of C (and especially for a large enough number of voters). In other words, it is pragmatic and technically more tractable to let the number of voters vary. In this, our approach is the same as Smith (1973), who studied some of these questions for PSR with a variable number of voters.

3.4.1 Cardinal voting systems

Proposition 3.9

1. *Range voting, Majority judgment, and Approval voting meet* **MajBal**.
2. *Let $C \geq 3$. Assume that, for each voter v and for each strict total order of preference p_v on the candidates, voter v has at least at her disposal:*
   - *A state ω_v such that $\mathrm{P}_v(\omega_v) = p_v$ and where she attributes the maximal grade to her most-liked candidate and the minimal grade to all the others,*
   - *And a state ω'_v such that $\mathrm{P}_v(\omega'_v) = p_v$ and where she attributes the maximal grade to her two most-liked candidates and the minimal grades to all the others.*

   Then these voting systems do not meet **MajFav**.

Proof. 1. Let c be a candidate. Consider a ballot consisting of attributing the maximal grade to c and the minimal grade to the other candidates. If a strict majority of voters use this ballot, then c is elected. So, these voting systems meet **MajBal**. If we define these voting systems reasonably, in an anonymous electoral space, then we just proved that they even meet **MajUniBal**.

2. Consider the following configuration where candidates are denoted $c, d_1, \ldots, d_{C-1}$ and where minimal and maximal grades are respectively 0 and 1

[4] To respect a mathematically rigorous terminology, we should say a *family* of voting systems, because by varying the number of voters, we consider systems that are defined over several different electoral spaces. However, we will avoid this coquetry of language and we will keep on talking about *voting systems*, even in this case.

by convention.

2	1
$c : 1$	$d_1 : 1$
$d_1 : 1$	$d_2 : 0$
$d_2 : 0$	$\vdots$
$\vdots$	$d_{C-1} : 0$
$d_{C-1} : 0$	$c : 0$

Then candidate c is the majority favorite but d_1 is elected. □

Property 2 is also true for $C = 2$ but we preferred not to mention it in the proposition because in that case, the assumption is debatable. Indeed, if there are only two candidates and if the binary relations of preference are strict total orders, then there is a natural sincerity function, since each voter has a dominant strategy: attribute the maximal grade to her most-liked candidate and the minimal grade to the other. With this sincerity function, these voting systems become equivalent to simple majority voting (mentioned in the introduction) and they trivially meet **MajFav** (and even **Cond**). In contrast, as soon as $C \geq 3$, the assumption that it is possible to sincerely vote for one's most-liked candidate or for one's two most-liked candidates is quite natural in practice, and compliant with the spirit of Approval voting, Range voting, and Majority judgment.

In all the rest of this section 3.4, dedicated to usual ordinal voting systems, we work in the electoral space of strict total orders.

As a reminder, Baldwin, Condorcet-Borda (Black's method), Condorcet-dean, Condorcet-dictatorship, CSD, Dodgson, IRVD, Kemeny, Maximin, Nanson, RP, and Schulze's method meet **Cond**, which immediately determines their position in the inclusion diagram of Figure 3.1.

3.4.2 Plurality with one or several rounds

Even if we will deal with Plurality again as a particular case of Proposition 3.13 about PSR, IRV as a particular case of Proposition 3.16 about IPSR-1, and IRVA as a particular case of Proposition 3.21 about IPSR-A, we will first study these voting systems on their own and seize the opportunity to examine the Two-round system as well. On the one hand, these voting systems are widely used in practice (except IRVA) and they all present some similarities of principle. On the other hand, this will lead us to prove, in a simple context, some results that will be useful later.

Proposition 3.10

1. *Plurality, the Two-round system (TR or ITR), IRV, and IRVA meet* **MajFav**.
2. *They meet* **rCond** *iff $C \leq 4$.*
3. *They do not meet* **Cond** *(except in the trivial case $C \leq 2$).*

In order to prove this proposition, we will use the following lemma.

Lemma 3.11

Consider the electoral space of strict total orders and assume that some candidate c is the resistant Condorcet winner.

If $C \leq 4$, then the Plurality score of c is strictly the highest one.

If $C \geq 5$, then the Plurality score of c can be strictly the lowest one.

Proof. For $C \leq 3$, a resistant Condorcet winner is, by definition, a majority favorite, hence the affirmation is obvious. So, let us assume $C = 4$. Name the candidates c, d_1, d_2, d_3 and denote γ (resp. δ_1, δ_2, δ_3) the number of voters who have c (resp. d_1, d_2, d_3) on top of their order of preference. We then have the following profile.

γ	δ_1	δ_2	δ_3
c	d_1	d_2	d_3
Others	Others	Others	Others

Since c is the resistant Condorcet winner, a strict majority of voters prefer her to d_1 and d_2 simultaneously. But we know that at least $\delta_1 + \delta_2$ voters do not meet this condition. As a consequence, we have $\gamma + \delta_3 > \delta_1 + \delta_2$. Similar relations are obtained by permuting the coefficients δ_i.

Assume that a candidate, for example d_1, has a Plurality score greater than or equal to that of c, i.e. we have $\delta_1 \geq \gamma$. By combining this relation with the previous ones, we successively obtain $\delta_1 + \delta_2 \geq \gamma + \delta_2 > \delta_1 + \delta_3 \geq \gamma + \delta_3 > \delta_1 + \delta_2$, which is a contradiction.

For $C \geq 5$, consider a profile of the following type.

1	2	2	2	2
c	d_1	d_2	d_3	d_4
	c	c	c	c
Others	Others	Others	Others	Others

Candidate c is preferred to any pair of opponents by at least 5 voters out of 9, hence she is the resistant Condorcet winner. However, her Plurality score is the lowest one. □

Lemma 3.12

Consider the electoral space of strict total orders and assume that some candidate c is the Condorcet winner.

If $C \geq 3$, the Plurality score of c can be the lowest one.

Proof. This classic property is a particular case of a lemma by Smith (1973), showing the same result not only for Plurality but for all PSR except Borda. We will come back to it in Lemma 3.17. For the moment, it is sufficient to consider the following profile.

1	2	2
c	d_1	d_2
	c	c
Others	Others	Others

□

We now have all the elements to prove Proposition 3.10.

Proof. 1. It is clear that Plurality, the Two-round system, IRV, and IRVA meet **MajFav**.

2. Lemma 3.11 proves that with 4 candidates or less, a resistant Condorcet winner always has the best Plurality score. In the Two-round system, it implies that she is selected for the second round; since she is the Condorcet winner, she wins this last duel. In IRV or IRVA, it implies that she has the best score during each round; in particular, she wins the election.

Lemma 3.11 also states that with 5 candidates or more, a resistant Condorcet winner can have the strictly worst Plurality score. So, she can lose in Plurality and she can be eliminated during the first round in the Two-round system, IRV, or IRVA.

3. Lemma 3.12 states that with 3 candidates or more, a Condorcet winner can have the strictly worst Plurality score, and as a consequence, she can also lose in the Two-round system, IRV, or IRVA. □

3.4.3 Positional scoring rules (PSR)

We already know which criteria are satisfied by Plurality. We now generalize our investigation to PSR, which comprise in particular Borda's method and Veto.

In studying PSR, and iterated PSR later, it will often be convenient to state the results using the arithmetic mean of all or some of the weights. We use the following convention: given real numbers $x_1, \dots, x_n$, we denote $\text{mean}(x_1, \dots, x_n)$ their arithmetic mean.

Proposition 3.13
Let f be a PSR with weight vector $\mathbf{x}$.

1. *For f to meet* **InfMC***, it is necessary that for all $k \in [\![2, C]\!]$, we have:*

$$\text{mean}(x_1, \dots, x_k) + \text{mean}(x_{C-k+1}, \dots, x_C) \leq x_1 + x_C.$$

2. *For f to meet* **InfMC***, it is sufficient that for all $k \in [\![2, C]\!]$, we have:*

$$x_k + x_{C-k+2} \leq x_1 + x_C,$$

and that at least $j - 1$ such inequalities are strict, where j is the largest integer such that $x_j = x_1$.

3. *f meets* **IgnMC** *iff it is Plurality. In that case, it also meets* **MajFav**.

4. *f meets* **rCond** *iff it is Plurality with $C \leq 4$.*

5. *f does not meet* **Cond** *(except in the trivial case $C \leq 2$).*

First of all, note that the particular case $k = 1$ is not mentioned in property 1. Indeed, it would be useless because the condition becomes $x_1 + x_C \leq x_1 + x_C$, which is always satisfied.

For property 2, we consider the sum of x_k and x_{C-k+2}, i.e. we add x_2 and x_C, or x_3 and x_{C-1}, etc. The particular case $k = 1$ would not even be defined, since the weight x_{C+1} does not exist.

Proof. 1. We will deal with the particular case $k = C$ last. Assume that there exists $k \in [\![2, C - 1]\!]$ such that:

$$\text{mean}(x_1, \dots, x_k) + \text{mean}(x_{C-k+1}, \dots, x_C) > x_1 + x_C.$$

Let α be a positive integer. Consider α sincere voters who always put some candidates $d_1, \dots, d_k$ on top (not necessarily in the same order) and candidate c last. If $\alpha + 1$ additional voters wish to make c win, it is necessary that c has a better score than candidates $d_1, \dots, d_k$. To achieve this necessary condition, manipulators cannot do better than put c on top and $d_1, \dots, d_k$ at the bottom of their ballots (not necessarily in the same order). But then, we have:

$$\begin{aligned}
&\text{mean}(\text{score}(d_1), \dots, \text{score}(d_k)) - \text{score}(c) \\
&= \alpha \,\text{mean}(x_1, \dots, x_k) + (\alpha+1)\,\text{mean}(x_{C-k+1}, \dots, x_C) - (\alpha+1)x_1 - \alpha x_C \\
&= (\alpha+1)\left[\frac{\alpha}{\alpha+1}\,\text{mean}(x_1, \dots, x_k) + \text{mean}(x_{C-k+1}, \dots, x_C) - x_1 - \frac{\alpha}{\alpha+1} x_C\right].
\end{aligned}$$

For α large enough, this quantity is positive, hence at least one candidate d_i has a better score than c, who therefore cannot be elected. Thus, f does not meet **InfMC**.

There remains the particular case $k = C$. Assume that f meet **InfMC**. We now know that it implies the condition that we gave for $k = C - 1$, which can be rewritten in the following way:

$$x_1 + 2x_2 + \ldots + 2x_{C-1} + x_C \leq (C-1)(x_1 + x_C).$$

By adding the trivial inequality $x_1 + x_C \leq x_1 + x_C$, we obtain $2\sum_i x_i \leq C(x_1 + x_C)$, which proves that the mentioned condition is also satisfied for $k = C$.

Note that, in order to prove that a given voting system meets all the mentioned necessary conditions, one does not need to test the particular case $k = C$, which is directly implied by the condition for $k = C - 1$. In contrast, to prove that a given system does not meet **InfMC**, it can be most convenient to prove that the condition for $k = C$ is false.

2. Let us prove that if the mentioned conditions are satisfied, then f meets **InfMC**. If there are α sincere voters and β manipulators in favor of some candidate c, with $\beta > \alpha$, separate the β manipulators into two groups: the α first of them will be used to give c a score that is greater than or equal to those of the other candidates and the $\beta - \alpha$ other manipulators will make this inequality strict.

The α first manipulators put themselves in bijection with the sincere voters and each of them casts a ballot that is the inverse order of her alter ego, except that she moves c on top of her ballot. Then, the assumed inequalities ensure that the other candidates have at most the same score as c and that at least $j - 1$ of them have a strictly lower score.

When an additional manipulator is added, she can attribute x_1 points to c and to these $j - 1$ candidates, and strictly less to the others, so candidate c has strictly the highest score. If $\beta - \alpha > 1$, the remaining manipulators simply put c on top and the inequality remains strict.

3. We now show that the only PSR meeting **IgnMC** is Plurality. Up to adding a constant to the weights and multiplying them by a positive constant, we can assume $x_1 = 1$ and $x_C = 0$.

Consider $\frac{V}{2} + \varepsilon$ voters who want to make c win, and $\frac{V}{2} - \varepsilon$ voters who vote after them and whose only goal is to avoid the election of c, with $\varepsilon > 0$. As for the position of c in the ballots, her proponents cannot do better than putting her on top and the others cannot do better than putting her at the bottom of their ballots. We then have $\text{score}(c) = \frac{V}{2} + \varepsilon$.

Each proponent of c distributes at least x_2 points to the other candidates, hence at least $\frac{x_2}{C-1}$ points in average to each of them. Among these candidates, let d be the one receiving the most points from the proponents of c. In order to prevent c from winning, the best strategy of the other voters is to put d on top of their ballots. Then we have:

$$\text{score}(d) \geq \left(\frac{V}{2} + \varepsilon\right) \frac{x_2}{C-1} + \left(\frac{V}{2} - \varepsilon\right).$$

For f to meet **IgnMC**, it is necessary that even when ε is negligible in front of V, we have $\text{score}(d) < \text{score}(c)$. Considering the limit $\varepsilon \to 0$, we obtain after simplification $x_2 \leq 0$, hence $x_2 = \ldots = x_C = 0$ (because weights are nonincreasing, cf. Definition 1.33). Thus, f is Plurality.

Conversely, Plurality meets **IgnMC** and even **MajFav** (Proposition 3.10).

4. For f to meet **rCond**, it is necessary that it meets **MajFav** hence, according the previous point, it is Plurality. But we know (Proposition 3.10) that Plurality meets **rCond** iff $C \leq 4$.

5. Finally, it is a classic property that f does not meet **Cond** (except if $C \leq 2$). A possible proof is the following. In order for f to meet **Cond**, it must meet **MajFav**, hence f is necessarily Plurality. But we know (Proposition 3.10) that Plurality does not meet **Cond** (except in the trivial case $C \leq 2$). □

The fact that the only PSR meeting **MajFav** is Plurality was already proven by Lepelley and Merlin (1998). Point 3 shows that in fact, it is the only PSR meeting the criterion **IgnMC** (*a priori* less demanding).

In order to complete Proposition 3.13 about PSR, the ideal would be to find a necessary and sufficient condition for a PSR to satisfy **InfMC**. According to our ongoing research on this subject, we conjecture that it is possible to find a sufficient condition whose form is close to that of the necessary condition that we have given. This question is left open for future work.

From Proposition 3.13, we immediately deduce the two following corollaries for Borda's method and Veto.

Corollary 3.14

1. *Borda's method meets* **InfMC**.
2. *It does not meet* **IgnMC** *(except in the trivial case $C \leq 2$).*

Corollary 3.15
Veto does not meet **InfMC** *(except in the trivial case $C \leq 2$).*

In both cases, it is possible to prove the result directly, without resorting to Proposition 3.13. For Veto, we can give an especially short proof. Let $C \geq 3$ and $V = 7$. Assume that 3 sincere voters vote against candidate c. Then c has more vetoes than the average (which is $\frac{V}{C}$), hence she cannot win, regardless of the ballots of the other 4 voters (although they form a strict majority).

3.4.4 Iterated PSR with simple elimination

We have already studied the case of IRV in Proposition 3.10. We now extend this result to the class of IPSR-1, which includes the methods of Baldwin and Coombs.

Proposition 3.16
Let f be an IPSR-1 with weight vectors $(\mathbf{x}^k)_{k \leq C}$.

1. *f meets* **MajUniBal**.

2. *f meets* **MajFav** *iff for all $k \in [\![3, C]\!]$:*

$$\text{mean}(x_1^k, x_k^k) \geq \text{mean}(x_1^k, \dots, x_k^k).$$

If this condition is met and if $C \leq 4$, then f also meets **rCond**.

3. *f meets* **Cond** *iff it is Baldwin's method.*

The condition stated for an IPSR-1 to meet **MajFav** is proven by Lepelley and Merlin (1998). The condition for meeting **Cond** is due to Smith (1973). We are still going to mention a sketch of proof in order to give a complete overview of Proposition 3.16.

In order to prove these results and, later, to study IPSR-A, we will use the following lemma several times.

Lemma 3.17 (Smith)
Let f be a PSR (for a fixed C and variable V).
If f is Borda's method, then a Condorcet winner necessarily has a score strictly higher than the average score.
Otherwise, a Condorcet winner may have a score that is strictly the lowest.

The first statement above is just a reminder: it follows from the fact that a candidate's Borda score is the sum of the scores of her row in the weighted majority matrix. The second statement is due to Smith (1973).

The following lemma provides a similar result for the majority favorite.

Lemma 3.18
Let f be a PSR with weight vector $\mathbf{x}$ (with $C \geq 2$).
Consider the following condition:

$$\text{mean}(x_1, x_C) \geq \text{mean}(x_1, \dots, x_C).$$

If it is satisfied, then a majority favorite necessarily has a score strictly higher than the average score.
Otherwise, a majority favorite may have a score that is strictly the lowest.

Proof. For a majority favorite c, the minimal possible score is $\frac{V}{2}(x_1 + x_C) + \varepsilon(x_1 - x_C)$, where $\varepsilon > 0$ can be made negligible in front of V. And the average score is $V \cdot \text{mean}(x_1, \dots, x_C)$. In order for the score of c to be always strictly greater than the average score, it is necessary and sufficient that $\frac{1}{2}(x_1 + x_C) \geq \text{mean}(x_1, \dots, x_C)$.

If the condition is not met, note that the score of a majority favorite c may be, not only less than or equal to, but strictly less than the average score (by taking ε small enough). Consider such a profile, make $C - 1$ copies of it by a circular permutation of the other candidates, and join the profiles we obtain this way. Then all the opponents have the same score, whereas the score of c is strictly lower than the average: therefore, her score is strictly the lowest. However, c is still the majority favorite. □

Finally, for $C \leq 4$, the following lemma shows that the same condition yields a similar result for the resistant Condorcet winner.

Lemma 3.19
Let f a PSR with weight vector $\mathbf{x}$.

Consider the following condition:

$$\text{mean}(x_1, x_C) \geq \text{mean}(x_1, \dots, x_C).$$

If it is satisfied and if $C \leq 4$, then a resistant Condorcet winner necessarily has a score strictly higher than the average score.

If it is not met (for any C), then a resistant Condorcet winner may have a score that is strictly the lowest.[5]

Proof. The second statement is easier: if the condition does not hold, then by Lemma 3.18, a majority favorite (who is in particular a resistant Condorcet winner) may have a score that is strictly the lowest. As for the first statement, if $C \leq 3$, it also follows from Lemma 3.18 because the notions of majority favorite and resistant Condorcet winner are then equivalent.

We still have to prove the first statement for $C = 4$. Up to subtracting x_4 from each weight, we can assume $x_4 = 0$. Let us translate the condition satisfied by assumption:

$$\frac{x_1 + x_2 + x_3 + 0}{4} \leq \frac{x_1 + 0}{2},$$

i.e. $x_2 + x_3 \leq x_1$.

For a voter v and a pair $\{d, e\}$ of distinct candidates, we say that $(v, \{d, e\})$ gives a *supremacy* to c iff v puts simultaneously c before d and e. To be a resistant Condorcet winner, it is necessary to have strictly more than $\frac{3}{2}V$ supremacies, because a strict majority is necessary against each pair of opponents.

If a voter puts c on top, she gives three supremacies to c: one over the pair of opponents $\{d_1, d_2\}$, one over $\{d_1, d_3\}$, and one over $\{d_2, d_3\}$. If a voter puts c in second position, she gives her only one supremacy, over the two candidates at the bottom of her ballot. If a voter puts c in third or fourth position, she gives her no supremacy. So, the "unit price" of a supremacy (in points of score) is $\frac{1}{3}x_1$ when c is in first position and x_2 when c is in second position.

Case 1 Assume $\frac{1}{3}x_1 \leq x_2$. Then, to be a resistant Condorcet winner while having as few points as possible, the optimum is to be in first position for a strict majority of voters (never in second position). In other words, it amounts to being a majority favorite. According to Lemma 3.18, the score of c score is then strictly greater than the average score.

Case 2 Assume $x_2 < \frac{1}{3}x_1$. Now, to be a resistant Condorcet winner while having as few points as possible, the optimum is to be in second position as often as possible. However, being always in second position is not enough, because it gives only V supremacies (instead of the $\frac{3}{2}V$ supremacies needed). To have enough supremacies, one must be in first position for $\frac{V}{4} + \varepsilon$ voters, for some $\varepsilon > 0$. With this notation, one must be in second position for at least $\frac{3V}{4} - 3\varepsilon$ voters (so, there remains 2ε voters whose ballots are left free). We then have, denoting mean(score) the average score of all the candidates:

$$\begin{aligned}
\text{score}\,(c) - \text{mean(score)} &\geq \frac{V}{4}x_1 + \frac{3V}{4}x_2 + \varepsilon\,(x_1 - 3x_2) - V\,\text{mean}(x_1, \dots, x_4) \\
&\geq \frac{V}{4}x_2 + \frac{V}{4}(x_2 - x_3) + \varepsilon\,(x_1 - 3x_2)\,.
\end{aligned}$$

[5] The condition can be violated only if $C \geq 3$.

Now the two first terms are nonnegative by definition of a PSR and that the third one is positive by assumption. Therefore, the score of c is strictly greater than the average score. □

The first statement of Lemma 3.19 cannot be extended to $C \geq 5$: in that case, a resistant Condorcet winner can even have a score that is strictly lower than the average score. Indeed, consider Plurality: $\mathbf{x} = (1, 0, \ldots, 0)$. Condition $\text{mean}(x_1, x_C) \geq \text{mean}(x_1, \ldots, x_C)$ is clearly satisfied. However, we have already shown in Lemma 3.11 that a resistant Condorcet winner can have a Plurality score that is strictly the lowest.

We now have all the elements to prove Proposition 3.16 about IPSR-1.

Proof. 1. Let us prove that f meets **MajUniBal**. If a strict majority coalition wishes to make candidate c win, it is sufficient that they put c on top of their ballots and all other candidates in a common arbitrary order. During the counting round starting with k remaining candidates, denote d_k the candidate put last by these proponents of c. Even if all other voters put d_k first and c last in their ballots, we have $\text{score}_k(c) > \frac{V}{2}(x_1^k + x_k^k) > \text{score}_k(d_k)$, hence c cannot be eliminated. Therefore, c wins the election.

2. From Lemma 3.18, we immediately deduce that f meets **MajFav** iff for all $k \in [\![3, C]\!]$, $\text{mean}(x_1^k, x_k^k) \geq \text{mean}(x_1^k, \ldots, x_k^k)$. If this condition is met and if $C \leq 4$, then Lemma 3.19 ensures that a resistant Condorcet winner cannot be eliminated, hence f meets **rCond**.

3. The fact that the only IPSR-1 meeting **Cond** is Baldwin's method was proven by Smith (1973): it is a direct consequence of Lemma 3.17. □

In order to complete Proposition 3.16 about IPSR-1, it remains to find a general criterion for **rCond**, but it seems compromised to obtain a simple relation valid for every number of candidates. We have a first taste of the problem in the proof of Lemma 3.19: the more candidates there are, the more ways there are to gain supremacies. Depending on the values of the weights, it may be cheaper (in points of score) to be more often placed first, second, etc. Moreover, if it is cheaper to be more often in k-th position for a certain $k > 1$, we may have to complete with voters who put the candidate in highest positions, as we did in the second case of the proof. Then, there is a sub-distinction of cases, depending on the unit price of supremacy in these different positions. One can therefore fear that it is necessary to distinguish all the nonempty subsets of the first $C - 2$ first positions as potential sources of supremacies, which may lead, in the worst case, to $2^{C-2} - 1$ inequalities.

Anyway, we know that for $C \geq 5$, an IPSR-1 may satisfy **MajFav** without satisfying **rCond**, since it is the case for IRV (Proposition 3.10).

From Proposition 3.16, we immediately deduce the following corollary about Coombs' method.

Corollary 3.20

1. *Coombs' method meets* **MajBal**.

2. *It does not meet* **MajFav** *(except in the trivial case $C \leq 2$).*

As well as for Corollary 3.15, we can prove the last result more concisely than by resorting to Proposition 3.16. Even if it is well known that Coombs' method does not satisfy **MajFav**, here is a counter-example for the record.

2	2	3
a	a	b
b	c	c
Others	Others	Others
c	b	a

Candidate a is the majority favorite but she is eliminated during the first round.

3.4.5 Iterated PSR with elimination based on the average

We have already studied IRVA in Proposition 3.10. We now study IPSR-A in general, which comprise the methods of Nanson and Kim-Roush. To this end, we will exploit some of the lemma we gave when we studied IPSR-1.

Proposition 3.21

Let f_C be an IPSR-A, with weight vectors $(\mathbf{x}^k)_{k\leq C}$. For all $C' < C$, denote $f_{C'}$ the IPSR-A of vectors of weights $(\mathbf{x}^k)_{k\leq C'}$.

1. *The following conditions are equivalent.*
   (a) *For all $C' \leq C$, $f_{C'}$ meets* **InfMC**.
   (b) *For all $k \in [\![3, C]\!]$:*
   $$\mathrm{mean}(x_1^k, x_k^k) \geq \mathrm{mean}(x_1^k, \ldots, x_k^k).$$
   When they are met, each $f_{C'}$ also meets **MajFav** *and, for $C' \leq 4$,* **rCond**.
2. *f_C meets* **Cond** *iff it is Nanson's method.*

It is worth noting that, if we weaken condition 1a by demanding only that f_C meets **InfMC**, then condition 1b may not be met. Indeed, let $C = 4$, $\mathbf{x}^4 = (1, 0, 0, 0)$ (Plurality), and $\mathbf{x}^3 = (1, 1, 0)$ (Veto). Then condition 1b is not met for $k = 3$. However, we will show that f meets **MajFav**, hence **InfMC**. Assume that a strict majority of voters put c on top of their ballots. Since there are 4 candidates in the beginning of the counting process, a candidate must have at least $\frac{V}{4}$ votes to get to the following round: since c has a strict majority of votes, at most one opponent of c may reach that threshold, hence at least two opponents are eliminated during the first round. As a consequence, there is no round for $k = 3$ and c is ensured to win. Therefore, f_C meets **MajFav** and weaker criteria, in particular **InfMC**.

Proof. not 1b $\Rightarrow$ not 1a: Let k be such that $\mathrm{mean}(x_1^k, x_k^k) < \mathrm{mean}(x_1^k, \ldots, x_k^k)$. Consider the voting system $f_{C'}$, with $C' = k$. Assume that $\frac{V}{2} - \varepsilon$ sincere voters put candidate c in last position, with $\varepsilon > 0$. In order to avoid an immediate elimination of c, the $\frac{V}{2} + \varepsilon$ manipulators cannot do better than putting her on top of their ballots. Then, denoting $\mathrm{mean}(\mathrm{score}_k)$ the average score of all candidates:

$$\begin{aligned}\mathrm{mean}(\mathrm{score}_k) - \mathrm{score}_k(c) &= V\,\mathrm{mean}(x_1^k, \ldots, x_k^k) - \left[\left(\frac{V}{2} - \varepsilon\right) x_k^k + \left(\frac{V}{2} + \varepsilon\right) x_1^k\right] \\ &= V\big[\mathrm{mean}(x_1^k, \ldots, x_k^k) - \mathrm{mean}(x_1^k, x_k^k)\big] - \varepsilon\big[x_1^k - x_k^k\big].\end{aligned}$$

If ε is small enough compared to V, then the above quantity is positive, hence c is eliminated. Therefore, $f_{C'}$ does not meet **InfMC**.

1b $\Rightarrow$ 1a: By Lemma 3.18, we know that condition 1b implies that a majority favorite has always a score strictly higher than the average, hence she cannot be eliminated. Therefore, each $f_{C'}$ meets **MajFav** and, as a consequence, weaker criteria, in particular **InfMC**.

Now, assume that these conditions are met. For $C' \leq 4$, Lemma 3.19 ensures that a resistant Condorcet winner has always a score strictly higher than the average, so she cannot be eliminated. Therefore, $f_{C'}$ meets **rCond**.

2. The fact that the only IPSR-A meeting **Cond** is Nanson's method is a direct consequence of Lemma 3.17, proven by Smith (1973). □

There remains to find a general criterion for **rCond** but, as for IPSR-1, we find it optimistic to hope for a simple relation valid for any number of candidates C. Here again, we know anyway that an IPSR-A can meet **MajFav** without meeting **rCond**, since it is the case of IRVA for $C \geq 5$ (Proposition 3.10).

To finish our study of IPSR-A, we examine the particular case of the Kim-Roush method.

Proposition 3.22
The Kim-Roush method does not meet **InfMC** *(except in the trivial case $C \leq 2$).*

It is not really a corollary of Proposition 3.21 (which would only lead to conclude that for every given $C \geq 3$, there exists $C' \leq C$ such that the Kim-Roush method on C' candidates does not meet **InfMC**). So, we have to prove this result directly.

Proof. We use the same counterexample as for Veto (Corollary 3.15). For $C \geq 3$ and $V = 7$ voters, assume that 3 voters vote against a given candidate c. Then, regardless of the ballots of the majority consisting of the other 4 voters, c is eliminated during the first round because she receives more vetoes than the average, equal to $\frac{V}{C} \leq \frac{7}{3}$. □

3.4.6 Simple or Iterated Bucklin's method

Proposition 3.23

1. *Bucklin's method meets* **MajFav**.
2. *It meets* **rCond** *iff $C \leq 3$.*
3. *It does not meet* **Cond** *(except in the trivial case $C \leq 2$).*

Proof. 1, 2. It is easy to prove the cases where these criteria are met. For **MajFav**, it is sufficient to note that when there is a majority favorite, her median rank is 1, whereas for every other candidate, her median rank is strictly greater. For **rCond**, it is sufficient to note that, for $C \leq 3$, a resistant Condorcet winner is necessarily a majority favorite (in the electoral space of strict total orders). It now remains to prove the cases where some criteria are not met.

2. Consider the following profile, where $C \geq 4$.

49	11	6	6	14	14
c	a	d_1	d_2	a	a
a	c	c	c	d_1	d_2
d_1	d_1	a	a	c	c
d_2	d_2	d_2	d_1	d_2	d_1
Others	Others	Others	Others	Others	Others

Candidate c is preferred to any pair (a, d_i) by 55 voters and to the pair (d_1, d_2) by 60 voters out of 100, hence she is the resistant Condorcet winner. But, in the sense of Bucklin, we have score$(c) = (2, 72)$ and score$(a) = (2, 88)$, hence a wins. Therefore, for $C \geq 4$, Bucklin's method does not meet **rCond**.

3. It is well known that Bucklin's method does not meet **Cond** (except in the trivial case $C \leq 2$) but we are going to give a counter-example for the record.

40	15	15	30
c	a	b	a
a	c	c	b
b	b	a	c
Others	Others	Others	Others

Candidate c is the Condorcet winner, but score$(c) = (2, 70)$ and score$(a) = (2, 85)$. Hence, a is elected. □

Proposition 3.24

1. *The Iterated Bucklin's method meets* **MajFav**.
2. *It meets* **rCond** *iff* $C \leq 4$.
3. *It does not meet* **Cond** *(except in the trivial case* $C \leq 2$*).*

To prove this proposition, we will use the following lemma.

Lemma 3.25
Consider the electoral space of strict total orders with $C \leq 4$ and assume that some candidate c is the resistant Condorcet winner.

Then there exists a candidate d whose median rank (in the sense of Bucklin) is strictly worse than the median rank of c. In particular, the Bucklin score of d is strictly worse than that of c.

Proof. If $C \leq 3$, it follows from the fact that a resistant Condorcet winner is necessarily a majority favorite (in the electoral space of strict total orders): hence, her median rank is 1 and she is the only one candidate with this property.

If $C = 4$, denote the candidates c, d_1, d_2, and d_3. Since c is the resistant Condorcet winner, she is preferred by a strict majority of voters to d_1 and d_2 simultaneously: so, this majority of voters put c at rank 1 or 2. As a consequence, the median rank of c (in the sense of Bucklin) is 2 at worst, i.e. 1 or 2.

Assume that no candidate has a strictly worse median rank. For this, it is necessary that each of the 4 candidates (including c) occupies strictly more than $\frac{V}{2}$ positions in ranks 1 or 2 of the V voters: so, there are strictly more than $2V$ pigeons for $2V$ holes, which is a contradiction. □

We can now prove Proposition 3.24.

Proof. 1, 2. It is clear that IB meets **MajFav**. Lemma 3.25 proves that for $C \leq 4$, it meets **rCond**. Now, let us prove the cases where some criteria are not met.

2. For $C = 5$, denote the candidates $\{c, d_1, \ldots, d_4\}$. Let $\alpha = 18$, $\beta = 4$, and $\gamma = 15$. Consider the following profile. For the first column, for example, our notation below means that, for each permutation σ of integers from 1 to 4, there are α voters who prefer c then $d_{\sigma(1)}$, $d_{\sigma(2)}$, $d_{\sigma(3)}$, and finally $d_{\sigma(4)}$. Overall, for the first column, there are $4! \times \alpha = 24\alpha$ voters who put c in first position. In total, there are $24(\alpha + \beta + \gamma) = 888$ voters.

24α	24β	24γ
c	•	•
•	•	•
•	c	•
•	•	•
•	•	c

Candidate c is preferred to any pair (d_i, d_j) by $24\alpha + 4\beta = 448$ voters, hence she is the resistant Condorcet winner. In the sense of Bucklin, we have $\text{score}(c) = (3, 24\alpha + 24\beta) = (3, 528)$ and for every other candidate d_i, we have $\text{score}(d_i) = (3, 12\alpha + 12\beta + 18\gamma) = (3, 534)$. Hence, c is eliminated. Therefore, IB does not meet **rCond**.

3. For $C = 3$, let us prove that IB does not meet **Cond**.

24	24	4	4	22	22
c	c	d_1	d_2	d_1	d_2
d_1	d_2	c	c	d_2	d_1
d_2	d_1	d_2	d_1	c	c

Candidate c is preferred to any other candidate d_i by 52 voters, hence she is the Condorcet winner. But, in the sense of Bucklin, we have $\text{score}(c) = (2, 56)$ and $\text{score}(d_i) = (2, 72)$, so c is eliminated.

To adapt these counterexamples to more candidates, simply add other candidates in a common order at the end of each order of preference: these dummy candidates will be eliminated in the first rounds of counting and we will be brought back to the above counterexamples. □

3.5 Informational aspect of the majoritarian criteria

To conclude this chapter, we propose a qualitative interpretation of the criteria we studied. This is an informal discussion, whose goal is simply to provide a thought experiment to understand the information and communication problems that may arise to vote strategically in a given voting system and potentially find an SNE or, at least, find the same candidate as if we had found an SNE.

For the sake of simplicity, we consider an electoral space where there is no semi-Condorcet configuration, for example the electoral space of strict total orders with an odd number of voters: thus, we can identify **RSNEA** and **RSNEC** on the one hand, **XSNEA** and **XSNEC** on the other.

We limit our discussion to voting systems satisfying **InfMC**, since these seem to be favored by practice in the cases of application where is it desired to have a certain equality between the voters on the one hand, and between candidates on the other hand (not that we neglect the interest of other types of systems, which we will study in the next chapter). With this assumption, we have seen that **RSNEA**, in that case equivalent to **RSNEC**, is also satisfied. So the only configurations ω that might have an SNE are the Condorcet configurations, and the only possible winner of an SNE is the Condorcet winner.

Finally, we assume that the electoral space is finite, which ensures the existence of a voting system of minimal manipulability (in the set-theoretic sense) within any class of voting systems, and in particular in the class **InfMC**.[6]

The objective of our thought experiment is to illustrate the following fact: the more criteria are satisfied by a given voting system in the inclusion diagram of Figure 3.1, the more we progress on the following aspects:

- More often there is at least one SNE,
- It is easier to achieve *the same result as* an SNE (i.e. elect the Condorcet winner if there is one),
- It is easier to achieve an SNE.

When we use the expression "easier", it is from the point of view of the quantity of information to be exchanged between the agents: the higher it is, the more difficult the task.

Here is our thought experiment, the conclusions of which are summarized in Table 3.1. Imagine that an external coordinator wants to help voters find the only possible outcome of a possible SNE (i.e. the Condorcet winner with our assumptions). It is assumed that there is a strict majority of voters, which we call here *the* coalition, who wish to collaborate with the coordinator to achieve this objective. Using the **InfMC** criterion, we can imagine the following protocol, where we exploit some supernatural gifts of the coordinator.

1. The coordinator has the magic power to know the voters' preferences, which allows her to identify which candidate c is the Condorcet winner. So, this is the target candidate that we want to get elected.
2. The coalition members send her a message informing her of their identity and that they will follow her instructions.
3. The coordinator has the magic power to know the ballots of the other voters.
4. By exploiting **InfMC**, the coordinator determines ballots for the coalition members which make c win and she sends this information to them.

To carry out this protocol, there are four aspects of information to be managed: on the one hand, the coordinator must know the target candidate c, the identity of the coalition members, and the ballots of the other voters; on the other hand, she must send to the coalition members their ballot assignments.

Obviously, we do not claim that such a protocol is actually possible. The purpose of this thought experiment is simply to illustrate what information and communication issues can arise in a population of strategic voters looking for a

[6] We will see in Chapter 5 another important sufficient condition which guarantees the existence of an optimum (in the probabilistic sense, this time) in certain particular cultures, even in infinite electoral spaces, typically those used for certain cardinal systems.

	InfMC	IgnMC	XSNEC	MajBal	MajFav	rCond	Cond
Property of SNE							
Maximize the set of all ω with an SNE			✓	✓	✓	✓	✓
There might exist a system with minimal manipulability						✓	✓
There exists a system with minimal manipulability							✓
Any SNE winner coincides with the sincere winner							✓
Information and communication issues							
Target candidate c	✗	✗	✗	✗	✗	✗	
Assignments of ballots	✗	✗	✗	✗			
Members of the coalition	✗	✗	✗				
Ballots of the other voters	✗						

Table 3.1 – Informational aspect of the majoritarian and equilibrium criteria. Each column illustrates the properties of the set of voting systems that satisfy a criterion but not the more demanding criteria.

strong Nash equilibrium or, at least, seeking to elect the same candidate as if they had reached an equilibrium.

Let us now see how the situation improves if we require voting systems that satisfy criteria stronger than **InfMC**. The thoughts below are summarized in Table 3.1.

If the voting system satisfies **IgnMC**, then the coordinator no longer needs to know the ballots of the voters not belonging to the coalition.

If the voting system satisfies **XSNEC**, the exchanges of information are *a priori* the same (lower part of the table). But the system maximizes the set of configurations of preference where an SNE exists, since we guarantee that every Condorcet configuration has an SNE and that the criterion **InfMC**, equivalent to **RSNEC** under our hypotheses, impose that these are the only ones. Thus, **XSNEC** does no modify the protocol used, but maximizes the set of preference configurations where it achieves a result that is the same as that of an SNE.

If the voting system satisfies **MajBal**, then up to broadcasting a ballot assignment to all the voters and not only to the coalition members, the coordinator does not need to know in advance which voters will obey her instructions. In practice, all common voting systems satisfying **MajBal** also satisfy **MajUniBal** (such as the IPSR-1, like Coombs' method), so the coordinator can just broadcast a single ballot and not use individualized messages for each voter. This is the case, for example, of Approval voting, for which Laslier (2009) shows, through a different approach, that an equilibrium can be reached relatively cheaply.

If the voting system satisfies **MajFav**, then there is no need to broadcast a ballot assignment to the voters, the coordinator can just tell them the name of candidate c instead: obedient voters just need to put c at the top of their ballot.

If we place ourselves in the class **rCond**, information issues are *a priori* the same (lower part of the table). But it is not excluded that one of these voting systems possesses minimal manipulability (in the set-theoretic sense), whereas it is impossible for a voting system not satisfying **rCond**, as mentioned in the corollary 2.22 of the Condorcification theorems. Such a system with minimal manipulability maximizes the set of non-manipulable situations, i.e. the set of configurations where an SNE can be found without any exchange of information, simply by sincere voting.

If the voting system satisfies **Cond**, then when there is an SNE, its result coincides with that of the sincere vote. The coordinator can then exchange no information with the voters: if they vote sincerely, they can always find the same result as a possible SNE without any exchange of information.

Moreover, in class **Cond**, the existence of a voting system with minimal manipulability is not only possible but guaranteed by Corollary 2.22. If such an optimal system is used, then voters have the ability to find an SNE as often as possible without any exchange of information.

Chapter 4

Generalized Condorcification

So far, we have focused on voting systems that satisfy **InfMC**, which is a criterion related to the notion of majority. As we discussed in the introduction of this memoir, the majoritarian principle stems from simple majority voting and, as formalized by May's theorem (May, 1952), it stems directly from principles of anonymity and neutrality.

However, there are practical applications where a voting system violates anonymity, neutrality, or both, for reasons that can be defended as legitimate. For example, a meeting of co-owners or an assembly of shareholders of a company is generally not symmetrical between the voters. When the French Constitution is revised, the new version must obtain two thirds of the votes, which creates an asymmetry between the candidates: indeed, the other candidate is the old version of the Constitution, which only needs to exceed one third of the votes.

In the general case, criteria like **InfMC** are not necessarily satisfied by the voting system used and we cannot therefore apply the weak Condorcification theorem 2.9: thus, we have no guarantee not to increase manipulability by using the usual Condorcification, based on the notion of majority. On the other hand, one can hope that a similar theorem is valid with a notion inspired by Condorcification. For this, we use an approach inspired by the theory of *simple games* and we define *generalized Condorcification*, then we explore its connections with manipulability.

In Section 4.1, we define the notion of *family*, which is central to this chapter. To each candidate c, a family $\mathcal{M}$ associates a set $\mathcal{M}_c$ of coalitions. This type of object will then make it possible to describe, for each candidate c, a set of coalitions that have the power to make c win, in a certain sense that we will choose (more specifically, either in an ignorant or informed manner). We use this notion to extend the notions of Condorcet winner, Condorcet-admissible candidate, and majority favorite.

In Section 4.2, we generalize certain majority criteria, limiting ourselves to those that will be useful for the theorems of this chapter. In particular, we generalize **InfMC** into a criterion denoted by $\mathcal{M}$**InfC**. In addition, we define two entirely new criteria, $\mathcal{M}$**InfC-A** and $\mathcal{M}$**IgnC-A**, which require that a coalition of $\mathcal{M}_c$ (respectively informed or ignorant) can not only make candidate c win, but can also do so while ensuring that c is *admissible* in the sense of the family $\mathcal{M}$ considered. These technical criteria will allow us, then, to compose generalized Condorcifications using different families. Furthermore, as we did for majoritarian criteria, we establish implication relationships between the criteria under study.

In Section 4.3, we naturally define the $\mathcal{M}$-Condorcification of a voting system f, which we denote by $f^{\mathcal{M}}$. It is then easy to adapt the proof of the weak Condorcification theorem 2.9 to show that, if a voting system meets $\mathcal{M}$**InfC**, then $f^{\mathcal{M}}$ is at most as manipulable as f: it is the *generalized Condorcification theorem* 4.18.

For the sake of conciseness, we do not go further into the parallel with Chapter 2: in particular, we do not generalize the notion of resistant Condorcet winner and the strong Condorcification theorem 2.20. Rather, we prefer to focus of the new possibilities offered by generalized Condorcification.

In particular, this leaves the choice of the family $\mathcal{M}$ used to define $f^{\mathcal{M}}$. In Section 4.4, we compare Condorcifications performed using two families $\mathcal{M}$ and $\mathcal{M}'$, and we introduce the *compared Condorcification theorem* 4.21 which makes it possible to prove, under certain assumptions, that $f^{\mathcal{M}'}$ is at most as manipulable as $f^{\mathcal{M}}$.

In Section 4.5, we address the following natural question: among the families $\mathcal{M}$ that make it possible to benefit from the generalized Condorcification theorem 4.18, is there one that decreases manipulability the most? To answer it, we define the *maximal family* of f: for each candidate c, we consider all the coalitions that are able to make c win when they are informed of the ballots of the other voters. We call *maximal Condorcification* of f the system $f^{\mathcal{M}'}$, where $\mathcal{M}'$ is the maximal family. Then we prove the *maximal Condorcification theorem* 4.25 which states, under certain assumptions, that among the Condorcifications satisfying the assumptions of Theorem 4.18, the maximal Condorcification is the least manipulable.

In Section 4.6, we examine various examples of applications of generalized Condorcification. For classical voting systems satisfying **InfMC**, we show that the maximal Condorcification is the Condorcification in the usual sense, that is to say, the majoritarian one (Section 4.6.1). Then we study the maximal Condorcification of Veto. Finally, we study the generalized Condorcification of various voting systems that violate anonymity, neutrality, or both.

4.1 Family of collections of coalitions

As a reminder, the theory of *simple games* is used to study some cooperative games in a framework that is abstract and relatively simplified. We have a subset of players who, in our case, are voters. A *simple game* is defined by a set of coalitions $\mathcal{M}$, i.e. each of these is a subset of voters. If a coalition M belongs to $\mathcal{M}$, it is called a *winning* coalition; otherwise, a *losing* coalition.

The spirit of this model is that, if members of a winning coalition M manage to coordinate (according to some kind of modality), then they win the game, generally at the disadvantage of the other players. Depending on the set of coalitions $\mathcal{M}$ that define the simple game studied, a variety of questions can be investigated, such as the respective powers of the players and the way rewards can be shared between the winners (which leads to define notions like the *Shapley value*).

In the case of voting systems, the coalitions that can make a given candidate c win *a priori* depend on c: in the example of changing the French Constitution, one third of the voters is enough to keep the old version, while two thirds are necessary to switch to the new one. These observations lead to define the notion of *family*, which is the central object of this chapter.

4.1.1 Definition of a family and basic properties

Definition 4.1 (family of collections of coalitions)
We call *family of collections of coalitions*, or *family*, a function:

$$\mathcal{M} : \left| \begin{array}{lcl} \mathcal{C} & \to & \mathcal{P}(\mathcal{P}(\mathcal{V})) \\ c & \to & \mathcal{M}_c. \end{array} \right.$$

Intuitively, we suggest to think of a family as an object giving the following information: for each candidate c, it describes what coalitions of voters can make c win, in several senses that we will give later. So, for a coalition of voters $M \in \mathcal{P}(\mathcal{V})$, we say that M is an *$\mathcal{M}$-winning coalition for c* iff $M \in \mathcal{M}_c$. For the moment, it is only a convention of language. We will soon see why it is convenient.

Definition 4.2 (basic properties of a family)
We say that $\mathcal{M}$ is *monotonic* iff $\forall c \in \mathcal{C}, \forall (M, M') \in \mathcal{P}(\mathcal{V})^2$:

$$M \in \mathcal{M}_c \text{ and } M \subseteq M' \Rightarrow M' \in \mathcal{M}_c.$$

We say that $\mathcal{M}$ is *exclusive* iff $\forall (c, d) \in \mathcal{C}^2, \forall (M, M') \in \mathcal{P}(\mathcal{V})^2$, if $c \neq d$, then:

$$M \in \mathcal{M}_c \text{ and } M' \in \mathcal{M}_d \Rightarrow M \cap M' \neq \varnothing.$$

We say that $\mathcal{M}$ is *neutral* iff $\forall (c, d) \in \mathcal{C}^2, \mathcal{M}_c = \mathcal{M}_d$.
We say that $\mathcal{M}$ is *anonymous* iff $\forall c \in \mathcal{C}, \forall \sigma \in \mathfrak{S}_{\mathcal{V}}, \forall M \in \mathcal{P}(\mathcal{V}) : M \in \mathcal{M}_c \Leftrightarrow \sigma(M) \in \mathcal{M}_c$, where $\sigma(M)$ denotes the coalition obtained by considering the images of the members of coalition M by a permutation σ of the voters.

Monotonicity means that if M is a winning coalition for c, then any coalition M' containing M is a winning coalition for c. With the interpretation we gave, it is a quite natural assumption.
Exclusivity means that if two coalitions M and M' are disjoint, then they cannot be winning, respectively, for two distinct candidates c and d. This notion is similar to the notion of *proper* simple game, which demands that a coalition and its complement cannot be both winning. If we assume that voters in M and those in M' have the same powers, then it is natural to consider that the two coalitions cannot make c and d win simultaneously, which is the case for an ignorant manipulation. In contrast, if we consider informed manipulation, then exclusivity is not obvious: indeed, an informed coalition might manipulate for some candidate a; then a disjoint informed coalition, knowing these ballots, might change their own ballots to make another candidate b win; but then, the first coalition, knowing these new ballots, might change their ballots to make a win again, etc. We will see an example of this phenomenon with parity voting.
Finally, the meaning of anonymity and neutrality is obvious. About anonymity, it is clearly equivalent to say that for a coalition M, belonging to $\mathcal{M}_c$ depends only on its cardinality.

In order to have a convenient recurring example for the rest of the chapter, we now define a simple particular case: *threshold families*.

Definition 4.3 (threshold family)
For $\alpha \in [0, V]$, we call *family of threshold α* the neutral family consisting of all coalitions with strictly more than α voters. Formally:

$$\mathcal{M} : \left| \begin{array}{lcl} \mathcal{C} & \to & \mathcal{P}(\mathcal{P}(\mathcal{V})) \\ c & \to & \{M \in \mathcal{P}(\mathcal{V}) \text{ s.t. } \mathrm{card}(M) > \alpha\}. \end{array} \right.$$

For $x \in [0, 100]$, we will also write *family of x %* to designate the family of threshold $V \cdot x$ %.

We call *majoritarian family* the family of 50 %, that is, the one consisting of all coalitions with strictly more than half of the voters.

It is easy to see that threshold families are the only neutral, anonymous, and monotonic families. Moreover, the majoritarian family is exclusive. Among the threshold families, it is the maximal exclusive family (maximal in the sense of inclusion, i.e. with the minimal threshold). For example, the family of 90 % (which is also exclusive) does not contain a coalition with only 70 % of the voters, whereas the majoritarian family does.

4.1.2 Victories and generalized Condorcet notions

We now quickly follow the same steps as in the previous chapters, where we were focusing on the majoritarian family. Fortunately, having authorized preferences that can fail to be antisymmetric has already lead us, in the particular case, to adopt definitions that are quite general and can be easily extended to the framework of this chapter.

We saw in Chapter 2, and especially in Section 2.5, that the notion of absolute victory $\mathrm{P}_{\mathrm{abs}}$ is the one naturally leading to Condorcification theorems, unlike the relation $\mathrm{P}_{\mathrm{rel}}$. So, we will focus on generalizing the former.

Definition 4.4 ($\mathcal{M}$-victory, $\mathcal{M}$-defeat)

For $\omega \in \Omega$, we denote $\mathrm{P}_{\mathcal{M}}(\omega)$, or in short $\mathrm{P}_{\mathcal{M}}$, the binary relation on candidates defined by $\forall (c, d) \in \mathcal{C}^2$:

$$c \,\mathrm{P}_{\mathcal{M}}\, d \Leftrightarrow \{v \in \mathcal{V} \text{ s.t. } c \,\mathrm{P}_v\, d\} \in \mathcal{M}_c.$$

In other words, this relation means that the set of voters who prefer c to d is a winning coalition for c (in the sense of family $\mathcal{M}$).

When this relation is met, we say that c has an *$\mathcal{M}$-victory* against d in ω, or that d has an *$\mathcal{M}$-defeat* against c in ω.

We say that c has a *strict $\mathcal{M}$-victory* (resp. *strict $\mathcal{M}$-defeat*) against d in ω iff she has an $\mathcal{M}$-victory and no $\mathcal{M}$-defeat (resp. an $\mathcal{M}$-defeat and no $\mathcal{M}$-victory) against d.

In the case of the majoritarian family, we saw (as a consequence of Proposition 1.25) that if preferences are antisymmetric, then the victory relation is as well. But this is not necessarily the case in general. Indeed, consider the family of 40 % and the following profile:

45	10	45
c	c, d	d
d		c

Then c has a $\mathcal{M}$-victory against d but it is not strict, since d also has an $\mathcal{M}$-victory against c. However, it is easy to give a sufficient condition for the victory relation to be antisymmetric.

Proposition 4.5

Let $\omega \in \Omega$. Assume that:

- *The family $\mathcal{M}$ is exclusive,*
- *The relations P_v are antisymmetric.*

Then the relation $\mathrm{P}_\mathcal{M}$ is antisymmetric. In other words, between two distinct candidates c and d, there cannot be mutual $\mathcal{M}$-victories.

Proof. If two distinct candidates c and d have mutual $\mathcal{M}$-victories, then by definition:

$$\begin{cases} \{v \in \mathcal{V} \text{ s.t. } c \, \mathrm{P}_v \, d\} \in \mathcal{M}_c, \\ \{v \in \mathcal{V} \text{ s.t. } d \, \mathrm{P}_v \, c\} \in \mathcal{M}_d. \end{cases}$$

Hence, by exclusivity, $\{v \in \mathcal{V} \text{ s.t. } c \ \mathrm{P}_v \ d\} \cap \{v \in \mathcal{V} \text{ s.t. } d \ \mathrm{P}_v \ c\} \neq \varnothing$, which contradicts the antisymmetry of relations P_v. □

For the majoritarian family, we already noted that the victory relation is not necessarily complete. On one hand, with an even number of voters, it can be the case that exactly half of the voters prefer c to d and the other half prefer d to c (even if the preferences meet quite strong assumption, such as being strict total orders). On the other hand, if preferences are strict weak orders, for example, it is possible that 45 % of the voters prefer c to d and that the same number of voters have the opposite opinion: then, none of the two candidates has a victory against the other. That said, the following proposition gives a sufficient condition for the victory relation to be complete.

Proposition 4.6

Let $\omega \in \Omega$. We assume that:

- *The family $\mathcal{M}$ is such that for each pair (c, d) of distinct candidates and for each pair $(M, M') \in \mathcal{P}(\mathcal{V})^2$, if $M \cup M' = \mathcal{V}$, then $M \in \mathcal{M}_c$ or $M' \in \mathcal{M}_d$;*
- *The relations P_v are complete.*

Then the relation $\mathrm{P}_\mathcal{M}$ is complete. In other words, between two distinct candidates c and d, there cannot be an absence of $\mathcal{M}$-victory.

In this proposition, the assumption on the family $\mathcal{M}$ is to be compared to the notion of *strong* simple game, which imposes that among a coalition and its complement, at least one of them must be winning.

Proof. Let c and d be two distinct candidates. Denote $M = \{v \in \mathcal{V} \text{ s.t. } c \, \mathrm{P}_v \, d\}$ and $M' = \{v \in \mathcal{V} \text{ s.t. } d \ \mathrm{P}_v \ c\}$. Since relations P_v are complete, $M \cup M' = \mathcal{V}$. Hence, from the assumption on $\mathcal{M}$, we deduce $M \in \mathcal{M}_c$ or $M' \in \mathcal{M}_d$. □

From Proposition 4.6, it follows that for each family containing the majoritarian family, then the victory relation $\mathrm{P}_\mathcal{M}$ is complete. But it is not necessarily antisymmetric. For example, for the family of 30 %, if preferences are complete, then in any duel, at least one candidate has a victory against the other, but it is possible that the converse is simultaneously true.

Since we have extended the notion of victory, it is immediate to do the same with the Condorcet winner and Condorcet-admissible candidates.

Definition 4.7 ($\mathcal{M}$-Condorcet, $\mathcal{M}$-admissible)

Let $\omega \in \Omega$ and $c \in \mathcal{C}$.

We say that c is $\mathcal{M}$-*Condorcet* in ω iff c has an $\mathcal{M}$-strict victory against any candidate d, i.e.:

$$\forall d \in \mathcal{C} \setminus \{c\}, \begin{cases} \{v \in \mathcal{V} \text{ s.t. } c \, \mathrm{P}_v \, d\} \in \mathcal{M}_c, & (4.1) \\ \{v \in \mathcal{V} \text{ s.t. } d \, \mathrm{P}_v \, c\} \notin \mathcal{M}_d. & (4.2) \end{cases}$$

We say that c is $\mathcal{M}$-*admissible* in ω iff c has no $\mathcal{M}$-defeat, i.e.:

$$\forall d \in \mathcal{C} \setminus \{c\}, \{v \in \mathcal{V} \text{ s.t. } d \, \mathrm{P}_v \, c\} \notin \mathcal{M}_d.$$

If the family $\mathcal{M}$ is exclusive and if relations P_v are antisymmetric, then any victory is strict (Proposition 4.5) hence condition 4.2 may be omitted: it is implied by condition 4.1. In the particular case of the majoritarian family, we had already noted this simplification in the definition 1.26 of the Condorcet winner.

The following proposition extends Proposition 1.31, and its proof is immediate from the definitions.

Proposition 4.8
Let $\omega \in \Omega$. If a candidate is an $\mathcal{M}$-Condorcet in ω, then:

- *She is $\mathcal{M}$-admissible,*
- *No other candidate is $\mathcal{M}$-admissible.*

In particular, if there is an $\mathcal{M}$-Condorcet candidate, then she is unique.

Actually, the motivation of condition (4.2) in Definition 4.7 is precisely to ensure that in the general case, on one hand, any $\mathcal{M}$-Condorcet candidate is also $\mathcal{M}$-admissible, and on the other hand, that the $\mathcal{M}$-Condorcet candidate, when she exists, is unique.[1]

4.1.3 $\mathcal{M}$-favorite candidate

We now extend the notion of majority favorite. It will not be directly used in the generalized Condorcification theorem 4.18, but it will be a convenient tool to establish a connection between generalized Condorcet notions and manipulation. We will see that this notion requires additional caution, compared to the particular case of the majority favorite.

Definition 4.9 ($\mathcal{M}$-favorite)
For $\omega \in \Omega$ and $c \in \mathcal{C}$, we say that c is *$\mathcal{M}$-favorite* in ω iff:

$$\{v \in \mathcal{V} \text{ s.t. } \forall d \in \mathcal{C} \setminus \{c\}, c \,\mathrm{PP}_v\, d\} \in \mathcal{M}_c.$$

If we consider the family of 40 %, then it is clear that the $\mathcal{M}$-favorite is not always unique: indeed, 45 % of the voters may prefer a certain candidate c, and as many voters may prefer another candidate d. The following proposition gives a sufficient condition for the $\mathcal{M}$-favorite to be unique and it shows that, under quite natural assumption, this condition is necessary.

Proposition 4.10
We consider the following conditions.

1. *$\mathcal{M}$ is exclusive.*
2. *In any configuration with an $\mathcal{M}$-favorite candidate, she is unique.*

We have 1 $\Rightarrow$ 2.
If we assume that the electoral space allows any candidate as most liked and that $\mathcal{M}$ is monotonic, then 2 $\Rightarrow$ 1.

[1] Incidentally, one can notice that in the case of a neutral family, its *Nakamura number* (Nakamura, 1979) provides a necessary and sufficient condition on the number of candidates C such that any profile of transitive preferences has at least one $\mathcal{M}$-admissible candidate.

Proof. The implication 1 $\Rightarrow$ 2 being easy, we will only prove not 1 $\Rightarrow$ not 2.

Assume that $\mathcal{M}$ is not exclusive. Then there are two distinct candidates c and d, two disjoint coalitions M and M', such that $M \in \mathcal{M}_c$ and $M' \in \mathcal{M}_d$. Since $\mathcal{M}$ is monotonic, we have $\mathcal{V} \setminus M \in \mathcal{M}_d$. Since the electoral space allows any candidate as most liked, there exists a configuration where members of M claim that c is their favorite and members of $\mathcal{V} \setminus M$ claim that d is their favorite. Then candidate c and d are both $\mathcal{M}$-favorite. $\square$

As we have already mentioned in the previous chapters, it is clear that a majority favorite is necessarily a Condorcet winner. The following proposition, whose proof is immediate from the definitions, extends this observation if the family under consideration is monotonic and exclusive, as is the majoritarian family.

Proposition 4.11

Assume that $\mathcal{M}$ is monotonic and exclusive.

For $\omega \in \Omega$ and $c \in \mathcal{C}$, if c is $\mathcal{M}$-favorite in ω, then c is $\mathcal{M}$-Condorcet in ω.

4.2 Criteria associated to a family

4.2.1 Definitions

We now extend the criteria **Cond**, **MajFav**, **IgnMC**, and **InfMC**, which we had defined in the particular case of the majoritarian family. We also define two brand new criteria, $\mathcal{M}$**IgnC-A** and $\mathcal{M}$**InfC-A**. Later, we will show that they make it possible to combine and compare the generalized Condorcifications based on different families $\mathcal{M}$ and $\mathcal{M}'$ for a given voting system. For the sake of conciseness, we will not mention the generalization of the other majoritarian criteria.

Definition 4.12 (criteria associated to a family $\mathcal{M}$)

We say that f *meets the $\mathcal{M}$-Condorcet criterion* ($\mathcal{M}$**Cond**) iff, for all $\omega \in \Omega$ and $c \in \mathcal{C}$, if c is $\mathcal{M}$-Condorcet in ω, then $f(\omega) = c$.

We say that f *meets the $\mathcal{M}$-favorite criterion* ($\mathcal{M}$**Fav**) iff, for all $\omega \in \Omega$ and $c \in \mathcal{C}$, if c is $\mathcal{M}$-favorite in ω, then $f(\omega) = c$.

We say that f *meets the ignorant $\mathcal{M}$-coalition criterion* ($\mathcal{M}$**IgnC**) iff $\forall c \in \mathcal{C}, \forall M \in \mathcal{M}_c, \exists \omega_M \in \Omega_M$ s.t.:

$$\forall \omega_{\mathcal{V}\setminus M} \in \Omega_{\mathcal{V}\setminus M}, f(\omega_M, \omega_{\mathcal{V}\setminus M}) = c.$$

We say that f *meets the admissible ignorant $\mathcal{M}$-coalition criterion* ($\mathcal{M}$**IgnC-A**) iff $\forall c \in \mathcal{C}, \forall M \in \mathcal{M}_c, \exists \omega_M \in \Omega_M$ s.t.:

$$\forall \omega_{\mathcal{V}\setminus M} \in \Omega_{\mathcal{V}\setminus M}, \begin{cases} f(\omega_M, \omega_{\mathcal{V}\setminus M}) = c, \\ c \text{ is } \mathcal{M}\text{-admissible in } (\omega_M, \omega_{\mathcal{V}\setminus M}). \end{cases}$$

We say that f *meets the informed $\mathcal{M}$-coalition criterion* ($\mathcal{M}$**InfC**) iff $\forall c \in \mathcal{C}, \forall M \in \mathcal{M}_c, \forall \omega_{\mathcal{V}\setminus M} \in \Omega_{\mathcal{V}\setminus M}$:

$$\exists \omega_M \in \Omega_M \text{ s.t. } f(\omega_M, \omega_{\mathcal{V}\setminus M}) = c$$

We say that f *meets the admissible informed $\mathcal{M}$-coalition criterion* ($\mathcal{M}$**InfC-A**) iff $\forall c \in \mathcal{C}, \forall M \in \mathcal{M}_c, \forall \omega_{\mathcal{V}\setminus M} \in \Omega_{\mathcal{V}\setminus M}$:

$$\exists \omega_M \in \Omega_M \text{ s.t. } \begin{cases} f(\omega_M, \omega_{\mathcal{V}\setminus M}) = c, \\ c \text{ is } \mathcal{M}\text{-admissible in } (\omega_M, \omega_{\mathcal{V}\setminus M}). \end{cases}$$

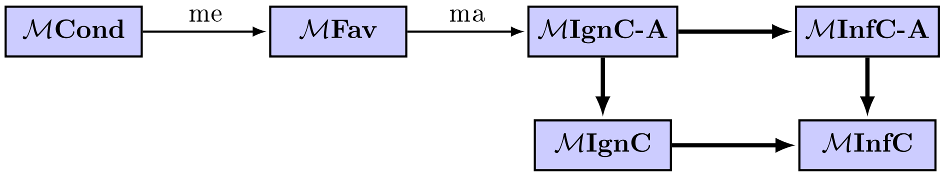


Figure 4.1 – Implications between the criteria associated to a family $\mathcal{M}$ (m: monotonic family, e: exclusive family, a: the electoral space allows any candidate as most liked).

The criteria $\mathcal{M}$**Cond**, $\mathcal{M}$**Fav**, $\mathcal{M}$**IgnC**, and $\mathcal{M}$**InfC** are quite natural generalizations of the corresponding criteria for the majoritarian family. The criterion $\mathcal{M}$**InfC-A** (resp. $\mathcal{M}$**IgnC-A**) is a stronger version of $\mathcal{M}$**InfC** (resp. $\mathcal{M}$**IgnC**) which requires that the informed (resp. ignorant) coalition, in addition to succeeding in making a given candidate c win, be able to ensure that c appears as $\mathcal{M}$-admissible. For usual voting systems, it is a natural condition: using the notations of the definition, if the members of M can make c win, then they can generally do so by casting a ballot where c is strictly preferred over other candidates; in that case, c is $\mathcal{M}$-favorite in the configuration obtained by manipulation and, under the common assumption that $\mathcal{M}$ is monotonic and exclusive, this implies that c is $\mathcal{M}$-Condorcet (Proposition 4.11) and *a fortiori* $\mathcal{M}$-admissible.

As we have already noted informally, the exclusivity assumption is natural when it comes to ignorant manipulation. The following proposition formalizes this observation.

Proposition 4.13
If a voting system satisfies $\mathcal{M}$**IgnC**, *then the family* $\mathcal{M}$ *is exclusive.*

Proof. If two disjoint coalitions have the respective power to make two distinct candidates c and d win in an ignorant way, then they can make c and d win simultaneously, which contradict the uniqueness of the output of f. □

In contrast, the exclusivity assumption is not obvious when it comes to informed manipulation: when a voting system meets $\mathcal{M}$**InfC**, it may be the case that $\mathcal{M}$ is not exclusive.

Indeed, consider parity voting: if there are an odd (resp. even) number of black balls, then candidate a (resp. b) is declared the winner. Consider the neutral family $\mathcal{M}$ consisting of all nonempty coalitions (i.e. the family of threshold 0): then parity voting meets $\mathcal{M}$**InfC**. Indeed, denoting v and v' two distinct voters (we assume $V \geq 2$), the coalitions $\{v\}$ and $\{v'\}$ are distinct and both of them are winning, hence the family is not exclusive.

In this voting system, if v votes last and knows the other ballots, then she can choose the output; and so does v'. So, the non-exclusivity is not contradictory with the uniqueness of the output.

4.2.2 Implications between the criteria associated to a family

The following proposition gives the implication relationships between the different criteria, i.e. inclusions between the corresponding sets of voting systems. All these results are summarized in the implication graph of Figure 4.1.

Proposition 4.14

If $\mathcal{M}$ is monotonic and exclusive, then $\mathcal{M}$**Cond** $\subseteq$ $\mathcal{M}$**Fav**.

If $\mathcal{M}$ is monotonic and if the electoral space allows any candidate as most liked, then $\mathcal{M}$**Fav** $\subseteq$ $\mathcal{M}$**IgnC-A**.

We have the following inclusions:

- $\mathcal{M}$**IgnC-A** $\subseteq$ $\mathcal{M}$**IgnC**,
- $\mathcal{M}$**IgnC-A** $\subseteq$ $\mathcal{M}$**InfC-A**
- $\mathcal{M}$**IgnC** $\subseteq$ $\mathcal{M}$**InfC**,
- $\mathcal{M}$**InfC-A** $\subseteq$ $\mathcal{M}$**InfC**.

However, in general, we have neither one of the two following inclusions:

- $\mathcal{M}$**IgnC** $\subseteq$ $\mathcal{M}$**InfC-A**,
- $\mathcal{M}$**InfC-A** $\subseteq$ $\mathcal{M}$**IgnC**.

Proof. $\mathcal{M}$**Cond** $\subseteq$ $\mathcal{M}$**Fav**: Assume that a voting system meets $\mathcal{M}$**Cond**. If c is $\mathcal{M}$-favorite in ω, then since $\mathcal{M}$ is monotonic and exclusive, c is $\mathcal{M}$-Condorcet in ω (Proposition 4.11) hence she is elected.

$\mathcal{M}$**Fav** $\subseteq$ $\mathcal{M}$**IgnC-A**: Assume that a voting system meets $\mathcal{M}$**Fav**. Let $c \in \mathcal{C}$ and $M \in \mathcal{M}_c$. Members of M can just claim that they strictly prefer c over other candidates, which is possible because the electoral space allows any candidate as most liked. Regardless of the ballots of the other voters, the set of voters claiming that c is their most liked candidate contains M, so it belongs to $\mathcal{M}_c$ (by monotonicity), hence c is $\mathcal{M}$-favorite and gets elected. This proves, in the same time, that the system meets $\mathcal{M}$**IgnC**, therefore (Proposition 4.13) that the family $\mathcal{M}$ is exclusive.

Let us continue with our manipulation for c: she is $\mathcal{M}$-favorite, we now know that the family $\mathcal{M}$ is exclusive and it is also monotonic by assumption, hence c is $\mathcal{M}$-Condorcet (Proposition 4.11) and, *a fortiori*, $\mathcal{M}$-admissible. Therefore, the voting system meets $\mathcal{M}$**IgnC-A**.

The other implications follow immediately from the definitions.

To show that in general, we have neither $\mathcal{M}$**IgnC** $\subseteq$ $\mathcal{M}$**InfC-A** nor $\mathcal{M}$**InfC-A** $\subseteq$ $\mathcal{M}$**IgnC**, consider the majoritarian family.

In Corollary 3.14 of Proposition 3.13 about PSR, we have already shown that Borda's method meets $\mathcal{M}$**InfC**. But in the proof, the manipulators always place candidate c on top of their ballots, hence Borda's method meets $\mathcal{M}$**InfC-A**. However, Corollary 3.14 also states that in general, it does not meet $\mathcal{M}$**IgnC**.

Finally, in the electoral space of strict total orders, with $C = 2$ and V odd, consider the following voting system: inverted majority voting. Each voter communicates an order of preference, but candidate a wins iff b is on top of a majority of ballots (and vice versa). If a majority coalition wants to make a candidate win, it is sufficient that they put this candidate at the bottom of their ballots. But, by definition of this strange voting system, a winning candidate never appears as a Condorcet-admissible candidate. $\square$

4.3 Generalized Condorcification theorem

We now generalize the weak Condorcification theorem 2.9 to our general framework using families of collections of coalitions. Since the approach is essentially the same, we will only give short explanations.

The following lemma generalizes Lemma 2.4. As in the initial lemma, no assumption is made on the possible criteria satisfied by f.

Lemma 4.15
Let $(\omega, \psi) \in \Omega^2$. Let $\mathrm{w} = f(\omega)$ and $c = f(\psi)$. Assume that f is manipulable in ω to ψ. It is further assumed that $\mathcal{M}$ is monotonic.

If w is $\mathcal{M}$-admissible in ω, then c cannot have an $\mathcal{M}$-victory against w in ψ; in particular, c is not $\mathcal{M}$-Condorcet in ψ.

Proof. Since w is $\mathcal{M}$-admissible in ω, we have: $\{v \in \mathcal{V}, c\, \mathrm{P}_v(\omega_v)\, \mathrm{w}\} \notin \mathcal{M}_c$. But, by definition of manipulability, $\{v \in \mathcal{V}, c\, \mathrm{P}_v(\psi_v)\, \mathrm{w}\} \subseteq \{v \in \mathcal{V}, c\, \mathrm{P}_v(\omega_v)\, \mathrm{w}\}$. Therefore, since the family $\mathcal{M}$ is monotonic, $\{v \in \mathcal{V}, c\, \mathrm{P}_v(\psi_v)\, \mathrm{w}\} \notin \mathcal{M}_c$. ☐

Lemma 4.15 also leads to extending Lemma 2.5: if the family $\mathcal{M}$ is monotonic and if the voting system meets the $\mathcal{M}$-Condorcet criterion, then a configuration with an $\mathcal{M}$-Condorcet candidate cannot be manipulable to another configuration with an $\mathcal{M}$-Condorcet candidate.

The following lemma generalizes Lemma 2.7.

Lemma 4.16
*Let $\omega \in \Omega$. Assume that f meets $\mathcal{M}$**InfC**.*

If $f(\omega)$ is not $\mathcal{M}$-admissible in ω, then f is manipulable in ω.

Proof. Denote $\mathrm{w} = f(\omega)$. Since w is not $\mathcal{M}$-admissible, then there exists another candidate c with a victory against w:

$$\exists c \in \mathcal{C} \setminus \{\mathrm{w}\} \text{ s.t. } \{v \in \mathcal{V} \text{ s.t. } c\, \mathrm{P}_v\, \mathrm{w}\} \in \mathcal{M}_c.$$

Denote $M = \mathrm{Manip}(\mathrm{w} \to c)$ the coalition for c against w. Using $\mathcal{M}$**InfC**, we then have:

$$\exists \psi_M \in \Omega_M \text{ s.t. } f(\omega_{\mathcal{V}\setminus M}, \psi_M) = c.$$

Hence, f is manipulable in ω (in favor of candidate c). ☐

As with Lemma 2.7, we can deduce that, if a configuration ω is not $\mathcal{M}$-admissible, then any voting system meeting $\mathcal{M}$**InfC** is manipulable in ω.

We now generalize the definition 2.8 of Condorcification.

Definition 4.17 ($\mathcal{M}$-Condorcification)
We call *$\mathcal{M}$-Condorcification of f* the SBVS:

$$f^{\mathcal{M}} : \left| \begin{array}{lcl} \Omega & \to & \mathcal{C} \\ \omega & \to & \left| \begin{array}{l} \text{if } \omega \text{ has an } \mathcal{M}\text{-Condorcet candidate } c, \text{then } c, \\ \text{otherwise, } f(\omega). \end{array} \right. \end{array} \right.$$

By definition, the $\mathcal{M}$-Condorcification of f satisfies the $\mathcal{M}$-Condorcet criterion. In particular, if the family $\mathcal{M}$ is monotonic and exclusive, and if the electoral space allows any candidate as most liked, then it meets $\mathcal{M}$**InfC** (Proposition 4.14).

In the context of this chapter, we call *majoritarian Condorcification* the usual Condorcification f^* (Definition 2.8), i.e. that based on the majoritarian family.

We now have all the tools needed to generalize the weak Condorcification theorem 2.9.

Theorem 4.18 (generalized Condorcification)
Let f be an SBVS and $\mathcal{M}$ a family. Assume that:

- *$\mathcal{M}$ is monotonic,*
- *f satisfies $\mathcal{M}$***InfC***.*

Then $f^{\mathcal{M}}$ is at most as manipulable as f:

$$\mathrm{CM}_{f^{\mathcal{M}}} \subseteq \mathrm{CM}_f.$$

Proof. Assume that $f^{\mathcal{M}}$ is manipulable in ω to ψ, but that f is not manipulable in ω.

Since f is not manipulable in ω, Lemma 4.16 ensures that $f(\omega)$ is $\mathcal{M}$-admissible in ω. If she is $\mathcal{M}$-Condorcet in ω, then $f^{\mathcal{M}}(\omega) = f(\omega)$. Otherwise, there is no $\mathcal{M}$-Condorcet candidate in ω (Proposition 4.8) hence, by definition of $f^{\mathcal{M}}$, we also have $f^{\mathcal{M}}(\omega) = f(\omega)$.

Denote $\mathrm{w} = f^{\mathcal{M}}(\omega) = f(\omega)$ and $c = f^{\mathcal{M}}(\psi)$. Since w is $\mathcal{M}$-admissible in ω, Lemma 4.15 (applied to $f^{\mathcal{M}}$) ensures that w is not $\mathcal{M}$-Condorcet in ψ. Hence, by definition of $f^{\mathcal{M}}$, we have $f^{\mathcal{M}}(\psi) = f(\psi)$.

We thus have $f(\omega) = f^{\mathcal{M}}(\omega)$ and $f(\psi) = f^{\mathcal{M}}(\psi)$. Therefore, f is manipulable en ω: a contradiction. □

4.4 Compared Condorcification theorem

Just as we generalized the weak Condorcification theorem 2.9, we could also generalize the strong Condorcification theorem 2.20 by defining a notion of resistant $\mathcal{M}$-Condorcet candidate which would extend the notion of resistant Condorcet winner (Definition 2.16). However, for the sake of conciseness, we will not discuss this point further. Instead, we will focus on an issue specific to generalized Condorcification: the choice of the family $\mathcal{M}$ used to condorcify.

Indeed, it is generally possible for a voting system to meet $\mathcal{M}$**InfC** for several possible choices of the family $\mathcal{M}$. In this case, one can consider using the generalized Condorcification theorem 4.18 with one of the other family. Natural questions then arise: in what cases can we "condorcify a Condorcification" while continuing to decrease (or nonincrease) manipulability? Does one Condorcification diminish manipulability as much as another? Finally, is there a family that makes it possible to decrease manipulability as much as all the others?

Unless otherwise specified, $\mathcal{M}$ and $\mathcal{M}'$ denote two families, with no specific assumption *a priori*.

Definition 4.19 (family at least as condorcifying)

We say that $\mathcal{M}'$ is *at least as condorcifying* as $\mathcal{M}$ iff an $\mathcal{M}$-Condorcet candidate is always also $\mathcal{M}'$-Condorcet, i.e.:

$$\forall \omega \in \Omega, \forall c \in \mathcal{C} : c \text{ is } \mathcal{M}\text{-Condorcet in } \omega \Rightarrow c \text{ is } \mathcal{M}'\text{-Condorcet in } \omega.$$

This notion leads to a particular case for composing two Condorcifications: if $\mathcal{M}'$ is at least as condorcifying as $\mathcal{M}$, then we have $(f^{\mathcal{M}})^{\mathcal{M}'} = f^{\mathcal{M}'}$. Indeed, for configurations where $f^{\mathcal{M}}$ and f have distinct winners, there is an $\mathcal{M}$-Condorcet candidate, who therefore is $\mathcal{M}'$-Condorcet; as a consequence, when we condorcify with the family $\mathcal{M}'$, the winner is that candidate, whatever the initial system.

The following lemma will allow the first Condorcification, performed with the family $\mathcal{M}$, to satisfy the assumptions that will allow it to benefit from the second Condorcification, performed with the family $\mathcal{M}'$. This lemma is actually the main motivation for defining the criterion $\mathcal{M}$**InfC-A**.

Lemma 4.20 (of the admissible informed coalition)
Assume that:

- *$\mathcal{M}'$ is at least as condorcifying as $\mathcal{M}$,*
- *f meets $\mathcal{M}'$**InfC-A**.*

*Then $f^{\mathcal{M}}$ meets $\mathcal{M}'$**InfC-A**.*

Let us interpret this somewhat obscure lemma: we suppose that f satisfies a good property ($\mathcal{M}'$**InfC-A**) for $\mathcal{M}'$, that is to say the second family that we are going to use to condorcify, and we conclude that $f^{\mathcal{M}}$ satisfies the same property ($\mathcal{M}'$**InfC-A**), still for this second family $\mathcal{M}'$. Intuitively, the motivation is as follows: later, we will take for $\mathcal{M}'$ a fixed family, very condorcifying, and it will suffice to test that f meets $\mathcal{M}'$**InfC-A** for this fixed family, so once and for all. Then, $\mathcal{M}$ can be any family at most as condorcifying as $\mathcal{M}'$: this lemma will allow us to show that, in all cases, $f^{\mathcal{M}}$ satisfies the property that makes it possible to continue with $\mathcal{M}'$-Condorcification while continuing to decrease (or nonincrease) manipulability; and this, without our needing to test conditions depending on $\mathcal{M}$ for each family $\mathcal{M}$ considered. In reality, it would suffice for us to have $f^{\mathcal{M}}$ satisfy $\mathcal{M}'$**InfC** in the conclusion of the lemma for the following applications; the fact that $f^{\mathcal{M}}$ also satisfies $\mathcal{M}'$**InfC-A** comes for free in the proof.

Proof. Let $\omega \in \Omega, c \in \mathcal{C}, M \in \mathcal{M}'_c$. Since f meets $\mathcal{M}'$**InfC-A**, there exists ballots $\psi_M \in \Omega_M$ for the coalition such that:

$$\begin{cases} f(\omega_{\mathcal{V}\setminus M}, \psi_M) = c, \\ c \text{ is } \mathcal{M}'\text{-admissible in } (\omega_{\mathcal{V}\setminus M}, \psi_M). \end{cases}$$

Denote $d = f^{\mathcal{M}}(\omega_{\mathcal{V}\setminus M}, \psi_M)$. If d is $\mathcal{M}$-Condorcet in $(\omega_{\mathcal{V}\setminus M}, \psi_M)$, then since $\mathcal{M}'$ is at least as condorcifying as $\mathcal{M}$, candidate d is also $\mathcal{M}'$-Condorcet, hence she is the only $\mathcal{M}'$-admissible candidate (Proposition 4.8), which leads to $d = c$. Alternately, if d is not $\mathcal{M}$-Condorcet in $(\omega_{\mathcal{V}\setminus M}, \psi_M)$, then by definition of $f^{\mathcal{M}}$, we also have $d = c$. As a consequence, M can make c win in an informed way in system $f^{\mathcal{M}}$, while guarantying that c is $\mathcal{M}'$-admissible. Therefore, $f^{\mathcal{M}}$ meets $\mathcal{M}'$**InfC-A**. □

The examples we will see in Section 4.6 will show that the assumptions of Lemma 4.20 are actually quite common in practice. Under these assumptions, we can now compare the Condorcifications obtained using the families $\mathcal{M}$ and $\mathcal{M}'$. This is the subject of the *compared Condorcification theorem*.

Theorem 4.21 (compared Condorcification)
Let f be an SBVS, $\mathcal{M}$ and $\mathcal{M}'$ two families.
Assume that:

- *$\mathcal{M}'$ is monotonic,*
- *$\mathcal{M}'$ is at least as condorcifying as $\mathcal{M}$,*
- *f meets $\mathcal{M}'$**InfC-A**.*

Then $f^{\mathcal{M}'}$ is at most as manipulable as $f^{\mathcal{M}}$:

$$\mathrm{CM}_{f^{\mathcal{M}'}} \subseteq \mathrm{CM}_{f^{\mathcal{M}}}.$$

Proof. By Lemma 4.20, we know that $f^{\mathcal{M}}$ meets $\mathcal{M}'$**InfC**. It suffices then to apply the generalized Condorcification theorem 4.18 to $f^{\mathcal{M}}$ and $\mathcal{M}'$, and to remember that $(f^{\mathcal{M}})^{\mathcal{M}'} = f^{\mathcal{M}'}$. □

It may not be obvious at first sight, but this theorem implicitly contains the assertion that $f^{\mathcal{M}'}$ is less manipulable than f (which we already know anyway, by the generalized Condorcification theorem 4.18). Indeed, consider the particular case of the family $\mathcal{M}$ such that for every candidate c, $\mathcal{M}_c = \varnothing$. Then, a candidate is always $\mathcal{M}$-admissible and never $\mathcal{M}$-Condorcet. In particular, every family is at least as condorcifying as $\mathcal{M}$ and we have $f^{\mathcal{M}} = f$. Hence, the conclusion of the theorem becomes: $\mathrm{CM}_{f^{\mathcal{M}'}} \subseteq \mathrm{CM}_f$.

With only the assumptions of this theorem, we do not have the guarantee that $f^{\mathcal{M}}$ is at most as manipulable as f: indeed, we have not assumed that $\mathcal{M}$ meets the assumptions of the generalized Condorcification theorem 4.18. That said, we know the most important thing: $f^{\mathcal{M}'}$ is at most as manipulable as f and as any system $f^{\mathcal{M}}$, where $\mathcal{M}$ is at most as condorcifying as $\mathcal{M}'$.

Reading this theorem, a question immediately arises: is the conclusion still true if we only assume that f meets $\mathcal{M}'$**InfC** instead of $\mathcal{M}'$**InfC-A**? We are going to see that it is not the case.

Consider the electoral space of strict total orders with $V = 3$ voters and $C = 3$ candidates named a, b, and c. Define the voting system f as follows.

1. If all voters put the same candidate at the top of their ballot, then c wins.
2. If at least one voter puts a on top and at least one voter puts c on top, then a wins.
3. In all other cases, b wins.

Denote $\mathcal{M}$ the *unanimous* family, i.e. the family which contains only the coalition of all the voters, and $\mathcal{M}'$ the majoritarian family. Since $\mathcal{M}'$ is a threshold family, it is monotonic, and it is easy to see that it is at least as condorcifying as $\mathcal{M}$: indeed, a candidate who is $\mathcal{M}$-Condorcet is preferred by all voters so she is also a Condorcet winner, i.e. $\mathcal{M}'$-Condorcet.

Let us show that f meets $\mathcal{M}'$**InfC**.

- To make c win, it is sufficient that 2 manipulators vote like the third voter (whether this one is sincere or not).
- To make a win, it is sufficient that a manipulator puts a on top and that another manipulator puts c on top.
- To make b win, if the sincere voter puts b on top, it is sufficient for the manipulators to put a and b on top; and if the sincere voter puts a or c on top, it is sufficient for the manipulators to put both b on top.

It should be noted that the proposed manipulations are not admissible. On the other hand, the manipulation for c necessarily creates a unanimous favorite (i.e. apparently preferred by all voters) and it is not always possible to choose that it is c: in particular, one cannot always ensure that c is Condorcet-admissible. Thus, f does not meet $\mathcal{M}'$**InfC-A**.

Now consider the following profile ω, which we know well: it is a minimal example of Condorcet paradox.

a	b	c
b	c	a
c	a	b

In f, candidate a wins by virtue of rule 2. Since there is neither $\mathcal{M}$-Condorcet (unanimously preferred candidate) nor $\mathcal{M}'$-Condorcet (usual Condorcet winner), candidate a also wins in $f^{\mathcal{M}}$ and $f^{\mathcal{M}'}$.

In $f^{\mathcal{M}'}$, i.e. the usual majoritarian Condorcification of f, the configuration ω is clearly manipulable because it is not admissible (Lemma 2.7). Incidentally, it can be noticed that the manipulation for c is not carried out in the same way in f and $f^{\mathcal{M}'}$. In f, manipulators must put a on top to benefit from rule 1. In $f^{\mathcal{M}'}$, they must put c on top so that c becomes the Condorcet winner.

On the opposite, in $f^{\mathcal{M}}$ (using the unanimous family), we are going to show that the configuration is not manipulable.

- To manipulate in favor of c, the last two voters need c to be at the top of all the ballots, which they cannot achieve.

- To manipulate in favor of b, only the second voter is interested. For her to succeed, either b must be at the top of all the ballots in the final configuration (to benefit from $\mathcal{M}$-Condorcification), or there must not be two ballots with respectively a and c on top (to benefit from the original rule f, avoiding case 2). But in both cases, it is impossible.

To summarize: all the assumptions of the theorem are satisfied, except the fact that instead of satisfying $\mathcal{M}'$**InfC-A**, the voting system f only meets $\mathcal{M}'$**InfC**. And we exhibited a configuration where $f^{\mathcal{M}'}$ is manipulable but $f^{\mathcal{M}}$ is not manipulable. Thus, the conclusion of the theorem is no longer valid if we only assume that f satisfies $\mathcal{M}'$**InfC**. This counter-example therefore motivates the use of the assumption $\mathcal{M}'$**InfC-A**.

4.5 Maximal Condorcification theorem

Now, we want to know if there exists a "best" family to apply Condorcification, i.e. to diminish manipulability using the generalized Condorcification theorem 4.18. The natural idea is to use the family which, to each candidate c, associates all the coalition which have the power to make c win when they are informed of the ballots of the other voters: that is what we will call the *maximal family* of a voting system. In this section, we examine under which conditions this family leads to a generalized Condorcification which, in some sense, is optimal.

Definition 4.22 (maximal family of informed winning coalitions)

We call *maximal family of winning informed coalitions* of f, or *maximal family* of f, the largest family $\mathcal{M}'$ (in the sense of inclusion) such that f meets $\mathcal{M}'$**InfC**, i.e. such that $\forall c \in \mathcal{C}, \forall M \in \mathcal{P}(\mathcal{V})$:

$$M \in \mathcal{M}'_c \Leftrightarrow \left[\forall \omega_{\mathcal{V}\setminus M} \in \Omega_{\mathcal{V}\setminus M}, \exists \omega_M \in \Omega_M \text{ s.t. } f(\omega_M, \omega_{\mathcal{V}\setminus M}) = c\right].$$

With this notation, the voting system $f^{\mathcal{M}'}$ is called the *maximal Condorcification* of f.

By definition, the maximal family is the largest family with which we can consider applying the generalized Condorcification theorem 4.18. One can therefore wonder if it is also the one that most diminishes manipulability by generalized Condorcification.

Another possible convention would be to call *maximal family* the largest family $\mathcal{M}'$ such that f satisfies $\mathcal{M}'$**InfC-A**, which is, by definition, included in the maximal family in the sense above. As we will see, the maximal Condorcification theorem 4.25 will deal with reasonable voting system, where these two notions are the same. However, in practice, it is easier to identify the maximal family $\mathcal{M}'$ (in the sense above) then to check that f also satisfies $\mathcal{M}'$**InfC-A**, rather than to identify the largest family such that f satisfies $\mathcal{M}'$**InfC-A**, then to check that it is also maximal for the notion $\mathcal{M}'$**InfC**.

The following proposition follows from the definition: indeed, if an informed coalition is always able to make c win, then any larger informed coalition (in the sense of inclusion) is also able to make c win.

Proposition 4.23
The maximal family of f is monotonic.

As a consequence, we can apply the generalized Condorcification theorem 4.18 to the maximal family $\mathcal{M}'$, which proves that $f^{\mathcal{M}'}$ is at most as manipulable as f. We will show that, under certain assumptions, the maximal Condorcification $f^{\mathcal{M}'}$ is the least manipulable of all $f^{\mathcal{M}}$, where $\mathcal{M}$ is a family such that f satisfies $\mathcal{M}$**InfC**.

By Proposition 4.23, we know that the maximal family is monotonic. But it is not necessarily exclusive. Indeed, for parity voting (with $V \geq 2$), it is easy to show that the maximal family consists of all nonempty coalitions, which we have already noted is not exclusive.

In order to apply the compared Condorcification theorem 4.21, we will show in the following proposition that, under certain assumptions, any other family $\mathcal{M}$ that can benefit from the generalized Condorcification theorem 4.18 is at most as condorcifying as the maximal family $\mathcal{M}'$.

Proposition 4.24
Let $\mathcal{M}'$ be the maximal family of f and $\mathcal{M}$ a family.
Assume that:

- *Relations P_v are always antisymmetric,*
- *$\mathcal{M}'$ is exclusive.*

*If f meets $\mathcal{M}$**InfC**, then $\mathcal{M}'$ is at least as condorcifying as $\mathcal{M}$.*

Proof. Let $\phi \in \Omega$ and $c \in \mathcal{C}$. Assume that c is $\mathcal{M}$-Condorcet in ϕ. We are going to show that she is also $\mathcal{M}'$-Condorcet, i.e. in the sense of the maximal family of f.

For all $d \in \mathcal{C} \setminus \{c\}$, candidate c has an $\mathcal{M}$-victory against d:

$$M = \{v \in \mathcal{V} \text{ s.t. } c \,\mathrm{P}_v\, d\} \in \mathcal{M}_c.$$

Since $M \in \mathcal{M}_c$ and f meets $\mathcal{M}$**InfC**, we have:

$$\forall \omega_{\mathcal{V}\setminus M} \in \Omega_{\mathcal{V}\setminus M}, \exists \omega_M \in \Omega_M \text{ s.t. } f(\omega_M, \omega_{\mathcal{V}\setminus M}) = c.$$

By definition of the maximal family $\mathcal{M}'$ of f, this implies that $M \in \mathcal{M}'_c$. Thus, c has an $\mathcal{M}'$-victory against d.

Since $\mathcal{M}'$ is exclusive and since preferences are antisymmetric, Proposition 4.5 ensures that this $\mathcal{M}'$-victory is strict. $\square$

In practice, we generally consider antisymmetric preferences, so the assumption of Proposition 4.24 which is more likely not to be satisfied is the exclusivity of the maximal family $\mathcal{M}'$. Let us show that, in that case, the conclusion of Proposition 4.24 is not true.

For this, consider parity voting (black and white balls). It is easy to see that the maximal family $\mathcal{M}'$ is neutral and contains all nonempty coalitions. This family is not exclusive (assuming $V \geq 2$) because two distinct voter singletons are winning coalitions for candidates a and b respectively. For candidate a to be $\mathcal{M}'$-Condorcet, it is necessary and sufficient that she has a victory, that is, at least one voter prefers her to b, and that she has no defeat, that is, no voter prefers b to a; in other words (assuming that preferences are strict total orders), it is necessary and sufficient that a is the preferred candidate of all voters. If we now consider the majoritarian family $\mathcal{M}$, for a to be $\mathcal{M}$-Condorcet, it is necessary and sufficient that she is preferred by a strict majority of voters, which is a less demanding condition. So, it is false that $\mathcal{M}'$ is at least as condorcifying as $\mathcal{M}$. In this case, we even have the opposite: $\mathcal{M}$ is at least as condorcifying as $\mathcal{M}'$.

It only remains to group together the known properties of the maximal family in the following theorem.

Theorem 4.25 (maximal Condorcification)
Let f be an SBVS and $\mathcal{M}'$ its maximal family.
*Assume that f meets $\mathcal{M}'$***InfC-A***.*

1. *The maximal Condorcification $f^{\mathcal{M}'}$ is at most as manipulable as f:*

$$\mathrm{CM}_{f^{\mathcal{M}'}} \subseteq \mathrm{CM}_f.$$

2. *For each family $\mathcal{M}$ at most as condorcifying as $\mathcal{M}'$, the maximal Condorcification $f^{\mathcal{M}'}$ is at most as manipulable as $f^{\mathcal{M}}$:*

$$\mathrm{CM}_{f^{\mathcal{M}'}} \subseteq \mathrm{CM}_{f^{\mathcal{M}}}.$$

3. *Moreover, assume that preferences P_v are always antisymmetric and that the maximal family $\mathcal{M}'$ is exclusive. Then, for each family $\mathcal{M}$ such that f meets $\mathcal{M}$***InfC***, the maximal Condorcification $f^{\mathcal{M}'}$ is at most as manipulable as $f^{\mathcal{M}}$:*

$$\mathrm{CM}_{f^{\mathcal{M}'}} \subseteq \mathrm{CM}_{f^{\mathcal{M}}}.$$

Proof. 1. By Proposition 4.23, we know that the maximal family $\mathcal{M}'$ is monotonic. And since f meets $\mathcal{M}'$**InfC-A**, it meets *a fortiori* $\mathcal{M}'$**InfC** (Proposition 4.14). We can therefore use the generalized Condorcification theorem 4.18.

2. Since the maximal family $\mathcal{M}'$ is monotonic, we can use the compared Condorcification theorem 4.21.

3. By Proposition 4.24, the family $\mathcal{M}'$ is at most as condorcifying as $\mathcal{M}$ so we can use point 2 of the present theorem. □

4.6 Examples of generalized Condorcification

We now apply generalized Condorcification to several voting systems in order to show how this notion behaves in practice.

4.6.1 Usual voting systems meeting InfMC

In the case of the usual voting systems meeting **InfMC**, we already know that they can benefit from majoritarian Condorcification (Theorem 2.9). The objective of this section is to examine whether it is their maximal Condorcification.

First, note that most usual voting systems meet $\mathcal{M}$**IgnC** and even $\mathcal{M}$**IgnC-A** for the majoritarian family. Indeed, in these reasonable voting systems, if some manipulators (in particular, those belonging to an ignorant majority coalition) can make some candidate c win, then they can do so while putting c at the top of their ballots.

To deal with these voting systems, the following proposition will be useful. Its key point is conclusion 1, which establishes an implication between $\mathcal{M}'$**IgnC-A** and the maximality of $\mathcal{M}'$ in the case of the majoritarian family. Conclusions 2, 3 and 4 simply reword the conclusions from the maximal Condorcification theorem in this particular case.

Proposition 4.26

Let f be an SBVS. Assume that:

- *Relations P_v are always antisymmetric,*
- *The number of voters V is odd,*
- *f meets the admissible ignorant majority coalition criterion.*

Then the following conclusions hold.

1. *The maximal family $\mathcal{M}'$ of f is the majoritarian family.*
2. *The majoritarian Condorcification $f^{\mathcal{M}'} = f^*$ is at most as manipulable as f:*
$$\mathrm{CM}_{f^*} \subseteq \mathrm{CM}_f.$$
3. *For each family $\mathcal{M}$ that is at most as condorcifying as the majoritarian family $\mathcal{M}'$, the majoritarian Condorcification f^* is at most as manipulable as $f^{\mathcal{M}}$:*
$$\mathrm{CM}_{f^*} \subseteq \mathrm{CM}_{f^{\mathcal{M}}}.$$
4. *For each family $\mathcal{M}$ such that f meets $\mathcal{M}$**InfC**, the majoritarian Condorcification f^* is at most as manipulable as $f^{\mathcal{M}}$:*
$$\mathrm{CM}_{f^*} \subseteq \mathrm{CM}_{f^{\mathcal{M}}}.$$

Proof. Since f meets the ignorant majority coalition criterion, it is easy to see that its maximal family is the majoritarian one: with more than a majority, a coalition can always choose the winner; with less, the other voters can always do so (because V is odd, hence they have a strict majority). This proves point 1.

Since f meets $\mathcal{M}'$**IgnC-A** by assumption, it also satisfies $\mathcal{M}'$**InfC-A** (Proposition 4.14), hence we can use the maximal Condorcification theorem 4.25. We can then conclude immediately about points 2 and 3.

Since preferences are always antisymmetric by assumption and since the majoritarian family $\mathcal{M}'$ is exclusive (Definition 4.3), the maximal Condorcification theorem 4.25 also proves point 4. $\square$

Corollary 4.27

Assume that relations P_v are always antisymmetric and that V is odd.

Then the conclusions of Proposition 4.26 are true for Approval voting, Baldwin, Borda, Bucklin, Condorcet-Borda (Black's method), Coombs, CSD, Dodgson, IB, IRV, IRVD, IRVA, ITR, Kemeny, Majority judgment, Maximin, Nanson, Plurality, Range voting, RP, and Schulze's method.

Proof. All the voting systems mentioned, except Borda's method, satisfy the admissible ignorant majority coalition criterion. For these, Proposition 4.26 therefore allows us to conclude.

If the voting system f is Borda's method, we cannot apply Proposition 4.26 because this voting system does not satisfy **IgnMC** in general (Corollary 3.14 of Proposition 3.13). However, we will see that the result remains true, as a corollary of the maximal Condorcification theorem 4.25.

Indeed, f meets the admissible informed majority coalition criterion. Let us show that the majoritarian family is maximal: if a coalition is not a majority, then the other voters are strictly a majority (because V is odd); if they put a certain candidate d first and c last in their ballots, then c cannot be elected, regardless of the ballots of the manipulators in the minority coalition. We can therefore apply the maximal Condorcification theorem 4.25. □

For all systems meeting the Condorcet criterion, Condorcification obviously does not modify the voting system. Proposition 4.26 and Corollary 4.27 mean that, even in their original state, we cannot hope to diminish their manipulability by applying the generalized Condorcification theorem 4.18. In other words, they are their own maximal Condorcification.

4.6.2 Veto

The voting system Veto, which does not satisfy **InfMC**, cannot benefit from the weak (majoritarian) Condorcification theorem 2.9. Now, we can determine its maximal family and apply the maximal Condorcification theorem 4.25.

Proposition 4.28

Let f be Veto. Assume that preferences are always antisymmetric and that $\mathrm{mod}(V, C) = C - 1$. Then the following assertions hold.

1. *The maximal family $\mathcal{M}'$ of Veto is the threshold family whose coalitions have a cardinality greater than or equal to $(C-1)\left\lceil \frac{V}{C} \right\rceil$.*
2. *Veto satisfies $\mathcal{M}'$**IgnC-A**.*
3. *The maximal family $\mathcal{M}'$ is exclusive.*

Therefore, all the conclusions of the maximal Condorcification theorem 4.25 hold.

Just as we assumed an odd number of voters in Proposition 4.26 and Corollary 4.27 to avoid questions of ties in voting systems linked to the notion of majority, Proposition 4.28 uses a modulo assumption to simplify the study: $\mathrm{mod}(V, C) = C - 1$. We can notice that for $C = 2$, Veto becomes equivalent to simple majority voting; then, this equation simply means that V is odd.

In the limit where V is large compared to C, we can consider as a first approximation that the maximal family of Veto is that of $\left(1 - \frac{1}{C}\right) V$, up to rounding and tie-breaking issues. In other words, to choose the winner in Veto, a large coalition is necessary, all the larger as C is large.

Proof. Denote $\alpha = (C-1)\left\lceil \frac{V}{C} \right\rceil$ and $\mathcal{M}'$ the threshold family whose coalitions have a cardinality greater than or equal to α. We are going to show that $\mathcal{M}'$ meets the mentioned properties and, especially, that it is the maximal family of f. During the beginning of this proof, we will not resort to the assumption $\mathrm{mod}(V, C) = C - 1$ but only to the weaker assumption that C does not divide V.

3. This weaker assumption is sufficient to have $\alpha > (C-1)\frac{V}{C} \geq \frac{V}{2}$. Therefore, the family $\mathcal{M}'$ is exclusive.

2. If a coalition of cardinality $(C-1)\left\lceil \frac{V}{C} \right\rceil$ wants a certain candidate c to win, it suffices that $\left\lceil \frac{V}{C} \right\rceil$ coalition members vote against each other candidate. Then as C does not divide V, each of these other candidates receives a number of vetoes strictly greater than the average, hence only c can be elected. Thus, Veto satisfies $\mathcal{M}'$**IgnC**.

Coalition members can perform this manipulation while putting c at the top of their ballots. As the family $\mathcal{M}'$ is exclusive, this ensures that c appears as an $\mathcal{M}'$-admissible candidate (as for every other candidate d, she cannot be placed α times before c so she cannot have an $\mathcal{M}'$-victory against c). Therefore, Veto satisfies $\mathcal{M}'$**IgnC-A**.

1. Since Veto meets $\mathcal{M}'$**IgnC-A**, we already know that it meets $\mathcal{M}'$**InfC-A** (Proposition 4.14). To show that $\mathcal{M}'$ is maximal, it only remains to show that a coalition not belonging to $\mathcal{M}'$ does not have the capacity to decide the winner, even in an informed way. This is the moment where we use the assumption $\mathrm{mod}(V, C) = C - 1$. Note that in the general case:

$$\left\lceil \frac{V}{C} \right\rceil = \frac{V + (C - \mathrm{mod}(V, C))}{C}.$$

We then obtain $\alpha = \frac{(C-1)(V+1)}{C}$.

Consider a coalition not belonging to $\mathcal{M}'$, that is to say of size strictly lower than α. Then the cardinality of its complement is at least:

$$V + 1 - \alpha = \frac{V+1}{C} > \frac{V}{C}.$$

So, if all the voters in the complement vote against candidate c, then she has a greater number of vetoes than the average and she cannot be elected. So the considered coalition is not always able to make c win, even in an informed way, which proves that $\mathcal{M}'$ is the maximal family of Veto. ☐

We now show that the maximal Condorcification of Veto can be *strictly* less manipulable than Veto. For this, consider $V = 7$ voters and $C = 4$ candidates. We have $\mathrm{mod}(V, C) = C - 1$, hence the assumptions of Proposition 4.28 are met and the maximal family $\mathcal{M}'$ is the one of coalitions with 6 voters or more. Consider the following configuration ω.

3	3	1
c	c	a
a	a	b
b	d	d
d	b	c

$D(\omega)$	a	b	c	d
a	–	7	1	7
b	0	–	1	4
c	6	6	–	6
d	0	3	1	–

In Veto, candidate a is elected and the configuration is manipulable in favor of c: the 6 manipulators just have to cast 2 vetoes against each of the candidates a, b, and d. On the other hand, in the maximal Condorcification of Veto, candidate c is elected and it is easy to see that it is not manipulable: indeed, c is not only

$\mathcal{M}'$-Condorcet but also $\mathcal{M}'$-favorite; as a consequence, the voters preferring c, who therefore never want to manipulate, are sufficient to ensure that she stays $\mathcal{M}'$-Condorcet.

As a curiosity, we can also show that the usual majoritarian Condorcification of Veto does not work, in the sense that it can be manipulable in some configurations where Veto is not. For that, consider the following configuration ω, again with $V = 7$ voters and $C = 4$ candidates.

1	2	1	1	2
c	c	d	a	a
a	a	c	b	b
d	b	a	c	d
b	d	b	d	c

$D(\omega)$	a	b	c	d
a	–	7	3	6
b	0	–	3	5
c	4	4	–	4
d	1	2	3	–

Candidate a is the only one without veto, so she is elected. Let us show that Veto is not manipulable in ω. No voter prefer b to a. Candidates c and d receive at least 2 vetoes each, which is more than the average number of vetoes $\frac{V}{C} = \frac{7}{4}$, hence it is impossible to manipulate in their favor. As a consequence, Veto is not manipulable in ω.

Now consider the majoritarian Condorcification of Veto. Candidate c is the Condorcet winner in ω, so she is elected. Consider the following configuration ψ, which is an attempt to manipulate in favor of a.

1	2	1	1	2
c	c	d	a	a
a	a	c	b	b
d	b	a	$\mathbf{d}$	d
b	d	b	$\mathbf{c}$	c

$D(\psi)$	a	b	c	d
a	–	7	3	6
b	0	–	3	5
c	4	4	–	3
d	1	2	4	–

Then, there is no Condorcet winner anymore so a is elected. Thus, the majoritarian Condorcification of Veto is manipulable in ω to ψ in favor of a, whereas Veto is not manipulable in ω.

In the case of Veto, the interest of maximal Condorcification is not above all its use in practice: indeed, the probability of having an $\mathcal{M}'$-Condorcet candidate seems relatively low in reasonable cultures, since she must have approximately $\left(1 - \frac{1}{C}\right) V$ supporters against any opponent. On the other hand, it makes it possible to better understand the general phenomenon and to give a measure of the power of the voters in this voting system.

4.6.3 Parity voting

We now examine a case where the maximal Condorcification theorem 4.25 cannot be used. Consider parity voting: if there is an odd (resp. even) number of black balls, then candidate a (resp. b) is elected. In addition to her ball, each voters also communicates her order of preference, but it is not taken into account.

We have already noticed that the maximal family of this system is that of all nonempty coalitions and that this family is not exclusive. It is therefore not possible to use the maximal Condorcification theorem 4.25.

However, we can notice that for any monotonic family $\mathcal{M}$ such that f satisfies $\mathcal{M}$**InfC**, f also satisfies $\mathcal{M}$**InfC-A**. By compared Condorcification theorem 4.21, if one of these families $\mathcal{M}'$ is at least as condorcifying as another of these families $\mathcal{M}'$, then $f^{\mathcal{M}'}$ is at most as manipulable as $f^{\mathcal{M}}$.

The relation "be at most as condorcifying" is clearly a (partial) order, in particular on the set of monotonic families $\mathcal{M}$ such that f meets $\mathcal{M}$**InfC**. The maximal elements of this order lead to optimal Condorcifications (in terms of manipulability), but we do not *a priori* know how to compare between them from the point of view of manipulability. Let us look at two examples.

- Assuming V odd, consider the majoritarian family. Then, the generalized Condorcification is simple majority voting, which we know is not manipulable.
- Now, let $\mathcal{M}_a$ be the set of the coalitions of more than one third of the voters and $\mathcal{M}_b$ the set of the coalition of at least two thirds of the voters. In the generalized Condorcification, b wins if she has at least two thirds of the votes; otherwise, it is a. Also in this case, the generalized Condorcification is not manipulable.

These Condorcifications are two different ways of decreasing manipulability, obtained with two families which are not comparable, that is to say such that neither of them is at least as condorcifying as the other.

On the other hand, we can examine the Condorcifications of parity voting using threshold families (i.e. neutral, anonymous, and monotonic).

First of all, note that for the generalized Condorcification, using for example the family of 30 % or that of 70 % is equivalent (assuming that V is not divisible by 10 to avoid nuances between strict and non-strict inequalities). Indeed, in the family of 30 %, a certain candidate c has a strict absolute victory against a certain d iff more than 30 % prefer c to d (victory) and less than 30 % prefer d to c (no defeat); since preferences are strict total orders, this amounts to saying that more than 70 % of the voters prefer c to d, i.e. that c has a strict absolute victory against d in the sense of the family of 70 %.

Still under the assumption that binary relations are strict total orders, we can therefore observe that the majoritarian family is, among the threshold families, the one that accepts the most victories (since even in the family of 30 %, it is necessary to have victories with more than 50 % of the voters). It is therefore the most condorcifying of monotonic, neutral, and anonymous families.

As a consequence, we can find simple majority voting as the optimal Condorcification of parity voting among those obtained with monotonic, neutral, and anonymous families.

To illustrate the above points, consider the following configuration ω, where all voters use a white ball.

60	39
a	b
b	a

Since there is an even number of black balls (equal to 0), candidate b is elected. If we use the family $\mathcal{M}$ of 70 %, then there is no $\mathcal{M}$-Condorcet candidate: no candidate has a victory against the other. If we use the family of 30 %, then there is no $\mathcal{M}$-Condorcet either: candidates a and b have mutual victories against each other (so they are not even $\mathcal{M}$-admissible). In these two systems, it suffices that a candidate preferring a replaces her white ball with a black ball, without modifying her order of preference, to manipulate in favor of a.

In contrast, if we use the majoritarian family, then the Condorcification of parity voting is simple majority voting: a is elected and this system is never manipulable (sincere voting, consisting of voting for one's most liked candidate, is a dominant strategy).

4.6.4 Vote of a law

Now we will focus on voting systems that violate anonymity, neutrality, or both. Throughout the rest of this chapter, we denote $\mathcal{M}'$ the maximal family of the system f studied. We assume for simplicity that binary relations of preference are strict total orders.

We will often implicitly use Proposition 4.13: if a system satisfies $\mathcal{M}$**IgnC**, then $\mathcal{M}$ is exclusive. Since preferences are antisymmetric, we recall that then the definition 4.7 of an $\mathcal{M}$-Condorcet is reduced to relation (4.1): it suffices to check that the candidate concerned has a victory against any other candidate, and it is not needed to check that no other candidate has a victory against her.

Let us examine a first system, which can be used to pass a law. One could think, for example, of a revision of the French Constitution. The following three options are put to the vote: two versions a and b of the bill, and $\varnothing$ which represents the status quo, i.e. the fact that neither of the two versions is adopted. The considered voting system is as follows:

1. Voters choose between a and $\varnothing$. If a receives at least two thirds of the votes, then a is elected and the process is over.

2. Otherwise, voters choose between b and $\varnothing$. If b receives at last two thirds of the votes, then b is elected. Otherwise, $\varnothing$ is elected.

This is a *general voting system* in several rounds (Section 1.4). The *game form* used poses a problem *a priori* to define the sincerity function:[2] if a voter has the order of preference $b \succ a \succ \varnothing$, it is not clear what should be her ballot in the first round because actually, the choice is not between a and $\varnothing$ but rather between a and the fact of carrying out an election opposing b and $\varnothing$. The *voting system* we are going to study uses the sincerity function implied by our initial formulation: for the first round, sincere voting consists of voting for a iff one prefers a to $\varnothing$.

The first thing we will do is instead consider the state-based version of this system, which is at most as manipulable (Proposition 1.4.2): each voter communicates her order of preference, then the original system is emulated.

Let us now examine its maximal family.

For each coalition M, we have:

$$M \in \mathcal{M}'_{\varnothing} \Leftrightarrow \mathrm{card}(M) > \frac{V}{3}.$$

For each coalition M and each candidate $c \neq \varnothing$, we have:

$$M \in \mathcal{M}'_{c} \Leftrightarrow \mathrm{card}(M) \geq \frac{2V}{3}.$$

It is easy to check that the voting system satisfies $\mathcal{M}'$**IgnC-A**. So the maximal family $\mathcal{M}'$ is exclusive.

[2]This example is very similar to that of the *non-alcoholic party* of Gibbard (1973), which is precisely designed to illustrate the issue of defining sincere voting.

To compare the manipulability of f and $f^{\mathcal{M}'}$, consider the following profile ω.

70	30
b	a
a	b
$\varnothing$	$\varnothing$

In the original system (or its state-based version), candidate a wins on the first vote, at least if voters are sincere. But the configuration is manipulable in favor of b: indeed, its supporters can swap a and $\varnothing$ in their order of preference; in that case, a is rejected, then b is accepted.

In the maximal Condorcification, b is $\mathcal{M}'$-Condorcet so it is elected and it is not manipulable: indeed, candidate b is $\mathcal{M}'$-favorite, hence the voters who have b as their favorite ensure by themselves that b is $\mathcal{M}'$-Condorcet.

In this example, the maximal Condorcification therefore leads to a voting system that is strictly less manipulable than the original.

4.6.5 Plurality with an imposition power

Renaud[3] and his friends consider going to the ball (b), to the temple (t), to visit Germaine (g) or the Pépette (p), or staying in the same place and just discussing ($\varnothing$). Plurality is used with the following exception. Renaud, who owns the car, has the right to impose candidate $\varnothing$: if he votes for $\varnothing$, then this candidate is automatically elected.

For each coalition M, we have:

$$M \in \mathcal{M}'_{\varnothing} \Leftrightarrow \text{Renaud} \in M \text{ or } \operatorname{card}(M) > \frac{V}{2}.$$

For each coalition M and each candidate $c \neq \varnothing$, we have:

$$M \in \mathcal{M}'_{c} \Leftrightarrow \text{Renaud} \in M \text{ and } \operatorname{card}(M) > \frac{V}{2}.$$

It is easy to check that the voting system meets $\mathcal{M}'$**IgnC-A**. Hence, the maximal family $\mathcal{M}'$ is exclusive.

Consider the following profile ω with $V = 9$ voters.

1 (Renaud)	1	1	1	1	1	1	1	1
g	$\varnothing$	b	b	b	p	p	t	t
$\varnothing$	g	$\varnothing$	$\varnothing$	t	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
t	t	g	t	$\varnothing$	g	g	g	g
p	p	p	g	g	t	t	b	b
b	b	t	p	p	b	b	p	p

$D(\omega)$	$\varnothing$	b	t	g	p
$\varnothing$	–	6*	6*	8	7*
b	3	–	3	3	5
t	3	6*	–	4	6*
g	1*	6*	5*	–	7*
p	2	4*	3	2	–

In the weighted majority matrix above, we conventionally mark with a star each set of voters which contains Renaud.

[3] Note for the English version of this memoir: Renaud is a popular anarchist French singer. Among his songs, in “Tu vas au bal?”, he narrates a humorous conversation with a friend where they consider going to the ball, to the church, or to the prostitutes, but mostly stay in the same place, talking about it for hours. Finally, he ends up dancing in the church with the prostitutes. In other songs, he mentions recurring characters named Germaine (in the song of the same name and in “Mon H.L.M.”) and the Pépette (in “Près des auto-tamponneuses” and “Le Retour de la Pépette”).

In the original voting system, we have $f(\omega) = b$, but Renaud can manipulate in favor of $\varnothing$ using his imposition power.

If we use the majoritarian Condorcification f^*, then the sincere winner is $\varnothing$ because it is the Condorcet winner. But the profile remains manipulable: indeed, consider the following configuration ψ, which is an attempt of manipulation for b.

1 (Renaud)	1	1	1	1	1	1	1	1
g	$\varnothing$	b	b	b	p	p	t	t
$\varnothing$	g	$\mathbf{g}$	$\mathbf{t}$	t	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
t	t	$\mathbf{t}$	$\boldsymbol{\varnothing}$	$\varnothing$	g	g	g	g
p	p	$\boldsymbol{\varnothing}$	g	g	t	t	b	b
b	b	$\mathbf{p}$	p	p	b	b	p	p

$D(\psi)$	$\varnothing$	b	t	g	p
$\varnothing$	–	6*	4*	6	7*
b	3	–	3	3	5
t	5	6*	–	4	7*
g	3*	6*	5*	–	7*
p	2	4*	2	2	–

There is no Condorcet winner anymore and b wins, hence the majoritarian Condorcification f^* is manipulable in ω to ψ in favor of b.

Finally, if we use the maximal Condorcification, then the sincere winner in ω is also $\varnothing$ because it is $\mathcal{M}'$-Condorcet. If we wish to manipulate in favor of g, then only Renaud is interested, but he cannot prevent $\varnothing$ from staying $\mathcal{M}'$-Condorcet because it will still have a strict majority against any other candidate; so manipulation fails. If we wish to manipulate in favor of a candidate other than g, then Renaud does not participate in the manipulation, so his vote alone ensures that $\varnothing$ keeps an $\mathcal{M}'$-victory against t, p, and b. The only hope of the manipulators is therefore to prevent $\varnothing$ from having an $\mathcal{M}'$-victory against g. If we consider a manipulation for b (resp. t, p), then we can read in the weighted majority matrix that there are 3 manipulators (resp. 3, 2), hence the number of votes for $\varnothing$ against g, which is 8 in sincere voting, is, after manipulation, greater than or equal to $8 - 3 = 5$ (resp. 5, 6), hence $\varnothing$ keeps a victory against g, and manipulation fails. Thus, $f^{\mathcal{M}'}$ is not manipulable in ω, whereas f and f^* are.

4.6.6 Election of Secretary-General of the United Nations

We now study a slightly more complex example, but one that comes from international politics. We will only give the maximal family, without studying in details the voting system obtained by maximal Condorcification.

The election of the Secretary-General of the United Nations can be modeled as follows. It uses Range voting, where the authorized grades are -1, 0, and 1; but each member country of the Security Council, a subset of $\mathcal{V}$ denoted CS, has a right of rejection on any person who is candidate. In fact, there are therefore as many candidates as persons running for the position, plus one special candidate $\varnothing$ which means: “no actual candidate is elected” (which has various consequences in practice).

Thus, each member of SC has a right to reject any candidate, except the special candidate $\varnothing$. This is a stronger power that Renaud's power of imposition seen in Section 4.6.5: indeed, a member country of the Security Council can not only impose candidate $\varnothing$ as winner (just as Renaud can), but it can also, in a more nuanced way, choose to veto certain other candidates but not all. For example, it can veto all candidates except a certain actual person c and the special candidate $\varnothing$, which restricts the choice of other voters to these two candidates: Renaud does not have such power.

For each coalition M, we have:

$$M \in \mathcal{M}'_{\varnothing} \Leftrightarrow \exists v \in \mathrm{CS} \text{ s.t. } v \in M.$$

For each coalition M and each candidate $c \neq \varnothing$, we have:

$$M \in \mathcal{M}'_c \Leftrightarrow \mathrm{CS} \subseteq M \text{ and } \operatorname{card}(M) > \frac{V}{2}.$$

It is easy to check that the voting system satisfies $\mathcal{M}'$**IgnC-A**, hence the maximal family $\mathcal{M}'$ is exclusive.

We can therefore apply the maximal Condorcification theorem 4.25. For the sake of conciseness, we will give no example of application this time. Note that, for an actual person to be $\mathcal{M}'$-Condorcet, she must not only be the Condorcet winner in the usual majoritarian sense but also the preferred candidate for each member of the Security Council. So, for actual persons, being $\mathcal{M}'$-Condorcet is a very demanding condition. On the other hand, for the special candidate $\varnothing$ to be $\mathcal{M}'$-Condorcet, it suffices that for every actual person c, there exists a member of the Security Council who prefers $\varnothing$ to c, which is a particularly weak condition. The analysis of the maximal family shows that, in such a system, the special outcome $\varnothing$ is much easier to obtain than any actual candidate running for the position, both in the initial system and in its maximal Condorcification.

If the voters involved have various other reasons to avoid the special outcome $\varnothing$, one can imagine, and that is what happens in practice, that there are processes external to the voting system which allow candidates to be pre-selected in such a way as not to lead to a deadlock during the vote itself. In this case, the members of the Security Council carry out preliminary negotiations in order to identify a sufficiently consensual candidate and then submit this candidate to the vote of the General Assembly (of which they are also members). To our knowledge, in each past election of the UN Secretary-General (at the time of this writing), the Security Council pre-selected one candidate, and only one, for the final election involving all members of the UN General Assembly.

Chapter 5

Slicing

In Chapters 2 and 3, we examined the links of majority notions (in particular the absolute Condorcet winner) with manipulability and, more generally, with the ability to find strong Nash equilibria. In Chapter 4, we extended the Condorcification results to decrease (or nonincrease) manipulability for systems where the notion of majority does not play a particular role.

In this chapter, we study the link between the ordinality of a voting system and manipulability. As a reminder, we say that a voting system is *ordinal* iff the outcome depends only on the binary relations of preference of the voters, even if they are not orders. Among non-ordinal systems, the qualifier *cardinal* is informally used for systems that use a numerical score, such as Approval voting, Range voting, and Majority judgment. Some other systems are not ordinal, such as parity voting, which uses black and white balls with no particular ordinal or cardinal meaning.

Regarding cardinal systems, intuition can suggest that they tend to be more manipulable than ordinal systems: indeed, not only can a voter lie about the relative positions of candidates in her preferences (ordinal aspect), but she also has an additional degree of freedom that consists in exaggerating her scores, upwards or downwards (cardinal aspect).

In this chapter, we will see that it is indeed possible to give a precise meaning to this intuition and to generalize it to all types of systems that are not ordinal. Most of this work is presented by Durand et al. (2014e,d).

In Section 5.1, we define the *slices* of a voting system f, each of which is an ordinal voting system. Intuitively, each slice is of the following form: considering the profile P of the ordinal preferences of the voters, we complete it in an arbitrary way to a fictitious state of preference ω' and we apply the original system f to the obtained state. For example, Borda's method is a slice of Range voting, among the infinity its possible slices: it consists in "placing" arbitrarily the grades $C-1, C-2, \ldots, 0$ on the order of preference of each voter.

An intuitive illustration of the slicing metaphor is given in Figure 5.1. In general, if we have a real function of two variables $(x, y) \to f(x, y)$, we can represent it in a 3-dimensional space (where it defines a surface). Fixing y amounts to considering a 2-dimensional slice of this space (in which the function $x \to f(x, y)$ defines a curve). In this analogy, x corresponds to the profile P of the ordinal preferences and y is the complementary information (for example the list of grades) which makes it possible to reconstitute a complete configuration. Informally, a slice of the voting system can be seen as a function $\mathrm{P} \to f(\mathrm{P}, y)$, where the additional information y is arbitrarily fixed.

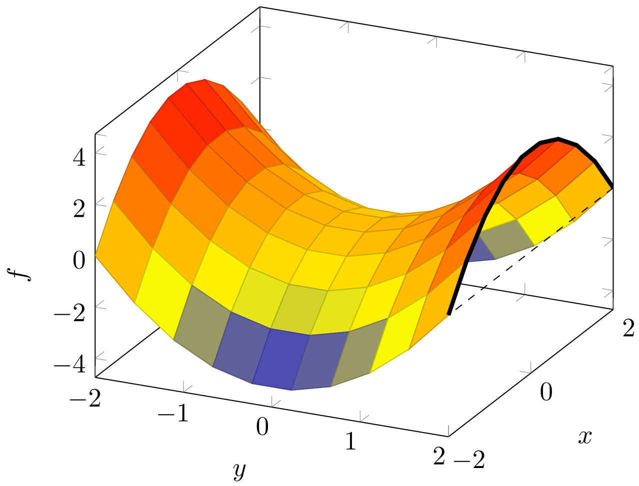


Figure 5.1 – In thick black line, the slice $y = 2$ of a real function f of two variables.

In Section 5.2, we introduce the notion of *decomposable* probabilized electoral space (PES), which will then serve as a hypothesis for the slicing theorem 5.9. Intuitively, the state of a voter is characterized by two pieces of information: her ordinal preferences and "complementary information" about her state, for example a list of grades (without their correspondence with the candidates). If we have both pieces of information, then we know her complete state. The initial idea of decomposability is to require that the two pieces of information be independent in the probabilistic sense. But we will see that if we define the space of complementary information cleverly, we can extend this notion to a wider variety of cases. In particular, we show that if the voters are mutually independent, then the space is always decomposable: this is not obvious, since this property holds even if, for each voter, her ordinal preferences and the other information about her state do not seem at all independent at first glance. We also show that voter independence is not necessary and we exhibit a larger class of models that are decomposable.

In Section 5.3, we prove the central result of this chapter, the *slicing theorem* 5.9: if the electoral space is decomposable, then there exists a slice of f whose manipulability rate is less than or equal to that of f. The slice concerned is, by definition, an ordinal system.

The fact that this system is a slice of f will ensure that it retains certain properties of f. In particular, we will see in Section 5.4 that if f satisfies the Condorcet criterion, then any slice of f also satisfies it. Therefore, the slicing theorem 5.9 can be combined with the Condorcification theorems. Starting from a voting system f, one can consider its Condorcification and then slice it to obtain a system at most as manipulable (in the probabilistic sense).

In particular, this leads us in Section 5.5 to the *optimality theorem* 5.15, which can be seen as the conclusion of these first five theoretical chapters: in the class **InfMC**, if the electoral space is decomposable, then there exists a voting system that is ordinal, satisfies the Condorcet criterion, and whose manipulability rate is minimal.

Sections 5.6 and 5.7 can be considered as technical appendices to this chapter. In Section 5.6, we extend the notion of decomposability to a probability space that is not necessarily an electoral space and we prove some technical results that were only mentioned in Section 5.2.

In Section 5.7, we discuss the assumptions of the slicing theorem 5.9 and show in which sense the theorem is sharp. In particular, we show that one can only hope to decrease manipulability in the probabilistic sense and not in the set-theoretic sense. We also show that it is not possible to remove the decomposability hypothesis purely and simply. On the other hand, we do not know if it is possible to replace it by a weaker condition. Finding a necessary and sufficient condition for any system to admit a less manipulable slice (in the probabilistic sense) is left as an open question.

5.1 Slices of a voting system

In the introduction to this chapter, we gave a first intuition of slicing. In more detail, each voter v communicates her binary relation of preference $p_v = \mathrm{P}_v(\omega_v)$. Then we use a predefined method, denoted y_v, to reconstitute a fictitious state ω'_v that is consistent with p_v. Finally, we apply f to the fictitious configuration $(\omega'_1, \ldots, \omega'_V)$. The whole process defines a voting system f_y which is *ordinal*: it depends only on the binary relations of preference.

For example, in the reference electoral space where $\omega_v = (p_v, u_v, a_v)$, consider $y_v(p_v) = (p_v, u'_v, a'_v)$, where u'_v is the vector of Borda scores[1] associated with p_v and a'_v is a vector of approval values with 1 for each candidate. In this particular case, the functions y_v are the same for each voter v, but note this is not mandatory in general.

Let us examine the *slice* of Range voting by $y = (y_1, \ldots, y_V)$. Once voters have communicated binary relations of preference $p = (p_1, \ldots, p_V)$, we use y in order to reconstitute fictitious states ω'_v: in particular, each u'_v is, now, a vector of Borda scores. Finally, we apply Range voting to this fictitious configuration: the candidate with the highest total score is declared the winner. To summarize, the *slice* of Range voting by y is Borda's method. An infinity of other slices can be defined, depending on the choice of y.

More generally, any PSR can be obtained as a slice of Range voting.[2] On the other hand, the opposite is not true: in a slice of Range voting, it is possible to complete the state of a voter with Plurality scores (1 for the most liked candidate, 0 for the others) and the state of another voter with Borda scores. In this case, the resulting slice is not a PSR. Incidentally, we can notice that slicing does not preserve the possible anonymity of the initial voting method. However, we will see that it preserves a certain balance of powers in Section 5.4.

We now give the formal definitions.

Notations 5.1 (space $\mathcal{Y}$)
For each voter v, denote:

$$\mathcal{Y}_v = \{y_v : \mathrm{P}_v(\Omega_v) \to \Omega_v \text{ s.t. } \mathrm{P}_v \circ y_v = \mathrm{Id}\}.$$

[1] Borda score of a candidate c for voter v (reminder): c receives one point for each candidate d such that v prefers c to d, and half a point for each candidate that v considers incomparable or mutually preferable to c. In this example, we then divide this score by $C - 1$ to obtain a value in the interval $[0, 1]$.

[2] Strictly speaking, this statement is true if we have made the hypothesis that the electoral space allows a voter to give the same score to two candidates while preferring one over the other. Otherwise, we must restrict ourselves to vectors $\mathbf{x}$ of strictly decreasing weights.

It is the set of functions y_v which, to each possible p_v, associate a fictitious state $\omega'_v = y_v(p_v)$ that is consistent with p_v, in the sense that $\mathrm{P}_v(\omega'_v) = p_v$. Mathematically, this is the set of right inverses of P_v (restricted to its image), i.e. the functions that choose an element ω'_v of $(\mathrm{P}_v)^{-1}(p_v)$. Intuitively, $\mathcal{Y}_v$ is the set of the possible "complementary pieces of information" for voter v.

Let $\mathcal{Y} = \prod_{v=1}^{V} \mathcal{Y}_v$. A function $y = (y_1, \ldots, y_V) \in \mathcal{Y}$ defines a complementary piece of information for each voter, i.e. a slicing method:[3] to each possible profile $p = (p_1, \ldots, p_V) \in \prod_{v \in \mathcal{V}} \mathrm{P}_v(\Omega_v)$, it associates a configuration $\omega' = y(p) = (y_1(p_1), \ldots, y_V(p_V))$ which is consistent with p.

Definition 5.2 (slice)

For an SBVS f and for $y \in \mathcal{Y}$, we call *slicing of f by y* the voting system f_y defined by:

$$f_y : \left| \begin{array}{lcl} \Omega & \to & \mathcal{C} \\ \omega & \to & f(y(\mathrm{P}(\omega))). \end{array} \right.$$

We now present a lemma that gives a central idea for the slicing theorem 5.9: in any configuration of the particular form $y(p)$, the voting systems f and f_y return the same result; but for the manipulators, the possibilities of expression they have in f_y are included in those they have in f, therefore they have less power in the former than in the latter.

For example, consider a very particular configuration ω where for each voter v, her vector of sincere scores u_v is equal to the vector of Borda scores associated to her binary relation p_v. Clearly, if voters cast their ballots sincerely, then both Range voting and Borda's method return the same result. We simply notice that if Borda's method is manipulable in ω, then so is Range voting: manipulators can use the same strategies as they would in Borda's method.

Lemma 5.3

For every profile $p \in \mathrm{P}(\Omega)$ and every slicing method $y \in \mathcal{Y}$, if f_y is manipulable in $y(p)$, then f is manipulable in $y(p)$. Reformulating this with the manipulability indicator, we always have:

$$\mathrm{CM}_{f_y}(y(p)) \leq \mathrm{CM}_f(y(p)).$$

Proof. Suppose that the slice f_y is manipulable in the configuration $\omega = y(p)$. By definition, there exist ballots $\psi \in \Omega$ such that $f_y(\psi) \neq f_y(\omega)$ and:

$$\forall v \in \mathrm{Sinc}(f_y(\omega) \to f_y(\psi)), \psi_v = \omega_v.$$

The slice and the original voting system have the same sincere result in ω: indeed, by expanding the definition of $f_y(\omega)$ and using the relation $\mathrm{P} \circ y = \mathrm{Id}$, we have $f_y(\omega) = f(y(\mathrm{P}(y(p)))) = f(y(p)) = f(\omega)$.

Let $\phi = y(\mathrm{P}(\psi))$: these are the ballots which are effectively taken into account by f_y (in our example, the conversion of strategic bulletins ψ into Borda format). By simple rewriting, we have:

$$\forall v \in \mathrm{Sinc}(f(\omega) \to f(\phi)), \psi_v = \omega_v.$$

If $\psi_v = \omega_v$, then $\phi_v = y_v(\mathrm{P}_v(\psi_v)) = y_v(\mathrm{P}_v(\omega_v)) = y_v(\mathrm{P}_v(y_v(p_v))) = y_v(p_v) = \omega_v$. Therefore:

$$\forall v \in \mathrm{Sinc}(f(\omega) \to f(\phi)), \phi_v = \omega_v.$$

Now we have seen that $f(\phi) = f_y(\psi) \neq f_y(\omega) = f(\omega)$. Thus, f is manipulable in ω to ϕ. $\square$

[3] By writing shortcut, we associate a V-tuple of functions to the corresponding multivariate function.

5.2 Decomposable electoral space

By choosing an appropriate pair (p, y), any configuration ω can be expressed as $\omega = y(p)$. For example, if the true state ω_v of each voter v corresponds to Borda scores, then it can be represented by her binary relation of sincere preference p_v and the slicing method y_v given as an example earlier. Therefore, in this configuration $\omega = y(p)$, Lemma 5.3 makes it possible to compare the manipulability of f and that of f_y.

The idea of decomposability is the following: by independently drawing p and y with well-chosen probability distributions, we would like to reconstitute a configuration ω with the correct probability measure π. If this is possible, then we will see (when proving the slicing theorem 5.9) that the manipulability rate of f can be compared to some average of those of all possible slices f_y.

We first give the formal definition of decomposability, then an interpretation and an example. Recall that μ denotes the distribution of the random variable P (in culture π).

Definition 5.4 (decomposability)
We say that the PES (Ω, π) is P-*decomposable*, or just *decomposable*, if there exists a probability measure ν on $\mathcal{Y}$ such that for every event A on Ω:

$$\pi(\omega \in A) = (\mu \times \nu)\big(y(p) \in A\big).$$

In what follows, when (Ω, π) is decomposable, we always denote ν an arbitrary measure on $\mathcal{Y}$ among those which satisfy this property.

This definition requires that π be the image measure of $\mu \times \nu$ by the operator which, to p and y, associates $y(p)$. In other words, by independently drawing p and y (with measures μ and ν respectively), then considering $\omega = y(p)$, we draw ω with the correct probability measure π. Decomposability is related to the notion of complementary information about the states of the voters: this idea is further developed in Section 5.6.

For convenience of language, when the PES (Ω, π) is decomposable, we sometimes say that the electoral space Ω or the culture π is decomposable.

Example 5.5
Consider $V = 2$ voters and $C = 2$ candidates named a and b. Suppose that the state of each voter is a pair consisting of a strict total order of preference on the candidates and a complementary information, "apple" or "banana". In a real case study, these two values could have a non-food meaning, such as "prefer passionately" and "prefer a little", but this is not important for our example.

Let π be the culture that equiprobably draws one of the following two configurations:

1. Each voter is in the state $\mathcal{A} = (a \succ b, \text{apple})$,
2. Each voter is in the state $\mathcal{B} = (b \succ a, \text{banana})$.

To show that this PES is decomposable, consider the measure ν that surely draws two identical functions y_1 and y_2 such that for every voter v, $y_v(a \succ b) = \mathcal{A}$ and $y_v(b \succ a) = \mathcal{B}$.

By drawing the profile p with the distribution μ, we have equiprobably $p = (a \succ b, a \succ b)$ or $p = (b \succ a, b \succ a)$. Then, drawing y with the (deterministic) distribution ν, we have equiprobably $y(p) = (\mathcal{A}, \mathcal{A})$ or $y(p) = (\mathcal{B}, \mathcal{B})$, which is exactly culture π.

In short, this PES (Ω, π) can be emulated by drawing the profile $p = (p_1, p_2)$ with distribution μ (which is directly defined by culture π), drawing $y = (y_1, y_2)$ with distribution ν (which we exhibited), and then considering $\omega = y(p)$. Therefore, it is decomposable.

In general, it is not trivial to decide whether a given electoral space is decomposable or not. For this reason, we are going to provide some sufficient or necessary conditions.

Proposition 5.6
If voters $(\omega_1, \ldots, \omega_V)$ are independent, then (Ω, π) is decomposable.

We are not going to prove Proposition 5.6 for now. It is a consequence of Proposition 5.18, which we will see in Section 5.6 in a more general setting.
However, independence is not a necessary condition. Indeed, in Example 5.5, voters are not independent: either they are both in state $\mathcal{A}$, or both in state $\mathcal{B}$; but the PES is decomposable nevertheless.

Another sufficient condition is satisfied by an important class of models. For example, consider an electoral space where the state of each voter v consists of a strict total order of preference p_v and an integer $k_v \in [\![0, C]\!]$. This integer can have the following meaning: voter v "approves" of the first k_v candidates of her order of preference (whatever the exact meaning of this term).
The culture π considered is as follows. We draw $(p_1, \ldots, p_V)$ according to a certain probability distribution μ on $(\mathcal{L}_\mathcal{C})^V$. Independently, we draw $(k_1, \ldots, k_V)$ according to a certain probability distribution ξ on $[\![0, C]\!]^V$.
It is worth noting that, for both distributions μ and ξ, the voters may not be independent. But the draws by μ and ξ are independent by hypothesis. The following proposition proves that such a PES is decomposable.

Proposition 5.7
For all $v \in \mathcal{V}$, let $\mathcal{P}_v$ be a nonempty part of $\mathcal{R}_\mathcal{C}$, let $\mathcal{K}_v$ be a nonempty measurable set, and let $\Omega_v = \mathcal{P}_v \times \mathcal{K}_v$. Let P_v be the function defined by $\mathrm{P}_v(p_v, k_v) = p_v$ and K_v the function defined by $K_v(p_v, k_v) = k_v$. Let π be a culture on $\Omega = \prod_{v \in \mathcal{V}} \Omega_v$.
If the two random variables $\mathrm{P} = (\mathrm{P}_1, \ldots, \mathrm{P}_V)$ and $K = (K_1, \ldots, K_V)$ are independent, then (Ω, π) is decomposable.

Proof. To k_v, we associate the function $\mathrm{concat}_v(k_v) \in \mathcal{Y}_v$ which consists in concatenating p_v and k_v in order to reconstitute a state ω_v:

$$\mathrm{concat}_v(k_v) : \left| \begin{array}{lll} \mathrm{P}_v(\Omega_v) & \to & \Omega_v \\ p_v & \to & \big(\mathrm{concat}_v(k_v)\big)(p_v) = (p_v, k_v). \end{array} \right.$$

To $k \in \mathcal{K}$, we associate $\mathrm{concat}(k) = (\mathrm{concat}_1(k_1), \ldots, \mathrm{concat}_V(k_V)) \in \mathcal{Y}$: for each profile p, this function simply juxtaposes the vector k in order to return a configuration.
Then, denoting ξ the distribution of K, the image measure ν of ξ by concat is clearly suitable to prove decomposability. □

The condition of Proposition 5.7 is sufficient for decomposability but it is not necessary: even if we assume that each set Ω_v is defined as a Cartesian product $\mathcal{P}_v \times \mathcal{K}_v$ (where $\mathcal{P}_v \subseteq \mathcal{R}_\mathcal{C}$), it is possible for the random variables P and K to be non-independent, but for the electoral space to be decomposable anyway. Indeed, in Example 5.5, if $\mathrm{P} = (a \succ b, a \succ b)$, then we know with certainty that

$K = (\text{apple}, \text{apple})$, whereas if $\mathrm{P} = (b \succ a, b \succ a)$, then $K = (\text{banana}, \text{banana})$; therefore, P and K are not independent. However, as we have seen, this PES is decomposable.

Like the didactic example 5.5, the following one shows that, even without satisfying the sufficient conditions of Propositions 5.6 and 5.7, the PES can be decomposable. This time though, we use a more elaborate model that is more interesting for practical applications.

Draw $(p_1, \ldots, p_V)$ according to a certain distribution μ on $(\mathcal{W}_\mathcal{C})^V$. For each voter v, let k_v be the number of indifference classes in her strict weak order of preference p_v. For example, if she has no ties, the number of her indifference classes is simply equal to the number of candidates. Then draw k_v scores according to a certain distribution $\xi_v(k_v)$ on $[0,1]^{k_v}$. This distribution $\xi_v(k_v)$ is chosen in such a way that the k_v scores are almost surely all distinct.

Given a strict weak order p_v and a vector of scores $(u_1, \ldots, u_{k_v})$, we construct a state ω_v by assigning scores to candidates in the order of p_v.

For example, if $p_v = (1 \sim 2 \succ 3 \succ 4)$, then there are $k_v = 3$ equivalence classes: $\{1,2\}$, $\{3\}$, and $\{4\}$. So we use the distribution $\xi_v(3)$ to draw 3 grades, for example $(0.1, 0.8, 0.2)$. Finally, we obtain $\omega_v = (1 \sim 2 \succ 3 \succ 4, (0.8, 0.8, 0.2, 0.1))$.

To prove that this PES is decomposable, let us construct a suitable measure ν. Simultaneously for each integer $k_v \in [\![1, C]\!]$, draw k_v grades according to $\xi_v(k_v)$. This perfectly defines y_v, a transformation which associates to each p_v a state ω_v. By definition, by drawing p with distribution μ and y in this way, $\omega = y(p)$ is drawn with the desired probability measure.

Finally, we present a necessary condition for a PES to be decomposable.

Proposition 5.8

If (Ω, π) is decomposable, then for each subset of the voters $\mathcal{V}'$, for each event A on $\prod_{v \in \mathcal{V}'} \Omega_v$ (i.e. concerning only the voters of $\mathcal{V}'$), for each profile $p = (p_1, \ldots, p_V) \in \mathcal{R}$ of nonzero probability:

$$\pi\big(\omega_{\mathcal{V}'} \in A \mid P_{\mathcal{V}'} = p_{\mathcal{V}'}\big) = \pi\big(\omega_{\mathcal{V}'} \in A \mid P = p\big). \tag{5.1}$$

This is a consequence of Proposition 5.19, which we will see in Section 5.6 in a more general framework. The intuitive interpretation is that, if one knows the ordinal preferences p_v of a subset of voters $\mathcal{V}'$ and if one wishes to reconstitute their states ω_v probabilistically, then knowing the relations p_v of the other voters does not provide any information. Note that it is not excluded, on the other hand, that knowing the complete states ω_v of the other voters (and not only their relations p_v) gives useful information to reconstitute the states of the voters of $\mathcal{V}'$ from their ordinal preferences.

The advantage of this condition is that in practice it is more convenient to test than decomposability. In Section 5.6, we will show that unfortunately it is not sufficient to ensure decomposability.

5.3 Slicing theorem

We now have all the elements to prove the *slicing theorem*.

Theorem 5.9 (slicing)

Let (Ω, π) be a PES and f be an SBVS whose manipulability rate is well defined (i.e. CM_f is measurable, cf. Appendix A).

If (Ω, π) is decomposable, then there exists a slicing method $y \in \mathcal{Y}$ such that the slice of f by y has a manipulability rate less than or equal to that of f:

$$\tau_{\mathrm{CM}}(f_y) \leq \tau_{\mathrm{CM}}(f).$$

Proof. First, we show that the manipulability rate of any slice f_y can be expressed by considering an ordinal electoral space (where each voter is described only by her binary relation of preference), equipped with the probability distribution μ. Indeed, by definition:

$$\tau_{\mathrm{CM}}(f_y) = \int_{\omega \in \Omega} \mathrm{CM}_{f_y}(\omega) \pi(\mathrm{d}\omega).$$

Since $\mathrm{CM}_{f_y}(\omega)$ depends only on the profile $\mathrm{P}(\omega)$ and since $y(\mathrm{P}(\omega))$ has the same profile, we have:

$$\tau_{\mathrm{CM}}(f_y) = \int_{\omega \in \Omega} \mathrm{CM}_{f_y}(y(\mathrm{P}(\omega))) \pi(\mathrm{d}\omega).$$

Hence, by substitution:

$$\tau_{\mathrm{CM}}(f_y) = \int_{p \in \mathrm{P}(\Omega)} \mathrm{CM}_{f_y}(y(p)) \mu(\mathrm{d}p).$$

Second, let us study the manipulability of f. Using decomposability, we have by substitution:

$$\tau_{\mathrm{CM}}(f) = \int_{(p,y) \in \mathrm{P}(\Omega) \times \mathcal{Y}} \mathrm{CM}_f(y(p)) (\mu \times \nu)(\mathrm{d}p, \mathrm{d}y).$$

The Fubini-Tonelli theorem yields:

$$\tau_{\mathrm{CM}}(f) = \int_{y \in \mathcal{Y}} \left(\int_{p \in \mathrm{P}(\Omega)} \mathrm{CM}_f(y(p)) \mu(\mathrm{d}p) \right) \nu(\mathrm{d}y).$$

Lemma 5.3 ensures that $\mathrm{CM}_f(y(p)) \geq \mathrm{CM}_{f_y}(y(p))$, which leads to:

$$\begin{aligned} \tau_{\mathrm{CM}}(f) &\geq \int_{y \in \mathcal{Y}} \left(\int_{p \in \mathrm{P}(\Omega)} \mathrm{CM}_{f_y}(y(p)) \mu(\mathrm{d}p) \right) \nu(\mathrm{d}y) \\ &\geq \int_{y \in \mathcal{Y}} \tau_{\mathrm{CM}}(f_y) \nu(\mathrm{d}y). \end{aligned}$$

Thus, the manipulability rate of f is greater than or equal to the average (with weighting ν) of the manipulability rates of the slices f_y. Therefore, there exists at least one slice f_y such that $\tau_{\mathrm{CM}}(f_y) \leq \tau_{\mathrm{CM}}(f)$. □

In Section 5.7, we will discuss whether the decomposability assumption is necessary for this theorem and whether one can expect a stronger conclusion, namely the decrease of manipulability in the set-theoretic sense and not only in the probabilistic one.

For the time being, we will rather focus on the consequences of the slicing theorem. Combining it with Proposition 5.6, we obtain the following corollary.

Corollary 5.10

If voters $(\omega_1, \ldots, \omega_V)$ are independent, then there exists $y \in \mathcal{Y}$ such that $\tau_{\mathrm{CM}}(f_y) \leq \tau_{\mathrm{CM}}(f)$.

At this point, we do not have a clear idea of what the voting system f_y whose existence is guaranteed by Theorem 5.9 or its corollary 5.10 might look like: we only know that it is a slice of f. But it could have undesirable properties: for example, it is not explicitly excluded that it is dictatorial, which would be a particularly trivial and uninteresting way to decrease the manipulability rate. To avoid this kind of pitfall, we will now see that these slicing results are particularly relevant when combined with the Condorcification theorems.

5.4 Condorcification and slicing

It is obvious, but essential, that if f satisfies the Condorcet criterion, then so does any slice f_y. Indeed, if there is a Condorcet winner in ω, then there is also a Condorcet winner in $\omega' = y(\mathrm{P}(\omega))$, since the profile is the same in both configurations ω and ω'. This leads to another corollary of Theorem 5.9.

Corollary 5.11
If (Ω, π) is decomposable and if f satisfies the Condorcet criterion, then there exists an ordinal SBVS f', satisfying the Condorcet criterion, such that:

$$\tau_{\mathrm{CM}}(f') \leq \tau_{\mathrm{CM}}(f).$$

Indeed, it suffices to consider a slice $f' = f_y$ whose manipulability rate is less than or equal to that of f. Its existence is guaranteed by the slicing theorem 5.9.

Combining the weak Condorcification theorem 2.9 and the corollary 5.11 of the slicing theorem 5.9, we obtain the following theorem.

Theorem 5.12 (Condorcification and slicing)
Let (Ω, π) be a PES and f be an SBVS (such that CM_f and CM_{f^*} are measurable, cf. Appendix A).

Assume that:

- *(Ω, π) is decomposable,*
- *f satisfies* **InfMC**.

Then there exists an ordinal SBVS f', satisfying the Condorcet criterion, such that:

$$\tau_{\mathrm{CM}}(f') \leq \tau_{\mathrm{CM}}(f).$$

Proof. By the Condorcification theorem 2.9, we know that $\tau_{\mathrm{CM}}(f^*) \leq \tau_{\mathrm{CM}}(f)$. Applying Corollary 5.11 to f^*, we obtain a suitable voting system f'. □

For example, consider Range voting again. We have just proved that in any decomposable culture, there exists a voting system f' that does not use scores but only binary relations of preference, that satisfies Condorcet's criterion and that is at most as manipulable as Range voting (in the probabilistic sense).

The power of the electoral space formalism lies in the fact that this result is not limited, for example, to Range voting: it applies to voting systems whose ballots are ratings, approval values, multi-criteria ratings, apples, bananas, or any other kind of object (cf. Section 1.1.4 for various examples of electoral spaces). Moreover, there are absolutely no assumptions about the binary relations of preference: they can violate transitivity and even antisymmetry.

5.5 Optimality theorem

The proof of the slicing theorem 5.9 is not constructive: therefore, in the Condorcification and slicing theorem 5.12, we unfortunately do not know how to exhibit a voting system f' that is ordinal, Condorcet, and whose manipulability rate is less than or equal to that of the original voting system f.

That said, we believe that the above results have deeper consequences. Indeed, if the electoral space is decomposable and if the goal is to find a voting system in **InfMC** that is as little manipulable as possible, then these results mean that investigation can be restricted to ordinal Condorcet voting systems.

Moreover, since there exists a finite number of such voting systems (for a given value of V and C), this observation guarantees the existence of a voting system whose manipulability rate is minimal in **InfMC**. We are now going to formalize these findings.

Definition 5.13 (ordinal image of a PES)
For a PES $(V, C, \Omega, \mathrm{P}, \pi)$, its *ordinal image* is the PES $(V, C, \Omega', \mathrm{P}', \mu)$, where:

- For each voter v, her state space Ω'_v is constituted by the ordinal preferences authorized in Ω_v: we have $\Omega'_v = \mathrm{P}_v(\Omega_v) \subseteq \mathcal{R}_\mathcal{C}$.
- For each voter v, the function P'_v is the identity function.
- The distribution μ is the distribution of the random variable P in the original PES $(V, C, \Omega, \mathrm{P}, \pi)$.

Proposition 5.14
Consider an ordinal *electoral space (V, C, Ω', P'): for every voter v, we have $\Omega'_v \subseteq \mathcal{R}_\mathcal{C}$ and P'_v is the identity function. Let μ be a culture on Ω'.*

Then there exists a Condorcet SBVS g that is least manipulable in the probabilistic sense:

$$\tau^{\mu}_{\mathrm{CM}}(g) = \min_{\mathbf{Cond}(\Omega')}(\tau^{\mu}_{\mathrm{CM}}).$$

*We then say that g is τ^{μ}_{CM}-*optimal among the Condorcet voting systems of Ω'.

Proof. There exists a finite number of functions $g : \Omega \to \mathcal{C}$, *a fortiori* if we require that they satisfy the Condorcet criterion. Therefore, at least one of them minimizes $\tau^{\mu}_{\mathrm{CM}}(g)$. □

Theorem 5.15 (optimality)
Let (Ω, π) be a PES. Let g be a τ^{μ}_{CM}-optimal system among the Condorcet voting systems of the ordinal image of (Ω, π).

If (Ω, π) is decomposable, then:

$$\tau^{\pi}_{\mathrm{CM}}(g \circ \mathrm{P}) = \min_{\mathbf{InfMC}(\Omega)}(\tau^{\pi}_{\mathrm{CM}}).$$

In other words, g is optimal, not only among ordinal Condorcet voting systems, but among the broader class of all voting systems that satisfy the informed majority coalition criterion and that may not be ordinal. As we have seen, this includes a significant portion of the usual voting systems in the literature and in real life.

Proof. Let $f \in$ **InfMC**. By the Condorcification and slicing theorem 5.12, there exists an ordinal SBVS f' (formally defined on Ω, but depending only on the profile P) which satisfies the Condorcet criterion and such that $\tau^{\pi}_{\mathrm{CM}}(f') \leq \tau^{\pi}_{\mathrm{CM}}(f)$. And since g is τ^{μ}_{CM}-optimal, $\tau^{\pi}_{\mathrm{CM}}(g \circ \mathrm{P}) \leq \tau^{\pi}_{\mathrm{CM}}(f') \leq \tau^{\pi}_{\mathrm{CM}}(f)$. □

Therefore, in the interest of finding a voting system that minimizes the manipulability rate in **InfMC**, investigation can be restricted to ordinal Condorcet voting systems, which exist in finite number. We will intensively exploit this property in Chapter 10, dedicated to finding voting systems that are optimal in this sense.

5.6 Decomposability of any probabilized set

In Section 5.2, we defined a decomposable electoral space and stated some propositions without proof, in order to give an overview of these properties and to quickly arrive at the slicing theorem 5.9 without dwelling too much on technical aspects.

In this section, we generalize the notion of decomposability to probabilized sets which are not necessarily electoral spaces and we prove some results related to this notion in this general framework.

5.6.1 One-dimensional case: the complementary random variable lemma

Before defining a decomposable space in all generality, we will consider the special case where there is only one dimension, which corresponds to a single voter for the application to electoral spaces. To do so, we are going to prove what we call the *complementary random variable* lemma, which shows that in this case, there always exists a measure ν satisfying a condition similar to that of Definition 5.4 of decomposability. This will allow us, later on, to deal with the case where there are several voters who are independent.

Consider a probabilized space (Ω, π) and a random variable X with values in a finite measurable set $\mathcal{X}$ equipped with the *discrete sigma-algebra* (cf. Appendix A). Denote μ the distribution of X. In order to establish the parallel with electoral spaces, the reader can picture $\omega \in \Omega$ as the state of the unique voter and X as her binary relation of preference.

When the random experiment is carried out, the state of the resulting system is described by the mathematical object ω. The value $x = X(\omega)$ is a partial information about this state: if we only know x, we generally lack information to know ω perfectly. Imagine that there exists a space $\mathcal{Y}$ that makes it possible to express this complementary information: it means that the pair (x, y) represents ω without ambiguity.

Imagine, moreover, that the random variables x and y are independent: in general, this is a powerful property, since it allows to treat the two variables separately. In particular, it is such a property of independence, generalized to the case of several voters, that allowed us to prove the slicing theorem 5.9.

The construction we are going to consider is a generalization of this notion of complementary information. Indeed, we have a very important freedom: we can choose the set $\mathcal{Y}$. In order to treat the question in all generality, we always choose the set of functions $y : X(\Omega) \to \Omega$ that are consistent with X, in the sense that $X \circ y = \mathrm{Id}$. Indeed, this is the general framework so that giving an x and a y perfectly defines a state ω that is consistent with x.

The following lemma then shows that it is always possible to choose a y that satisfies the desired properties. In practice, we are going to anticipate each possible value of x and assign to it one of the values of ω consistent with this x, as we did in the "fruitful" example 5.5. As the different values of x are never reached at the same time (the voter always has a well determined value of x), we will see that they can be treated independently.

Lemma 5.16 (complementary random variable)

Let (Ω, π) be a probabilized space, X a random variable with values in a finite set $\mathcal{X}$ equipped with the discrete sigma-algebra, μ the distribution of X, and $\mathcal{Y} = \{y : X(\Omega) \to \Omega \text{ s.t. } X \circ y = Id\}$.

Then there exists a measure ν on $\mathcal{Y}$ such that for each event A on Ω:

$$\pi(\omega \in A) = (\mu \times \nu)\big(y(x) \in A\big).$$

To give the idea of the proof, imagine for example that we have three possible states and two corresponding values of X:

Probability	ω	X
π^1	ω^1	x^a
π^2	ω^2	x^b
π^3	ω^3	

To reconstitute an ω from an X, the simplest and most general way is that y is a function from $\mathcal{X}$ to Ω. We just need to find a good way to randomly draw the values of this function in x^a and x^b. When X is x^a, it is easy: we know that $\omega = \omega^1$. So we can decide that for all y, we will have $y(x^a) = \omega^1$. When X is x^b, we have $\omega = \omega^2$ with probability $\frac{\pi^2}{\pi^2+\pi^3}$ and $\omega = \omega^3$ the rest of the time. So we will choose $y(x^b) = \omega^2$ with probability $\frac{\pi^2}{\pi^2+\pi^3}$ and $y(x^b) = \omega^3$ the rest of the time. In summary, we can write informally:

$$y : \left| \begin{array}{l} x^a \to \omega^1 \\ x^b \to \left| \begin{array}{l} \omega^2 \text{ with probability } \frac{\pi^2}{\pi^2+\pi^3}, \\ \omega^3 \text{ with probability } \frac{\pi^3}{\pi^2+\pi^3}. \end{array} \right. \end{array} \right.$$

Drawing $y(x^a)$ (here, deterministically) and $y(x^b)$ (here, with some randomness) makes it possible to draw a function y that matches the required conditions.

If there were another image value x^c, we would draw the value of $y(x^c)$, independently of the values of $y(x^a)$ and $y(x^b)$, among the possible antecedents of x^c, with the probability distribution conditional to the fact that $X = x^c$.

Let us generalize and formalize this.

Proof. For all $x \in X(\Omega)$:

- If $\pi(X = x) > 0$, let π_x be the conditional probability measure knowing $X = x$ (restricted to $X^{-1}(x)$);

- If $\pi(X = x) = 0$, choose an $\omega_x \in X^{-1}(x)$ arbitrarily[4] and denote π_x the probability measure that returns ω_x with certainty.

Identifying any function $y \in \mathcal{Y}$ with the list of its values for each possible argument x, define ν as the product measure of all π_x.

[4] There is no problem with the axiom of choice because it suffices to make a finite number of choices.

Then, for each event A on Ω:

$$\begin{aligned}(\mu \times \nu)\big(y(x) \in A\big) &= \sum_{x \in X(\Omega)} \mu(\{x\}) \cdot \nu(y(x) \in A) \\ &= \sum_{\pi(X=x)>0} \pi(X = x) \cdot \pi(\omega \in A \mid X = x) \\ &= \pi(\omega \in A).\end{aligned}$$

We thank Anne-Laure Basdevant and Arvind Singh for fruitful discussions about this lemma. □

5.6.2 Multidimensional case: decomposability in general

We now deal with decomposability in the general multidimensional setting, which corresponds to the case with multiple voters in the application to electoral spaces.

Let $V \in \mathbb{N} \setminus \{0\}$. For each $v \in [\![1, V]\!]$, let Ω_v be a measurable set and $X_v : \Omega_v \to \mathcal{X}_v$ a measurable function, where $\mathcal{X}_v$ is a finite set equipped with the discrete sigma-algebra.

Let π be a probability measure on the universe $\Omega = \prod_{v=1}^{V} \Omega_v$. Let $X = (X_1, \ldots, X_V)$ and μ be the distribution of X.

For each v, let $\mathcal{Y}_v = \{y_v : X_v(\Omega_v) \to \Omega_v \text{ s.t. } X_v \circ y_v = \mathrm{Id}\}$. Denote $\mathcal{Y} = \prod_{v=1}^{V} \mathcal{Y}_v$. For $(x, y) \in X(\Omega) \times \mathcal{Y}$, denote $y(x) = (y_1(x_1), \ldots, y_V(x_V))$.

Definition 5.17 (decomposability in the general case)

We say that (Ω, π) is X-*decomposable* iff there exists a measure ν on $\mathcal{Y}$ such that for each event A on Ω:

$$\pi(\omega \in A) = (\mu \times \nu)\big(y(x) \in A\big).$$

The difficulty arises from our requirement that the complementary random variables y_v be *individual*: y cannot be any function from $X(\Omega)$ to Ω, but must be a V-tuple of functions, where each y_v is defined from $X_v(\Omega_v)$ to Ω_v. Indeed, in the proof of Lemma 5.3, which is the cornerstone of the slicing theorem 5.9, we need individual random variables to deal with sincere voters. If we were to ask for a *collective* random variable y which, starting from x, reconstitutes state ω with the desired probability distribution, this would always be possible, by a direct application of Lemma 5.16 of the complementary random variable.

We can now generalize and prove Proposition 5.6, which stated that if voters are independent, then the electoral space is decomposable.

Proposition 5.18

If the random variables $(\omega_1, \ldots, \omega_V)$ are independent, then (Ω, π) is X-decomposable.

Proof. It suffices to apply Lemma 5.16 for each $v \in [\![1, V]\!]$, which defines a measure ν_v on each set $\mathcal{Y}_v$. We then define ν as the product measure of ν_v. □

The following proposition generalizes and proves Proposition 5.8, which gives a necessary condition for decomposability.

Proposition 5.19

If (Ω, π) is X-decomposable then, for each subset $\mathcal{V}'$ of $[\![1, V]\!]$, for each event A on $\prod_{v \in \mathcal{V}'} \Omega_v$, for each $x = (x_1, \ldots, x_V)$ of nonzero probability:

$$\pi\big(\omega_{\mathcal{V}'} \in A \mid X_{\mathcal{V}'} = x_{\mathcal{V}'}\big) = \pi\big(\omega_{\mathcal{V}'} \in A \mid X = x\big). \tag{5.2}$$

Proof. On the one hand:

$$\begin{aligned}\pi\big(\omega_{\mathcal{V}'} \in A \mid X = x\big) &= (\mu \times \nu)\big(y_{\mathcal{V}'}(x_{\mathcal{V}'}) \in A \mid X = x\big)\\ &= \nu\big(y_{\mathcal{V}'}(x_{\mathcal{V}'}) \in A\big).\end{aligned}$$

On the other hand, we obtain in a similar way:

$$\pi\big(\omega_{\mathcal{V}'} \in A \mid X_{\mathcal{V}'} = x_{\mathcal{V}'}\big) = \nu\big(y_{\mathcal{V}'}(x_{\mathcal{V}'}) \in A\big).$$

□

However, condition (5.2) does not ensure that (Ω, π) is X-decomposable. As a counterexample, consider $V = 2$. The state ω_v (with $v = 1$ or $v = 2$) can take 4 values, denoted $\omega_v^1, \ldots, \omega_v^4$. The variable X_v can take 2 values, x_v^a and x_v^b. The following table defines the correspondence between the states ω_v and the variables x_v, as well as the measure π.

π	$\omega_1^1 \to x_1^a$	$\omega_1^2 \to x_1^a$	$\omega_1^3 \to x_1^b$	$\omega_1^4 \to x_1^b$
$\omega_2^1 \to x_2^a$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	0
$\omega_2^2 \to x_2^a$	$\frac{1}{16}$	$\frac{1}{16}$	0	$\frac{1}{8}$
$\omega_2^3 \to x_2^b$	$\frac{1}{8}$	0	0	$\frac{1}{8}$
$\omega_2^4 \to x_2^b$	0	$\frac{1}{8}$	$\frac{1}{8}$	0

This table reads as follows. In the header of the first column of data, we read that if $\omega_1 = \omega_1^1$, then $x_1 = x_1^a$. Similarly, in the header of the first row of data, we read that if $\omega_2 = \omega_2^1$, then $x_2 = x_2^a$. At the intersection of this column and this row, we see that the probability of the state (ω_1^1, ω_2^1) is $\frac{1}{16}$.

It is tedious but easy to check that condition (5.2) is satisfied. Indeed, for example:

$$\pi(\omega_1^1 \mid x_1^a \wedge x_2^a) = \frac{1}{2} = \pi(\omega_1^1 \mid x_1^a).$$

Now suppose that (Ω, π) is X-decomposable and let ν be a suitable measure for decomposition. For $(\alpha, \beta, \gamma, \delta) \in \{1, 2\} \times \{3, 4\} \times \{1, 2\} \times \{3, 4\}$, we adopt the following notation shortcut:

$$\nu(\alpha, \beta, \gamma, \delta) = \nu\Big(y_1(x_1^a) = \omega_1^\alpha \ \wedge\ y_1(x_1^b) = \omega_1^\beta \ \wedge\ y_2(x_2^a) = \omega_2^\gamma \ \wedge\ y_2(x_2^b) = \omega_2^\delta\Big).$$

For example, $\nu(\mathbf{1}, \mathbf{4}, \mathbf{2}, \mathbf{4})$ represents the probability that, when y is drawn with the probability measure ν, we have $y_1(x_1^a) = \omega_1^{\mathbf{1}}$ and $y_1(x_1^b) = \omega_1^{\mathbf{4}}$ and $y_2(x_2^a) = \omega_2^{\mathbf{2}}$ and $y_2(x_2^b) = \omega_2^{\mathbf{4}}$: this value of y corresponds to the choice of columns **1** and **4** and rows **2** and **4** in the table.

We note the following facts.

- $0 = \pi(\omega_1^3 \wedge \omega_2^3 \mid x_1^b \wedge x_2^b) = \sum_{(\alpha,\gamma)} \nu(\alpha, 3, \gamma, 3)$. Since all the terms of this sum are nonnegative, they are all zero and we have in particular $\nu(1, 3, 1, 3) = 0$.
- $0 = \pi(\omega_1^1 \wedge \omega_2^4 \mid x_1^a \wedge x_2^b) = \sum_{(\beta,\gamma)} \nu(1, \beta, \gamma, 4)$, hence $\nu(1, 3, 1, 4) = 0$ and $\nu(1, 4, 1, 4) = 0$.
- $0 = \pi(\omega_1^4 \wedge \omega_2^1 \mid x_1^b \wedge x_2^a) = \sum_{(\alpha,\delta)} \nu(\alpha, 4, 1, \delta)$, hence $\nu(1, 4, 1, 3) = 0$.

Therefore, $\frac{1}{4} = \pi(\omega_1^1 \wedge \omega_2^1 \mid x_1^a \wedge x_2^a) = \sum_{(\beta,\delta)} \nu(1,\beta,1,\delta) = \nu(1,3,1,3) + \nu(1,3,1,4) + \nu(1,4,1,3) + \nu(1,4,1,4) = 0$: this contradiction proves that (Ω,π) is not decomposable. Thus, condition (5.2) is necessary but not sufficient for (Ω,π) to be X-decomposable.

In the above counterexample, it can be shown that there exists a *signed measure* ν that satisfies the relation defining decomposability. Unfortunately, we need a measure in the usual sense, i.e. a positive measure: indeed, the proof of the slicing theorem 5.9 uses the growth property of the integral, which is itself based on its positivity property.

5.7 Discussing the assumption and conclusions of the slicing theorem

In this section, we examine whether the slicing theorem 5.9 is sharp or whether it can be generalized, either by strengthening the conclusions or by weakening the assumptions.

5.7.1 Diminishing manipulability in the probabilistic or set-theoretic sense?

In the slicing theorem 5.9, we conclude that there exists a slice f_y that is at most as manipulable as the original system f in the probabilistic sense, i.e., such that $\tau_{\mathrm{CM}}^{\pi}(f_y) \leq \tau_{\mathrm{CM}}^{\pi}(f)$ for a given culture π.

The conclusion would be stronger if, like for the Condorcification theorems 2.9 and 2.20, the resulting voting system were at most as manipulable as f in the set-theoretic sense, i.e. if we had $\mathrm{CM}_{f_y} \subseteq \mathrm{CM}_f$. With this stronger conclusion, Theorem 5.9 would take the form of the "pseudo-theorem" 5.20 below.

Pseudo-theorem 5.20 (slicing, with reduced manipulability in the set-theoretic sense)
Let (Ω,π) be a PES and f be an SBVS whose manipulability rate is well defined (i.e., CM_f is measurable, cf. Appendix A).

If (Ω,π) is decomposable, then there exists a slicing method $y \in \mathcal{Y}$ such that the slice of f by y is at most as manipulable as f:

$$\mathrm{CM}_{f_y} \subseteq \mathrm{CM}_f.$$

In this pseudo-theorem, it is likely that the measurability and decomposability hypotheses are useless, since the conclusion is not probabilistic, but we will keep them just in case they would be of any use. If this pseudo-theorem were true, then we would immediately have the following corollary, which is obtained in the same way as the Condorcification and slicing theorem 5.12.

Pseudo-theorem 5.21 (Condorcification and slicing, with reduced manipulability in the set-theoretic sense)
Let (Ω,π) be a PES and f be an SBVS (such that CM_f and CM_{f^*} are measurable, cf. Appendix A).

Assume that:

- *(Ω,π) is decomposable,*
- *f satisfies* **InfMC**.

Then there exists an ordinal SBVS f', satisfying the Condorcet criterion, such that:

$$\mathrm{CM}_{f'} \subseteq \mathrm{CM}_f.$$

We are going to show that Pseudo-theorem 5.21 is false and, consequently, that Pseudo-theorem 5.20 is also false.

Consider the following electoral space. There are $V = 3$ voters and $C = 3$ candidates a, b, c. The state of each voter is given by a strict total order of preference and a bit whose value is 0 or 1. This electoral space is similar to the decentralized random generator of Example 1.9. In order to avoid measurability problems, we equip the electoral space with the discrete sigma-algebra (cf. Appendix A), which ensures that any function is measurable. Let us define culture π: for each voter independently, we draw her state uniformly at random. By Proposition 5.6, this implies that the PES (Ω, π) is decomposable.

Here is the SBVS f that we consider.

1. If there is a Condorcet winner, then she is elected.

2. If the configuration is non-Condorcet and if at least two bits are 1, then c is elected.

3. If the configuration is non-Condorcet and if at least two bits are 0, then b is elected.

Since f satisfies the Condorcet criterion, it also satisfies **InfMC**. Thus, the hypotheses of Pseudo-theorem 5.21 are all met. We are going to see that, however, the conclusion is not satisfied.

Consider the following three configurations ω, ϕ, and ψ.

Configuration	Voter			Condorcet winner
	1	2	3	
ω	a c , 0 b	b c , 1 a	c a , 1 b	c
ϕ	a b , 1 c	b a , 0 c	c a , 1 b	a
ψ	a b , 0 c	b c , 0 a	c b , 1 a	b

The voting system f is not manipulable in ω. Indeed, on the one hand, in order to manipulate in favor of b, the only interested voter cannot prevent a or c from being the Condorcet winner. On the other hand, to manipulate in favor of a, the only interested voter cannot make a the Condorcet winner because she will always have a victory against c, and there is no other way to make a win.

Similarly, it can be shown that f is not manipulable in either ϕ or ψ.

Now consider an ordinal system f' satisfying the Condorcet criterion, and the following family of configurations χ (the bits of the voters do not matter for f').

Configuration	Voter			Condorcet winner
	1	2	3	
χ	a b c	b c a	c a b	None

If $f'(\chi) = a$ (resp. b, c), then f' is manipulable in ω (resp. ϕ, ψ) to χ. Therefore, f' is necessarily manipulable in at least one of the three configurations ω, ϕ, ψ, and it is therefore impossible to have $\text{CM}_{f'} \subseteq \text{CM}_f$. Therefore, Pseudo-theorem 5.21 is false, which implies that Pseudo-theorem 5.20 is also false.

Thus, in the slicing theorem 5.9 and the Condorcification and slicing theorem 5.12, the conclusion cannot be strengthened by requiring that the resulting voting system be at most as manipulable as the original system f in the set-theoretic sense. We cannot have more than obtaining a system that is at most as manipulable as f in the probabilistic sense.

5.7.2 Decomposability assumption

In the previous section, we saw that the conclusion of Theorems 5.9 and 5.12 cannot be strengthened by requiring a decrease in manipulability in the set-theoretic sense. We now examine whether it is possible to generalize these theorems in another way: by weakening the assumptions, and in particular by removing the central assumption of decomposability for the PES (Ω, π). Without this assumption, the slicing theorem 5.9 takes the form of Pseudo-theorem 5.22, as follows.

Pseudo-theorem 5.22 (slicing, without decomposability)

Let (Ω, π) be a PES and f be an SBVS whose manipulability rate is well defined (i.e., CM_f is measurable, cf. Appendix A).

Then there exists a slicing method $y \in \mathcal{Y}$ such that the slice of f by y has a manipulability rate less than or equal to that of f:

$$\tau_{\text{CM}}(f_y) \leq \tau_{\text{CM}}(f).$$

If this pseudo-theorem were true, then one could immediately generalize the Condorcification and slicing theorem 5.12 into the following pseudo-theorem.

Pseudo-theorem 5.23 (Condorcification and slicing, without decomposability)

Let (Ω, π) be a PES and f be an SBVS (such that CM_f and CM_{f^*} are measurable, cf. Appendix A).

Assume that f satisfies **InfMC**.

Then there exists an ordinal SBVS f', satisfying the Condorcet criterion, such that:

$$\tau_{\text{CM}}(f') \leq \tau_{\text{CM}}(f).$$

To prove that this result is false, take up the counterexample of Section 5.7.1. But now, define the culture π by $\pi(\omega) = \pi(\phi) = \pi(\psi) = \frac{1}{3}$. Then the hypotheses of Pseudo-theorem 5.23 are all satisfied.

The initial voting system f has a manipulability rate equal to zero: although it is manipulable in some configurations, it is almost surely non-manipulable because all the probability is concentrated on configurations ω, ϕ, and ψ, which are non-manipulable. However, we saw that in any ordinal Condorcet system f', at least one of these configurations is manipulable, hence f' has a manipulability rate of at least $\frac{1}{3}$. This proves that the conclusion of Pseudo-theorem 5.23 is not true.

Therefore, Pseudo-theorem 5.23 is false, which proves that Pseudo-theorem 5.22 is also false.

Thus, the slicing theorem 5.9 and its sequel, the Condorcification and slicing theorem 5.12, are no longer valid if we remove the decomposability hypothesis purely and simply. However, a natural question remains: do Theorems 5.9 and 5.12 remain true with a weaker assumption than decomposability, for example the necessary condition (5.1) presented in Proposition 5.8? We leave this question open for future work.

Part II

Computer-Aided Study of Manipulability

Chapter 6

S∀AMP: *Simulator of Various Voting Algorithms in Manipulating Populations*

As mentioned earlier, the history of voting theory has been marked by the discovery of several paradoxes, in particular that of Condorcet (1785) or "paradox of voting" on pairwise comparisons of candidates; the paradox of Arrow (1950), a key point of which is the independence of irrelevant alternatives (IIA); and the impossibility theorem on manipulability of Gibbard (1973) and Satterthwaite (1975).

Since no non-trivial voting system can avoid these paradoxes completely, the probability of occurrence of these pathological cases in various types of cultures has been the subject of intense investigation (Campbell and Tullock, 1965; Chamberlin et al., 1984; Nitzan, 1985; Lepelley and Mbih, 1987; Kelly, 1993; Aleskerov and Kurbanov, 1999; Smith, 1999; Lepelley and Valognes, 2003; Favardin and Lepelley, 2006; Pritchard and Wilson, 2007; Aleskerov et al., 2008; Reyhani et al., 2009; Jennings, 2011; Reyhani, 2013; Green-Armytage, 2014; Wang et al., 2014). However, there remain important open questions in the field, particularly about the relative performances of the voting systems under different criteria and various sets of assumptions about voter preferences.

Recently, thanks to the development of the community of computational social choice, interesting results have been published about algorithmic issues related to voting systems and their manipulation. One one hand, complexity problems were investigated, following the line of research initiated by Bartholdi et al. (1989b) then Bartholdi and Orlin (1991), and it was proven that deciding manipulability was $\mathcal{NP}$-hard for some voting systems. On the other hand, some authors, like Xia et al. (2009); Walsh (2010a); Zuckerman et al. (2009, 2011); Gaspers et al. (2013), have developed explicit algorithms to deal with manipulability issues.

However, to our knowledge, there was no publicly available software that relies on these existing techniques, especially for the study of manipulability. This observation led us to develop S∀AMP (*Simulator of Various Voting Algorithms in Manipulating Populations*), a Python package designed to the study of voting systems and their manipulability. It is easy to use, supports a wide choice of models as well as importation of real-life data for the preferences of the voters, and implements a variety of voting systems. It is simple to extend S∀AMP by adding new models of populations and new voting systems.

SVVAMP is free software, under the GNU General Public License version 3. Its documentation includes the installation procedure, tutorials, a reference guide and instructions for contributors. It is available at:

https://svvamp.readthedocs.org.

After a basic example intended to give a brief overview of SVVAMP, we present in Section 6.1 the tools for the Population class, representing a set of voters with preferences on a set of candidates. In Section 6.2, we present the abstract class Election, from which all voting systems inherit. In Section 6.3, we illustrate our implementation choices by focusing on the example of coalitional manipulability (CM), whose study is one of the main goals of the software. In Section 6.4, we indicate the specific algorithms used for some voting systems. Finally, in Section 6.5, we focus on the implementation of IRV and its variants: indeed, we will see in the following chapters that this voting system is especially resistant to manipulation. This is why it has been the subject of particular attention.

Let us give a brief overview of SVVAMP through a basic example.

After loading SVVAMP, we define a random population of $V = 5$ voters with preferences over $C = 4$ candidates by using the *Spheroid* model, an extension of the usual *impartial culture* (Definition 1.14) that we will present with more details in Section 6.1.3.

```
>>> import svvamp
>>> pop = svvamp.PopulationSpheroid(V=5, C=4)
```

This creates an object `pop` of class Population. We can then sort the voters lexicographically on their preference orders and display these orders.

```
>>> pop.ensure_voters_sorted_by_rk()
>>> print(pop.preferences_rk)
[[0 2 1 3]
 [1 3 2 0]
 [1 3 2 0]
 [2 0 1 3]
 [3 2 0 1]]
```

Each row of the attribute matrix `preferences_rk` represents the strict total order of preference of a voter. For example, the first voter prefers candidate 0, then 2, then 1, then 3. Following the usual convention in Python, the numbering of candidates and voters starts at 0.

Let us create an election with this population of voters, using the voting system Plurality, and determine the sincere winner. This one, denoted w in SVVAMP and throughout this chapter, is candidate 1 in our example.

```
>>> election = svvamp.Plurality(pop)
>>> print(election.w)
1
```

SVVAMP can tell us that the election is CM and give us details on this type of manipulation.

```
>>> is_CM, log_CM, candidates_CM = election.CM_with_candidates
    ()
>>> print(is_CM)
True
>>> print(candidates_CM)
[ 1.  0.  1.  0.]
```

Candidates 0 or 2 can benefit from CM, but not candidate 3. Candidate 1, i.e. w, is excluded by convention from the definition of CM.

6.1 Population

In this section, we present how to study the properties of a population by continuing our first example (Section 6.1.1). Then, we show that SVVAMP is not limited to strict total preference orders but can also exploit utilities and weak preference orders (Section 6.1.2). Finally, we present a panel of tools to define populations (Section 6.1.3).

6.1.1 Get information about a population

An object of class Population has several methods and attributes that can be used to study its properties.

The Borda scores assigned by voters to candidates are directly implemented as an attribute in the Population class, because they are used by several voting systems.

```
>>> print(pop.preferences_borda_rk)
[[3 1 2 0]
 [0 3 1 2]
 [0 3 1 2]
 [2 1 3 0]
 [1 0 2 3]]
```

The total Borda score and the Plurality score[1] of each candidate are available in the same way, using the attributes `borda_score_c_rk` and `plurality_scores_rk` respectively.

Other attributes provide the *weighted majority matrix* (*matrix of duels*), the *matrix of victories*,[2] and the identity of the Condorcet winner.

```
>>> print(pop.matrix_duels_rk)
[[0 3 1 2]
 [2 0 2 4]
 [4 3 0 2]
 [3 1 3 0]]
>>> print(pop.matrix_victories_rk)
[[ 0.  1.  0.  0.]
 [ 0.  0.  0.  1.]
 [ 1.  1.  0.  0.]
 [ 1.  0.  1.  0.]]
>>> print(pop.condorcet_winner_rk)
nan
```

In this example, the last value returned is `nan` (*not a number*), which means by convention that there is no Condorcet winner.

[1] For the definition of Borda and Plurality scores, see Section 1.6.2.
[2] For each pair of distinct candidates (c, d), `matrix_victories_rk[c, d]` is:

- 1 iff `matrix_duels_rk[c, d]` $> V/2$,
- 0.5 iff `matrix_duels_rk[c, d]` $= V/2$,
- 0 iff `matrix_duels_rk[c, d]` $< V/2$.

By convention, the diagonal coefficients are equal to 0.

The matrix of duels, the matrix of victories, and the Condorcet winner have several variants implemented in SVVAMP, which depend on tie-breaking choices. In particular, the notions of Condorcet-admissible candidate, weak Condorcet winner, and relative Condorcet winner are implemented. The interested reader can refer to the documentation of SVVAMP for more details.

6.1.2 Work with utilities

Most of the studies on the likelihood of a manipulation, cited in the introduction to this chapter, concern only ordinal voting systems. However, some interesting voting systems are cardinal. A special case is Approval voting, which is an important topic of research (Brams and Fishburn (1978); Fishburn and Brams (1981); Brams (1994, 2003); Laslier and Van der Straeten (2008); Laslier (2009)). It is also used in real-life elections: for example, it was chosen by social choice theorists for their actual elections in the Society of Social Choice and Welfare.

One of the objectives of SVVAMP is to make it possible to compare ordinal and cardinal voting systems on the same population of voters. To this end, voters do not only have preference orders, but also utilities over the candidates. Therefore, it is possible to define a population by providing such utilities.

```
>>> pop1 = svvamp.Population(preferences_ut=
                              [[1,  1,  0],
                               [2,  0, -1]])
```

Utilities extend the set of possible preferences, compared to strict total orders of preference. They provide a measure of intensity and they allow a voter to be indifferent between several candidates, which gives a convenient way to handle weak orders of preference. In `pop1`, if candidate 0 is the sincere winner of an election, then the first voter is not interested in a manipulation in favor of candidate 1, and vice versa.

However, for some voting systems, voters are forced to provide a strict total order of preference: in this case, each voter must break the ties in her own preferences. For this reason, if a population has been defined only by its utilities, the ranking of each voter is obtained by breaking its ties at random, once and for all, the first time the attribute `preferences_rk` is called.

```
>>> print(pop1.preferences_rk)
[[1 0 2]
 [0 1 2]]
```

The purpose of this specification choice is that the comparison of voting systems based on strict total orders is done solely on the basis of their specific behavior, without introducing a bias due to sincere voters giving different ballots to different voting systems.

Conversely, if a population is defined only by rankings, SVVAMP can automatically populate the utility matrix. Of course, choosing utilities consistent with the rankings is necessarily arbitrary; by convention, SVVAMP uses utilities equal to Borda scores in this case.

```
>>> pop2 = svvamp.Population(preferences_rk=
                              [[0, 1, 2, 3],
                               [2, 0, 1, 3]])
>>> print(pop2.preferences_ut)
[[3 2 1 0]
 [2 1 3 0]]
```

Finally, it is possible to provide utilities and rankings simultaneously, the latter indicating how each voter breaks her own ties in her sincere ballot, when a voting system based on rankings is used.

```
>>> pop3 = svvamp.Population(
            preferences_ut=[[1, 1,  0],
                            [2, 0, -1]],
            preferences_rk=[[0, 1, 2],
                            [0, 1, 2]])
```

Rankings must be consistent with utilities, in the sense that if a voter v places a candidate c before a candidate d in her ranking, then her utility for c must be greater than or equal to her utility for d. If this is not the case, SVVAMP raises an error.

Some of the attributes mentioned in Section 6.1.1 differ depending on whether utilities or rankings are considered. For example, the attribute `preferences_borda_ut` is utility-based and treats indifferences as such.[3] In contrast, the attribute `preferences_borda_rk` is based on rankings.

```
>>> print(pop3.preferences_borda_ut)
[[ 1.5  1.5  0. ]
 [ 2.   1.   0. ]]
>>> print(pop3.preferences_borda_rk)
[[2 1 0]
 [2 1 0]]
```

Similar variants with the suffixes `_rk` (based on strict total orders) and `_ut` (based on utilities and allowing weak orders) are implemented for the matrix of duels, the matrix of victories, and the Condorcet notions.

6.1.3 Create a Population object

So far, apart from our introductory example, we have defined populations manually, using the initialization method of the class Population. This is an easy way to define simple examples. But in order to do studies at a larger scale, it is also possible to import a population from an external file or to use a panel of probabilistic models to generate random populations.

Importing a population from an external file is immediate: SVVAMP can read simple CSV files containing the population utilities, or files in the format of the PrefLib database (Mattei and Walsh, 2013), which we will return to in Chapter 9.

To generate artificial populations, a variety of cultures, i.e. random generators using a probabilistic model, are implemented.

Spheroid and *Cubic Uniform*: These two models extend the impartial culture (Definition 1.14) to utilities. For rankings, these two models are equivalent to impartial culture. We will study the spheroidal model in Chapter 7.

Ladder: Like in the impartial culture, voters are independent and the model is anonymous and neutral (in the sense that it treats voters symmetrically, and the same for candidates), but a voter can be indifferent between several candidates. This model can thus be seen as an extension of the impartial culture using weak orders. In practice, each voter's utility for each candidate is drawn independently

[3]For each voter v and candidate c, `preferences_borda_ut[v, c]` is the sum of:

- 1 point for each other candidate d such that v has strictly greater utility for c than for d,
- 0.5 point for each other candidate d such that v has the same utility for c and for d.

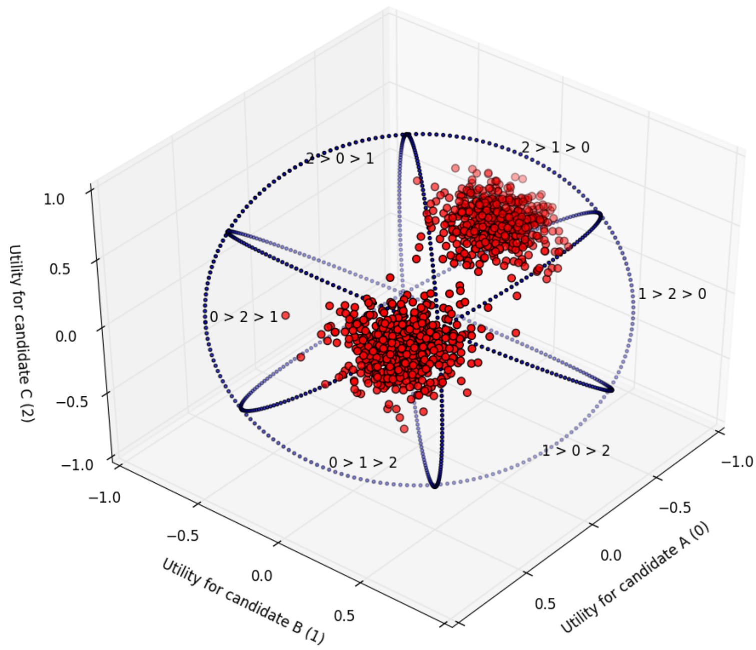


Figure 6.1 – A population using the Von Mises–Fisher model. $V = 1000$ voters, $C = 3$ candidates, two random poles of the same concentration $\kappa = 50$.

and uniformly in a finite set of values (the "rungs" of the ladder). In the development phase, this culture is often used to check that SVVAMP has the desired behavior when voters are indifferent between multiple candidates. In order not to unnecessarily lengthen the following chapters, we will not make a detailed study of it in this thesis.

Gaussian Well and *Euclidean Box*: Both models assign random positions to voters and candidates in a Euclidean space called the *political spectrum*. The utility of a voter for a candidate is a decreasing function of the distance between them. If the dimension of the political spectrum is equal to 1, then the preferences are necessarily *single-peaked* (cf. Definition 1.11 and Black, 1958). These models will be the subject of Chapter 8.

Von Mises–Fisher: This model is similar to the model of Mallows (1957), but it is adapted to populations defined by their utilities. There is a special point in the utility space, the *pole*, where the probability density is maximum, and a concentration parameter κ which allows SVVAMP to control the shape of the distribution (Watson and Williams, 1956). It is possible to create a population consisting of several subgroups using different poles with different concentrations. An example of such a population is given in Figure 6.1, itself generated by SVVAMP. Each red dot represents the utility vector of a voter. The Von-Mises Fisher distributions are generated using the algorithm of Ulrich (1984) modified by Wood (1994). We will study this model in Chapter 7.

6.2 Elections

In SVVAMP, Election is an abstract class. The end user always uses one of its subclasses implementing elections in a particular voting system (Plurality, Approval voting, etc) and inheriting the attributes and methods of the class Election.

In this section, we show how SVVAMP makes it possible to study the result of an election, various notions of manipulation, and the independence of irrelevant alternatives (IIA).

6.2.1 Result of the election

Various attributes are dedicated to studying the outcome of an election when voters are sincere.

```
>>> pop = svvamp.PopulationSpheroid(V=3, C=5)
>>> print(pop.preferences_rk)
[[0 2 3 4 1]
 [0 2 4 3 1]
 [2 1 3 4 0]]
>>> election = svvamp.Plurality(pop)
>>> print(election.ballots)
[0 0 2]
>>> print(election.scores)
[2 0 1 0 0]
>>> print(election.w)
0
```

We obtain the sincere ballot of each voter, the score of each candidate, and the winner of the election. Tie-breaking issues will be discussed in Section 6.3.3. SVVAMP also provides the attributes `candidates_by_scores_best_to_worst` and `scores_best_to_worst` to sort the candidates based on their score in the election.

The types of the attributes `ballots`, `scores`, and `scores_best_to_worst` depend on the voting system. For example, for Approval voting, `ballots` is a $V \times C$ matrix of Booleans and `ballots[v, c]` is **True** iff voter v approves of candidate c.

Please refer to the documentation for an exhaustive list of attributes and more details on each voting system.

6.2.2 Coalitional manipulation (CM)

To illustrate coalitional manipulation, consider the following example. Recall that *IRV* is the acronym for Instant runoff voting.

```
>>> pop = svvamp.PopulationSpheroid(V=1000,
                                    C=10)
>>> election = svvamp.IRV(pop)
```

Let us look at the coalitional manipulability of this election.

```
>>> is_CM, log_CM = election.CM()
>>> print(is_CM)
nan
```

For each method, SVVAMP defaults to its most accurate algorithm among those running in polynomial time. In the case of IRV, deciding CM is $\mathcal{NP}$-complete (Bartholdi and Orlin, 1991), hence this polynomial algorithm is not exact. For this reason, even if `is_CM` is typically a Boolean (whether the election is manipulable or not), it can also take the conventional value `nan`, meaning that the algorithm was not able to decide.

We can choose an exact algorithm by modifying the attribute `CM_option`.

```
>>> election.CM_option = 'exact'
>>> is_CM, log_CM = election.CM()
>>> print(is_CM)
True
```

In the case of voting systems for which there is an exact algorithm running in polynomial time, `'exact'` is the only available option.

As we saw in the introduction of this chapter, SWAMP can specify in favor of which candidates the election is manipulable.

```
>>> is_CM, log_CM, candidates_CM = election.CM_with_candidates
    ()
>>> print(candidates_CM)
[ 0.  0.  0.  1.  0.  0.  0.  0.  0.  0.]
```

6.2.3 Variants of coalitional manipulation (ICM, UM, TM)

SWAMP also implements three variants of CM.

We say that an election is *ignorant-coalition manipulable* (*ICM*) in favor of a certain candidate $c \neq \mathrm{w}$ iff the voters who prefer c to the sincere winner w can use strategies such that, regardless of the strategies of the other voters, candidate c is declared the winner. It is thus a notion of manipulation inspired by the criterion **IgnMC**.

We say that an election is *unison-manipulable* (*UM*) in favor of candidate c iff there is a strategy such that, when all voters who strictly prefer c to the sincere winner w use this same strategy (while the other voters continue to vote sincerely), candidate c is declared the winner. The terminology "unison" is borrowed from Walsh (2010a). One can draw a parallel with the criterion **MajUniBal** for the unison aspect, but it is important to point out the differences. On the one hand, it is imposed that all interested voters participate in the manipulation and use the same ballot (above all to allow for easier computation). On the other hand, this is an informed manipulation (i.e. with given ballots of the sincere voters), whereas the criterion **MajUniBal** concerns an ignorant manipulation (which must work regardless of the ballots of the sincere voters).

Finally, an election is said to be *trivially manipulable* (*TM*) in favor of candidate c iff, when all voters who strictly prefer c to w use their *trivial strategy* (while other voters continue to vote sincerely), candidate c wins.

What we call the *trivial strategy* for voter v in favor of candidate c against candidate w depends on the type of ballot. For ordinal voting systems, it is the ballot where c is ranked first ("compromise"), w is ranked last ("burial"), and where the other candidates remain in the sincere preference order of v. For score-based voting systems, it is the ballot where v assigns the maximum allowed score to c (default 1) and the minimum score to the other candidates (default -1).

The trivial strategy seems natural when voter v wants to make c win and knows that w is a strong opponent, without any other clue about the chances of the other candidates. It requires little coordination between the manipulators. For this reason, when CM is possible, TM is a good indicator of whether the manipulation is easy or not, in terms of information exchange between the manipulators. The other obvious advantage of this notion is that it can be computed in polynomial time (provided that computing the winner of an election is polynomial, which is generally the case, except for some voting systems like Kemeny's method).

These three variants of CM — ICM, UM, and TM — use the same syntax as CM. For example, UM can be decided with the methods `UM` and `UM_with_candidates`, and its algorithm can be chosen with `UM_option`. Like for CM, their default option is to use the most accurate polynomial algorithm available.

6.2.4 Individual manipulation (IM)

An election is said to be *individually manipulable* (*IM*) iff a voter v, by casting an insincere ballot, can make a certain candidate c win whom she strictly prefers to the sincere winner w (while the other voters continue to vote sincerely).

The method that provides the most information about IM is `IM_full`.

```
>>> pop = svvamp.PopulationSpheroid(V=4, C=3)
>>> election = svvamp.IRV(pop)
>>> election.w
2
>>> election.IM_option = 'exact'
>>> is_IM, log_IM, candidates_IM, voters_IM, v_IM_for_c =
    election.IM_full()
>>> print(is_IM)
True
>>> print(candidates_IM)
[ 1.  0.  0.]
>>> print(voters_IM)
[ 0.  0.  1.  1.]
>>> print(v_IM_for_c)
[[ 0.  0.  0.]
 [ 0.  0.  0.]
 [ 1.  0.  0.]
 [ 1.  0.  0.]]
```

In this example, the election is IM. More precisely, only the first candidate can benefit from it. This manipulation can be accomplished by any of the last two voters, while the first two cannot or are not interested.

If only a subset of this information is required, dedicated methods can be used. They perform only the necessary computation, which makes them faster than `IM_full`.

6.2.5 Independence of irrelevant alternatives (IIA)

Recall that the election is *independent of irrelevant alternatives* (*IIA*) iff, when the election is run with the same voters and any subset of candidates containing the sincere w, she remains the winner.

Here we study IIA for a specific instance of election using a given population, not IIA as a property of a voting system in general.

```
>>> not_IIA, log_IIA, example_subset_IIA, example_winner_IIA =
     election.not_IIA_full()
>>> print(not_IIA)
True
>>> print(example_subset_IIA)
[ True  True False]
>>> print(example_winner_IIA)
0
```

In this example, `not_IIA` is **True**, which means that the election violates the IIA property (by consistency with manipulation methods like CM, **True** corresponds to the "undesirable" behavior). When this is the case, `example_subset_IIA` provides a subset of candidates that violates IIA: here, the subset consisting of candidates 0 and 1. `example_winner_IIA` provides the corresponding winner, in this case candidate 0.

If the election does not violate IIA, the counterexample variables return the conventional value `nan`.

6.2.6 Criteria met by a voting system

Voting systems have special attributes that represent various properties. The end user can access these attributes through any Election object, but they relate to the voting system itself, not to a particular instance of an election. They are especially useful for developers of new voting systems, who can fill in these attributes to help speed up computation (cf. Section 6.3.4).

For example, the following attribute means that IRV is based on strict rankings, not utilities.

```
>>> election = svvamp.IRV(pop)
>>> print(election.is_based_on_rk)
True
```

The following attribute means that the voting system is utility-based and that, for a single manipulator or for a coalition of manipulators in favor of candidate c, it is optimal to act as if they had a utility of 1 for c and -1 for the other candidates. For example, this is the case for Approval voting and Range voting in their default configuration.[4] Of course, this attribute is **False** for IRV.

```
>>> print(election.is_based_on_ut_minus1_1)
False
```

The following four criteria are intimately related and are used in the calculation of CM and its variants. They are the main majority criteria studied in Chapter 3.

```
>>> print(election.meets_Condorcet_c_rk)
False
>>> print(election.meets_majority_favorite_c_rk)
True
>>> print(election.meets_IgnMC_c)
True
>>> print(election.meets_InfMC_c)
True
```

In SWAMP, several variations of these criteria exist, depending on the usual distinction between ranking and utilities, and on tie-breaking issues between candidates. See the documentation for more details.

SWAMP manages the implications between these criteria: For example, a developer may simply inform SWAMP that a certain voting system meets a given variant of **MajFav**. SWAMP then knows which other variants of **MajFav** and which declensions of **IgnMC** and **InfMC** this implies.

[4]In the default configuration for Range voting and Majority judgment, we consider that utility vectors are defined up to a positive affine transformation, as in the classical model of Von Neumann and Morgenstern (Appendix B). So to claim that one has a utility of 1 for c and -1 for the others is to say that one has strictly the best utility for c and the worst utility for the others.

In the default configuration of Approval voting, we consider that utility vectors are defined up to one positive multiplicative constant, as in the model with approval threshold of Section B.7, and that the sincere ballot of a voter depends only on the sign of the utilities. So to claim that one has a utility of 1 for c and -1 for the others is to say that one approves of c, and only c.

6.3 Implementation of CM

In this section, we focus on the implementation of CM, which best illustrates the general techniques used in SWAMP.

6.3.1 Minimizing computation

When computing CM and its variants, if the election is not manipulable, it is usually necessary to loop over all candidates to prove it. But if the election is manipulable, it is sufficient to prove the manipulation in favor of one candidate. For this reason, it is interesting to guess which candidates can most likely benefit from manipulation and to test these candidates first. In the generic methods of the Election class, a simple heuristic is used: the candidates are examined in the decreasing order of the corresponding number of manipulators. This behavior can easily be redefined when implementing a particular voting system.

In general, SWAMP tries to be as lazy as possible. This means for example: 1. Avoid unnecessary calculations and 2. Not do the same calculation twice.

This can be illustrated by the following example.

```
>>> is_CM, log_CM = election.CM()
>>> print(is_CM)
True
>>> is_CM, log_CM, candidates_CM = election.CM_with_candidates
    ()
>>> print(candidates_CM)
[0. 0. 1. 0. 1.]
```

1. When `CM` is called, here is a possible execution: SWAMP examines candidate 0 first, because it has the most manipulators, but the manipulation is proved impossible. Then SWAMP examines candidate 2 and finds a manipulation. SWAMP stops computation and decides that `is_CM` is **True**.

2. When `CM_with_candidates` is called, SWAMP remembers previous results for candidates 0 and 2, and only performs the calculation for the other candidates.

6.3.2 Number of manipulators

When UM or TM is computed for a given candidate c, there is a fixed set of manipulators: the voters who prefer c to the sincere winner w. However, CM and ICM work differently. In fact, to study CM or ICM in favor of a candidate c, SWAMP addresses the following question. Given the sub-population consisting of the sincere voters, i.e., those who do not strictly prefer c to w, what is the minimal number of manipulators that must be added to perform manipulation in favor of c?[5] Denote x_c this integer.

During computation, SWAMP maintains a lower bound and an upper bound of x_c. As soon as the lower bound becomes larger than the number of manipulators, manipulation for c is proved impossible. Conversely, as soon as the upper bound becomes less than or equal to the number of manipulators, manipulability in favor of c is proved.

[5] This question is called the *Constructive Coalitional Unweighted Optimization* problem by Zuckerman et al. (2009). It is sufficient to decide CM for all the voting systems currently implemented in SWAMP.

This mechanism allows S𝕍AMP to combine different algorithms and to speed up computation, even when the attribute `CM_option` has the value `'exact'`. For example, when using IRV in exact mode, S𝕍AMP starts by using an approximate polynomial algorithm to decrease the interval between the bounds (we will come back to this in Section 6.5). If it is sufficient to decide CM, computation stops. Otherwise, the non-polynomial exact algorithm is run and it exploits the already known bounds to speed up computation.

6.3.3 Anonymity and resoluteness

The mechanism of adding manipulators that we have just described explains an important choice made for S𝕍AMP: the voting systems must be anonymous, in the sense that they treat all voters in the same way. This choice allows S𝕍AMP to work on the number of manipulators, not their identity. If voters had different weights, many easy problems would become difficult: for example, even for a fixed number of 3 candidates, deciding CM with weighted voters for Borda, Maximin, IRV, or Veto is $\mathcal{NP}$-complete (Conitzer and Sandholm, 2002; Conitzer et al., 2003, 2007), whereas with unweighted voters, the problem is in $\mathcal{P}$ (relatively to the number of voters V).

We also assume that the voting systems are deterministic and resolute: they elect a single candidate. Otherwise, defining the manipulation is a question in itself, because it is necessary to specify how preferences over candidates are lifted to preferences over sets of candidates (or probability distributions on candidates), which can be done in an infinite number of ways. Examples are given by Gärdenfors (1976); Gibbard (1978); Duggan and Schwartz (2000). Using resolute voting systems allows S𝕍AMP to decide CM and its variants by looping over the candidates who might benefit from a manipulation (i.e. all except the sincere winner), considering for each one the set of interested voters. On the other hand, if the voting system returns a subset of candidates (resp. a probability distribution over the candidates), then it is unreasonable (resp. impossible) to loop over all possible outcomes.

Since the voting systems are anonymous and resolute, they are necessarily non-neutral in general (cf. Proposition 1.16): indeed, they must return a unique winner, even in totally symmetric configurations like the usual minimal example of Condorcet's paradox (with $V = 3$ and $C = 3$). In S𝕍AMP, the usual tie-breaking rule is that candidate with smaller indices are favored. For example, if there is a tie between candidates 0 and 1, then candidate 0 is declared the winner. We chose this rule for its simplicity: with more complicated rules, some manipulation problems usually in $\mathcal{P}$ become $\mathcal{NP}$-hard (Obraztsova et al., 2011; Obraztsova and Elkind, 2012; Aziz et al., 2013). However, there is no architectural limitation in S𝕍AMP that prevents the implementation of voting systems using other tie-breaking rules.

6.3.4 CM auxiliary methods

S𝕍AMP is written in a modular way, using an object-oriented approach that is natural in the Python language. For example, the method `CM` (or its detailed version `CM_with_candidates`) defined in the class Election actually calls a number of specific auxiliary methods. Each of them can be redefined in the subclass that implements a particular voting system, while keeping the default implementation of the other auxiliary methods.

As a rough guide, the auxiliary methods can be divided into three categories: general pre-tests, candidate-dedicated pre-tests, computational core.

General preliminary tests are intended to decide CM immediately if possible.

- If the voting system satisfies **MajFav** (Section 6.2.6) and if w is a majority favorite,[6] then CM is **False**.
- If the voting system satisfies **Cond** and if w is the resistant Condorcet winner, then CM is **False**.
- If the voting system satisfies **InfMC** and if w is not Condorcet-admissible, then CM is **True**.
- If TM, UM, or ICM is **True**, then CM is **True**.

If necessary, SVVAMP performs preliminary tests which are dedicated to the manipulation in favor of a given candidate c.

- If the voting system satisfies **InfMC**, then it is sufficient to have more manipulators than sincere voters.
- If the voting system satisfies **MajFav**, then it is necessary to have enough manipulators to prevent the sincere winner w from appearing as a majority favorite.
- If the voting system satisfies **Cond**, then it is necessary to have enough manipulators to prevent w from appearing as a Condorcet winner (one can make the parallel between this test and the notion of a resistant Condorcet winner).
- If TM or UM is possible for c, then CM is possible for c.
- The number of manipulators sufficient for ICM is also sufficient for CM.

At this stage, if the attribute `CM_option` has the value `'lazy'`, the algorithm stops, whether the solution has been found or not. In the latter case, the conventional value `nan` is returned.

If the attribute `CM_option` has the value `'exact'`, then the class Election provides a default algorithm, which can be redefined in each voting system for more efficiency.

If the attribute `is_based_on_rk` is **True**, the default exact algorithm of the generic class Election resorts to brute force: the manipulators try all possible ballots. For a candidate c, denoting n_c the number of manipulators for c against w, this default algorithm has a time complexity of the order of $(C!)^{n_c}$ (multiplied by a polynomial and by the time needed to compute the winner of an election), which makes it usable only for small configurations.[7]

If the attribute `is_based_on_ut_minus1_1` is **True**, the default exact algorithm is obvious: the manipulators simply need to test the sincere ballot corresponding to a utility of 1 for c and a utility of -1 for all the other candidates. This runs in polynomial time (multiplied by the time needed to compute the winner of an election).

In the other cases (i.e., for a system that does not rely on rankings and where the optimal manipulation strategy for a certain candidate c is not to act as if one

[6]For each pre-test, there are several variants that depend on tie-breaking issues. For the sake of brevity and clarity, we do not describe these variants here. More details can be found in the code of the class Election.

[7]The complexity is actually slightly lower because we test the possible ballots up to a permutation of the manipulators.

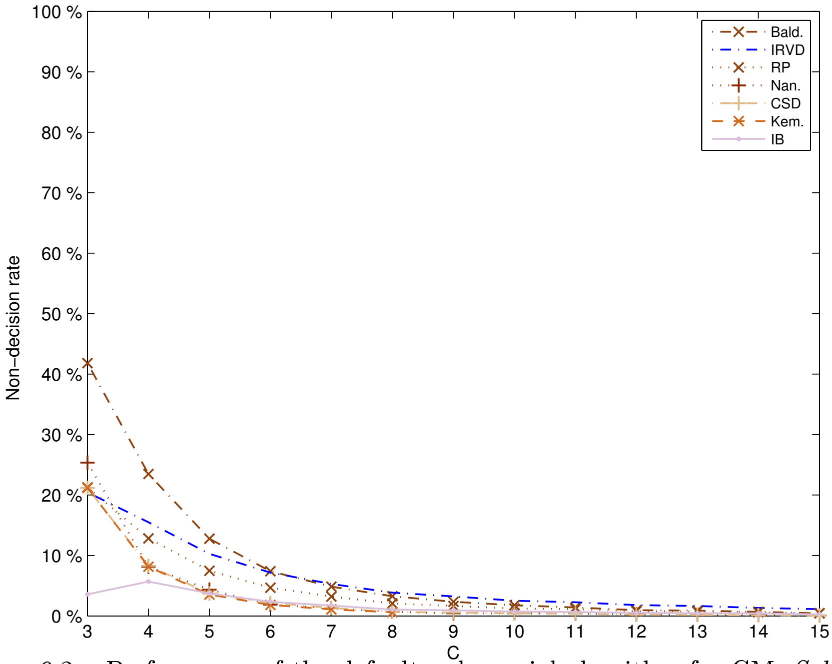


Figure 6.2 – Performance of the default polynomial algorithm for CM. *Spheroid* model, $V = 33$.

has a utility of 1 for c and -1 for the other candidates), this method raises a NotImplementedError.

In the subclass that implements a particular voting system, the auxiliary method implementing the exact algorithm is frequently redefined to take advantage of a more efficient algorithm than brute force.

6.3.5 Performance of the generic polynomial algorithm

Using the `'lazy'` option, the generic algorithm for CM runs in polynomial time but may not decide. Figure 6.2 gives its measured efficiency on voting systems for which no specific algorithm is implemented in SWAMP.[8] For these voting systems, only the winner calculation rule is provided to SWAMP, as well as the basic properties of the voting system (`is_based_on_rk`, `meets_Condorcet_c`, etc). Therefore, the generic algorithm `'lazy'` relies above all on the preliminary tests exposed in Section 6.3.4, the most important of which is TM.

To obtain Figure 6.2, for each value of C, 10,000 random populations were generated by the *Spheroid* model with $V = 33$. The value on the y-axis is the ratio of populations for which the method `CM` returned the value `nan`, meaning that it was unable to prove either CM or its negation.

We see that the generic lazy algorithm is quite efficient for deciding CM. Empirically, by comparing with the exact algorithm, we noticed that the lazy algorithm is efficient to prove CM, but less efficient to prove the negation of CM. In order not to make false advertisement, we must mention that in the simulations leading to this figure, the frequencies of CM were relatively high (as we will see

[8] Among the systems represented in Figure 6.2, it is known that the manipulation problem is $\mathcal{NP}$-hard for the methods of Baldwin and Nanson (Narodytska et al., 2011; Davies et al., 2014), RP (Xia et al., 2009; Hemaspaandra et al., 2013; Parkes and Xia, 2012), and Kemeny's method. For IRVD, CSD, and IB, the question is open.

by taking up this example in Section 7.3), which explains the good performances of the generic lazy algorithm.

6.4 Algorithms for specific voting systems

For some voting systems, specific algorithms are implemented. In this section, we give an overview of these algorithms.

For Range voting, Majority judgment, Approval voting, Bucklin's method, Plurality, the Two-round system, and Veto, CM is computed exactly in polynomial time. For Bucklin's method, we use an algorithm of Xia et al. (2009). For the others, the manipulation algorithm is obvious.

For Schulze's method, we use an algorithm of Gaspers et al. (2013), which runs in polynomial time. The original paper proves that this algorithm is correct for the multi-winner problem, i.e., if the manipulators aim to have a certain candidate c included in the set of potential winners (cf. the definition of this term in the definition of Schulze's method, Section 1.6.7). The same is true for the single-winner problem, i.e., if the manipulators aim to have a certain candidate c be the only potential winner (i.e., winner for each possible tie-breaking rule). In SVVAMP, it has a margin of error of one manipulator because of the tie-breaking rule. Therefore, it may happen (relatively rarely) that the simulator returns `nan` when asked to decide manipulability.

For Borda's method, deciding CM is $\mathcal{NP}$-complete, even when there are only 2 manipulators (Betzler et al., 2011; Davies et al., 2011). We use an approximation algorithm of Zuckerman et al. (2009), which has a margin of error of one manipulator.

For Maximin, deciding CM is $\mathcal{NP}$-complete, even when there are only 2 manipulators (Xia et al., 2009). We use an approximation algorithm of Zuckerman et al. (2011), which has a multiplicative error factor of $\frac{5}{3}$ on the number of manipulators needed.

For Coombs' method, there are two available options for CM: `'fast'` and `'exact'`. In exact mode, our algorithm is similar to the one developed by Walsh (2010a) for IRV. Its time complexity is of order $C!$. In fast mode, we use an original heuristic that we will not describe in detail but whose principle is similar to the one used for IRV, which we will describe in Section 6.5. However, the question is simpler than for IRV because for Coombs' method, we can notice that CM is equivalent to UM: indeed, if the manipulators have the ability to eliminate candidates $d_1, \dots, d_{C-1}$ successively, then they can also do it by using the common order of preference ($c \succ d_{C-1} \succ \dots \succ d_1$).

6.5 Specific algorithms for IRV and its variants

IRV was one of the first voting systems for which deciding IM (and *a fortiori* CM) was proven to be $\mathcal{NP}$-complete (Bartholdi and Orlin, 1991). Moreover, it seems particularly resilient to manipulation (Chamberlin et al., 1984; Bartholdi and Orlin, 1991; Lepelley and Mbih, 1994; Lepelley and Valognes, 2003; Green-Armytage, 2011, 2014). To the best of our knowledge, the fastest manipulation algorithm for individual manipulation is in $O(2^C V C^k)$ (Walsh, 2010b) and the fastest one for coalition manipulation is in $O(C!(V + Ck))$ (Coleman and Teague, 2007).

For these reasons, we have dedicated special attention to the methods for IRV and for two of its variants, EB and CIRV, which we will see in the following chapters are also quite robust to manipulation. For CIRV, we know from the Condorcification theorems 2.9 and 2.20 that it is strictly less manipulable than IRV, and it would be interesting to quantitatively measure the manipulability gap. For EB, we know that it is strictly more manipulable than IRV (Section 1.4.2), but it has the advantage of often being simpler than IRV and allowing faster computation. For example, when we want to prove that IRV is not manipulable, it is sufficient to show that EB is not, and we will see that this is generally faster.

6.5.1 CM algorithms for EB

Algorithm 1 Exact CM in favor of c for EB

$SetStart = \mathcal{C}$
$PossibleSetsStart = \{SetStart\}$
for $t = 1 \to C - 1$ **do**
  $PossibleSetsEnd = \varnothing$
  for all $SetStart \in PossibleSetsStart$ **do**
    for all $d \in SetStart \setminus \{c\}$ **do**
      $test$ = under the assumption that the elimination round t is carried out with the candidates in $SetStart$, can the manipulators choose their ballots so as to eliminate d?
      if $test$ **then**
        $SetEnd = SetStart \setminus \{d\}$
        $PossibleSetsEnd = PossibleSetsEnd \cup \{SetEnd\}$
  $PossibleSetsStart = PossibleSetsEnd$
return $(PossibleSetsEnd \neq \{\})$

In exact mode, SVVAMP uses Algorithm 1, which is an adaptation for EB of an algorithm originally designed for IRV by Coleman and Teague (2007). At each round, we determine the set of situations reachable by the manipulators at the end of the round, i.e., the candidates remaining in the race (ignoring situations where c is eliminated). It is clear that this algorithm terminates and decides manipulability: at the end of round $C-1$, the set $PossibleSetsEnd$ is nonempty iff a manipulation is possible in favor of c. In this case, the set $PossibleSetsEnd$ necessarily contains only the singleton $\{c\}$.

The main difference with the algorithm of Coleman and Teague (2007) is the following: in IRV, the ballot used by a manipulator in round t in favor of a candidate d is blocked until candidate d is eliminated. For this reason, eliminating a certain candidate d and then e or the reverse is not necessarily equivalent for the manipulators: even if the same candidates remain at the beginning of the next round, the blocked manipulated ballots are not necessarily the same. For this reason, one may need to explore all possible orders of elimination, which causes a complexity in $C!$.

On the other hand, in EB, the situation of the manipulators at the beginning of a round is entirely characterized by the candidates remaining in the race. Thus, Algorithm 1 explores, at worst, all subsets of $\mathcal{C}$ and thus has a complexity of order 2^C “only”.

In fast mode (obtained with `CM_option = fast`), SVVAMP uses Algorithm 2, which is an original contribution. At the beginning of each round, the manipulators examine all the candidates (distinct from c) that they are able to eliminate. Among them, they choose the best candidate d_{best}, in the sense of a well-chosen

Algorithm 2 Fast CM in favor of c for EB

$NonEliminated = \mathcal{C}$
for $t = 1 \to C - 1$ **do**
    $Eliminable = \{d \in NonEliminated \backslash \{c\}$ s.t., under the hypothesis that the elimination round t is carried out with the candidates in $NonEliminated$, the manipulators can choose their ballots in order to eliminate $d\}$.
    if $Eliminable = \varnothing$ **then**
        return Maybe
    else if $t = C - 1$ **then**
        return true
    $d_{\text{best}} = \arg\max\{\text{Situation}(d), d \in Eliminable\}$
    $NonEliminated = NonEliminated \setminus \{d_{\text{best}}\}$

function denoted Situation, which is in charge of evaluating the situation after the elimination of each possible opponent d. It is thus a greedy algorithm.

An important choice for this algorithm is therefore that of the function Situation. In SVVAMP, we use the following function:

$$\text{Situation}(d) = \text{score}(c \mid \cancel{d}) - \max_{e \neq c}(\text{score}(e \mid \cancel{d})),$$

where $\text{score}(e \mid \cancel{d})$ denotes the score assigned to a candidate e, by sincere voters only, in the case where d is eliminated, i.e. in a round carried out with the members of $NonEliminated \setminus \{d\}$.

It is clear that this algorithm terminates in polynomial time and that if it returns **True**, then the profile is manipulable in favor of candidate c. On the other hand, if it returns **Maybe**, it is impossible to conclude.

Figure 6.3 shows the performance of SVVAMP in fast mode. We consider an impartial culture, $V = 33$ voters, and a variable number of candidates. Each point is drawn from 10,000 random populations. We plot the CM rate found by the exact algorithm for reference.

The upper bound of the CM rate provided in fast mode is given by the cases where the preliminary tests were able to prove that the profile is non-manipulable, for example when the sincere winner is a majority favorite. We see that, in this type of culture, for $C \geq 4$, the preliminary tests are only able to prove the non-manipulability in an extremely low proportion of the cases.[9]

On the other hand, the lower bound on the CM rate provided in fast mode is very close to the exact rate. For reference, the trivial manipulation is also shown: indeed, a good index of performance is the proportion of manipulations that are not found by TM but are found by the fast algorithm. For each value of C in this figure, the fast algorithm always finds at least 96 % of the manipulations not found by TM (99 % of manipulations in total).

In summary, the fast mode allows to prove manipulability in a high proportion of cases where it is possible, thanks to the fast algorithm 2, but rarely proves non-manipulability.

Figure 6.4 shows the time saving obtained by the fast algorithm. In practice, the average complexity observed on these curves is $1.6 \times C$ ms/population for the

[9] On the experiments resulting from real elections of Chapter 9, we will see, however, that for many voting systems, the preliminary tests are able to conclude in a non-negligible proportion of the cases, which motivates their implementation.

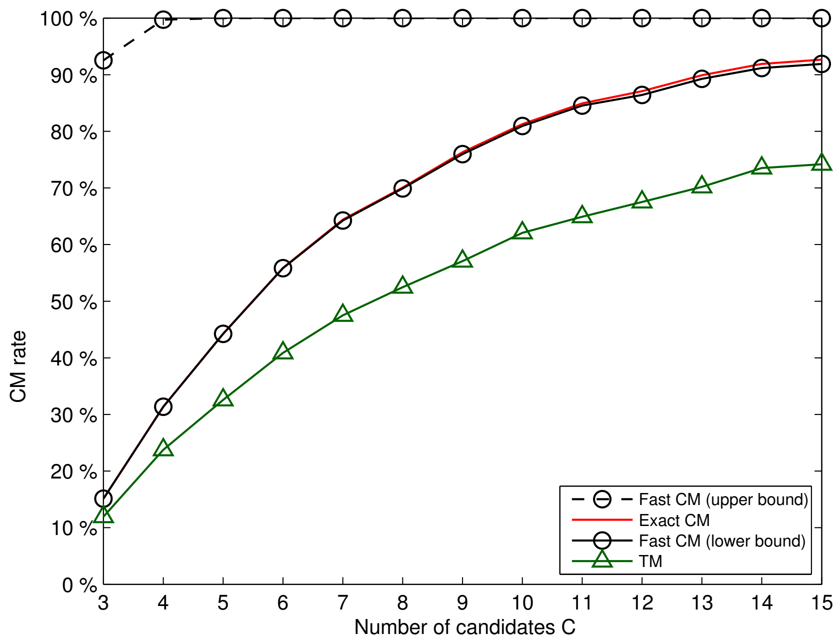


Figure 6.3 – EB: CM rate found by the different algorithms. Impartial culture, $V = 33$.

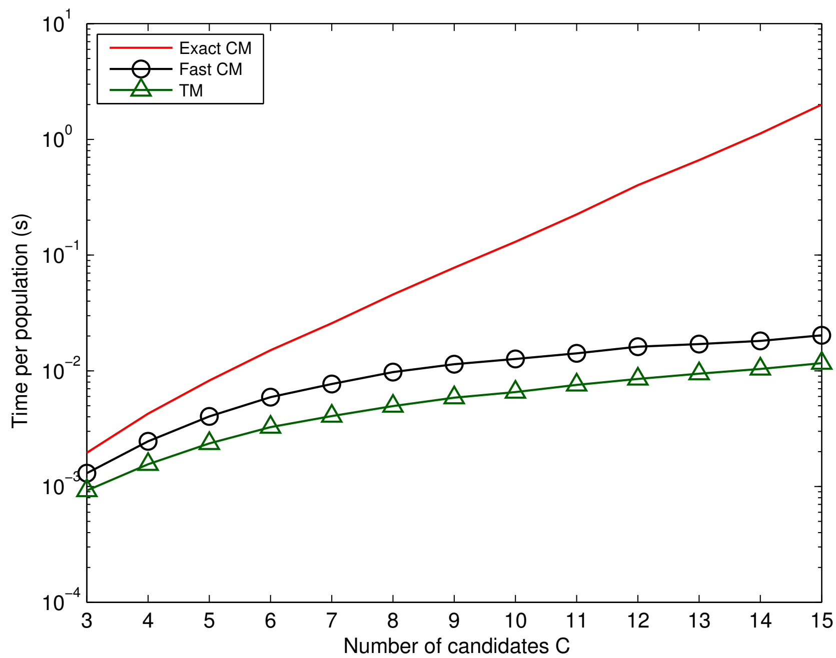


Figure 6.4 – EB: computation time of the different algorithms. Impartial culture, $V = 33$.

fast mode, and 0.5×1.75^C ms/population for the exact mode of SVVAMP.[10] Note that, for the exact algorithm, the complexity is not in 2^C, which can be explained by several causes: first, the theoretical behavior in 2^C expresses a worst-case complexity, whereas we measure the average complexity. Second, the CM rate is not constant as a function of C, and manipulable configurations are usually faster to process, especially because the fast algorithm can often conclude before the exact algorithm is called. Third, SVVAMP has some additional optimizations, in particular the preliminary tests which lead to conclude quickly in a number of cases.

Thus, our approximate algorithm 2 for the CM of EB is efficient in proving manipulability and much faster than the exact algorithm. Its main limitation is that it is unable to prove non-manipulability if necessary. In practice, in the fast mode of SVVAMP, Algorithm 2 is supplemented by preliminary tests that can prove non-manipulability but whose success rate may be very low in some cultures. For future work, it would be of particular interest to have a polynomial-time algorithm that is capable of proving non-manipulability with a good success rate, as a complement to our algorithm which is relatively efficient at handling manipulability cases.

6.5.2 CM algorithms for IRV

For IRV, as we mentioned, the problem is more complex than for EB: since the manipulators are immobilized by their past votes in favor of candidates not yet eliminated, it is necessary to take into account the history of the previous rounds.

In exact mode, SVVAMP uses the algorithm of Coleman and Teague (2007), whose complexity is in $C!$.

In fast mode, we use a greedy approach similar to Algorithm 2: at each round, among the candidates that can be eliminated, we choose the one that maximizes a function evaluating the obtained situation. In this case, we have chosen the same function as for EB.

The calculation of CM for IRV is therefore based on two algorithms, fast and exact. But in practice, in SVVAMP, the CM of IRV has three options.

`fast`: After the preliminary tests, only the fast algorithm is used.

`slow`: In addition, one determines if EB is manipulable by using the exact algorithm of EB. If it is false, we conclude that IRV is not manipulable. If it is true, we take the opportunity to test the elimination path allowing the manipulation in EB, in case it would allow to manipulate also IRV.

`exact`: If the above processes do not allow to conclude, then we use the exact algorithm of IRV.

The option `slow` thus offers an intermediate possibility, faster than the exact option, but more powerful than the fast option, in particular to decide the cases of non-manipulability.

Figure 6.5 shows the performance of the different options for IRV, under the same conditions as Figure 6.3 for EB. For each value of C in this figure, the fast mode always finds at least 88 % of the manipulations not found by TM (97 % of manipulations in total). On the other hand, like for EB, the upper bound found

[10] The machine used is a Dell Precision M6600, Intel Core I7-2820QM at 2.30 GHz, 8 Mb of cache, 16 Gb of RAM at 1.33 GHz in DDR3.

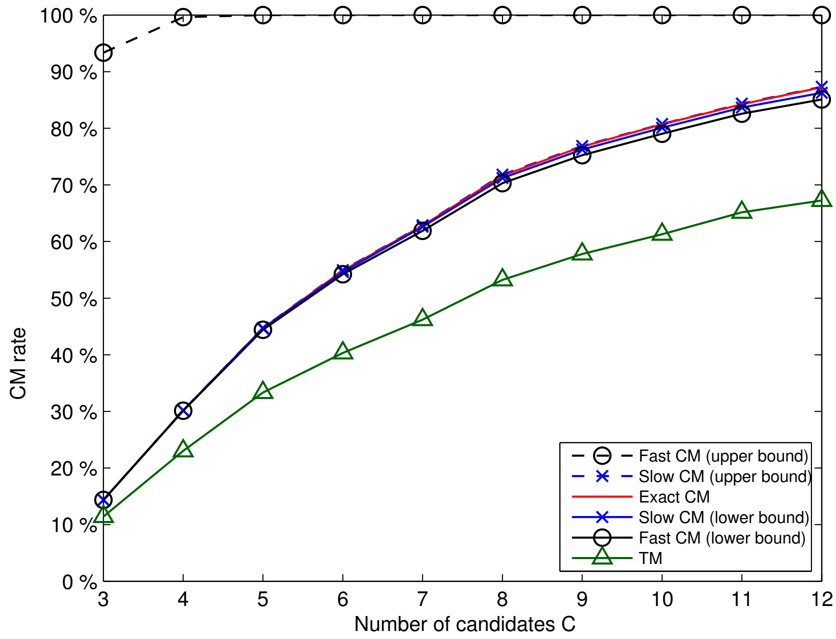


Figure 6.5 – IRV: CM rate found by the different algorithms. Impartial culture, $V = 33$.

in fast mode is usually close to 1: this mode is not very well suited to prove non-manipulability.

In the slow mode, the lower bound is better: for each value of C in this figure, the slow mode always finds at least 95 % of the manipulations not found by TM (99 % of the manipulations in total). More importantly, the upper bound provided by the slow mode is very close to the exact CM curve, which means that we are able to prove many cases of non-manipulability. In fact, for each value of C, the slow mode always finds at least 99 % of the non-manipulability cases. Thus, the difference between the lower and upper bounds found in slow mode is at most about 1 %, which means that the conventional value `nan` was returned in only about 1 % of the cases in slow mode.

Finally, Figure 6.6 presents the computation times for the different algorithms in the case of IRV. In practice, the average complexity observed on this figure is $2.0 \times C$ ms/population for the fast mode, 0.5×1.75^C ms/population for the slow mode and $4.1 \times (0.23C)^{0.57C}$ ms/population for the exact mode.[11] For the slow mode, it is the same asymptotic behavior as for the exact mode of EB, which is natural given our way of proceeding: indeed, the additional computations to be performed for the adaptation to IRV are negligible in front of those necessary for the exact algorithm of EB (which require an exponential time in C).

6.5.3 CM algorithms for CIRV

For CIRV, we use a heuristic that we will describe briefly.

[11] Theoretical worst-case behavior is of the order of $C!$, i.e. of the order of C^C using the Stirling formula. By adjusting the parameters, we obtain the approximate behavior stated.

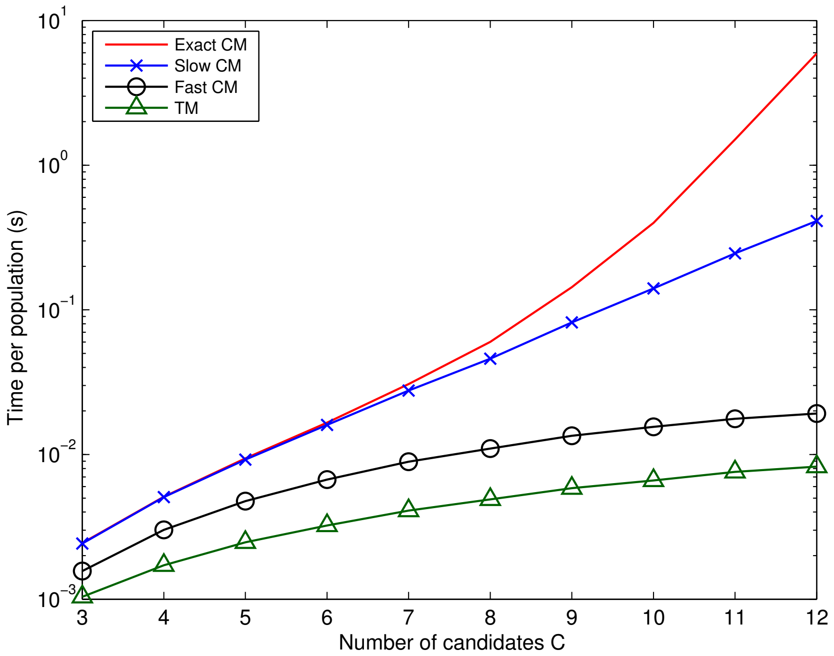


Figure 6.6 – IRV: computation time of the different algorithms. Impartial culture, $V = 33$.

First, for each candidate $c \neq$ w, we determine whether IRV is manipulable using one of the available options (fast, slow, or exact). Then, using the resulting elimination path, we try to tune the ballots of the manipulators to prevent a Condorcet winner distinct from c. If the initial profile is Condorcet, we know that there should be no Condorcet winner at all (Lemma 2.5).

To do this, we first place candidate c as high as possible in the ballots, and then the candidates who are already known to have a defeat in the weighted majority matrix. In doing so, it is possible that other candidates acquire defeats and we can then iterate the process. For more details, see the code of the corresponding class in SVAMP.

If the process succeeds, i.e. if there is no Condorcet winner distinct from c at the end of the algorithm, then CM is proved. Otherwise, we are left with uncertainty: indeed, we have only tested one elimination path among those that allow to manipulate IRV, and the process used to prevent the appearance of a Condorcet winner has been arbitrarily chosen. Thus, this algorithm provides only a lower bound on manipulability.

However, there is also an upper bound: if IRV is not manipulable, then CIRV is not either, by the weak Condorcification theorem 2.9. In the following chapters, we will see that in practice, the gap between the lower bound and the upper bound is in general very small, which ensures, on the one hand, that the heuristic used for CIRV is relatively efficient, and will show, on the other hand, that the manipulability gap between IRV and CIRV, although nonzero by the strong Condorcification theorem 2.20, is in fact relatively small.

Chapter 7

Simulations in Spheroidal Cultures

In this chapter and the next two, we study two issues that were central to our motivation for developing S∀AMP.

1. Whereas manipulability is a theoretical necessity by the theorem of Gibbard (1973) and Satterthwaite (1975), is it a common phenomenon in practice?

2. How do different voting systems compare in this regard?

These are the main motivations for this chapter and the next two. A secondary motivation is to illustrate the possibilities offered by S∀AMP.

In general, we focus on coalitional manipulability (CM), which is the central subject of this memoir and which we also call *manipulability* without further precision. However, since S∀AMP does not have dedicated exact CM algorithms for each voting system, we will also use trivial manipulation (TM) as a point of comparison between voting systems. As we discussed in Section 6.2.3, TM has the advantage of being easy to compute (in polynomial time, up the time needed to compute the winner). In addition, when manipulation is possible, TM gives an indicator of the cases where it is relatively inexpensive to implement in terms of communication.

Given the variety of cultures studied, reading Chapters 7 and 8 might seem too long for the impatient reader, but this an issue we did not manage to avoid. For a quicker overview, we suggest reading Section 7.1 which defines the cultures used in this chapter, Section 7.2 which presents our reference scenario, Section 7.3 which introduces the conventions used in the plots, Section 7.9 which concludes this chapter, the introduction to Chapter 8, Sections 8.6 and 8.7 which conclude it, before moving on to Chapter 9 which analyzes data from real-life elections.

7.1 Presentation of the spheroidal cultures

In voting theory, it is classical to consider the impartial culture (Definition 1.14), which is defined for preferences that are strict total orders. Since we wish to study not only ordinal but also cardinal voting systems, the first question that naturally arises is how we can extend this model to voters characterized by their utilities over the candidates. In this section, we briefly describe our approach, which is detailed and presented more formally in Appendix B and in the paper by Durand et al. (2015).

Without adopting all the assumptions of the expected utility model of Von Neumann and Morgenstern (1944), we will nevertheless draw inspiration from it. We do not necessarily assume that voter preferences are characterized by expected utilities, since there is no random element once the population is fixed: the voting systems studied compute the winner deterministically, and manipulators only use pure strategies, not mixed strategies.

However, we keep an important property of this model: first, the utility vector $\mathbf{u}_v \in \mathbb{R}^C$ of a voter over the different candidates is defined up to an additive constant and a positive multiplicative constant, which means that vector $\mathbf{u}_v$ represents the same preferences as vector $\alpha \mathbf{u}_v + \beta \mathbf{1}$, where α is a positive real number, β a real number, and where $\mathbf{1}$ denotes the vector whose all coordinates are equal to 1. This choice is motivated by the fact that each voter has no canonical way, neither to fix her point 0 of utility, nor to choose the "unit of measurement" used to evaluate her utility values: she is only able to measure their respective intensities.

Secondly, in order to have a complete model for Approval voting, we add the assumption that utility 0 is an *approval threshold* (cf. Section B.7): a voter approves of candidates who have nonnegative utility and disapproves of those who have negative utility. With this additional assumption, the utility vector of a voter is defined up to only a positive multiplicative constant α. The utility space is therefore the one of the half-lines of $\mathbb{R}^C$. All the vectors of the same half-line represent the same preferences in practice and are experimentally indistinguishable, even by the voter herself.

Intuitively, a natural way to represent this space is to normalize each utility vector so that its Euclidean norm is equal to 1. The utility space, i.e. the set of half-lines, is then represented by the unit sphere of $\mathbb{R}^C$. In Appendix B, we prove, by arguments from projective and differential geometry, that this representation is essentially the only one with good properties.[1]

We then have a canonical way to generalize the impartial culture: it consists in drawing a vector on the surface of the unit sphere of $\mathbb{R}^C$ with the uniform probability distribution (in the sense of the usual Euclidean measure): this is what we call the *spherical culture*. Each voter almost surely has a strict total order of preference, voters are independent, and the culture is neutral: as a consequence, the ordinal image of this culture (Definition 5.13) is indeed the impartial culture. Thus, for ordinal voting systems, this culture is simply equivalent to the impartial culture. But in addition, it makes it possible to study ordinal and cardinal systems within a common natural framework.

Thereafter, it is interesting to enrich this model by introducing correlation between voters. For this, we use the Von Mises-Fisher model, or VMF (Downs, 1966). This type of culture makes it possible to model populations that tend to have similar preferences but with a certain dispersion. First, a unit vector $\mathbf{n}$ is fixed and called the *pole* of the distribution. Then, independently for each voter, a unit vector $\mathbf{u}$ is drawn in $\mathbb{R}^C$ according to a VMF distribution:

$$p(\mathbf{u}) = X_\kappa e^{\kappa \langle \mathbf{u} \,|\, \mathbf{n} \rangle},$$

where $\mathbf{n}$ is the pole of the distribution, κ its concentration, X_κ a normalization constant, and where $\langle \mathbf{u} \,|\, \mathbf{n} \rangle$ denotes the canonical inner product.

[1] More precisely, in the model with approval threshold, the suitable representations are spheroids that are images of the unit sphere by a dilatation along direction $\mathbf{1}$, which therefore constitute a family with one real parameter. The choice of this parameter has an influence on Approval voting only: with extreme values of the dilatation factor, one favors either utility vectors with a lot of components of the same sign, or utility vectors whose sum of the components is zero and therefore always have elements of both signs. To simplify, we always consider the spherical model, which is intermediary between these two extreme cases.

Qualitatively, the VMF model is similar to Mallows' model (Mallows, 1957), which is used for ordinal preferences.[2] Besides the fact that the former is better suited to cardinal preferences than the latter, it has other advantages which are discussed in Section B.5. In particular, the VMF model is characterized by the following property: among the distributions on the sphere sharing a given mean direction and concentration,[3] the VMF distribution maximizes the entropy just as, in Euclidean space, the Gaussian distribution maximizes the entropy among the distributions having a given mean and standard deviation (Mardia, 1975). Therefore, in the absence of additional information, it is the "natural" distribution that should be used.

This is indeed a generalization of the spherical culture: for $\kappa = 0$, the VMF model is equivalent to the spherical culture. For $\kappa = +\infty$, we obtain the other degenerate case of VMF culture: a Dirac peak on the pole $\mathbf{n}$.

We group the VMF model and the spherical culture, which is a particular case of it, under the name *spheroidal*, because of the underlying spherical model used for the utility space.

7.2 Reference scenario

In this section, we present our methodology and our reference scenario. It is the spherical culture with $V = 33$ voters and $C = 5$ candidates. We chose an odd number of voters to avoid certain questions of ties, for example the subtleties between the different Condorcet notions: in particular, the notions of Condorcet winner and Condorcet-admissible candidate are then equivalent. To choose the exact values of V and C, we explored the values $V = 2^k + 1$ for $k \in [\![1, 10]\!]$ (thus for V between 3 and 1025) and the values $C \in [\![3, 15]\!]$. In Sections 7.3 and 7.4, we will see that, for most voting systems, in spherical culture, the manipulability rate seems to increase and tend to 1 when either one of these parameters increases. Our choice of $V = 33$ and $C = 5$ leads to manipulability rates that are reasonably far from 1 and therefore facilitates the comparison between the different voting systems under study.

Figure 7.1 illustrates the performances of various voting systems in this reference scenario. One can visualize CM and its variants TM, UM, and ICM. For this figure, 10,000 random populations were drawn with the culture considered.

The proportion of CM (resp. TM, UM, ICM) configurations for a voting system out of these 10,000 random draws provides an estimator of the CM (resp. TM, UM, ICM) rate for the culture considered. This induces a margin of uncertainty that we will call *statistical*, of the order of $\frac{1}{\sqrt{10{,}000}} = 1\%$ on the CM (resp. TM, UM, ICM) rate. By convention, this part of the uncertainty is not represented in Figure 7.1 or in the following ones.

[2] To model populations that have a certain homogeneity, one can also use Pólya-Eggenberger urn models, as do for example Lepelley and Valognes (2003). The disadvantage of these models is that they can only generate ordinal preferences and have no natural extension to the utility space.

[3] More precisely, we consider the distributions having the same *resultant vector*. In spherical statistics, the resultant vector is defined as the mean of the (unit) vectors of the distribution. For example, consider the unit sphere of dimension 1, that is, the unit circle of $\mathbb{C}$. If a distribution deterministically draws the unit vector of polar angle $\frac{\pi}{2}$, then its resultant vector is $1 \cdot e^{i\frac{\pi}{2}}$. But if it draws equiprobably the unit vectors of polar angles $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, then the resultant vector is $\frac{1}{2} \cdot e^{i\frac{\pi}{2}}$: the mean direction is the same, but the norm is lower. In general, the direction of the resulting vector should be seen as the mean direction of the distribution, and its norm as a measure of concentration (playing the same role, qualitatively, as the inverse of a standard deviation).

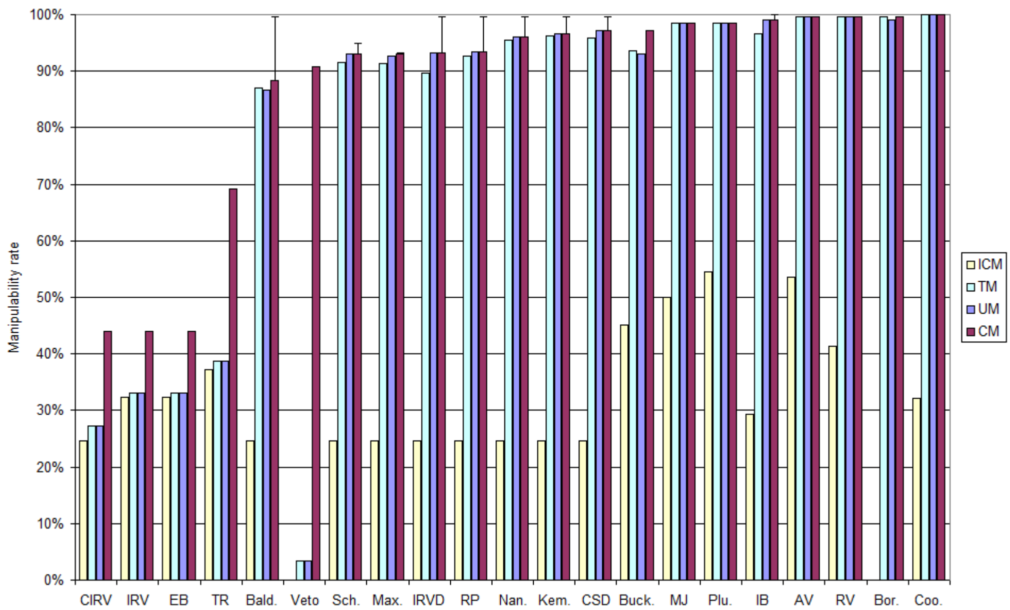


Figure 7.1 – Coalitional manipulation (CM) and variants. Spherical culture, $V = 33$, $C = 5$.

In addition, for some of the voting systems, the algorithm used to decide CM is not exact. Histograms in Figure 7.1 represent a lower bound of the proportion of manipulable configurations, which is given by the cases where SVVAMP was able to prove CM. The uncertainty bar (for example for Baldwin's method) indicates an upper bound of this proportion, which is provided by the cases where SVVAMP was able to prove that CM is impossible. We will call the difference between these bounds the *algorithmic uncertainty*.

Let us take this opportunity to examine the performances of the algorithms implemented in SVVAMP for CM.

For Maximin, Schulze, and Borda, we use an approximate algorithm but it seems that most of time, SVVAMP is in practice able to decide if these systems are manipulable or not: in Figure 7.1, the algorithmic uncertainty is 1.8 % for Schulze's method and it is zero for Maximin and Borda.

For Baldwin, IRVD, RP, Nanson, CSD, Kemeny, and IB,[4] SVVAMP only uses its generic manipulation methods, essentially consisting of preliminary tests and TM. In that case, we essentially have a lower bound of manipulability, which will however suffice to prove that these systems are more manipulable that some others in certain cases. In the reference scenario of Figure 7.1, for example, we have the guarantee that these systems are more manipulable than CIRV and IRV.

For EB, we will always use the exact algorithm. For IRV, we use the exact algorithm as well, except in Section 7.3, where we replace it with the slow option

[4]In this and subsequent chapters, we do not include Kim-Roush and IRVA, which were implemented in SVVAMP after performing these simulations and which also use generic manipulation algorithms. For reasons of computation time and to avoid adding last-minutes typos, we thought it wiser to stick to what we had, and we apologize to the reader. That said, according to the first tests carried out, the performances of Kim-Roush in terms of manipulability seem quite mediocre. In contrast, for IRVA, the results are promising and we will come back to this in the conclusion of this memoir. Finally, we also exclude Copeland's method, for the equality problems exposed in Section 1.6..

(Section 6.5.2) when the number of candidates is greater than or equal to 9 in order to limit the computation time. For CIRV, we use the heuristic presented in Section 6.5.3: although it is not an exact algorithm, we see in the particular case of Figure 7.1 that the algorithmic uncertainty is equal to zero. This also means that in practice, CIRV has a manipulability rate very close to that of IRV, even if we know that it is, in general, strictly lower by the strong Condorcification theorem 2.20.

In Figure 7.1, one can however note a non-negligible difference: CIRV is significantly less vulnerable to ICM, TM, and UM than IRV, even if its CM rate is very close. So, manipulation looks more difficult in CIRV than in IRV.

On the other hand, IRV and EB present very similar performances, even if the example of Table 1.1 (Section 1.4.2) shows that EB is strictly more manipulable than IRV (since they are respectively equivalent to TR and ITR in this example with 3 candidates).

Veto is the only voting system for which the TM rate (3%) is very different from the CM rate (91 %). This is easy to explain, since the trivial strategy is very bad in Veto: of course, it is optimal to avoid a victory of the sincere winner w, but the manipulators abandon all control on the winning candidate. On the contrary, the optimal strategy consists in coordinating to balance the total number of negative votes on all candidates, except the one desired by the manipulators.

Lastly, we note that IRVD is significantly more manipulable than CIRV, IRV, and EB. However, at first glance, this voting system is based on a similar principle. We can try to understand why with an example where there are 3 candidates (possibly after some eliminations of other candidates). If the sincere winner w is not a Condorcet winner, then the profile is manipulable anyway, so the interesting case is when w is a Condorcet winner. Denote c the candidate for which we want to manipulable and d the third candidate.

In IRVD, c must never face w in a duel, otherwise she loses. So here is the only possible scenario: w faces d and loses, then c faces d and wins. Thus, the candidates selected for the first electoral duel must be w and d: for this, the best strategy is to defend c by placing her at the top of each manipulator's ballot. Moreover, w must lose against d: for that, the best strategy is to attack w by placing her at the bottom of the ballot. So, the different imperatives, namely defending c and attacking the dangerous candidate w, are perfectly compatible.

Now, let us examine the situation in EB or IRV. In each elimination round, some manipulators must defend c to prevent her from being eliminated. Moreover, as long as w is not eliminated, no sincere voter is counted in favor of c: the manipulators can only rely on themselves for that. Furthermore, manipulators must avoid a last round between c and w, for the same reasons as before. So they must attack w to eliminate her immediately. But for that, they may need to divide their votes between c and d, which is *a priori* more difficult than in IRVD, where it was enough to defend c.

This leads to another remark: although IRV is the IPSR-1 of Plurality, it has a similarity with Veto (and, conversely, Coombs' method has a similarity with Plurality). Indeed, the choice of the eliminated candidate in IRV is made in Veto: each voter, by voting for one candidate, issues a veto against her elimination and the candidate with least vetoes against her elimination (i.e. with the fewest votes) is designated (i.e. eliminated). However, we have seen that the maximal family in Veto is, up to questions of ties, the family of threshold $V\frac{C-1}{C}$ (Proposition 4.28): it is therefore especially difficult to choose the eliminated candidate in

IRV, which may explain the low manipulability of this voting system, although it meets **InfMC** if we consider the system as a whole and not each round separately.

7.3 Spherical culture: number of candidates C

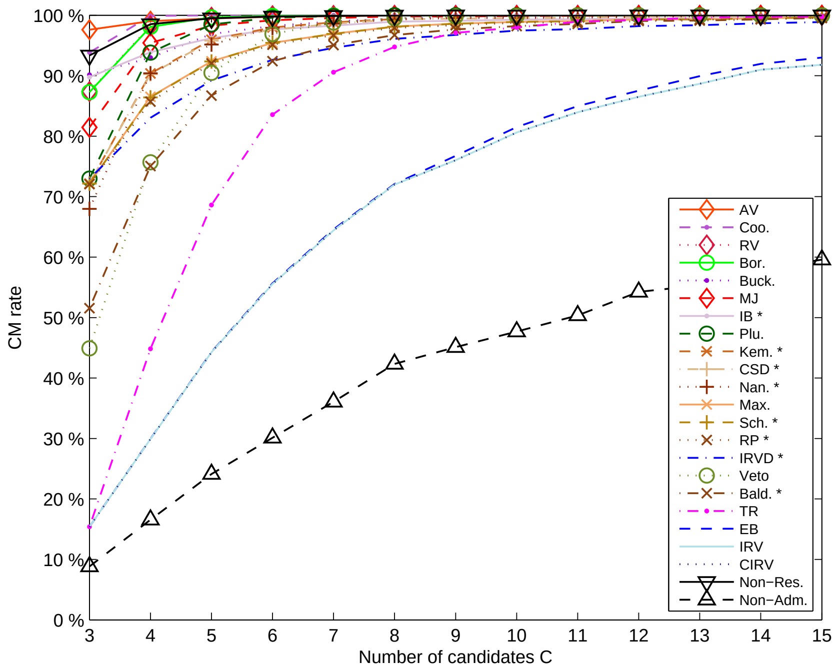


Figure 7.2 – CM rate as a function of the number of candidates C. Spherical culture, $V = 33$.

In this section and the following, we will consider variants of the reference scenario, each time varying a specific parameter. To begin with, we keep the spherical culture of the reference scenario, i.e. the particular case where the concentration κ is equal to zero, we leave constant the number of voters $V = 33$, and we represent in Figure 7.2 the CM rate found as a function of the number of candidates C.

This first plot is an opportunity to specify our graphic conventions for the remainder of this chapter and the next one. When the name of a voting system is accompanied by a star in the legend, it means that, for certain points of the curve, the approximate algorithm used by S∀AMP found a difference between the lower and the upper bounds greater than 1 %. In this case, given the number of voting systems under study, we only plot the CM rate found by S∀AMP, i.e. the lower bound. For example, this is the case of Baldwin's method in Figure 7.2.

When there is no star in the legend, it does not necessarily mean that the algorithm is exact but that, for each point of the curve, S∀AMP found an algorithmic uncertainty lower than 1 %. Thus, for CIRV or Maximin, although approximate algorithms are used, the algorithmic uncertainty is lower than 1 % for each point in Figure 7.2.

To facilitate the comparison of the voting systems across the different figures, each voting system is assigned a unique graphic style. These styles are grouped by family: systems based on grades use shades of red with diamond markers; PSR, shades of green with circle markers; "natural" Condorcet systems, shades of brown with cross markers; systems of the IRV family, shades of blue without point marker; other systems, shades of mauve and pink with large dot markers.

To easily visualize the correspondence with the curves, the legend displays the voting systems by decreasing order on the manipulability rate found (on average over the points of the curve).

For reference, we represent the proportion of non-resistant configurations (i.e. where there is no resistant Condorcet winner), in black with triangles pointing downwards. According to Theorem 2.17, it provides an upper bound of manipulability for all voting systems meeting the Condorcet criterion. Similarly, we represent the proportion of non-admissible configurations (i.e. where there is no Condorcet-admissible candidate), in black with triangles pointing upwards, which gives a lower bound of manipulability for all voting systems meeting **InfMC** (Lemma 2.7).

When the number of voters is even, we will also represent the proportion of non-Condorcet configurations (i.e. where there is no Condorcet winner), as a black dotted line with the same triangle marker. However, when the number of candidates is odd, as in Figure 7.2 and most of the following ones, it is useless to plot this curve: since we consider cultures where preferences are almost surely strict total orders, the notions of Condorcet-admissible candidate and Condorcet winner are equivalent, which is not the case for an even number of voters.

The first observation from Figure 7.2 is the following conjecture.

Conjecture 7.1
In spherical culture, for $V \geq 3$, the manipulability rate of each voting system considered here is an increasing function of the number of candidates C.

To understand this fact qualitatively, let us also observe that the probability of having a non-admissible configuration, i.e. a non-Condorcet configuration since V is odd, seems to increase with the number of candidates.

Kelly (1974) conjectured that, for $V = 3$ or $V \geq 5$ voters, the probability that there exists a weak Condorcet winner (i.e. a Condorcet-admissible candidate) is a decreasing function of C. This has been proven by Fishburn et al. (1979) for $V = 3$ voters. Similarly, Kelly (1974) conjectured that, for $V \geq 3$, the probability that there exists a Condorcet winner decreases. This is proven for $V \to +\infty$ (Gehrlein, 2006). To the best of our knowledge, the others cases remain conjectures. In the same vein, Figure 7.2 leads us to the following conjecture and proposition.

Conjecture 7.2
In spherical culture, for $V \geq 3$, the probability that there exists a resistant Condorcet winner is a decreasing function of the number of candidates C.

Proposition 7.3
In spherical culture, for $V \geq 3$ and $C \to +\infty$, the probability that there exists a resistant Condorcet winner tends to 0.

It is easy to prove this proposition with a previous result: indeed, in impartial culture and with a constant number of voters, the probability that there exists a Condorcet winner tends to 0 when C tends to $+\infty$ (May, 1971). It is therefore also true for the resistant Condorcet winner.

In Figure 7.2, we also observe the following phenomenon, which we will be able to demonstrate in part.

Conjecture 7.4
In spherical culture, for $V \geq 3$ and $C \to +\infty$, the manipulability rate of any voting system meeting **InfMC** *tends to* 1.

The assumption **InfMC** concerns all voting systems considered here, except Veto, to which we will come back in a moment. It is easy to prove Conjecture 7.4 for an odd number of voters. Indeed, in impartial culture and with a constant number of voters, we have already recalled that the probability that there exists a Condorcet winner tends to 0 when C tends to $+\infty$ (May, 1971). But, for odd V, the notions of Condorcet winner and of Condorcet-admissible candidate are equivalent. So, the sincere winner w is not Condorcet-admissible with high probability. For a voting system meeting **InfMC**, Lemma 2.7 then ensures that the configuration is manipulable.

For an even number of voters, to the best of our knowledge, it is not proven that the probability of having a Condorcet-admissible candidate (i.e. a weak Condorcet winner, given the assumptions) tends to 0 when C tends to $+\infty$. If this is true, then our proof of Conjecture 7.4 is also valid for an even number of voters.

On the other hand, although the CM rate of Veto also seems to tend to 1 in Figure 7.2, we can prove that this limit behavior is not true for Veto, when equipped with the tie-breaking rule implemented in SVVAMP.

Proposition 7.5
In impartial culture, for constant V and $C \to +\infty$, the manipulability rate of Veto (equipped with the lexicographical tie-breaking rule on candidates) does not tend to 1.

Proof. For $C > V$, consider the restriction of the impartial culture for C to the $V + 1$ candidates with lowest indexes: it is also the impartial culture. Therefore, with probability equal to $\frac{1}{((V+1)!)^V}$, all voters have the following order of preference over the first $V + 1$ candidates: $(1 \succ 2 \succ \ldots \succ V + 1)$. Their relations of preferences over the other candidates do not matter for our proof.

With such preferences, no voter casts a ballot against candidate 1, hence she is elected. There is no possible manipulation for candidates $2, \ldots, V+1$ because no voter is interested. Finally, because of the tie-breaking rule, none of the candidates $V + 2, \ldots, C$ can be the winner: even if each of the first V candidates receives a veto, it is the candidate $V + 1$ who is elected. Thus, manipulation is impossible.

As a consequence, with a probability at least equal to $\frac{1}{((V+1)!)^V}$, the profile is not manipulable. □

In Figure 7.2, we also see that CIRV, IRV, and EB are significantly less manipulable than the other voting systems and that the Two-round system presents intermediate performances between these three variants of IRV and the other voting systems, except for large values of C. Indeed, the above theoretical results prove that TR is (slightly) more manipulable than Veto for a large enough number of candidates.

We also observe that the curves for CIRV, IRV, and EB are almost the same. The difference of manipulability between CIRV and IRV is undetectable in this figure: the difference in the proportion of observed manipulable configurations is always equal to zero. As for that between IRV and EB, SVVAMP finds slightly less manipulations for IRV for $C = 9$ and more: this corresponds to the values for which we used the approximate algorithm for IRV, whereas we continued to use the exact algorithm for EB. The algorithmic uncertainty for IRV, of the order of 1 %, corresponds precisely to the difference between the curves of IRV and EB. This difference should not be over-interpreted: it is a combination, in unknown proportions, of a (small) difference of manipulability between IRV and EB and a (slight) drop in performance of the approximate algorithm for IRV when the number of candidates increases.

Thus, even if it is theoretically proven that CIRV is strictly less manipulable than IRV (by the corollary 2.21 of the Condorcification theorems) and that IRV itself is strictly less manipulable that EB (Section 1.4.2), these differences are, in fact, very small, especially when compared with the other voting systems studied here.

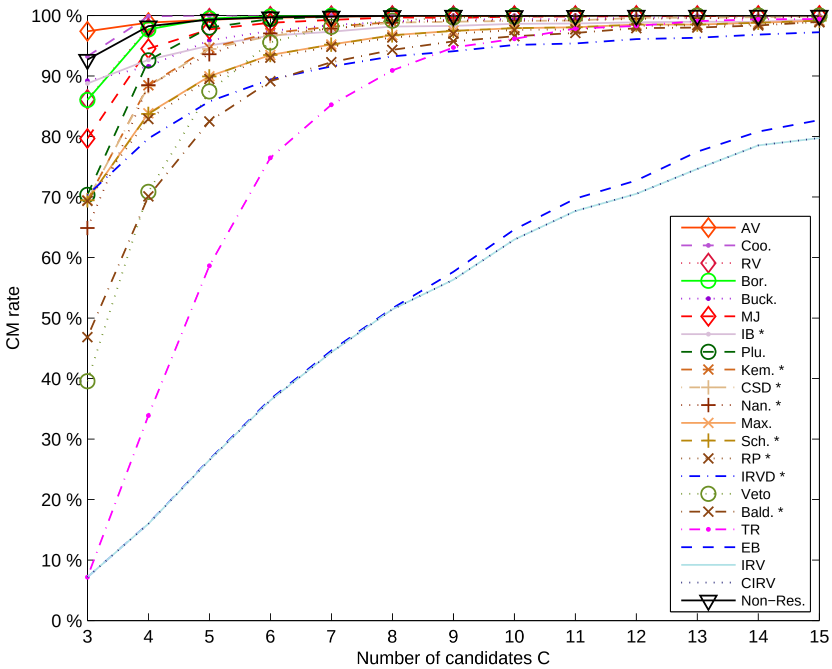


Figure 7.3 – Normalized CM rate (relative to admissible configurations) as a function of the number of candidates C. Spherical culture, $V = 33$.

To explain all these phenomena, one can propose the explanation that they are essentially linked to the degradation of the "quality" of the population, i.e. to its growing disunity. In particular, for voting systems meeting **InfMC** (all those studied here except Veto), we know, by Lemma 2.7, that non-admissible configurations are doomed to be manipulable. One can therefore wonder whether the increase of the rate of non-admissible configurations is the only factor explaining the growth of the manipulability rates. To test this hypothesis, we have indicated in Figure 7.3 the *normalized CM rate*, i.e. relative to the proportion of admissible configurations. For voting systems meeting **InfMC**, it is their manipulability rate for the restriction of the impartial culture to the admissible configurations.[5] The interesting finding is that, even so, the previously mentioned phenomena still hold true, which leads us to the following conjecture.

Conjecture 7.6

Consider the spherical culture with $V \geq 3$.

The normalized manipulability rate of each voting system considered here is an increasing function of the number of candidates C.

For $C \to +\infty$, the normalized manipulability rate of each voting system considered here, except Veto, tends to 1*.*

This conjecture does not invalidate our intuitive explanation regarding the increase of the manipulability rate. The fact that there are fewer and fewer Condorcet-admissible candidates is, in our opinion, only a symptom of a more general phenomenon of growing disagreement, which give an intuitive picture explaining that some voting systems that does not meet **InfMC**, such as Veto, are also concerned.

7.4 Spherical culture: number of voters V

7.4.1 Odd number of voters V

In Figure 7.4, we plot CM rates for odd numbers of voters V. The parity of the number of voters can create particular phenomena, which we will examine in Section 7.4.2.

For most voting systems, the CM rate seems to be an increasing function of the number of voters V (on the set of odd values). In the particular case of EB, the apparent non-monotonicity between $V = 9$ and $V = 13$ is lower than the statistical uncertainty of 1 %, so even in that case, it is not excluded that the CM rate is an increasing function of the number of voters.

For CIRV, IRV, and EB, we observe in Figure 7.4 that the CM rates are, again, significantly lower than for the other voting systems and that the differences between these three systems are very small. This time, we used the exact algorithm for IRV. Since we know that IRV is at most as manipulable as EB in the set-theoretic sense (Section 1.4.2), the slight difference between IRV and EB (at most slightly greater than 1 %) is only due to an actual difference of manipulability, i.e. to profiles where EB is manipulable but where IRV is not. This gap seems to tend to 0 when the number of voters tends to $+\infty$. As for the difference between CIRV and IRV, it is again extremely small.[6]

[5] Gehrlein pointed out that ideally it would be interesting to compare the different voting systems ignoring the set of configurations where they are all manipulable: these only increase all manipulability rates, without bringing discrimination between the different systems. However, such a process generally depends on the set of voting systems studied.

The normalization that we propose here, based on admissible configurations, has the advantage of being universal for all voting systems satisfying **InfMC** and, on this class, it achieves exactly the objective formulated by Gehrlein. Indeed, any non-admissible configuration is necessarily manipulable. And for each admissible configuration, there exists a voting system satisfying **InfMC** (and even **Cond**) where it is not manipulable: it suffices to choose arbitrarily one of its Condorcet-admissible candidates a and to consider the Condorcet-dean voting system, where the dean is a; non-manipulability then follows from Lemma 2.4.

[6] Our approximate algorithm for CIRV was not able to decide manipulability for 2 cases out of 10,000 for $V = 5$, and 1 case out of 10,000 for $V = 13$ and $V = 17$. In all other cases, the manipulability, true or false, could be decided and it is equal to that of IRV.

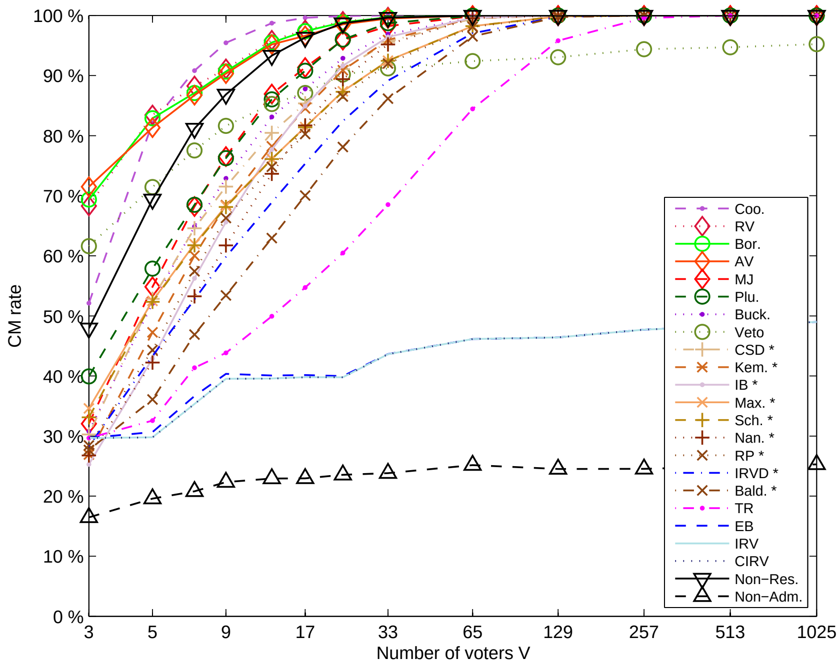


Figure 7.4 – CM rate as a function of the number of voters V. Spherical culture, $C = 5$, large odd values of V.

Apart from these voting systems, the Two-round system behaves better than the other systems, except for about a hundred voters or more, where Veto becomes less manipulable.

For all voting systems except CIRV, IRV, EB, and Veto, the CM rate seems to tend to 1 when the number of voters V tends to $+\infty$. But contrary to the case $C \to +\infty$, the probability of existence of a Condorcet winner does not tend to 0 when V tends to $+\infty$ (Gehrlein, 2006), hence we cannot prove this observation by the same means. However, there exists theoretical results on this matter. On the one hand, Kim and Roush (1996) showed the following results.

- For Veto, if $C = 3$ and $V \to +\infty$, the CM rate tends to $\frac{1}{2}$.
- For Veto, if $C > 3$ and $V \to +\infty$, the CM rate tends to a limit strictly between 0 and 1.
- For all PSR except Veto, if $C \geq 3$ and $V \to +\infty$, the CM rate tends to 1.
- For Maximin, if $C \geq 3$ and $V \to +\infty$, the CM rate tends to 1.
- For Coombs' method, if $C = 3$ and $V \to +\infty$, the CM rate tends to 1.

On the other hand, Lepelley and Valognes (1999) showed that for EB, if $C = 3$ and $V \to +\infty$, the CM rate tends to 0.16887 (approximate value): in particular, it is a value strictly between 0 and 1.

In light of these theoretical results and the curves of Figure 7.4, we propose the following conjectures.

Conjecture 7.7

In impartial culture, for $C \geq 3$ and $V \to +\infty$, the CM rate tends to 1 for

Baldwin, Bucklin, Coombs, CSD, IB, IRVD, Kemeny, Nanson, RP, and Schulze's method. For $C > 3$ and $V \to +\infty$, it is also the case for the Two-round system.

In spherical culture, the same is true for Range voting, Approval voting, and Majority judgment.

Recall that this result is proven for Maximin and all PSR except Veto..

Conjecture 7.8

In impartial culture, for $C \geq 3$ and $V \to +\infty$, the CM rate tends to a limit strictly between 0 and 1 for CIRV, IRV, and EB.

Recall that this result is already proven for Veto.

In spherical culture, it would therefore seem that CIRV, IRV, EB, and Veto are the only voting systems, among those studied here, whose manipulability rate does not tend to 1 when the number of voters V tends to $+\infty$.

7.4.2 Parity of V

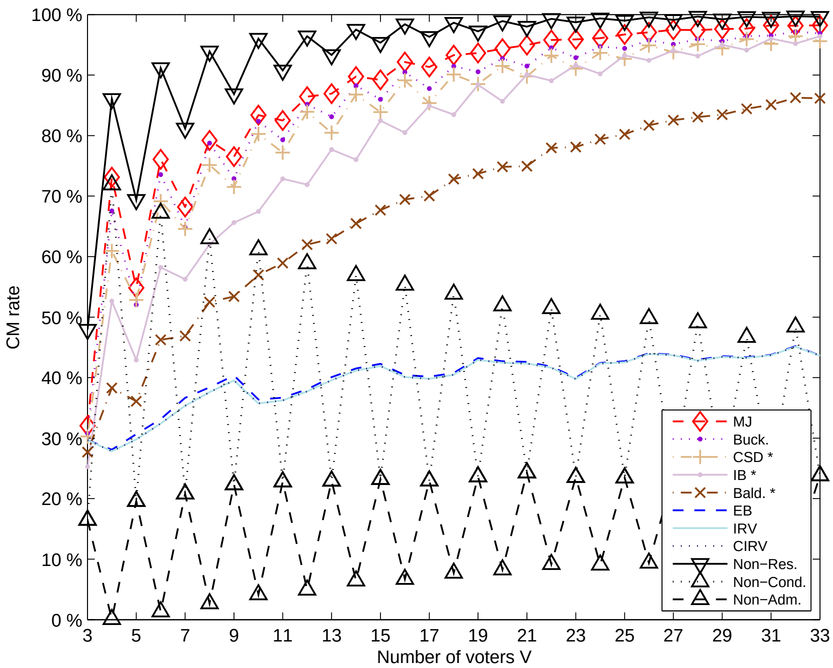


Figure 7.5 – CM rate as a function of the number of voters V for voting systems with oscillatory behaviors. Spherical culture, $C = 5$, small values of V.

If we use odd and even values of V, we observe phenomena of non-monotonicity for the CM rate of certain voting systems. In Figure 7.5, we consider all the values of the number of voters from 3 to 33 and we represent the CM rate of the voting systems concerned. The others are omitted in order to lighten the figure.

First, let us examine Majority judgment or Bucklin's method, for which the CM rate exhibits oscillations without algorithmic uncertainty between even and odd values of V.

This oscillatory behavior could be related to the fact that the probabilities of existence of a Condorcet winner and of a Condorcet-admissible candidate also oscillate between the even and odd values of V. However, as Gehrlein (2006) proves by theoretical arguments, these oscillations are in phase opposition. When V is odd, the two notions are equivalent. When V goes from an odd value to an adjacent even value, the probability of having at least one Condorcet-admissible candidate increases, whereas the probability of having a Condorcet winner decreases. On these curves obtained by computer simulation, we also observe oscillations for the probability of existence of a resistant Condorcet winner, in phase with the Condorcet winner.[7] And, as we have seen, the probability of existence of a Condorcet-admissible candidate, of a Condorcet winner (which was equivalent in the previous curves), or of a resistant Condorcet winner are indicators with an impact on manipulability.

However, if this explanation is correct, it is surprising that other voting systems meeting **InfMC** do not exhibit the same kind of oscillatory behavior. We can formulate the hypothesis that there are two competing effects: the higher probability of having a Condorcet-admissible candidate protects better against manipulation for V even, but the probability of having a Condorcet winner or even a resistant one is a better protection for V odd. Thus, the two effects may partially cancel each other out, which could explain why few voting systems exhibit oscillations. We will se that this oscillatory phenomenon is much more widespread in a Gaussian well culture and we will explain why (Section 8.2.2).

This explanation raises an additional question. For voting systems meeting **InfMC**, the usual lower bound on the manipulability rate is given by the proportion of non-admissible configurations, so one might think that the oscillations of these curves will be in phase. But we observe the opposite: the oscillations of Majority judgment and Bucklin's method are in phase with the proportion of Condorcet configurations.

For these reasons, we propose another explanation, which does not exclude the previous one. Both Majority judgment and Bucklin's method are based on the notion of median, which has a slightly different definition depending on the parity of the number of voters. Specifically, let us examine what can happen with an even number of voter, if there are exactly $\frac{V}{2}$ voters who prefer a certain candidate c to the sincere winner w. Generally, the typical sincere voter gives w neither the maximal rank nor the maximal grade, and she gives c neither the minimal rank nor the minimal grade. After manipulation, we obtain the kind of profile represented in a simplified way in Table 7.1.

In Majority judgment or Bucklin's method, by convention, the unfavorable median is used, i.e. the lower median grade in Majority judgment and the upper median rank in Bucklin's method. Thus, the median grade taken into account by Majority judgment (resp. the median rank taken into account by Bucklin's method) for w is 0 (resp. C), whereas the median grade (resp. the median rank) of c is 0.2 (resp. $C-1$ for example). Thus, a coalition formed by a weak majority of voters can generally manipulate. On the other hand, if we consider a profile similar to the simplified example above but with an odd number of voters, a strict majority of voters is necessary to make c win. This may explain the fact that the manipulability rate is higher for an even number of voters.

[7] It is also known (Gehrlein, 2006) that for a fixed parity, the probability of existence of a Condorcet winner is monotonic. At first glance, this does not seem to be verified in Figure 7.5 between $V = 28$, $V = 30$, and $V = 32$. However, the difference observed between $V = 30$ and $V = 32$ is less than 2 % and is solely of statistical origin. Remember that this uncertainty, which is of the order of 1 % for one value and of $\sqrt{2} \times 1$ % for the difference between two values, is only an order of magnitude, which may sometimes be exceeded.

$\frac{V}{2}$ (sincere)	$\frac{V}{2}$ (manipulators)
Misc : 1	c : 1
w : 0.9	$\vdots$
Misc : 0.5	Misc : 0.5
c : 0.2	$\vdots$
Misc : 0	w : 0

Table 7.1 – Explanatory example for Majority judgment and Bucklin's method.

For CSD, the oscillatory phenomenon needs to be confirmed, because the curve plotted is only a lower bound of the CM rate (in practice, it is the TM rate). We can still notice that the counting rule of CSD has a behavior that depends on the parity of V: indeed, the penalty for a defeat of c against d is $1 + D_{dc} - \frac{V}{2}$ when V is even, but only $\frac{1}{2} + D_{dc} - \frac{V}{2}$ when V is odd.

For CIRV, IRV, and EB, we also observe non-monotonic effects, the amplitude of which is greater than the margin of uncertainty. However, they do not seem to have an obvious pseudo-period. Given the way these voting systems are counted, we propose the explanation that several competing effects combine, depending on the modulo of V by all integers from 2 to C.

For IB, like for CSD, the curve only provides a lower bound, but it can be commented on as a curve of the TM rate. This time, there seems to be a transition about $V = 8$ or $V = 9$: the oscillations remain but the phase of the curve reverses! For smaller values of V, the manipulability is greater for V even (than for adjacent odd values). For larger values, the manipulability is larger for V odd. Since it is a multiround voting system, it is possible to imagine effects of modulo like for CIRV, IRV, and EB, and parity effects like for Bucklin's method. In any case, the complex rule of this voting system makes the perspective of a simple explanation quite remote.

Lastly, we notice that the non-monotonic phenomena decrease in amplitude when the number of voters V increases. This is easy to explain: when the number of voters is large, phenomena of ties, implying exactly half of the voters, are quite rare. Amusingly, this observation can by summed up by the adage: "infinity is odd". Or: "when infinity is even, it does not notice it."

7.5 Monopolar culture: concentration κ

We now consider non-degenerate VMF cultures, i.e. with a nonzero concentration κ. Recall that we first choose a unit vector $\mathbf{n}$, called the *pole* of the distribution. Then, independently for each voter, we draw a unit vector $\mathbf{u}$ in $\mathbb{R}^C$ according to a VMF distribution:

$$p(\mathbf{u}) = X_\kappa e^{\kappa \langle \mathbf{u} \mid \mathbf{n} \rangle},$$

where X_κ is a normalization constant.

Now that we have studied the impact of the number of candidates or the number of voters, we will study the effect of the concentration and the position of the pole. Then we will extend the model to several poles in order to represent several social groups, each with a typical opinion.

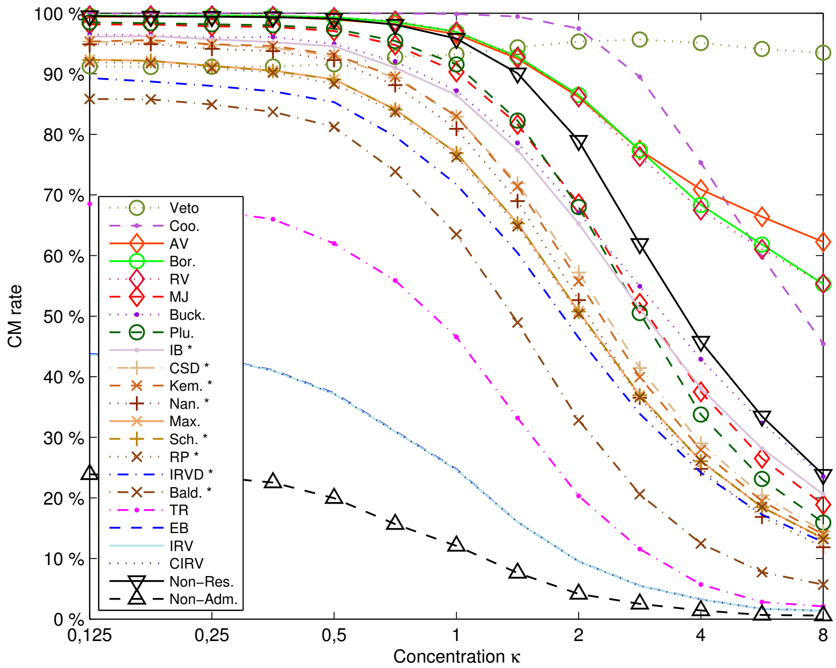


Figure 7.6 – CM rate as a function of concentration κ. VMF culture, $V = 33$, $C = 5$, one pole of random position.

In Figure 7.6, for each population, the pole $\mathbf{n}$ is drawn at random, then the voters are drawn with a certain concentration κ. Note that voters are not independent. More precisely, *once the pole is chosen*, they are independent; but, *on the entire set of possible populations* (with all possible positions of the pole), they are not independent. Indeed, *ex ante*, a voter has a uniform probability on the unit sphere of $\mathbb{R}^C$; but if we know, for example, that the first $V - 1$ voters are close to a given point of the utility space, then the last voters has a higher probability of being close to this point.

As expected, the higher the concentration κ, the lower the CM rates, at least for most voting systems. In the degenerate case $\kappa = +\infty$, all voters always have the same utility vector, so every unanimous voting system (Section 1.2.2) is non-manipulable: it is the case for all the voting systems studied here, except Veto (see below).

However, some voting systems seem to be significantly less responsive than others to an increase of the concentration κ: Veto, Approval voting, Borda's method, Range voting, and, to a lesser extent, Coombs' method. This means that, even when voters have relatively similar opinions, these voting systems still have high manipulability.

For Veto, the CM is not even monotonic. And we are going to show that it does not tend to 0, contrary to what happens for unanimous voting systems.

Proposition 7.9

We consider Veto, equipped with a tie-breaking rule that uses no information on preferences other than the candidate against whom each voter casts a ballot.

Assume $V \geq C - 1$.

In VMF culture with a pole drawn uniformly at random, when $\kappa \to +\infty$, the manipulability rate tends to $1 - \dfrac{1}{C-1}$.

The assumption made on the tie-breaking rule is met, in particular, when ties are broken by lexicographical order on the candidates, as is the case in SVAMP. In Figure 7.6, since $C = 5$, the limit CM rate is $\frac{3}{4}$, but it is far from being reached: convergence seems therefore to be relatively slow.

Proof. In the limit where all voters have the same order of preference, they all vote against the same candidate and it is only the tie-breaking rule that designates the winner among the other candidates. Since the pole is drawn neutrally, there is one chance out of $C - 1$ that the elected candidate is the one preferred by the voters. In all other cases, all the voters can form a coalition and manipulate to make their preferred candidate win: indeed, since $V \geq C - 1$, it is possible that at least one voter casts a ballot against each other candidate. □

Once again, we notice the low and extremely similar CM rates for CIRV, IRV, and EB, followed by the intermediate performances of TR, itself less manipulable than the other voting systems.

7.6 Monopolar culture: position of the pole

Now we will work with a constant concentration κ and vary the position of the voters. We keep the same order of preference on candidates $1 \succ 2 \succ \ldots \succ 5$ but we vary the shape of the opinion of the typical voter (i.e. located exactly at the pole of the distribution). First, we consider the limit case where the typical voter prefers candidate 1 and is indifferent between the other candidates; and we transition to a relatively balanced state where her utilities are Borda scores (Figure 7.7). Then, we move the typical voter from a Borda-like state to another limit case where she hates candidate 5 and is indifferent between the other candidates (Figure 7.8).

In Figure 7.7, concentration $\kappa = 2$ is thus constant, but the position of the pole $\mathbf{n}$ is imposed and no longer drawn at random. Consider two unit utility vectors $\mathbf{n}_0 = \frac{1}{\sqrt{20}}(4, -1, -1, -1, -1)$ and $\mathbf{n}_1 = \frac{1}{\sqrt{10}}(2, 1, 0, -1, -2)$. Denoting $\theta_{\max}$ the angle between $\mathbf{n}_0$ and $\mathbf{n}_1$, we have $\theta_{\max} = 45°$. We explore the geodesic of the unit sphere going from $\mathbf{n}_0$ to $\mathbf{n}_1$, taking as parameter the polar angle θ since $\mathbf{n}_0$. When $\theta = 0$, we have $\mathbf{n} = \mathbf{n}_0$: the typical voters prefers candidate 1 and is indifferent between the other candidates. When $\theta = \theta_{\max}$, we have $\mathbf{n} = \mathbf{n}_1$: the typical voter has a well-established order of preference $1 \succ 2 \succ 3 \succ 4 \succ 5$, with utilities that are Borda scores up to normalization.

We observe that all voting systems become more manipulable when the pole of the distribution approaches $\mathbf{n}_1$ (with very different amplitudes for this phenomenon, depending on the voting system). We propose the following explanation. With a given value of κ, when the pole is $\mathbf{n}_0$, the population is strongly polarized in favor of candidate 1, hence it is unlikely that a voter would prefer another candidate to candidate 1. But, when the pole is $\mathbf{n}_1$, it is closer to points of the hypersphere where voters prefer candidate 2 to candidate 1. Therefore, coalitions in favor of candidate 2 have more members and are more likely to succeed in manipulation.

As before, we note the good performances of CIRV, IRV, and EB, followed by those of the Two-round system.

In Figure 7.8, we continue our investigation by moving toward the pole $\mathbf{n}_2 = \frac{1}{\sqrt{20}}(1, 1, 1, 1, -4)$. Denoting $\theta'_{\max}$ the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$, we again have $\theta'_{\max} = 45°$. We take θ' as parameter, the polar angle traveled since $\mathbf{n}_1$. For $\theta' = 0$, we have $\mathbf{n} = \mathbf{n}_1$: the typical voter has a well-established order of preference

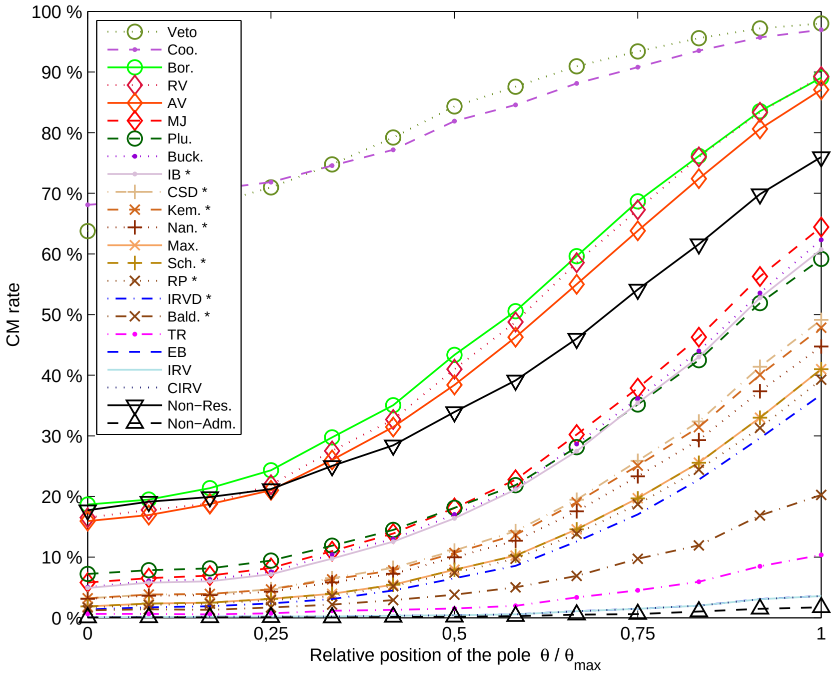


Figure 7.7 – CM rate as a function of the position of the pole $\mathbf{n}$. VMF culture, $V = 33$, $C = 5$, one pole of concentration $\kappa = 2$. For $\theta = 0$, we have $\mathbf{n} = \mathbf{n}_0 = \frac{1}{\sqrt{20}}(4, -1, -1, -1, -1)$. For $\theta = \theta_{\max}$, we have $\mathbf{n} = \mathbf{n}_1 = \frac{1}{\sqrt{10}}(2, 1, 0, -1, -2)$.

$1 \succ 2 \succ 3 \succ 4 \succ 5$. For $\theta' = \theta'_{\max}$, we have $\mathbf{n} = \mathbf{n}_2$: the typical voter has an aversion for candidate 5 and is indifferent between the other candidates.

Note that the path traveled from $\mathbf{n}_0$ to $\mathbf{n}_1$ then from $\mathbf{n}_1$ to $\mathbf{n}_2$ is not a geodesic of the sphere:[8] indeed, as shown in Appendix B, the geodesics in the utility space are the zones of unanimity. However, in $\mathbf{n}_0$ and $\mathbf{n}_2$, the typical voter is indifferent between candidates 2, 3, and 4, which is not the case in $\mathbf{n}_1$. The path traveled by the pole $\mathbf{n}$ on the sphere between Figure 7.7 and Figure 7.8 is therefore a broken line from $\mathbf{n}_0$ to $\mathbf{n}_1$ then from $\mathbf{n}_1$ to $\mathbf{n}_2$.

In Figure 7.8, we can see that the CM rates and the rates of non-admissible configurations and non-resistant configurations continue to increase. The culture becomes similar to a uniform culture on the four credible candidates (all but candidate 5), except the fact that there is an additional candidate less liked by all voters. This one has no chance of winning in all reasonable voting systems (even Veto) but she may disrupt voting systems that are the most sensitive to the addition of an irrelevant candidate, for example Borda's method: for a manipulator, this candidate makes it possible to add one point to the score difference between the desired candidate c and the sincere winner w, so she increases the possibility of manipulation, compared to a uniform culture for 4 candidates.

[8] For information, if we continue the arc of circle from $\mathbf{n}_0$ to $\mathbf{n}_1$ up to $\theta = 90°$, we obtain $\mathbf{n} = \frac{1}{\sqrt{20}}(0, 3, 1, -1, -3)$, which does not represent the same order of preference.

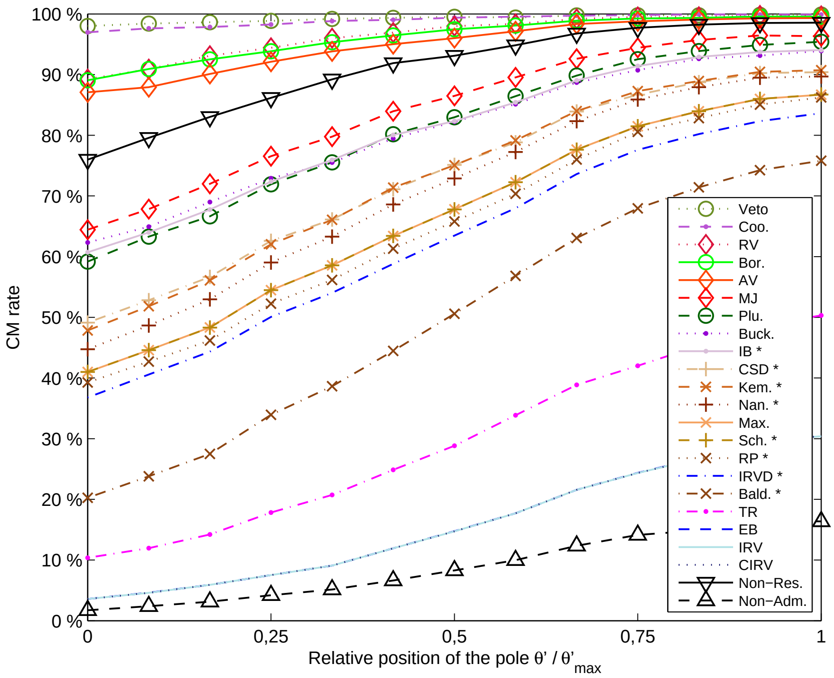


Figure 7.8 – CM rate as a function of the position of the pole $\mathbf{n}$ (continued). VMF culture, $V = 33$, $C = 5$, one pole of concentration $\kappa = 2$. For $\theta' = 0$, we have $\mathbf{n} = \mathbf{n}_1 = \frac{1}{\sqrt{10}}(2, 1, 0, -1, -2)$. For $\theta' = \theta'_{\text{max}}$, we have $\mathbf{n} = \mathbf{n}_2 = \frac{1}{\sqrt{20}}(1, 1, 1, 1, -4)$.

7.7 Multipolar culture: number of poles

So far, we have considered cultures with a single pole $\mathbf{n}$. However, in practice, a population of agents does not always have preferences centered around one typical opinion. In particular, one can consider that there are several social groups which each possess a certain cohesion. We use the expression *social group* in a very broad sense: for example, in a political election, it can be a socio-professional class or a community of interest; in other contexts, such as a professional or associative organization, it may be a group of individuals with a certain cohesion, whether based on ideological, personal, or other reasons.

In Figure 7.9, we therefore extend the VMF model by considering a multipolar culture made up of several social groups. Formally, we take a parameter k which represents the number of groups. Here is how we draw a population. First, we draw k unit vectors $\mathbf{n}_1, \ldots, \mathbf{n}_k$ independently and uniformly which will be the poles of the distribution, i.e. the utility vector of the typical voter of each social group. Then, for each voter independently, we draw an integer i equiprobably between 1 and k, which represents her social group, then we draw her utility vector according to a VMF distribution of pole $\mathbf{n}_i$ and concentration κ. In order to simplify the study, we consider only social groups of comparable size (since i is drawn with equiprobability) and of the same constant concentration κ, even if SVVAMP also makes it possible to vary these parameters.

As we expected, we see in Figure 7.9 that CM rates increase with the number of poles (except for Veto).

With V and C fixed, when the number of poles k tends to $+\infty$, the probability that there exists a pair of voters who are in the same social group tends to 0, so

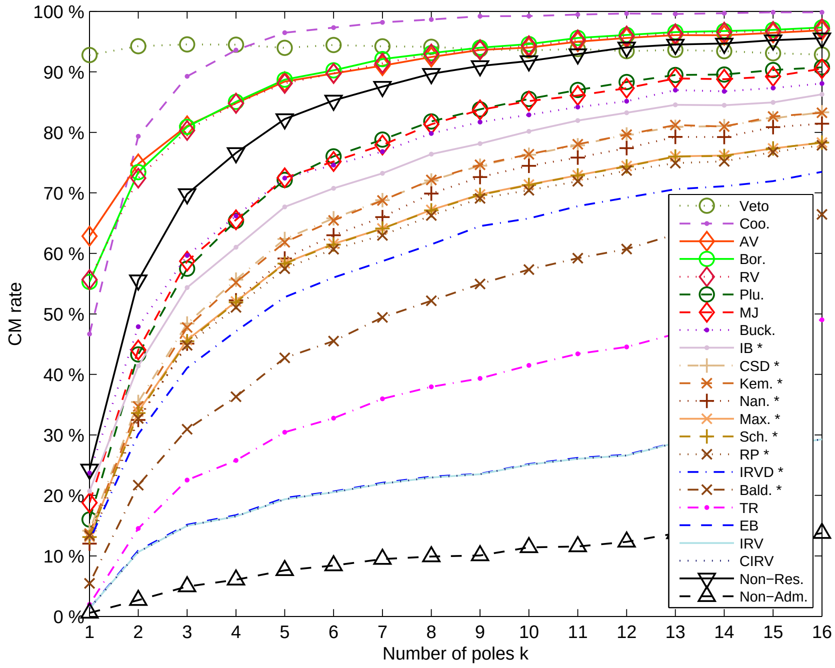


Figure 7.9 – CM rate as a function of the number of poles. VMF culture, $V = 33$, $C = 5$, equiprobable poles with random positions, $\kappa = 2$.

the distribution becomes uniform on the hypersphere: thus, the limit distribution is simply the spherical culture. The curves confirm this theoretical remark: for $k = 16$, CM rates are slightly smaller but already comparable to those observed in a spherical culture with $V = 33$ and $C = 5$ (cf. Figure 7.2 for example).

7.8 Multipolar culture: relative positions of the poles

Finally, we will investigate the relative positions of the social groups in the utility space. The simplest example to study this phenomenon consists in considering two social groups of similar size (each voter is equiprobably in one or the other) and of the same concentration κ.

Consider the two utility vectors $\mathbf{n}_0 = \frac{1}{\sqrt{20}}(4, -1, -1, -1, -1)$ and $\mathbf{n}_1 = \frac{1}{\sqrt{20}}(-1, 4, -1, -1, -1)$. Denoting $\theta_{\max}$ the angle between $\mathbf{n}_0$ and $\mathbf{n}_1$, we have $\cos(\theta_{\max}) = -\frac{1}{4}$ therefore $\theta_{\max} \simeq 104°$.

The first pole we take is always $\mathbf{n}_0$. The second pole explores the arc of the unit circle from $\mathbf{n}_0$ to $\mathbf{n}_1$, using the angle θ as parameter. When $\theta = 0$, the two poles coincide at $\mathbf{n}_0$: the typical voter prefers candidate 1 and is indifferent between the other candidates. When $\theta = \theta_{\max}$, the second pole is $\mathbf{n}_1$: the typical voter of the second social group prefers candidate 2 and is indifferent between the other candidates.

We observe in Figure 7.10 that the CM rate increases for all the voting systems considered when θ increases. This is not surprising: when disagreement grows in the population, the motivation to manipulate also grows.

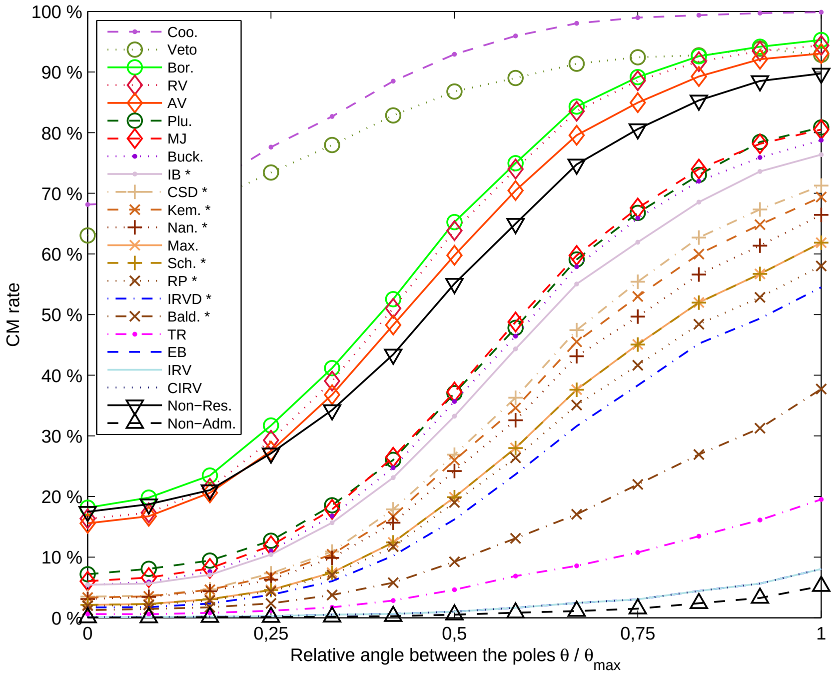


Figure 7.10 – CM rate as a function of the angle between poles. VMF culture, $V = 33$, $C = 5$, two equiprobable poles, $\kappa = 2$. The first pole is $\mathbf{n}_0 = \frac{1}{\sqrt{20}}(4, -1, -1, -1, -1)$. For $\theta = 0$, the second pole is the same. For $\theta = \theta_{\max}$, the second pole is $\mathbf{n}_1 = \frac{1}{\sqrt{20}}(-1, 4, -1, -1, -1)$.

Again, note the remarkable performances of CIRV, IRV, and EB, followed by that of the Two-round system.

7.9 Meta-analysis in spheroidal cultures

In order not to make this study any longer, we spare the meritorious reader the variation of other parameters: thus, we will not vary the typical relative sizes of the social groups (i.e. the relative probabilities that a voter belongs to each of them) and we will not study the case where the concentrations are different from one group to another. The results found are similar and anyone can reproduce such experiments with SVVAMP.

From all the previous curves, a general trend seems to emerge. Some voting systems are almost always less manipulable than others: CIRV, IRV, and EB. The only exception was obtained for $V = 3$ voters (Figure 7.4), where the lower bound found for some voting systems is lower than the rate found for CIRV, IRV, and EB. However, it should be noted that with 3 voters, the tie-breaking rule plays such an important role that it is difficult to draw definitive conclusions about the voting systems themselves. We will therefore choose to ignore the particular case $V = 3$ in the rest of this section.

In this chapter and the following ones, we will often resort to a *meta-analysis*, which makes it possible to present in a compact way the results obtained for a set of cultures which may be heterogeneous and to examine the compared performances of different voting systems.

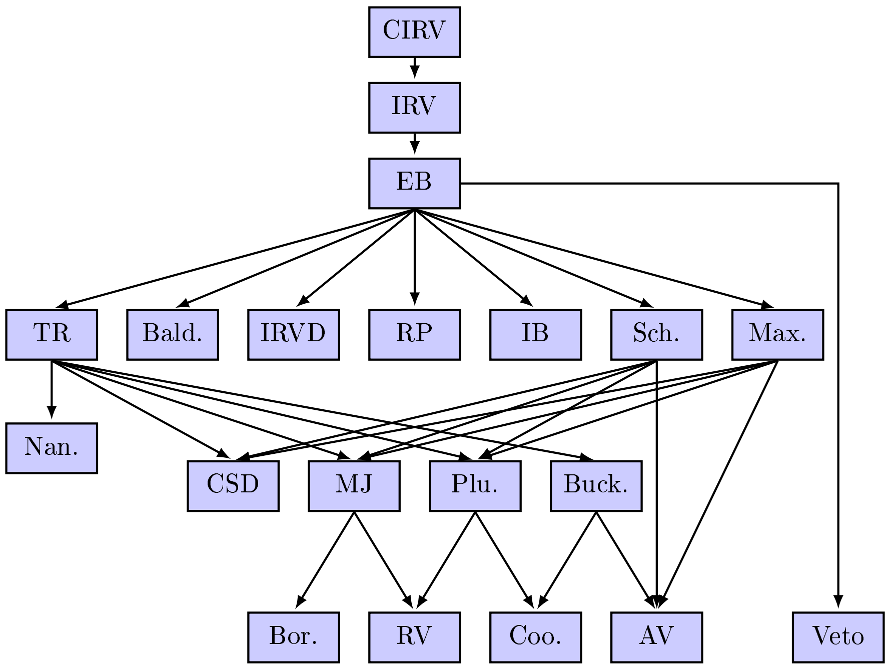


Figure 7.11 – Meta-analysis of CM in all spheroidal cultures under study (except $V = 3$).

7.9.1 Meta-analysis of CM in spheroidal cultures

The graph in Figure 7.11 illustrates this method for the CM rate. Each vertex represents a voting system. An edge from a voting system f to another one g means that, for all the cultures observed in this chapter (except $V = 3$), i.e. for each point of each curve of this chapter, the voting system f is proven at most as manipulable as g.

More precisely, for a voting system f and a given culture (i.e. a fixed set of parameters), let $\underline{\tau_{\text{CM}}(f)}$ denote the lower bound found by SVVAMP and $\overline{\tau_{\text{CM}}(f)}$ the upper bound. For two distinct voting systems f and g, we draw an edge from f to g iff we always have $\overline{\tau_{\text{CM}}(f)} \leq \underline{\tau_{\text{CM}}(g)}$.

In all our meta-analyses, we exclude statistical uncertainty, i.e. we consider the *proportion of manipulable configurations* on each draw of 10,000 experiments, and not the exact *manipulability rate* in the underlying culture. The motivation is as follows. Imagine that a voting system is generally much less manipulable than another one, but that for a specific culture, the two voting systems have a CM rate close to 100 %. Because of this last case, since there is a statistical uncertainty of the order of 1 %, it is impossible to conclude that the first voting system always has a lower manipulability rate than the second in the cultures considered. However, this case is quite common, so we could not conclude for a large part of the pairs of voting systems. By considering the proportion of manipulable configurations in the actual draws, we can conclude more often. So, one need to be aware that this meta-analysis gives results on a set of random experiments, not on a set of cultures. Despite this limitation, it provides a compact representation of the results that gives a qualitative indication about what happens in the cultures studied.

The other advantage of ignoring the statistical uncertainty is that the meta-analysis graph obtained is necessarily transitive (otherwise, we could have a binary relation of the same type as in Example 1.7). For the sake of readability, we represent only the minimal set of edges that makes it possible to deduce the whole graph by transitivity (i.e. its *transitive reduction*).

For some voting systems, like Baldwin's method, the approximate algorithm essentially provides a lower bound of manipulability, so it is not possible to establish that the voting system is less manipulable than another. On the other hand, it is possible to show that is is more manipulable: for this reason, Baldwin's method, for example, has incoming edges but no outgoing edges, although its lower bound of manipulability, represented in the curves, is relatively low in general. For Maximin, Schulze, and Borda, the upper bound is sufficiently precise to establish their superiority compared to certain other voting systems, even if the case does not happen in practice for Borda's method.

The main conclusion from Figure 7.11 is that CIRV, IRV, and EB are generally less manipulable that the other voting systems. Although we know, by theoretical arguments, that CIRV is strictly less manipulable than IRV, itself strictly less manipulable than EB, the difference observed are generally very small: for each draw of 10,000 experiments carried out for the curves of this chapter, the difference in the proportions of manipulable profiles is at most 0,02 % between CIRV and IRV and at most 1,3 % between CIRV and EB.

In our simulations, these three voting systems are followed by the Two-round system, which is generally (but not always) less manipulable than the other voting systems under study. For example, the difference of manipulability in favor of IRVD against the Two-round system is never more than 1,3 %; on the opposite, the difference of manipulability in favor of the Two-round system against IRVD may exceed 57 %. The Two-round system can only be clearly dominated by Veto, and only for a large number of voters. We can then informally distinguish several groups, gathered by line in the figure.

For Baldwin, IRVD, RP, and IB, the manipulability rates found are in general fairly comparable, but these are only lower bounds which only make it possible to establish their defeats against EB and their quasi-inferiority against the Two-round system. As for Nanson's method, it is always more manipulable than the Two-round system in all the curves of this chapter. Maximin and Schulze's method have very similar performances (the observed differences are always less than 1 %) and better than those of Plurality, CSD, and Majority judgment. They are also often better than Bucklin's method (the difference of manipulability, when it is in favor of the latter, never exceeds 1 %).

The approximate algorithm used for CSD does not make it possible to establish if it is less manipulable than the voting systems in the lower group. We simply note that it has not proved more manipulable than Plurality, Majority judgment, or Bucklin's method, hence its indicative position in the figure.

Voting systems with the worst performances are Borda's method (more manipulable than Majority judgment), Range voting (more manipulable than Majority judgment or Plurality), Coombs' method (more manipulable than Plurality or Bucklin's method), and Approval voting (more manipulable than Bucklin's method). In many figures of this chapter, the manipulability curves of these four voting systems are above the proportion of non-resistant configurations: in the cases concerned, this means that any Condorcet voting system is less manipulable.

Veto may be considered as a group on its own, since it behaves very differently depending on the cases studied. On the one hand, it has a rare advantage: in impartial culture, its manipulability rate does not tend to 1 when the number

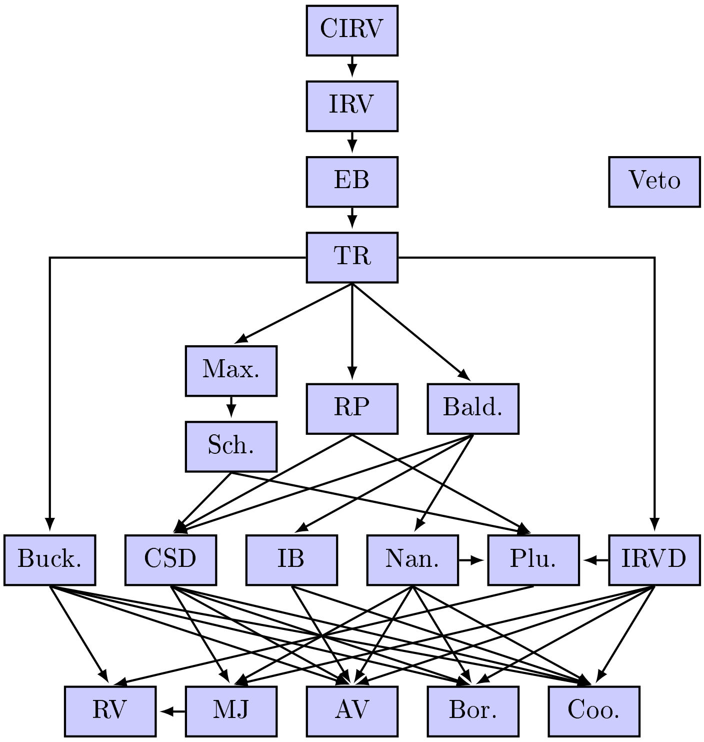


Figure 7.12 – Meta-analysis of TM in all spheroidal cultures under study (except $V = 3$).

of voters V tends to $+\infty$, unlike all the other PSR and Maximin. Based on our simulations, we even conjecture that, among the voting systems under study here, the only ones to share this desirable property are CIRV, IRV, and EB. On the other hand, Veto has a very unfortunate drawback: in a population where all voters have the same preferences, its manipulability rate does not tend to 0, unlike all the voting systems studied here and more generally all unanimous ones.

7.9.2 Meta-analysis of TM in spheroidal cultures

Figure 7.11 has the advantage of dealing with CM in general terms, but with the disadvantage of not being able to provide complete conclusions concerning certain voting systems using an approximate algorithm. In order to have an additional indication of manipulability for all voting systems, we represent a similar graph for TM in Figure 7.12. Moreover, TM gives an indicative measure of realistic manipulations, which can be performed with a limited exchange of information. For TM, we spare the reader the detailed curves and we go directly to the meta-analysis.

Let us start by noticing that most of the edges of Figure 7.11 (explicitly drawn or implied by transitivity) are included in those of Figure 7.12: thus, TM provides a reasonable indicator about the hierarchy of manipulability between voting systems.

We observe that Veto forms an isolated connected component: it is sometimes more TM and sometimes less TM than any other voting system studied here.

For TM as for CM, the voting systems CIRV, IRV, and EB, in that order, are always less manipulable than all the order voting systems studied, except Veto.

They are always followed by the Two-round system, which has intermediate performances. We can then indicatively distinguish three groups, by increasing TM:

- Maximin and Schulze's method (with very similar performances), RP and Baldwin's method;
- CSD, IB, Nanson's method, and Plurality (even if Nanson's method is less TM than Plurality);
- Range voting, Majority judgment, Approval voting, Borda, and Coombs (while noticing that Majority judgment is less TM than Range voting).

Bucklin's method is a bit isolated in the figure: it is always always more TM than the Two-round system and always less than Range voting, but it is not comparable to any voting system in the intermediate groups. The case of IRVD is similar: always more TM than the Two-round system, less TM than Plurality and all voting systems of the last group, but not comparable to the others.

Chapter 8

Simulations in Cultures Based on a Political Spectrum

In this chapter, we study cultures based on a *political spectrum*, which we just mentioned in Section 6.1.3. Our reference model will be the *Gaussian well* and we will briefly discuss another similar model, the *Euclidean box*, in Section 8.5.

In these models, we use an n-dimensional space (for a given integer n), called the *political spectrum*. In Gaussian well, we take as a parameter a vector of nonnegative real numbers $(\sigma_1, \ldots, \sigma_n)$: each number σ_i is called the *characteristic length* (in short, *length*) of the political spectrum along the axis i. For each voter v (resp. each candidate c), we randomly draw a position $\mathbf{x}_v = (x_v^1, \ldots, x_v^n)$ (resp. $\mathbf{y}_c = (y_c^1, \ldots, y_c^n)$). Each coordinate x_v^i (resp. y_c^i) is drawn independently according to a centered normal distribution of standard deviation σ_i. The utility of a voter v for a candidate c is $A - \delta(\mathbf{x}_v, \mathbf{y}_c)$, where δ denotes the usual Euclidean distance and where A is a constant such that the average utility is zero. Note that this constant only has an influence on Approval voting.

When the political spectrum is one-dimensional ($n = 1$), the culture is obviously single-peaked (Definition 1.11). If the number of voters is odd, there is always a Condorcet winner, who is the preferred candidate of the median voter (on the political spectrum). It the number of voters is even, then Condorcet-admissible candidates are the preferred candidates of the two median voters; if it is the same candidate, then she is Condorcet winner. Consequently, the proportion of non-admissible configurations is zero and it cannot provide us with an exploitable lower bound of manipulability, contrary to the previous chapter. For the record and for the sake of homogeneity in the presentation of the results, we will, however, continue to represent it in the figures.

Figuratively, in a one-dimensional political spectrum, Condorcet-admissible candidates will be called *centrists*.[1] Similarly, candidates with a higher (resp. lower) abscissa in the political spectrum will be called *right* (resp. *left*) candidates.

In the one-dimensional case, it is useless to specify the value of σ: indeed, up to changing the unit of length in the political spectrum, all the models obtained are equivalent (because utility vectors are defined up to a multiplicative constant, cf. Appendix B). On the other hand, as soon as there are two dimensions, the

[1] This is just a language convention based on an analogy. Even if one accepts the assumption that voter preferences in a political election are approximately single-peaked, the Condorcet winner is not necessarily a centrist candidate, in the usual political sense of this word. By the way, we will have the opportunity to discuss the experimental validity of this assumption by examining the results obtained in Chapter 9.

relative lengths of the political spectrum along the different axes are important. The experimental validity of such multidimensional models for political elections is demonstrated in particular by Laslier (2004, 2006).

Finally, recall that even when the preferences are assumed to always be single-peaked, if the reference order P_{ref} is not *a priori* fixed, the conclusion of the theorem of Gibbard-Satterthwaite remains true: any non-trivial system is manipulable (Section 1.3).

8.1 One-dimensional Gaussian well: number of candidates C

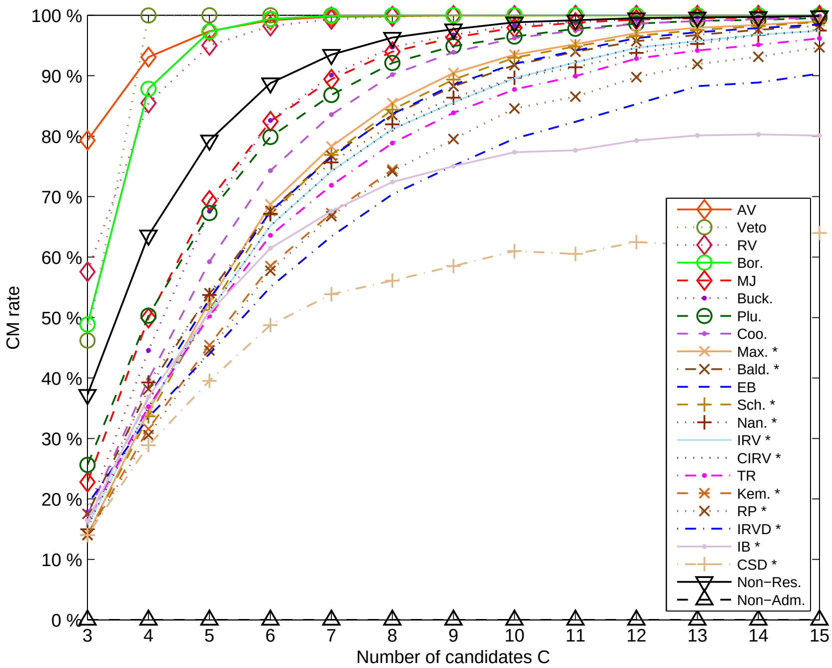


Figure 8.1 – CM rate as a function of the number of candidates C. Gaussian well, $V = 33$, $n = 1$.

In Figure 8.1, we plot the CM rate as a function of the number of candidates C in a one-dimensional Gaussian well.

The first conclusion from Figure 8.1 is that the manipulability rates seem to tend to 1, just like in spherical culture. For CSD and IB, it is difficult to decide on the possible limit, since we only have lower bounds of manipulability which do not obviously tend to 1. In general, the convergences to a manipulability rate equal to 1 seem slower than in spherical culture. In the particular case of Veto, the manipulability rate also seems to tend to 1 but we can again prove that it is false (with the lexicographic tie-breaking rule): indeed, we can use the same reasoning as in Proposition 7.5 because the restriction of a Gaussian well culture to candidates $1, \ldots, V+1$ is also a one-dimensional Gaussian well culture.

We observe that the compared performances of the different voting systems are not the same as those discussed in Chapter 7. In particular, now it is no longer clear that CIRV, IRV, and EB have lower manipulability rates than the

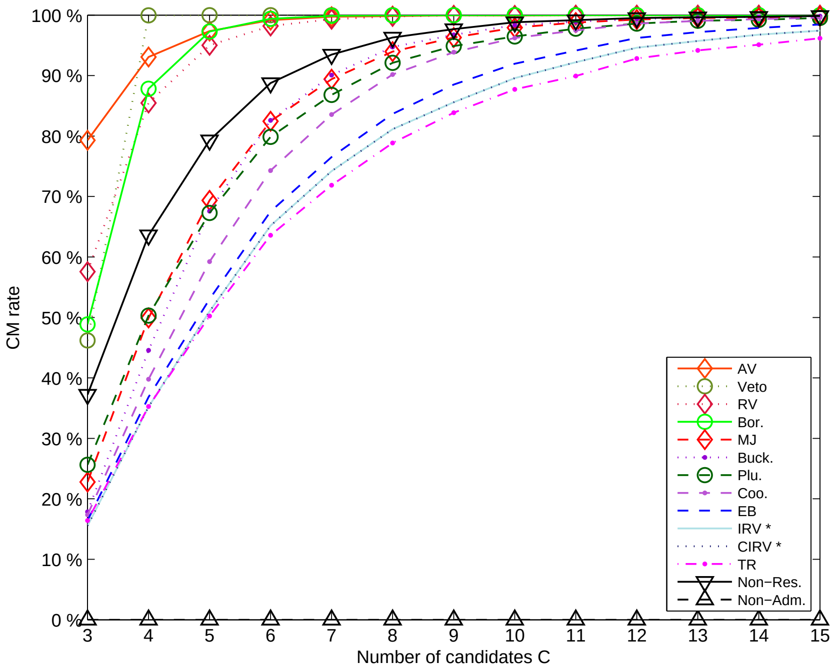


Figure 8.2 – CM rate as a function of the number of candidates C, for a selection of voting systems. Gaussian well, $V = 33$, $n = 1$.

other voting systems. In order to confirm this observation, Figure 8.2 presents the same curves, with a focus on these three voting systems and those for which the algorithmic uncertainty is less than 1 % for each point (which includes Borda's method, although the algorithm used is not exact).

This time, the margins of uncertainty for CIRV and IRV are sometimes greater than 1 % because we used the option `fast` of SWAMP when the number of candidates C is greater than or equal to 9 (hence the star indicated in the legend of Figure 8.2). However, the exact algorithm was used for EB, so the curve for EB provides an upper bound for those of CIRV and IRV.

In Figure 8.2, we observe that the Two-round system is (slightly) better than CIRV, IRV, and EB, contrary to what happens in spherical or VMF cultures. A qualitative explanation can be proposed. In IRV, once far-left and far-right candidates are eliminated, their supporters vote for moderate left and right candidates, and the centrist risks being eliminated because she receives only few transferred votes. Since IRV meets **InfMC**, this implies that the configuration is manipulable. In contrast, in the Two-round system, moderate left and right candidates do not benefit from transferred votes at the time when the selection for the final round (i.e. second round) is made, so the centrist has more chances of reaching the second round.

It should however be noted in Figure 8.2 that CIRV, IRV, and EB remain less manipulable than Coombs' method, itself less manipulable than Plurality, Bucklin's method, and Majority judgment, themselves even less manipulable than Borda's method, Range voting, Veto, and Approval voting. Thus, the hierarchy observed previously for these voting systems seems to be confirmed, except for Coombs' method, which seems to behave better here than in spheroidal cultures.

This could be linked to the fact that in a single-peaked culture, Coombs' method meets the Condorcet criterion (Grofman and Feld, 2004).

For the comparison between CIRV, IRV, EB, and the other voting systems, the question remains open. In future works, it would be interesting to develop more efficient algorithms for these voting systems, in order to establish whether they are less manipulable than CIRV, IRV, and EB in such a culture. In the meta-analyses of Sections 8.6 and 8.7, using TM will allow us to say more about this comparison.

8.2 One-dimensional Gaussian well: number of voters V

8.2.1 Number of voters V odd

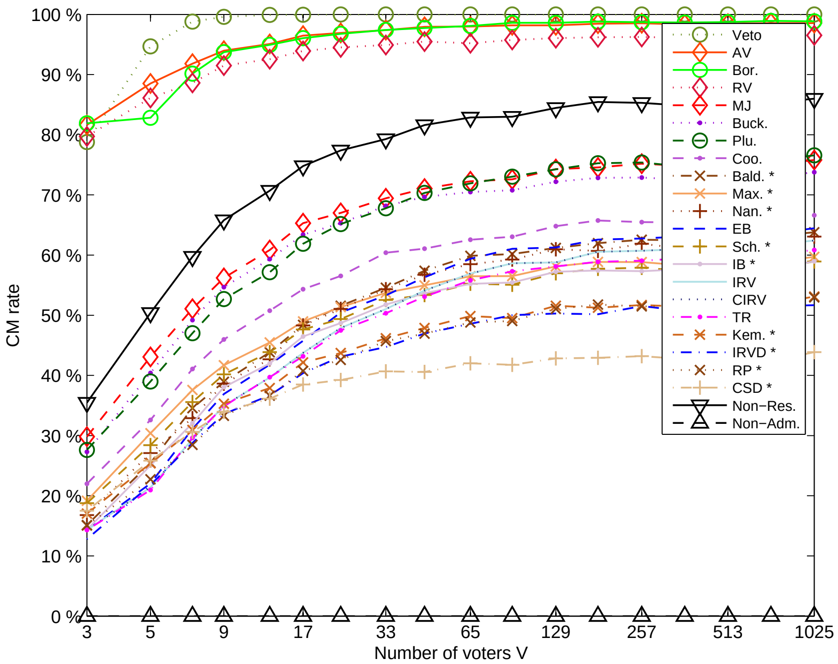


Figure 8.3 – CM rate as a function of the number of voters V. Gaussian well, $C = 5$, $n = 1$, large odd values of V.

Similarly to we did in spherical culture, Figure 8.3 represents the CM rate as a function of the number of voters V for large odd values, still for a one-dimensional Gaussian well. As in spherical culture, manipulability rates seem to be increasing functions of the number of voters V (on the set of odd values). However, they do not seem to tend to 1 for most voting systems, which leads us to the following conjecture.

Conjecture 8.1

In one-dimensional Gaussian well, for $C \geq 3$ and $V \to +\infty$:

- *The CM rate tends to 1 for Veto,*

- *But it tends to a limit strictly between 0 and 1 for all the other voting systems studied here.*

8.2.2 Parity of V

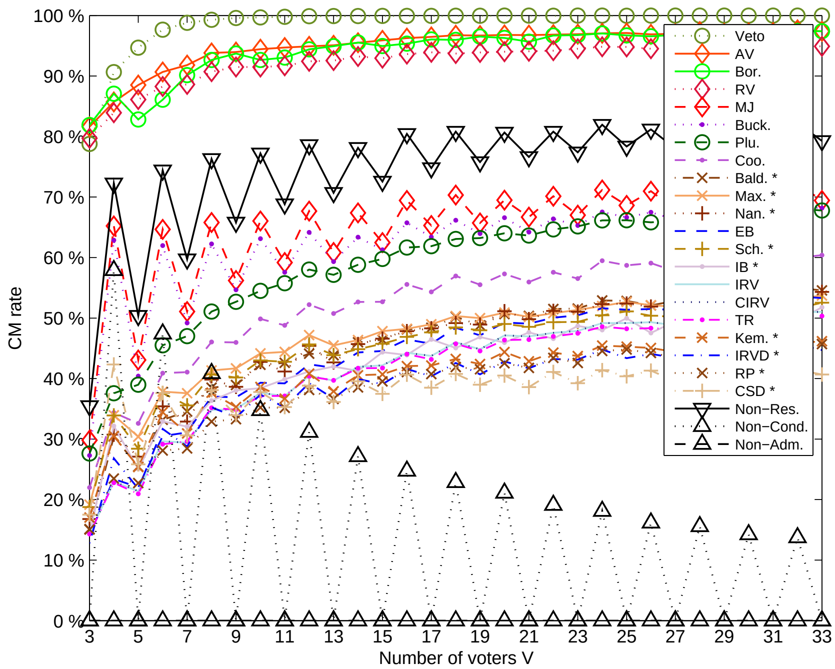


Figure 8.4 – CM rate as a function of the number of voters V. Gaussian well, $C = 5$, $n = 1$, small values of V.

In spherical culture, we have seen in Figure 7.5 that, for certain voting systems, the manipulability rate presents oscillations depending on the parity of V (and sometimes more complicated non-monotonicity phenomena). In order to examine this kind of phenomenon in a one-dimensional Gaussian well, Figure 8.4 represents the CM rate as a function of the number of voters V for all values between 3 and 33. The oscillatory phenomenon is much more general here than in spherical culture: it affects all the voting systems studied apart from Veto, Approval voting, Borda's method, and Range voting.

It can be explained by the fact that, contrary to the spherical case, there are essentially no competing effects anymore: the probabilities of existence of a Condorcet winner or of a resistant Condorcet winner, exhibiting oscillations which are in phase, make odd values of V less prone to manipulation. In particular, for V odd, recall that there is always a Condorcet winner. For Majority judgment and Bucklin's method, their specific mechanisms generate an effect of parity that goes in the same direction, as we explained in Section 7.4.2. As for the probability of existence of a Condorcet-admissible candidate, it is constant (and equal to 1), contrary to the spherical case where its oscillations, in phase opposition with respect to the other effects, were partially competing with them.

In single-peaked culture, we can also notice that the notion of resistant Condorcet winner is equivalent to that of majority favorite. Indeed, if she is not an extreme candidate, a majority of voters must simultaneously prefer her to the two candidates immediately to her left and to her right, hence to all the other candidates. If she is an extreme-left candidate (for example), the mere fact of being the Condorcet winner requires that a majority of voters prefer her to the candidate immediately to her right, hence to all other candidates.

For V even, we also observe that the probability of existence of a Condorcet winner increases with V, which is easy to explain: with a constant number of candidates, the more voters there are, the less probable it is that the boundary between two most-liked candidates is located precisely between the two median voters. Generally and not surprisingly, parity effects decrease when the number of voters increases, as in spherical culture.

8.3 One-dimensional Gaussian well: shift y_0

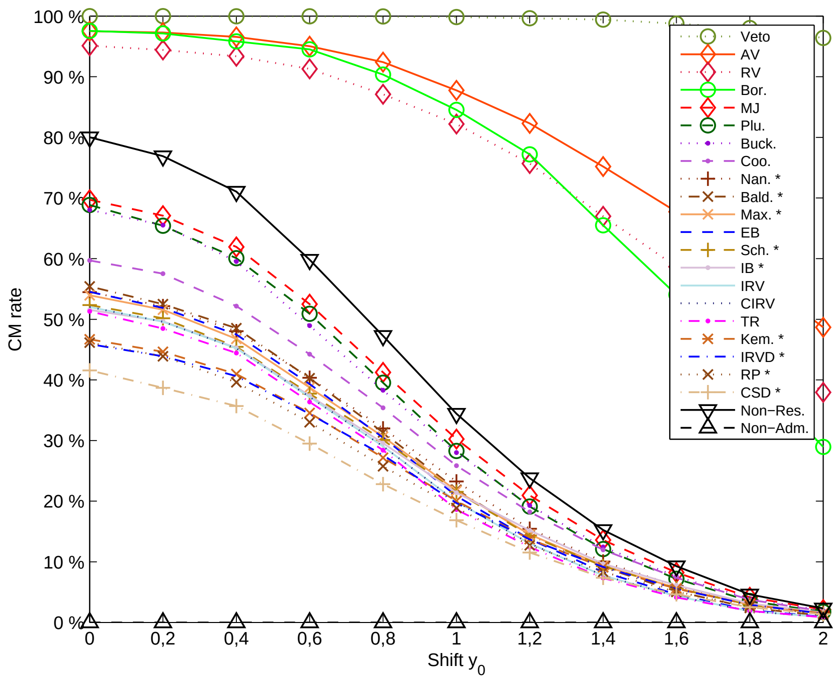


Figure 8.5 – CM rate as a function of the shift between voters and candidates. Gaussian well, $V = 33$, $C = 5$, $n = 1$, $\sigma = 1$.

In Figure 8.5, we also consider a one-dimensional Gaussian well of characteristic length $\sigma = 1$. But we add a shift y_0 between the distribution of candidates and the one of voters: the normal distribution is used for candidates is no longer centered in 0 but in y_0. When this shift tends to $+\infty$, each candidate is to the right of each voter. In particular, the leftmost candidate is preferred by all voters.[2] Such a situation is not very realistic because, in this case, some candidates would have a strategic interest in moving their political offer to attract more voters. While being aware of this limitation, we study this case to explore qualitatively the influence of the different parameters of the model.

We explore what happens for a shift varying from 0 to 2. As expected, voting systems tend to be less manipulable when the shift increases, i.e. when voters agree more.

What is interesting is the similarity with what we observed in Figure 7.6, describing a VMF culture where the concentration κ increases. In particular,

[2]There is no hidden message in this paragraph. We could have taken the opposite convention, but we had to choose.

some voting systems are significantly less reactive to the increasing agreement between voters, in the sense that their manipulability rates decrease more slowly: Approval voting, Range voting, Borda's method, and especially Veto. It is the same list of voting systems as for Figure 7.6, except for Coombs' method, which behaves better in Gaussian well.

From a theoretical point of view, the arguments set out in Section 7.5 remain valid: as the culture tends to perfect agreement between voters, the manipulability rate of each unanimous voting system tends to 0 and the manipulability rate of Veto tends (slowly) to $1 - \frac{1}{C-1}$.

8.4 Multidimensional Gaussian well: number of dimensions n

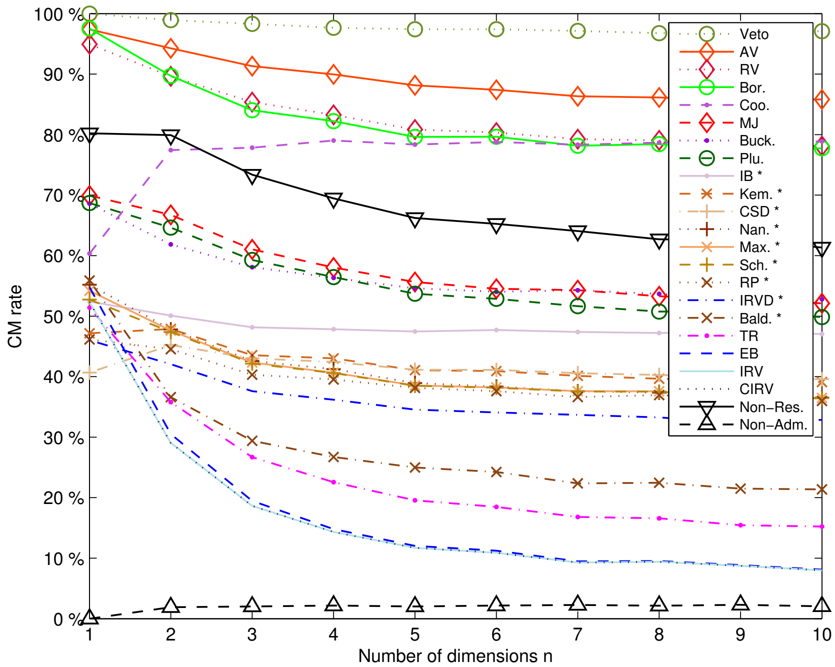


Figure 8.6 – CM rate as a function of the number of dimensions n. Gaussian well, $V = 33$, $C = 5$, $\sigma = (1, \ldots, 1)$.

So far, we have considered one-dimensional political spectra. In Figure 8.6, we generalize this model by considering a multidimensional Gaussian well, with dimension n varying from 1 to 10.

At first glance, one might think that if n grows, then the culture tends to a spherical culture, as for a VMF culture with an infinite number of poles (Section 7.7). If this were true, then the CM rates should increase. However, such reasoning would be wrong. Indeed, even with a multidimensional political spectrum, a candidate whose position is close to the origin is more likely to be preferred by the voters, hence the latter are not independent, even in the limit $n \to +\infty$: this is enough to prove that the limit culture is not the spherical culture.

As a matter of fact, we observe in Figure 8.6 that when n increases, the CM rate decreases for most voting systems (except for Coombs' method, as well as IB whose variation is small).

Starting from $n = 2$, CIRV, IRV, and EB become less manipulable than the other voting systems. Therefore, it seems that the (relatively) underperformance of these three voting systems is deeply linked to having a one-dimensional political spectrum.

They may even exhibit performance that is better than that observed in the spheroidal cultures of Chapter 7. By way of comparison, in the reference scenario (Figure 7.1), the manipulability rates of these three voting systems were 44 %. With the same numbers of voters and candidates, in Gaussian well with $n = 10$, the rates found are about 8 %.

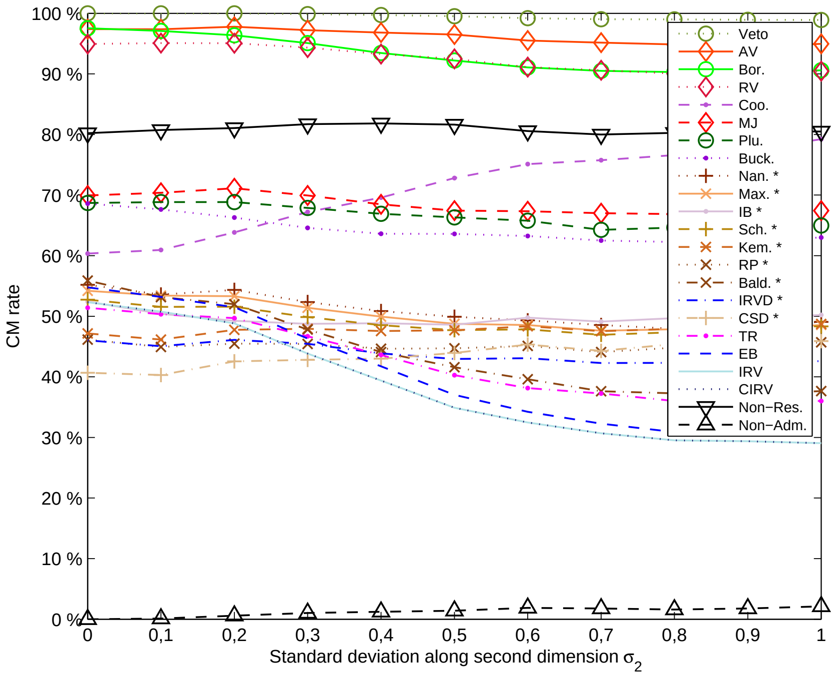


Figure 8.7 – CM rate as a function of σ_2. Gaussian well, $V = 33$, $C = 5$, $\sigma_1 = 1$.

In order to complete the previous observation, we explore in Figure 8.7 the transition from a one-dimensional culture to a two-dimensional culture. For this, we consider a two-dimensional Gaussian well with characteristic lengths $(1, \sigma_2)$. For $\sigma_2 = 0$, the culture is one-dimensional. For $\sigma_2 = 1$, we have a "square" culture: the two dimensions are equally important.

It is useless to continue the figure for $\sigma_2 \in [1, +\infty]$ because we would get the exact symmetry of the curves presented. Indeed, up to changing to unit of length and to inverting the axes of the political spectrum, the culture $\sigma = (1, \sigma_2)$ is equivalent to the culture $\sigma' = (1, \frac{1}{\sigma_2})$.

These curves seem to naturally interpolate what happens between dimensions 1 and 2 in Figure 8.6: manipulability rates decrease. Like in Figure 8.6, only that of Coombs' method is clearly increasing. For $\sigma_2 \geq 0.4$ approximately, CIRV, IRV, and EB become less manipulable than the other voting systems, as in spheroidal cultures. Similarly, for $\sigma_2 \geq 0.5$ approximately, the Two-round system has an intermediate manipulability between these three voting systems and the others.

8.5 Comparison with a Euclidean box

One can wonder if the phenomena observed in Gaussian wells are a qualitative consequence of a culture based on a one-dimensional political spectrum or if they depend on the Gaussian distribution used for candidates and voters.

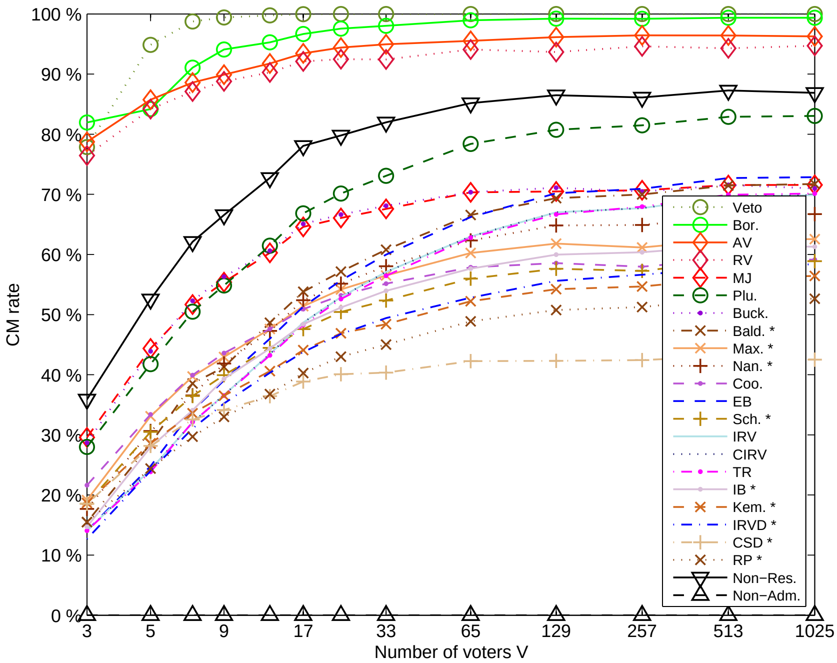


Figure 8.8 – CM rate as a function of the number of voters V. Euclidean box, $C = 5$, $n = 1$, large odd values of V.

As an example, Figure 8.8 presents the manipulability rates in a one-dimensional Euclidean box, for odd values of the number of voters: instead of using a normal distribution, the positions of the voters and candidates are drawn uniformly in a segment $[-1, 1]$.

There is strong similarity with Figure 8.3, which seems to indicate that our findings have certain general validity for cultures based on a one-dimensional political spectrum. The main difference concerns the relative performances of TR and IRV (or CIRV), which are even closer than in Gaussian well.

8.6 Meta-analysis in one-dimensional culture

8.6.1 Meta-analysis of CM in one-dimensional culture

Similarly to Figure 7.11 for spheroidal cultures, Figure 8.9 presents the comparison of CM between voting systems in a culture based on a one-dimensional political spectrum, whether in Gaussian well or in Euclidean box. For the moment, we focus on this case because we saw in Section 8.4 that in multidimensional culture, the behaviors observed are relatively similar to those observed in Chapter 7. In Section 8.7, we will synthesize all the cultures studied, including those from the previous chapter.

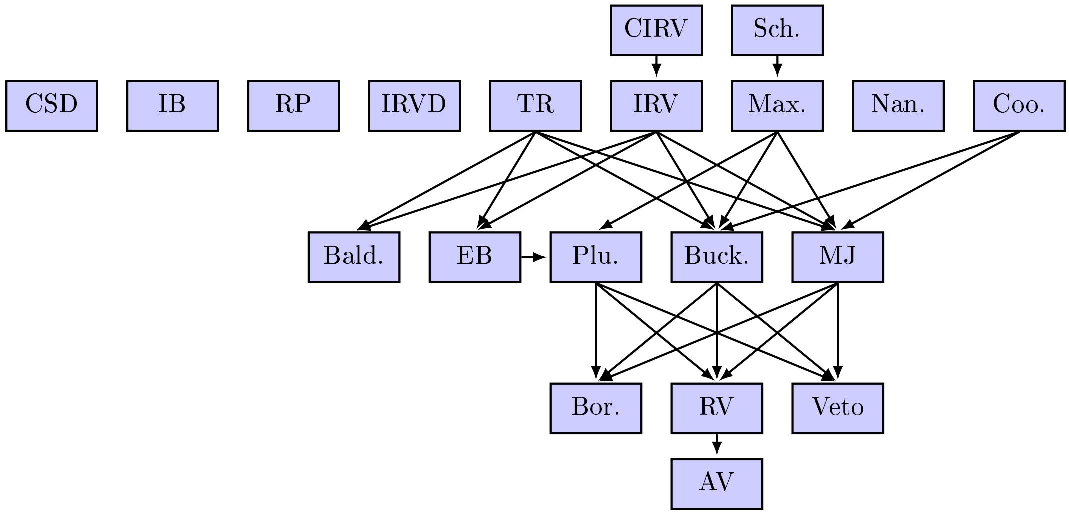


Figure 8.9 – Meta-analysis of CM in all one-dimensional cultures under study.

On the graph of Figure 8.9, we see that in one-dimensional culture, the Two-round system becomes competitive with CIRV and IRV, and even dominates EB in the cultures we studied. With regard to CSD, IB, RP, IRVD, Schulze, and Nanson, the approximate algorithms do not always make it possible to conclude but these voting systems present promising results. Again, Schulze's method and Maximin show very similar performances (the gap is at most 2 %). Baldwin's method is more manipulable than the Two-round system.

The Two-round system, IRV, Exhaustive ballot (itself more manipulable than the two previous ones), Maximin (itself more manipulable than Schulze's method), and Coombs' method serve as a manipulability reference to compare to the following two groups, in increasing order of manipulability:

- Plurality, Bucklin's method, and Majority judgment;
- Borda's method, Range voting, Veto, and Approval voting (itself more manipulable than Range voting).

Like in spheroidal culture, these last four voting systems often have a higher manipulability rate than the proportion of non-resistant configurations, which then makes them more manipulable than any Condorcet voting system. Coombs' method, which seems to behave better in a one-dimensional political spectrum than in spheroidal culture, no longer has this type of bad behavior. This is *a priori* not obvious: even if we recalled that it satisfies the Condorcet criterion for single-peaked preferences, in particular those obtained by sincere voting in this model, it does not necessarily verify it for configurations obtained by manipulation.

In general, some Condorcet voting systems, in particular Schulze's method for which the algorithmic uncertainty is limited, seem to behave better in a one-dimensional political spectrum than in a spherical culture. Intuitively, we can understand why: in spherical culture, the probability of existence of a Condorcet-admissible candidate is relatively low and we know that it even tends to 0 for V odd and $C \to \infty$ (Gehrlein, 2006). On the other hand, in a single-peaked culture and in particular in a one-dimensional Gaussian well, this probability (equal to that of existence of a Condorcet winner for V odd) is equal to 1. A non-Condorcet

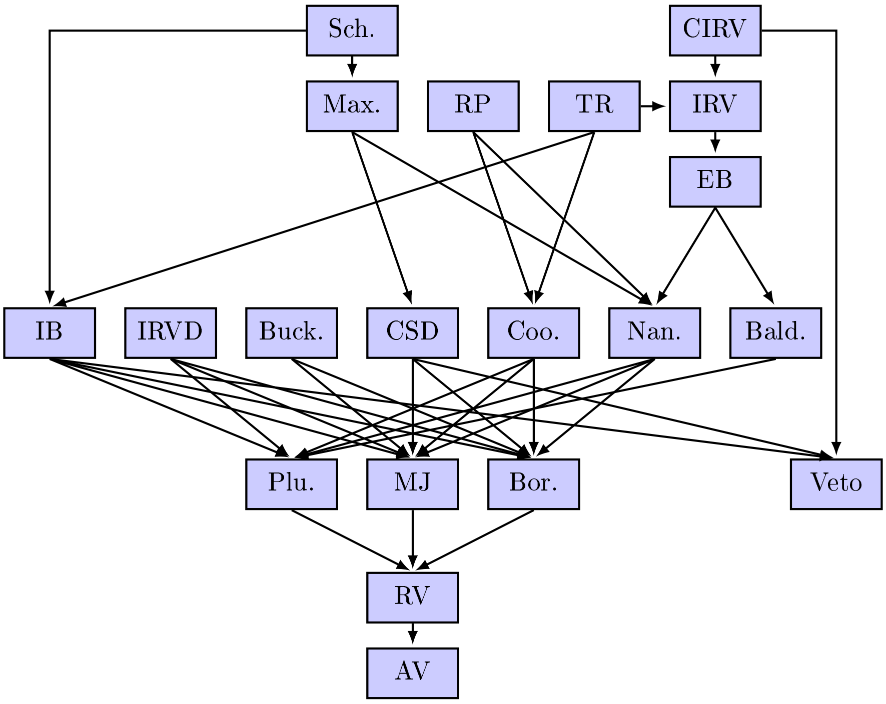


Figure 8.10 – Meta-analysis of TM in all one-dimensional cultures under study.

voting system therefore comes with a major handicap: for V odd, each time the winner is a non-Condorcet candidate, the configuration is manipulable.

8.6.2 Meta-analysis of TM in one-dimensional culture

As we did for spheroidal cultures, we present in Figure 8.10 a similar graph for TM. Since there is, then, no algorithmic uncertainty, we can conclude in all cases. Thus, if there is no edge between two voting systems (explicit or implied by transitivity), it means that one is sometimes less TM than the other and that it is sometimes the opposite.

Voting systems with no incoming edge are the best performers from a TM perspective: Schulze's method (with performance similar to Maximin, albeit slightly better), RP, the Two-round system, CIRV, IRVD, and Bucklin's method.

Voting systems with the worst performances are Plurality, Majority judgment, and Borda's method, themselves less TM than Range voting, itself less TM than Approval voting.

Veto is again a special case: it is dominated by certain voting systems, such as Schulze's method, the Two-round system, or CIRV, but it is not comparable to many others.

8.7 Meta-analysis across all cultures studied

8.7.1 Meta-analysis of CM across all cultures studied

In order to synthesize this chapter and the previous one, we can wonder if certain voting systems are less manipulable than others in all cultures studied

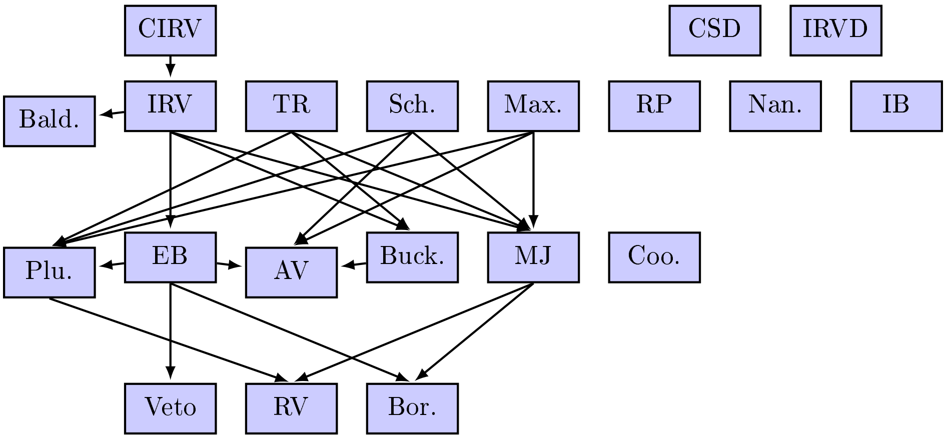


Figure 8.11 – Meta-analysis of CM across all cultures from Chapters 7 and 8 (except spherical culture with $V = 3$).

in Chapters 7 and 8. This is the object of the graph in Figure 8.11, which uses all the previous curves, except the spherical culture with $V = 3$, for the reasons mentioned in Section 7.9.

Voting systems with no incoming edge are CIRV, the Two-round system, Schulze's method and Maximin (which still show very similar performance), Coombs, CSD, IRVD, RP, Nanson's method, and IB. Regarding the latter five, the approximate algorithm essentially provides a lower bound of manipulability, which makes them essentially incomparable across the set of experiments. As usual, we will complete these results with a meta-analysis of TM.

On the contrary, we have repeatedly noticed the very poor performance of Borda's method, Range voting, and Approval voting, whose manipulability rates are often higher than the proportion of non-resistant configurations. In the cases concerned, this means that these voting systems are more manipulable than any voting system meeting the Condorcet criterion.

Among the voting systems that are widespread in practice, it is worth noting that Plurality is dominated by voting systems such as CIRV, IRV, EB, the Two-round system, Schulze's method, or Maximin. From the point of view of manipulability, this seems to plead for a limited use of Plurality in actual elections and its replacement by one of these voting systems.

8.7.2 Meta-analysis of TM across all cultures studied

Similarly, Figure 8.12 makes it possible to compare TM for the different voting systems in all the cultures from this chapter and the previous one (except the spherical culture with $V = 3$). An edge from a voting system to another therefore means that the first is better from the point of view of TM. Once again, we see that the voting systems with the best performance are CIRV, the Two-round system, Schulze's method, Maximin, and Bucklin's method. IRVD has no incoming edge either, but its difference of manipulability with the Two-round system is never favorable by a large amplitude in the cultures studied (it is always less than 2 %), whereas we have seen that TR is sometimes much less TM than IRVD.

All other voting systems, except Veto, are dominated by at least one of these voting systems. Veto constitutes a connected component on its own, confirming

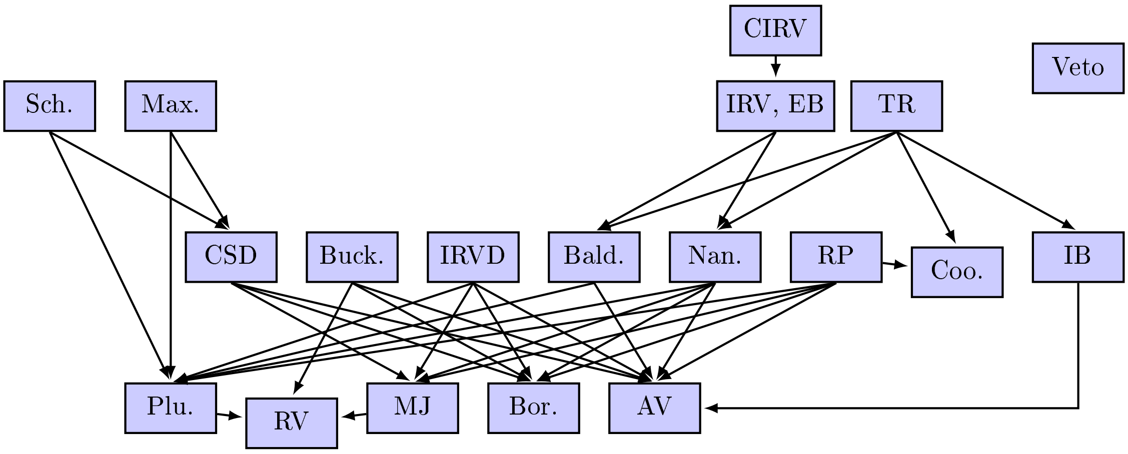


Figure 8.12 – Meta-analysis of TM across all cultures from Chapters 7 and 8 (except spherical culture with $V = 3$).

its non-comparable character with the other voting systems studied, amplified by its resistance to trivial manipulation.

At last, note the mediocre performance of the following voting systems: Coombs' method, Plurality, Range voting, Majority judgment, Borda's method, and Approval voting.

However, as we discussed in the introduction of this memoir, Approval voting has the advantage of proposing a fairly natural strategy in practice, the *Leader rule* (Laslier, 2009), which makes it possible to reach equilibria with limited exchange of information. Conversely, the complexity to establish a strategic ballot in IRV seems, experimentally, to discourage voters from undertaking such a computation and to incite them to vote sincerely (Van der Straeten et al., 2010), which can be seen as an advantage or a disadvantage, depending on the point of view.

Chapter 9

Analysis of Experimental Data

In previous chapters, we have studied the manipulability of various voting systems in artificial culture, either spheroidal (Chapter 7) or based on a political spectrum (Chapter 8).

An important conclusion of these chapters was the low manipulability of CIRV, IRV, and EB, except in cultures based on a one-dimensional political spectrum, where some other voting systems, such as the Two-round system and Schulze's method, showed interesting performances. It is therefore natural to wonder what happens in the real world and this is the subject of this chapter.

We base this study on a corpus of 168 experiments from different contexts. In order to be able to use the exact algorithm for IRV while keeping a reasonable computation time, we limit ourselves to elections with 3 to 14 candidates. Kemeny's method will be excluded from the analyses of this chapter due to the computation time needed to determine the winner.

9.1 Presentation of the experiments

9.1.1 Realized experiments

The three following datasets were obtained using relatively similar methods, with our direct participation in establishing the modalities of the election and collecting the ballots.

Algotel During the Algotel 2012 conference,[1] the program committee pre-selected 5 papers, named here arbitrarily A, B, C, D, and E, for the election of the best conference paper. To decide between them, each participant was asked to attribute a grade between 0 and 10 to each paper. It was possible to grade only part of the papers, the absence of grade being counted as 0. Out of 72 participants of the conference, there were 57 ballots cast, 1 blank vote, and 2 invalid ballots.

Participants had been warned that their ballots would be tested on several voting systems, but they did not know which one would ultimately be used to designate the winning paper. In the event of a disagreement between the voting systems, we would have had the possibility to reward all the papers designated by at least one tested voting system. However, in practice, we will see that all the voting systems designated the same winner, so the question did not arise.

[1] http://algotel2012.ens-lyon.fr/

Grade	French appreciation	English translation
5 stars	Culte!	Awesome!
4 stars	Franchement bien	Really good
3 stars	Pas mal	Quite good
2 stars	Bof, sans plus	Blah, neither good nor bad
1 star	Vraiment pas aimé!	I really disliked it!

Table 9.1 – Scale of grades and appreciations of the website www.bdtheque.com.

The conclusions of this experiment, which are partially exposed in this chapter, are also available in the article by Durand et al. (2014a).

Bordeaux During the day of the Doctoral School of Mathematics and Computer Science of Bordeaux in November 2014,[2] a vote was organized to elect the best poster among 11 posters proposed by doctoral students in their final year. The modalities were similar to Algotel 2012, but the authorized grades ranged from 0 to 20. There were 86 ballots cast for as many participants.

Paris VII In April 2015, an internal poll was organized to guide the choice of the new name for the computer science research department at the University Paris VII–Paris Diderot.[3] 10 possible names were proposed. This time, the voters did not assign grades but appreciations: *Good*, *Quite good*, *Neither good nor bad*, *Quite bad*, or *Bad*. There were 95 ballots cast for as many participants.

We thank the organizers of these events for making this experiments possible.

9.1.2 Website www.bdtheque.com

The website www.bdtheque.com is a collaborative site dedicated to comics. Users are invited to rate series of comics according to the scale presented in Table 9.1, which is both a scale of grades and a scale of appreciations (most of which are idiomatic French expressions that are difficult to translate accurately into English).

In June 2012, the site's webmaster, Alix Bergeret, whom we thank, was kind enough to send us the site's database, on the condition that we preserve the anonymity of the users.

We drew 12 experiments from it as follows. For each integer $C \in [\![3, 14]\!]$, we wanted to choose C candidate series and select the intersection of the users who rated them all. In addition, our goal was to have a significant number of voters or even to maximize it if possible. This problem, known as the *Maximum Subset Intersection Problem*, was recently proven $\mathcal{NP}$-complete by Xavier (2012).

As an approximation, we used a greedy algorithm. We start from $C = 0$ (i.e. $\mathcal{C} = \varnothing$) and we initialize the set $\mathcal{V}$ to the set of all users. At each increment of C, we select the series with the most grades among the users still present in $\mathcal{V}$ and we add it to the set of candidates $\mathcal{C}$, then we eliminate from $\mathcal{V}$ the users who did

[2] http://www.math.u-bordeaux1.fr/ED/ecole_doctorale/
[3] http://www.univ-paris-diderot.fr/

C	3	4	5	6	7	8	9	10	11	12	13	14
V	33	24	21	19	18	15	14	14	13	13	12	12

Table 9.2 – Bdtheque: number of candidates and voters.

not rate the last series added. Table 9.2 shows the correspondence between the number of candidate series and the number of users obtained by this algorithm.[4]

The particularity of this dataset is that when assigning each grade, the user is not placed in an election context: indeed, the purpose of the operation is not to collectively select a single option among a set of candidate options, as it is the case when it comes to awarding a prize or electing a person to a position. Moreover, this evaluation has no stake or practical consequence on the lives of the voters, unlike a political election for example. Finally, there is a way to limit and to some extent control strategic voting: indeed, the site's policy requires that any grade be accompanied by a sufficiently substantial comment to justify it. If a user wishes to assign an artificially low or high grade to a series for purely strategic reasons, she must therefore pay a cognitive cost and a time cost to do so. For all these reasons, we can hope — but we can only hope — that this dataset is relatively untainted by strategic voting.

9.1.3 Judgment of Paris

In 1976, 11 experts (9 French and 2 American) met for two blind wine tastings: first, 10 white wines made with Chardonnay grape, then 10 red wines based on Cabernet Sauvignon. Each of these two tastings led to a rating by the experts and a prize list.

Grades ranged from 0 to 20, with half-points allowed. The experts knew in advance that Range voting would be used, which is a difference with the experiments we mentioned above. Indeed, for Algotel, Bordeaux, and Paris VII, voters had been informed that their ballots "would be tested on several voting systems", without further details. And for bdtheque, as we said, it is not really an election situation.

To everyone's surprise, in each of the two categories, it was a Californian wine that won first place, although the jury was mostly French. This victory led to many comments in the press on the terms of the vote and had a significant impact on the reputation and development of American wines.

The dataset we use for the Judgment of Paris comes from the page `www.liquidasset.com/lindley.htm`.

9.1.4 PrefLib dataset

PrefLib (`http://www.preflib.org/`) is a collaborative database that collects datasets of collective preferences in order to make them available to social choice specialists (Mattei and Walsh, 2013). Unlike the experiments presented above, it contains exclusively ordinal data: it can be weak orders (with ties) or strict

[4] For the curious reader: the series obtained are, in order, *XIII*, *Lanfeust of Troy*, *Blacksad*, *The Quest for the Time-Bird*, *Universal War One*, *The Third Testament*, *Largo Winch*, *De Cape et de Crocs*, *The adventures of Tintin*, *Thorgal*, *Asterix*, *Peter Pan* (by Loisel), *Wake*, and *Lanfeust of the Stars*.

orders, complete or incomplete. When the orders are incomplete, we assume that all the unranked candidates are placed behind all the ranked candidates.[5]

For the present study, we consider *a priori* all election files from PrefLib, with the following exceptions.

- In order to be able to use the exact algorithm for IRV while keeping a reasonable computation time, we exclude elections with strictly more than 14 candidates.
- We exclude elections whose preferences derive from cardinal preferences (grades or approval values), since this cardinal information is not provided in PrefLib. Indeed, we would need to arbitrarily extrapolate grades from ordinal preferences, as we will see in Section 9.2, and these grades would *a priori* not conform to the original experiment.
- We exclude elections obtained by random sampling from large datasets (PrefLib 4, 11, and 15), considering that the particular realization of such a random drawing is specific to its authors.

In PrefLib, files are organized into datasets that group elections held in a similar context and coming from the same source. For example, the dataset PrefLib 1 gathers political elections held in Dublin in 2002.

In this study, we use 151 experiments from the political field (PrefLib 1, 5, 8, 16 to 23), the professional or associative world (PrefLib 2, 7, 9, and 12), or cognition experiments (PrefLib 24 and 25). PrefLib 1 is a donation from Jeffrey O'Neill, who administrates the website `http://www.openstv.org/`. PrefLib 5, 8, and 16 to 23 come from this same site. PrefLib 2 comes from `http://www.debian.org/vote/`. PrefLib 7, 9, and 12 are respectively donations from Nicolaus Tideman, Piotr Faliszewski, and Carleton Coffrin. PrefLib 24 and 25 come from Mao et al. (2013). For more information on these datasets, we encourage the reader to consult the website `http://www.preflib.org/`.

Table 9.3 presents an overview of all the experiments used in this chapter. In total, we base ourselves on 168 experiments, from 10 to nearly 300,000 voters and from 3 to 14 candidates.

9.2 Methodology

Consider a given experiment, for example, the election of the best Algotel paper. As illustrated in Figure 9.1, compared to the raw ballots, we add to the grades a uniform random noise, the amplitude of which is negligible compared to the differences between grades. Once this noise is added, we obtain a configuration for the entire population, which we call a *realization*. Thus, if a voter was putting a candidate strictly before another one in her original ballot (for example D before E in Figure 9.1), this order is still respected because the amplitude of noise is negligible. On the other hand, if a voter was putting several candidates in a tie (for example A and B in Figure 9.1), these are placed in a random order after the addition of noise.

[5]When only incomplete preference are available, another possibility, studied by Konczak and Lang (2005), consists of considering the set of all possible completions to complete orders. We can then determine all possible Condorcet winners, all possible winners for a given voting system, and so on. However, the authors show that it leads to difficult problems of algorithmic complexity, with however important exceptions such as the determination of the possible winners of a PSR or the possible Condorcet winners.

Data	Experiments	V	C	Ballots
Algotel	1	57	5	Grades 0–10
Bordeaux	1	86	11	Grades 0–20
Paris VII	1	95	10	Appreciations
Bdtheque	12	12–33	3–14	Grades 1–5
Judgment of Paris	2	11	10	Grades 0–20 *
PrefLib 1: Dublin	3	29,988–64,081	9–14	SOI
PrefLib 2: Debian	8	143–504	4–9	SOI
PrefLib 5: Burlington	2	8,980–9,788	6	TOI
PrefLib 7: ERS	75	32–3,419	3–14	SOI
PrefLib 8: Glasgow	21	5,199–12,744	8–13	SOI
PrefLib 9: AGH Course	2	146–153	7–9	SOC
PrefLib 12: T-shirt	1	30	11	SOC
PrefLib 16: Aspen	2	2,487–2,528	5–11	TOI
PrefLib 17: Berkley	1	4,189	4	TOI
PrefLib 18: Minneapolis	2	32,086–36,655	7–9	SOI
PrefLib 19: Oakland	7	11,358–145,443	4–11	TOI
PrefLib 20: Pierce	4	40,031–299,664	4–7	TOI
PrefLib 21: San Francisco	11	24,180–184,046	4–10	TOI
PrefLib 22: San Leandro	3	22,539–25,564	4–7	TOI
PrefLib 23: Takoma	1	204	4	TOI
PrefLib 24: MT Dots	4	794–800	4	SOC
PrefLib 25: MT Puzzle	4	793–797	4	SOC
Total	168	10–299,664	3–14	

Table 9.3 – Summary of analyzed experiments. SOC: strict orders, complete list. SOI: strict orders, incomplete list. TOI: orders with ties, incomplete list. * Half-points allowed.

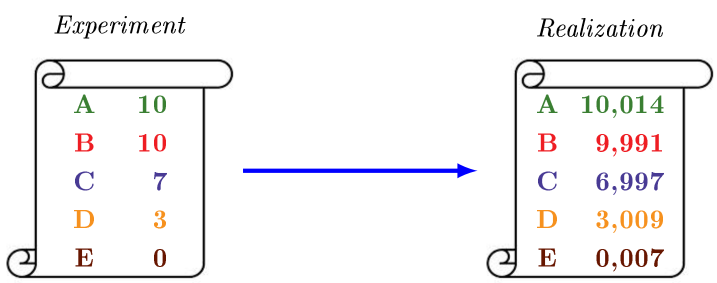


Figure 9.1 – Addition of random noise to the ballots.

For each experiment, we draw several random realizations. The objective is twofold: evaluate the consistency of our results on a space made richer than the single original experiment, and break ties to simplify the analysis of voting systems based on orders of preference. So, when we mention, for example, the CM rate of Plurality for the Algotel experiment, we are actually talking about the CM rate of Plurality for a culture that consists of drawing a random configuration in a small neighborhood of this experiment.

For the purely ordinal data from each PrefLib experiment, we perform an additional pre-processing step: first, ordinal preferences are converted to cardinal preferences using Borda scores. Note that this particular choice only matters for Approval voting and Range voting. Indeed, for ordinal voting systems, our technique is equivalent anyway to keeping the strict preferences of the voters and break their indifferences in an impartial manner, i.e. symmetrically with respect to the candidates. As for Majority judgment, only the topological order on the space of grades or appreciations counts; numerical values have no impact.

For the experiment Paris VII, a pre-processing step is also added. Each of the five possible appreciations is converted into an integer value between -2 and $+2$. Again, this particular choice only matters for Approval voting and Range voting.

In this chapter, we work at two levels:

- We use the election of the best Algotel paper as a *recurring example*. When we analyze this example, we use 10,000 realizations.

- We also conduct *meta-analyses* on all experiments: in that case, we rely on 100 realizations for each of the 168 experiments.

In both cases, the randomness of the noise induces a statistical uncertainty of the order of 1 % on the measured rates ($\sqrt{1/10,000}$ or $\sqrt{1/16,800}$).

For the meta-analyses, we will give, on the one hand, histograms representing the average rates over the 168 experiments carried out. These results should not be interpreted as precise quantitative conclusions but rather as qualitative indications: indeed, giving the same weight to the 168 experiments considered is an arbitrary choice in itself. In particular, it gives a relatively large weight (45 %) to the experiments of the PrefLib 7 dataset. Conversely, giving equal weight to each PrefLib dataset would lead to giving as much importance to datasets that contain only one experiment, such as PrefLib 12 or PrefLib 17, as to PrefLib 7, from which we analyze 75 elections. So there is no perfect solution.

On the other hand, we will establish graphs of meta-analysis (cf. Section 7.9). They will indicate that one voting system is less manipulable than another in all the experiments, which is independent of any weighting on them. We will then see that similar trends emerge across all the experiments.

Lastly, it is impossible to determine *a posteriori* if a ballots corresponds to a sincere opinion or to a more elaborate strategy. We are therefore obliged to assume that the participants were not too far from sincere voting and that ballots (with noise) are sincere. Without this assumption, it would be very difficult to quantitatively analyze the impact of manipulations. That said, as noted, some datasets are more likely to be exempt from strategic voting, in particular the bdtheque experiments.

	A	B	C	D	E	Borda score	Top of ballot
A	—	**35,5**	**36,0**	**36,5**	**39,0**	147,0	16,5
B	21,5	—	**29,5**	**33,0**	**33,0**	117,0	14,7
C	21,0	27,5	—	**32,0**	**31,5**	111,9	9,8
D	20,5	24,0	25,0	—	**29,5**	99,0	10,3
E	18,0	24,0	25,5	27,5	—	95,1	5,7

Table 9.4 – Algotel: average weighted majority matrix, Borda scores, and tops of ballots.

9.3 "Sincere" results and Condorcet notions

For the Algotel experiment, Table 9.4 gives the weighted majority matrix, the Borda scores, and the number of ballots where each candidate is ranked first, on average on the ballots with added noise. In the weighted majority matrix, victories are represented in bold. We observe, in particular, that A wins all its duels on average: it is therefore the average Condorcet winner. In fact, we even observe a stronger property: A is the Condorcet winner in all the realizations we tested.

As for the other candidates, A is followed by B which only loses against A, then C which only loses against the first two, and so on. We therefore have on average a *strict total Condorcet order*: $A \succ B \succ C \succ D \succ E$. In practice, this strict total order was observed in 99 % of the realizations.

So, it is tempting to say that A should get elected, and it happens in almost all the cases. The main exception is Plurality, which elects B in 18% of cases: since B is ranked first almost as many times as A (Table 9.4), random noise can change the result. Much more marginally, the other exceptions relate to Bucklin's method, the Two-round system, and IRV, which can elect B or C with very low probability (less than 1‰ of the tested realizations).

These results are relatively unexpected: since this was the first time we conducted an experiment of this type, we initially thought that there would be more diversity in the possible winners.[6] They show that if the ballots are sincere, A appears to be a canonical winner that attests a clear choice from the participants.

We can therefore wonder whether these phenomena are exceptional. To address this question, let us move on to meta-analysis: as a reminder, it involves studying the 168 experiments, each one with 100 realizations. As shown in Figure 9.2, there is, actually, almost always at least one Condorcet-admissible candidate (99 %) and it is very common to have a Condorcet winner (96 %). These results confirm and extend those of Tideman (2006), based on datasets which are now grouped in PrefLib 7 and which are therefore included in our own study.[7]

[6] For example, in impartial culture with 3 candidates, Merlin et al. (2000) show that the probability that a certain collection of classical voting systems would choose the same winner is about 50 %. Since the assumption $C = 3$ seem rather favorable from this point of view, the difference between this theoretical result and our observation shows to what extent the impartial culture must above all be considered as a worst case.

[7] The results of Tideman are obtained with a different methodology, without adding random noise. Our results, leading to the same qualitative conclusions, are therefore complementary. For comparison, if we exclude the data from PrefLib 7 in order to have an independent study, the existence rates obtained are similar: 98 % for a Condorcet-admissible candidate and 95 % for a Condorcet winner.

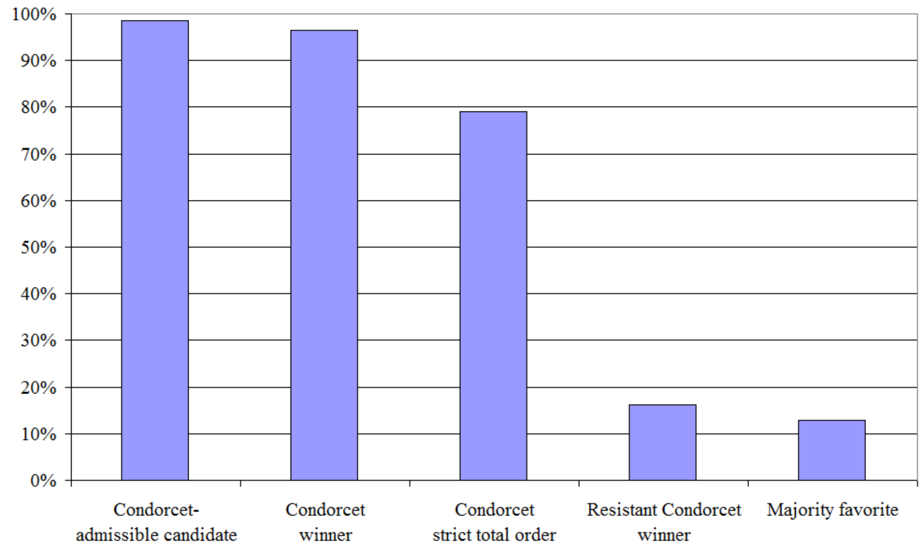


Figure 9.2 – Meta-analysis: rate of apparition for certain structures in the preferences of the population.

We have also computed the appearance rate of stronger phenomena (in the sense that each of them implies the existence of a Condorcet winner). In 79 % of the realizations, there is a strict total Condorcet order, which makes it a fairly frequent phenomenon despite the very structuring nature of this property for the population.[8] The rate of existence of a resistant Condorcet winner is not negligible (16 %), which ensures that in all the cases concerned, any Condorcet voting system is not manipulable. Finally, there is a majority favorite in 13 % of cases: most of the resistant Condorcet winners observed in practice are therefore majority favorites, but not all.

As in the Algotel experiment, proponents of the Condorcet criterion will therefore estimate that most of the time, there is a canonical winner who should be elected. So one can wonder to what extent each voting system is likely to violate the Condorcet criterion. Figure 9.3 shows the probability that a Condorcet winner exists but is not elected by a given voting system (in "sincere" voting). As this rate of violation is necessarily zero for the Condorcet voting systems, these are not represented.

[8]The probability of existence of an intransitivity paradox (absence of Condorcet winner, absence of strict total Condorcet order) has been the subject of intense investigation by the means of theory or computer simulations. Besides the reference book by Gehrlein (2006), mention may be made of Ruben (1954); Campbell and Tullock (1965); Garman and Kamien (1968); Niemi and Weisberg (1968); DeMeyer and Plott (1970); Pomeranz and Weil Jr (1970); Tullock and Campbell (1970); Gehrlein and Fishburn (1976); Fishburn et al. (1979); Gehrlein (1981); Jones et al. (1995); Gehrlein (1999); Maassen and Bezembinder (2002); Merlin et al. (2002); Tsetlin et al. (2003). Except in single-peaked models, these works generally conclude that the absence of a Condorcet winner occurs with a relatively large probability. The study of these probabilities in real experiments, without being completely new, is currently developing thanks to better access to datasets and seems on the contrary to indicate that it is very frequent to have a Condorcet winner, as our study confirms. Regenwetter et al. (2006) also show that with realistic preferences, it is quite unlikely to have a cycle in the majority preference relation.

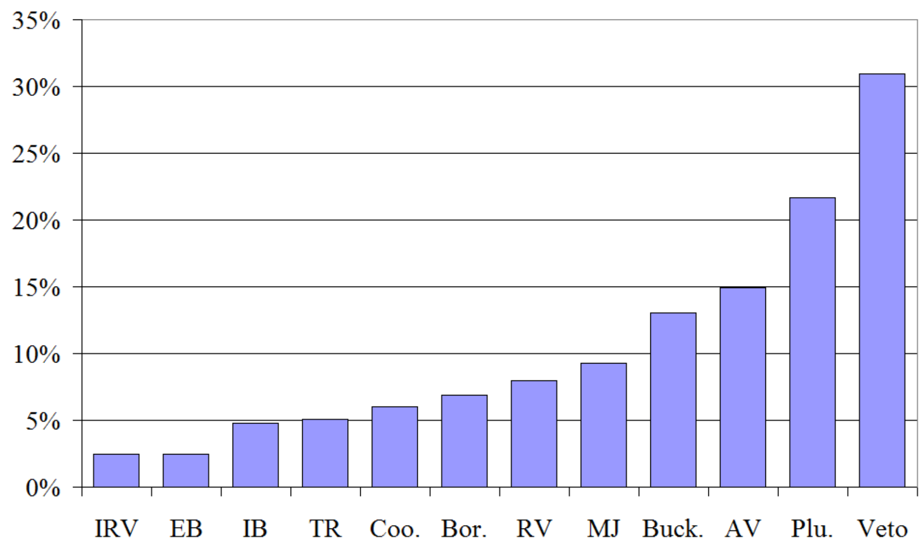


Figure 9.3 – Meta-analysis: rate of violation of the Condorcet criterion.

Among the non-Condorcet voting systems, it should be noted that IRV (equivalent to EB in sincere voting) presents the best performances: in the dataset used, it violates the Condorcet criterion in only 2 % of the cases. It is followed by IB and the Two-round system (5 %). The worst performances are achieved by Approval voting (15 %), Plurality (22 %), and Veto (31 %).

For all the voting systems presented here except Veto, each case of violation of the Condorcet criterion is also a case of trivial manipulability in favor of the Condorcet winner: thus, we already know, for example, that the TM rate of Plurality is at least 22 %, if only for that reason.

9.4 Coalitional manipulation

9.4.1 Average CM rates

Let us return to the Algotel experiment. In Figure 9.4, we represent the CM rate for each voting system. The graph reads as follows: for example, the Two-round system (TR) is manipulable in 30 % of the realizations. As we said before, it is the CM rate of this voting system in a culture which consists in drawing a random configuration in a small neighborhood of the actual experiment in the space of preferences.

Since some algorithms used are approximate, an uncertainty bar is indicated for the corresponding voting systems: for example, the CM rate of Schulze's method is between 25 % and 37 %. For certain other voting systems, such as Maximin and Borda's method, the algorithm used is not exact in theory,[9] but we note that the uncertainty bar is of zero size: in the particular case of this experiment, the algorithm was able to decide manipulability for all the realizations.

[9]For these voting systems, we used the `fast` option of SVVAMP.

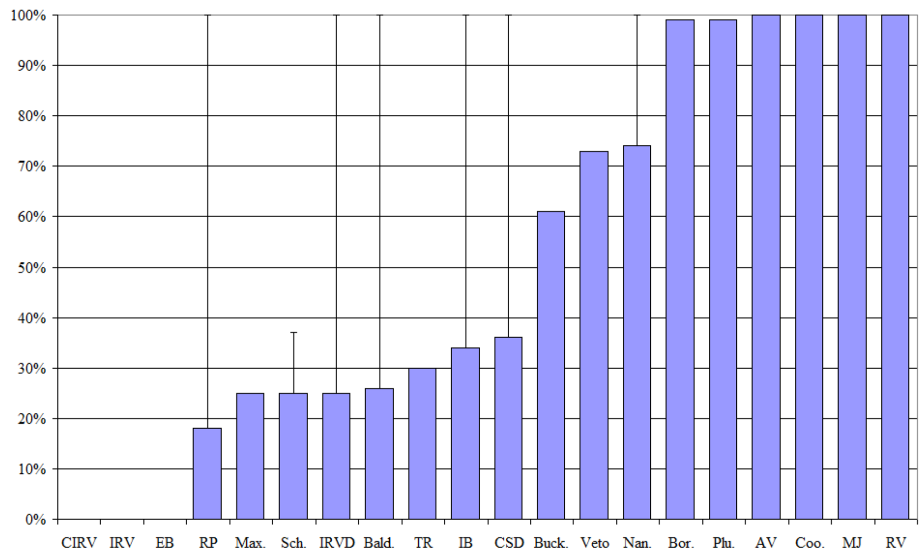


Figure 9.4 – Algotel: CM rates.

For this Algotel experiment, we see that CIRV, IRV, and EB are better than all the other voting systems, with a CM rate lower than 1 % (the statistical uncertainty). On the opposite, voting systems with the worst proven performances are Borda's method, Plurality, Approval voting, Coombs' method, Majority judgment, and Range voting, with CM rates close to 100 %.

We now turn to meta-analysis. Average CM rates are represented in Figure 9.5. For voting systems without a dedicated algorithm (Baldwin, IRVD, etc), the upper bounds found by SVVAMP are significantly lower than 100 %. To explain this, recall that there is a resistant Condorcet winner in 16 % of the realizations: in all these cases, SVVAMP knows that Condorcet voting systems are not manipulable (Section 6.3.4). For IB, SVVAMP uses a similar result involving the majority favorite. Note that the algorithms used for CIRV, Maximin, and Borda's method, despite being approximate in theory, fail to decide manipulability in less than 1 % of the realizations studied here.

Concerning the average CM rate, there is again a strikingly good performance of CIRV, IRV, and EB (9%). Even if we know, from theory, that CIRV is strictly less manipulable than IRV (Corollary 2.21 of the Condorcification theorems), which is itself strictly less manipulable than EB (Section 1.4.2), the difference is lower than 1 % on this corpus of experiments.[10] This result is similar to what we observed on artificial populations in Chapters 7 and 8, except that in one-dimensional cultures, there was more frequently a slight difference between IRV and EB.

All the other voting systems are clearly more manipulable, even the Two-round system with a CM rate of 37 %. Qualitatively, this difference is closer to what was observed for spheroidal or multidimensional political spectrum cultures than for one-dimensional cultures.

[10] Across all tested realizations, SVVAMP finds an average CM rate of 9,32 % for EB, 9,28 % for IRV, and between 9,20 % and 9,28 % for CIRV. The differences are therefore detectable but their amplitudes, below the statistical uncertainty, should only be considered as indicative.

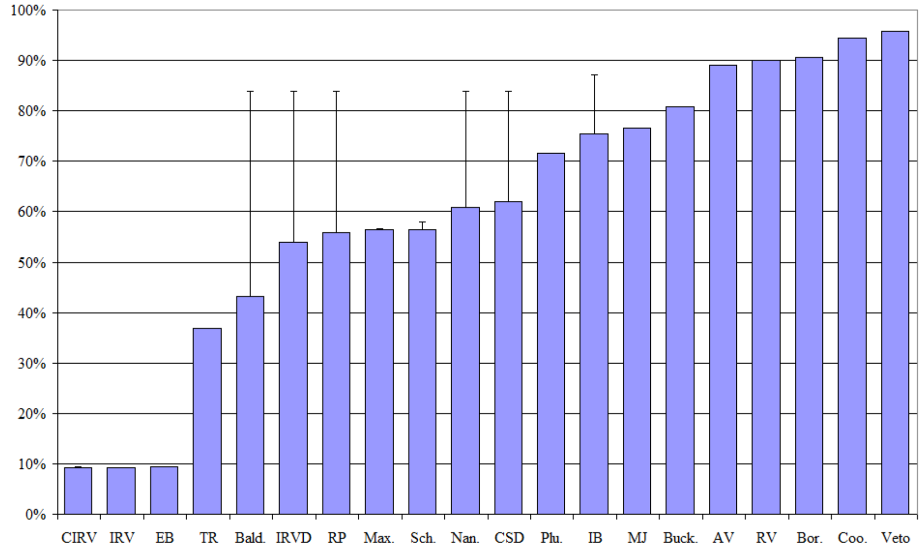


Figure 9.5 – Meta-analysis: CM rates.

The most manipulable on average, with rates close to 90 % or even higher, are Approval voting, Range voting, Borda's method, Coombs' method, and Veto. These results are similar to those of Chapters 7 and 8, except for Coombs' method whose bad behavior is closer to the results obtained in spheroidal or multidimensional political spectrum cultures than in one-dimensional cultures.

We know that, in all the voting systems considered here except Veto, if the winner is not Condorcet-admissible (either that there is not one, or that the voting does not designate her), the configuration is necessarily manipulable (Lemma 2.7). This has led some authors (such as Lepelley and Valognes, 2003) to think that Condorcet voting systems might be less manipulable. While this seems to be generally true, we must qualify this idea. Indeed, there are great discrepancies: whereas CIRV has a CM rate of 9%, Maximin and Schulze's method have a CM rate between 56 % and 58 %, which places them significantly behind IRV (9 %), which however does not respect the Condorcet criterion. On this point also, the results obtained are closer to those of spheroidal or multidimensional political spectrum cultures than those of one-dimensional cultures.

9.4.2 Comparing the CM of all voting systems

	CIRV	IRV	EB	TR	Max.	Sch.	Bald.	IRVD	RP	Nan.	Plu.	CSD	MJ	IB	Buck.	AV	Bor.	Veto	RV	Coo.
CIRV	—	0 (4)	**0 (4)**	**40**	**62**	**62**	**49 (81)**	**60 (81)**	**61 (81)**	**63 (81)**	**70**	**64 (81)**	**76**	**78 (83)**	**80**	**86**	**88**	**92**	**86**	**90**
IRV	0	—	**3**	**40**	**62**	**62**	**49 (81)**	**60 (81)**	**61 (81)**	**63 (81)**	**70**	**64 (81)**	**76**	**78 (83)**	**80**	**86**	**88**	**92**	**86**	**90**
EB	0	0	—	**40**	**62**	**62**	**49 (81)**	**60 (81)**	**61 (81)**	**63 (81)**	**70**	**64 (81)**	**76**	**78 (83)**	**80**	**86**	**88**	**92**	**86**	**90**
TR	0	0	0	—	**35**	**35 (37)**	**18 (55)**	**30 (55)**	**33 (55)**	**37 (55)**	**45**	**39 (55)**	**50**	**51 (57)**	**55**	**60**	**62**	**67**	**60**	**64**
Max.	0	0	0	5	—	0 (14)	1 (38)	0 (38)	1 (38)	**14 (38)**	**27**	**19 (38)**	**33**	**31 (40)**	**38**	**43**	**45**	**49**	**43**	**47**
Sch.	0	0	0	3 (5)	0 (4)	—	1 (38)	0 (38)	0 (38)	**10 (38)**	**26 (27)**	**13 (38)**	**32 (33)**	**27 (40)**	**33 (38)**	**42 (43)**	**44 (45)**	**49**	**42 (43)**	**46 (47)**
Bald.	0	0	0	1 (10)	0 (30)	0 (31)	—	0 (50)	0 (50)	0 (50)	2 (39)	0 (50)	4 (45)	0 (52)	3 (49)	**16 (55)**	**18 (57)**	**23 (61)**	**16 (55)**	**20 (59)**
IRVD	0	0	0	1 (8)	0 (17)	0 (18)	0 (40)	—	0 (40)	0 (40)	2 (29)	0 (40)	4 (35)	0 (42)	3 (39)	**16 (45)**	**18 (46)**	**23 (51)**	**16 (45)**	**20 (49)**
RP	0	0	0	1 (7)	0 (11)	0 (15)	0 (38)	0 (38)	—	0 (38)	2 (27)	0 (38)	4 (33)	0 (40)	3 (38)	**16 (43)**	**18 (45)**	**23 (49)**	**16 (43)**	**20 (47)**
Nan.	0	0	0	1 (2)	0 (5)	0 (10)	0 (34)	0 (34)	0 (34)	—	2 (23)	0 (34)	4 (28)	0 (36)	3 (33)	**16 (39)**	**18 (40)**	**23 (45)**	**16 (39)**	**20 (43)**
Plu.	0	0	0	0	0	0 (2)	0 (23)	0 (23)	0 (23)	0 (23)	—	1 (23)	**15**	13 (24)	**15**	**26**	**29**	**34**	**27**	**32**
CSD	0	0	0	1	0	0 (6)	0 (33)	0 (33)	0 (33)	0 (33)	2 (20)	—	4 (27)	0 (35)	3 (31)	**16 (38)**	**17 (40)**	**23 (45)**	**16 (38)**	**20 (42)**
MJ	0	0	0	1	0	0	0 (17)	0 (17)	0 (17)	1 (17)	4	1 (17)	—	9 (19)	**11**	**23**	**23**	**27**	**23**	**27**
IB	0	0	0	0 (3)	0 (3)	0 (7)	0 (21)	0 (21)	0 (21)	0 (21)	0 (16)	0 (21)	2 (17)	—	0 (19)	**10 (26)**	**8 (29)**	**14 (33)**	**10 (26)**	**11 (31)**
Buck.	0	0	0	1	0	0 (3)	0 (13)	0 (13)	0 (13)	1 (13)	10	2 (13)	**11**	1 (15)	—	**17**	**20**	**25**	**18**	**23**
AV	0	0	0	1	0	0	0 (6)	0 (6)	0 (6)	0 (6)	3	0 (6)	1	2 (7)	4	—	**8**	**11**	**5**	**10**
Bor.	0	0	0	1	0	0	0 (1)	0 (1)	0 (1)	0 (1)	1	0 (2)	2	0 (5)	0	5	—	**10**	**4**	**9**
Veto	0	0	0	2	4	4	3 (8)	4 (8)	4 (8)	5 (8)	5	4 (8)	8	5 (8)	5	10	9	—	**10**	**10**
RV	0	0	0	1	0	0	0 (2)	0 (2)	0 (2)	0 (2)	2	0 (2)	1	1 (3)	1	2	**4**	8	—	**6**
Coo.	0	0	0	0	0	0	0 (1)	0 (1)	0 (1)	0 (1)	0	0 (1)	1	0 (3)	1	5	1	7	4	—

Table 9.5 – Meta-analysis: Pairwise comparison of CM of the voting systems (in percents).

As we did in the meta-analyses of Chapters 7 and 8, we now compare the voting systems by pairs on the corpus of experiments. Since it is difficult to give detailed results for all the experiments, unlike the previous chapters where curves made it possible to visualize the compared results for the various voting systems, we provide Table 9.5, which establishes the meta-analysis graph.

This table reads as follows. In cell (TR, Sch.) for example, the first percentage means that it is *sure* that TR has a CM rate strictly lower than that of Schulze's method in 35 % of the experiments (in the sense that the upper bound for the first is strictly lower than the lower bound for the second). The percentage in parentheses means that it is *possible* that it is the case in 37 % of the experiments

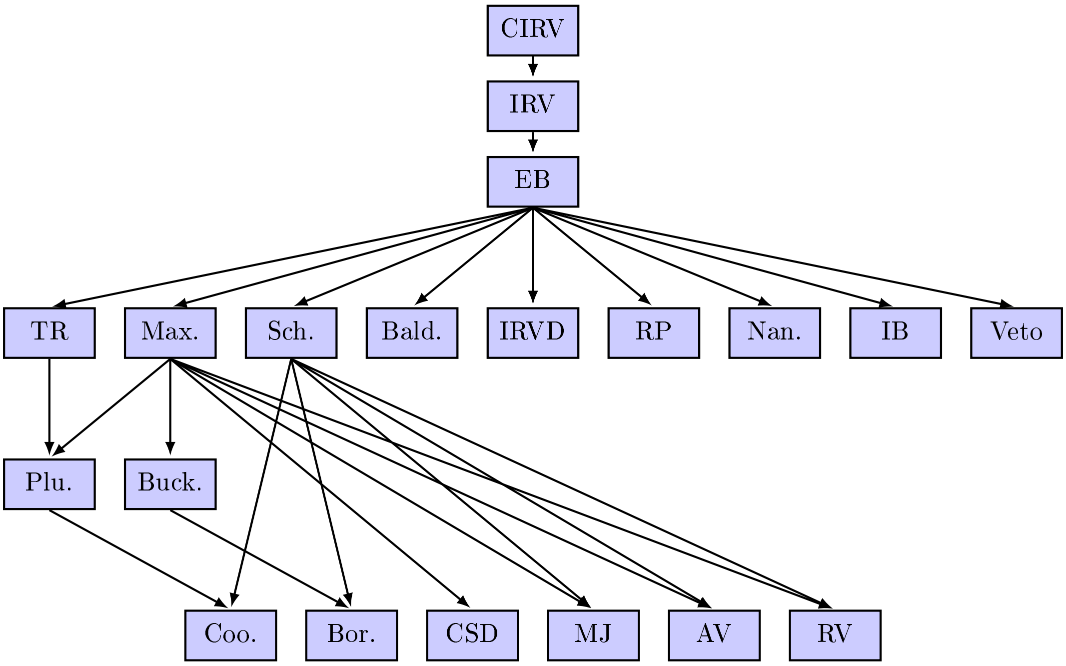


Figure 9.6 – Meta-analysis of CM in the experiments.

(in the sense that the lower bound for the first is strictly lower than the upper bound for the second).

In contrast, in the cell (Sch., TR) of the table, one can read that it is sure that Schulze's method has a CM rate lower than that of TR in 3 % of the experiments, and that it is possible in 5 % of the experiments.

With a uniform weighting on the 168 experiments, we can conclude that this duel is won by TR. Indeed, even in the less favorable hypothesis for TR, it has a CM rate lower than that of Schulze's method in 35 % of the experiments, whereas Schulze's method has a lower CM rate than TR in 5 % of the experiments. For information, such victories, based on uniform weighting, are indicated in bold in the table. Due to uncertainty for some voting systems, we are unable to provide results for all duels and we will supplement our results with a meta-analysis of TM.

As we did in the previous chapter, we present in Figure 9.6 the meta-analysis graph, based on victories in all the experiments: an edge from a voting system f to another one g means that it is sure that g does not have a CM rate strictly lower than that of f in any experiment. This property is therefore independent of any kind of weighting chosen on the 168 experiments.

The most striking conclusion is that CIRV dominates IRV, which dominates EB (which we already knew by theory), and that the latter dominates all other voting systems studied in all experiments. Maximin and Schulze's method, whose approximate algorithm provide limited uncertainty, constitute a good point of comparison with other voting systems. We note in particular that the poorest performances are achieved by Coombs, Borda, CSD, Majority judgment, Approval voting, and Range voting. The poor performances of Borda's method, Majority judgment, Approval voting, and Range voting are similar to those observed in Chapters 7 and 8. The following points are closer to the behaviors seen in spheroidal and multidimensional political spectrum cultures: the supremacy

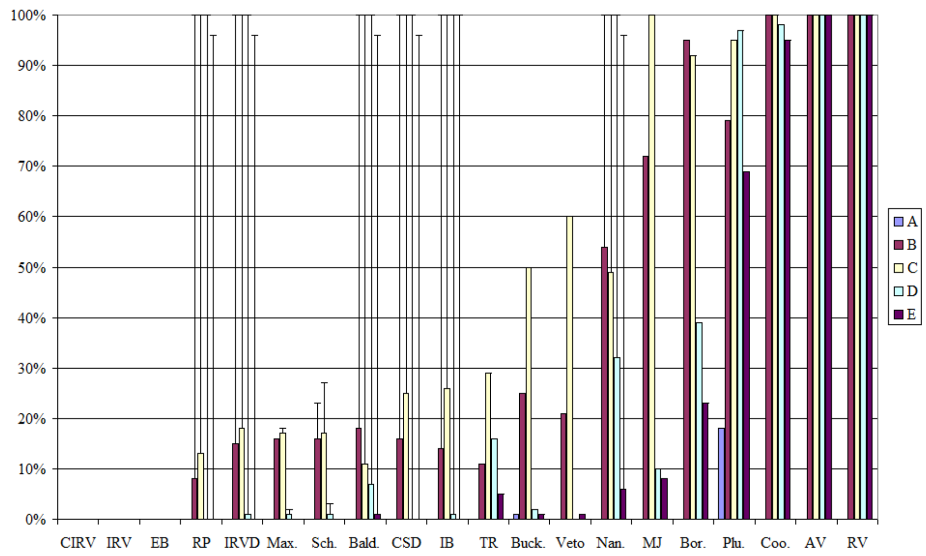


Figure 9.7 – Algotel: CM rate in favor of each candidate.

of CIRV, IRV, and EB; the intermediate performances of Maximin and Schulze's method (whereas they are quite good in one-dimensional cultures); and the poor performances of Coombs' method and CSD (whereas they are correct in one-dimensional cultures).

9.4.3 CM by candidate

To end this study of CM, we will focus on the manipulability per candidate for the Algotel experiment. In Figure 9.7, we represent the manipulability rate in favor of each candidate.

Without carrying out an exhaustive analysis, a certain number of observations seem especially relevant on this figure. First of all, the low manipulability for A just expresses the fact that A is almost always the sincere winner, and its supporters have nothing special to do. The cases of manipulability for A, especially in Plurality, correspond to the cases where B, or even C, is the sincere winner. There is then always a manipulation for A.

The worst results are obtained for two relatively natural voting systems, Approval voting and Range voting, which are manipulable in favor of the four rivals of A in all realizations! Majority judgment, Borda's method, Plurality, and Coombs' method also present a high risk of manipulation for a variety of candidates.

9.5 Trivial manipulation

As we have often mentioned, trivial manipulability (TM) is an interesting criterion for two reasons. On the one hand, it retains only the cases where manipulation is relatively easy to identify and coordinate, which makes its occurrence more credible in practice. On the other hand, its algorithmic simplicity allows exact computation for all the voting systems studied here, so that we can supplement the comparison between voting systems based on CM.

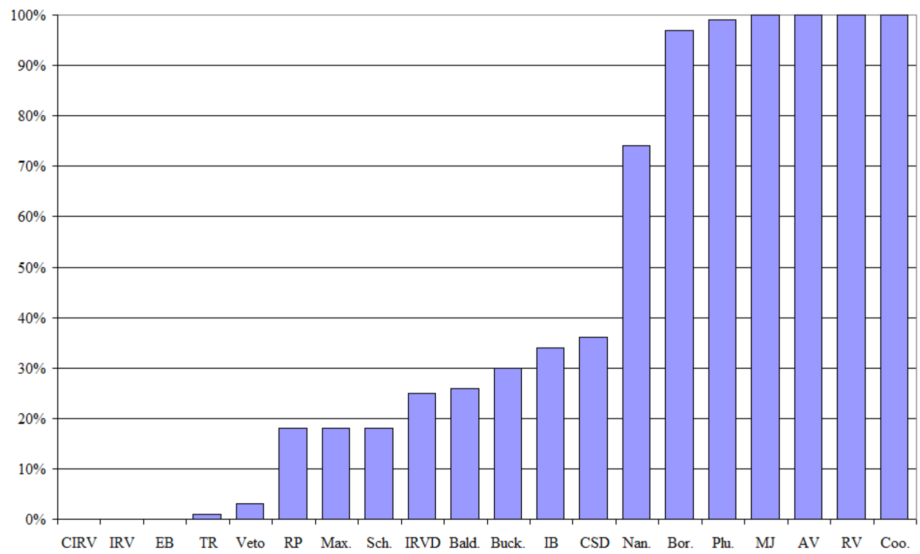


Figure 9.8 – Algotel: TM rates.

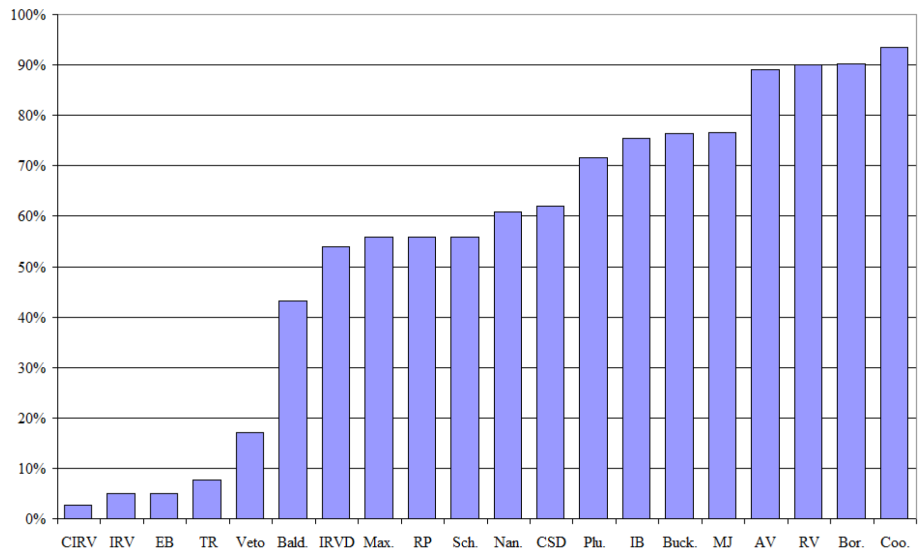


Figure 9.9 – Meta-analysis: TM rates.

9.5.1 Average TM rates

In Figure 9.8, we represent the TM rate for each voting system in the Algotel experiment. Since conclusions of Figures 9.8 and 9.9 are qualitatively similar, we are going to comment directly on the meta-analysis of Figure 9.9. Four voting systems are clearly distinguished by average rates below 10 %: CIRV (3 %), IRV and EB[11] (5 %), and the Two-round system (8 %). Next come Veto (17 %) and

[11] Recall that IRV and EB are equivalent from the point of view of trivial manipulation.

Baldwin's method (43 %), then a large number of voting systems between 50 % and 80 %. Finally, four voting systems have a TM rate that approaches or exceeds 90 %: Approval voting, Range voting, and the methods of Borda and Coombs.

As we have already noticed and explained in Section 7.2, Veto is the only voting system for which the TM rate (17%) is very different from the CM rate (96 %).

9.5.2 Comparing the TM rates of all voting systems

	CIRV	IRV	EB	TR	Veto	Bald.	IRVD	RP	Max.	Sch.	Nan.	CSD	IB	Plu.	Buck.	MJ	AV	Bor.	Coo.	RV
CIRV	—	**14**	**14**	**16**	**38**	**53**	**64**	**65**	**66**	**66**	**67**	**68**	**82**	**74**	**82**	**80**	**90**	**92**	**94**	**90**
IRV	0	—	**0**	**12**	**38**	**51**	**63**	**64**	**64**	**64**	**65**	**67**	**80**	**73**	**80**	**78**	**88**	**90**	**92**	**88**
EB	0	**0**	—	**12**	**38**	**51**	**63**	**64**	**64**	**64**	**65**	**67**	**80**	**73**	**80**	**78**	**88**	**90**	**92**	**88**
TR	0	1	1	—	**36**	**50**	**61**	**63**	**63**	**63**	**64**	**65**	**79**	**71**	**79**	**77**	**87**	**89**	**91**	**87**
Veto	5	8	8	10	—	**46**	**57**	**57**	**58**	**58**	**60**	**61**	**73**	**68**	**73**	**73**	**80**	**81**	**84**	**80**
Bald.	0	0	0	0	30	—	**25**	**28**	**29**	**29**	**32**	**33**	**46**	**39**	**47**	**45**	**55**	**57**	**59**	**55**
IRVD	0	0	0	1	25	5	—	**15**	**16**	**16**	**20**	**21**	**37**	**29**	**38**	**35**	**45**	**46**	**49**	**45**
RP	0	0	0	1	25	1	3	—	**8**	**8**	**18**	**20**	**34**	**27**	**36**	**33**	**43**	**45**	**47**	**43**
Max.	0	0	0	1	24	2	1	5	—	**2**	**19**	**20**	**35**	**28**	**37**	**34**	**44**	**46**	**48**	**44**
Sch.	0	0	0	1	24	2	1	4	0	—	**18**	**19**	**34**	**27**	**36**	**33**	**43**	**45**	**47**	**43**
Nan.	0	0	0	0	23	0	1	0	2	2	—	**14**	**24**	**23**	**30**	**28**	**39**	**40**	**43**	**39**
CSD	0	0	0	0	23	0	0	0	0	0	2	—	**23**	**20**	**27**	**27**	**38**	**40**	**42**	**38**
IB	0	0	0	1	17	1	0	1	1	1	7	8	—	**16**	**17**	**17**	**26**	**29**	**31**	**26**
Plu.	0	0	0	0	20	0	0	0	0	0	0	1	13	—	**13**	**15**	**26**	**29**	**31**	**27**
Buck.	0	0	0	0	17	1	1	1	1	1	2	5	7	11	—	**13**	**21**	**24**	**27**	**23**
MJ	0	0	0	0	17	0	0	0	0	0	1	1	9	4	8	—	**23**	**23**	**25**	**23**
AV	0	0	0	0	5	0	0	0	0	0	0	0	2	3	2	1	—	**8**	**10**	**5**
Bor.	0	0	0	0	7	0	0	0	0	0	0	0	0	1	0	2	6	—	**8**	**5**
Coo.	0	0	0	0	7	0	0	0	0	0	0	0	0	0	1	1	8	5	—	**7**
RV	0	0	0	0	5	0	0	0	0	0	0	0	1	2	0	1	2	4	6	—

Table 9.6 – Meta-analysis: Comparison of TM by pair of voting systems (in percents).

Like for CM, we provide Table 9.6 for the meta-analysis of TM. This time, there is no algorithmic uncertainty, which makes it easier to read the table and provides more precise results.

In particular, the voting system with best performances, CIRV, is never outperformed by any other voting system, except Veto. IRV and EB show similar performances, although they are beaten by the Two-round system in 1 % of the experiments (they are however better in 12 % of the cases). Among the other voting systems, the Two-round system is the one with the best results. Although it is strictly outperformed by CIRV (for TM) in 16 % of the experiments and strictly outperformed by IRV or EB in 12 % of the experiments (against 1 % for the converse), it is strictly better than Veto in 36 % of the experiments (against 10 % for the converse) and compared to any other voting system studied, it is only outperformed in 0 to 1 % of the experiments.

As before, the meta-analysis graph in Figure 9.10 is based on domination in all experiments and it therefore independent of any weighting. It should be noted in particular that, in the corpus of experiments considered, CIRV is strictly better than all the other voting systems studied, with the exception of Veto, which

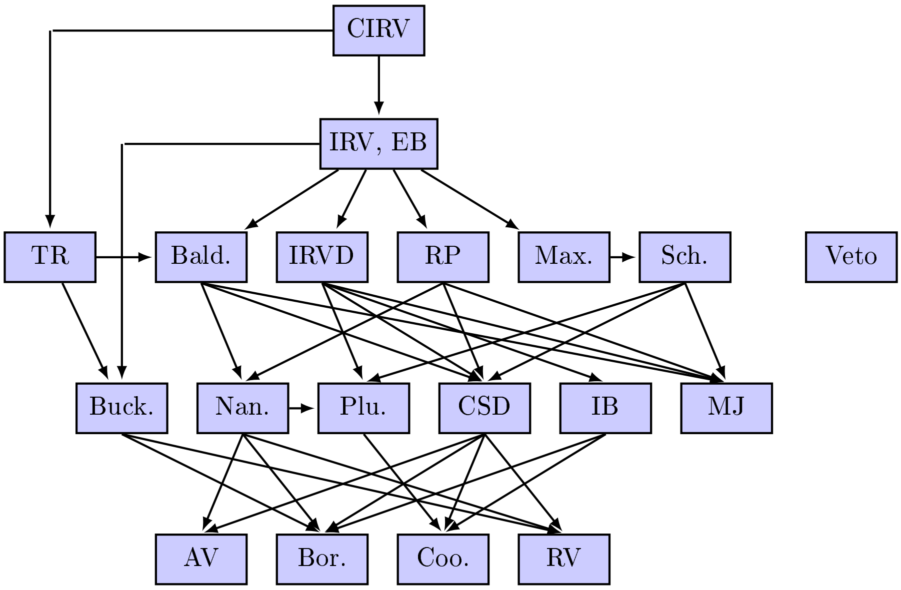


Figure 9.10 – Meta-analysis of TM in the experiments.

constitutes, on its own, an isolated connected component. In particular, IRV, EB, and TR are more manipulable than CIRV. Then, we can, as a indication, distinguish three groups, by ascending order of TM.

1. Baldwin's method, IRVD, RP, Maximin, and Schulze's method. Again, the later two show very similar performance (although with a slight advantage for Maximin).

2. The methods of Bucklin and Nanson, Plurality, CSD, IB, and Majority judgment. Among these, Nanson is less TM than Plurality.

3. Approval voting, Borda, Coombs, and Range voting.

9.5.3 TM by candidate

Like for CM, we examine in Figure 9.11 the manipulability per candidate for the Algotel experiment. Overall, the results are similar to those obtained for CM in Figure 9.7, in particular the superiority of CIRV, IRV, and EB. However, we observe the following differences.

Whereas TR could sometimes be CM for a variety of candidates, it is only TM (and quite rarely) in favor of candidate C. Thus, manipulations in the Two-round system generally require relatively sophisticated strategies.

Veto is much less TM than CM, which we have already noticed and explained, since the trivial strategy is very sub-optimal in this voting system.

Unsurprisingly, the results are the same in CM and TM for Majority judgment, Plurality, Approval voting, and Range voting: for these voting systems, the trivial strategy is optimal in the sense that if a manipulation is possible, then this strategy achieves it. As we discussed from the introduction of this memoir, this can be seen as a defect or as a quality of these voting systems: manipulation being simple to implement makes it possible to seek equilibria with a limited exchange of information. However, we think that the property of being non-manipulable

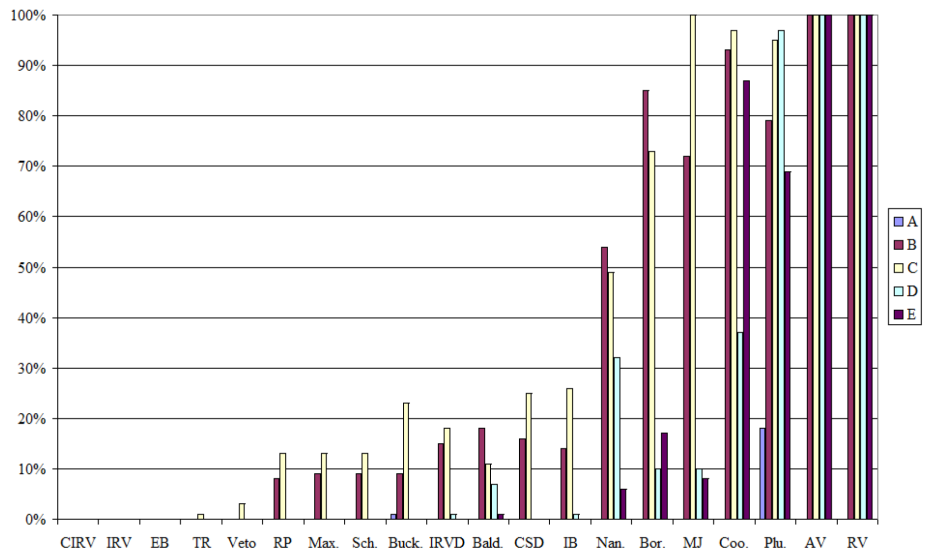


Figure 9.11 – Algotel: TM rate in favor of each candidate.

is even more interesting (which is the case for CIRV, IRV, and EB in the Algotel experiment, cf. Figure 9.7 for example), since it makes it possible to reach an equilibrium with zero exchange of information, as we discussed notably in Chapter 3.

9.6 Summary of results

In this chapter, we have analyzed the results of 168 experiments from the real world, by adding a random noise that allows to explore the space of preference in the neighborhood of each experiment.

We were able to establish that there is almost always a Condorcet-admissible candidate, very often a Condorcet winner, and that the presence of a Condorcet strict total order, which is nevertheless a very structuring property for a population, is, in fact, a relatively frequent phenomenon.

Among the non-Condorcet voting systems, IRV and EB violate the Condorcet criterion the least often, followed by IB and the Two-round system. Those who violate it most often are Approval voting, Plurality, and Veto. Despite these differences, we find that even non-Condorcet voting systems designate the possible Condorcet winner in a majority of cases (of very variable size depending on the voting system).

We have found that manipulability is not just a theoretical concept, but a concrete reality likely to appear even in a context where the result seems *a priori* obvious: indeed, an important part of the voting systems studied are manipulable in a large proportion of the experiments.

Even if the voting systems relatively often designate the same winner in "sincere" voting (the Condorcet winner when she exists, even for the non-Condorcet voting systems), the choice of the voting system is therefore also crucial for another reason: the observed CM rates range from less than 10 % to more than 90 %, depending on the voting system.

CIRV, IRV, and EB are the least manipulable. By theory, we know that CIRV is strictly less manipulable than IRV, itself strictly less manipulable than EB; but in practice, the gap is very small. For practical applications, it is therefore very conceivable to prefer IRV to CIRV, because the counting of IRV is easier if no computer resources are involved. Among the other voting systems, the Two-round system is generally the least manipulable. The most manipulable ones are Approval voting, Borda, Coombs, Range voting, and Veto.

The results observed are globally closer to those obtained in the spheroidal cultures of Chapter 7 or in the cultures based on a multidimensional political spectrum of Chapter 8 than in one-dimensional cultures (Chapter 8 as well), despite the fact that our corpus of experiments includes elections from the political world, where the assumption of single-peakedness would seem *a priori*, if not realistic, at least an interesting first approximation of reality. Without totally validating spheroidal or multidimensional political spectrum models, this conclusion encourages, at least, to pursue their study in future works.

Chapter 10

Optimal Voting Systems

In this chapter, we focus on voting systems whose manipulability rate is minimal within the set of those meeting **InfMC**: for convenience of language, we will say that such a system is *optimal*.

Thanks to the corollary 2.23 of the Condorcification theorems, we know that to look for such a voting system, we can restrict ourselves to Condorcet systems. By the optimality theorem 5.15, we know that if the culture is decomposable, in particular if voters are independent, then there exists an optimal system and we can restrict ourselves to systems which are ordinal and satisfy the Condorcet criterion.

In Section 10.1, we present the technique used. In Section 10.1.1, we define the *opportunity graph* of an electoral space. Independent of the voting system studied, this indicates whether one configuration is susceptible of being manipulable towards another, depending on the candidate who is declared the winner in the two configurations by the voting system used subsequently. In this representation, a culture simply corresponds to a weighting on the vertices of the graph.

In Sections 10.1.2, 10.1.3, and 10.1.4, we show that the problem of finding an optimum is simplified if we restrict ourselves to Condorcet systems, if we assume that semi-Condorcet configurations come with zero probability and if we restrict ourselves to ordinal systems.

In Section 10.1.5, we present a greedy algorithm which makes it possible to find an approximate optimum, but we will see that it is not exact. In Section 10.1.6, we show that searching a optimum, with the previous assumptions, can be reduced to an integer linear programming optimization problem, which the dedicated software CPLEX can process for moderate values of the parameters.

In Section 10.2, we restrict our investigation to the impartial culture with an odd number of voters, which verifies all the previous assumptions. We discuss the solutions found for $V \in \{3, 5, 7\}$ and $C = 3$, and for $V = 3$ and $C = 4$ (Sections 10.2.1 to 10.2.4).

In Section 10.3, we conclude this chapter by comparing the minimal manipulability rates found (within **InfMC**) to those of classical voting systems.

10.1 Opportunity graph

10.1.1 Definition

As usual, we work in an electoral space Ω. The *opportunity graph* that we define in this section is entirely defined by Ω: it expresses a structure of the electoral space which is independent of the voting system used subsequently. Within this framework, we will then see that a culture is simply represented by a weighting on the vertices of the graph.

Before defining the opportunity graph, we first need to define (w, c)-*pointing*.

Definition 10.1
Let ω and ψ be two distinct configurations, w and c two distinct candidates. We say that ω (w, c)-*points* to ψ iff:

$$\forall v \in \mathrm{Sinc}_\omega(\mathrm{w} \to c), \omega_v = \psi_v.$$

In other words:

$$\forall v \in \mathcal{V}, \omega_v \neq \psi_v \Rightarrow c \, \mathrm{P}_v(\omega_v) \, \mathrm{w}.$$

If a voter presents a different state in ω and ψ, then she prefers c to w in ω.

Intuitively, this property means that if w wins in ω and if c wins in ψ, then ω is manipulable to ψ in favor of c: indeed, voters who prefer c to w can change their ballots to produce configuration ψ.

For example, consider the following configurations.

$$\omega : \begin{array}{c|c|c} a & b & c \\ b & c & b \\ c & a & a \end{array} \qquad\qquad \psi : \begin{array}{c|c|c} a & b & c \\ b & c & \mathbf{a} \\ c & a & \mathbf{b} \end{array}$$

Configuration ω (b, c)-points to ψ: indeed, the only voter who changes state prefers c to b in ω. It would therefore be harmful for a voting system to designate b in ω and c in ψ: indeed, since ω (b, c)-points to ψ, this would imply that configuration ω is manipulable.

In the above example, configuration ω (a, c)-points and (a, b)-points to ψ as well. This (a, b)-pointing gives us the opportunity to insist on the fact that each modified voter (the unique modified voter, in this example) prefers by assumption b to a in the starting configuration ω, but not necessarily in the target configuration ψ. Thus, if a voting system designated a in ω and b in ψ, then the last voter could manipulate configuration ω in favor of b by lowering b in her order of preference! Such a situation cannot be ruled out at all: certain widely used voting systems, such as the Two-round system or IRV, present such shortcomings of monotonicity.

More generally, we do not have only an implication from pointing to manipulability but the following equivalence, which is a simple translation of the definition of manipulability.

Proposition 10.2
Let $f : \Omega \to \mathcal{C}$ be an SBVS, ω and ψ two distinct configurations.
The voting system f is manipulable in ω to ψ iff ω $(f(\omega), f(\psi))$-points to ψ.

This remark will allow us to reformulate questions of manipulability as graph problems.

Definition 10.3
Consider the labeled multigraph $(\Omega, E, e : E \to \mathcal{C}^2)$, defined as follows.

- The vertices are the states $\omega \in \Omega$.
- E is the set of edges.

- e is a function that, to each edge, associates a label which is a pair of candidates.
- A vertex ω has an edge labeled (w, c) to a vertex ψ iff configuration ω (w, c)-points to ψ.

This labeled multigraph is called the *opportunity graph* of the electoral space Ω.

This is a multigraph: *a priori*, it is entirely possible that a vertex ω has several edges to a vertex ψ with different labels, as we have already seen.

An SBVS $f : \Omega \to \mathcal{C}$ is then seen as a function which, to each vertex ω of the opportunity graph, associates an element of $\mathcal{C}$. The manipulability indicator of f at ω, which we have already denoted $\mathrm{CM}_f(\omega)$, is a Boolean function on the vertices which equals 1 at a vertex ω iff there exists at least one vertex ψ towards which vertex ω has an edge labeled $(f(\omega), f(\psi))$.

The opportunity graph is a representation which makes it possible to study the manipulability rate: if we consider a probability distribution π on the electoral space, it suffices to give each vertex ω a weight $\pi(\omega)$. The manipulability rate of f is then the total weight of the vertices that are manipulable according to f:

$$\tau_{\mathrm{CM}}(f) = \sum_{\omega \in \Omega} \mathrm{CM}_f(\omega)\pi(\omega).$$

Looking for a voting system with minimal manipulability rate and respecting a given constraint (for example **Cond**) means looking for a function f which minimizes the total weight of the manipulable vertices among those satisfying this constraint.

The problem gets simpler at least in the following three cases, which make it possible to "clean up" the graph, i.e. to remove irrelevant edges without altering its validity for the search of an optimum.

- If the constraint studied implies that a configuration ω designates a certain winner w_0, then for $\mathrm{w} \neq \mathrm{w}_0$, it is useless to consider the edges (w, c) outgoing from ω.
- If we know that a configuration ω is necessarily manipulable (because of the constraint studied), then it is useless to consider the outgoing edges of ω.
- If an event $A \subset \Omega$ has zero probability, then for each configuration $\omega \in A$, it is useless to consider its outgoing edges: indeed, making ω manipulable does not increase the manipulability rate. In contrast, its incoming edges are important, because ω could make manipulable certain configurations ω' of nonzero probability which point to ω.

10.1.2 Restriction to Condorcet voting systems

By the corollary 2.23 of the Condorcification theorems, we know that it is interesting to look for a voting system which, within the class of those meeting **Cond**, has minimal manipulability. Indeed, such a voting system will also exhibit minimal manipulability in the larger class of voting systems meeting **InfMC**.

For this specific constraint (**Cond**), the first two simplifications seen above become the following.

- If ω has a Condorcet winner w_0, we know that w_0 is necessarily elected in ω so for $\mathrm{w} \neq \mathrm{w}_0$, it is useless to consider the outgoing edges (w, c).

<table>
<tr><th></th><th>ψ Condorcet</th><th>ψ semi-Condorcet</th><th>ψ non-admissible</th></tr>
<tr><td>ω Condorcet (winner w_0)</td><td>no</td><td colspan="2">for $\mathrm{w} = \mathrm{w}_0$</td></tr>
<tr><td>ω semi-Condorcet</td><td colspan="3">for any w</td></tr>
<tr><td>ω non-admissible</td><td colspan="3">no</td></tr>
</table>

Table 10.1 – Searching an optimal Condorcet voting system: (w, c)-pointings to consider from ω to ψ.

- If ω is a non-admissible configuration, we know that it is necessarily manipulable, so it is useless to consider its outgoing edges.

Moreover, if ω and ψ both have a Condorcet winner, then ω cannot be manipulable to ψ, so it is useless to consider edges from ω to ψ.

Table 10.1 summarizes which edges should be considered to look for a Condorcet voting system of minimal manipulability.

In summary, Condorcet configurations simplify because the winner is already known; and non-admissible configurations, because we already know their manipulability (equal to True). Moreover, we can notice that any resistant configuration is also a simple case, since we also know its manipulability (equal to False); in practice, this results in such a configuration having no outgoing edge, once the graph is cleaned up as indicated.

Semi-Condorcet configurations (with at least one Condorcet-admissible candidate, but without a Condorcet winner) are the trickiest: indeed, neither their winner nor their manipulability are known *a priori*.

10.1.3 Semi-Condorcet configurations with zero probability

Now we add the assumption that the set of semi-Condorcet configurations has zero probability. This is the case, in particular, if the number of voters is odd and preferences are almost surely strict total orders. Concerning this last assumption, this may be so either because other kinds of binary relations are considered impossible in the electoral space studied, like that of strict total orders, or because they are authorized but of zero probability in the culture studied: in particular, it is the case if preferences derive from utilities and if the probability of having two equal utilities is zero, like in all the cultures of Chapters 7 and 8. This is therefore an important particular case.

As we remarked in Section 10.1.1, this implies that we can ignore all outgoing edges from semi-Condorcet configurations, since their manipulability has no influence on the manipulability rate.

However, *a priori*, a semi-Condorcet configuration ψ could make manipulable a configuration ω that has nonzero probability. In order to circumvent this risk, we add the following constraint: for each semi-Condorcet configuration, the winner is an arbitrary Condorcet-admissible candidate. Thus, a semi-Condorcet configuration cannot make manipulable either a non-admissible configuration (which is manipulable anyway) nor a Condorcet configuration (because the candidate who would benefit from such a manipulation would still have a defeat against the Condorcet winner, so she could not be Condorcet-admissible, Lemma 2.6). By imposing this additional property, one can also ignore the incoming edges of the semi-Condorcet configurations and finally totally ignore these configurations.

For the search of a voting system with minimal manipulability rate, this assumption in no way alters the optimality of the voting system found: indeed, for

	ψ Condorcet	ψ semi-Condorcet	ψ non-admissible
ω Condorcet (winner w_0)	no		for $\mathrm{w} = \mathrm{w}_0$
ω semi-Condorcet	no		
ω non-admissible	no		

Table 10.2 – Searching for an optimal Condorcet voting system: (w, c)-pointings to consider from ω to ψ, if semi-Condorcet configurations have zero probability. Without altering optimality, we impose that the winner in each semi-Condorcet configuration be an arbitrary Condorcet-admissible candidate.

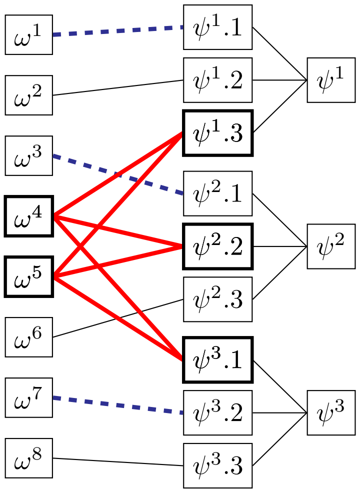


Figure 10.1 – Type of graph used to find an optimal Condorcet voting system, in a discrete electoral space where semi-Condorcet configurations have zero probability. Dashed blue line: a possible result of the greedy algorithm. Thick red line: the global optimum.

each optimal voting system, the previous considerations show that, if we modify it by designating a Condorcet-admissible candidate in each semi-Condorcet configuration, then the resulting system has exactly the same manipulability rate.

Table 10.2 summarizes the edges that we can simply consider with these new assumptions. It is now a bipartite labeled graph, since edges come only from Condorcet configurations and go only to non-admissible configurations.

When ω (w, c)-points to ψ, it is now unnecessary to specify w, since in the cleaned graph, the considered candidate is necessarily the Condorcet winner in ω. Therefore, we will simply say that ω c-points to ψ (which implies that ω is a Condorcet configuration and ψ a non-admissible configuration).

Figure 10.1 shows a representation of such an opportunity graph. This is a simplified example, which does not necessarily correspond to a particular electoral space. Every configuration ω^i is Condorcet, every configuration ψ^j is non-admissible. Semi-Condorcet or resistant configurations are not represented because they have no impact on our problem. In order to graphically represent the edge labeled c from a configuration ω to a configuration ψ, we equip ψ^j with a "socket" denoted $\psi^j.c$ and we "plug" configuration ω^i to the socket $\psi^j.c$.

This graph reads as follows: for example, in configuration ψ^2, if candidate 2 is declared the winner by a certain voting system, then configurations ω^4 and ω^5 become manipulable: we will also say that they are *contaminated*. The problem then comes down to choosing exactly one socket for each configuration ψ while minimizing the total weight of the contaminated configurations (or more simply their cardinality, if the Condorcet configurations are finite in number and endowed with uniform probability).

10.1.4 Restriction to ordinal voting systems

The slicing theorems (5.9 and 5.10) and their combinations with Condorcification theorems (5.12 and 5.15) suggest that we seek an optimum among the voting systems which not only respect the Condorcet criterion, but are also ordinal (in the sense that they only depend on binary relations of preference, even if these are not orders). Indeed, if the culture is decomposable, such an optimum will exhibit a minimal manipulability rate in the larger class of voting systems which satisfy **InfMC** and which are not necessarily ordinal (optimality theorem 5.15). We can therefore add the assumption that the opportunity graph is finite. In that case, there exists a finite number of voting systems, so there necessarily exists at least one voting system whose manipulability rate is minimal, as we have already noticed in Proposition 5.14.

10.1.5 Greedy algorithm

It is easy to design a greedy minimization algorithm which proceeds by local optimization: for each non-admissible configuration ψ^j, we choose the socket $\psi^j.c$ which has the fewest incoming edges. Such an approximate solution is represented by a dashed blue line in Figure 10.1. In this example, we would choose candidate 1 or 2 for configuration ψ^1 (which contaminates ω^1 or ω^2), 1 or 3 for configuration ψ^2 (which contaminates ω^3 or ω^6), and 2 or 3 for configuration ψ^3 (which contaminates ω^7 or ω^8). Thus, 3 configurations would be contaminated.

However, this algorithm is not optimal for a general graph of this type: in Figure 10.1, we have represented in red bold line the optimal solution, which is strictly better. By choosing candidate 3 for ψ^1, 2 for ψ^2, and 1 for ψ^3, we contaminate only 2 configurations, ω^4 and ω^5.

In the general case of a labeled multigraph of the type presented in Figure 10.1, it is likely that the problem to be solved is $\mathcal{NP}$-difficult. This does not exclude, *a priori*, that the problem can be simplified, for example, for the subset of the graphs obtained as opportunity graphs of each electoral space of strict total orders (for all V odd and all C). But it would still be surprising if we escaped pathological situations like the one in Figure 10.1, and we can already expect that the greedy algorithm is not optimal.

10.1.6 Exact approach: integer linear programming optimization

To sum up, we are looking for voting systems meeting **InfMC** and whose manipulability rate is minimal. We can restrict the search to ordinal voting systems (if the culture is decomposable) and verifying the Condorcet criterion. Furthermore, we assume that semi-Condorcet configurations have zero probability and, in these, we impose that the winner is an arbitrary Condorcet-admissible candidate, without altering the optimality of the voting system we will obtain.

In this context, our problem can be reduced to an integer linear programming optimization problem, as follows.

- For each pair (ψ, c), where ψ is a non-admissible configuration and c a candidate, we declare the integer variable $W(\psi, c)$, whose value is 1 if c is declared the winner in ψ and 0 otherwise.
- For each Condorcet configuration ω, we declare the integer variable $\mathrm{CM}(\omega)$, whose value is 1 if ω is manipulable and 0 otherwise.
- For each non-admissible configuration ψ, there is a unique winner, which is expressed by the constraint $\sum_c W(\psi, c) = 1$.
- For each triple (ω, ψ, c) such that ω c-points to ψ, we know that if $W(\psi, c) = 1$, then $\mathrm{CM}(\omega) = 1$, which is expressed by the constraint $\mathrm{CM}(\omega) \geq W(\psi, c)$.
- The objective is to minimize $\sum_\omega \pi(\omega)\mathrm{CM}(\omega)$.

The advantage of this formulation is that there are generic software packages that implement powerful algorithms to solve integer linear programming problems. At various stages of our work, we used AIMMS first, then IBM ILOG CPLEX Optimization Studio, both of which use the CPLEX engine.

The disadvantage of our problem is that it requires a large number of variables. For example, in the electoral space of strict total orders with V odd, it lies between $(C!)^V$ and $C \times (C!)^V$. Indeed, there is a variable for each Condorcet configuration and C variables for each non-admissible configuration (there is no semi-Condorcet configuration in this case). The problem is therefore only reasonably treatable for very small values of the parameters. However, we will see that the cases we can exploit in practice are already rich in lessons on optimal voting systems in general.

10.2 Optimal voting systems for small values of V and C

Now we consider the impartial culture: voters almost surely have a strict total order of preference, they are independent, and the culture is neutral (and anonymous). This culture therefore verifies all previous assumptions. In this particular case, there is no semi-Condorcet configuration, hence non-admissible configurations are exactly the non-Condorcet configurations. Moreover, configurations are equiprobable, hence minimizing the total weight of the contaminated configurations amounts to minimizing their cardinality.

10.2.1 $V = 3$ and $C = 3$: a lot of optima

For $V = 3$ voters and $C = 3$ candidates, there are $(C!)^V = 216$ configurations in the electoral space and $C^{\mathrm{card}(\Omega)} \simeq 10^{103}$ possible voting systems in total (including those that do not satisfy the Condorcet criterion). However, it is possible to deal with the problem manually, which is interesting for forging intuition and which will allow us to prove a more general result than the simple identification of an optimum by the brute force of CPLEX.

Lemma 10.4
Out of the $(C!)^V = 216$ possible profiles, there are 12 non-Condorcet profiles.

This is the following one:

$$\psi : \begin{array}{c|c|c} a & b & c \\ b & c & a \\ c & a & b \end{array}$$

and its variants by permutation of voters and/or candidates.

Configuration ψ is a minimal example of the classic Condorcet paradox: a defeats b, who defeats c, who defeats a.

Proof. For a profile to be non-Condorcet, the candidates at the tops of the ballots must be distinct: otherwise the candidate placed on top twice is a Condorcet winner. Up to exchanging b and c, assume that the voter who puts a on top prefers b to c. By also counting the voter who puts b on top, b is thus assured of a victory against c. Therefore, it is necessary to have a victory of a against b and one of c against a. It is then easy to conclude that the configuration is ψ. □

To define a Condorcet voting system, one only has to choose the winner in each non-Condorcet profile. Thus, there are "only" $C^{12} = 531,441$ Condorcet voting systems.

Before going further on the case $V = 3$ and $C = 3$, we notice the following easy lemma, which is true in general, not only for these values of the parameters.

Lemma 10.5
For a configuration ω, with a Condorcet winner w*, to c-point to a configuration ψ, it is necessary that* w *be preferred to c in ψ by a strict majority of voters.*

Proof. This is a simple reformulation of Lemma 2.6. In ω, a majority of voter prefer w to c. However, these voters cannot change state in ψ. □

We are now going to proceed with a series of fairly simple lemmas to deal with the case $V = 3$ and $C = 3$.

Lemma 10.6
The configuration ψ above is only a-pointed by the following profile:

$$\omega : \begin{array}{c|c|c} a & b & c \\ c & c & a \\ b & a & b \end{array},$$

whose Condorcet winner is c.

Proof. Let ω be a profile that a-points to ψ and w its Condorcet winner. By virtue of Lemma 10.5, w must have a victory against a in ψ: hence, this candidate w is necessarily c. In ω, there are therefore at least two voters who prefer c to a, and by a-pointing, their ballots do not change in ψ. Since only the last two voters prefer c to a in ψ, it is necessarily those: they therefore have the same ballot in ω. It remains to determine the ballot of the first voter, knowing that, by assumption, she prefers a to c. If she puts b on top, then b is Condorcet winner (which is excluded), and if she puts b in the middle of her list, then it is configuration ψ. Consequently, she necessarily puts b at the bottom of her ballot. □

We immediately deduce that ψ is only b-pointed or c-pointed by one profile, obtained from ω by permutation of the roles.

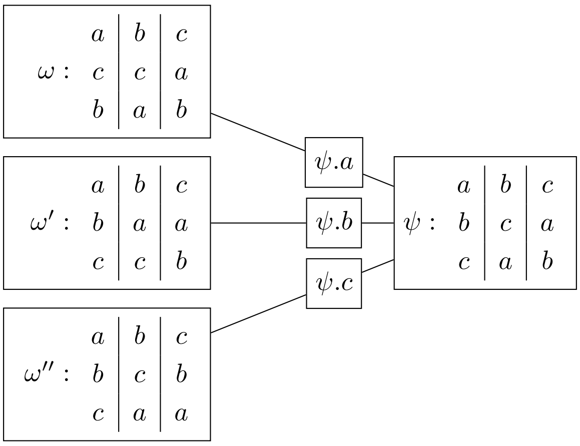


Figure 10.2 – A connected component of the opportunity graph for $V = 3$ and $C = 3$.

Lemma 10.7

Using the previous notations, configuration ω, whose Condorcet winner is c, only points to configuration ψ.

Proof. Assume that ω b-points to some configuration. In this one, the first and the last voter, who prefer c to b, are unmodified. Since their ballots are not circular permutations of each other, it is impossible to obtain a non-Condorcet profile this way (Lemma 10.4).

Suppose, now, that ω a-points to a certain configuration. In this one, the last two voters, who prefer c to a, are unmodified. But we know (Lemma 10.4) that to obtain a non-Condorcet configuration, the ballots must be circular permutations of one another, hence the only possible configuration is ψ. □

Proposition 10.8

In impartial culture, for $V = 3$ and $C = 3$, all SBVS satisfying the Condorcet criterion have the same manipulability rate: $\frac{24}{216} = \frac{1}{9} \simeq 11.11$ % .

Proof. The connected component of profile ψ contains exactly three Condorcet profiles, ω, ω', and ω'', as represented in Figure 10.2. Indeed, only these three profiles point to ψ (Lemma 10.6) and none of them points to another profile (Lemma 10.7).

Therefore, the graph contains exactly 12 non-singleton connected components. The other non-trivial components are obtained from Figure 10.2 by permutation of voters and/or candidates.

In the component shown, whatever winner is chosen for profile ψ, it contaminates one, and exactly one, Condorcet profile. Thus, the manipulable profiles are the 12 non-Condorcet profiles and the 12 contaminated Condorcet profiles: the manipulability rate is $\frac{24}{216}$. □

In Section 2.8, we asserted without proof that, in the electoral space of strict total orders, there is no Condorcet SBVS which reaches the upper bound of manipulability that we gave, i.e. which is manipulable in all non-resistant configurations. Figure 10.2 proves this fact: indeed, profile ψ cannot contaminate at the same time ω, ω', and ω''. Since these profiles have no other way to be contaminated (Lemma 10.7), they cannot all be manipulable. Moreover, it is easy to prove that these profiles are non-resistant, since by virtue of the graph of Figure 10.2, each of them can be manipulable.

10.2.2 $V = 5$ and $C = 3$: CIRV and that's it!

For $V = 5$ voters and $C = 3$ candidates, it is difficult to certify the global optimum by hand, but we can, at least, manually perform the greedy algorithm. In fact, we will see that, in this case, the result is the same. The manual approach will therefore allow us to have a better understanding of the optimal voting system.

Among the $(C!)^V = 7,776$ possible profiles, it is easy to show, by manual enumeration up to symmetries, that 540 profiles are non-Condorcet. On the one hand, we have the following profile:

$$\psi : \begin{array}{c|c|c|c|c} a & a & b & b & c \\ b & b & c & c & a \\ c & c & a & a & b \end{array}$$

and its 180 variants by permutation of voters and/or candidates. On the other hand, we have the following profile:

$$\chi : \begin{array}{c|c|c|c|c} a & a & b & b & c \\ b & c & c & c & a \\ c & b & a & a & b \end{array}$$

and its 360 variants by permutation of voters and/or candidates. The first profile ψ has half as many variants as the second χ because it has an additional internal symmetry, consisting in exchanging the first two voters.

By applying the same techniques as for $V = 3$ voters, it is humanly possible to prove that there is only one profile which a-points to ψ, only one profile that a-points to χ, and that it is the same profile ω:

$$\omega : \begin{array}{c|c|c|c|c} a & a & b & b & c \\ c & c & c & c & a \\ b & b & a & a & b \end{array} ,$$

whose Condorcet winner is c.

It is hardly more difficult to show that, on the other hand, there are strictly more than one profile that b-point (or c-point) to ψ (or χ). Thus, when using the greedy algorithm, one must have $f(\psi) = f(\chi) = a$, which makes only the profile ω manipulable. Moreover, this profile ω is also made manipulable by the profile obtained by exchanging the first two voters of χ. Consequently, the greedy algorithm contaminates ω and its variants by permutation, i.e. 180 Condorcet profiles. In total, there are therefore $360 + 180 + 180 = 720$ manipulable profiles and the manipulability rate is $\frac{720}{7,776} = \frac{5}{54} \simeq 9.26$ %.

Using CPLEX, we establish that it is, in fact, a global optimum. To find out if this optimum is unique, CPLEX is then asked to find the optimum either with the additional constraint $W(\psi, a) = 0$, or with the additional constraint $W(\chi, a) = 0$, i.e. that candidate a is forbidden to win in ψ or in χ. In both cases, we see that the optimal manipulability rate is strictly greater than $\frac{720}{7,776} = \frac{5}{54}$. It is therefore necessary, for an optimal voting system, to elect a in ψ and in χ. By argument of symmetry, for each optimal voting system, the possible winner is also unique in any profile obtained from ψ or χ by permutation of voters and/or candidates. The optimum shown above is therefore unique.

Moreover, we note that the voting system obtained coincides with CIRV: indeed, it suffices to check that, in the profiles ψ and χ above, the voting system returns the same result as IRV, which is the case. For $V = 5$ and $C = 3$, since there is never a tie between several candidates in CIRV (or IRV), the question of the tie-breaking rule vanishes: CIRV therefore defines the solution in an unambiguous way, and this solution is anonymous and neutral. We summarize all these observations in the following proposition and its immediate corollary.

Proposition 10.9

Consider the electoral space of strict total orders for $V = 5$ and $C = 3$, equipped with the impartial culture.

Among the SBVS meeting the Condorcet criterion, CIRV is the only system with a minimal manipulability rate.

Corollary 10.10

Consider a probabilized electoral space with $V = 5$ and $C = 3$. Assume that the culture is decomposable (for example, because voters are independent) and that the probability distribution induced on the profile P *is the impartial culture.*

Among the voting systems meeting **InfMC**, *CIRV has the minimal manipulability rate, which is equal to $\frac{720}{7{,}776} = \frac{5}{54} = \simeq 9.26$ %. It is the only system which meets this property, is an SBVS, and respects the Condorcet criterion. Moreover, it is anonymous and neutral (cf. Section 1.2.2).*

Moreover, we are going to see that the optimal voting system cannot, in this case, be based solely on the weighted majority matrix. Indeed, the profile χ seen above has the following weighted majority matrix:

$D(\chi)$	a	b	c
a	–	3	2
b	2	–	3
c	3	2	–

This matrix is invariant when the circular permutation $(a \to b \to c \to a)$ is applied to the candidates. In particular, it is the same weighted majority matrix for this profile:

$$\chi' : \begin{array}{c|c|c|c|c} b & b & c & c & a \\ a & c & a & a & b \\ c & a & b & b & c \end{array},$$

which is obtained from χ by this permutation.

The optimal voting system, which we have proved to be CIRV for these parameters, cannot be based solely on the weighted majority matrix: it designates a in χ and b in χ'.

More formally, consider a voting system f whose result depends only on the weighted majority matrix. This voting system can use a tie-breaking rule based on the identity of the candidates and/or the weighted majority matrix, but not on the detailed preferences of the voters: for example, it can break ties by lexicographical order on the candidates, as in SVAMP. Such a voting system f must then designate the same winner in χ and in χ', and it is therefore not optimal.

We think that an important consequence of this case study is that in the class **InfMC**, no voting system whose outcome depends only on the weighted majority matrix can be optimal for $V = 5$ and $C = 3$. Indeed, we have just proved this assertion for Condorcet voting systems. For a non-Condorcet voting system which depends only on the weighted majority matrix, it suffices to notice that its Condorcification (which we know is at most as manipulable as the original) only depends on the weighted majority matrix as well, so it cannot be optimal. This observation notably excludes Condorcet-dean, Baldwin, Borda, Black (Condorcet-Borda), CSD, Kemeny, Maximin, Nanson, RP, and Schulze's method.

10.2.3 $V = 7$ and $C = 3$: choose the tie-breaking rule for CIRV, and choose wisely

For $V = 7$ voters and $C = 3$ candidates, it becomes extremely painful to deal with the problem manually. There are $(C!)^V = 279,936$ profiles in total. And above all, there are 7 types of non-Condorcet profiles (up to permutation of voters and/or candidates), instead of one or two types in the previous cases. Recall that, for each of these non-Condorcet profiles, all the Condorcet profiles pointing to it must be considered.

We therefore resort to computer calculations. The greedy algorithm finds a voting system with a manipulability rate of 13.46 %. CPLEX finds an optimal solution with a rate of $\frac{31,920}{279,936} = \frac{665}{5832} \simeq 11.40$ %. The greedy algorithm is therefore not optimal in this case.

In order to study the properties of the set of solutions, one can use CPLEX by imposing additional constraints. For example, we can impose the value $W(\psi, c) = 0$ for a certain ψ and a certain c: if the optimum is no longer reachable in this case, this means that c must win in ψ.

We can also look for voting systems that are optimal in the smaller class of those that are anonymous, neutral, or both. For example, to find an anonymous optimum, we quotient the opportunity graph by the equivalence relation which consists, for two configurations, in being deduced one from the other by permutation of some voters. To find a neutral optimum, we quotient the opportunity graph by permutation of the candidates; care must then be taken to modify c-pointing accordingly.

The minimal manipulability rate found by CPLEX for an anonymous and neutral voting system is $\frac{31,920}{279,936} = \frac{665}{5832} \simeq 11.40$ %: it is therefore an optimum, even among voting systems that are not necessarily anonymous and/or neutral.

Table 10.3 summarizes the main conclusions of this study. For each of the 7 types of non-Condorcet profile (up to permuting voters and/or candidates), we give a necessary condition on the winner for the voting system to be optimal, and we do the same with the additional constraint of anonymity and of neutrality.

Let us take the example of profile ψ_4. For an optimal voting system, we read in the column f of Table 10.3 that the winner must be a or b. If, in addition, we require that the voting system be anonymous and neutral, then we read in column $f_{\text{a\&n}}$ of Table 10.3 that the winner must be b. This is far from obvious, because profile ψ_4 has no symmetry with respect to the candidates. However, if we impose a as the winner for an anonymous and neutral voting system, then we observe, by using CPLEX, that it is no longer possible to achieve minimal manipulability. Moreover, we will see that the conditions of the column $f_{\text{a\&n}}$ are, in fact, sufficient conditions to obtain an optimum.

For any optimum (without any requirement of symmetry), the conditions indicated in the column f of Table 10.3 are only necessary: it is possible that all these conditions are satisfied, as well as their variants by permutation of voters

Non-Condorcet profile	Weighted majority matrix	f	$f_{\text{a\&n}}$	CIRV
$\psi_1 : \begin{array}{c\|c\|c\|c\|c\|c\|c} a & a & a & b & c & c & c \\ c & c & c & a & b & b & b \\ b & b & b & c & a & a & a \end{array}$	$\begin{array}{\|c\|c\|c\|c\|}\hline D(\psi_1) & a & b & c \\ \hline a & - & 3 & 4 \\ \hline b & 4 & - & 1 \\ \hline c & 3 & 6 & - \\ \hline \end{array}$	a	a	a
$\psi_2 : \begin{array}{c\|c\|c\|c\|c\|c\|c} a & a & a & b & c & c & c \\ b & c & c & a & b & b & b \\ c & b & b & c & a & a & a \end{array}$	$\begin{array}{\|c\|c\|c\|c\|}\hline D(\psi_2) & a & b & c \\ \hline a & - & 3 & 4 \\ \hline b & 4 & - & 2 \\ \hline c & 3 & 5 & - \\ \hline \end{array}$	a	a	a
$\psi_3 : \begin{array}{c\|c\|c\|c\|c\|c\|c} a & a & a & b & c & c & c \\ b & b & c & a & b & b & b \\ c & c & b & c & a & a & a \end{array}$	$\begin{array}{\|c\|c\|c\|c\|}\hline D(\psi_3) & a & b & c \\ \hline a & - & 3 & 4 \\ \hline b & 4 & - & 3 \\ \hline c & 3 & 4 & - \\ \hline \end{array}$	a	a	a
$\psi_4 : \begin{array}{c\|c\|c\|c\|c\|c\|c} a & a & a & b & b & c & c \\ c & c & c & a & c & b & b \\ b & b & b & c & a & a & a \end{array}$	$\begin{array}{\|c\|c\|c\|c\|}\hline D(\psi_4) & a & b & c \\ \hline a & - & 3 & 4 \\ \hline b & 4 & - & 2 \\ \hline c & 3 & 5 & - \\ \hline \end{array}$	a, b	b	a, b
$\psi_5 : \begin{array}{c\|c\|c\|c\|c\|c\|c} a & a & a & b & b & c & c \\ b & c & c & a & c & b & b \\ c & b & b & c & a & a & a \end{array}$	$\begin{array}{\|c\|c\|c\|c\|}\hline D(\psi_5) & a & b & c \\ \hline a & - & 3 & 4 \\ \hline b & 4 & - & 3 \\ \hline c & 3 & 4 & - \\ \hline \end{array}$	a, b	b	a, b
$\psi_6 : \begin{array}{c\|c\|c\|c\|c\|c\|c} a & a & a & b & b & c & c \\ c & c & c & a & a & b & b \\ b & b & b & c & c & a & a \end{array}$	$\begin{array}{\|c\|c\|c\|c\|}\hline D(\psi_6) & a & b & c \\ \hline a & - & 3 & 5 \\ \hline b & 4 & - & 2 \\ \hline c & 2 & 5 & - \\ \hline \end{array}$	a, b	a, b	a, b
$\psi_7 : \begin{array}{c\|c\|c\|c\|c\|c\|c} a & a & a & b & b & c & c \\ b & c & c & a & a & b & b \\ c & b & b & c & c & a & a \end{array}$	$\begin{array}{\|c\|c\|c\|c\|}\hline D(\psi_7) & a & b & c \\ \hline a & - & 3 & 5 \\ \hline b & 4 & - & 3 \\ \hline c & 2 & 4 & - \\ \hline \end{array}$	a, b	a, b	a, b

Table 10.3 – Optimal solutions for $V = 7$ and $C = 3$. f: any optimum. $f_{\text{a\&n}}$: anonymous and neutral optimum. Reading: in order for f to be an optimum, $f(\psi_4)$ must be either a or b, but the conditions presented are not sufficient. For $f_{\text{a\&n}}$ to be an anonymous and neutral optimum, it is necessary and sufficient to verify the conditions of the column $f_{\text{a\&n}}$ (and their variants by anonymity and neutrality).

and/or candidates, but that the voting system obtained is not optimal. This is the case, for example, of CIRV equipped with the lexicographical tie-breaking rule used in SVVAMP, as we will see in Table 10.4 of Section 10.3. These necessary conditions are sharp in the following sense: for each of them, there exists an optimum satisfying it. For example, there exists at least one optimum f such that $f(\psi_4) = a$ and there exists at least one such that $f(\psi_4) = b$. These necessary conditions allow us to establish the following proposition.

Proposition 10.11

We consider the electoral space of strict total orders for $V = 7$ and $C = 3$, equipped with the impartial culture.

Regardless of the tie-breaking rule used, the following voting systems are not optimal among Condorcet voting systems: Condorcet-Bucklin, Condorcet-Borda (Black's method), Condorcet-Coombs, and Condorcet-IB.

No voting system based solely on the weighted majority matrix can be optimal.

Any optimal Condorcet voting system has a manipulability rate equal to $\frac{31{,}920}{279{,}936} = \frac{665}{5832} \simeq 11.40\ \%$ *and can be seen as a variant of CIRV, equipped with an adequate tie-breaking rule.*

Proof. According to Table 10.3, for the voting system to be optimal, candidate a must be elected in ψ_1. But in Condorcet-Bucklin or Condorcet-Borda (Black's method), c is declared the winner.

Similarly, a must win in ψ_2. But in Condorcet-Coombs or IB, c wins.

Still according to Table 10.3, candidate a must win in ψ_3. This profile has a weighted majority matrix symmetric by the permutation $(a \to b \to c \to a)$, but the profile itself presents no symmetry with respect to the candidates. Let ψ'_3 denote the profile obtained by applying this circular permutation to ψ_3. If the voting systems depends only on the weighted majority matrix, then it must also designate a in ψ'_3. But if it is optimal, then the winner must be the image of a by this permutation, namely b. These two assumptions are therefore incompatible.

In Table 10.3, we see that the possible winners for an optimal voting systems are always exactly the same as in CIRV, the actual winner depending on the tie-breaking rule. This proves that every optimum is a variant of CIRV (but not the converse). □

For each anonymous and neutral optimum, CPLEX ensures that the conditions presented in the column $f_{\text{a\&n}}$ of Table 10.3 are not only necessary but also sufficient. From this, we deduce the following proposition.

Proposition 10.12

Consider the electoral space of strict total orders for $V = 7$ and $C = 3$, equipped with the impartial culture.

Regardless of the anonymous and neutral tie-breaking rule used, the following voting systems are not optimal among the Condorcet systems: Baldwin, Condorcet-Plurality, CSD, IRVD, Kemeny, Maximin, Nanson, RP, and Schulze's method.

Among the Condorcet SBVS, there are four which are optimal, anonymous, and neutral. Each of them can be seen as CIRV, equipped with an adequate tie-breaking rule.

Proof. According to Table 10.3, for an anonymous and neutral voting system to be optimal, b must win in ψ_4. But this is the case neither in IRVD (b faces c in the first duel and loses), nor in RP (the first validated victory is that of c against b), nor in CSD or Maximin (b has the lowest score), nor in Condorcet-Plurality (a wins), nor in Baldwin or Nanson's method (b is eliminated during the first round), nor in Schulze's method (c is *better* than b), nor in Kemeny's method (b is necessarily behind c).

In order to define an optimal, anonymous, and neutral voting system, it is necessary and sufficient to choose the winner among two possibilities for profiles ψ_6 and ψ_7. So, four voting systems are possible. All other non-Condorcet profiles are deduced from those in Table 10.3 by anonymity and neutrality. The fact that they are all variants of CIRV is a simple particular case of Proposition 10.11 which we recall for the sake of exhaustiveness. □

Propositions 10.11 and 10.12 exclude all the Condorcet voting systems that we have studied except CIRV, regardless of the tie-breaking rule used if it is anonymous and neutral. CIRV, equipped with an anonymous and neutral tie-breaking rule, is optimal iff the tie-breaking rule meets the conditions indicated in the column $f_{\text{a\&n}}$ of Table 10.3.

It would be interesting to formulate such a tie-breaking rule, not by an exhaustive enumeration of the cases, but by a property that we could try to apply to other cases than $V = 7$ and $C = 3$ (intensive definition). This would perhaps make it possible to identify a version of CIRV particularly resistant to manipulation in the general case.

Let us give it a first try with the following rule: in the event of a tie between two candidates in an elimination round, we organize a virtual duel and we eliminate the loser.[1] But this rule does not work. Indeed, in profiles ψ_4, ψ_5, ψ_6, and ψ_7, candidates b and c are tied for elimination in the first round and b, the loser of the duel, is eliminated; then a is declared the winner. By permutation of voters and/or candidates, we deduce that this tie-breaking rule unambiguously determines the winner in all non-Condorcet configurations and that the voting system obtained is anonymous and neutral. But, in that case, the winner in ψ_4 should be b and not a (Table 10.3). Therefore, the voting system cannot be optimal.

Now consider another tie-breaking rule: if two winners are possible by CIRV, then we organize a virtual duel to choose the final winner of the election.[2] In ψ_4, ψ_5, ψ_6, and ψ_7, the winner obtained is b, so the voting system is indeed optimal.

However, this solution is not entirely satisfactory. For an anonymous and neutral voting system to be optimal, b must win in ψ_4 and ψ_5. But in ψ_6 or ψ_7, choosing a is also fine. The tie-breaking rule exhibited is therefore sufficient but not necessary: to exactly cover the set of anonymous and neutral solutions, it would be necessary to identify a "natural" tie-breaking rule (i.e. defined in an intensive way, and as simple as possible) that would choose candidate b in profiles ψ_4 and ψ_5 but remains undecided between a and b in profiles ψ_6 and ψ_7. We leave this question for future work.

10.2.4 $V = 3$ and $C = 4$: a complicated set of optima

For $V = 3$ candidates and $C = 4$ candidates, one might think that the problem is simpler, because there are "only" $(C!)^V = 13,824$ profiles in total. However, there are 12 types of non-Condorcet profiles (up to permutation of voters and/or candidates), instead of 7 in the previous case. Furthermore, we will see that there

[1] In the event of a tie between more than two candidates, we can choose the eliminated candidate by using a tie-breaking rule of which the electoral duel is a particular case: for example, eliminate the candidate with the lowest Borda score in the weighted majority matrix restricted to the candidates tied for elimination.

[2] If there is a tie between more than two candidates, we can choose a tie-breaking rule of which the electoral duel is a particular case: for example, we designate the candidate with highest Borda score in the weighted majority matrix restricted to the candidates tied for victory.

is no anonymous and neutral optimum and that the set of solutions is therefore more difficult to explore.

The greedy algorithm finds a voting system with a manipulability rate of 21.09 %. CPLEX finds an optimal solution with a rate of $\frac{2{,}688}{13{,}824} = \frac{7}{36} \simeq 19.44$ %. The greedy algorithm is therefore again not optimal in this case. By a detailed study of the solution that CPLEX exhibits by default, we were able to show that it was neither anonymous nor neutral. As a consequence, the solution is not unique: indeed, any voting system which is deduced from it by a permutation of the candidates is distinct from it and is also optimal.

If we impose that the solution be neutral, we find the same manipulability rate. If we impose that the solution be anonymous, we obtain a rate of $\frac{2{,}712}{13{,}824} = \frac{113}{576} \simeq$ 19.62 %. If we impose that the solution be anonymous and neutral, we obtain a rate of $\frac{3{,}264}{13{,}824} = \frac{17}{72} \simeq 23.61$ %.

It is difficult to study the set of all arbitrary solutions in this case (i.e. with no symmetry assumption). Indeed, because of their multiplicity, there are generally 2 or even 3 possible winners for each non-Condorcet profile.[3]

However, more can be said about neutral solutions. For this, consider the following profiles.

$$\psi_1 : \begin{array}{c|c|c} a & c & b \\ b & a & c \\ c & b & a \\ d & d & d \end{array} \qquad \psi_2 : \begin{array}{c|c|c} a & b & c \\ b & c & a \\ c & a & b \\ d & d & d \end{array}$$

By permutation of candidates, these two profiles cover all cases where voters place the same candidate last and realize a minimal example of Condorcet paradox between the three other candidates. These profiles are deduced from each other by permutation of the two voters on the right.

By using CPLEX, we see that an optimum cannot elect d in ψ_1, which is quite intuitive since it is the candidate placed last by all the voters. By symmetry of this profile, any candidate a, b, or c can be designated in an optimal voting system. This amounts to favoring voter 1, 2, or 3 in this profile ψ_1 and those deduced from it by permutation of the candidates. To fix ideas, suppose that a is declared the winner. This designates voter 1 as privileged in ψ_1 and also breaks the symmetry between the two other voters: voter 2 (resp. 3) is the one for whom the winning candidate is in second (resp. third) position.

In profile ψ_2, CPLEX informs us that there are then only two possible winners: a or b. In other words, we favor either the same voter as in profile ψ_1, or the voter whose second preferred candidate is elected in ψ_1. When one option or the other option is fixed, CPLEX ensures that the neutral optimum is unique.

Since there a 3 possible choices for ψ_1 then 2 choices for ψ_2, there are 6 possible solutions. Seeing this number, one might think that it is a certain voting system and its variants for the $3! = 6$ permutations of the voters. But it is not the case. Indeed, consider the unique neutral optimum f obtained for $f(\psi_1) = a$ and $f(\psi_2) = a$. If we exchange the last two voters, then the optimal voting system obtained is the same. Thus, the orbit of f by permutation of the voters has only 3 elements and not 6. Similarly, if we consider the unique neutral optimum g obtained with $g(\psi_1) = a$ and $g(\psi_2) = b$, its orbit has only 3 elements by permutation of the voters. There are therefore exactly two distinct neutral solutions, up to permutation of the voters.

[3] In an optimal voting system, there are 2 possible winners in 8 types of non-Condorcet profiles, and 3 possible winners in 4 types of non-Condorcet profiles. This remains true even if we impose the neutrality of the solution.

Profiles ψ_1 and ψ_2 above also make it possible to intuitively understand why the anonymous and neutral optimum is significantly more manipulable than the general optimum. Indeed, if we impose anonymity and neutrality, then candidate d must win in these two profiles, whereas she is the least liked by all voters. This idea can be linked to the fact that, for these values of the parameters, there exists no neutral, anonymous and efficient voting system (cf. Section 1.2.2).

The following proposition will summarize the observations made at this stage.

Proposition 10.13

Consider the electoral space of strict total orders for $V = 3$ and $C = 4$, equipped with the impartial culture.

Any optimal Condorcet voting system has a manipulability rate of $\frac{2{,}688}{13{,}824} = 19.44$ %. *There exists at least one non-neutral optimum (and it is not unique).*

There are exactly 6 optimal voting systems which are neutral. Up to permutation of the voters, there are 2 distinct solutions. Each has 3 variants by permutation of the voters.

There is no optimum that is anonymous.

We are now going to present some results which relate these solutions to usual voting systems.

Proposition 10.14

Consider the electoral space of strict total orders for $V = 3$ and $C = 4$, equipped with the impartial culture.

Regardless of the neutral tie-breaking rule used, the following voting systems are not optimal among Condorcet systems: Baldwin, Condorcet-Borda (Black's method), Condorcet-Coombs, CIRV, Condorcet-Plurality, CSD, IRVD, Kemeny, and Nanson's method.

Proof. As above, let f be the neutral optimum obtained with $f(\psi_1) = a$ and $f(\psi_2) = a$, and g the neutral optimum obtained for $g(\psi_1) = a$ and $g(\psi_2) = b$.

Consider the following profile.

$$\psi : \begin{array}{c|c|c} b & a & d \\ c & c & a \\ d & b & b \\ a & d & c \end{array}$$

$D(\psi)$	a	b	c	d
a	–	2	2	1
b	1	–	2	2
c	1	1	–	2
d	2	1	1	–

Thanks to the optimization performed by CPLEX, we were able to determine that the winner must be c, both for f and for g.

However, she is elected neither in CIRV (c is eliminated during the first round), nor in Condorcet-Plurality (c has no vote), nor in CSD (c has the lowest score), nor in Condorcet-Borda or Nanson's method (c has the lowest Borda score), nor in Kemeny's method (if we rearrange the weighted majority matrix with c in first position, there are two defeats with an amplitude of one vote above the diagonal, instead of only one defeat of one vote if we keep the alphabetical order on the candidates).

In IRVD, since c has no votes and all the others have at least one, c participates in all elimination duels. Since she is not a Condorcet winner, she cannot win all of them. Hence c cannot be elected.

In Baldwin's method or in Condorcet-Coombs, for c to win, she must face d during the last round because it is the only candidate that she can defeat in an electoral duel. Hence, in the first round, it is necessary that neither c nor d is eliminated.

However, in Condorcet-Coombs, there is a tie between a, c, and d during the first round. But since we must eliminate neither c nor d, a must be eliminated. In this case, d is eliminated, then c and finally, b is the winner. So c cannot win.

In Baldwin's method, c or d are the only ones who can be eliminated during the first round. So c cannot win. □

Proposition 10.15

Consider the electoral space of strict total orders for $V = 3$ and $C = 4$, equipped with the impartial culture.

Any optimal Condorcet voting system can be seen as Condorcet-Bucklin, Condorcet-IB, Schulze's method, RP, or Maximin, equipped with an adequate tie-breaking rule.

Proof. For the sake of conciseness, we simply explain the method followed. For each of the 12 types of non-Condorcet profiles (up to permutation of voters and/or candidates), we determine with CPLEX which candidates can be winners in an optimal voting system, as we did in Table 10.3 for $V = 7$ and $C = 3$.

In each case, we observe in practice that the set of possible winners is the same with or without the assumption of neutrality and that it is equal to the set of candidates who can win in Condorcet-Bucklin. Manually, we check that this set is included in the set of candidates who can win in Condorcet-IB (resp. Schulze's method, RP, Maximin). We emphasize that this means that every optimum is Condorcet-Bucklin equipped with an adequate tie-breaking rule, but not necessarily that any tie-breaking rule makes Condorcet-Bucklin optimal (as we will see in Table 10.4). □

The previous proposition can be interpreted as an argument in favor of the five voting systems mentioned. However, some of them lead to more ties than others. Condorcet-Bucklin leaves the least leeway to the tie-breaking rule: out of the 12 types of non-Condorcet profiles (up to permutation of voters and/or candidates), it causes a 3-candidate tie in 4 profiles and a 2-candidate tie in 8 profiles. IB always causes a 3-candidate tie. Schulze's method and RP cause a 4-candidate tie (i.e. a tie between all candidates!) in 6 profiles and a 3-candidate tie in 6 profiles. Finally, Maximin causes a 4-candidate tie in 7 profiles and a 3-candidate tie in 5 profiles. It is therefore not so surprising that any optimal voting system can be seen as Maximin equipped with an adequate tie-breaking rule, because Maximin is especially irresolute in the case $V = 3$ and $C = 4$.

Thus, we were able to determine the minimal manipulability rate for a Condorcet voting system, and more generally for a voting system meeting **InfMC**, in impartial culture, in four cases: $V = 3, 5$, or 7 voters and $C = 3$ candidates; $V = 3$ voters and $C = 4$ candidates. For more voters or candidates, the problem is too large to handle on the machine we used.[4]

10.3 Comparison between the optimum and the usual voting systems

Now we will compare the optimal manipulability rate (within the class **InfMC**) to that of various voting systems. Note that as soon as we have the

[4]Dell Precision M6600, Intel Core I7-2820QM at 2.30 GHz, 8 Mb of cache, 16 Gb of RAM at 1.33 GHz in DDR3.

opportunity graph, we can easily determine the manipulability rate of any Condorcet voting system: indeed, it suffices to report in the graph the winners of all non-Condorcet configurations and to contaminate the corresponding Condorcet configurations.

In Table 10.4, we have indicated the exact manipulability rates of a variety of voting systems.

- The optimum is obtained by integer linear programming in CPLEX, as well as the optimum in the restricted class of Condorcet systems that are anonymous, neutral, or both. For $V = 3$ and $C = 3$, it it not possible to have an anonymous and neutral voting system (cf. Proposition 1.16).
- The approximate optimum is obtained by the greedy algorithm presented previously.
- For the voting systems satisfying the Condorcet criterion, we use the opportunity graph to determine their exact manipulability. For those implemented in SWAMP, one can also use it with the option `CM_option = 'exact'`, which provides an additional check of consistency for the results obtained.
- For the other voting systems, such as Borda or Veto, we use SWAMP with the option `CM_option = 'exact'`.

In this table, we stress on the fact that the tie-breaking rules are important, because we consider a small number of voters. Thus, it is exaggerated to say that we have represented the manipulability rate of IRV (for example): it is that of IRV, equipped with the tie-breaking rule used in SWAMP, i.e. by lexicographical order on candidates. Thus, we found that some known voting systems (especially CIRV for $V = 7$ and $C = 3$, Condorcet-Bucklin for $V = 3$ and $C = 4$) could be optimal when equipped with an adequate tie-breaking rule, but this does not appear in Table 10.4. For future works, it would be interesting to vary this tie-breaking rule.

For each value of the parameters, we also indicated the rate of non-Condorcet profiles, which provided a first lower bound of the manipulability rate for voting systems satisfying **InfMC**, before we knew the optimal rate.

Each manipulability rate is written in bold and green iff it is equal to the minimal manipulability rate in **InfMC**.

For $V = 3$ and $C = 3$, many voting systems are optimal, in particular all those meeting the Condorcet criterion, as we showed in Section 10.2.1.

For $V = 5$ and $C = 3$, the optimal voting systems are CIRV (which coincides with the result of the greedy algorithm and which is the only Condorcet optimum, as we showed in Section 10.2.2), and also IRV and EB (which coincides with the Two-round system, because there are 3 candidates). This does not contradict what we saw in Section 10.2.2: indeed, we showed that, among the voting systems meeting **InfMC**, CIRV is the unique optimum which is a Condorcet SBVS. But IRV and EB do not satisfy the Condorcet criterion and EB is not an SBVS. It it therefore impossible for them to be less manipulable than CIRV, but it is not forbidden for them to have the same manipulability. Moreover, Green-Armytage et al. (2014) proved that, when there are 3 candidates, CIRV and IRV have the same manipulability.

For $V = 7$ and $C = 3$, we have shown that there exist 4 voting systems which are optimal among Condorcet systems and which are also anonymous and neutral. Each of them can be seen as a CIRV, equipped with an adequate tie-breaking rule.

Populations				
Number of voters V	3	5	7	3
Number of candidates C	3	3	3	4
Non-Condorcet profiles (%)	5.56	6.94	7.50	11.11
CM rate (%)				
Optimum	**11.11**	**9.26**	**11.40**	**19.44**
Neutral optimum	**11.11**	**9.26**	**11.40**	**19.44**
Anonymous optimum	**11.11**	**9.26**	**11.40**	19.62
Anonymous neutral optimum	—	**9.26**	**11.40**	***23.61***
Approximate optimum (greedy)	**11.11**	**9.26**	13.46	21.09
CIRV	**11.11**	**9.26**	12.38	21.35
IRV	**11.11**	**9.26**	12.38	21.35
EB	**11.11**	**9.26**	12.53	21.35
TR	**11.11**	**9.26**	12.53	21.35
RP	**11.11**	18.52	26.56	21.09
Bald.	**11.11**	17.75	24.26	22.27
CBuck.	**11.11**	18.52	24.98	22.22
CDean	**11.11**	20.06	27.31	22.83
Nan.	**11.11**	***20.83***	***27.86***	***22.92***
CDict.	**11.11**	***20.37***	27.28	***23.96***
Kem.	**11.11**	***21.60***	***29.26***	***22.92***
CSD	**11.11**	***21.60***	***29.26***	***22.92***
Sch.	**11.11**	***21.60***	***29.26***	***23.09***
IRVD	**11.11**	***22.38***	***30.76***	22.57
CCoo.	**11.11**	***22.38***	***36.16***	22.22
Max.	**11.11**	***21.60***	***29.26***	***24.05***
CPlu.	**11.11**	***22.38***	***30.76***	***24.83***
IB	**11.11**	***22.38***	***36.16***	***23.26***
CBor.	**11.11**	***27.01***	***37.76***	***27.78***
Plu.	***16.67***	***26.23***	***33.01***	***30.21***
Buck.	***13.89***	***28.16***	***40.13***	***23.61***
Veto	***26.39***	***32.83***	***35.54***	***50.20***
Coo.	***22.22***	***38.58***	***53.41***	***39.84***
Bor.	***23.61***	***44.37***	***53.36***	***51.39***

Table 10.4 – Exact CM rates, in percents (impartial culture). We use the lexicographical tie-breaking rule, which is the one implemented in S$\forall$AMP.

For $V = 3$ and $C = 4$, we have seen that CIRV cannot be optimal, even with a well-chosen tie-breaking rule. Any optimal Condorcet voting system can be seen as Condorcet-Bucklin, Condorcet-IB, Schulze's method, RP, or Maximin, equipped with an adequate tie-breaking rule.

Since the Condorcet-dean system is intuitively the simplest of Condorcet systems (along with Condorcet-dictatorship), it seems a good reference for manipulability: indeed, if a voting system is more manipulable than this one, it has the harmful property of having worst performances than a very unsophisticated voting system. As an indication, manipulability rates higher than that of Condorcet-dean are written in italic and red characters in the table. Many voting systems (Nanson, Condorcet-dictatorship, Kemeny, CSD, Schulze, IRVD, Condorcet-Coombs, Maximin, Condorcet-Plurality, IB, Condorcet-Borda, Plurality, Bucklin, Veto, Coombs, and Borda) are generally more manipulable than Condorcet-dean for the parameters considered. The only voting systems that always have a lower manipulability rate in this table are those obtained by the greedy algorithm, CIRV, IRV, EB, the Two-round system, RP, Baldwin's method, and Condorcet-Bucklin.

Conclusion and Perspectives

At the time of concluding this memoir, we would like to come back to various notions encountered, with particular attention to perspectives they suggest for future work.

In the introduction to this dissertation, we recalled what we think are the founding paradoxes of voting theory. To the relatively traditional triptych consisting of the paradox of Condorcet (1785), the theorem of Arrow (1950), and the theorem of Gibbard (1973) and Satterthwaite (1975), it seems to us that one must add the various impossibility results on the interpersonal comparison of utilities, which limit one of the only conceivable escapes from Arrow's theorem and, thereby, reinforce the negative result that it states. We recalled that the result of Gibbard (1973) does not only apply to ordinal voting systems, contrary to a weakened formulation of the theorem which is sometimes presented. We also discussed Gibbard's nondeterministic theorems (Gibbard, 1977, 1978), which fully characterize non-manipulability for voting systems involving an element of chance. But we believe that the obtained systems (in particular *random dictatorship*) should be limited to applications where elections are frequent and where the result has a moderate impact. In contrast, our work concerns deterministic voting systems.

We insisted, as does Gibbard (1973), on the fact that the deep problem with manipulability is the lack of *straightforwardness*, that is, the fact that voters cannot always optimally defend their opinion without knowing the ballots of other voters. This causes many doubts when voting, a *de facto* asymmetry between informed and uninformed voters, problems with the legitimacy of the outcome, and a questionable power granted to information sources such as polling institutes. But we also recalled that low manipulability can be seen as an approximate straightforwardness, and that the study of manipulability can therefore be approached as a technical means of tackling straightforwardness.

While respecting the point of view that condemns the action of *manipulation*, we have defended a perhaps less orthodox view which consists in defending the strategic attitude of voters but condemning *manipulability*, not because there is a risk of obtaining a manipulated result, but because, due to a lack of information or strategic approach from the voters, there is a risk of getting a sincere result which is not a strong Nash equilibrium (SNE) and which would imply regret for some voters and question the legitimacy of the result. We have thus argued that an easy manipulation is more desirable than a difficult one; but that the easiest "manipulation" occurs when sincere voting is strategically optimal, that is, when the configuration is not manipulable.

In Chapter 1, we defined our formalism, in particular the notion of *electoral space*, which makes it possible to study in all generality voting systems based on orders but also on approval values, grades, or any other type of information. We gave some examples of electoral spaces in order to show the richness of such a framework, but this list is far from exhaustive. We defined *general voting systems*, inspired by the framework of *game forms* introduced by Gibbard (1973). We focused on *state-based voting systems*, which overcome the issue of a problematic definition of a canonical sincere voting. Indeed, we showed that, in order to limit manipulability, investigation can be restricted to SBVS, thus generalizing a result of Moulin (1978) which applied to ordinal electoral spaces only. We showed how to go back and forth between general voting systems and SBVS by the state-based version of a general voting system and by the canonical implementation of an SBVS.

In Section 1.5, we recalled the usual notions of relative Condorcet winner and weak Condorcet winner, and we defined the concepts of absolute Condorcet winner and Condorcet-admissible candidate, of which the rest of this memoir showed the relevance for the study of manipulability.

In Section 1.6, we presented the voting systems studied in this memoir. In particular, we defined the *Iterated Bucklin's method* (IB). The system *IRV with duels* (IRVD), a variant of *Instant-Runoff Voting*, is a contribution of Laurent Viennot. We also defined the systems *Condorcet-Dean* and *Condorcet-dictatorship* which, by their simplicity, provide good examples and counter-examples for various theoretical results.

Among the different concepts introduced in Chapter 1, we would like to discuss one last time two of them in particular.

First, through anonymity (Section 1.2.2), and later through the majority unison ballot criterion **MajUniBal** (Section 3.1), we saw that one could define *group actions of the permutations over an electoral space* and *isomorphisms of voting systems*. In general, it seems interesting to define notions which are stable by isomorphism in order to have properties that are intrinsic and not related to the labeling of objects. Several applications could be obtained by such an approach.

1. For example, we believe that for some voting systems, but not all, it is possible to define a canonical sincere ballot as a strategy that is optimal in all neutral cultures, i.e. stable by a certain class of isomorphisms. After clearing up the theoretical difficulties to rigorously define this notion in any electoral space, it would be interesting to study which voting systems have such a canonical sincere ballot and for which this is not the case.

2. Furthermore, the notion of isomorphism could be extended to a more general notion of morphism of voting systems that is not necessarily an isomorphism, i.e. not necessarily bijective. This opens up prospects for examining transformations of voting systems or for establishing correspondences between certain known systems, which could explain certain similarities of behavior.

Secondly, in the context of this memoir, even though we use the word *preference* for brevity, the exact meaning of the binary relation $c \mathrm{P}_v \mathrm{w}$ is as follows: if candidate w is the sincere winner, then voter v is inclined to act so as to make candidate c the winner instead. By allowing preferences that are not antisymmetric but arbitrary binary relations, we were able to model a wide variety of behaviors. We mainly saw examples of preferences that violate transitivity, or even negative-transitivity (Examples 1.6, 1.7, 1.8). Another possible advantage of our framework is to combine manipulability in the usual sense (where voters manipulate for the outcome to be more in line with their personal opinions) and bribery (where some voters can manipulate in one direction or the other, depending on whether or not they receive a bribe). We only touched on this possibility briefly in Section 1.3 but there is a much broader field of study in this direction. One can imagine, for example, that for each voter and each pair of candidates (w, c), there is a minimum amount of money for which the voter agrees to participate in a manipulation for c against w. Thus, the matrix of her binary relation is replaced by a matrix of nonnegative real costs, in which $+\infty$ corresponds to **False** (never interested) and 0 to **True** (always interested, even without financial incentive). We can then examine with what total budget it is possible to make a given candidate win, as is customary in bribery problems.

In Chapter 2, we defined the *informed majority coalition criterion* (**InfMC**), which is the weakest of the majority criteria that we defined thereafter. We proved the weak and strong Condorcification theorems 2.9 and 2.20: if a voting system satisfies **InfMC**, then its Condorcification is at most as manipulable; if, furthermore, it does not satisfy the *resistant-Condorcet criterion* (**rCond**), then its Condorcification is strictly less manipulable. As we have seen, the strong version of the theorem applies to most of the usual voting systems.

To demonstrate the strong Condorcification theorem 2.20, we introduced and characterized the notion of resistant Condorcet winner. This concept provides an upper bound (in the sense of inclusion) of manipulability for Condorcet systems, which we showed to be tight for $C \geq 6$, but which we later saw was not tight for $C = 3$ (Chapter 10). It would be interesting to study the intermediate values of C in order to also have a tight upper bound in this case.

We showed that, in the general case, it is the *absolute* Condorcification that makes the Condorcification theorems work. However, we also saw that a *relative* Condorcification can, for Plurality or the Two-round system, decrease manipulability, even compared to the absolute Condorcification. On the other hand, relative Condorcification does not decrease manipulability for IRV, Majority judgment, Approval voting, or Range voting. It would be interesting to extend this investigation to all classical voting systems and, if possible, to identify a condition (applying to as many systems as possible) sufficient for the relative Condorcification to be at most as manipulable as the initial voting system, or even at most as manipulable as the absolute Condorcification.

As we saw later in the simulations (Chapters 7 to 10), the manipulability reduction offered by Condorcification is not necessarily very large quantitatively, especially when comparing IRV and its Condorcification CIRV. It seems that the main consequences of these Condorcification theorems are rather to be found in Corollaries 2.22 and 2.23: to find a voting system of minimal manipulability within the class **InfMC**, investigation can be restricted to Condorcet systems.

In Chapter 3, we established a hierarchy of various majoritarian criteria for a voting system and we showed their links with the existence and a certain form of uniqueness of strong Nash equilibria. In particular, we extended the results of Sertel and Sanver (2004) and Brill and Conitzer (2015) by showing that the criterion **InfMC** is equivalent to the *criterion of restriction of possible SNE to Condorcet-admissible candidates* **RSNEA**, according to which any winner of an SNE is a Condorcet-admissible candidate ("uniqueness" of the equilibrium). We also studied the *criterion of existence of an SNE for any Condorcet winner* **XSNEC**, according to which there exists an SNE in any Condorcet configuration of preferences (existence of an equilibrium). We showed that the class **XSNEC** is included in the class **IgnMC** of systems satisfying the *ignorant majority coalition criterion* and contains the class **MajBal** of systems meeting the *majority ballot criterion* (within reasonable electoral spaces), and that these inclusions are generally strict. It would be interesting, in the future, to have a simple characterization of the voting systems that satisfy **XSNEC**.

Moreover, we showed that the *criterion of restriction of possible SNE to Condorcet winners* **RSNEC** and the *criterion of existence of an SNE for any*

Condorcet-admissible candidate **XSNEA** are less "natural" than their weaker versions, **RSNEA** and **XSNEC** respectively, because they are met by few classical voting systems, because they do not have simple inclusion relations with the other criteria, and because these two criteria are generally incompatible with each other.

We examined the criteria satisfied by the usual voting systems, by compiling classic results from the literature and by presenting original results, particularly, on the one hand, on the criteria **InfMC** and **rCond** and, on the other hand, on all the criteria for the Iterated Bucklin's method. Despite our efforts, a few questions remain open: finding a simple necessary and sufficient condition under which a positional scoring rule (PSR) meets **InfMC** and conditions under which an iterated PSR (IPSR-1, with simple elimination, or IPSR-A, with elimination based on the average) meets **rCond**.

In Section 3.5, we commented informally on the links between the majority criteria and the concept of information exchange, in particular to achieve an equilibrium or, at least, to find the same winning candidate as in an equilibrium. Here again, there is a whole field of possible investigation, in connection with the field of distributed algorithms. For example, one can ask the following question: given a voting system, is there an algorithm that achieves an SNE when one exists? In synchronous execution without failure, this is obvious, since it is possible to emulate a non-distributed behavior. But what about asynchronous execution? Is it possible to establish an algorithm which is in a sense resistant to failures, that is, in this context, to the non-cooperation of certain voters? And above all, what is the best complexity in exchange of information that such an algorithm can have? In worst case? On average?

In Chapter 4, we defined *generalized Condorcification* based on the notion of *family*, which is inspired by the theory of simple games. This enabled us, in particular, to apply a transformation inspired by Condorcification to voting systems that do not satisfy a majority criterion in particular (generalized Condorcification theorem 4.18). We compared the manipulability of Condorcifications performed with two different families (compared Condorcification theorem 4.21). Finally, we defined the *maximal family* of a voting system, that is to say, the family of coalitions that can manipulate in an informed way for such or such candidate. We showed that, under certain assumptions, the Condorcification using this maximal family is the least manipulable of the Condorcifications performed with a family $\mathcal{M}$ such that the system meets $\mathcal{M}$**InfC** (maximal Condorcification theorem 4.25). We were thus able to establish that, for the usual systems which satisfy **InfMC**, the majoritarian Condorcification is the maximal Condorcification, and therefore that it is optimal in some sense. In particular, we showed that (for an odd number of voters and antisymmetric preferences), Condorcet systems are their own maximal Condorcification. We also provided examples of generalized Condorcification for various voting systems non marginally violating anonymity and/or neutrality, which showed a broad field of application for the theorems of this chapter.

To extend these results, the ideal would be to find a theorem of the following form: the maximal Condorcification of f is a minimally manipulable voting system among those which share a certain well-chosen property with f. The maximal Condorcification theorem we have is, in a way, of this form, but the common property is to be a generalized Condorcification of f by a family $\mathcal{M}$ such that f meets **InfMC**. It would be interesting to have a simpler and more intuitive property.

Moreover, the families that we consider are indexed by a candidate c and describe the coalitions that can make c win (usually in an informed way). This approach has the advantage of allowing a direct translation of the reasoning carried out for the weak Condorcification theorem 2.9 into the generalized Condorcification theorem 4.18. But one could also consider generalizing this *vector* of collections of coalitions, indexed by c, to a *matrix* of collections of coalitions, indexed by the winner w to dethrone and the candidate c for which we want to manipulate.

In Chapter 5, we showed the slicing theorem 5.9: if the culture is *decomposable*, then for each voting system, there is one of its slices which is at most as manipulable as the original system (in the probabilistic sense). In particular, we showed that this is the case when voters are independent by proving a more general probability result, the lemma of the complementary random variable 5.16. We examined the possible generalizations of this theorem and showed that one cannot demand a decrease in manipulability in the sense of inclusion, and that it is impossible to purely and simply suppress the decomposability assumption. The main open question that we left is whether one can give a weaker assumption than decomposability. In particular, it would be interesting to know if condition (5.1) from Proposition 5.8, which is necessary but not sufficient for decomposability, is still sufficient to imply the same conclusions as the slicing theorem. The ideal would even be to have a sharp version of the theorem, that is, to find a condition on the culture, not only sufficient, but also necessary so that any SBVS admits a slice at most as manipulable as the original system.

The slicing theorem 5.9 is not constructive, which does not lead to an immediate practical application. But, in our opinion, its main consequence is the optimality theorem 5.15: if the culture is decomposable, then a system that is optimal (in the probabilistic sense) among ordinal Condorcet voting systems is also optimal in the much broader class of systems which verify **InfMC** and which can fail to be ordinal. If the objective is to reduce manipulability, this means that systems such as Range voting, Majority judgment, or Approval voting are *a priori* unpromising, even before examining their results in simulations. That said, if the objective is a certain algorithmic simplicity in the identification of a strategic vote, Approval voting has interesting properties, which we have already mentioned.

In Chapter 6, we presented SVVAMP, *Simulator of Various Voting Algorithms in Manipulating Populations*, a Python package dedicated to the study of voting systems and especially their manipulability. Voting populations can be characterized by strict weak orders or by utilities. It is possible to import them from external files or generate them by a variety of random models. This simulator implements coalitional manipulation (CM) and several variants (informed coalitional manipulation ICM, unison manipulation UM, trivial manipulation TM), as well as individual manipulation (IM), independence of irrelevant alternatives (IIA), and the Condorcet notions presented in this memoir.

Generic methods make it possible to quickly implement new voting systems and dedicated algorithms are implemented for some of them. We gathered algorithms corresponding to the state of the art and we developed original methods, especially for IRV and its variants, given the particular interest of these voting systems to achieve low manipulability.

Such software is by nature a constantly evolving project, where developments are always possible. Regarding populations, it would be possible to implement more general models, in particular non-transitive preferences. We also plan to implement the models of Mallows (1957) and Pólya-Eggenberger (Lepelley and Valognes, 2003), in order to compare the results with those obtained in the VMF model. For populations from real data, we also want to implement modules that test whether a given population is close to a given model (*single-peaked*, VMF, etc) in order to test the degree of realism of these assumptions.

It would also be interesting to make it possible to vary the tie-breaking rule used for each voting system, but one should be aware that this option entails significant difficulties for the manipulation algorithms. Finally, the main source of improvement would be to implement dedicated manipulation algorithms for voting systems that do not already have any and currently use generic algorithms. Regarding IRV, the main improvement would be to find a polynomial heuristic that allows to certify the non-manipulability in a high proportion of non-manipulable configurations, in the same way that our heuristic identifies a significant proportion of manipulable configurations .

Given the good performance of IRV in terms of manipulation, it would be interesting to implement several variants of IRV in SVVAMP in order to compare their manipulability. We have already studied IRVD and of course CIRV. Currently, two other variants of IRV are implemented in SVVAMP: IRVA, which is the IPSR-A associated with Plurality, and another system called *Instant-Condorcet Runoff Voting* (ICRV). We preferred not to include these systems in our study for the following reason: since these systems usually have a very low TM rate (like IRV), the generic algorithm for CM usually gives a pretty high algorithmic uncertainty, which does not make it possible to compare the manipulability of these systems to the others. In the future, it would be interesting to implement dedicated algorithms for these systems and to add other variants of IRV, such as those mentioned by Green-Armytage (2011) (which include ICRV).

One of the objectives of SVVAMP, in the long term, is to finely measure manipulability, not only for a culture, but for a given profile. For this, we have developed a methodology by adding random noise, but it would be interesting to develop other metrics. For example, a family of manipulability indicators proposed in the literature is based on the size of coalitions (which is already partially implemented in SVVAMP). Another approach seems interesting, inspired by problems of corruption and reflections on the analysis of real experiences: the notion of *manipulation threshold*. For each candidate c different from the winner w and for each real number ε, we can ask the following question: by considering the voters for whom c provides a greater utility improvement than ε compared to w, is this coalition able to make c win? By authorizing negative values of ε, the coalition can contain all voters, so there is a threshold ε below which it is actually possible.[5] This threshold not only contains the information of the manipulability for c (which is true iff the threshold is positive) but it also measures the ease of manipulation or a form of distance to a manipulable configuration.

Note that SVVAMP is designed to perform a wide range of simulations, only part of which has been exploited herein. The most important opportunities, which we have ignored in order to devote ourselves to manipulation by coalition, concern individual manipulation. Indeed, dedicated algorithms are implemented for almost as many voting systems as in the case of manipulation by coalition: this is the case for Majority judgment, Maximin, Plurality, the Two-round system (TR), Veto, Approval voting, Range voting, Exhaustive ballot (EB), IRV, and the methods of Borda, Bucklin, Coombs, and Schulze. It would therefore be interesting to use SVVAMP to study individual manipulation.

In Chapters 7 and 8, we used SVVAMP to study the manipulability of various voting systems in spheroidal cultures, using for the first time the Von Mises-Fisher

[5] Here, we make the implicit assumption that the voting system is surjective, that is to say that any candidate is actually eligible. This condition is met by all reasonable voting systems (except possibly Veto if $C > V + 1$).

model, then in cultures based on a political spectrum. This allowed us to verify some results known in the literature and to propose a number of conjectures about the monotonicity and the limit of the manipulability rates with respect to certain parameters, particularly C and V. While we have proven some of them, others remain conjectures.

Among these conjectures, the one that seems the most accessible is that the manipulability rate of reasonable voting systems (**InfMC**) tends to 1 when $C \to \infty$ in spherical culture (Conjecture 7.4). We have demonstrated this result for an odd number of voters. To extend the result to V even, it would be sufficient to prove that in impartial culture, the probability of having a Condorcet-admissible candidate (that is to say, a weak Condorcet winner) tends to 0 when $C \to \infty$. Moreover, we proved that the CM rate of Veto (with the tie-breaking rule used in SVAMP) does not tend to 1 when $C \to +\infty$ in impartial culture, even if the simulations suggest a limit close to 1.

It also seems interesting to examine the limits of the manipulability rates in spherical culture when the number of voters is very large (Conjectures 7.7 and 7.8). Although the spherical culture is not intended to be a descriptive model, it offers a normative baseline which can be seen as a worst case, since it is the most disorderly (the entropy is maximum). Thus, it is interesting to know which voting systems do not have a manipulability rate tending to 1 when $V \to \infty$ in this culture *a priori* rather unfavorable.[6] We conjectured that, among the voting systems studied, only IRV, EB, CIRV, and Veto have a CM rate that tends to a limit that is different from 1 when $V \to +\infty$ in impartial culture.

We observed oscillatory phenomena for CM rates, which are more pronounced in cultures of one-dimensional political spectrum than in spheroidal culture, and we have proposed a qualitative explanation for it. For Majority judgment and Bucklin's method, we have put forward the explanation that this behavior is amplified by the fact of considering an unfavorable median, which tends to make them more manipulable when the number of voters is even. For future work, it would be interesting to examine variants of these voting systems where the favorable median is used, to see if it reduces the manipulability in this case.

When the population tends to become uniform (Sections 7.5, 7.6, 7.8, and 8.3), we observed, unsurprisingly, that most voting systems become less manipulable. However, we saw that some systems are much less responsive in terms of manipulability reduction in this case, especially Veto, Approval voting, Range voting, Borda's method, and Majority judgment. Regarding Veto (equipped with the tie-breaking rule used in SVAMP), we even showed that its manipulability does not tend to 0 when voters have identical preferences, contrary to all unanimous voting systems.

We confirmed the importance of the resistant Condorcet winner for the non-manipulability of Condorcet voting systems, especially through the upper bound given in Section 2.8. For this reason, it would be interesting to prove the decrease of its probability of existence with respect to C in spherical culture. In general, it would be interesting to study the probability of existence of a resistant Condorcet winner in various contexts, as Gehrlein (2006) and others did for a Condorcet winner or a weak Condorcet winner.

Finally, these simulations provided a good overview of the compared performances of different voting systems. In spheroidal cultures and in multidimensional political spectrum models, CIRV, IRV, and EB show the best performances. In one-dimensional political spectrum cultures, Schulze's method, Maximin, the Two-round system, Bucklin's method, and IRVD show promising results, which

[6] About this idea, see notably Tsetlin et al. (2003).

encourage further study of these systems in the future. Generally speaking, Range voting, Approval voting, and Borda's method are very manipulable. In particular, they are often more manipulable than all Condorcet systems, as can be seen by comparing with the rate of existence of a resistant Condorcet winner.

In Chapter 9, we analyzed the results of 168 actual experiments, including 17 from original experiments and others from the PrefLib database. Among these datasets, it is particularly interesting to consider:

- Those based on cardinal data because they allow to deepen the comparison between ordinal and cardinal systems and they also help to naturally consider issues based on the notion of utility, like the manipulability threshold described above;

- Those that do not relate to an election scenario and whose strategic stake is *a priori* low (such as the *bdtheque* experiments) because these datasets can be expected to have opinions as sincere as possible.

For future work, it would be interesting to expand this corpus.

In all experiments analyzed, CIRV, IRV, and EB are distinguished by CM rates that are always lower than that of the other voting systems. In TM, the same result holds for CIRV and almost always for IRV and EB. However, we have seen, in Chapter 8, that the performances of these voting systems are not as good (compared to other voting systems) in cultures based on a one-dimensional political spectrum than in other cultures. We therefore see some opposition between the performance of IRV-type systems in our real datasets and in one-dimensional spectra. It would be interesting to carry out more experiments on large-scale political elections to test the resistance of CIRV, IRV, and EB to manipulation in these application cases that are *a priori* rather unfavorable, and to check to what extent the assumption of one-dimensional spectrum is verified, especially in terms of manipulability.

It is troubling that among the studied voting systems, the least manipulable in practice, i.e. IRV and its variants, are also systems that violate other important properties, in particular monotonicity. This leads to two interrelated questions.

1. Is it possible to define the «monotonization» of a voting system, as we have defined Condorcification, in such a way as to obtain a monotonic system (and preserving some other reasonable properties, for example **InfMC**) without making the system more manipulable? From a technical point of view, defining such monotonization is far from immediate.

2. Is there necessarily a trade-off between low manipulability and monotonicity? One can advance the argument that a Condorcet system, for example CIRV, is already monotonic on the non-manipulable configurations, since these are included in the non-Condorcet configurations.

This last remark takes on its full meaning when we observe, as we have done, that it is very common experimentally to have a Condorcet winner (confirming, in this, various previous results). In practice, this limits the seriousness of Condorcet's paradox, Arrow's theorem, and non-monotonicity problems. Indeed, there is a canonical way of synthesizing voter preferences that is anonymous, neutral, and has good monotonicity properties: it is the victory relation in the weighted majority matrix; the only flaw in this relation is that it is not necessarily transitive. But, in the experiments, we find that it often has a maximal element (the Condorcet winner), which makes it a natural winner and has good properties like IIA. For this reason, we advance the idea that the behavior of a voting system in non-Condorcet configurations should not be examined in terms of the relevance of the elected candidate when such a configuration occurs in sincere voting (because in this case, there is no perfect solution), but rather by the impact it has on the possibilities of manipulating Condorcet configurations.

This is precisely what we did in Chapter 10, where we introduced the *opportunity graph* of an electoral space and we used it to study optimal voting systems, that is to say, whose manipulability rate is minimal within a certain class. We considered **InfMC**, so we could restrict ourselves to systems meeting **Cond** thanks to the Condorcification theorems, but the reasoning we used is valid in all generality.

This object raises several theoretical questions. On the set of all labeled and weighted multigraphs like the typical graph of Figure 10.1, is it true that solving the minimal contamination problem is $\mathcal{NP}$-hard? If we restrict the question to the opportunity graphs obtained for a certain class of electoral spaces, such as those of strict total orders for all values of C and odd values of V, does the problem remain $\mathcal{NP}$-hard? Can we devise a polynomial algorithm that allows a better approximation than the greedy algorithm we presented?

In the case of $C = 3$ candidates and $V = 3, 5$, or 7 voters, we found that CIRV, provided it is equipped with an adequate tie-breaking rule, is always optimal. It is therefore natural to ask whether the same is true for every number of voters and $C = 3$. For $C = 4$ candidates, we already know that this is not the case in general, since we established that this is not true for $V = 3$ and $C = 4$.

It would be interesting to extend the search for optimal systems to other parameter values. In our view, the main purpose of this approach is not necessarily to use these systems in practice: one can rarely guarantee in advance that there will be a number of voters and a number of candidates for which we are able to solve the problem. The goal is rather to use the few cases where the optimal systems can be exhibited to better observe how manipulability behaves and understand what can make a system less manipulable.

In summary, we introduced a unified formalism to study ordinal and non-ordinal voting systems. We proposed various tools to transform a voting system in order to achieve lower manipulability: reduction of a voting system to its state-based version, Condorcification, relative Condorcification in some cases, generalized Condorcification, and slicing. These tools helped in particular to establish optimality theorems that suggest further research on ordinal Condorcet systems. Furthermore, we established various criteria for a voting system and showed the deep connections between these criteria, manipulability, Nash equilibria, and the notion of information exchange. We have laid the first stones of the edifice of optimal Condorcet voting systems, thus providing a yardstick of manipulability for moderate values of the parameters and offering a better insight of the reasons that make a system little manipulable. In this, we have made some steps on the path leading us "towards less manipulable voting systems".

With our software package SVVAMP, we were also able to study manipulability from a quantitative point of view and propose some answers to the problems posed in the introduction. In particular: yes, manipulability is a frequent phenomenon in artificial cultures and in the real world, it is not a mere theoretical possibility. It raises problems which turn out to be much more frequent than the Condorcet paradox. We were also able to measure the very different vulnerabilities of the various voting systems to this phenomenon, and we have showed the particularly spectacular supremacy of CIRV, IRV, and EB in all real-life experiments studied, which encourages to keep on studying other voting systems from this family.

All these results are not necessarily intended to be used immediately in large-scale elections, even if, in the case of IRV, we have a system that is already used in various countries for political elections. Advances in electronic voting, particularly in terms of security, will soon make is possible to envision using voting systems with more complex counting than what is possible with manual voting. When this time comes, it seems important that social choice theorists be able to offer as complete a picture as possible of the advantages and disadvantages of each voting system in order to enable informed decisions. Furthermore, the profusion of associative and professional structures, of foundations, of Internet organizations ignoring borders, offers a formidable field of experimentation which allows real human groups to benefit from the most recent results of social choice and, possibly, to pass on their experience in this area to all forms of human organization seeking to improve their democratic functioning.

Appendices

Appendix A

Probabilistic Spaces and Measurability

In this appendix, we discuss the issue of *measurability*, which is implied in all probabilistic notions (e.g. the definition of the manipulability rate) and which is mainly used in Chapter 5 on slicing.

In order to define the probabilistic notions in a rigorous way, we consider measurable sets, which consist of a set E and a sigma-algebra Σ_E on E. Such a measurable set is denoted (E, Σ_E), or simply E when there is no ambiguity. An element of Σ_E is called an *event* on E.

We always equip the set $\mathcal{R}_\mathcal{C}$ of binary relations on $\mathcal{C}$ with its discrete sigma-algebra, which we denote $\Sigma_{\mathcal{R}_\mathcal{C}}$.

When we consider a Cartesian product E of measurable sets (E_v, Σ_{E_v}), we always equip it with its product sigma-algebra. In particular, the set $\mathcal{R} = (\mathcal{R}_\mathcal{C})^V$ is provided with the product sigma-algebra $\Sigma_\mathcal{R} = (\Sigma_{\mathcal{R}_\mathcal{C}})^V$, which is simply its discrete sigma-algebra.

We say that Ω is a *measurable electoral space* iff each Ω_v is equipped with a sigma-algebra Σ_{Ω_v} and each function $\mathrm{P}_v : \Omega_v \to \mathcal{R}_\mathcal{C}$ is measurable. When we consider a probabilized electoral space (PES), we always make the implicit assumption that it is a measurable electoral space.

For example, consider the reference electoral space, where $\omega_v = (p_v, u_v, a_v)$. Equip each Ω_v with the product sigma-algebra of the discrete sigma-algebra on $\mathcal{R}_\mathcal{C}$, the Lebesgue sigma-algebra on $[0,1]^C$, and the discrete sigma-algebra on $\{0,1\}^C$. Then each function P_v is obviously measurable. Therefore, Ω is a measurable electoral space.

In the context of Chapter 5, each set $\mathcal{Y}_v$ of slicing methods has a canonical sigma-algebra. Indeed, this is the case for the set ${\Omega_v}^{\mathrm{P}_v(\Omega_v)}$ of functions from $\mathrm{P}_v(\Omega_v)$ to Ω_v: by associating each function with the list of its values, it suffices to consider the product sigma-algebra $\Sigma_{\Omega_v} \times \ldots \times \Sigma_{\Omega_v}$, with a number of factors equal to the cardinality of $\mathrm{P}_v(\Omega_v)$. The space $\mathcal{Y}_v$, as a subset of ${\Omega_v}^{\mathrm{P}_v(\Omega_v)}$, inherits this sigma-algebra.

The following lemma solves the measurability issues for the slicing theorem 5.9 and the optimality theorem 5.15.

Lemma A.1

Let Ω be a measurable electoral space, E a measurable set, and $g : \Omega \to E$.

Assume that g depends only on binary relations of preference:

$$\forall(\omega, \psi) \in \Omega^2, \mathrm{P}(\omega) = \mathrm{P}(\psi) \Rightarrow g(\omega) = g(\psi).$$

Then g est measurable.

Proof. Since g depends only on $\mathrm{P}(\omega)$, we can define $h : (\mathcal{R}_{\mathcal{C}})^V \to E$ such that $g = h \circ \mathrm{P}$. Since $(\mathcal{R}_{\mathcal{C}})^V$ is equipped with the discrete measure, h is measurable. And by definition of a measurable electoral space, P is measurable. Therefore, $g = h \circ \mathrm{P}$ is measurable. □

In the slicing theorem 5.9, this lemma ensures that, for all y, the manipulability indicator CM_{f_y} is measurable. Therefore, $\tau^{\pi}_{\mathrm{CM}}(f_y)$ is well defined.

Similarly, in the optimality theorem 5.15, this lemma ensures that $\mathrm{CM}_{g \circ \mathrm{P}}$ is measurable. Therefore, $\tau^{\pi}_{\mathrm{CM}}(g \circ \mathrm{P})$ is well defined.

Appendix B

Geometry on the Utility Space

In this appendix, which results from a collaboration with Benoît Kloeckner, we examine the geometric properties of the space of expected utilities over a finite number of options, which is commonly used to model the preferences of an agent under uncertainty. We focus on the case where the model is neutral with respect to the available options, i.e. it treats them, *a priori*, as symmetric to one another. In particular, we prove that the only Riemannian metric that respects the geometric properties and natural symmetries of the utility space is the round metric. This canonical metric allows us to define a uniform probability on the utility space and to generalize the impartial culture to a model involving expected utilities. In general, it is the basis of all the so-called *spheroidal* cultures that we study in Chapter 7. This appendix takes up and develops the ideas presented by Durand et al. (2015).

In the traditional Arrovian social choice literature (Arrow, 1950), and particularly in voting theory, an agent's preferences are often represented by ordinal information only: a strict total ordering over the available options, or sometimes a binary relation of preference that may not be a strict total order (e.g. if indifference is allowed). However, it may be interesting to consider cardinal preferences, at least for two reasons.

On the one hand, some voting systems are not only based on ordinal information, such as Approval voting or Range voting.

On the other hand, voters may be in a situation of uncertainty, either because the rule of the voting system implies an element of chance, or because each voter has incomplete information about the preferences of other voters and the ballots they are about to choose. In order to express preferences in a situation of uncertainty, a classical and elegant model is that of expected utilities (Von Neumann and Morgenstern, 1944; Fishburn, 1970; Kreps, 1990; Mas-Colell et al., 1995). For each agent, a numerical utility is associated with each option. The utility of a lottery on options is computed as an expectation. Therefore, the utility vector $\mathbf{u}$ representing the preferences of an agent is an element of $\mathbb{R}^C$, where C is the number of options or *candidates*.

For a wide range of applications in economics, the options under consideration are financial rewards or quantities of one or more economic goods, which has an important consequence: there is a natural structure for the space of options. For example, if the options are financial rewards, then there is a natural ordering on the possible amounts, which is defined before the preferences of the agents.

We consider the opposite scenario, where the options are *a priori* symmetrical. This symmetry assumption is particularly relevant in voting theory, because of a normative principle of neutrality, but it can also be applied to other contexts of choice under uncertainty when there is no pre-existing natural structure on the space of available options.

The motivation for this chapter came from the possible generalizations for impartial culture to agents with expected utilities. Impartial culture is a classical probabilistic model in voting theory where each agent independently draws her strict total order of preference with uniform probability over all possible orders.

The difficulty is not to define a probability distribution on utilities such that its projection on the ordinal information is the impartial culture. Indeed, it is sufficient to define a distribution where the voters are independent and all the candidates are treated symmetrically. The problem is to choose one in particular: an infinite number of distributions satisfy these conditions and one can ask whether one of these generalizations has canonical reasons to be chosen over the others.

In order to answer this question, we need to take into account an important technical point. Under the assumptions of the Von Neumann-Morgenstern theorem (which we will recall), the utility vector of an agent is defined up to two constants, and choosing a particular normalization is arbitrary. As a consequence, the utility space is a quotient space of $\mathbb{R}^C$ and, *a priori*, there is no canonical way to push a metric from $\mathbb{R}^C$ to this quotient space. Therefore, at first sight, it seems that there is no natural definition of a uniform probability distribution on this space.

More broadly, seeking a natural generalization of the impartial culture to the utility space leads to exploring various aspects of the geometry of this quotient space and to better understanding its properties related to algebra, topology, and measure theory.

The rest of this appendix is organized as follows. In Section B.1, we introduce the Von Neumann–Morgenstern formalism and define the utility space. In Section B.2, we show that the utility space can be viewed as the quotient of the dual of the space of lottery pairs over the candidates. In Section B.3, we naturally define an opposite operator, which corresponds to inverting preferences while preserving their intensities, and a sum operator, which is characterized by the fact that it preserves unanimous preferences.

Since the utility space is a variety, it is natural to want to equip it with a Riemannian metric. In Section B.4, we prove that the only Riemannian representation which preserves the natural projective properties and the symmetry *a priori* between the candidates is the round metric. In Section B.5, we use this result to give a canonical generalization of the impartial culture and to suggest the use of the Von Mises–Fisher model to represent polarized cultures. In Section B.6, we briefly summarize our findings for the classical Von Neumann-Morgenstern model.

In Section B.7, we generalize these results to a utility model with *approval threshold*, i.e., the utility value 0 has a particular meaning. In this case, the utility vector of an agent is defined up to one positive multiplicative constant, but there is no longer any degree of freedom in translation. We show that, in this model, the only representations that respect the natural properties of the utility space and neutrality constitute a one-parameter family of spheroids.

B.1 Von Neumann–Morgenstern model

In this section, we define some notations and recall the classical Von Neumann-Morgenstern theorem which makes it possible to represent the preferences of an agent on probabilized options by introducing the notion of expected utility.

Let $C \in \mathbb{N} \setminus \{0\}$. We will consider C mutually exclusive options called *candidates*, each of them being represented by an index belonging to $\mathcal{C} = [\![1, C]\!]$.

A *lottery* on candidates is a C-tuple $(L_1, \ldots, L_C) \in {\mathbb{R}_+}^C$ such that $\sum_{c=1}^{C} L_c = 1$. The set of lotteries is denoted $\mathbb{L}$. For every pair of lotteries $L = (L_1, \ldots, L_C)$ and $M = (M_1, \ldots, M_C)$, for every $\lambda \in [0, 1]$, we naturally define their *reduced compound lottery* as a barycenter:

$$\lambda L + (1 - \lambda)M = \Big(\lambda L_1 + (1 - \lambda)M_1, \ldots, \lambda L_C + (1 - \lambda)M_C\Big).$$

The preferences of an agent on lotteries are represented by a binary relation $\precsim$ on $\mathbb{L}$. We note $\prec$ the strict relation associated to $\precsim$, defined by: $L \prec M \Leftrightarrow L \precsim M$ and not $M \precsim L$.

Recall the assumptions of the Von Neumann–Morgenstern theorem. We say that the relation $\precsim$ is *complete* iff $\forall (L, M) \in \mathbb{L}^2$: $L \precsim M$ or $M \precsim L$. It is *transitive* iff $\forall (L, M, N) \in \mathbb{L}^3$, $L \precsim M$ and $M \precsim N \Rightarrow L \precsim N$. It is *Archimedean* iff $\forall (L, M, N) \in \mathbb{L}^3$:

$$L \prec M \text{ and } M \prec N \Rightarrow \exists \varepsilon \in]0, 1[\text{ s.t. } (1 - \varepsilon)L + \varepsilon N \prec M \prec \varepsilon L + (1 - \varepsilon)N.$$

And it is *independent of irrelevant alternatives* iff $\forall (L, M, N) \in \mathbb{L}^3, \forall \lambda \in]0, 1]$:

$$L \prec M \Rightarrow \lambda L + (1 - \lambda)N \prec \lambda M + (1 - \lambda)N.$$

We say that $\mathbf{u} = (u_1, \ldots, u_C) \in \mathbb{R}^C$ is a *utility vector* representing $\precsim$ iff following the relationship $\precsim$ is equivalent to maximizing expected utility in the sense of $\mathbf{u}$; that is, iff for every pair of lotteries (L, M):

$$L \precsim M \Leftrightarrow \sum_{c=1}^{C} L_c u_c \leq \sum_{c=1}^{C} M_c u_c.$$

Theorem B.1 (Von Neumann and Morgenstern)
For a binary relation $\precsim$ on $\mathbb{L}$, the following conditions are equivalent.

1. *The relation $\precsim$ is complete, transitive, Archimedean, and independent of irrelevant alternatives.*
2. *There exists a utility vector $\mathbf{u} \in \mathbb{R}^C$ representing $\precsim$.*

When they are satisfied, $\mathbf{u}$ is defined up to one additive constant and one positive multiplicative constant. Formally, let $\mathbf{u} \in \mathbb{R}^C$ be a utility vector representing $\precsim$. A vector $\mathbf{v}$ is also a utility vector representing $\precsim$ iff $\exists \alpha \in]0, +\infty[, \exists \beta \in \mathbb{R}$ s.t. $\mathbf{v} = \alpha \mathbf{u} + \beta \mathbf{1}$, where $\mathbf{1}$ denotes the vector whose C coordinates are equal to 1.

It is beyond the scope of this chapter to give a proof of this theorem (Von Neumann et al., 2007; Mas-Colell et al., 1995; Kreps, 1990) or to discuss whether its hypotheses are experimentally valid (Fishburn, 1988; Mas-Colell et al., 1995).

In order to define the utility space, all vectors representing the same preferences must be identified at a single point. We denote $\approx$ the equivalence relation defined by $\forall (\mathbf{u}, \mathbf{v}) \in (\mathbb{R}^C)^2$:

$$\mathbf{u} \approx \mathbf{v} \Leftrightarrow \exists \alpha \in]0, +\infty[, \exists \beta \in \mathbb{R} \text{ s.t. } \mathbf{v} = \alpha \mathbf{u} + \beta \mathbf{1}.$$

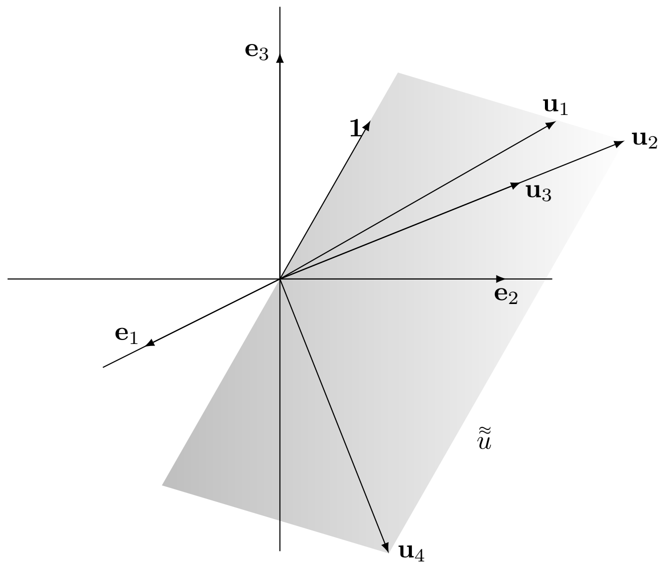


Figure B.1 – Space $\mathbb{R}^3$ of the utility vectors for 3 candidates (without passing to the quotient).

The *utility space* on $\mathcal{C}$, denoted $\mathbb{U}$, is defined as the quotient set $\mathbb{R}^C/\approx$. The following function is called the *canonical projection* from $\mathbb{R}^C$ to $\mathbb{U}$:

$$\tilde{\tilde{\pi}} : \left| \begin{array}{lcl} \mathbb{R}^C & \to & \mathbb{U} \\ \mathbf{u} & \to & \tilde{\tilde{u}} = \{\mathbf{v} \in \mathbb{R}^C \text{ s.t. } \mathbf{v} \approx \mathbf{u}\}. \end{array} \right.$$

For every $\mathbf{u}$, we unambiguously denote $\precsim_{\tilde{\tilde{u}}}$ the binary relation on $\mathbb{L}$ represented by $\mathbf{u}$ in the sense of Theorem B.1 (Von Neumann–Morgenstern).

Figure B.1 represents the space $\mathbb{R}^3$ of utility vectors for 3 candidates, before projection onto the quotient space. The canonical basis of $\mathbb{R}^3$ is denoted by $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. The utility vectors $\mathbf{u}_1$ to $\mathbf{u}_4$ represent the same preferences as any other vector in the half-plane $\tilde{\tilde{u}}$, shown in gray. More generally, any non-trivial point $\tilde{\tilde{u}}$ of the quotient space corresponds to a half-plane of $\mathbb{R}^C$ bounded by the line $\text{vect}(\mathbf{1})$, the linear span of $\mathbf{1}$. The only exception is the point of total indifference $\tilde{\tilde{0}}$. In $\mathbb{R}^C$, it does not correspond to a half-plane but to the line $\text{vect}(\mathbf{1})$ itself.

In order to manipulate utilities, both theoretically and practically, it would be advantageous to have a canonical representative $\mathbf{u}$ for each equivalence class $\tilde{\tilde{u}}$. In Figure B.2, for each non-indifferent $\tilde{\tilde{u}}$, we choose its representative satisfying $\min(u_i) = -1$ and $\max(u_i) = 1$. The utility space $\mathbb{U}$ (except the indifference point) is represented in $\mathbb{R}^3$ by six edges of the unit cube. In Figure B.3, we choose the representative satisfying $\sum u_i = 0$ and $\sum {u_i}^2 = 1$. In this case, $\mathbb{U} \setminus \{\tilde{\tilde{0}}\}$ is represented in $\mathbb{R}^3$ by the unit circle of the vector plane which is orthogonal to $\mathbf{1}$.

If we choose such a representation, the quotient space $\mathbb{U}$ can inherit the Euclidean distance from $\mathbb{R}^C$. For example, one can evaluate the distances along the edges of the cube in Figure B.2, or along the unit circle in Figure B.3. But it is clear that the result will depend on the chosen representation. Therefore, an interesting question is whether either of these two representations, or yet another, has canonical grounds for use. But before answering this question, we need to explore more generally the geometric properties of the utility space.

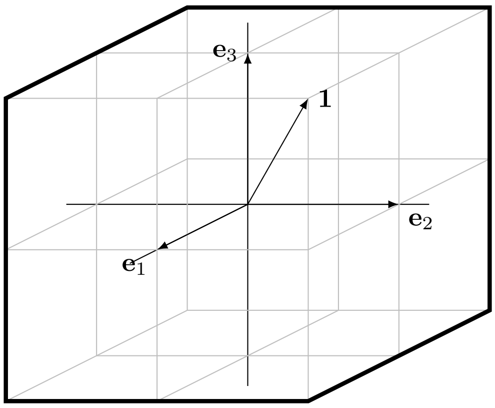


Figure B.2 – With 3 candidates, representation of $\mathbb{U}$ as edges of the unit cube in $\mathbb{R}^3$.

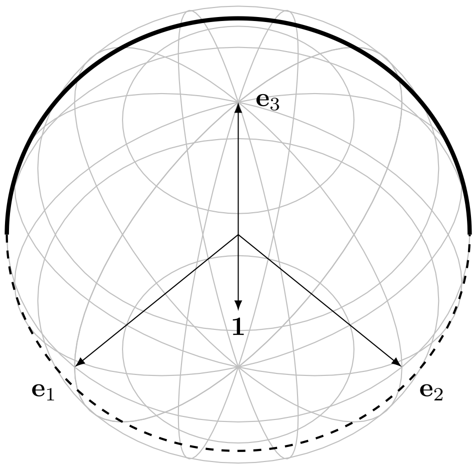


Figure B.3 – With 3 candidates, representation of $\mathbb{U}$ as a circle in $\mathbb{R}^3$.

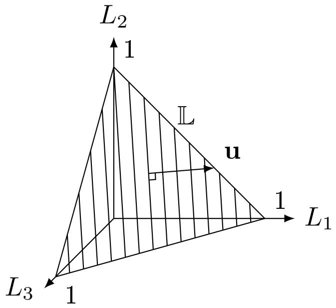


Figure B.4 – Space $\mathbb{L}$ of the lotteries for 3 candidates.

B.2 Duality with the tangent hyperplane of lotteries

In this section, we note that the space of utilities is a dual of the space of lottery pairs. Not only does this give a different view of the utility space (which we find interesting in itself), but it will also be useful in proving Theorem B.4, which characterizes the sum operator which we will define in Section B.3.

In the example shown in Figure B.4, we consider $C = 3$ candidates and $\mathbf{u} = (\frac{5}{3}, -\frac{1}{3}, -\frac{4}{3})$. The large triangle, or simplex, is the space $\mathbb{L}$ of the lotteries. The hatches are the indifference lines of the agent: she is indifferent between any pair of lotteries lying on the same indifference line (cf. Mas-Colell et al. (1995), section 6.B). The utility vector $\mathbf{u}$ represented here is in the plane of the simplex but this is not mandatory: indeed, $\mathbf{u}$ can be chosen arbitrarily in its equivalence class $\tilde{\tilde{u}}$. However, this is a rather natural choice, since the component of $\mathbf{u}$ in the direction $\mathbf{1}$ (orthogonal to the simplex) is of no importance for the preferences.

The utility vector $\mathbf{u}$ can be seen as a gradient of preference:[1] at each point, it indicates in which directions to find lotteries that the agent prefers. However, only the orientation and the sense of $\mathbf{u}$ are important, while its norm has no particular meaning. Therefore, the utility space is not quite a dual but a quotient of dual, as we will formalize.

For every pair of lotteries $L = (L_1, \ldots, L_C)$ and $M = (M_1, \ldots, M_C)$, we call *bipoint* from L to M the vector: $\overrightarrow{LM} = (M_1 - L_1, \ldots, M_C - L_C)$. The set $\mathcal{T}$ of bipoints of $\mathbb{L}$ is called the *tangent polytope* of $\mathbb{L}$.

We call *tangent hyperplane* of $\mathbb{L}$:

$$\mathcal{H} = \{(\Delta_1, \ldots, \Delta_C) \in \mathbb{R}^C \text{ s.t. } \sum_{c=1}^{C} \Delta_c = 0\}.$$

In Figure B.4, the tangent polytope $\mathcal{T}$ is the set of bipoints of the large triangle, seen as a part of vector space (whereas $\mathbb{L}$ is morally affine). The hyperplane $\mathcal{H}$ is the whole vector plane which contains $\mathcal{T}$.

Let $\langle\, \mathbf{u} \,|\, \mathbf{v} \,\rangle$ be the canonical inner product of $\mathbf{u}$ and $\mathbf{v}$. We call *positive half-hyperplane associated to* $\mathbf{u}$ the set $\mathbf{u}^{+\mathcal{H}} = \{\mathbf{\Delta} \in \mathcal{H} \text{ s.t. } \langle\, \mathbf{u} \,|\, \mathbf{\Delta} \,\rangle \geq 0\}$. By definition, a lottery M is preferred to a lottery L iff the bipoint $\overrightarrow{LM}$ belongs to this positive half-hyperplane:

$$L \precsim_{\tilde{\tilde{u}}} M \Leftrightarrow \langle\, \mathbf{u} \,|\, \overrightarrow{LM} \,\rangle \geq 0 \Leftrightarrow \overrightarrow{LM} \in \mathbf{u}^{+\mathcal{H}}.$$

[1] More precisely, it is a uniform gradient: it is the same at each point of the simplex.

Let $\mathcal{H}^\star$ be the dual space of $\mathcal{H}$, that is, the set of linear forms on $\mathcal{H}$. For all $\mathbf{u} \in R^C$, we call *linear form associated to* $\mathbf{u}$ the following element of $\mathcal{H}^\star$:

$$\langle\, \mathbf{u} \,| : \left| \begin{array}{lcl} \mathcal{H} & \to & \mathbb{R} \\ \boldsymbol{\Delta} & \to & \langle\, \mathbf{u} \,|\, \boldsymbol{\Delta} \,\rangle. \end{array} \right.$$

We have already observed that the utility vector can be seen as a gradient, except that only its orientation and sense matter, not its norm. Let us formalize this idea. For $(f, g) \in (\mathcal{H}^\star)^2$, denote $f \approx g$ iff these two linear forms are positive multiples of each other, that is, iff $\exists \alpha \in \,]0, +\infty[$ s.t. $g = \alpha f$. Denote $\widetilde{\widetilde{\pi}}(f) = \{g \in \mathcal{H}^\star \text{ s.t. } f \approx g\}$: this is the equivalence class of f up to a positive multiplication.

Proposition B.2
For every pair $(\mathbf{u}, \mathbf{v}) \in (\mathbb{R}^C)^2$*, we have:*

$$\mathbf{u} \approx \mathbf{v} \Leftrightarrow \langle\, \mathbf{u} \,| \approx \langle\, \mathbf{v} \,|.$$

The following map is a bijection:

$$\Theta : \left| \begin{array}{lcl} \mathbb{U} & \to & \mathcal{H}^\star / \approx \\ \widetilde{\widetilde{\pi}}(\mathbf{u}) & \to & \widetilde{\widetilde{\pi}}(\langle\, \mathbf{u} \,|). \end{array} \right.$$

Proof. $\mathbf{u} \approx \mathbf{v}$
$\Leftrightarrow \exists \alpha \in (0, +\infty), \exists \beta \in \mathbb{R}$ s.t. $\mathbf{v} - \alpha \mathbf{u} = \beta \mathbf{1}$
$\Leftrightarrow \exists \alpha \in (0, +\infty)$ s.t. $\mathbf{v} - \alpha \mathbf{u}$ is orthogonal to $\mathcal{H}$
$\Leftrightarrow \exists \alpha \in (0, +\infty)$ s.t. $\forall \boldsymbol{\Delta} \in \mathcal{H}, \langle\, \mathbf{v} \,|\, \boldsymbol{\Delta} \,\rangle = \alpha \langle\, \mathbf{u} \,|\, \boldsymbol{\Delta} \,\rangle$
$\Leftrightarrow \langle\, \mathbf{u} \,| \approx \langle\, \mathbf{v} \,|$.
The implication $\Rightarrow$ proves that Θ is well defined: indeed, if $\widetilde{\widetilde{\pi}}(\mathbf{u}) = \widetilde{\widetilde{\pi}}(\mathbf{v})$, then $\widetilde{\widetilde{\pi}}(\langle\, \mathbf{u} \,|) = \widetilde{\widetilde{\pi}}(\langle\, \mathbf{v} \,|)$. The implication $\Leftarrow$ ensures that Θ is injective. Finally, Θ is clearly surjective. $\square$

The utility space can thus be seen as a quotient of the dual $\mathcal{H}^\star$ of the tangent space $\mathcal{H}$ of the lotteries $\mathbb{L}$. A utility vector can be seen, up to a positive constant, as a uniform gradient, i.e. a linear form on $\mathcal{H}$ which reveals, for every point in the space of lotteries, in which directions the agent can find lotteries which she prefers.

B.3 Inversion and sum operators

As a quotient of $\mathbb{R}^C$, the utility space inherits natural operations on $\mathbb{R}^C$, the inversion and the sum. We will see that these two quotient operations have intuitive meaning regarding preferences. The sum will also allow us to define *lines* in Section B.4, which will be a key notion for Theorem B.6, characterizing the suitable Riemannian metrics for the utility space.

We define the *inversion* operator of $\mathbb{U}$ by:

$$- : \left| \begin{array}{lcl} \mathbb{U} & \to & \mathbb{U} \\ \widetilde{\widetilde{\pi}}(\mathbf{u}) & \to & \widetilde{\widetilde{\pi}}(-\mathbf{u}). \end{array} \right.$$

This operator is correctly defined and it is a bijection: indeed, we have for example $\widetilde{\widetilde{\pi}}(\mathbf{u}) = \widetilde{\widetilde{\pi}}(\mathbf{v})$ iff $\widetilde{\widetilde{\pi}}(-\mathbf{u}) = \widetilde{\widetilde{\pi}}(-\mathbf{v})$. Considering the opposite corresponds to reversing the agent's preferences, without changing their relative intensities.

We now want to push the sum operator of $\mathbb{R}^C$ to the quotient $\mathbb{U}$. We use a generic method to push an operator to the quotient space: given $\widetilde{\widetilde{u}}$ and $\widetilde{\widetilde{v}}$ in $\mathbb{U}$, we

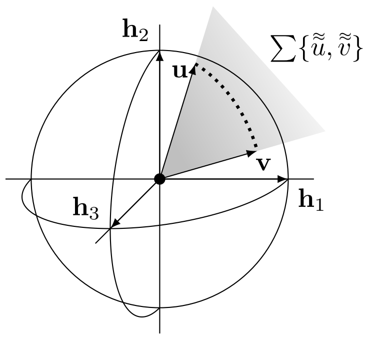


Figure B.5 – Sum of two utility vectors in the utility space for 4 candidates.

take their antecedents in $\mathbb{R}^C$ thanks to $\tilde{\tilde{\pi}}^{-1}$, compute the sum in $\mathbb{R}^C$, and then the result is converted back to the quotient space $\mathbb{U}$ thanks to $\tilde{\tilde{\pi}}$.

However, there is no uniqueness of the result. Indeed, designate arbitrarily representatives $\mathbf{u} \in \tilde{\tilde{u}}$ and $\mathbf{v} \in \tilde{\tilde{v}}$. To compute the sum, we can consider taking any representatives. So the possible sums are $\alpha\mathbf{u} + \alpha'\mathbf{v} + \beta + \beta'$, where α and α' are positive and $\beta + \beta'$ is any real number. Going back to the quotient, we obtain for example $\tilde{\tilde{\pi}}(2\mathbf{u} + \mathbf{v})$ and $\tilde{\tilde{\pi}}(\mathbf{u} + 3\mathbf{v})$, which are generally distinct. Therefore, the result is not a point of the utility space $\mathbb{U}$ but a set of points, i.e. an element of $\mathcal{P}(\mathbb{U})$.

This example shows how we can define the sum of two elements $\tilde{\tilde{u}}$ and $\tilde{\tilde{v}}$. For more generality, we will define the sum of any number of elements of $\mathbb{U}$. Therefore, we will also take $\mathcal{P}(\mathbb{U})$ as our input set.

We define the *sum* operator of $\mathbb{U}$ as:

$$\sum : \left| \begin{array}{ccl} \mathcal{P}(\mathbb{U}) & \to & \mathcal{P}(\mathbb{U}) \\ A & \to & \left\{ \tilde{\tilde{\pi}} \left(\sum_{i=1}^{n} \mathbf{u}_i \right), n \in \mathbb{N}, (\mathbf{u}_1, \ldots, \mathbf{u}_n) \in \left(\tilde{\tilde{\pi}}^{-1}(A) \right)^n \right\}. \end{array} \right.$$

Example B.3

Consider $\mathbb{U}$ for $C = 4$ candidates. In Figure B.5, for visualization purposes, we represent its projection in $\mathcal{H}$, which is made possible by the choice of the normalization constant β. Since $\mathcal{H}$ is a 3-dimensional space, we can consider an orthonormal basis $(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3)$.

For two non-trivial utility vectors $\tilde{\tilde{u}}$ and $\tilde{\tilde{v}}$, the choice of the normalization multipliers α allows to choose representatives $\mathbf{u}$ and $\mathbf{v}$ whose Euclidean norm is 1.

In this representation, the sum $\sum\{\tilde{\tilde{u}}, \tilde{\tilde{v}}\}$ consists of the utilities corresponding to the vectors $\alpha\mathbf{u}+\alpha'\mathbf{v}$, where α and α' are nonnegative real numbers. Indeed, we have taken a representation in $\mathcal{H}$, so all the normalization constants β disappear. Moreover, α, α', or both can be zero because our definition allows to ignore $\mathbf{u}$, $\mathbf{v}$, or both. As long as we take unit norm representatives for non-trivial utility vectors, let us notice that the sum $\sum\{\tilde{u}, \tilde{v}\}$ can be represented by the dotted line and the indifference point $\mathbf{0}$.

Geometrically, the sum is the quotient of the convex hull of the arguments. Note that this convex hull is, in fact, a convex cone. We will see later its interpretation in terms of preferences.

Given our definition of the sum operator, we consider the closed cone: for example, the arguments themselves (e.g. $\tilde{\tilde{u}}$) satisfy our definition, as does the indifference point $\tilde{\tilde{\pi}}(\mathbf{0})$. This would generally not be the case if we had taken $\tilde{\tilde{\pi}}(\alpha\mathbf{u} + \alpha'\mathbf{v} + \beta\mathbf{1})$, where $\alpha > 0$ and $\alpha' > 0$. This convention has the avowed purpose of obtaining a concise statement for Theorem B.4, which we are going to state.

We will now show that, if A is the set of utility vectors of a population, then $\sum A$ is the locus of utility vectors that respect the unanimous preferences of the population.

Theorem B.4 (characterization of the sum)
Let $A \in \mathcal{P}(\mathbb{U})$ and $\widetilde{\widetilde{v}} \in \mathbb{U}$.
The following conditions are equivalent.

1. $\widetilde{\widetilde{v}} \in \sum A$.
2. $\forall (L, M) \in \mathbb{L}^2$: $\left(\forall \widetilde{\widetilde{u}} \in A, L \precsim_{\widetilde{\widetilde{u}}} M\right) \Rightarrow L \precsim_{\widetilde{\widetilde{v}}} M$.

Proof. First, notice that the tangent polytope $\mathcal{T}$ generates the tangent hyperplane $\mathcal{H}$ by positive multiplication. That is:

$$\forall \mathbf{\Delta} \in \mathcal{H}, \exists \overrightarrow{LM} \in \mathcal{T}, \exists \lambda \in]0, +\infty[\text{ s.t. } \mathbf{\Delta} = \lambda \overrightarrow{LM}.$$

Indeed, $\mathcal{T}$ contains a neighborhood of the origin in the vector space $\mathcal{H}$.

Let $\mathbf{v} \in \widetilde{\widetilde{\pi}}^{-1}(\widetilde{\widetilde{v}})$. We have the following equivalences.

- $\forall (L, M) \in \mathbb{L}^2, \left(\forall \widetilde{\widetilde{u}} \in A, L \precsim_{\widetilde{\widetilde{u}}} M\right) \Rightarrow L \precsim_{\widetilde{\widetilde{v}}} M$,
- $\forall \overrightarrow{LM} \in \mathcal{T}, \left(\forall \mathbf{u} \in \widetilde{\widetilde{\pi}}^{-1}(A), \langle\, \mathbf{u} \,|\, \overrightarrow{LM} \,\rangle \geq 0\right) \Rightarrow \langle\, \mathbf{v} \,|\, \overrightarrow{LM} \,\rangle \geq 0$,
- $\forall \mathbf{\Delta} \in \mathcal{H}, \left(\forall \mathbf{u} \in \widetilde{\widetilde{\pi}}^{-1}(A), \langle\, \mathbf{u} \,|\, \mathbf{\Delta} \,\rangle \geq 0\right) \Rightarrow \langle\, \mathbf{v} \,|\, \mathbf{\Delta} \,\rangle \geq 0$ (because $\mathcal{T}$ spans $\mathcal{H}$),
- $\bigcap_{\mathbf{u} \in \widetilde{\widetilde{\pi}}^{-1}(A)} \mathbf{u}^{+\mathcal{H}} \subset \mathbf{v}^{+\mathcal{H}}$,
- $\mathbf{v}$ is in the convex cone of $\widetilde{\widetilde{\pi}}^{-1}(A)$ (because of the duality seen in Section B.2),
- $\widetilde{\widetilde{v}} \in \sum A$.

$\square$

Example B.5
Consider a non-indifferent $\widetilde{\widetilde{u}}$ and examine the special case of the sum of $\widetilde{\widetilde{u}}$ and its opposite $-\widetilde{\widetilde{u}}$. By direct application of the definition, we see that the sum consists of $\widetilde{\widetilde{u}}$, $-\widetilde{\widetilde{u}}$, and $\widetilde{\widetilde{0}}$.

Now, we have just proved that the sum is the locus of utility vectors that preserve unanimous preferences over lotteries. Intuitively, one might think that, since $\widetilde{\widetilde{u}}$ and $-\widetilde{\widetilde{u}}$ never seem to agree, any utility vector $\widetilde{\widetilde{v}}$ respects the empty set of their common preferences; hence, their sum should be the entire space. But this intuition is not correct.

Indeed, consider the example of $\mathbf{u} = (1, 0, \ldots, 0)$. For two lotteries L and M, the two opposite opinions $\widetilde{\widetilde{u}}$ and $-\widetilde{\widetilde{u}}$ are in agreement iff $L_1 = M_1$: in this case, both are indifferent between L and M. The only points in the utility space that satisfy this common property are $\widetilde{\widetilde{u}}$ and $-\widetilde{\widetilde{u}}$ themselves, as well as the indifference point $\widetilde{\widetilde{0}}$.

B.4 Riemannian representation of the utility space

Since the utility space is a variety, it is natural to want to equip it with a Riemannian metric. In this section, we prove that there is a limited choice of metrics that are consistent with the natural properties of the space and the symmetry between the candidates.

First of all, note that the indifference point $\tilde{\tilde{0}}$ must be excluded. Indeed, its only open neighborhood is $\mathbb{U}$ as a whole, and no distance is consistent with this property.[2] On the other hand, $\mathbb{U} \setminus \{\tilde{\tilde{0}}\}$ has the same topology as the sphere of dimension $C-2$, so we can equip it with a distance.

A natural property of the distance would be that its geodesics coincide with the unanimity segments defined by the sum. Indeed, consider an agent with utility vector $\tilde{\tilde{u}}_0$ — say $(-10, 0, 1, 2)$, who is progressively persuaded by another agent with utility vector $\tilde{\tilde{u}}_1$ — say $(10, 0, 1, 2)$, along a parameterized curve $\tilde{\tilde{u}}_t$. If the geodesics coincide with the unanimity segments, this means that if the agent follows the shortest path for the chosen metric, her preferences will gradually change about their points of disagreement, without ever deviating about the points on which they already agree (in the example, all the lottery pairs that have the same probability for candidate 1). Such a deviation could occur in real life, but may reasonably not be considered a shortest path.

Now let us define the round metric. The quotient $\mathbb{R}^C / \operatorname{vect}(\mathbf{1})$ is identified with $\mathcal{H}$ and equipped with the inner product inherited from the canonical one of $\mathbb{R}^C$. The space of utilities $\mathbb{U} \setminus \{\tilde{\tilde{0}}\}$ is identified with the unit sphere of $\mathcal{H}$ and equipped with the induced Riemannian structure. We note ξ_0 this Riemannian metric on $\mathbb{U} \setminus \{\tilde{\tilde{0}}\}$.

To get an intuitive picture, we can represent any position $\tilde{\tilde{u}}$ by a vector $\mathbf{u}$ which satisfies $\sum u_i = 0$ and $\sum {u_i}^2 = 1$. We obtain the unit sphere of $\mathcal{H}$, of dimension $C-2$, and we consider the metric induced by the canonical Euclidean metric of $\mathbb{R}^C$. That is, we measure the distances on the surface of the sphere using on each tangent space the restriction of the canonical inner product. For $C = 3$, such a representation has already been illustrated in Figure B.3.

With this representation in mind, we can give a formula for computing distances in the sense of ξ_0. Let I be the identity matrix of size C, J the matrix of size $C \times C$ whose coefficients are all equal to 1, and D_0 the matrix of the orthogonal projection on $\mathcal{H}$:

$$D_0 = I - \frac{1}{C} J.$$

The canonical Euclidean norm of $\mathbf{u}$ is denoted by $\|\mathbf{u}\|$. For two non-indifferent utility vectors $\tilde{\tilde{u}}$ and $\tilde{\tilde{v}}$, the distance between $\tilde{\tilde{u}}$ and $\tilde{\tilde{v}}$ in the sense of the metric ξ_0 is:

$$d(\tilde{\tilde{u}}, \tilde{\tilde{v}}) = \arccos \left\langle \frac{D_0 \mathbf{u}}{\|D_0 \mathbf{u}\|} \,\middle|\, \frac{D_0 \mathbf{v}}{\|D_0 \mathbf{v}\|} \right\rangle.$$

If $\mathbf{u}$ and $\mathbf{v}$ are already unit vectors of $\mathcal{H}$, i.e., canonical representatives of their equivalence classes $\tilde{\tilde{u}}$ and $\tilde{\tilde{v}}$, then the formula simplifies to $d(\tilde{\tilde{u}}, \tilde{\tilde{v}}) = \arccos\langle \mathbf{u} \,|\, \mathbf{v} \rangle$.

We will now prove that for $C \geq 4$, the spherical representation is the only one which is consistent with the natural properties of the utility space and which respects the symmetry between the candidates.

[2] Technically, this remark proves that $\mathbb{U}$ (equipped with the quotient topology) is not a T_1 *space* (Guénard and Lelièvre, 1985).

Theorem B.6 (Riemannian representation of the utility space)
We assume that $C \geq 4$. Let ξ be a Riemannian metric on $\mathbb{U} \setminus \{\tilde{\tilde{0}}\}$. The following conditions 1 and 2 are equivalent.

1. (a) *For every non-antipodal pair of points $\tilde{\tilde{u}}, \tilde{\tilde{v}} \in \mathbb{U} \setminus \{\tilde{\tilde{0}}\}$ (i.e. $\tilde{\tilde{v}} \neq -\tilde{\tilde{u}}$), the set $\sum\{\tilde{\tilde{u}}, \tilde{\tilde{v}}\}$ of elements respecting the unanimous preferences of $\tilde{\tilde{u}}$ and $\tilde{\tilde{v}}$ is a segment of geodesic of ξ; and*

   (b) *for every permutation σ of $[\![1, C]\!]$, the action Φ_σ induced on $\mathbb{U} \setminus \{\tilde{\tilde{0}}\}$ by*

$$(u_1, \ldots, u_C) \to (u_{\sigma(1)}, \ldots, u_{\sigma(C)})$$

   is an isometry.

2. *$\exists \lambda \in]0, +\infty[$ s.t. $\xi = \lambda \xi_0$.*

Proof. The implication 2 $\Rightarrow$ 1 being obvious, we will prove 1 $\Rightarrow$ 2. The underlying deep result is a classical theorem of Beltrami, which dates from the middle of the nineteenth century (Beltrami, 1866, 1869).

The image of a 2-dimensional subspace of $\mathcal{H}$ in $\mathbb{U} \setminus \{\tilde{\tilde{0}}\}$ by the canonical projection $\tilde{\tilde{\pi}}$ is called a *line*[3] of $\mathbb{U} \setminus \{\tilde{\tilde{0}}\}$. This notion is deeply related to the sum operator: indeed, the sum of two non-antipodal points of $\mathbb{U} \setminus \{\tilde{\tilde{0}}\}$ is a segment of the line which connects them. Condition 1a means precisely that the geodesics of ξ are the lines of $\mathbb{U} \setminus \{\tilde{\tilde{0}}\}$. Beltrami's theorem ensures, then, that $\mathbb{U} \setminus \{\tilde{\tilde{0}}\}$ has constant curvature. Note that this result is, in fact, more subtle in dimension 2 (i.e., for $C = 4$) than in higher dimensions. See Spivak (1979a), Theorem 1.18 and Spivak (1979b), Theorem 7.2 for proofs.

Since $\mathbb{U} \setminus \{\tilde{\tilde{0}}\}$ is a topological sphere, this constant curvature is necessarily positive. Up to multiplying ξ by a constant, we can suppose that this constant curvature is 1. Therefore, there exists an isometry $\Psi : \mathcal{S}_{C-2} \to \mathbb{U} \setminus \{\tilde{\tilde{0}}\}$, where $\mathcal{S}_{C-2}$ is the unit sphere of $\mathbb{R}^{C-1}$ with its usual round metric. Since the function Ψ obviously maps geodesics to geodesics, we will deduce the following lemma.

Lemma B.7
There exists a linear application $\Lambda : \mathbb{R}^{C-1} \to \mathcal{H}$ which induces Ψ, that is, such that:

$$\Psi \circ \Pi = \Pi \circ \Lambda,$$

where Π denotes both the projection $\mathbb{R}^{C-1} \to \mathcal{S}_{C-2}$ and the projection $\mathcal{H} \to \mathbb{U} \setminus \{\tilde{\tilde{0}}\}$.

Proof. First, Ψ maps any pair of antipodal points of $\mathcal{S}_{C-2}$ to a pair of antipodal points of $\mathbb{U} \setminus \{\tilde{\tilde{0}}\}$: indeed, in both cases, the antipodal pairs are characterized by the fact that there is more than one geodesic that contains both of them. It follows that Ψ induces a map Ψ' of the projective space $\mathbb{P}(\mathbb{R}^{C-1})$ (which is the set of lines passing through the origin in $\mathbb{R}^{C-1}$, identified with the set of pairs of antipodal points of $\mathbb{S}_{C-2}$) to the projective space $\mathbb{P}(\mathcal{H})$ (which is the set of lines passing through the origin in $\mathcal{H}$, identified with the set of pairs of antipodal points of $\mathbb{U} \setminus \{\tilde{\tilde{0}}\}$).

The fact that Ψ sends geodesics of $\mathcal{S}_{C-2}$ to geodesics of $\mathbb{U} \setminus \{\tilde{\tilde{0}}\}$, combined with condition 1a, implies that Ψ' sends projective lines to projective lines.

Now it is well known that a one-to-one map $\mathbb{R}^n \to \mathbb{R}^n$ that sends lines onto lines is necessarily an affine map. A similar result is true in projective geometry,

[3] The reader not expert in projective geometry can familiarize herself with Kloeckner (2012).

which implies that Ψ' is a projective map. See for example Audin (2003) for the two results mentioned.

The fact that Ψ' is projective means precisely that there is a linear map $\Lambda : \mathbb{R}^{C-1} \to \mathcal{H}$ which induces Ψ', which in turn induces $\Psi : \mathcal{S}_{C-2} \to \mathbb{U} \setminus \{\tilde{\tilde{0}}\}$. $\square$

Using Λ to push the canonical inner product of $\mathbb{R}^{C-1}$, we deduce that there exists an inner product $(\mathbf{u}, \mathbf{v}) \to \phi(\mathbf{u}, \mathbf{v})$ on $\mathcal{H}$ which induces ξ, in the sense that ξ is the Riemannian metric obtained by identifying $\mathbb{U} \setminus \{\tilde{\tilde{0}}\}$ with the unit sphere defined in $\mathcal{H}$ by ϕ and restricting ϕ to it.

It remains to prove that ϕ is the inner product induced by the canonical inner product of $\mathbb{R}^C$. Note that the hypothesis 1b is mandatory, since any inner product on $\mathcal{H}$ does induce on $\mathbb{U} \setminus \{\tilde{\tilde{0}}\}$ a Riemannian metric which satisfies 1a.

Each vector $\mathbf{e}_c = (0, \ldots, 1, \ldots, 0)$ of the canonical basis defines a point of $\mathbb{U} \setminus \{\tilde{\tilde{0}}\}$ and a half-line ℓ_c of $\mathcal{H}$. The condition 1b guarantees that these half-lines are permuted by some isometries of $(\mathcal{H}, \phi)$. In particular, there exist vectors $\mathbf{u}_c \in \ell_c$ whose pairwise distances are all equal (in the sense of ϕ).

Lemma B.8

The family $\mathbf{u}_1, \ldots, \mathbf{u}_{C-1}$ is, up to multiplication by a scalar, the only basis of $\mathcal{H}$ such that $\mathbf{u}_c \in \ell_c$ and $\sum_{c<C} \mathbf{u}_c \in -\ell_C$.

Proof. First of all, by definition of the $\mathbf{u}_c$, these vectors form a regular simplex and $\sum_c \mathbf{u}_c = \mathbf{0}$. It follows that $\mathbf{u}_1, \ldots, \mathbf{u}_{C-1}$ has the desired property and it remains to show uniqueness.

Suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_{C-1}$ is a basis of $\mathcal{H}$ such that $\mathbf{v}_c \in \ell_c$ and $\sum_{c<C} \mathbf{v}_c \in -\ell_c$. Then there exist positive scalars $a_1, \ldots, a_C$ such that $\mathbf{v}_c = a_c \mathbf{u}_c$ for all $c < C$, and $\sum_{c<C} \mathbf{v}_c = a_C \sum_{c<C} \mathbf{u}_c$.

Then $\sum_{c<C} a_c \mathbf{u}_c = \sum_{c<C} a_C \mathbf{u}_c$, and since the $\mathbf{u}_c$ form a basis, we necessarily have $a_c = a_C$ for all c. $\square$

Now consider the canonical inner product ϕ_0 on $\mathcal{H}$ induced by the canonical inner product of $\mathbb{R}^C$. Since permutations of coordinates are isometries, we deduce that the vectors $\mathbf{v_c} = \Pi(\mathbf{e}_c)$ (where Π now denotes the orthogonal projection from $\mathbb{R}^C$ onto $\mathcal{H}$) form a regular simplex for ϕ_0, such that $\sum_c \mathbf{v_c} = \mathbf{0}$. Therefore, $\mathbf{u}_c = \lambda \mathbf{v_c}$ for some $\lambda > 0$ and for all c. We deduce that the vectors $\mathbf{u}_c$ form a regular simplex for both ϕ and ϕ_0, which are therefore multiples of each other. This completes the proof of Theorem B.6. $\square$

In contrast, the implication 1 $\Rightarrow$ 2 of the theorem is false for $C = 3$. For each non-indifferent utility vector, consider its representative which satisfies $\min(u_i) = 0$ and $\max(u_i) = 1$. We thus identify $\mathbb{U} \setminus \{\tilde{\tilde{0}}\}$ to edges of the unit cube of $\mathbb{R}^3$, as in Figure B.2. We use this identification to equip $\mathbb{U} \setminus \{\tilde{\tilde{0}}\}$ with the metric induced on these edges by the canonical inner product of $\mathbb{R}^3$. Then conditions 1a and 1b of the theorem are satisfied, but not condition 2.

This theorem leads to another important remark. Since we have a canonical representative $\mathbf{u}$ for each equivalence class $\tilde{\tilde{u}}$ which is a unit vector of $\mathcal{H}$, it might be tempting to use it to compare utilities of different agents.

We emphasize that this representation cannot be used for interpersonal comparisons of utility differences.

For example, for two agents, consider the following representatives:

$$\begin{cases} \mathbf{u} = (0.00, 0.71, -0.71)\,, \\ \mathbf{v} = (0.57, 0.22, -0.79)\,. \end{cases}$$

The fact that $v_3 < u_3$ does not mean that an agent with preferences $\tilde{\tilde{v}}$ has more aversion to candidate 3 than an agent with preferences $\tilde{\tilde{u}}$. Similarly, when we switch from candidate 1 to candidate 2, the gain for agent $\tilde{\tilde{u}}$ (+0.71) cannot be compared to the loss for agent $\tilde{\tilde{v}}$ (−0.35).

Theorem B.6 contains no message for the interpersonal comparison of utilities. Indeed, utilities belonging to two agents are essentially incomparable in the absence of additional assumptions (Hammond, 1991). Taking canonical representatives in the $(C-2)$-dimensional sphere is only used to compute distances between two points in the utility space.

B.5 Application: probability measures on the utility space

Once the space is equipped with a metric, it is also equipped with a natural probability measure: the uniform measure in the sense of this metric (which is possible because the space has a finite total measure). We will denote μ_0 this measure, which is thus the normalized Riemannian volume defined by the metric ξ_0.

In practice, to draw vectors with a uniform probability on $\mathbb{U}\setminus\{\tilde{\tilde{0}}\}$, it is sufficient to use a uniform distribution on the unit sphere of $\mathcal{H}$. In other words, once we have identified $\mathbb{U}\setminus\{\tilde{\tilde{0}}\}$ with the unit sphere of $\mathcal{H}$, then μ_0 is exactly the usual uniform measure.

In this case, the fact that the round sphere has many symmetries implies additional nice properties for μ_0, which we can summarize in one proposition.

Proposition B.9

Let μ be a probability measure on $\mathbb{U}\setminus\{\tilde{\tilde{0}}\}$.

1. *Suppose that for all $r > 0$, μ gives the same probability to all the balls of $\mathbb{U}\setminus\{\tilde{\tilde{0}}\}$ of radius r (according to the metric ξ_0). Then $\mu = \mu_0$.*

2. *If μ is invariant by any isometry of the metric ξ_0, then $\mu = \mu_0$.*

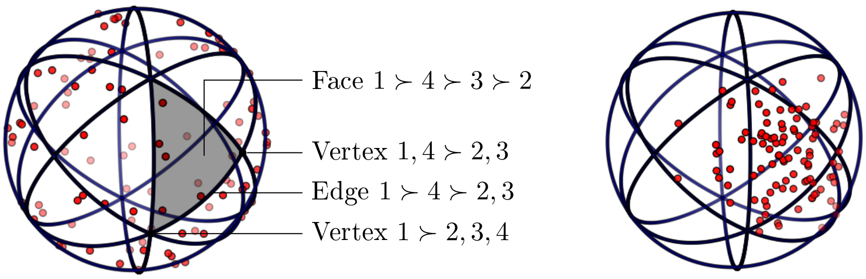
Both characterizations are classical. The first is (a strong version of) the definition of the Riemannian volume. The second one follows from the first one and from the fact that any two points of the round sphere can be sent to each other by an isometry.

In Figure B.6(a), made with SVAMP, we represent a distribution of 100 agents drawn uniformly and independently on the sphere, with 4 candidates. Like for Figure B.5, we represented only the unit sphere of $\mathcal{H}$.

The blue lines form the *permutohedron*, a geometric figure representing the ordinal aspect of these preferences. Each *face* consists of all points that share the same strict total order of preference. A utility vector belongs to an *edge* iff it has only 3 distinct utilities: for example, if an agent prefers candidate 1 to 4, 4 to 2 and 3, but is indifferent between candidates 2 and 3. Finally, a point is a *vertex* iff it has only two distinct utilities: for example, if the agent prefers candidate 1 to all the others but is indifferent between them.

In this distribution, each agent almost surely has a strict total order of preference. Each order has the same probability and the agents are independent, so this distribution is a natural generalization of the impartial culture when considering expected utilities.

Since the point $\tilde{\tilde{0}}$ is a geometric singularity, it is difficult to include it naturally in such a measure. If we want to take it into account, the simplest solution is to



(a) Uniform (impartial culture).

(b) Von Mises–Fisher.

Figure B.6 – Two distributions of 100 agents on $\mathbb{U}$ with 4 candidates.

draw it with a certain probability and to use the distribution on $\mathbb{U} \setminus \{\tilde{\tilde{0}}\}$ in the other cases. However, we have just noticed that all other strict non-total orders have a measure equal to 0; so for a canonical theoretical model, a natural choice is to also assign a measure 0 to the indifference point.

Having a distance, and thus a uniform measure, also allows to define other measures by their density with respect to the uniform measure. Here is an example of a distribution defined by its density. Given a vector $\mathbf{u}_0$ of the unit sphere of $\mathcal{H}$ and κ a nonnegative real number, the Von Mises–Fisher (VMF) distribution of pole $\mathbf{n}$ and concentration κ is defined by the following probability density with respect to the uniform distribution of the unit sphere of $\mathcal{H}$:

$$p(\mathbf{u}) = X_\kappa e^{\kappa \langle\, \mathbf{u} \,|\, \mathbf{u}_0 \,\rangle},$$

where X_κ is a normalization constant. Given the mean resultant vector of a distribution on the sphere, the VMF distribution maximizes entropy just as, in Euclidean space, the Gaussian distribution maximizes entropy among distributions with a given mean and standard deviation (Mardia, 1975). Therefore, in the absence of additional information, it is the "natural distribution" which should be used. This culture is implemented in SVVAMP (Section 6.1.3) and is studied in detail in Chapter 7. Figure B.6(b) represents such a distribution, with the same conventions as in Figure B.6(a). To draw a VMF distribution, we use the Ulrich algorithm revised by Wood (Ulrich, 1984; Wood, 1994).

Qualitatively, the VMF model is similar to the Mallows model, which is used for ordinal preferences (Mallows, 1957). In the latter, the probability of a preference order σ is:

$$p(\sigma) = X'_\kappa e^{-\kappa d(\sigma, \sigma_0)},$$

where σ_0 is the mode of the distribution, $d(\sigma, \sigma_0)$ a distance between σ and σ_0 (typically Kendall's tau distance), κ a nonnegative real number (concentration), and X'_k a normalization constant. Both the VMF and Mallows models describe a culture where the population is polarized, i.e. dispersed around a central point, with more or less concentration.

However, there are several differences.

- The VMF distribution draws a particular point in the utility space, whereas the Mallows distribution selects only one face of the permutohedron.
- In particular, the pole of a VMF distribution can be located at any point on this continuum. For example, even if the pole is on the face $1 \succ 4 \succ 3 \succ 2$,

it may be closer to the face $1 \succ 4 \succ \mathbf{2} \succ \mathbf{3}$ than to the face $\mathbf{4} \succ \mathbf{1} \succ 3 \succ 2$. Such a nuance is not possible in the Mallows model.

- In the vicinity of the pole, the probability of VMF decreases as the exponential of the square of the distance (because the inner product is the cosine of the distance on the sphere), while the probability of Mallows decreases as the exponential of the distance (without the square).

- VMF is the maximum entropy distribution, given the spherical mean and dispersion (just like a Gaussian distribution in a Euclidean space).

The existence of a canonical measure makes it possible to define other probability measures easily, in addition to the two measures just described. Such measures can generate artificial populations of agents for simulation purposes. They can also be used to match data from real experiments to a theoretical model and serve as a neutral comparison point for such data.

To develop this last point, let us insist on the fact that given a (reasonably regular) distribution μ on a space such as $\mathbb{U} \setminus \{\tilde{\tilde{0}}\}$, there is no way to define *a priori* what it means for an element $\tilde{\tilde{u}}$ to be more probable than another element $\tilde{\tilde{v}}$. Indeed, both have a null probability and what would make sense is to compare the probabilities of being close to $\tilde{\tilde{u}}$ or to $\tilde{\tilde{v}}$. But then we have to compare neighborhoods of the same size, which only makes sense if we have a metric. In other words, if we have a reference distribution like μ_0, then it makes sense to consider the density $p = \frac{\mathrm{d}\mu}{\mathrm{d}\mu_0}$, which is a (say continuous) function on $\mathbb{U} \setminus \{\tilde{\tilde{0}}\}$. Then, we can compare $p(\tilde{\tilde{u}})$ and $p(\tilde{\tilde{v}})$ to say if one of these elements is more probable than the other according to μ. Note that in this case, comparing the probability of r-neighborhoods for the metric ξ_0 or the density relative to μ_0 gives the same result in the limit $r \to 0$, which is precisely the definition of the Riemannian volume.

B.6 Conclusion

We have studied the geometric properties of the classical model of expected utilities introduced by Von Neumann and Morgenstern, when the proposed candidates are considered symmetric. We have noticed that the space of utilities can be seen as a dual of the space of lotteries, that the operators inversion and sum inherited from $\mathbb{R}^C$ have a natural interpretation in terms of preferences, and that the space has a spherical topology when we remove the indifference point.

We have proved that the only Riemannian representation which respects the projective lines naturally defined by the sum operator and the symmetry between the candidates is a round sphere.

All our considerations are based on the principle of adding the minimum amount of information in the system, in particular by respecting the symmetry between the candidates. This does not imply that the spherical representation of the utility space $\mathbb{U}$ is the most suitable for the study of this or that particular situation. Indeed, as soon as we have additional information (for example, a model of candidate placement on a political spectrum), it seems natural to take it into account in the chosen model. However, if one wishes, for example, to study a voting system in all its generality, without concentrating on its application in a particular domain, it seems natural to consider a utility space with a metric as neutral as possible, such as the one defined in this chapter by the spherical representation.

B.7 Extension: utility space with approval threshold

The model we have just studied corresponds to a situation where there are two degrees of freedom in the choice of utility representation. In this section, which has not yet been published, we present a slightly different variation, where the utility value 0 has a particular meaning, which removes the degree of freedom in translation. The main motivation for this model is to provide a natural framework to study not only ordinal voting systems and Range voting but also Approval voting.

B.7.1 Model with virtual options

Given an agent's position $\widetilde{\widetilde{u}}$ in the utility space, one can choose any vector $\mathbf{u} \in \widetilde{\widetilde{u}}$ to represent her preferences. This freedom offers the opportunity, by choosing a particular $\mathbf{u}$, to store additional information that one considers relevant. We will give several methods: in some particular applications, they allow one to choose a specific representative $\widetilde{\widetilde{u}}$ in a way that is relevant in the context. This will lead us to the notion of virtual options and finally to the model with *approval threshold*.

Before formalizing these concepts, we present some examples where the choice of a specific utility vector makes it possible to store additional information.

Example B.10 (funfair)
At a funfair booth, one can win nothing ($\varnothing$), a lynx (L) or platypus (P) stuffed animal, or both (LP). The set of candidates is $\mathcal{C} = \{\varnothing, \mathrm{L}, \mathrm{P}, \mathrm{LP}\}$.

If we write the utility vector of a player, up to adjusting the parameters α and β of Theorem B.1, and assuming that every player prefers to win both plushies rather than nothing, we can adopt the convention that the utility of winning nothing will be 0 and that of winning both plushies will be 1. For example, we can have for a particular player:

$$(u_\varnothing, u_\mathrm{L}, u_\mathrm{P}, u_\mathrm{LP}) = (0, 0.3, 0.9, 1)\,.$$

In this case, we chose a normalization convention using two options that are actual (i.e. available in the game) and fixed: if we were to express the preferences of a second player, we would use these same actual options as reference points.

Example B.11 (road junction)
Alice and Bob are planning to travel together. They are at a road junction where the possible directions are Northeast (NE), Southeast (SE), and West (W).

In absolute terms, the best option for Alice would be South (S) and the worst would be North (N). If the assumptions of Theorem B.1 are satisfied, then we can use it to represent her preferences on the set $\{\mathrm{NE}, \mathrm{SE}, \mathrm{O}, \mathrm{S}, \mathrm{N}\}$. Since the utilities are defined up to two constants, we can take for convention that the utility of the worst option (N) is 0 and the utility of the best option (S) is 1. For example:

$$(u_\mathrm{NE}, u_\mathrm{SE}, u_\mathrm{W}, u_\mathrm{S}, u_\mathrm{N}) = \left(\frac{1}{4}, \frac{3}{4}, \frac{1}{2}, 1, 0\right).$$

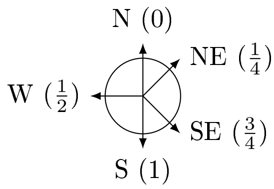


Figure B.7 – Road junction: Alice's utilities.

Finally, Alice's preferences on $\{\text{NE}, \text{SE}, \text{W}\}$ can be represented in $[0,1]^3$ by the vector $(u_{\text{NE}}, u_{\text{SE}}, u_{\text{W}}) = (\frac{1}{4}, \frac{3}{4}, \frac{1}{2})$, as represented in Figure B.7.

In this case, the utilities carry additional information compared to the situation in Theorem B.1. Indeed, in the framework of Theorem B.1, a utility vector like $(u_{\text{NE}}, u_{\text{SE}}, u_{\text{W}}) = (0, 1, \frac{1}{2})$ would have exactly the same meaning as $(\frac{1}{4}, \frac{3}{4}, \frac{1}{2})$: in both cases, the agent prefers SE then W then NE, and she is indifferent between W and an equiprobable combination of SE and NE. But, thanks to our particular choice, we also recorded, for example, that she is indifferent between NE and a probabilistic combination of her best imaginable option with probability $\frac{1}{4}$ and her worst imaginable option with probability $\frac{3}{4}$. In particular, Northeast is her worst *available* option, but not her worst *conceivable* option.

From a pure game-theoretic point of view, it is the same: whether we take the representation $(\frac{1}{4}, \frac{3}{4}, \frac{1}{2})$ or $(0, 1, \frac{1}{2})$, Alice's preferences on the lotteries of $\{\text{NE}, \text{SE}, \text{W}\}$ are identical, as are her strategic interests in a game with these possible outcomes.

However, the information we have added makes sense in an extended framework, not limited only to the proposed options and the possible strategically optimal behaviors. For example, it can be useful for modeling if we estimate that, in the problem under study, two people with respective utility vectors $(\frac{1}{4}, \frac{3}{4}, \frac{1}{2})$ and $(0, 1\,\frac{1}{2})$, with the above convention, do not behave in the same way, even if their strategic interests are the same.

Example B.12 (road junction, continued)
Let us continue with the previous example. Imagine, now, that for Bob, the best imaginable option is West (W) and the worst is East (E). If we use the same procedure as before, it is possible to obtain the utility vector $(v_{\text{NE}}, v_{\text{SE}}, v_{\text{W}}) = (\frac{1}{4}, \frac{1}{4}, 1)$, but these values are no longer relative to South and North (as they were for Alice), but to West and East.

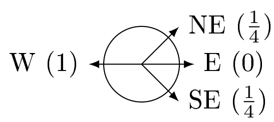


Figure B.8 – Road junction: Bob's utilities.

This double example illustrates what we call normalization with *best and worst virtual options*. Rather than taking two actual and common options as references, as in Example B.10, this normalization assumes that all players can understand and apply abstract notions that establish a unit of measurement for each of them, which is subjective but based on agreed upon concepts. Each agent considers the best and worst conceivable options, even if they are not available, and normalizes her utilities with respect to these two virtual and subjective options. Of course, the notion "conceivable" is not a mathematical property: before using this model for a particular problem, one must define what is considered "conceivable" by each player. Moreover, determining such virtual options may not be sufficient for consistent normalization, as the following example shows.

Example B.13 (game with financial reward)

We propose a game where one can lose 5 coins, or win 10 or 20 coins: $\mathcal{C} = \{-5, 10, 20\}$. We are interested in agents who always play according to the expectation of financial gain. These agents therefore have a utility for each option that is exactly (a positive linear function of) the monetary reward.[4]

Consider that, for an agent, the best conceivable option is to win an infinite number of coins and the worst is to lose an infinite number. Therefore, if we want to use a normalization with best and worst virtual options, we must have $u_{+\infty} = 1$ and $u_{-\infty} = 0$. But then, it is clearly impossible to normalize the utilities of -5, 10, and 20. Indeed, the differences of utility between -5, 10, and 20 are infinitesimal compared to that between $+\infty$ and $-\infty$, so that we would have the same utility for these three options: the agent's preferences would not be represented correctly. The core of the problem is that the agent's preferences, although Archimedean on $\{-5, 10, 20\}$, are not Archimedean on $\{-5, 10, 20, +\infty, -\infty\}$.

Even if infinity is not "conceivable", the ability to conceive of an arbitrarily large amount of money (won or lost) leads to the same impossibility of using a model with best and worst virtual options.

To summarize the lessons of the previous examples, a natural normalization appears whenever it is relevant to define two special virtual options for each agent (corresponding to the best and worst in our examples, but it is not mandatory) such as:

- The agent strictly prefers one special option to the other;
- Even when including these two special options, the preferences still respect the assumptions of Theorem B.1.

Then, we can choose a utility vector over the set of actual and special options such that the latter two receive the values 0 and 1. This gives a normalized representation of the utilities of the actual options in $\mathbb{R}^C$ (or simply $[0, 1]^C$ when the special options correspond to best and worst virtual options). This utility space $\mathbb{R}^C$ has an obvious natural geometry, so we will not study it further, preferring to focus on the model where there is only one special option, which we will introduce in the next example.

Example B.14 (election and approval threshold)

In an election, several candidates are proposed: $\mathcal{C} = \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$. Suppose that a voter has a utility vector $\mathbf{u} = (1, \frac{1}{3}, 0)$, up to normalization.

For each candidate, we ask the voter the following question: "Do you consider yourself overall favorable or unfavorable to this candidate?" Up to choosing a particular vector in $\approx{u}$ to represent the voter's preferences, it may be interesting to keep track of the answer to this question, for example if our goal is to study Approval voting. We can consider a simple convention: the utility is positive iff the answer is positive. For example, if the voter answers yes only for candidate A, we can choose as utility vector $(1, -1, -2)$.

[4] At least, they behave as if this were the case, which is the same from our point of view of observer.

We can go further: suppose that the voter is able to imagine an imaginary typical candidate S (special candidate) for which she would hesitate between a positive or negative answer. If her preferences on A, B, C, S respect the hypotheses of Theorem B.1, we can write utilities and adjust the β of the theorem to fix by convention the utility of the special candidate to 0. For example:

$$(u_{\mathrm{A}}, u_{\mathrm{B}}, u_{\mathrm{C}}, u_{\mathrm{S}}) = (3, -1, -3, 0).$$

The last coordinate does not bring any information since it is purely conventional: we can simply write $(u_{\mathrm{A}}, u_{\mathrm{B}}, u_{\mathrm{C}}) = (3, -1, -3)$. Note that with this convention, this vector no longer has the same meaning as $(1, -1, -2)$: for example, with $(3, -1, -3)$, the voter strictly prefers an equiprobable combination of A and B rather than the special candidate S.

We call this convention: normalization with the *approval threshold* as virtual option. In this framework, the utility vector is defined up to a positive multiplicative constant, as we will formalize below. Again, the information recorded by our choice does not change the optimal strategies of the player, but it can be used as behavioral information in a particular application context.

In general, we can extend these ideas to any situation where we consider it relevant to define a special virtual option for each agent, which can be subjective, such that by including this special option, her preferences still respect the assumptions of Theorem B.1.

In summary, the methods presented in this section are intended to take into account a variety of additional factors.

- Apart from its primary outcome, such as the candidate elected in an election, the game may have a secondary outcome, such as the "message" conveyed by voters, such as overall satisfaction or dissatisfaction with the candidates. Admittedly, one could take the Cartesian product of the candidate space and the message space as the outcome space and apply Theorem B.1, but this would complicate the model significantly. More simply, we can distinguish between a voter $(-1, -0.9, -0.8)$, wishing to express dissatisfaction, and a voter $(0.8, 0.9, 1)$, globally satisfied, even if their preferences on the candidate lotteries and their optimal strategies are the same.

- Players may be aware that other options could have been offered by the game organizers. This knowledge may influence how they play, regardless of the nature of their optimal strategies.

- Players may have a behavioral variable, such as a good or bad mood, that also influences how they play, regardless of their preferences about lotteries.

Thus, the proposed normalizations make it possible to add extra information to the model of the agents, in particular contextual or behavioral information, by exploiting the two degrees of freedom left by Theorem B.1.

To conclude this section, we will now give a mathematical framework of the model with approval threshold.

Let $\mathcal{C}' = [\![0, C]\!]$. The elements of $\mathcal{C} = [\![1, C]\!]$ are still called *candidates*. The element 0 is called *special candidate*.

Let $\mathbb{L}' = \{(L_0, \ldots, L_C) \in {\mathbb{R}_+}^{(C+1)} \text{ s.t. } \sum_{c=0}^{C} L_c = 1\}$. The elements of $\mathbb{L}$ are identified with those of $\mathbb{L}'$ whose coordinate of index 0 is equal to 0. The probabilistic mixture of two lotteries of $\mathbb{L}'$ is defined in a natural way.

For a binary relation $\precsim$ on $\mathbb{L}'$, the previously defined notions of completeness, transitivity, Archimedeanness, and independence of irrelevant alternatives extend in an obvious way.

We say that $\mathbf{u} = (u_1, \ldots, u_C) \in \mathbb{R}^C$ is a *utility vector with approval threshold* representing the binary relation $\precsim$ iff, for every pair of lotteries $L = (L_0, \ldots, L_C)$ and $M = (M_0, \ldots, M_C)$ of $\mathbb{L}'$:

$$L \precsim M \Leftrightarrow \sum_{c=1}^{C} L_c u_c \leq \sum_{c=1}^{C} M_c u_c.$$

Note that the sums do not take the index 0, which amounts to conventionally setting the utility of the special candidate to 0.

We will now translate the Von Neumann-Morgenstern theorem B.1 into this new model.

Proposition B.15

Let $\precsim$ be a binary relation on $\mathbb{L}'$.

The following conditions are equivalent.

1. *The binary relation $\precsim$ is complete, transitive, Archimedean, and independent of irrelevant alternatives.*

2. *There exists $\mathbf{u} \in \mathbb{R}^C$ a utility vector with approval threshold representing $\precsim$.*

When they are satisfied, $\mathbf{u}$ is defined up to a positive multiplicative constant: if $\mathbf{u} \in \mathbb{R}^C$ is a utility vector with approval threshold representing $\precsim$, then $\mathbf{v} \in \mathbb{R}^C$ is a utility vector with approval threshold representing $\precsim$ iff $\exists \alpha \in]0, +\infty[$ s.t. $\mathbf{v} = \alpha \mathbf{u}$.

Proof. This is a consequence of Theorem B.1: fixing the utility of the special candidate at 0 is possible thanks to the choice of β, but we lose the degree of freedom in translation. $\square$

In a similar way to the equivalence relation $\approx$ in the classical model, we now define the equivalence relation $\sim$ by $\forall (\mathbf{u}, \mathbf{v}) \in (\mathbb{R}^C)^2$:

$$\mathbf{u} \sim \mathbf{v} \Leftrightarrow \exists \alpha \in]0, +\infty[\text{ s.t. } \mathbf{v} = \alpha \mathbf{u}.$$

We call *space of utilities on C candidates with approval threshold*, and we denote $\mathbb{V}$, the quotient set $\mathbb{R}^C / \sim$. This is the space of half-lines of $\mathbb{R}^C$ (to which we add the indifference point), which is similar to the model of *oriented projective geometry* presented for example by Stolfi (1987); Laveau and Faugeras (1996).

We call *canonical projection* from R^C to $\mathbb{V}$ the function:

$$\tilde{\pi} : \left| \begin{array}{ccl} \mathbb{R}^C & \to & \mathbb{V} \\ \mathbf{u} & \to & \{\mathbf{v} \in \mathbb{R}^C \text{ s.t. } \mathbf{v} \sim \mathbf{u}\}. \end{array} \right.$$

For $\mathbf{u} \in \mathbb{R}^C$ and $\tilde{u} = \tilde{\pi}(\mathbf{u})$, we note unambiguously $\precsim_{\tilde{u}}$ the binary relation on $\mathbb{L}'$ represented by $\mathbf{u}$ in the sense of Proposition B.15.

B.7.2 Correspondence between utilities with approval threshold and affine forms

Just as the space of classical utilities corresponds to linear forms on bipoints of lotteries (Section B.2), we will show that the space of utilities with approval threshold corresponds to affine forms on lotteries.

For every pair of lotteries $L = (L_0, L_1, \dots, L_C)$ and $M = (M_0, M_1, \dots, M_C)$, we call *bipoint* from L to M the following vector of $\mathbb{R}^C$:

$$\overrightarrow{LM} = (M_1 - L_1, \dots, M_C - L_C).$$

We have chosen to define it as a vector of $\mathbb{R}^C$ and not of $\mathbb{R}^{C+1}$. Indeed, since $\sum_{c=0}^{C}(M_c - L_c) = 0$, the coordinate of index 0 would not provide any additional information.

Using Figure B.4 again, for every two lotteries L and M of $\mathbb{L}'$, the points L and M are identified with the points $(L_1, \dots, L_C)$ and $(M_1, \dots, M_C)$ of the solid tetrahedron which is bounded by the large triangle (simplex) and by the usual axes. A lottery belongs to $\mathbb{L}$ iff the corresponding point belongs to the simplex.

We call *tangent polytope* of $\mathbb{L}'$ the set $\mathcal{T}'$ of bipoints of $\mathbb{L}'$:

$$\mathcal{T}' = \{\overrightarrow{LM}, (L, M) \in \mathbb{L}'^2\}.$$

The tangent polytope $\mathcal{T}'$ is the set of bipoints of the tetrahedron, seen as a part of vector space.

Noting that the polytope $\mathcal{T}'$ contains a neighborhood of the origin in the vector space it generates, we obtain the following two observations.

Lemma B.16 ($\mathcal{T}'$ spans $\mathbb{R}^C$)
$\forall \mathbf{\Delta} \in \mathbb{R}^C, \exists \overrightarrow{LM} \in \mathcal{T}', \exists \lambda \in]0, +\infty[$ *s.t.* $\mathbf{\Delta} = \lambda \overrightarrow{LM}$.

Lemma B.17 ($\mathbb{L}$ spans $\mathbb{R}^C$)
Seeing $\mathbb{L}$ *as a part of the vector space* $\mathbb{R}^C$*, we have* $\mathrm{vect}(\mathbb{L}) = \mathbb{R}^C$.

Just as we denoted $\mathcal{H}^\star$ the set of linear forms on the vector space $\mathcal{H}$, let us denote $\mathbb{L}^\star$ the set of affine forms[5] on $\mathbb{L}$.

For $\mathbf{u} = (u_1, \dots, u_C) \in \mathbb{R}^C$, we call *affine form associated to* $\mathbf{u}$ the following element[6] of $\mathbb{L}^\star$:

$$\langle\, \mathbf{u} \,|: \left| \begin{array}{ccc} \mathbb{L} & \to & \mathbb{R} \\ L = (L_1, \dots, L_C) & \to & \langle\, \mathbf{u} \,|\, L \,\rangle = \sum_{c=1}^{C} u_c L_c. \end{array} \right.$$

We call *positive half-space associated to* $\mathbf{u}$:

$$\mathbf{u}^+ = \{\mathbf{\Delta} \in \mathbb{R}^C \text{ s.t. } \langle\, \mathbf{u} \,|\, \mathbf{\Delta} \,\rangle \geq 0\}.$$

By simply re-reading the definitions, we note that for $(L, M) \in \mathbb{L}^2$ and $\mathbf{u} \in \mathbb{R}^C$, we have the following properties.

1. $L \precsim_{\tilde{u}} M \Leftrightarrow \langle\, \mathbf{u} \,|\, \overrightarrow{LM} \,\rangle \geq 0 \Leftrightarrow \overrightarrow{LM} \in \mathbf{u}^{+\mathcal{H}}$.

[5] We make a slight abuse of vocabulary to simplify: $\mathbb{L}^\star$ is the set of restrictions to $\mathbb{L}$ of the forms of the affine space generated by $\mathbb{L}$. Since the affine forms and their restrictions are in obvious bijection, there is no risk of confusion.

[6] We use the same notation as for the linear form associated to $\mathbf{u}$. Since their expressions are identical, there is no risk of confusion.

2. $L \succsim_{\tilde{u}} (0, \ldots, 0) \Leftrightarrow \langle\, \mathbf{u} \,|\, L \,\rangle \geq 0$.

We can now state the following proposition, which presents a duality result similar to that given by Proposition B.2 for the classical model without approval threshold.

Proposition B.18

For $(f, g) \in (\mathbb{L}^\star)^2$*, denote* $f \sim g$ *iff* $\exists \alpha \in]0, +\infty[$ *s.t.* $g = \alpha f$*. Denote* $\tilde{\pi}(f) = \{g \in \mathbb{L}^\star, f \sim g\}$.

For $(\mathbf{u}, \mathbf{v}) \in (\mathbb{R}^C)^2$*, we have:*

$$\mathbf{u} \sim \mathbf{v} \Leftrightarrow \langle\, \mathbf{u} \,| \sim \langle\, \mathbf{v} \,|.$$

The following map is a bijection:

$$\Theta : \left| \begin{array}{ccc} \mathbb{V} & \to & \mathbb{L}^\star / \sim \\ \tilde{\pi}(\mathbf{u}) & \to & \tilde{\pi}(\langle\, \mathbf{u} \,|). \end{array} \right.$$

Proof. $\mathbf{u} \sim \mathbf{v}$
$\Leftrightarrow \exists \alpha \in]0, +\infty[$ s.t. $\mathbf{v} - \alpha \mathbf{u} = \mathbf{0}$
$\Leftrightarrow \exists \alpha \in]0, +\infty[$ s.t. $\forall L \in \mathbb{L}, \langle\, \mathbf{v} - \alpha \mathbf{u} \,|\, L \,\rangle = 0$ (cf. Lemma B.17)
$\Leftrightarrow \langle\, \mathbf{u} \,| \sim \langle\, \mathbf{v} \,|$.

The implication $\Rightarrow$ proves that Θ is well defined: indeed, if $\tilde{\pi}(\mathbf{u}) = \tilde{\pi}(\mathbf{v})$, then $\tilde{\pi}(\langle\, \mathbf{u} \,|) = \tilde{\pi}(\langle\, \mathbf{v} \,|)$. The implication $\Leftarrow$ ensures that Θ is injective. Finally, it is clear that Θ is surjective. □

The space of utilities with approval threshold can thus be seen as a quotient of the space $\mathbb{L}^\star$ of affine forms on lotteries $\mathbb{L}$. A utility vector with approval threshold can be seen, up to a positive constant, as an affine form determining, for every point in the space of lotteries, how strongly it is valued by the agent (relative to a level 0 which is the special candidate).

B.7.3 Inversion and sum operators in the utility space with approval threshold

As a quotient of $\mathbb{R}^C$, the utility space inherits natural operations on $\mathbb{R}^C$, the inversion and the sum. We will see that, as with the model without approval threshold, these two quotient operations have intuitive meaning regarding preferences.

In $\mathbb{V}$, we define the *inversion* operator by:

$$- : \left| \begin{array}{ccc} \mathbb{V} & \to & \mathbb{V} \\ \tilde{\pi}(\mathbf{u}) & \to & \tilde{\pi}(-\mathbf{u}). \end{array} \right.$$

The inversion operator is correctly defined and it is a bijection. Indeed, we have $\tilde{\tilde{\pi}}(\mathbf{u}) = \tilde{\tilde{\pi}}(\mathbf{v})$ iff $\tilde{\tilde{\pi}}(-\mathbf{u}) = \tilde{\tilde{\pi}}(-\mathbf{v})$. As in the usual utility space $\mathbb{U}$, considering the inverse corresponds to reversing the agent's preferences, without changing their relative intensities. In $\mathbb{V}$, moreover, the inversion preserves the relative intensities with respect to the approval threshold.

On the parts of $\mathbb{V}$, we define the *sum* operator by:

$$\sum : \left| \begin{array}{ccc} \mathcal{P}(\mathbb{V}) & \to & \mathcal{P}(\mathbb{V}) \\ A & \to & \left\{ \tilde{\pi}\left(\sum_{i=1}^n \mathbf{u}_i \right), n \in \mathbb{N}, (\mathbf{u}_1, \ldots, \mathbf{u}_n) \in \left(\tilde{\pi}^{-1}(A) \right)^n \right\}. \end{array} \right.$$

Similarly to the model without approval threshold, we will show that the sum is the locus of utility vectors that preserve the unanimous preferences of the population.

Theorem B.19 (characterization of the sum in the utility space with approval threshold)
Let $A \in \mathcal{P}(\mathbb{V})$ and $\tilde{v} \in \mathbb{V}$.
The following conditions are equivalent.

1. $\tilde{v} \in \sum A$.
2. $\forall (L, M) \in \mathbb{L}'^2$: $(\forall \tilde{u} \in A, L \precsim_{\tilde{u}} M) \Rightarrow L \precsim_{\tilde{v}} M$.

Proof. Let $\mathbf{v} \in \tilde{\pi}^{-1}(\tilde{v})$. We have the following equivalences.

- $\forall (L, M) \in \mathbb{L}'^2, (\forall \tilde{u} \in A, L \precsim_{\tilde{u}} M) \Rightarrow L \precsim_{\tilde{v}} M$,
- $\forall \overrightarrow{LM} \in \mathcal{T}', \left(\forall \mathbf{u} \in \tilde{\pi}^{-1}(A), \langle \mathbf{u} \mid \overrightarrow{LM} \rangle \geq 0\right) \Rightarrow \langle \mathbf{v} \mid \overrightarrow{LM} \rangle \geq 0$,
- $\forall \mathbf{\Delta} \in \mathbb{R}^C, \left(\forall \mathbf{u} \in \tilde{\pi}^{-1}(A), \langle \mathbf{u} \mid \mathbf{\Delta} \rangle \geq 0\right) \Rightarrow \langle \mathbf{v} \mid \mathbf{\Delta} \rangle \geq 0$ (cf. Lemma B.16),
- $\bigcap_{\mathbf{u} \in \tilde{\pi}^{-1}(A)} \mathbf{u}^+ \subset \mathbf{v}^+$,
- $\mathbf{v}$ is in the convex cone of $\tilde{\pi}^{-1}(A)$,
- $\tilde{v} \in \sum A$.

□

B.7.4 Riemannian representation of the utility space with approval threshold

Like for the classical model, we will exhibit the Riemannian representations which are consistent with the natural properties of the space of utilities with approval threshold and with the symmetry *a priori* between the candidates (except the special candidate).

For the same reasons as before, the indifference point $\tilde{0}$ must be excluded: indeed, its only open neighborhood is $\mathbb{V}$ as a whole and no distance is compatible with this property. On the other hand, $\mathbb{V}$ has the same topology as the sphere of dimension $C - 1$, so it can be equipped with a distance.

The situation being, here, less symmetrical than in the case of $\mathbb{U} \setminus \{\tilde{\tilde{0}}\}$, we need a more diverse family of metrics. Rather than considering only inner products proportional to the canonical inner product, we will consider all those which:

- Make $\mathcal{H}$ orthogonal to the direction $\mathbf{1}$,
- Have a restriction to $\mathcal{H}$ proportional to the restriction of the canonical product,

i.e. the $\lambda\psi_\rho$, where ψ_ρ is the inner product which coincides on $\mathcal{H}$ with the canonical product, such that $\mathbf{1}$ is orthogonal to $\mathcal{H}$ and that $\psi_\rho(\mathbf{1}, \mathbf{1}) = \rho^2$.

We note ζ_ρ the metric on $\mathbb{V} \setminus \{\tilde{0}\}$ induced by ψ_ρ.

To picture this, we need only imagine that the utility vectors are normalized to belong to the spheroid which is the image of the unit sphere by the orthogonal affinity of factor ρ along the direction $\mathbf{1}$. Then we measure the distance between two points, not using the metric induced by the canonical inner product of $\mathbb{R}^C$, but the metric induced by the inner product for which this spheroid is the sphere of diameter λ.

For example, the distance between two antipodal points will always be $\lambda\pi$ (where π denotes the usual number pi, i.e. the half-perimeter of a unit circle), which would not be the case with the metric derived from the canonical inner product. We will come back to this construction in Section B.7.5.

The real number $\rho \in]0, +\infty[$ is thus the distortion factor which defines the elongation or the flattening of the spheroid in the direction $\mathbf{1}$. The real $\lambda \in]0, +\infty[$ is a global dilation factor which multiplies all distances.

Theorem B.20 (Riemannian representation of the utility space with approval threshold)

We assume that $C \geq 3$. Let ζ be a Riemannian metric on $\mathbb{V} \setminus \{\widetilde{0}\}$.
The following conditions 1 and 2 are equivalent.

1. (a) *For every non-antipodal pair of points $\widetilde{u}, \widetilde{v} \in \mathbb{V} \setminus \{\widetilde{0}\}$ (i.e. $\widetilde{v} \neq -\widetilde{u}$), the set $\sum\{\widetilde{u}, \widetilde{v}\}$ of elements respecting the unanimous preferences of $\widetilde{u}$ and $\widetilde{v}$ is a segment of geodesic of ζ; and*

   (b) *for every permutation σ of $[\![1, C]\!]$, the action Φ_σ induced by $\mathbb{V} \setminus \{\widetilde{0}\}$ on*

   $$(u_1, \ldots, u_C) \to (u_{\sigma(1)}, \ldots, u_{\sigma(C)})$$

   is an isometry.

2. $\exists \rho \in]0, +\infty[, \exists \lambda \in]0, +\infty[$ *s.t.* $\zeta = \lambda \zeta_\rho$.

Proof. As in Theorem B.6, we deduce from the condition 1a the existence of an inner product ψ on $\mathbb{R}^C$ inducing ζ.

By the condition 1b, the direction $\mathbf{1}$ is orthogonal (in the sense of ψ) to $\mathcal{H}$. Indeed, the vector $(1, -1, 0, \ldots, 0)$ is sent by the transposition $\Phi_{(12)}$ on its opposite, and this one being an isometry, the angle between $\mathbf{1}$ and $(1, -1, 0, \ldots, 0)$ is right. The same argument applies to all permutations of this vector, which generate $\mathcal{H}$.

In particular, the orthogonal projections onto $\mathcal{H}$ in the sense of ψ and of the canonical inner product of $\mathbb{R}^C$ coincide. It follows that Φ_σ permutes the projections of the vectors of the canonical basis onto $\mathcal{H}$. We can therefore reason as in the proof of Theorem B.6 to conclude that the restriction of ψ to $\mathcal{H}$ is proportional to the canonical inner product. □

B.7.5 The Riemannian representation in practice

For every real number $\rho \neq 0$, denote D_ρ the matrix of the orthogonal affinity which dilates by a factor ρ in the direction of $\mathbf{1}$:

$$D_\rho = D_0 + \rho \frac{1}{C} J = I + \frac{\rho - 1}{C} J.$$

The spheroidal representation of Theorem B.20 is then the image of the unit sphere by D_ρ (whose reciprocal bijection is $D_{(\rho^{-1})}$).

For $(\mathbf{u}, \mathbf{v}) \in (\mathbb{R}^C \setminus \mathbf{0}\})^2$, the distance between $\widetilde{u}$ and $\widetilde{v}$ in the sense of the metric $\lambda\zeta_\rho$ of Theorem B.20 is:

$$d(\widetilde{u}, \widetilde{v}) = \lambda \arccos \left\langle \frac{D_{(\rho^{-1})}\mathbf{u}}{\|D_{(\rho^{-1})}\mathbf{u}\|} \,\middle|\, \frac{D_{(\rho^{-1})}\mathbf{v}}{\|D_{(\rho^{-1})}\mathbf{v}\|} \right\rangle.$$

Indeed, we have two possibilities:

- Compute the representative of $\tilde{u}$ which belongs to the spheroid then bring it back to the sphere by the affinity $D_{(\rho^{-1})}$;

- Calculate the image $\mathbf{u}'$ of $\mathbf{u}$ by the affinity $D_{(\rho^{-1})}$ and then normalize it, i.e. calculate the intersection of the half-line defined by $\mathbf{u}'$ with the unit sphere.

Since the unit sphere is the image of the spheroid by the affinity $D_{(\rho^{-1})}$, these two possibilities are equivalent. Choosing the second one immediately proves the proposed formula.

If we want to draw vectors following a uniform probability distribution on $\mathbb{V} \setminus \{\tilde{0}\}$ (equipped with the metric $\lambda\zeta_\rho$), it is sufficient to draw $\mathbf{u}$ according to a uniform distribution on the unit sphere of $\mathbb{R}^C$ then to return $D_\rho \mathbf{u}$.

In order to form a correct intuition of this spheroidal representation, the main difficulty is that we must not consider the canonical Euclidean metric on the spheroid in question, but a metric for which this ellipse is a sphere, as we will now illustrate.

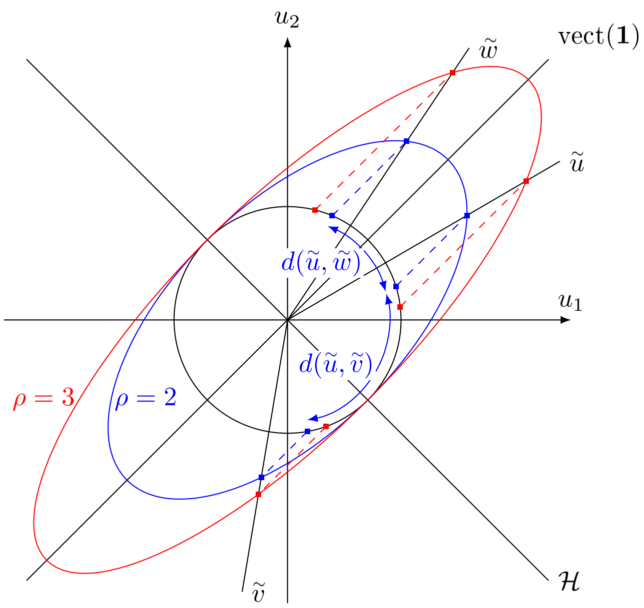


Figure B.9 – Spheroidal representations of the utility space with approval threshold.

In Figure B.9, we show how the distance is calculated, for example between $\tilde{u}$ and $\tilde{v}$. The case $\rho = 2$ is in blue and $\rho = 3$ in red. In each case, we take the intersection of the half-line $\tilde{u}$ with the spheroid, then we use the affinity $D_{(\rho^{-1})}$ to bring this point back on the sphere. We do the same with $\tilde{v}$, then we measure the distance to the surface of the sphere.

When ρ tends to infinity, we see that $d(\tilde{u}, \tilde{v})$ tends to 0 while $d(\tilde{u}, \tilde{w})$ tends to a positive real number (which is $\lambda\pi$). In this degenerate case, the distance between two points depends only on the relative utilities on the candidates, without considering the position with respect to the approval threshold: from this point of view, this resembles the $(C-2)$-dimensional spherical model. However, it also means that a distance λ is concentrated between $\tilde{u}$ and $\tilde{w}$, and in general between any pair of half-lines located on either side of $\mathrm{vect}(\mathbf{1})$: finally, if we use the uniform probability distribution, we almost surely obtain $-\mathbf{1}$ or $\mathbf{1}$.

When ρ tends to 0, we obtain the opposite behavior: the distance between two points depends only on the global approval or disapproval (i.e. in which half-space we are with respect to the hyperplane $\mathcal{H}$), not on the relative appreciations between candidates. However, if we use the uniform probability distribution, we almost certainly obtain a vector belonging to $\mathcal{H}$.

In the practical applications of Chapter 7, we always choose $\rho = 1$, i.e. a spherical representation. This is an intermediate case between the two limiting cases presented above, which favor either utilities of the same sign, or zero-sum utility vectors (hence almost surely comprising utilities of both signs).

Appendix C

Voting Systems Applied to Telecommunication Networks

We consider a simplified model where nodes in a network want to establish a path to transmit packets, with revenue shared equally among the participants and a cost that depends on the path used. We consider the possibility that nodes use a voting system to determine the path followed.

If nodes vote sincerely, Range voting gives by definition the economic optimum. But the nodes have an interest in voting tactically: in Range voting, we show that the performances then collapse. On the other hand, if IRV is used, then the system is much less manipulable and the chosen paths are still close to the economic optimum.

This appendix essentially takes up the work presented by Durand et al. (2013, 2014c).

The Internet has become an economic ecosystem with many competing actors. In order to make relevant decisions, while preventing economic actors from manipulating the natural outcome of the decision process, game theory is a natural framework. Voting systems represent an interesting alternative that, to our knowledge, has not yet been considered. They allow competing entities to decide between different options. In this chapter, we examine their use for the selection of a path from one point to another in a network involving several operators, and we analyze their manipulability by tactical voting and their economic efficiency. We show that IRV is much more efficient and resilient to tactical voting than the natural voting system for finding the economic optimum, i.e. Range voting.

The emergence of new services, coupled with the steady growth of existing services, has led to an explosion of the Internet traffic. The network of networks has become a huge economic ecosystem from which many companies derive revenue. This includes network operators, who interconnect their infrastructures to form the Internet, but also service providers, who monetize their services within the Internet. In this context, it is important to ensure the fairness of decisions that involve many competing actors, with the goal of achieving some form of global economic optimum.

Indeed, participants in a decentralized network often have to make global decisions based on particular interests. For example, Internet routes span several Autonomous Systems (AS) while they result from local and partly arbitrary decisions. Many fields of game theory have been proposed to better understand distributed decision making (Nisam et al., 2007). However, to our knowledge, one

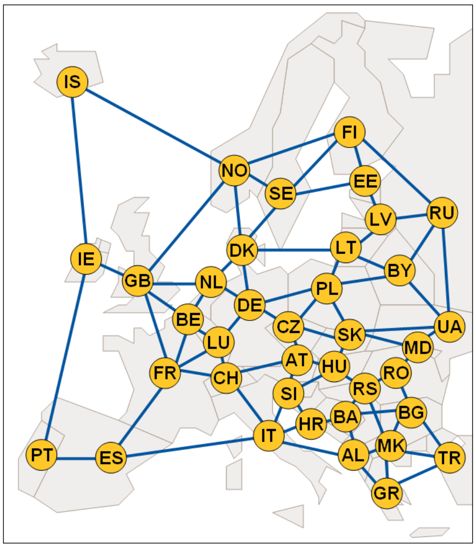


Figure C.1 – Example of a connection network between several operators.

of these fields remains to be explored: voting systems, which allow competing entities to make a decision between several options.

The main goal of this chapter is to examine the use of voting systems within the Internet economic ecosystem on a given use case. In particular, we focus on the issue of manipulability, which is crucial in a decision-making context. We know that, apart from degenerate cases, all voting systems are manipulable (Gibbard, 1973; Satterthwaite, 1975). However, voting systems are not equal in manipulability.

Section C.1 defines and models the use case of this appendix: the establishment of a path in a multi-operator network. It is a simplified model initially developed by Ludovic Noirie, whose goal is above all to qualitatively study the behaviors of the various voting systems without dwelling on the technological complexity of the system. Finally, in Section C.2, we analyze the results obtained, illustrating how and why voting systems can be very interesting for a practical use in an economic ecosystem such as a multi-operator network.

C.1 Presenting the model of multi-operator path establishment

In this section, we define the problem we wish to solve: the establishment of paths from one point to another by several operators. We explain how voting systems can be used for this purpose, and how candidate paths are pre-selected. Then we define cost and payoff modeling, and we present how voting systems are used and how manipulability is computed.

C.1.1 Voting systems for multi-operator path establishment

The problem to be solved is the following: in a multi-operator network, given an ingress and an egress node, which path should be chosen if we take into account the operators' preferences?

Figure C.1 represents an example of interconnection for a multi-operator network, with one operator per European country connected to members in a certain geographic neighborhood.[1]

We use voting systems with the following model:

- The voters are the operators.
- The candidates are the possible routing paths for a certain request, or a subset of them.
- The preferences of each operator are represented by a utility vector, from which a ballot is defined.
- The result of the election is the selected path for the client request. Any voting system can be used. Here we assume that there is a trusted and independent entity, called the *supervisor*, in charge of the election process. The supervisor collects the ballots from all operators and computes the counting function to decide the winning path (but other options are possible; for example, the operators can cooperatively participate in the entire counting process).

It is possible that some operators have information about the utilities of their competitors:

- By public knowledge of the utilities,
- By cooperation between some operators (coalition),
- By inference from previous polls (learning),
- By espionage or interception of information.

Operators can use their knowledge to lie about their own preferences in order to improve their profits, potentially at the expense of the general interest. Therefore, studying the manipulability of voting systems is important in this context.

C.1.2 Candidate paths

In our case study, we consider the interconnection of $V = 38$ operators in Figure C.1.

A *request* is a connection demand with the only constraint to start at a given operator (ingress) and to end at another given operator (egress).

For each request, if we consider the set of all possible paths without loop, the size of this set grows exponentially with the number of nodes in the network. To overcome this problem, we apply the following rule to limit the number of proposed paths. To apply this rule, only the knowledge of the network topology is necessary.

- The supervisor sets a minimal number of candidate paths C_{min} and an initial threshold ε_{min}.

[1]This example is not representative of an actual multi-operator network. For example, such a flat connection topology differs from the historical hierarchical BGP topology of the Internet. Nevertheless, our goal here is to give a simple model where costs are derived from some form of underlying metric. Therefore, basing on the geography of the network is a natural choice. Note that some recent studies show a move towards flatter topologies than in the past (Labovitz et al., 2010; Dhamdhere and Dovrolis, 2010).

- She knows the topology of the links (but not the costs). She calculates the minimal number of hops $h_{\min}$ necessary to satisfy the request.

- She considers all the paths without loop having a number of jumps less than or equal to $h_{\min} + \varepsilon$, where ε is the smallest threshold greater than or equal to $\varepsilon_{\min}$ which selects at least $C_{\min}$ candidates.

It is important to note that:

- The candidate paths are completely determined by the demand and by the parameters $(C_{min}, \varepsilon_{\min})$. Their exact number depends on the demand.

- For each candidate path c, only a subset $\mathcal{V}_c$ of the operators $v \in \mathcal{V}$ is concerned by the candidate path.

- Among the operators, some can be concerned by only a subset of the candidates.

For numerical evaluations, we use two sets of parameters: $(C_{\min}, \varepsilon_{\min}) \in \{(5, 0), (10, 1)\}$. The first setting gives an average of 9.94 candidate paths per query (minimum 5, maximum 43), and the second setting gives an average of 21.25 paths per query (minimum 10, maximum 127).

C.1.3 Multi-operator cost and gain modeling

We define the utilities of each operator as the difference between its gains and costs for each possible path. Various models of costs and gains could be defined, but the main purpose of this study is to evaluate the manipulability of voting systems in this context, so we use the following simplified models. If one wished to test models closer to reality, there would be no technical obstacle to doing so. Our intuition is that this would make only minor changes to the observed patterns, but it would be interesting to validate this intuition in future work.

Cost

The cost for an operator v of $\mathcal{V}_c$ to carry the path c is denoted $\alpha_{v,c}$. We define $\alpha_{v,c}$ as the sum of half the cost of the incoming connection (zero for the ingress operator) and half the cost of the outgoing connection (zero for the egress operator). For the cost of a connection between two adjacent operators v and v', we choose a linear function $X_0 + \delta(v, v')/\delta_0$ of the geographical distance $\delta(v, v')$ between v and v' (v and v' being the capitals of the countries in our example network). In our numerical study, we consider three options for the cost: dominated by the constant cost X_0 ($X_0 = 1$ and $\delta_0 = 100 \times \max(d(v, v'))$), purely linear ($X_0 = 0$, $\delta_0 = \max(\delta(v, v'))$), and intermediate ($X_0 = 1$, $\delta_0 = \max(\delta(v, v'))/3$).

Gain

Concerning the gain for the operators, we consider that the client pays a fixed amount A for a given request (*flat fare*). If the path c is selected, this amount A is distributed equally among the concerned operators (i.e. the members of $\mathcal{V}_c$). We set the value of A of the user price so that, when the least-cost path is considered for each query, the average overall revenue is 140% of the average overall cost (which translates into a profit of 40%).

Utility

The sincere utility value for the operator v to carry the candidate path c is defined as the net revenue (positive or negative) for the operator in the case that this candidate path is selected:

- $u_{v,c} = \dfrac{A}{card(\mathcal{V}_c)} - \alpha_{v,c}$ if $v \in \mathcal{V}_c$,
- $u_{v,c} = 0$ if $v \notin \mathcal{V}_c$ (an operator is thus indifferent between the paths in which it does not participate).

Global income

The total revenue of a given operator (per unit time) depends on the distribution of demands. In our study, we consider that requests are uniformly distributed over the pairs of ingress and egress operators.

C.1.4 Voting systems studied

Among the voting systems presented in Chapter 1, we consider two in particular, which we recall while indicating the specificities of their application to the establishment of a path in the network.

Range voting (RV)

With the utilities defined above, the most natural voting system is the one that maximizes the global income, i.e. the sum of the utilities of all the operators that contribute to the selected path. This corresponds exactly to Range voting, so this will be the natural reference system. In detail:

- The operators give their utilities to the supervisor,
- The supervisor computes the sum of the scores for each candidate path and selects the path with the maximum value.

One of the disadvantages of RV is that operators must give all information about their costs to the supervisor. Even though this is a reliable and independent entity, they may prefer to avoid disclosing this type of information.

IRV

Many other voting systems can be applied. Previous work (e.g. Chamberlin et al. (1984) or Walsh (2010b)), as well as that of this thesis, suggests that IRV is among the least manipulable of the known voting systems. This is why we use IRV as a second voting system for our case study.[2] Path selection with IRV works as follows.

- Each voter (operator) gives her order of preference on the candidate paths which concern her and, for each one of them, if she approves it (financial gain) or disapproves it (financial loss) so as to place them with respect to the paths in which she does not participate. In this way, each operator is forced to express an indifference between the paths in which she does not participate.

[2] In fact, we also tried other voting systems in experiments not presented here, and verified that IRV was the most promising voting system in terms of manipulability, apart from Condorcet-IRV which is more complex and has similar performances.

- The supervisor calculates the count according to the usual IRV rule, with the following rule in case of indifference between several candidates: in each round of IRV, for a given operator, her vote is divided equally between the candidates still in the race who are tied at the top of her ballot. In each round, the supervisor eliminates the candidate path with the fewest votes (the number of votes is not necessarily integer). In case of a tie, the candidate path with the highest index is eliminated.

C.1.5 Manipulation algorithms

First, for both voting systems, an operator's manipulation of the ballot is limited to the candidate paths that concern her: the operator cannot claim to like or dislike a candidate path for which she should clearly be indifferent.

RV

When voters preferring c to w try to make c win, their best strategy is obviously the trivial one: give the maximum score to c and the minimum score to the other candidates, except for the paths in which they do not participate, for which they are forced to give a score equal to 0. For this reason, coalitional manipulation (CM) and trivial manipulation (TM) are equivalent. In our study, the maximum score is set to A and the minimum score allowed is $-A$.

IRV

Deciding whether there is a way to manipulate IRV is much more difficult. In fact, the problem is known to be $\mathcal{NP}$-difficult (Bartholdi and Orlin, 1991). Therefore, we use simple methods to test the manipulability of a given request. When the tests are not conclusive for a given request, we can only answer *maybe*, but when we consider all requests, this allows us to give lower and upper bounds on manipulability.

In order to prove that IRV is manipulable, we try only trivial manipulation. This gives a lower bound of manipulability.

In order to prove that IRV is not manipulable, our algorithm is an adaptation of Coleman and Teague (2007). The difficulty arises from the fact that voters have imposed ties (for candidate paths in which they do not participate) and that it is possible that their vote is split between several candidates in a given round of counting. The idea of our adaptation is to use a variant of the voting system that gives more power to the manipulators and for which CM can be calculated exactly. If the modified system cannot be manipulated, then IRV cannot be manipulated either. In detail:

- In each round, the manipulators are allowed to change their vote (as in Exhaustive ballot). The point is that eliminating candidate c then d or d then c leads to the same situation, regardless of the history of getting there. This makes it easier to have an iterative approach instead of a recursive approach.

- In each round, each manipulator can split her vote between several candidates, including non-equally, e.g. $\frac{1}{3}$ votes for one candidate and $\frac{2}{3}$ votes for another. This allows for a water-filling approach and avoids backpacking problems.

- Voters are allowed to lie, including about paths in which they do not participate. Therefore, all manipulators are equal in right and we do not need to deal with them individually.

With these modifications, we can manage the group of manipulators globally: in each round, we can divide their votes as we wish between the candidates, in order to eliminate the desired candidate.

When manipulation is impossible with these adapted rules, it is also impossible with the rules given in C.1.4: this provides an upper bound for the manipulability of IRV.

As we will see in the next section, the lower and upper bounds tend to be close to each other, giving a good estimate of manipulability.

Remarque In fact, the previous algorithm remains very expensive (potentially of order 2^C). When there are more than 25 candidates and the trivial manipulation does not work, we do not try to prove the impossibility and we directly consider the test as inconclusive.

C.2 Manipulability and efficiency of the voting systems studied

In this section, we analyze the results obtained on the case of multi-operator networks described in Section C.1.

For both RV and IRV, we first focus on a specific scenario as a baseline and then extend the results by changing various parameters. We observe manipulability and cost-effectiveness, with sincere and non-sincere ballots.

C.2.1 Baseline scenario

Our baseline scenario is the case with $(C_{\min}, \varepsilon_{\min}) = (5, 0)$ (on average 9.94 candidate paths per request) and the intermediate cost model ($X_0 = 1$, $\delta_0 = \max(d(v, v'))/3$). For this scenario, we measure:

- *Sincere efficiency*: overall net income, as a percentage of optimal overall net income.
- *Manipulability*: proportion of requests that are CM. For Range voting, the calculation is exact. For IRV, we use TM as a lower bound and the variant presented in Section C.1.5 to give an upper bound.
- *Manipulated efficiency*: efficiency of the system when manipulations are allowed. For a given request, several manipulations can occur. We can measure the average manipulated efficiency, when a path with a possible manipulation is chosen at random, or the worst case if for each manipulable request, we choose the manipulation that minimizes the global net revenue (remember however that by definition, the net revenue of each manipulator is necessarily increased).

The results are presented in Table C.1. IRV has many advantages over Range voting:

- In sincere voting, RV gives the economic optimum by definition. Although IRV does not have the economic optimum as an intrinsic target, it manages to reach 95% of the optimum. This slight loss of efficiency can be seen as the price of robustness, as we shall see.

Voting system	RV	IRV
Sincere efficiency	100%	95 %
Manipulability	96%	< 20% (TM: 18%)
Manipulated efficiency (average)	37%	90%
Manipulated efficiency (worst case)	-75%	89%

Table C.1 – Main results for the baseline scenario. $(C_{\min}, \varepsilon_{\min}) = (5, 0)$, $(C_0, \delta_0) = (1, \max(d(v, v'))/3)$.

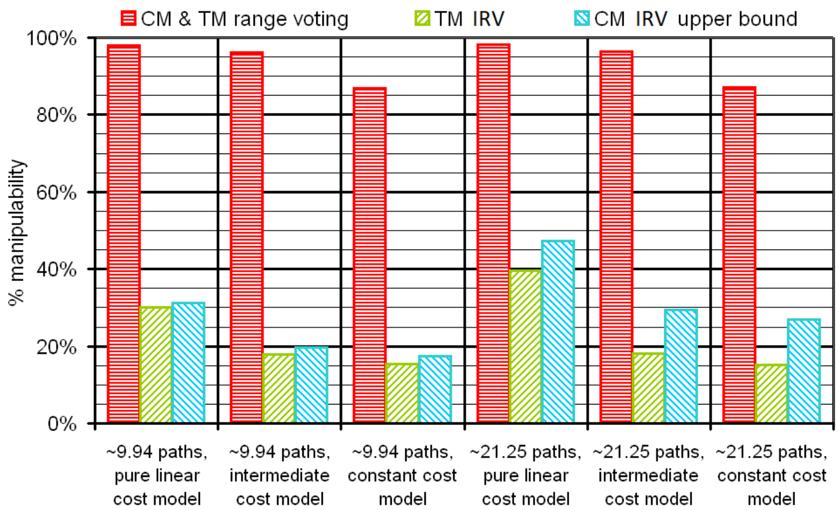


Figure C.2 – Manipulability of IRV compared to Range voting.

- Almost all requests (96%) can be manipulated if Range voting is used, compared to less than 20% for IRV.
- As for the impact on efficiency, the degradation is very high for Range voting: it goes down to 37% of the optimum on average, and to -75% in the worst case. For its part, IRV maintains a robust 90% on average (89% in the worst case scenario).

As a result, IRV is much less prone to manipulation than Range voting, and even when manipulation exists, its impact on the overall welfare is tolerable.

C.2.2 Impact of parameters

We proposed in Section C.1 two candidate path limitation parameters and three cost models, so we have six possible settings.

Manipulability

Figure C.2 shows the manipulability of the considered scenarios.

For both voting systems, the more candidates there are, the more manipulable it is. This is hardly surprising, since more candidates means that there are more paths that can be manipulated and also more space for expression in the ballots.

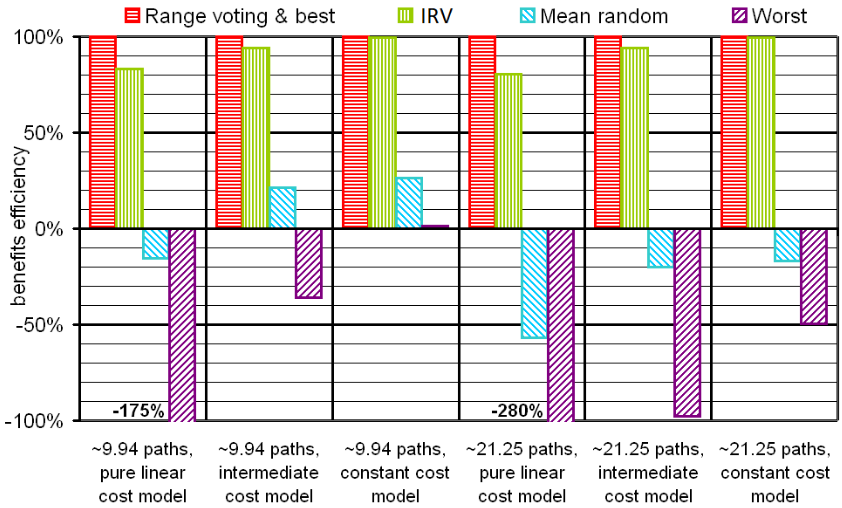


Figure C.3 – Efficiency of IRV and Range voting in sincere voting.

Therefore, it is advisable for the supervisor to limit the number of candidates offered in order to decrease the manipulability, while keeping enough candidates to allow for the fair and relevant selection of a path.

Regarding the cost model, we observe that the flatter it is, the lower the manipulability (for both voting systems). But the most interesting result is that the manipulability of IRV is much lower than that of Range voting in all cases. While the manipulability of RV is always higher than 85%, the manipulability of IRV is around 30% or less except for one scenario (between 40% and 48% for the largest number of candidate paths and with a purely linear cost).

Note that for IRV, the lower and upper bounds on manipulability are relatively close. The difference grows with the number of candidate paths, but this is due to the method used to compute the upper bound (cf. Section C.1), which consists in dropping the evaluation for requests that result in too many candidate paths.

Sincere economic efficiency

Figure C.3 gives the sincere efficiency of each voting system for the considered scenarios. By definition, RV gives 100% economic efficiency. But IRV gives an economic efficiency close to this optimum. Whether for a low or high number of candidate paths, this efficiency is about 80% for the purely linear cost model, about 95% for the intermediate cost model, and over 99% for the constant cost model. This confirms that, although it is possible that IRV yields a path that is not optimal for overall economic income, the selected path is relatively close to the optimal choice.

For reference, Figure C.3 also shows the efficiency obtained when the path is chosen randomly among the candidates, and when the worst candidate is chosen.

Efficiency after manipulation

Figure C.4 shows the manipulated efficiencies. For all scenarios, there is a dramatic gain by choosing IRV over Range voting. For IRV, the economic efficiency is only slightly degraded compared to the sincere case, and the worst case is never far from the average case. For Range voting, the economic efficiency is largely

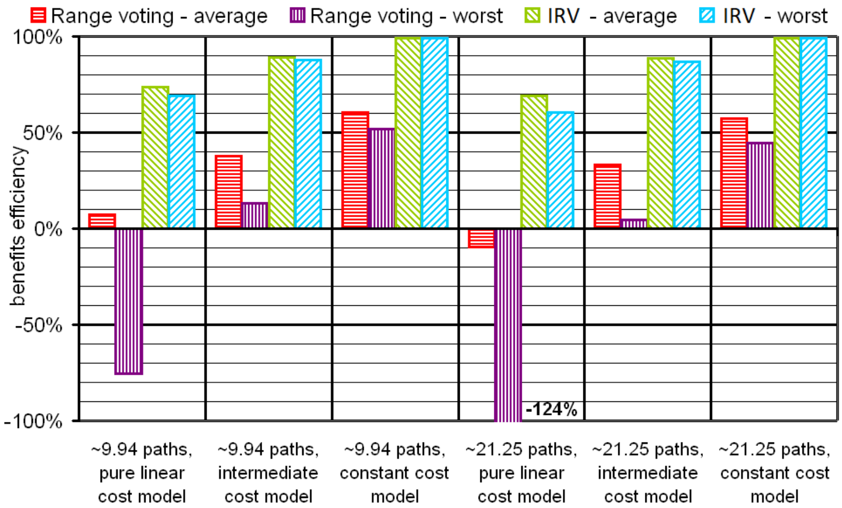


Figure C.4 – Efficiency of IRV and Range voting with manipulation.

degraded compared to the sincere case: while the sincere economic efficiency is by definition 100%, the manipulation brings it down to about 0% on average and large negative values in the worst case for the purely linear cost model; about 35% on average and about 8% in the worst case for the intermediate cost model; and to about 60% on average and 50% in the worst case for the constant cost model.

All of these results confirm that IRV is indeed safer for preserving economic benefits than Range voting.

C.3 Conclusion

Existing theoretical results show that, except for a few degenerate cases, any voting system is susceptible to manipulation in some scenarios, even by a single voter. In this chapter, we proposed to quantify manipulability in practical scenarios and measure its effects on overall welfare. We focused on establishing a path from one point to another in a multi-operator network. We compared two voting systems:

- *Range voting*: RV maximizes the global net revenue of the system, provided that the operators (the voters) give their sincere utilities (their net revenues) on the proposed candidate paths.
- *IRV*: based on weak orders of preference with successive rounds of elimination, IRV has the reputation of being less manipulable (Chamberlin et al., 1984; Walsh, 2010b), which we were able to confirm in our work.

Our study highlights the value of voting systems in the context of the Internet economic ecosystem where many competing players are involved, with the practical use case of a path selection from one point to another.

We also observed that not all voting systems are equivalent, quite the contrary. When operators can lie about their preferences, IRV is significantly better than Range voting at maximizing economic gain:

- The manipulability rate of IRV can be as low as 20% while, in the same scenario, RV is close to 100%.

- With IRV, operators do not need to give all the information, the order of preference with approval threshold is enough, while for RV, they have to give all the information on their costs.
- Although IRV is not inherently designed to aim for the economically optimal choice, it comes very close in practice. The price of non-manipulability is low (of the order of 5% in our study).
- As far as manipulations are concerned, the degradation of economic efficiency is limited with IRV, whereas it is very important with RV.

Future Work

The work presented in this appendix opens up many opportunities for future work:

- Analysis of the sensitivity of the results with respect to various parameters, cost and revenue models;
- Analysis of alternative voting systems;
- Identification of other use cases in the Internet context, with other types of ecosystems.

Indeed, the framework of voting systems can be useful in any situation where voters can be identified who need to decide between several options. We have shown in our case study that by choosing the voting system correctly, one can limit the manipulability by tactical voting of a coalition of voters and preserve revenue for the overall economic ecosystem.

Notations

Non-alphabetical symbols

$[\alpha, \beta[$	Real interval from α included to β excluded (French convention).
$[\![j, k]\!]$	Integer interval from j to k included.
$\lfloor \alpha \rfloor$	Floor function of the real number α.
$\lceil \alpha \rceil$	Ceiling function of the real number α.
$\lvert \mathcal{A}(v) \rvert$	Number of voters v satisfying assertion $\mathcal{A}(v)$.
$\pi(A \mid B)$	Conditional probability of event A knowing B.

Greek alphabet

μ	The distribution of variable P (unless otherwise stated).
π	A *culture* over the electoral space Ω. More generally, a probability measure.
$\tau^{\pi}_{\mathrm{CM}}(f)$	Coalitional manipulability rate of the voting system f in culture π.
Ω	Set $\prod_{v \in \mathcal{V}} \Omega_v$ of possible configurations ω. Also used as a shorthand for an electoral space $(V, C, \Omega, \mathrm{P})$.
Ω_M	Set of possible states ω_M for voters in a set M.
Ω_v	Set of possible states ω_v for voter v.

Latin alphabet

$C \in \mathbb{N} \setminus \{0\}$	Number of candidates.
$\mathcal{C}$	Set $[\![1, C]\!]$ of indexes of candidates.
$\mathrm{card}(E)$	Cardinality of set E.
$D(\omega)$	Weighted majority matrix (matrix of duels) of ω. The coefficient of indexes c and d is denoted $D_{cd}(\omega)$ or simply D_{cd}.

f	A state-based voting system (SBVS), i.e. a function $\Omega \to \mathcal{C}$. In the case of a general voting system, f denotes its counting function $\mathcal{S}_1 \times \ldots \times \mathcal{S}_V \to \mathcal{C}$.
f^*	Condorcification of f.
$f^{\text{adm}}, f^{!\text{adm}}$	Condorcification variants of f based on the notion of Condorcet-admissible candidate.
$f^{\text{weak}}, f^{!\text{weak}}$	Condorcification variants of f based on the notion of weak Condorcet winner.
f^{rel}	Relative Condorcification of f.
$f^{\mathcal{M}}$	$\mathcal{M}$-Condorcification of f.
f_y	Slice of f by a slicing method y.
$c \,\mathrm{I}_v\, d$	Voter v is indifferent between c and d.
Id	The identity function (the context precises in which set).
$\mathcal{L}_\mathcal{C}$	Set of strict total orders over $\mathcal{C}$.
$\mathcal{M}$	A family of collections of coalitions.
$\mathcal{M}_c \in \mathcal{P}(\mathcal{P}(\mathcal{V}))$	A collection of coalitions that are said *winning* for candidate c.
$\text{Manip}_\omega(\text{w} \to c)$	Set of voters preferring c to w. In short, $\text{Manip}(\text{w} \to c)$.
CM_f	Set of configurations ω where f is manipulable (or indicator function of this set).
$\text{mean}(x_1, \ldots, x_k)$	Arithmetical average of $x_1, \ldots, x_k$.
P	Function $\Omega \to \mathcal{R}$ that, to the state ω of the population, associates profile $\mathrm{P}(\omega) = (\mathrm{P}_1(\omega_1), \ldots, \mathrm{P}_V(\omega_V))$.
$c \,\mathrm{P}_v\, d$	Voter v prefers c to d.
$c \,\mathrm{P}_{\text{abs}}\, d$	c has an absolute victory against d: $\lvert c \,\mathrm{P}_v\, d \rvert > \frac{V}{2}$.
$c \,\mathrm{P}_{\text{rel}}\, d$	c has a relative victory against d: $\lvert c \,\mathrm{P}_v\, d \rvert > \lvert d \,\mathrm{P}_v\, c \rvert$.
$c \,\mathrm{P}_{\mathcal{M}}\, d$	c has an $\mathcal{M}$-victory against d: $\{v \text{ s.t. } c \,\mathrm{P}_v\, d\} \in \mathcal{M}_c$.
$c \,\mathrm{MP}_v\, d$	Voter v prefers c to d and vice versa (impossible if P_v is antisymmetric).
$c \,\mathrm{PP}_v\, d$	Voter v prefers c to d but not d to c (synonym of $c \,\mathrm{P}_v\, d$ if P_v is antisymmetric).
$\mathcal{R}$	Set ${\mathcal{R}_\mathcal{C}}^V$ of which an element (profile) represents binary relations of preference for the whole population of voters.
$\mathcal{R}_\mathcal{C}$	Set of binary relations over $\mathcal{C}$.
$\text{Sinc}_\omega(\text{w} \to c)$	Set of voters who do not prefer c to w. In short, $\text{Sinc}(\text{w} \to c)$.
$V \in \mathbb{N} \setminus \{0\}$	Number of voters.
$(V, C, \Omega, \mathrm{P})$	An electoral space. In short, Ω.
$\mathcal{V}$	Set $[\![1, V]\!]$ of indexes of voters.
$\text{vect}(E)$	Linear span of E, where E is a part of a vector space.
$\mathcal{W}_\mathcal{C}$	Set of strict weak orders over $\mathcal{C}$.
$\mathcal{Y}$	Set $\prod_{v \in \mathcal{V}} \mathcal{Y}_v$ of slicing methods y for the whole population of voters.
$\mathcal{Y}_v$	Set $\{y_v : \mathrm{P}(\Omega_v) \to \Omega_v \text{ s.t. } \mathrm{P}_v \circ y_v = \text{Id}\}$ of slicing methods y_v for voter v.

Acronyms and abbreviations

AV	Approval voting.
Bald.	Baldwin's method.
Bor.	Borda's method.
Buck.	Bucklin's method.
CIRV	Condorcification of IRV.
CM	Coalition manipulation / manipulable.
Cond	Condorcet criterion.
Coo.	Coombs' method.
CSD	Condorcet's method with sum of defeats.
EB	Exhaustive ballot.
IB	Iterated Bucklin's method.
ICM	Ignorant-coalition manipulation / manipulable.
iff	If and only if.
IgnMC	Ignorant majority coalition criterion.
IIA	Independence of irrelevant alternatives.
IM	Individual manipulation / manipulable.
InfMC	Informed majority coalition criterion.
IRV	Instant-runoff voting.
IRVA	Instant-runoff voting based on the average.
IRVD	Instant-runoff voting with duels.
ITR	Instant two-round system.
Kem.	Kemeny's method.
KR	Kim-Roush method.
MajBal	Majority ballot criterion.
MajFav	Majority favorite criterion.
MajUniBal	Majority unison ballot criterion.
Max.	Maximin.
MJ	Majority Judgment.
Nan.	Nanson's method.
Plu.	Plurality.
RP	Ranked Pairs method.
RV	Range voting.
s.t.	Such that.
SBVS	State-based voting system.
Sch.	Schulze's method.
SVVAMP	Simulator of Various Voting Algorithms in Manipulating Populations.
TM	Trivial manipulation / manipulable.

TR	Two-round system.
UM	Unison manipulation / manipulable.
VMF	Von Mises–Fisher.

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